Complex Analysis

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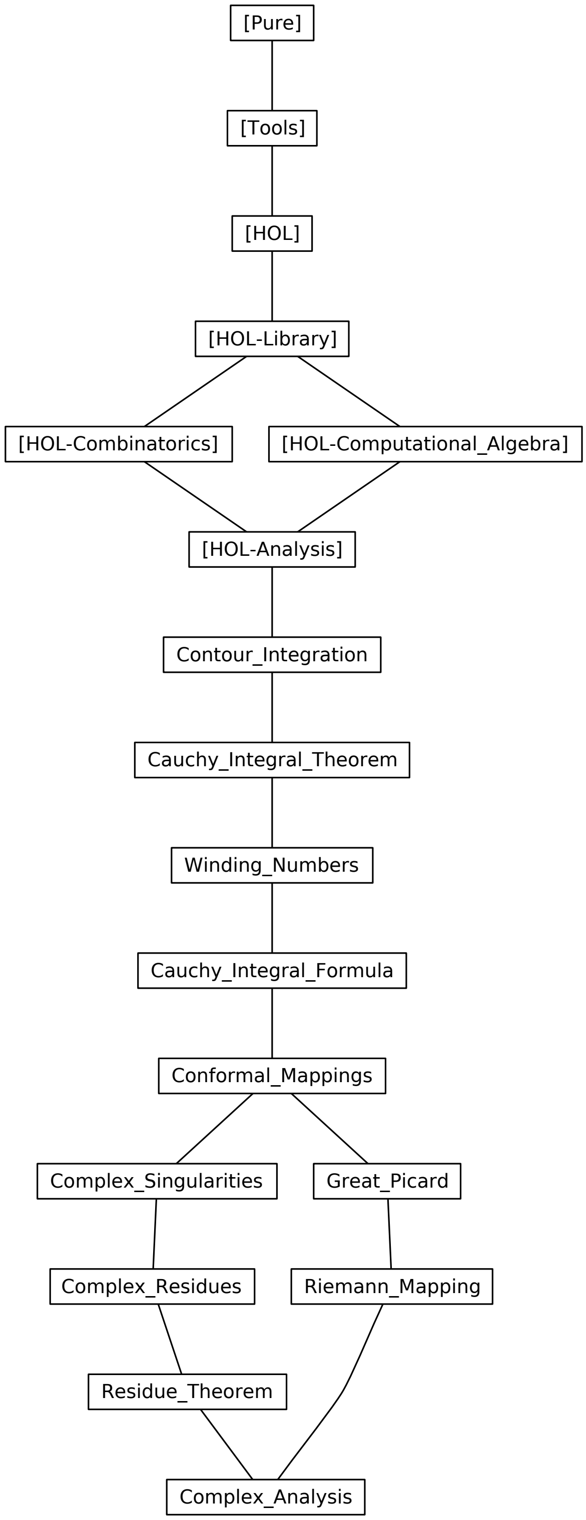
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1 Contour integration

theory *Contour_Integration*
imports *HOL−Analysis.Analysis*
begin

1.1 Definition

definition *has_contour_integral* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ (*real* ⇒ *complex*) ⇒ *bool*
(**infixr** *has′_contour′_integral 50*)
where (*f has_contour_integral i*) *g* ≡
((λ*x*. *f*(*g x*) ∗ *vector_derivative g* (*at x within* {*0..1*}))
has_integral i) {*0..1*}

definition *contour_integrable_on*
(**infixr** *contour′_integrable′_on 50*)
where *f contour_integrable_on g* ≡ ∃ *i*. (*f has_contour_integral i*) *g*

definition *contour_integral*
where *contour_integral g f* ≡ *SOME i*. (*f has_contour_integral i*) *g* ∨ ¬ *f contour_integrable_on g* ∧ *i=0*

