Graphical Models in Computer Vision

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MAX-PLANCK-GESELLSCHAFT

Syllabus

11.04.2016	Introduction
18.04.2016	Graphical Models 1
25.04.2016	Graphical Models 2 (Sand 6/7)
02.05.2016	Graphical Models 3
09.05.2016	Graphical Models 4
23.05.2016	Body Models 1
30.05.2016	Body Models 2
06.06.2016	Body Models 3
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20.06.2016	Stereo
27.06.2016	Optical Flow
04.07.2016	Segmentation
11.07.2016	Object Detection 1
18.07.2016	Object Detection 2

Todays topic

- Recap
  - Belief Networks
  - Markov Networks & Markov Random Fields
  - Filter View
  - Factor Graphs
  - Belief Propagation on Trees
- Approximate Inference
  - Loopy Belief Propagation on General Graphs
  - Sampling

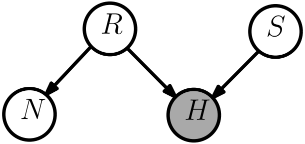
Belief Networks

Belief network

A belief network is a distribution of the form

$$p(x_1, \ldots, x_D) = \prod_{i=1}^{D} p(x_i \mid pa(x_i))$$

where $pa(x)$ denotes the parental variables of x

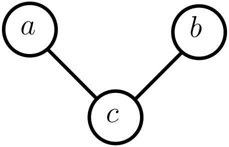


Markov Networks & Markov Random Fields

Markov Network

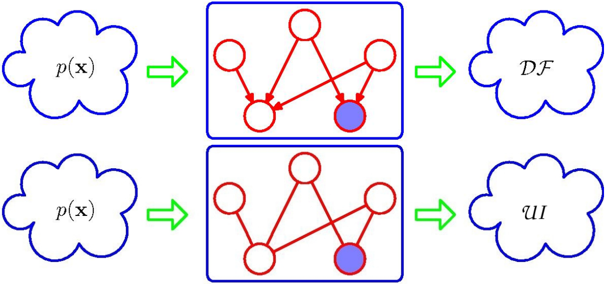
For a set of variables $\mathcal{X} = \{x_1, \ldots, x_D\}$ a Markov network is defined as a product of potentials over the maximal cliques $\mathcal{X}_c$ of the graph $\mathcal{G}$

$$p(x_1, \ldots, x_D) = \frac{1}{Z} \prod_{c=1}^{C} \phi_c(\mathcal{X}_c)$$



$$p(a, b, c) = \frac{1}{Z} \phi_{ac}(a, c) \phi_{bc}(b, c)$$

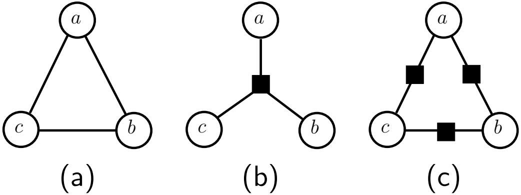
Filter View



- Each graph describes a family of probability distributions
- Extremes:
  - Fully connected, no constraints, all p pass
  - no connections, only product of marginals may pass

Factor Graphs

- Now consider we introduce an extra node (a square) for each factor:



- (a) Markov Network
- (b) Factor graph representation of $\phi(a, b, c)$
- (c) Factor graph representation of $\phi(a, b)\phi(b, c)\phi(c, a)$
- Both factor graphs have the same Markov network (b,c)$\Rightarrow$(a)

Factor Graphs

Factor Graph

Given a function

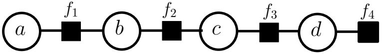
$$f(x_1, \ldots, x_n) = \prod_i \psi_i(\mathcal{X}_i)$$

the factor graph (FG) has a node (represented by a square) for each factor $\psi_i(\mathcal{X}_i)$ and a variable node (represented by a circle) for each variable x_j
When used to represent a distribution

$$p(x_1, \ldots, x_n) = \frac{1}{Z} \prod_i \psi_i(\mathcal{X}_i)$$

a normalization constant Z is assumed.

Belief Propagation on a Chain



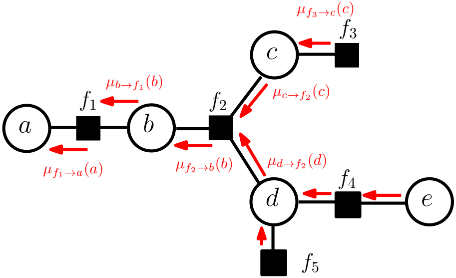
$$p(a,b,c,d) = f_1(a,b) f_2(b,c) f_3(c,d) f_4(d)$$

$$\begin{aligned} p(a,b,c) &= \sum_d p(a,b,c,d) \\ &= f_1(a,b) f_2(b,c) \underbrace{\sum_d f_3(c,d) f_4(d)}_{\mu_{d\to c}(c)} \end{aligned}$$

$$p(a,b) = \sum_c p(a,b,c) = f_1(a,b) \underbrace{\sum_c f_2(b,c) \mu_{d\to c}(c)}_{\mu_{c\to b}(b)}$$

