Functional Programming

WS 2010/11

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Today's Topics

- Evaluation Strategies
- Abstract Data Types
- Sets and Binary Search Trees

Evaluation Strategies

Recall - λ-Terms

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Examples

Conventions	Verbose	in Words
$x\ y$	$(x\ y)$	"x applied to y"
$\lambda x.\, x$	$(\lambda x.\, x)$	"lambda x to x" (identity function)
$\lambda xy.\, x$	$(\lambda x.\, (\lambda y.\, x))$	"lambda x y to x"
$\lambda x.\, x\ x$	$(\lambda x.\, (x\ x))$	"lambda x to x applied to x"
$(\lambda x.\, x)\ x$	$((\lambda x.\, x)\ x)$	"lambda x to x, applied to x"

Recall - β-Reduction

- term s (β-)reduces to term t in one step
- written: $s \rightarrow_\beta t$
- iff there is context C, variable x, and terms u and v, s.t.,
- $s = C[(\lambda x.\, u)\ v]$ and $t = C[u\{x/v\}]$

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Examples

$$K \stackrel{\text{def}}{=} \lambda xy.\, x$$

$$I \stackrel{\text{def}}{=} \lambda x.\, x$$

$$\Omega \stackrel{\text{def}}{=} (\lambda x.\, x\ x)\ (\lambda x.\, x\ x)$$

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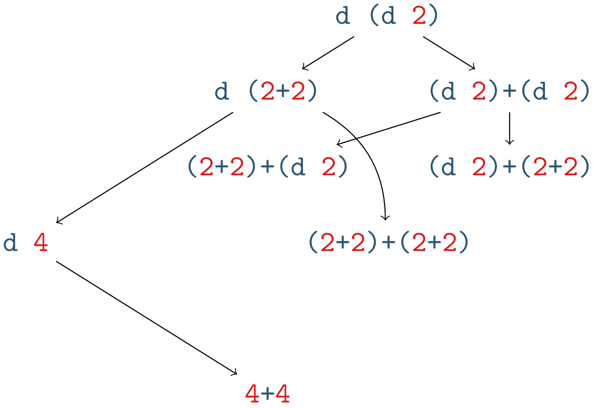

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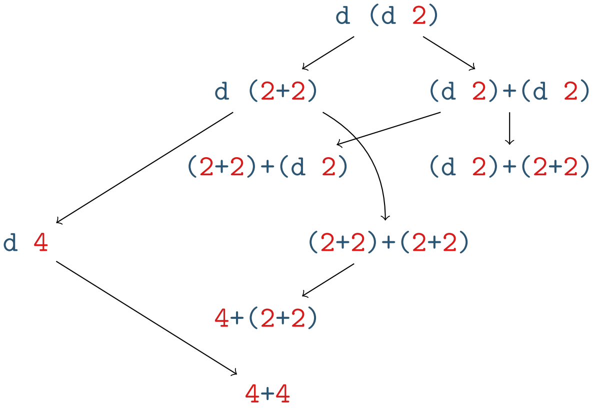

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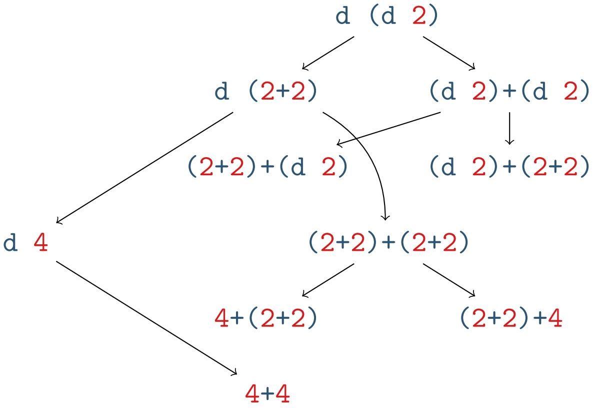