1.2 Relation to subpath construction

1.3 Cauchy's theorem where there's a primitive

corollary *Cauchy_theorem_primitive*:
assumes ⋀*x*. *x* ∈ *S* ⟹ (*f has_field_derivative f′ x*) (*at x within S*)
and *valid_path g path_image g* ⊆ *S pathfinish g* = *pathstart g*
shows (*f′ has_contour_integral 0*) *g*

1.4 Reversing the order in a double path integral

proposition *contour_integral_swap*:
assumes *fcon*: *continuous_on* (*path_image g* × *path_image h*) (λ(*y1*,*y2*). *f y1 y2*)
and *vp*: *valid_path g valid_path h*
and *gvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative g* (*at t*))
and *hvcon*: *continuous_on* {*0..1*} (λ*t*. *vector_derivative h* (*at t*))
shows *contour_integral g* (λ*w*. *contour_integral h* (*f w*)) =
contour_integral h (λ*z*. *contour_integral g* (λ*w*. *f w z*))

1.5 Partial circle path

definition *part_circlepath* :: [*complex*, *real*, *real*, *real*, *real*] ⇒ *complex*
where *part_circlepath z r s t* ≡ λ*x*. *z* + *of_real r* ∗ *exp* (i ∗ *of_real* (*linepath s t x*))

proposition *path_image_part_circlepath*:
assumes $s \le t$
shows *path_image* (*part_circlepath z r s t*) = {*z* + *r* ∗ *exp*(i ∗ *of_real x*) | *x*. $s \le x \wedge x \le t$}

1.6 Special case of one complete circle

definition *circlepath* :: [*complex*, *real*, *real*] ⇒ *complex*
where *circlepath z r* ≡ *part_circlepath z r 0* (*2*∗*pi*)

1.7 Uniform convergence of path integral

proposition *contour_integral_uniform_limit*:
assumes *ev_fint*: *eventually* (λ*n*::'*a*. (*f n*) *contour_integrable_on* γ) *F*
and *ul_f*: *uniform_limit* (*path_image* γ) *f l F*
and *noleB*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *norm* (*vector_derivative* γ (*at t*)) ≤ *B*
and γ: *valid_path* γ
and [*simp*]: ¬ *trivial_limit F*
shows *l contour_integrable_on* γ ((λ*n*. *contour_integral* γ (*f n*)) ⟶ *contour_integral* γ *l*) *F*

end

2 Complex Path Integrals and Cauchy's Integral Theorem

theory *Cauchy_Integral_Theorem*
imports
HOL−Analysis.Analysis
Contour_Integration
begin

proposition *Cauchy_theorem_triangle_interior*:
assumes *contf*: *continuous_on* (*convex hull* {*a*,*b*,*c*}) *f*
and *holf*: *f holomorphic_on interior* (*convex hull* {*a*,*b*,*c*})
shows (*f has_contour_integral 0*) (*linepath a b* +++ *linepath b c* +++ *linepath c a*)

2.1 Cauchy's theorem for a convex set

corollary *Cauchy_theorem_convex_simple*:
assumes *holf*: *f holomorphic_on S*
and *convex S valid_path g path_image g* $\subseteq$ *S pathfinish g* = *pathstart g*
shows (*f has_contour_integral 0*) *g*

2.2 Homotopy forms of Cauchy's theorem

proposition *Cauchy_theorem_homotopic_paths*:
assumes *hom*: *homotopic_paths S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

proposition *Cauchy_theorem_homotopic_loops*:
assumes *hom*: *homotopic_loops S g h*
and *open S* **and** *f*: *f holomorphic_on S*
and *vpg*: *valid_path g* **and** *vph*: *valid_path h*
shows *contour_integral g f* = *contour_integral h f*

end

3 Winding numbers

theory *Winding_Numbers*
imports *Cauchy_Integral_Theorem*
begin

3.1 Definition

definition *winding_number_prop* :: [*real* $\Rightarrow$ *complex*, *complex*, *real*, *real* $\Rightarrow$ *complex*, *complex*] $\Rightarrow$ *bool* **where**
winding_number_prop γ *z e p n* $\equiv$
valid_path p $\wedge$ *z* $\notin$ *path_image p* $\wedge$
pathstart p = *pathstart* γ $\wedge$
pathfinish p = *pathfinish* γ $\wedge$
($\forall$ *t* $\in$ {*0..1*}. *norm*(γ *t* − *p t*) < *e*) $\wedge$
contour_integral p (λ*w*. *1*/(*w* − *z*)) = *2* ∗ *pi* ∗ i ∗ *n*