Belief Propagation on a Tree

- Idea: compute messages



Belief Propagation: Finding Marginals

Sum-Product Algorithm for Trees

1. Initialize messages
2. Iterate from leaves of the tree to target variable:
   - Factor-to-variable messages ("sum-product")

$$\mu_{f \to x}(x) = \sum_{\mathcal{X}_f \setminus x} \phi_f(\mathcal{X}_f) \prod_{y \in \{\mathrm{ne}(f) \setminus x\}} \mu_{y \to f}(y)$$

   - Variable-to-factor messages (at target $\Rightarrow$ marginal!)

$$\mu_{x \to f}(x) = \prod_{g \in \{\mathrm{ne}(x) \setminus f\}} \mu_{g \to x}(x)$$

- $\mathcal{X}_f$: Variables that connect to factor f
- $ne(x)$: Factors that connect to variable x
- If all marginals are desired: 1) leaves $\to$ root 2) root $\to$ leaves

Belief Propagation: Find Most Likely State (MAP)

Max-Product Algorithm for Trees

1. Initialize messages
2. Iterate from leaves of the tree to target variable:
   - Factor-to-variable messages ("max-product")

$$\mu_{f \to x}(x) = \max_{\mathcal{X}_f \setminus x} \phi_f(\mathcal{X}_f) \prod_{y \in \{\text{ne}(f) \setminus x\}} \mu_{y \to f}(y)$$

   - Variable-to-factor messages (at target $\Rightarrow$ most likely state!)

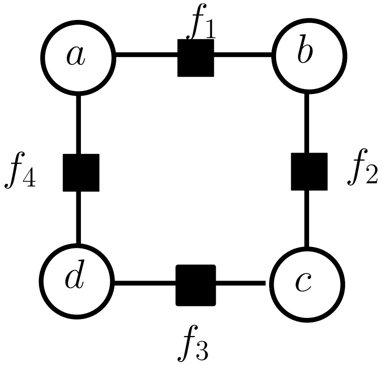
$$\mu_{x \to f}(x) = \prod_{g \in \{\text{ne}(x) \setminus f\}} \mu_{g \to x}(x)$$

- $\mathcal{X}_f$: Variables that connect to factor f
- $ne(x)$: Factors that connect to variable x
- If all states are of interest: 1) leaves $\to$ root 2) root $\to$ leaves

Fantastic, this is all very nice!

BUT ...

What if the graph is not singly connected?



$$p(a, b, c, d) = f_1(a, b) f_2(b, c) f_3(c, d) f_4(d, a)$$

What if the graph is not singly connected?

$$p(a, b, c, d) = f_1(a, b) f_2(b, c) f_3(c, d) f_4(d, a)$$

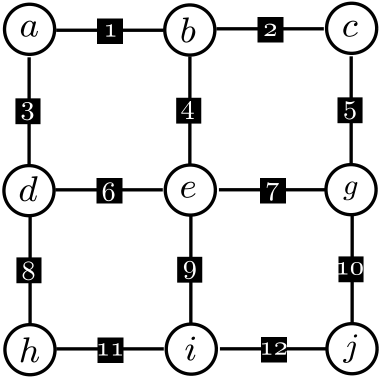
$$p(a, b, c) = \sum_d p(a, b, c, d) = f_1(a, b) f_2(b, c) \underbrace{\sum_d f_3(c, d) f_4(d, a)}_{\mu_{d \to a,c}(\color{red}{a,c})}$$

$$p(a, b) = \sum_c p(a, b, c) = f_1(a, b) \underbrace{\sum_c f_2(b, c)\, \mu_{d \to a,c}(\color{red}{a, c})}_{\mu_{c \to a,b}(\color{red}{a,b})}$$

$$p(a) = \sum_b p(a, b) = \sum_b f_1(a, b)\, \mu_{c \to a,b}(\color{red}{a, b})$$

2D messages now $\Rightarrow$ simply buy more RAM and wait a bit longer?

What if the graph gets bigger?



$$p(all) = f_1(a,b)f_2(b,c)f_3(a,d)f_4(b,e)f_5(c,g)f_6(d,e)$$
$$f_7(e,g)f_8(d,h)f_9(e,i)f_{10}(g,j)f_{11}(h,i)f_{12}(i,j)$$

What if the graph gets bigger?

$$p(all) = f_1(a,b)f_2(b,c)f_3(a,d)f_4(b,e)f_5(c,g)f_6(d,e) f_7(e,g)f_8(d,h)f_9(e,i)f_{10}(g,j)f_{11}(h,i)f_{12}(i,j)$$

$$p(all\backslash\{j\}) = f_1(a,b)f_2(b,c)f_3(a,d)f_4(b,e)f_5(c,g)f_6(d,e) f_7(e,g)f_8(d,h)f_9(e,i)f_{11}(h,i)\mu_{j\to i,g}(i,g)$$

$$p(all\backslash\{i,j\}) = f_1(a,b)f_2(b,c)f_3(a,d)f_4(b,e)f_5(c,g)f_6(d,e) f_7(e,g)f_8(d,h)\mu_{i\to e,h,g}(e,h,g)$$

3D messages now $\Rightarrow$ this is getting intractable!

How can we handle general loopy graphs?

