Functional Programming

WS 2010/11

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Today's Topics

- Mathematical Induction
- Induction Over Lists
- Structural Induction

Mathematical Induction

When to use Mathematical Induction?

- prove that some property P holds for all natural numbers
- more formally, prove:

$$\forall n.\, P(n) \qquad (\text{where } n \in \mathbb{N})$$

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- first prove base case

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assume $P(k)$ (induction hypothesis), show $P(k+1)$

Why does this Work?

- we have two ingredients:
  1. P is true for 0
  2. if P is true for arbitrary k it is also true for $k + 1$
- and want to show P for every natural number ($\forall n.\, P(n)$)

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Idea

- intuitively we can reach arbitrary n
- such that $P(n)$
- hence, $\forall n.\, P(n)$

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$$1 + 2 + \cdots + (k+1)$$

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- of course, the base case may be changed
- e.g., if base case $P(1)$, property holds for all $n \geq 1$

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- first domino will fall
- if a domino falls also its right neighbor falls

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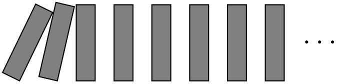

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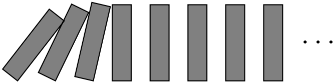

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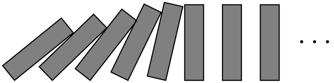

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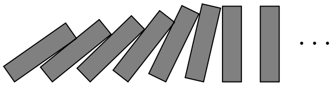

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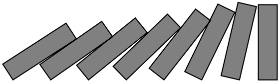

$\ldots$

Induction Over Lists

Recall

- type: `data [a] = [] | (:) a [a]`

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Notes

- lists are recursive structures
- base case: `[]`
- step case: x `:` xs

Induction Principle for Lists - Informally

- to show $P(xs)$ for all lists xs
- show base case: $P(\texttt{[]})$
- show step case: $P(xs) \longrightarrow P(x : xs)$ for arbitrary x and xs

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$$\longrightarrow \forall ls. P(ls)$$

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Remark

- for lists, P can be seen as function `p :: [a] -> Bool`

Exercise - Right Identity for List Append

- definition

```
  []    ++ ys  =  ys
(x:xs) ++ ys  =  x : (xs ++ ys)
```

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  []    ++ ys  =  ys
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- lemma: [] is a right identity of ++, i.e.,

$$xs \mathbin{++} [] = xs$$

Exercise - Associativity of Append

- recall

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- and '*xs* ++ [] = *xs*' for all lists *xs*

Exercise - Associativity of Append

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- and 'xs ++ [] $= xs$' for all lists xs
- lemma: ++ is associative, i.e.,

$$xs \mathbin{++} (ys \mathbin{++} zs) = (xs \mathbin{++} ys) \mathbin{++} zs$$

Exercise - Length and Append

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- definition

```
length []       =  0
length (_:xs)   =  1 + length xs
```

Exercise - Length and Append

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- lemma: sum of lengths is length of combined list, i.e.,

$$\texttt{length}\ xs + \texttt{length}\ ys = \texttt{length}\ (xs\ \texttt{++}\ ys)$$

Structural Induction

Example - Terms

```
data Term = Var String
          | Lab String Term
          | App Term Term
```

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General Structures - Induction Principle

- for every non-recursive constructor, show base case
  - base case: $P(\texttt{Var x})$
- for every recursive constructor, show step case
  - step case 1: $P(\texttt{t}) \longrightarrow P(\texttt{Lab x t})$
  - step case 2: $P(\texttt{s}) \wedge P(\texttt{t}) \longrightarrow P(\texttt{App s t})$

Example - Binary Trees

```
data BTree a = Empty
             | Node a (BTree a) (BTree a)
```

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```haskell
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Induction Principle for Binary Trees

$$(P(\texttt{Empty}) \land \forall x. \forall l. \forall r. (P(l) \land P(r) \longrightarrow P(\texttt{Node}\ x\ l\ r))) \\ \longrightarrow \forall t. P(t)$$

Exercise - Perfect Binary Trees

- a binary tree is perfect if all leaf nodes have same depth

```
perfect Empty          = True
perfect (Node x l r) =
  height l == height r && perfect l && perfect r

height Empty          = 0
height (Node _ l r) =
  max (height l) (height r) + 1

size Empty          = 0
size (Node _ l r) = size l + size r + 1
```

- lemma: a perfect binary tree t of height n has exactly $2^n - 1$ nodes, i.e.,

$$P(t) = (\texttt{perfect}\ t \longrightarrow \texttt{size}\ t = 2^{\texttt{height}\ t} - 1)$$

Exercises (for November 26th)

1. Prepare for the 1st test!
2. Prove the following equation by induction

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Prove $\texttt{rev}\ (xs \mathbin{+\!+} ys) = \texttt{rev}\ ys \mathbin{+\!+} \texttt{rev}\ xs$ for

```
rev []      =  []
rev (x:xs)  =  rev xs ++ [x]
```

using the equations

$$xs \mathbin{+\!+} [] = xs \qquad (\star)$$

$$(xs \mathbin{+\!+} ys) \mathbin{+\!+} zs = xs \mathbin{+\!+} (ys \mathbin{+\!+} zs) \qquad (\star\star)$$