A quick tour of differential geometry

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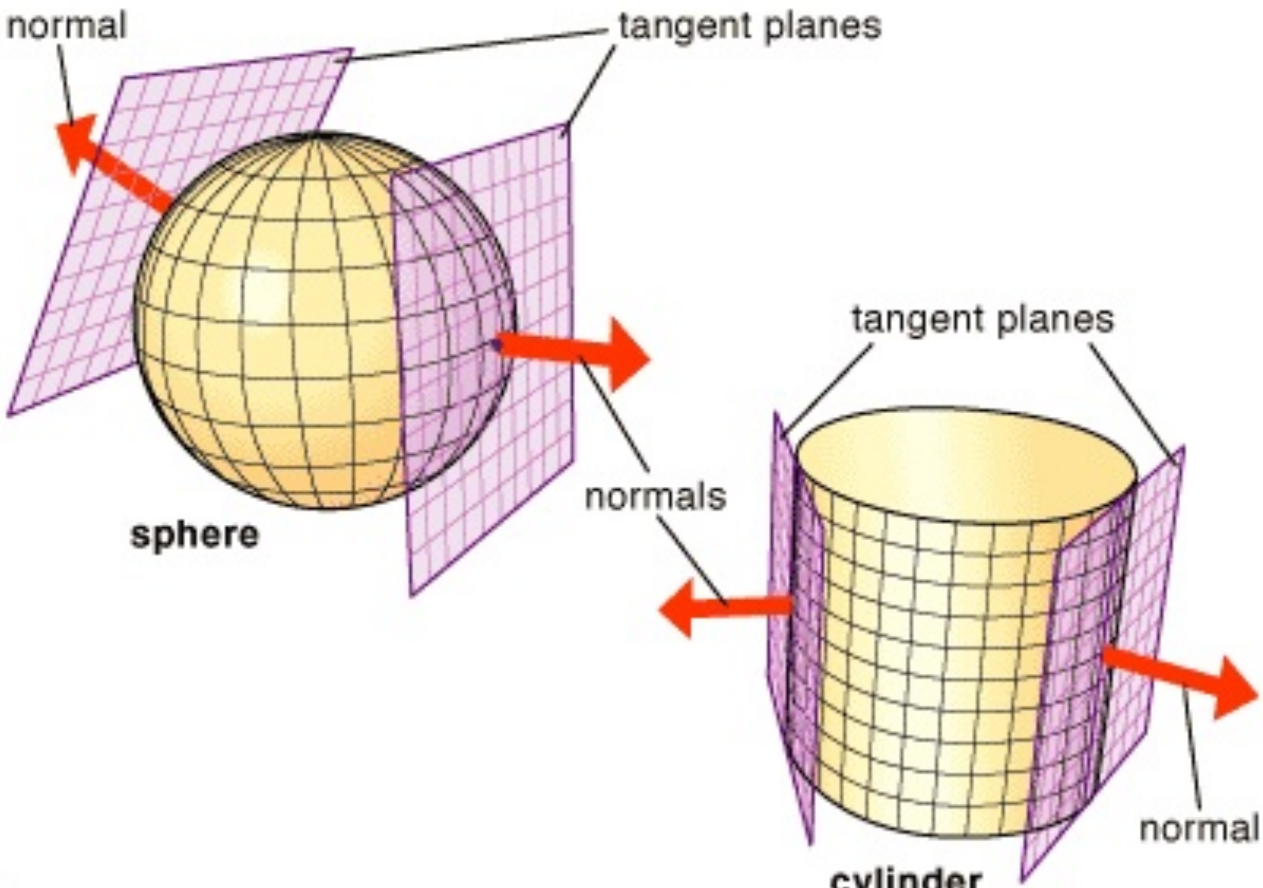
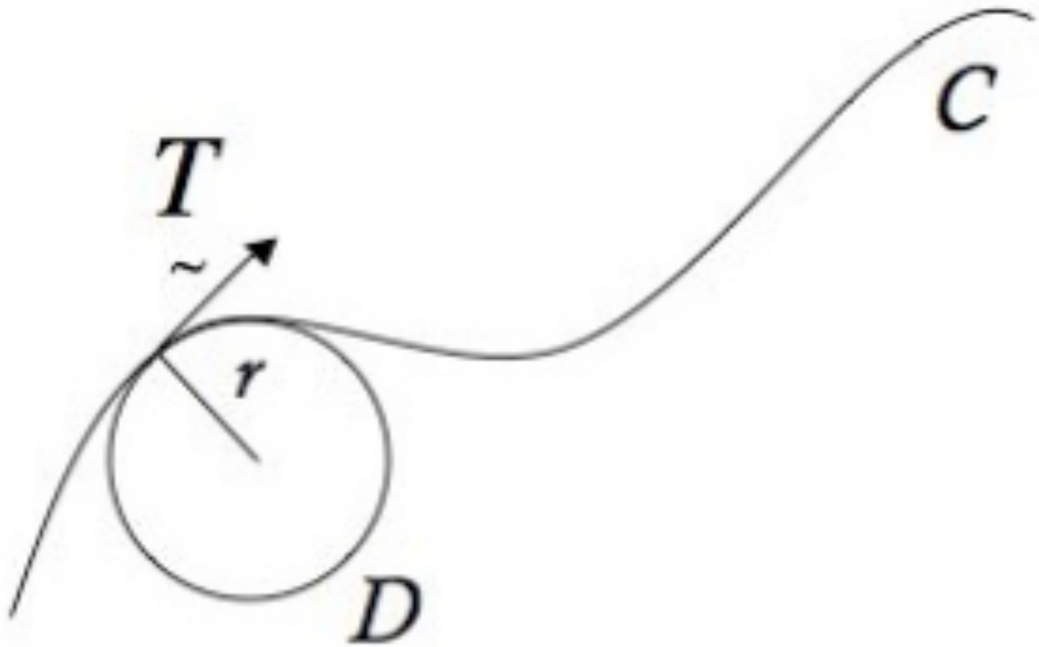