Loopy Belief Propagation

- Messages are well defined for loopy graphs:

$$\mu_{x\to f}(x) = \prod_{g\in\{\mathrm{ne}(x)\setminus f\}} \mu_{g\to x}(x)$$

$$\mu_{f\to x}(x) = \sum_{\mathcal{X}_f\setminus x} \phi_f(\mathcal{X}_f) \prod_{y\in\{\mathrm{ne}(f)\setminus x\}} \mu_{y\to f}(y)$$

- Simply apply them to loopy graphs as well
- We loose exactness ($\Rightarrow$ approximate inference)
- No guarantee of convergence [Yedida et al. 2004]
- But often works astonishingly well in practice
- Same algorithm works for trees (exact) as well as for loopy graphs (approximate) $\Rightarrow$ Programming exercise

Loopy Belief Propagation

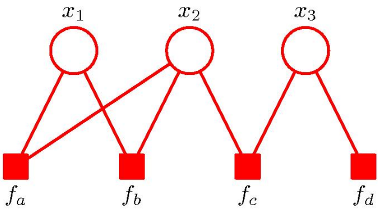
Outline of the algorithm:

- Initialize messages to fixed value (*e.g.*, uniform distribution)
- Perform message updates in fixed or random order
- After convergence: Calculate approximate marginals
- Note: LBP does not always converge
- There exist converging variants: TRW-S [Kolmogorov, PAMI 2006]

Loopy Belief Propagation

Which message passing schedule?

- Random or fixed order
- Popular choice:
  1. Factors $\rightarrow$ variables
  2. Variables $\rightarrow$ factors
  3. Repeat for N iterations
- Can be run in parallel as factor graph is bipartite:



Loopy Belief Propagation

Sum-Product Belief Propagation

- Goal: Compute marginals of distribution
- Multiplying many double-precision numbers is not a good idea
- Better use log messages $\lambda(x) = \log \mu(x)$:
  - Factor-to-variable messages:
    $\mu_{f\to x}(x) = \sum_{\mathcal{X}_f \setminus x} \phi_f(\mathcal{X}_f) \prod_{y\in\mathcal{X}_f \setminus x} \mu_{y\to f}(y)$
    $$\boxed{\lambda_{f\to x}(x) = \log \left(\sum_{\mathcal{X}_f \setminus x} \phi_f(\mathcal{X}_f) \exp \left\{ \sum_{y\in \mathrm{ne}(f)} \lambda_{y\to f}(y) \right\} \right)} \quad (1)$$
  - Variable-to-factor messages:
    $\mu_{x\to f}(x) = \prod_{g\in\{\mathrm{ne}(x)\setminus f\}} \mu_{g\to x}(x)$
    $$\boxed{\lambda_{x\to f}(x) = \sum_{g\in\{\mathrm{ne}(x)\setminus f\}} \lambda_{g\to x}(x)} \quad (2)$$
- $\sum_{\mathcal{X}_f \setminus x}$: Summation over all states in $\mathcal{X}_f \setminus x$
- $\sum_{y\in\mathrm{ne}(f)}$: Summation over all incoming messages
- To avoid numbers from getting too large, normalize $\lambda_{x\to f}(x)$ after the message update (Eq. 2), for example by subtracting its mean

Loopy Belief Propagation

Max-Product/Sum Belief Propagation

- Goal: Find most likely state (MAP state)
- Very similar to sum-product, only factor-to-variable message changes
- As before, we better use log messages $\lambda(x) = \log \mu(x)$:
  - Factor-to-variable messages:

    $\mu_{f\to x}(x) = \max_{\mathcal{X}_f \setminus x} \left[\phi_f(\mathcal{X}_f) \prod_{y\in\mathcal{X}_f\setminus x} \mu_{y\to f}(y) \right]$

    $$\boxed{\lambda_{f\to x}(x) = \max_{\mathcal{X}_f\setminus x} \left[\log \phi_f(\mathcal{X}_f) + \sum_{y\in \mathrm{ne}(f)} \lambda_{y\to f}(y) \right]} \quad (3)$$
  - Variable-to-factor messages:

    $\mu_{x\to f}(x) = \prod_{g\in\{\mathrm{ne}(x)\setminus f\}} \mu_{g\to x}(x)$

    $$\boxed{\lambda_{x\to f}(x) = \sum_{g\in\{\mathrm{ne}(x)\setminus f\}} \lambda_{g\to x}(x)} \quad (2)$$
- $\max_{\mathcal{X}_f\setminus x}$: Maximization over all states in $\mathcal{X}_f \setminus x$
- $\sum_{y\in\mathrm{ne}(f)}$: Summation over all incoming messages
- To avoid numbers from getting too large, normalize $\lambda_{x\to f}(x)$ after the message update (Eq. 2), for example by subtracting its mean

Loopy Belief Propagation

Unary and Pairwise Factor-to-Variable Messages

Factor-to-variable messages simplify as follows if you only consider unary or pairwise factors. Variable-to-factor messages don't simplify.

