On Transformations

Lecture 3

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Pose and Shape

- *Pose:* The position and orientation of the object with respect to the camera. This is specified by 6 numbers (3 for its translation, 3 for rotation). For example, we might consider the coordinates of the centroid of the object relative to the center of projection, and the rotation of a coordinate frame on the object with respect to that of the camera.
- *Shape:* The coordinates of the points of an object relative to a coordinate frame on the object. These remain invariant when the object undergoes rotations and translations.

Definition 1 *Euclidean transformations (also known as isometries) are transformations that preserve distances between pairs of points.*

$$||\psi(\mathbf{a}) - \psi(\mathbf{b})|| = ||\mathbf{a} - \mathbf{b}|| \tag{3.1}$$

Translations, $\psi(\mathbf{a}) = \mathbf{a} + \mathbf{t}$, are isometries, since

$$||\psi(\mathbf{a}) - \psi(\mathbf{b})|| = ||\mathbf{t} + \mathbf{a} - (\mathbf{t} + \mathbf{b})|| = ||\mathbf{a} - \mathbf{b}|| \tag{3.2}$$

We now define orthogonal transformations; these constitute another major class of isometries.

Definition 2 *A linear transformation:* $\psi(\mathbf{a}) = \mathbf{A}\mathbf{a}$, *for some matrix* $\mathbf{A}$.

Definition 3 *Orthogonal transformations are linear transformations which preserve inner products.*

$$\mathbf{a} \cdot \mathbf{b} = \psi(\mathbf{a}) \cdot \psi(\mathbf{b}) \tag{3.3}$$

Rotations and reflections are examples of orthogonal transformations

Rigid body motions
(Euclidean transformations / isometries)

- Theorem: Any rigid body motion can be expressed as an orthogonal transformation followed by a translation.

$$\psi(\mathbf{a}) = \mathbf{A}\mathbf{a} + \mathbf{t}$$

A is an orthogonal matrix

Property 1 *Orthogonal transformations preserve norms.*

$$\mathbf{a} \cdot \mathbf{a} = \psi(\mathbf{a}) \cdot \psi(\mathbf{a}) \implies ||\mathbf{a}|| = ||\psi(\mathbf{a})|| \tag{3.4}$$

Property 2 *Orthogonal transformations are isometries.*

$$(\psi(\mathbf{a}) - \psi(\mathbf{b})) \cdot (\psi(\mathbf{a}) - \psi(\mathbf{b})) \stackrel{?}{=} (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \tag{3.5}$$

$$||\psi(\mathbf{a})||^2 + ||\psi(\mathbf{b})||^2 - 2(\psi(\mathbf{a}) \cdot \psi(\mathbf{b})) \stackrel{?}{=} ||\mathbf{a}||^2 + ||\mathbf{b}||^2 - 2(\mathbf{a} \cdot \mathbf{b}) \tag{3.6}$$

By property 1,

$$||\psi(\mathbf{a})||^2 = ||\mathbf{a}||^2 \tag{3.7}$$

$$||\psi(\mathbf{b})||^2 = ||\mathbf{b}||^2. \tag{3.8}$$

By definition 3,

$$\psi(\mathbf{a}) \cdot \psi(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{3.9}$$

Thus, equality holds.

Orthogonal Matrices

Let ψ be an orthogonal transformation whose action we can represent by matrix multiplication, $\psi(\mathbf{a}) = \mathbf{A}\mathbf{a}$. Then, because it preserves inner products:

$$\psi(\mathbf{a}) \cdot \psi(\mathbf{b}) = \mathbf{a}^T \mathbf{b}\,. \tag{3.10}$$

By substitution,

$$\begin{aligned} \psi(\mathbf{a}) \cdot \psi(\mathbf{b}) &= (\mathbf{A}\mathbf{a})^T (\mathbf{A}\mathbf{b}) & (3.11)\\ &= \mathbf{a}^T \mathbf{A}^T \mathbf{A}\mathbf{b}\,. & (3.12) \end{aligned}$$

Thus,

$$\mathbf{a}^T\mathbf{b} = \mathbf{a}^T\mathbf{A}^T\mathbf{A}\mathbf{b} \implies \mathbf{A}^T\mathbf{A} = \mathbf{I} \implies \mathbf{A}^T = \mathbf{A}^{-1}\,. \tag{3.13}$$

Note that $\det(\mathbf{A})^2 = 1$ which implies that $\det(\mathbf{A}) = +1$ or -1. Each column of $\mathbf{A}$ has norm 1, and is orthogonal to the other column.