normal
tangent planes
sphere
normals
tangent planes
cylinder
normal

There are two possible definitions of curves in 2 dimensions:

1. Level sets (implicit equations): for a given mapping $f : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \mathbb{R}$, $f^{-1}(c)$ is the set of all points that map to the same number, c, and this set of points defines a curve. The only constraint is that $\vec{\nabla} f \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for points on the curve.
   - Example: for circles, $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto x^2+y^2 \implies f^{-1}(25)$ is the set $x^2+y^2 = 25$, which corresponds to a circle of radius 5
2. Parametric: we have $x(t)$ and $y(t)$, with the constraint that $\left\| \begin{matrix} \dot{x} \\ \dot{y} \end{matrix} \right\| \neq 0$
   - Example: a circle of radius a is given as $x(t) = a\cos t$ and $y(t) = a\sin t$
   - Example: an ellipse is given as $x(t) = a\cos t$ and $y(t) = b\sin t$



Use of osculating circle to determine curvature.

Surfaces in 3D can be defined in two ways:

1. Implicitly - level set definition

   For the mapping $f : \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \mapsto \mathbb{R}$, $f^{-1}(c)$ is a set which we define to be a surface; the only constraint is that $\vec{\nabla} f \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ for points on the surface

2. Parametric definition

   We define $\begin{bmatrix} X(u,v) \\ Y(u,v) \\ Z(u.v) \end{bmatrix}$ to be a 2D surface in 3D, with the constraint that