- **Sum-Product Belief Propagation:**
  - Unary factor $\phi_f(x)$:

$$\boxed{\lambda_{f\to x}(x) = \log \phi_f(x)} \tag{1}$$

  - Pairwise factor $\phi_f(x, y)$:

$$\boxed{\lambda_{f\to x}(x) = \log \left(\sum_y \phi_f(x, y) \exp \{\lambda_{y\to f}(y)\}\right)} \tag{1}$$

- **Max-Product Belief Propagation:**
  - Unary factor $\phi_f(x)$:

$$\boxed{\lambda_{f\to x}(x) = \log \phi_f(x)} \tag{3}$$

  - Pairwise factor $\phi_f(x, y)$:

$$\boxed{\lambda_{f\to x}(x) = \max_y [\log \phi_f(x, y) + \lambda_{y\to f}(y)]} \tag{3}$$

Note: The sum/max here run over all states of variable y!

Loopy Belief Propagation

Let's implement this now! Which data structures to use?

- A vector `variables` containing the #labels each variable can take
- A vector `factors`; each factor contains:
  - The variable id or id's of the variables it is connected to
  - A vector or matrix storing the factor values for all states
- A vector of `factor-to-variable messages` ($\lambda_{f \to x}$)
- A vector of `variable-to-factor messages` ($\lambda_{x \to f}$)
- Each message contains:
  - The id's of the involved `variables`, `factors` and input `messages` it depends on for enabling quick updates according to the formulas on the previous slide
  - The message log values themselves (a vector, length: #labels)
- `variables` and `factors` are the inputs to the algorithm
- `messages` are computed by the algorithm

Loopy Belief Propagation

Belief Propagation Algorithm (handles both cases)

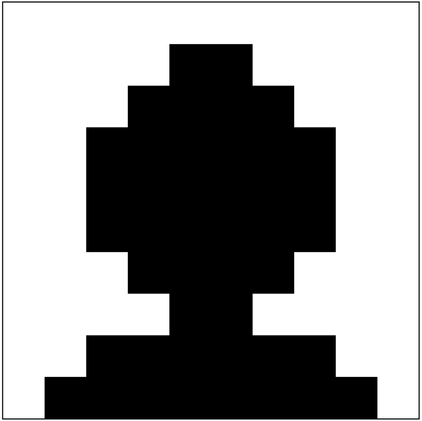
- Input: `variables` and `factors`
- Allocate all `messages`
- Initialize the `message` log values to 0 (=uniform distribution)
- For $N = 10$ iterations do
  - Update all `factor-to-variable messages` (Eq. 1 or Eq. 3)
  - Update all `variable-to-factor messages` (Eq. 2)
  - Normalize all `variable-to-factor messages`:
    $\mu_{x \to f}(x) \leftarrow \mu_{x \to f}(x) - \text{mean}\left(\mu_{x \to f}(x)\right)$
- Read off marginal or MAP state at each variable:

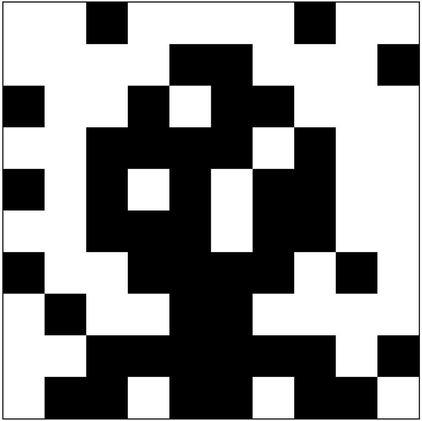
$$\lambda(x) = \sum_{g \in \{\text{ne}(x)\}} \lambda_{g \to x}(x)$$

$$p(x) = \exp\{\lambda(x)\} / \sum_x \exp\{\lambda(x)\}$$

$$x^* = \operatorname*{argmax}_x \sum_{g \in \{\text{ne}(x)\}} \lambda_{g \to x}(x)$$

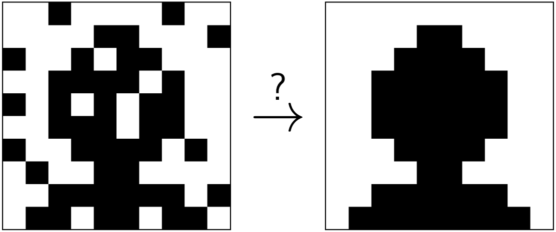
Imagine ...



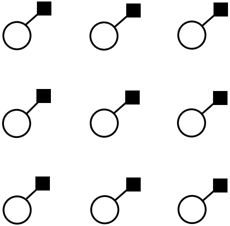


Denoising a Binary Image

Can we recover the original image from the noisy observation?

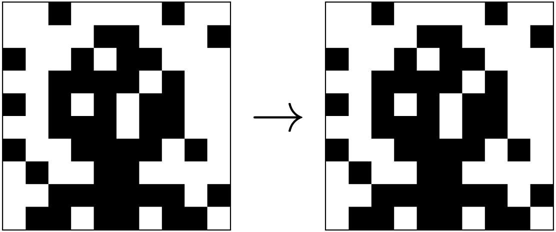


- Let us model this using a MRF!
- Variables: $x_1, \ldots, x_{100} \in \{0, 1\}$
- Unary potentials: $\psi_1(x_1), \ldots, \psi_{100}(x_{100})$
- $\psi_i(x_i) = [x_i = o_i]$ with observation o_i
- Log representation: $\psi_i(x_i) = \log f_i(x_i)$
  $p(x) = \frac{1}{Z} \prod_i f_i(x_i) = \frac{1}{Z} \exp\{\sum_i \psi_i(x_i)\}$



Denoising a Binary Image

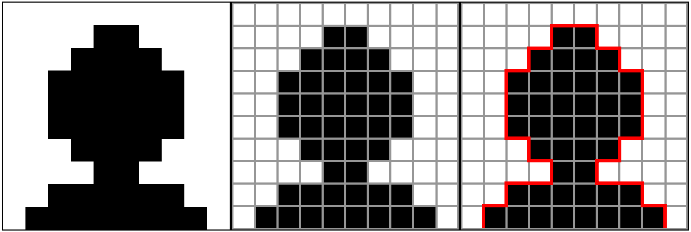
What will be the outcome of MAP inference with unary factors only?