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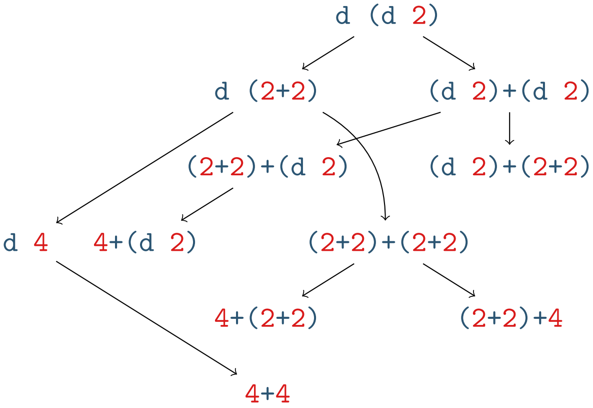

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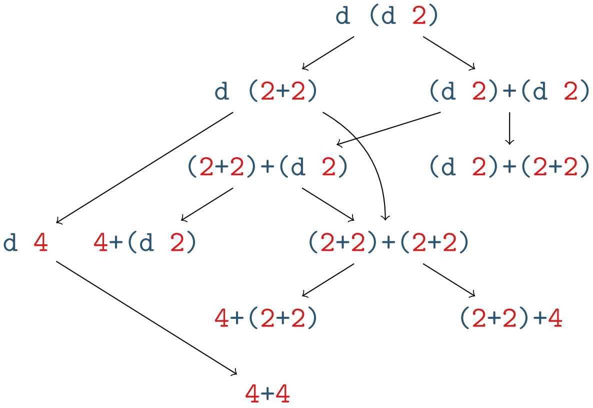

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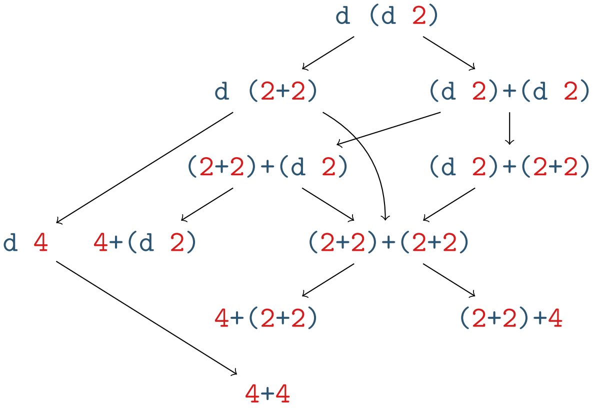

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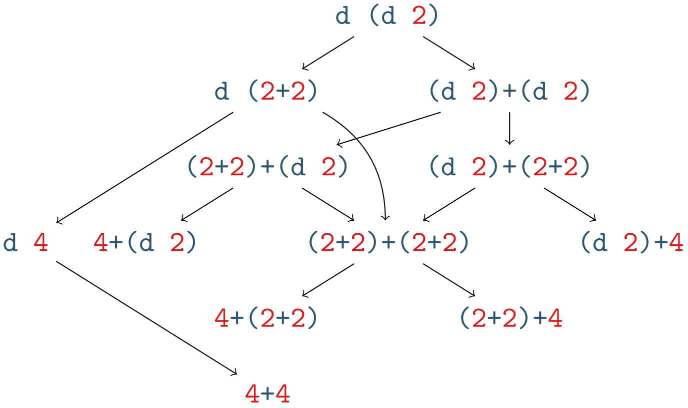

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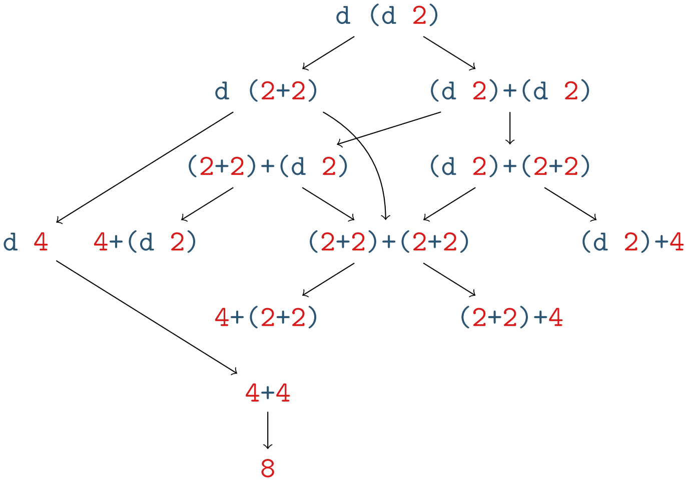