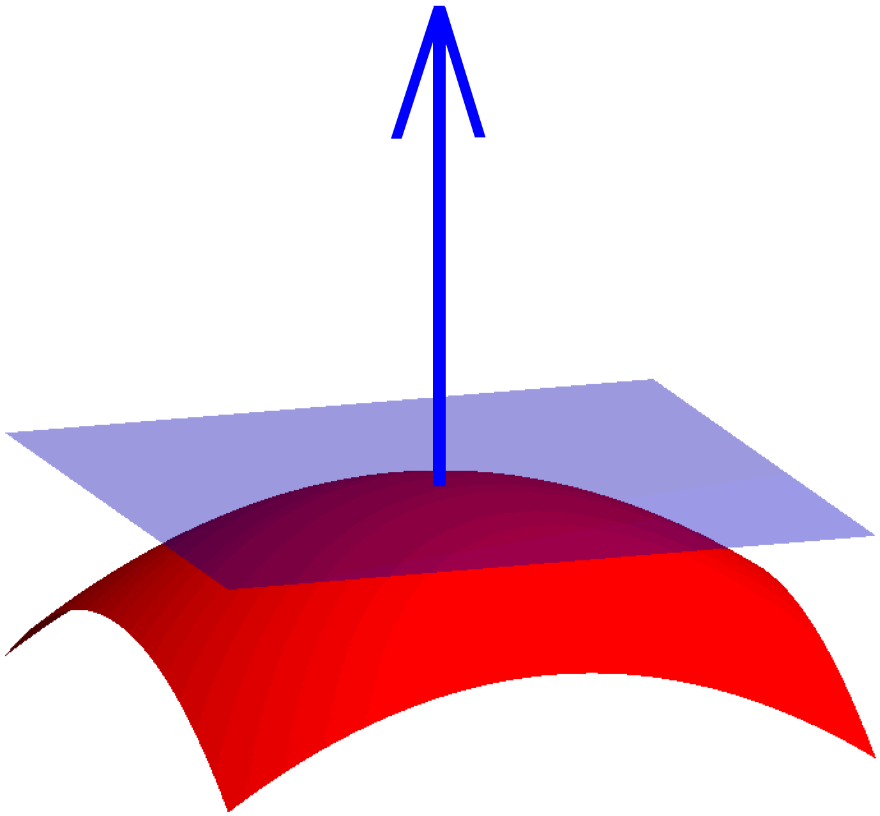
the derivatives with respect to u and with respect to v should not be in the same direction (so that the surface normal can be calculated).

Two ways of defining surfaces

Indeed we can use generalize these two ways of defining surfaces to n dimensions:

Level set definition of surfaces: A surface of dimension n in R^{n+1} is a non-empty subset S of R^{n+1} of the form $S = f^{-1}(c)$ where f is a smooth function $U \subset \mathrm{R}^{n+1} \to \mathrm{R}$ with the property that $\bigtriangledown f(p) \neq \mathbf{0}$. Picking $n = 1$ above defines plane curves. Picking $n = 2$ defines 2-surfaces.

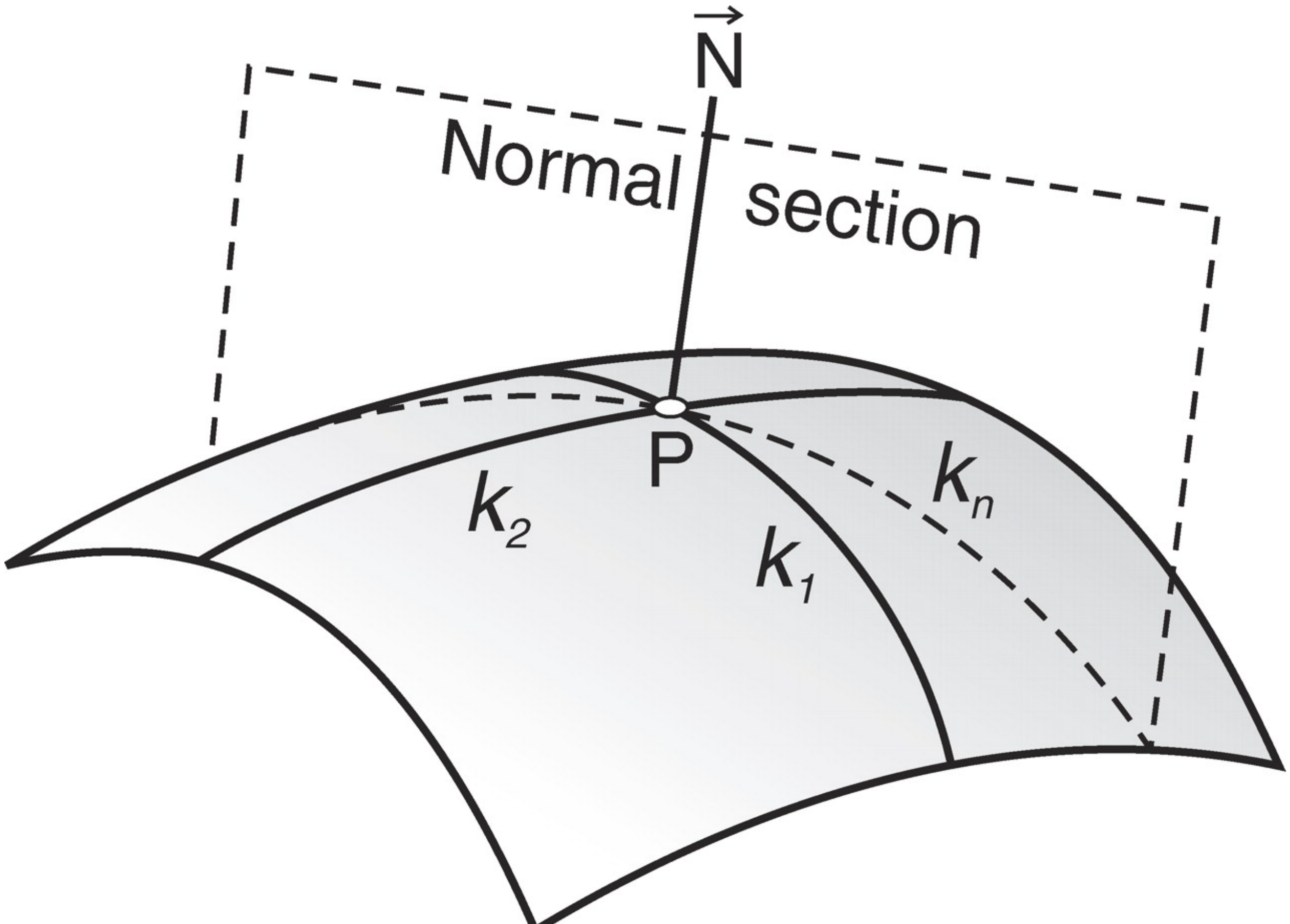
Parametrized patch definition of surfaces: A parametrized *n-surface* in R^{n+1} is the image of a smooth map $\varphi : U \to \mathrm{R}^{n+1}$, where U is a connected open set in R^n which is such that its derivative $d\varphi_p$ is non-singular (has rank n) for each $p \in U$. The image $d\varphi_p$ is the tangent space to φ corresponding to the point $p \in U$.