definition *winding_number*:: [*real* $\Rightarrow$ *complex*, *complex*] $\Rightarrow$ *complex* **where**
winding_number γ *z* $\equiv$ *SOME n*. $\forall$ *e* > *0*. $\exists$ *p*. *winding_number_prop* γ *z e p n*

proposition *winding_number_valid_path*:
assumes *valid_path* γ $z \notin$ *path_image* γ
shows *winding_number* γ $z = 1/(2*pi*\mathrm{i}) *$ *contour_integral* γ $(\lambda w.\ 1/(w - z))$

proposition *has_contour_integral_winding_number*:
assumes γ: *valid_path* γ $z \notin$ *path_image* γ
shows $((\lambda w.\ 1/(w - z))$ *has_contour_integral* $(2*pi*\mathrm{i}*winding\_number\ \gamma\ z))$ γ

3.2 The winding number is an integer

theorem *integer_winding_number*:
$[\![path\ \gamma;\ pathfinish\ \gamma = pathstart\ \gamma;\ z \notin path\_image\ \gamma]\!] \Longrightarrow winding\_number\ \gamma\ z \in \mathbb{Z}$

3.3 Continuity of winding number and invariance on connected sets

theorem *continuous_at_winding_number*:
fixes *z*::*complex*
assumes γ: *path* γ **and** *z*: $z \notin$ *path_image* γ
shows *continuous* (*at z*) (*winding_number* γ)

corollary *continuous_on_winding_number*:
path $\gamma \Longrightarrow$ *continuous_on* ($-$ *path_image* γ) ($\lambda w.$ *winding_number* γ *w*)

3.4 Winding number is zero "outside" a curve

proposition *winding_number_zero_in_outside*:
assumes γ: *path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ **and** *z*: $z \in$ *outside* (*path_image* γ)
shows *winding_number* γ $z = 0$

proposition *winding_number_part_circlepath_pos_less*:
assumes $s < t$ **and** *no*: $norm(w - z) < r$
shows $0 < Re$ (*winding_number*(*part_circlepath z r s t*) *w*)

proposition *winding_number_circlepath*:
assumes $norm(w - z) < r$ **shows** *winding_number*(*circlepath z r*) $w = 1$

3.5 Winding number for a triangle

proposition *winding_number_triangle*:
assumes *z*: $z \in$ *interior*(*convex hull* $\{a,b,c\}$)
shows *winding_number*(*linepath a b* +++ *linepath b c* +++ *linepath c a*) z =
(*if* $0 < Im((b - a) * cnj\ (b - z))$ *then* 1 *else* -1)

3.6 Winding numbers for simple closed paths

proposition *simple_closed_path_winding_number_inside*:
assumes *simple_path* γ
obtains $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* $\gamma\ z = 1$
| $\bigwedge z.\ z \in$ *inside*(*path_image* γ) $\Longrightarrow$ *winding_number* $\gamma\ z = -1$

3.7 Winding number for rectangular paths

proposition *winding_number_rectpath*:
assumes $z \in$ *box a1 a3*
shows *winding_number* (*rectpath a1 a3*) $z = 1$

proposition *winding_number_rectpath_outside*:
assumes *Re a1* $\leq$ *Re a3* *Im a1* $\leq$ *Im a3*
assumes $z \notin$ *cbox a1 a3*
shows *winding_number* (*rectpath a1 a3*) $z = 0$