Orthogonal Matrices in 2D

In 2D, these constraints put together force $\mathbf{A}$ to be one of two types of matrices.

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation, det}=+1} \text{or} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}}_{\text{reflection, det}=-1}$$

Under a rotation by angle θ,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

The reflection matrix above corresponds to reflection around the line with angle $\frac{\theta}{2}$ (verify). Note that two rotations one after the other give another rotation, while two reflections give us a rotation.

Orthogonal Matrices in 3D

Let us now construct some examples in 3D. Just as in 2D, rotations are characterized by orthogonal matrices with det $= +1$. For orthogonal matrices, each column vector has length 1, and the dot product of any two different columns is 0. This gives rise to six constraints (3 pairwise dot product constraints, and 3 length constraints), so for a 3 dimensional rotation matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.14}$$

with 9 total parameters, there are really only three free parameters. There

- Rotation about z-axis by θ:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotation about x-axis by θ:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Parameterizing Rotations in 3D

Recall that rotation matrices have the property that each column vector has length 1 and the dot product of any 2 different columns is 0. These 6 constraints leave only 3 degrees of freedom. Here are some alternative notations used to represent orthogonal matrices in 3-D:

- Euler angles which specify rotations about 3 axes
- Axis plus amount of rotation
- Quaternions which generalize complex numbers from 2-D to 3-D. (Note, a complex number can represent a rotation in 2-D)

We will use the axis and rotation as the preferred representation of an orthogonal matrix: s, θ, where s is the unit vector of the axis of rotation and θ is the amount of rotation.

Skew symmetric matrices can be used to represent "cross" products or vector products. Recall:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

We define $\widehat{\mathbf{a}}$ as:

$$\widehat{\mathbf{a}} \stackrel{def}{=} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

Thus, multiplying $\widehat{\mathbf{a}}$ by any vector gives:

$$\begin{aligned} \widehat{\mathbf{a}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} &= \begin{bmatrix} -a_3b_2 + a_2b_3 \\ a_3b_1 - a_1b_3 \\ -a_2b_1 + a_1b_2 \end{bmatrix} \\ &= \mathbf{a} \wedge \mathbf{b} \end{aligned}$$

Consider now, the equation of motion of a point q on a rotating body:

$$\dot{\mathbf{q}}(t) = \omega \wedge \mathbf{q}(t)$$

where the direction of ω specifies the axis of rotation and $\|\omega\|$ specifies the angular speed. Rewriting with $\widehat{\omega}$

$$\dot{\mathbf{q}}(t) = \widehat{\omega}\mathbf{q}(t)$$

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The solution to this differential equation uses the matrix exponential

$$\mathbf{q}(t) = e^{\widehat{\omega}t}\mathbf{q}(0)$$

Where,

$$e^{\widehat{\omega}t} = \mathbf{I} + \widehat{\omega}t + \frac{(\widehat{\omega}t)^2}{2!} + \frac{(\widehat{\omega}t)^3}{3!} + \ldots$$

Collecting the odd and even terms in the above equation, we get to **Roderigues Formula** for a rotation matrix $\mathbf{R}$.

$$\begin{aligned}\mathbf{R} &= e^{\phi \widehat{\mathbf{s}}} \\ &= \mathbf{I} + \sin\phi\, \widehat{\mathbf{s}} + (1 - \cos\phi)\widehat{\mathbf{s}}^2\end{aligned}$$

Here $\mathbf{s}$ is a unit vector along ω and $\phi = \|\omega\| t$ is the total amount of rotation. Given an axis of rotation, $\mathbf{s}$, and amount of rotation ϕ we can construct $\widehat{\mathbf{s}}$ and plug it in.

The composition of two isometries is an isometry

$$\psi_1(\mathbf{a}) = \mathbf{A}_1\mathbf{a} + \mathbf{t}_1 \qquad \psi_2(\mathbf{a}) = \mathbf{A}_2\mathbf{a} + \mathbf{t}_2.$$

$$\begin{aligned}
\psi_1 \circ \psi_2(\mathbf{a}) &= \mathbf{A}_1(\mathbf{A}_2\mathbf{a} + \mathbf{t}_2) + \mathbf{t}_1 \\
&= \mathbf{A}_1\mathbf{A}_2\mathbf{a} + \mathbf{A}_1\mathbf{t}_2 + \mathbf{t}_1 \\
&= (\mathbf{A}_1\mathbf{A}_2)\mathbf{a} + (\mathbf{A}_1\mathbf{t}_2 + \mathbf{t}_1) \\
&= \mathbf{A}_3\mathbf{a} + \mathbf{t}_3
\end{aligned}$$

where $\mathbf{A}_3 = \mathbf{A}_1\mathbf{A}_2$ and $\mathbf{t}_3 = \mathbf{A}_1\mathbf{t}_2 + \mathbf{t}_3$.