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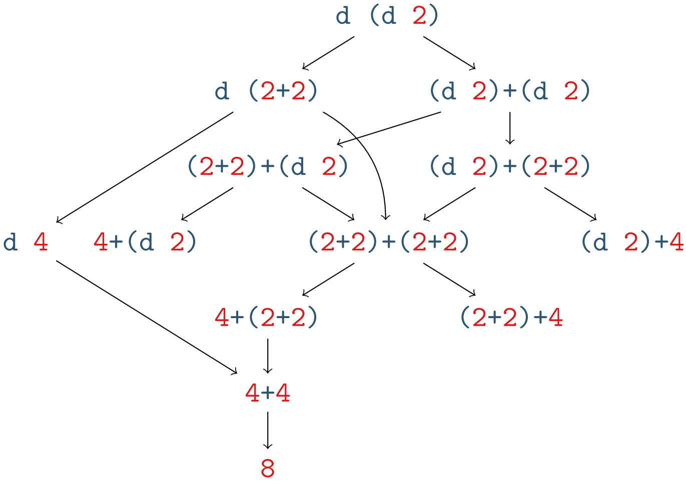

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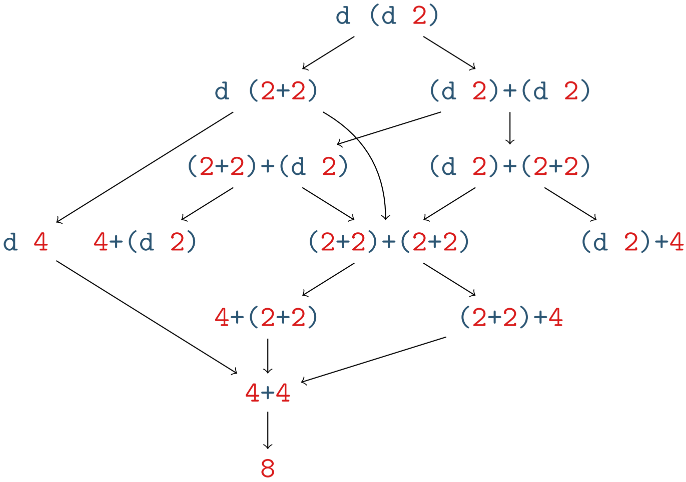

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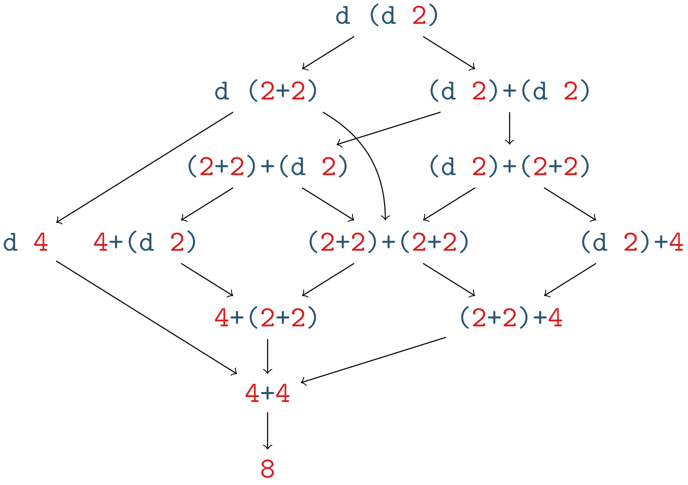

Strategies

- fix evaluation order
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Example

- call-by-value

$$
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\texttt{d (d 2)} &= \texttt{d (2+2)} \\
&= \texttt{d 4} \\
&= \texttt{4+4} \\
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$$

- call-by-name

$$
\begin{aligned}
\texttt{d (d 2)} &= \texttt{(d 2) + (d 2)} \\
&= \texttt{(2+2) + (d 2)} \\
&= \texttt{4 + (d 2)} \\
&= \texttt{4 + (2+2)} \\
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(Leftmost) Innermost Reduction