end

4 Cauchy's Integral Formula

theory *Cauchy_Integral_Formula*
imports *Winding_Numbers*
begin

4.1 Proof

theorem *Cauchy_integral_formula_convex_simple*:
assumes *convex S* **and** *holf*: *f holomorphic_on S* **and** $z \in$ *interior S* *valid_path* γ *path_image* $\gamma \subseteq S - \{z\}$
pathfinish γ = *pathstart* γ
shows (($\lambda w.\ f\ w\ /\ (w - z)$) *has_contour_integral* ($2*pi * \mathrm{i} *$ *winding_number* $\gamma\ z * f\ z$)) γ
theorem *Cauchy_integral_circlepath*:
assumes *contf*: *continuous_on* (*cball z r*) *f* **and** *holf*: *f holomorphic_on* (*ball z r*) **and** *wz*: $norm(w - z) < r$
shows (($\lambda u.\ f\ u/(u - w)$) *has_contour_integral* ($2 *$ *of_real pi* $* \mathrm{i} * f\ w$))
(*circlepath z r*)

4.2 Existence of all higher derivatives

proposition *derivative_is_holomorphic*:
assumes *open S*
and *fder*: $\bigwedge z.\ z \in S \Longrightarrow (f\ has\_field\_derivative\ f'\ z)\ (at\ z)$
shows f' *holomorphic_on S*

4.3 Morera's theorem

proposition *Morera_triangle*:
$[\![continuous\_on\ S\ f;\ open\ S;$
$\bigwedge a\ b\ c.\ convex\ hull\ \{a,b,c\} \subseteq S$
$\longrightarrow contour\_integral\ (linepath\ a\ b)\ f\ +$
$contour\_integral\ (linepath\ b\ c)\ f\ +$
$contour\_integral\ (linepath\ c\ a)\ f = 0]\!]$
$\Longrightarrow f\ analytic\_on\ S$

4.4 Combining theorems for higher derivatives including Leibniz rule

proposition *no_isolated_singularity*:
fixes *z::complex*
assumes *f*: *continuous_on S f* **and** *holf*: $f\ holomorphic\_on\ (S - K)$ **and** *S*: *open S* **and** *K*: *finite K*
shows *f holomorphic_on S*

proposition *Cauchy_integral_formula_convex*:
assumes *S*: *convex S* **and** *K*: *finite K* **and** *contf*: *continuous_on S f*
and *fcd*: $(\bigwedge x.\ x \in interior\ S - K \Longrightarrow f\ field\_differentiable\ at\ x)$
and *z*: $z \in interior\ S$ **and** *vpg*: $valid\_path\ \gamma$
and *pasz*: $path\_image\ \gamma \subseteq S - \{z\}$ **and** *loop*: $pathfinish\ \gamma = pathstart\ \gamma$
shows $((\lambda w.\ f\ w\ /\ (w - z))\ has\_contour\_integral\ (2*pi * i * winding\_number\ \gamma\ z * f\ z))\ \gamma$

corollary *Cauchy_contour_integral_circlepath*:
assumes $continuous\_on\ (cball\ z\ r)\ f\ f\ holomorphic\_on\ ball\ z\ r\ w \in ball\ z\ r$
shows $contour\_integral(circlepath\ z\ r)\ (\lambda u.\ f\ u/(u - w)\widehat{\ }(Suc\ k)) = (2 * pi * i) * (deriv\ \widehat{\ }\widehat{\ }\ k)\ f\ w\ /\ (fact\ k)$

4.5 A holomorphic function is analytic, i.e. has local power series

theorem *holomorphic_power_series*:
assumes *holf*: *f holomorphic_on ball z r*
and *w*: $w \in ball\ z\ r$
shows $((\lambda n.\ (deriv\ \widehat{\ }\widehat{\ }\ n)\ f\ z\ /\ (fact\ n) * (w - z)\widehat{\ }n)\ sums\ f\ w)$