- Maximizing a MRF with unary factors only is equivalent to maximizing each factor individually (no dependencies)
- Thus the result equals the observation

Denoising a Binary Image

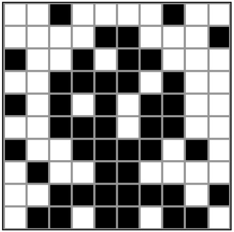
What can we do?



- Let us look at the clean image again!
- What prior knowledge do we have about this image?
- Smoothness! (Neighboring pixels tend to have the same label)
- Really? How many neighbors share / don't share their label?
- $10 \times 10 \times 2 - 20 = 180$ neighborhood relationships in total
- $34\times$ label transition $\Rightarrow$ $146\times$ same label

Denoising a Binary Image

Introducing a Smoothness Prior



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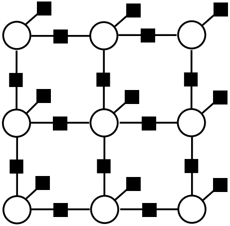


- Log representation:

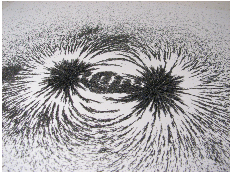
$$p(x) \propto \exp\left\{\sum_{i=1}^{100} \psi_i(x_i) + \sum_{i\sim j} \psi_{ij}(x_i, x_j)\right\}$$

- Variables: $x_1, \ldots, x_{100} \in \{0, 1\}$
- Unary potentials: $\psi_i(x_i) = [x_i = o_i]$ with pixel observation $o_i \in \{0, 1\}$
- Pairwise potentials: $\psi_{ij}(x_i, x_j) = \alpha \cdot [x_i = x_j]$
- Parameter α controls the strength of the smoothing / prior

Ising Model

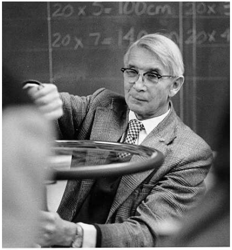
Ising Model (1924)

- Statistical mechanics
- Mathematical model of ferromagnetism
- Magnetic dipole moments of atomic spins
- Two states: $+1$ and -1, arranged in lattice
- Allows identification of phase transitions



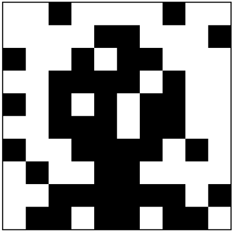
Ernst Ising (1900-1998)

- Studies in Göttingen, Bonn, Hamburg
- Investigated simple chain model
- Grid model solved in 1944 by Osanger
- School teacher (Caputh, Berlin)
- Escaped to US (Bradley University, Illinois)



Denoising a Binary Image

What will the MAP result look like?

 $\rightarrow$

- Programming exercise
- Play with smoothness parameters α
- How to set α in a principled fashion?
- Learn from training data! $\Rightarrow$ Next week ...
- Next: Approximate inference via sampling

So far:

- We learned about one particular deterministic approximation
- There are other deterministic techniques (overview at end of lecture)
- There is also another way of approaching approximate inference:

Sampling

Deterministic Approximation

- Approximate the model or inference procedure
- Retrieve a determ. solution to this approximation

Stochastic Approximation

- Use the true model / target distribution of interest
- Draw samples to approximate this distribution

Motivation: Sampling

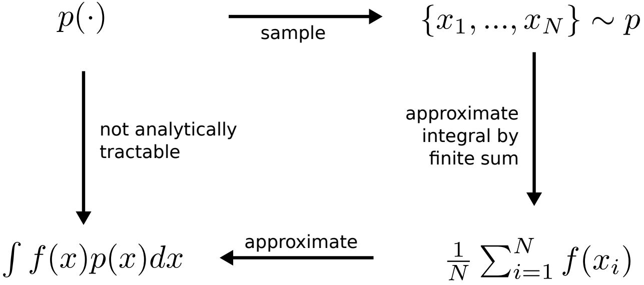
Many statistical problems involve solving analytically intractable integrals (for example in Bayesian inference with continuous variables and non-conjugate priors). Typical problems that can be solved with sampling:

- Normalization: $p(x|y) = \frac{p(y|x)p(x)}{\int p(y|x')p(x')dx'}$
- Marginalization: $p(x|y) = \int p(x, z|y)dz$
- Maximization: $x^* = \text{argmax}_x \, p(x|y)$ (no integral here)
- Expectation: $E_p(f(x)) = \int f(x)p(x)dx$