- always reduce leftmost innermost redex
- a redex u inside a term t is innermost if it does not contain any redexes as proper subterms, i.e.,

$$\nexists C\ s.\, u = C[s],\ C \neq \Box \text{ and } s \text{ is a redex}$$

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Example

- consider $t = (\lambda x.\, (\lambda y.\, y)\ x)\ z$
- $(\lambda y.\, y)\ x$ is innermost redex
- t is redex, but not an innermost redex

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- consider $t = (\lambda x.\, (\lambda y.\, y)\ x)\ z$
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Exercises

- consider the λ-terms
- $S = \lambda xyz.\, x\ z\ (y\ z)$
- $K = \lambda xy.\, x$
- $I = \lambda x.\, x$
- reduce $S\ K\ I$ to NF using leftmost innermost reduction
- reduce $S\ K\ I$ to NF using leftmost outermost reduction

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- corresponds to strict (or eager) evaluation
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Weak Head Normal Form

term t is in weak head normal form iff t is not an application

Abstract Data Types

Idea

- hide implementation details
- just provide interface
- allows to change implementation (e.g., make more efficient) without breaking client code

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Haskell

- consider module
```
module M (T, ...) where
type T = C1 |   | CN
```
- only name `T` is exported, but none of `C1` to `CN`
- thus we are not able to directly construct values of type `T`
- if we want to export `C1` to `CN`, we can use `T(..)` in export list

Set Characteristics

- order of elements not important
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Examples

$$\{1, 2, 3, 5\} = \{5, 1, 3, 2\}$$
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Set Operations

description	notation	Haskell
empty set	$\varnothing$	`empty :: Set a`
insertion	$\{x\} \cup S$	`insert :: a -> Set a -> Set a`
membership	$e \in S$	`mem :: a -> Set a -> Bool`
union	$S \cup T$	`union :: Set a -> Set a -> Set a`
difference	$S \setminus T$	`diff :: Set a -> Set a -> Set a`

Example - Sets as Lists

```
module Set (Set,empty,insert,mem,union,diff) where
import qualified Data.List as List
data Set a = Set [a]

empty :: Set a
empty = Set []

insert :: Eq a => a -> Set a -> Set a
insert x (Set xs) = Set (List.nub (x : xs))

mem :: Eq a => a -> Set a -> Bool
mem x (Set xs) = x `elem` xs

union :: Eq a => Set a -> Set a -> Set a
union (Set xs) (Set ys) = Set (List.nub (xs ++ ys))

diff :: Eq a => Set a -> Set a -> Set a
diff (Set xs) (Set ys) = Set (xs List.\\ ys)
```

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- **`import`** `qualified M` allows to access all functions defined in `M` using prefix `M.`
- **`import`** `qualified M as N`, same as **`import`** `qualified M` but additionally rename `M` to `N`

New Types

- in `Set` we use **`data`** with a single constructor `Set` to hide the fact that sets are implemented by lists
- this is a common special case
- we may use `newtype Set a = Set a` instead
- only difference: **`newtype`** has better performance than **`data`**

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Record Syntax

- for data type / new type `T`, instead of `C t1` ... `tN`, we may use
- `C {n1 :: t1, ..., nN :: tN}` as constructor
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Example

- **`data`** `Equation a = E { lhs :: a, rhs :: a }`
- ```
  ghci> let e1 = E "10" "5+5"
  ghci> let e2 = E { rhs = "5+5", lhs = "10" }
  ghci> lhs e1
  "10"
  ghci> rhs e2
  "5+5"
  ```
  ```

Sets and Binary Search Trees

The Type

- we want to use type BTree without prefix

```
import BTree (BTree)
```

- all other functions from BTree with prefix

```
import qualified BTree
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- the internal representation of a set is a binary tree