4.6 The Liouville theorem and the Fundamental Theorem of Algebra

proposition *Liouville_weak*:
assumes f *holomorphic_on UNIV* **and** $(f \longrightarrow l)$ *at_infinity*
shows $f\ z = l$

proposition *Liouville_weak_inverse*:
assumes f *holomorphic_on UNIV* **and** *unbounded*: $\bigwedge B.$ *eventually* $(\lambda x.$ *norm* $(f\ x) \geq B)$ *at_infinity*
obtains z **where** $f\ z = 0$
theorem *fundamental_theorem_of_algebra*:
fixes a :: *nat* $\Rightarrow$ *complex*
assumes $a\ 0 = 0 \lor (\exists\, i \in \{1..n\}.\ a\ i \neq 0)$
obtains z **where** $(\sum i{\le}n.\ a\ i * z\hat{\ }i) = 0$

4.7 Weierstrass convergence theorem

proposition *has_complex_derivative_uniform_limit*:
fixes z::*complex*
assumes *cont*: *eventually* $(\lambda n.$ *continuous_on* $($*cball* $z\ r)\ (f\ n) \land$
$(\forall\, w \in$ *ball* $z\ r.\ ((f\ n)$ *has_field_derivative* $(f'\ n\ w))$ $($*at* $w)))\ F$
and *ulim*: *uniform_limit* $($*cball* $z\ r)\ f\ g\ F$
and F: $\neg$ *trivial_limit* F **and** $0 < r$
obtains g' **where**
continuous_on $($*cball* $z\ r)\ g$
$\bigwedge w.\ w \in$ *ball* $z\ r \Longrightarrow (g$ *has_field_derivative* $(g'\ w))$ $($*at* $w) \land ((\lambda n.\ f'\ n\ w) \longrightarrow g'\ w)\ F$

4.8 On analytic functions defined by a series

corollary *holomorphic_iff_power_series*:
f *holomorphic_on ball* $z\ r \longleftrightarrow$
$(\forall\, w \in$ *ball* $z\ r.\ (\lambda n.\ ($*deriv* $\hat{}\hat{}\ n)\ f\ z\ /\ ($*fact* $n) * (w - z)\hat{\ }n)$ *sums* $f\ w)$

4.9 General, homology form of Cauchy's theorem

theorem *Cauchy_integral_formula_global*:
assumes S: *open* S **and** *holf*: f *holomorphic_on* S
and z: $z \in S$ **and** *vpg*: *valid_path* γ
and *pasz*: *path_image* $\gamma \subseteq S - \{z\}$ **and** *loop*: *pathfinish* γ = *pathstart* γ

and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow winding\_number\ \gamma\ w = 0$
shows $((\lambda w.\ f\ w\ /\ (w - z))\ has\_contour\_integral\ (2{*}pi * i * winding\_number\ \gamma\ z * f\ z))\ \gamma$

theorem *Cauchy_theorem_global*:
assumes *S*: *open S* **and** *holf*: *f holomorphic_on S*
and *vpg*: *valid_path* γ **and** *loop*: $pathfinish\ \gamma = pathstart\ \gamma$
and *pas*: $path\_image\ \gamma \subseteq S$
and *zero*: $\bigwedge w.\ w \notin S \Longrightarrow winding\_number\ \gamma\ w = 0$
shows $(f\ has\_contour\_integral\ 0)\ \gamma$

corollary *Cauchy_theorem_global_outside*:
assumes *open S f holomorphic_on S valid_path* γ $pathfinish\ \gamma = pathstart\ \gamma$ $path\_image\ \gamma \subseteq S$
$\bigwedge w.\ w \notin S \Longrightarrow w \in outside(path\_image\ \gamma)$
shows $(f\ has\_contour\_integral\ 0)\ \gamma$

4.10 Cauchy's inequality and more versions of Liouville

theorem *Liouville_theorem*:
assumes *holf*: *f holomorphic_on UNIV*
and *bf*: *bounded (range f)*
shows *f constant_on UNIV*