Examples for functions $f(x)$ in the latter case:

- The expectation: $\int xp(x)dx$
- The variance: $\int x^2 p(x)dx - \left(\int xp(x)dx\right)^2$
- The expected risk: $\int \text{risk}(x)p(x)dx$

Monte Carlo Approximation



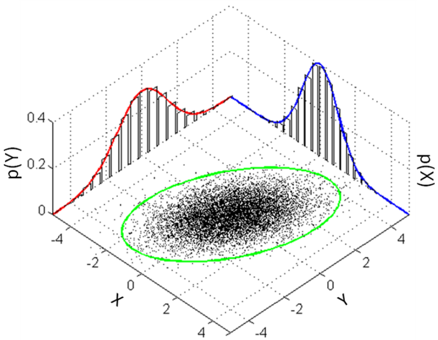
- The more samples we draw, the better the approximation:

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i) \xrightarrow{N\to\infty} \int f(x)p(x)dx$$

- The estimate is unbiased and will almost surely converge to the right value by the strong law of large numbers
- Difficulties: Obtaining uncorrelated samples for fast convergence

Basic Sampling Strategies

- For most (multivariate) standard distributions there exist good sampling algorithms that you can just call in Python/MATLAB
- Uniform, Gaussian, Poisson, Dirichlet, Discrete
- But those are usually not the distributions we are interested in
- Our distributions specified by a graphical model are more complex



So how to sample?

Let's look at the simple univariate case first

Discrete Case

- Assume distribution: $$p(x) = \begin{cases} 0.6 & x = 1 \\ 0.1 & x = 2 \\ 0.3 & x = 3 \end{cases}$$
- Calculate cumulant: $$c(y) = \sum_{x \leq y} p(x) = \begin{cases} 0.6 & y = 1 \\ 0.7 & y = 2 \\ 1.0 & y = 3 \end{cases}$$
- Draw $u \sim [0, 1]$ using pseudo-random number generator
- Find y such that: $c(y-1) < u \leq c(y)$
- Return state y as sample from p

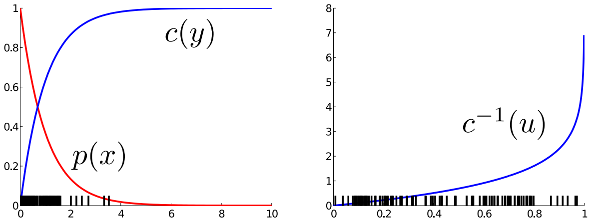
Continuous Case

- Similar to the discrete case
- Compute the cumulant function:

$$c(y) = \int_{-\infty}^{y} p(x)dx$$

- Sample $u \sim [0, 1] \Rightarrow$ compute $x = c^{-1}(u)$
- The integral $c(y)$ can be computed analytically or numerically

$$\text{For example:} \quad p(x) = \begin{cases} \exp(-x) & 0 \leq x, \\ 0 & \text{else} \end{cases}$$



Overview: Sampling Methods

- Inverse Transform
- Ancestral Sampling
- Rejection Sampling
- Importance Sampling
- Slice Sampling
- Markov Chain Monte Carlo
  - Metropolis-Hastings
  - Gibbs Sampling
  - Hybrid Monte Carlo

- Do I need to know them all?
- Yes! Most efficient technique depends on model/application
- Today "only" the ones in red ;)

Rejection Sampling

Rejection Sampling

- Suppose a $p(x)$ such that direct sampling is not tractable
- Furthermore assume we can evaluate $p(x)$ up to a constant (*e.g.*, Markov Networks!):

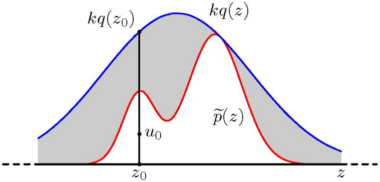
$$p(x) = \frac{1}{Z}\tilde{p}(x) = \frac{1}{Z}\prod_c \phi_c(\mathcal{X}_c)$$

- Sample from a proposal distribution $q(x)$
- Choose $q(\cdot)$ which we can easily sample and a k exists with

$$k\, q(x) \geq \tilde{p}(x)\ \forall x$$

Rejection Sampling

- Sample two random variables:
  1. $z_0 \sim q(x)$
  2. $u \sim [0, kq(z_0)]$ uniform
- Reject sample z_0 if $u_0 > \tilde{p}(z_0)$



- z_0 from q is accepted with probability $\tilde{p}(z)/kq(z)$

$$p(accept) = \int \frac{\tilde{p}(z)}{kq(z)} q(z)dz = \frac{1}{k} \int \tilde{p}(z)dz$$

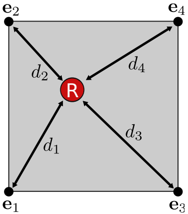
- $k = 1$ and $q(x) = p(x) \Rightarrow p(accept) = 1$
- But often: $p(accept \mid x) = \prod_{i=1}^{D} p(accept \mid x_i) = \mathcal{O}(\gamma^D)$