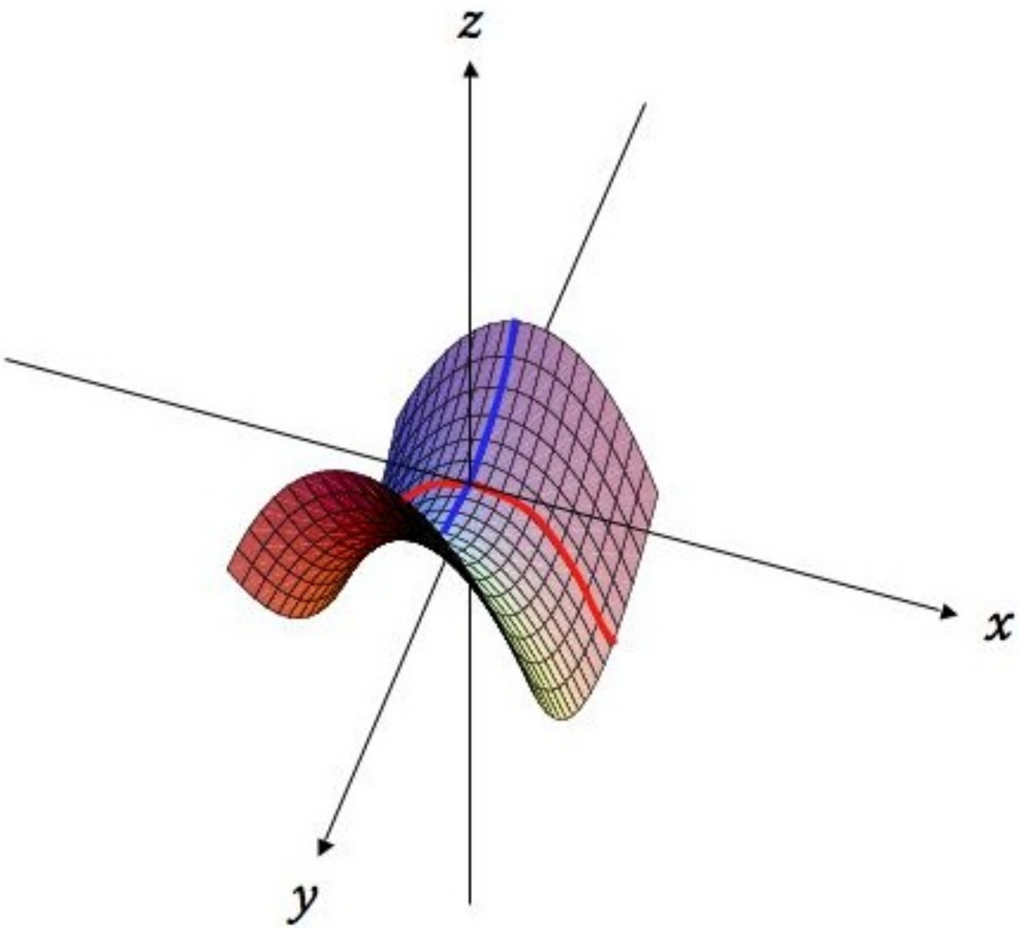
The tangent space has two equivalent characterizations:

- The plane at point $\mathbf{p}$ which has perpendicular $\mathbf{n}$.

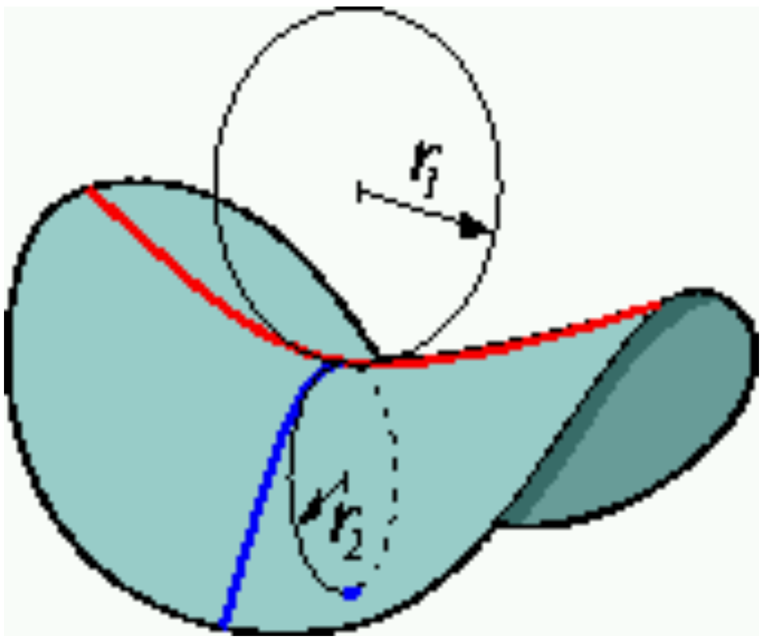
- The space of tangent directions of curves on the surface that go through $\mathbf{p}$. To illustrate this characterization, consider the parameterized curve $\alpha(\mathbf{t})$, where $\alpha(\mathbf{0}) = \mathbf{p}$ and $\alpha'(\mathbf{t})$ is in the direction of the tangent. If you consider multiple such curves, you will obtain a set of tangent vectors at $\mathbf{p}$ that all lie on a single plane.