4.11 Complex functions and power series

definition *fps_expansion* :: $(complex \Rightarrow complex) \Rightarrow complex \Rightarrow complex\ fps$
where
$fps\_expansion\ f\ z0 = Abs\_fps\ (\lambda n.\ (deriv\ \widehat{}\widehat{}\ n)\ f\ z0\ /\ fact\ n)$

end

5 Conformal Mappings and Consequences of Cauchy's Integral Theorem

theory *Conformal_Mappings*
imports *Cauchy_Integral_Formula*

begin

5.1 Analytic continuation

proposition *isolated_zeros*:
assumes *holf*: *f holomorphic_on S*
and *open S connected S* $\xi \in S$ $f\ \xi = 0$ $\beta \in S$ $f\ \beta \neq 0$
obtains *r* **where** $0 < r$ **and** $ball\ \xi\ r \subseteq S$ **and**
$\bigwedge z.\ z \in ball\ \xi\ r - \{\xi\} \Longrightarrow f\ z \neq 0$

proposition *analytic_continuation*:
assumes *holf*: f *holomorphic_on* S
and *open* S **and** *connected* S
and $U \subseteq S$ **and** $\xi \in S$
and ξ *islimpt* U
and *fU0* [*simp*]: $\bigwedge z.\ z \in U \Longrightarrow f\ z = 0$
and $w \in S$
shows $f\ w = 0$

corollary *analytic_continuation_open*:
assumes *open* s **and** *open* s' **and** $s \neq \{\}$ **and** *connected* s'
and $s \subseteq s'$
assumes f *holomorphic_on* s' **and** g *holomorphic_on* s'
and $\bigwedge z.\ z \in s \Longrightarrow f\ z = g\ z$
assumes $z \in s'$
shows $f\ z = g\ z$

corollary *analytic_continuation'*:
assumes f *holomorphic_on* S *open* S *connected* S
and $U \subseteq S$ $\xi \in S$ ξ *islimpt* U
and f *constant_on* U
shows f *constant_on* S

5.2 Open mapping theorem

theorem *open_mapping_thm*:
assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$
and *fne*: $\neg$ f *constant_on* S
shows *open* $(f$ ‘ $U)$

5.3 Maximum modulus principle

proposition *maximum_modulus_principle*:
assumes *holf*: f *holomorphic_on* S
and S: *open* S **and** *connected* S
and *open* U **and** $U \subseteq S$ **and** $\xi \in U$
and *no*: $\bigwedge z.\ z \in U \Longrightarrow norm(f\ z) \leq norm(f\ \xi)$
shows f *constant_on* S

proposition *maximum_modulus_frontier*:
assumes *holf*: f *holomorphic_on* (*interior* S)
and *contf*: *continuous_on* (*closure* S) f
and *bos*: *bounded* S
and *leB*: $\bigwedge z.\ z \in frontier\ S \Longrightarrow norm(f\ z) \leq B$

and $\xi \in S$
shows $norm(f\ \xi) \leq B$

5.4 Relating invertibility and nonvanishing of derivative

proposition *holomorphic_has_inverse*:
assumes *holf*: *f holomorphic_on S*
and *open S* **and** *injf*: *inj_on f S*
obtains *g* **where** *g holomorphic_on* (*f ‘ S*)
$\bigwedge z.\ z \in S \Longrightarrow deriv\ f\ z * deriv\ g\ (f\ z) = 1$
$\bigwedge z.\ z \in S \Longrightarrow g(f\ z) = z$

5.5 The Schwarz Lemma

proposition *Schwarz_Lemma*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
and ξ: $norm\ \xi < 1$
shows $norm\ (f\ \xi) \leq norm\ \xi$ **and** $norm(deriv\ f\ 0) \leq 1$
and $((\exists z.\ norm\ z < 1 \wedge z \neq 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\Longrightarrow \exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1$
(**is** $?P \Longrightarrow ?Q$)

corollary *Schwarz_Lemma'*:
assumes *holf*: *f holomorphic_on* (*ball 0 1*) **and** [*simp*]: $f\ 0 = 0$
and *no*: $\bigwedge z.\ norm\ z < 1 \Longrightarrow norm\ (f\ z) < 1$
shows $((\forall \xi.\ norm\ \xi < 1 \longrightarrow norm\ (f\ \xi) \leq norm\ \xi)$
$\wedge\ norm(deriv\ f\ 0) \leq 1)$
$\wedge\ (((\exists z.\ norm\ z < 1 \wedge z \neq 0 \wedge norm(f\ z) = norm\ z)$
$\vee\ norm(deriv\ f\ 0) = 1)$
$\longrightarrow (\exists \alpha.\ (\forall z.\ norm\ z < 1 \longrightarrow f\ z = \alpha * z) \wedge norm\ \alpha = 1))$