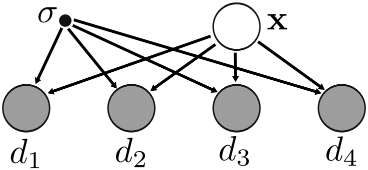
Rejection Sampling

Robot Localization Example

- You bought a vaccum robot for your living room (1×1 m)
- For proper cleaning, the robot needs to localize itself
- No prior knowledge on location: $\mathbf{x} \sim \mathcal{U}([0,1] \times [0,1])$
- Independent measurements: $d_i|\mathbf{x} \sim \mathcal{N}(\|\mathbf{x} - \mathbf{e}_i\|, \sigma^2)$

$$
\begin{aligned}
p(\mathbf{x}|d_1, d_2, d_3, d_4) &\propto p(\mathbf{x})p(d_1|\mathbf{x})p(d_2|\mathbf{x})p(d_3|\mathbf{x})p(d_4|\mathbf{x}) \\
&\propto [0 \leq x_1, x_2 \leq 1] \\
&\quad \times \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{4}[\|\mathbf{x} - \mathbf{e}_i\| - d_i]^2\right)
\end{aligned}
$$

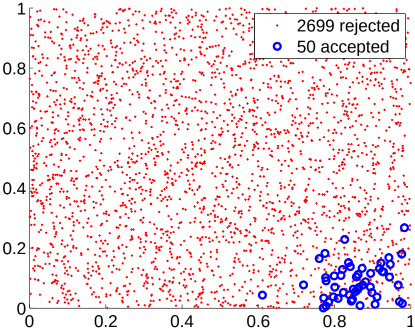


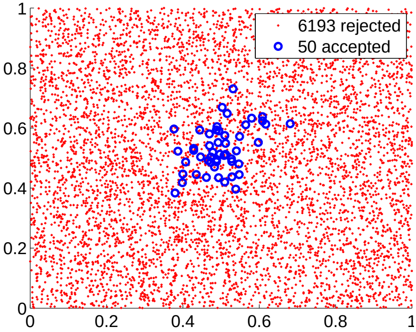


Rejection Sampling

Robot Localization Example

- The maximum of the unnormalized posterior is 1
- Thus we can choose: $q(\mathbf{x}) = [0 \leq x_1, x_2 \leq 1]$





Metropolis-Hastings Sampling

Metropolis-Hastings Sampling

Markov Chain

- Discrete random process with Markov property:

$$P(x_i|x_{i-1},...,x_1) = P(x_i|x_{i-1}) = P(x'|x)$$

Markov Chain Monte Carlo (MCMC)

- We want to sample from $p(x) = \frac{1}{Z}\tilde{p}(x)$ with Z unknown
- Idea: Establish a Markov chain with transition kernel $T(x' \mid x)$ and with stationary distribution $p(x)$:

$$p(x') = \int_x T(x' \mid x)\, p(x) dx$$

- Task: Find $T(x' \mid x)$ such that $p(x)$ is its stationary distribution!

Metropolis-Hastings Sampling

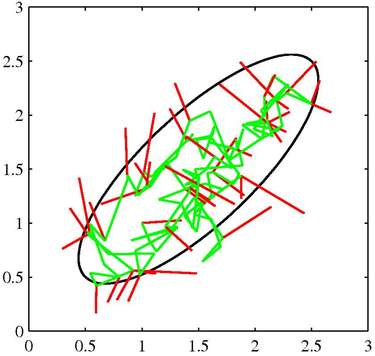
Metropolis-Hastings

- Initialize x and specify proposal distribution $q(x'|x)$
- Sample x' from $q(x'|x)$ and accept with probability

$$A(x', x) = \min\left(1, \frac{p(x')\, q(x|x')}{p(x)\, q(x'|x)}\right) = \min\left(1, \frac{\tilde{p}(x')\, q(x|x')}{\tilde{p}(x)\, q(x'|x)}\right)$$

- If accepted: $x \leftarrow x'$
- If not accepted: stay at x
- Iterate (sample again)

Example: 2D Gaussian



- 150 proposal steps, 43 are rejected (red)

Why does it work?

- Remember the acceptance probability:

$$A(x', x) = \min\left(1, \frac{p(x')\, q(x|x')}{p(x)\, q(x'|x)}\right)$$

- Let us write down the transition kernel $T(x'|x)$
  i.e., the probability to transition the state from x to x':

$$\begin{aligned} T(x'|x) &= q(x'|x)\, A(x', x) \\ &\quad + \delta(x' - x) \int q(\tilde{x}|x)\, [1 - A(\tilde{x}|x)]\, d\tilde{x} \end{aligned}$$

Why does it work?

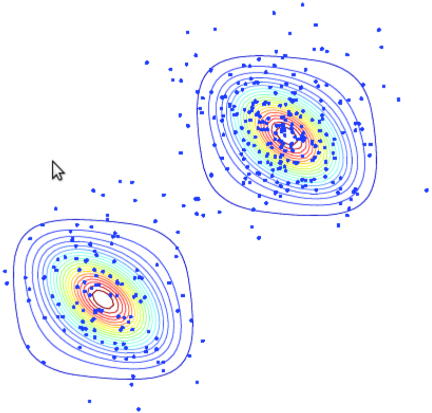
$$
\begin{aligned}
\int T(x'|x)p(x)dx &= \int \min\{p(x)q(x'|x), p(x')q(x|x')\}dx \\
&+ \int p(x')q(\tilde{x}|x')[1 - A(\tilde{x}|x')]d\tilde{x} \\
&= \int \min\{p(x)q(x'|x), p(x')q(x|x')\}dx \\
&+ p(x') \int q(\tilde{x}|x')\tilde{dx} \\
&- \int p(x')q(\tilde{x}|x')A(\tilde{x}|x')d\tilde{x} \\
&= \int \min\{p(x)q(x'|x), p(x')q(x|x')\}dx \\
&+ p(x') \\
&- \int \min\{p(x')q(\tilde{x}|x'), p(\tilde{x})q(x'|\tilde{x})\}d\tilde{x} \\
&= p(x')
\end{aligned}
$$

Why does it work?