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Note

- `newtype Set a = Set { rep :: BTree a }` is almost the same as writing `type Set a = BTree a`
- additionally the type system prevents us from "accidentally" (i.e., without the constructor `Set`) using `BTree`s as `Set`s
- no runtime penalty (in contrast to `data Set a = Set { rep :: BTree })`)
- reason: `newtype` restricted to single constructor (usually of same name as newly introduced type),
- whereas `data` may define arbitrary many constructors (e.g., `Empty` and `Node`)

Empty Set

```
empty :: Set a
empty = Set BTree.Empty
```

Empty Set

```
empty :: Set a
empty = Set BTree.Empty
```

Membership

```
mem :: Ord a => a -> Set a -> Bool
mem x s = x `memTree` (rep s)
  where memTree x Empty = False
        memTree x (Node y l r) =
          case compare x y of
            EQ -> True
            LT -> x `memTree` l
            GT -> x `memTree` r
```

Insertion

```haskell
insert :: Ord a => a -> Set a -> Set a
insert x s = Set (insertTree x (rep s))

insertTree :: Ord a => a -> BTree a -> BTree a
insertTree x Empty          = Node x Empty Empty
insertTree x (Node y l r) =
  case compare x y of
    EQ -> Node y l r
    LT -> Node y (insertTree x l) r
    GT -> Node y l (insertTree x r)
```

Union

```haskell
union :: Ord a => Set a -> Set a -> Set a
union s t = Set (rep s `unionTree` rep t)

unionTree :: Ord a => BTree a -> BTree a -> BTree a
unionTree Empty s         = s
unionTree (Node x l r) s =
  insertTree x (l `unionTree` r `unionTree` s)
```

Removing the Maximal Element

```
splitMaxTree :: BTree a -> Maybe (a,BTree a)
splitMaxTree Empty              = Nothing
splitMaxTree (Node x l Empty) = Just (x,l)
splitMaxTree (Node x l r)       =
  let Just (m,r') = splitMaxTree r
  in Just (m,Node x l r')
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The Maybe Type

- Prelude: `data Maybe a = Just a | Nothing`
- used for type-safe error handling
- if an error occurs, we return `Nothing`
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Example - Safe Head

```
safeHead (x:_) = Just x
safeHead _     = Nothing
```

Remove Given Element

```
removeTree :: Ord a => a -> BTree a -> BTree a
removeTree x Empty          = Empty
removeTree x (Node y l r) = case compare x y of
  LT -> Node y (removeTree x l) r
  GT -> Node y l (removeTree x r)
  EQ -> case splitMaxTree l of
    Nothing      -> r
    Just (m,l') -> Node m l' r
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Idea

- have binary search tree (BST)
- `x` smaller `y`: `x` can only occur in `l`
- `x` greater `y`: `x` can only occur in `r`
- `x` equals `y`: remove current node and
- combine `l` and `r` into new BST
- therefore, take maximum of `l` as new root
- guarantees that all other elements in `l` are smaller and
- that all elements in `r` are greater

Difference

```
diff :: Ord a => Set a -> Set a -> Set a
diff s t = Set (rep s `diffTree` rep t)

diffTree :: Ord a => BTree a -> BTree a -> BTree a
diffTree t Empty          = t
diffTree t (Node x l r) =
  removeTree x t `diffTree` l `diffTree` r
```

Exercises (for November 19th)

1. Read chapter 3 of Real World Haskell and the lecture notes about the lambda-calculus.
2. Reduce each of the following λ-terms to NF

$$
\begin{gathered}
(\lambda w.\, w)\ ((\lambda xy.\, y)\ (z\ z)) \\
(\lambda xy.\, x)\ (\lambda z.\, y\ z) \\
\lambda z.\, (\lambda x.\, x\ z\ y)\ (\lambda xy.\, y\ z) \\
\lambda xy.\, y\ (\lambda w.\, w)\ (\lambda yz.\, y\ x)
\end{gathered}
$$

3. Reduce ADD 3 2 to WHNF using leftmost innermost/outermost reduction.
4. Give λ-terms encoding `(&&)`, `(||)`, and `not`.
5. Implement safe versions (i.e., using `Maybe`) of `tail`, `init`, and `last`.
6. Implement the function `equals :: Ord a => Set a -> Set a -> Bool`, checking whether two sets are equal.