Isometries constitute a group.

identity

inverse

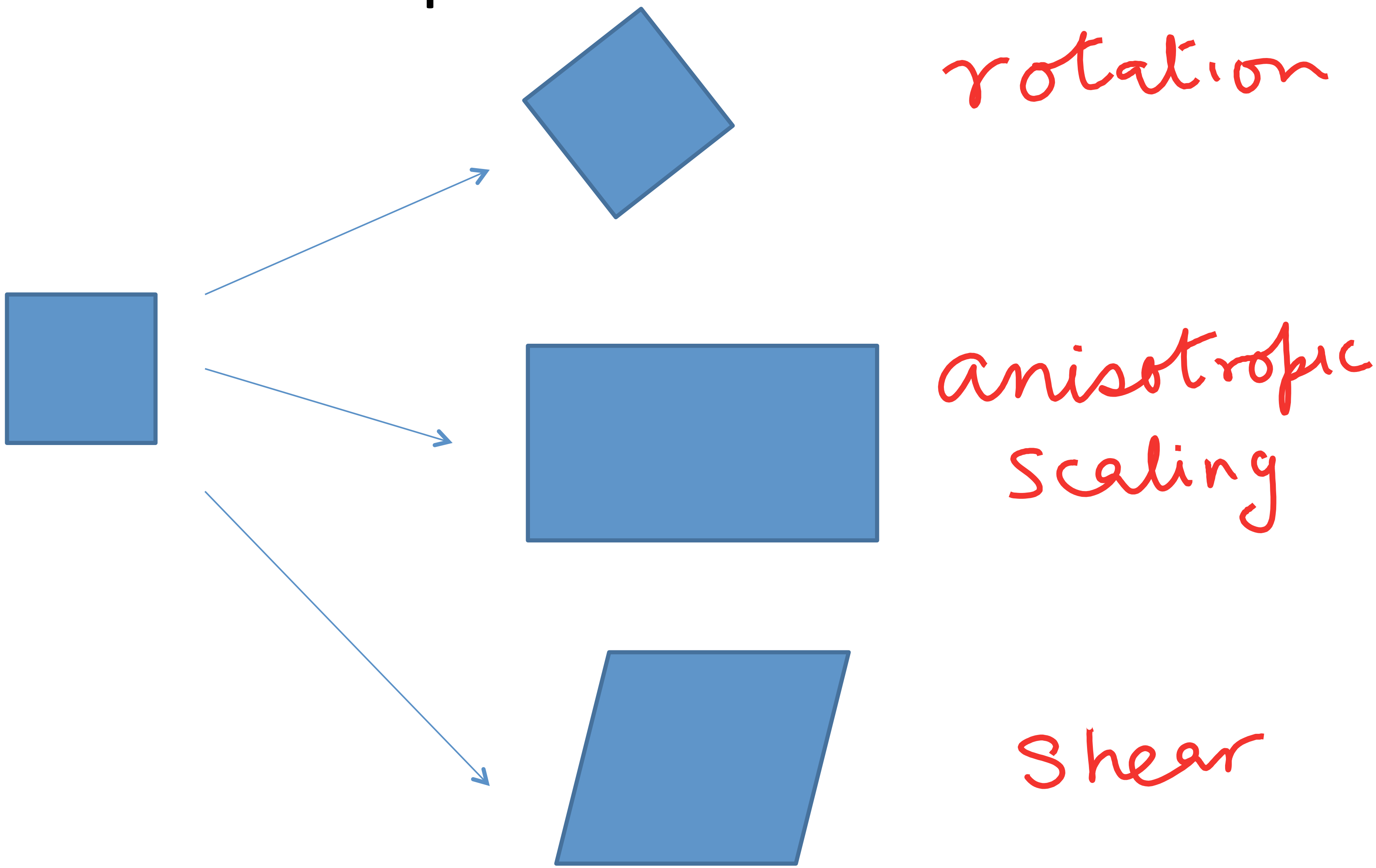
associative

Affine transformations

- Definition: An affine transformation is a nonsingular linear transformation followed by a translation.

$$\psi(\mathbf{a}) = \mathbf{A}\mathbf{a} + \mathbf{t}$$

Some examples of affine transforms…



Number of parameters required to specify isometry vs. affine transform

	ISOMETRY	AFFINE
In 2D	$1+2=3$	$4+2=6$
In 3D	$3+3=6$	$9+3=12$

Invariants under transformation (Properties that remain unchanged)