Other requirements that need to be fulfilled:

- **Irreducibility:** Any state x' can be reached by any other state x in a finite number of steps
- **Aperiodicity:** The occurrence of states is not restricted to periodic events (any state may occur at any time).

Example: Irreducibility



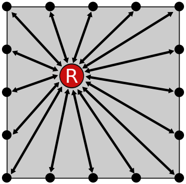
- $q(x'|x)$ needs to be able to bridge the gap

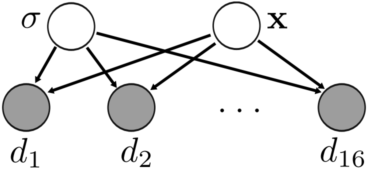
Metropolis-Hastings Sampling

Robot Localization Example

- Now inferring 2 variables: location $\mathbf{x}$ and sensor noise σ
- Uniform prior on location: $\mathbf{x} \sim \mathcal{U}([0,1] \times [0,1])$
- Uniform prior on sensor noise: $\sigma \sim \mathcal{U}(0.01, 0.5)$
- Measurements depend on σ: $d_i|\mathbf{x}, \sigma \sim \mathcal{N}(\|\mathbf{x} - \mathbf{e}_i\|, \sigma^2)$

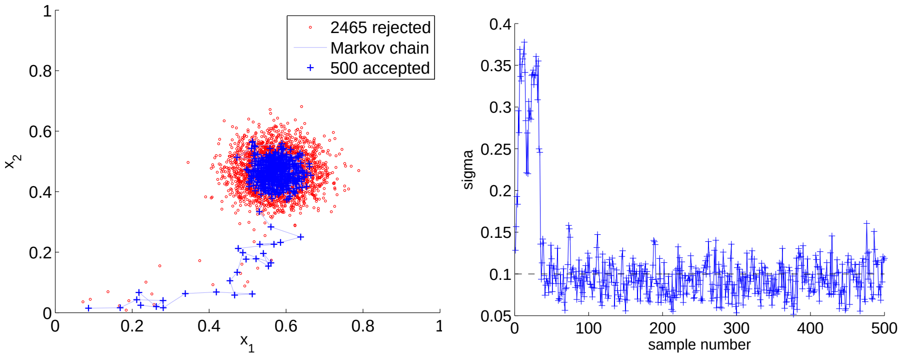
$$
\begin{aligned}
p(\mathbf{x}, \sigma | d_1, ... d_{16}) &\propto p(\mathbf{x}) p(\sigma) p(d_1|\mathbf{x}, \sigma) \cdots p(d_{16}|\mathbf{x}, \sigma) \\
&\propto [0 \leq x_1, x_2 \leq 1] \times [0.01 \leq \sigma \leq 0.5] \\
&\quad \times \frac{\exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{16} [\|\mathbf{x} - \mathbf{e}_i\| - d_i]^2\right)}{(2\pi\sigma^2)^8}
\end{aligned}
$$





Metropolis-Hastings Sampling

Robot Localization Example



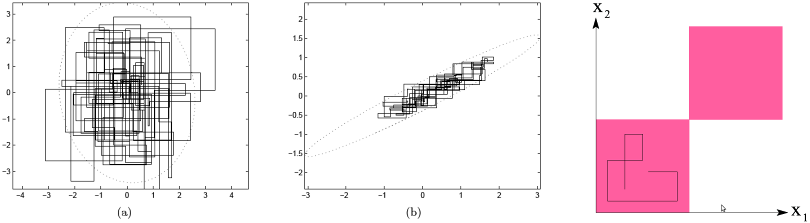
Gibbs Sampling

Special case of MH Sampling:

- Cyclic MH kernel that updates one variable at a time
- Sample directly from the full conditional distribution

$$q(x'|x) = p(x_k|x_1, ..., x_{k-1}, x_{k+1}, ..., x_D)$$

- Samples get accepted with probability 1 (exercise)
- But: conditionals must be easy to sample from!
- Danger of slow convergence and non-irreducibility:



Approximate Inference Overview

- Deterministic Inference
  - Junction Tree (not approximate but intractable)
  - Loopy Belief Propagation
  - Variational Approximation
  - Expectation Propagation
  - Mean field
  - Gradient Descent
  - ...
- Sampling
  - Rejection Sampling
  - Slice Sampling
  - Metropolis-Hastings Sampling
  - Gibbs Sampling
  - ...

Next Time ...

- Learning
- And after that: Computer Vision, finally!
- No more toy examples, but real stuff - promised ;)