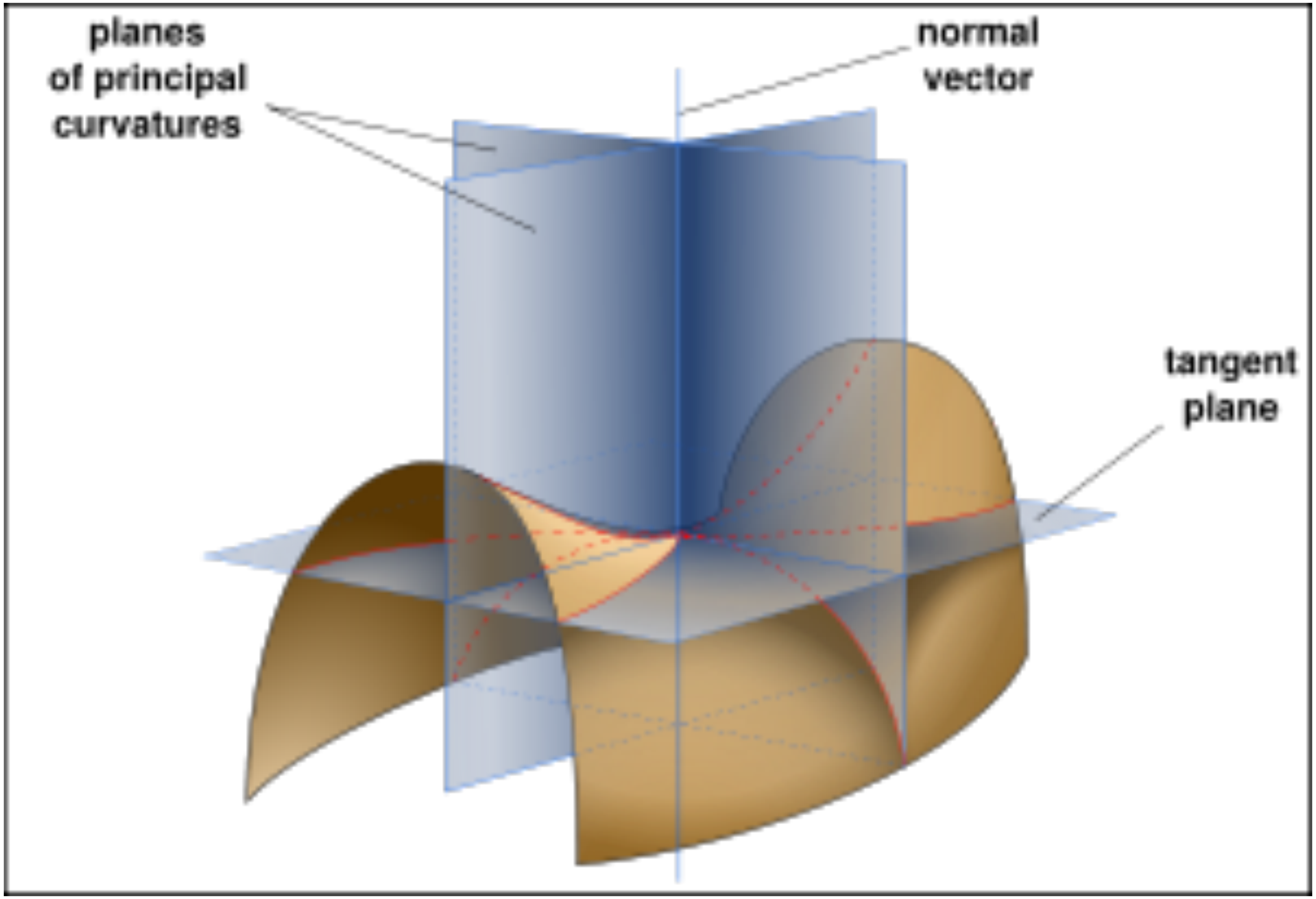


$\vec{N}$
Normal section
P
k_2
k_1
k_n


z
x
y

Defining normal curvature




planes
of principal
curvatures
normal
vector
tangent
plane

Principal curvatures for different surfaces

- Plane
- Cylinder
- Sphere
- Elliptic patch
- Hyperbolic patch

Principal curvatures for different surfaces

- Plane – both zero
- Cylinder – one zero, other is 1/r
- Sphere – both are 1/r
- Elliptic patch – both have same sign
- Hyperbolic patch – have opposite signs

Gaussian and Mean Curvature

The **Gaussian curvature** at a point on an embedded smooth surface given locally by the equation

$z = F(x,y)$

in $\mathbf{E}^3$, is defined to be the product of the principal curvatures at the point;[5] the **mean curvature** is defined to be their average. The principal curvatures are the maximum and minimum curvatures of the plane curves obtained by intersecting the surface with planes normal to the tangent plane at the point. If the point is (0, 0, 0) with tangent plane $z = 0$, then, after a rotation about the z-axis setting the coefficient on xy to zero, F will have the Taylor series expansion

$$F(x,y) = \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2 + ...$$

The principal curvatures are k_1 and k_2 in this case, the Gaussian curvature is given by

$$K = k_1 \cdot k_2.$$

and the mean curvature by

$$K_m = \tfrac{1}{2}(k_1 + k_2).$$

Surfaces of negative, zero and positive Gaussian Curvature