EUCLIDEAN

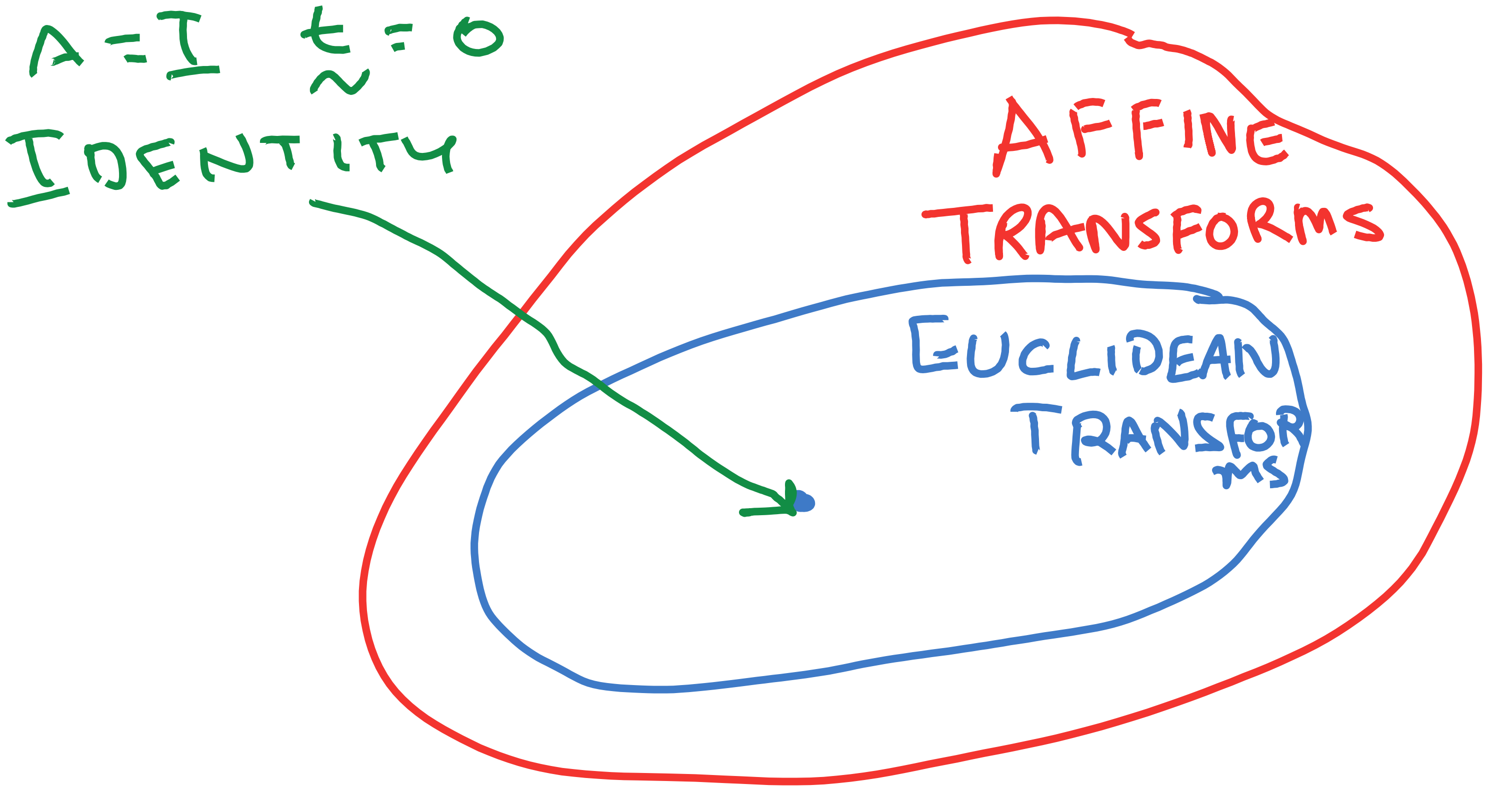
- Lengths
- Angles
- Area
- …

AFFINE

- Parallelism
- Midpoints

(LENGTHS, ANGLES & AREAS ARE NOT AFFINE INVARIANTS)

The big picture ...



But are affine transforms as general as we need to be?

Projective Transformations

- Under perspective projection, parallel lines can map to lines that intersect. Therefore, this cannot be modeled by an affine transform!
- Projective transformations are a more general family which includes affine transforms and perspective projections.
- Projective transformations are linear transformations using homogeneous coordinates.
- We will study them later in the course.