5.6 The Schwarz reflection principle

proposition *Schwarz_reflection*:
assumes *open S* **and** *cnjs*: $cnj\ ‘\ S \subseteq S$
and *holf*: *f holomorphic_on* $(S \cap \{z.\ 0 < Im\ z\})$
and *contf*: *continuous_on* $(S \cap \{z.\ 0 \leq Im\ z\})$ *f*
and *f*: $\bigwedge z.\ [\![z \in S;\ z \in \mathbb{R}]\!] \Longrightarrow (f\ z) \in \mathbb{R}$
shows $(\lambda z.\ if\ 0 \leq Im\ z\ then\ f\ z\ else\ cnj(f(cnj\ z)))$ *holomorphic_on S*

5.7 Bloch's theorem

proposition *Bloch_unit*:
  assumes *holf*: *f holomorphic_on ball a 1* **and** [*simp*]: *deriv f a* = *1*
  obtains *b r* **where** *1/12* < *r* **and** *ball b r* ⊆ *f* ‘ (*ball a 1*)

theorem *Bloch*:
  assumes *holf*: *f holomorphic_on ball a r* **and** *0* < *r*
    and *r′*: *r′* ≤ *r* ∗ *norm* (*deriv f a*) / *12*
  obtains *b* **where** *ball b r′* ⊆ *f* ‘ (*ball a r*)

corollary *Bloch_general*:
  assumes *holf*: *f holomorphic_on S* **and** *a* ∈ *S*
    and *tle*: ⋀*z*. *z* ∈ *frontier S* ⟹ *t* ≤ *dist a z*
    and *rle*: *r* ≤ *t* ∗ *norm*(*deriv f a*) / *12*
  obtains *b* **where** *ball b r* ⊆ *f* ‘ *S*

end
theory *Complex_Singularities*
  imports *Conformal_Mappings*
begin

5.8 Non-essential singular points

definition *is_pole* ::
  (*′a*::*topological_space* ⇒ *′b*::*real_normed_vector*) ⇒ *′a* ⇒ *bool* **where**
  is_pole f a = (*LIM x* (*at a*). *f x* :> *at_infinity*)
definition *zorder* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *int* **where**
  zorder f z = (*THE n*. (∃ *h r*. *r*>*0* ∧ *h holomorphic_on cball z r* ∧ *h z*≠*0*
    ∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w*−*z*) *powr* (*of_int n*)
    ∧ *h w* ≠*0*)))

definition *zor_poly*
  ::[*complex* ⇒ *complex*, *complex*] ⇒ *complex* ⇒ *complex* **where**
  zor_poly f z = (*SOME h*. ∃ *r*. *r* > *0* ∧ *h holomorphic_on cball z r* ∧ *h z* ≠ *0*
    ∧ (∀ *w*∈*cball z r* − {*z*}. *f w* = *h w* ∗ (*w* − *z*) *powr* (*zorder f z*)
    ∧ *h w* ≠*0*))

end
theory *Complex_Residues*
  imports *Complex_Singularities*
begin

5.9 Definition of residues

definition *residue* :: (*complex* ⇒ *complex*) ⇒ *complex* ⇒ *complex* **where**
  residue f z = (*SOME int*. ∃ *e*>*0*. ∀ ε>*0*. ε<*e*

$\longrightarrow$ (*f has_contour_integral 2*pi** i **int*) (*circlepath z* ε))

theorem *residue_fps_expansion_over_power_at_0*:
assumes *f has_fps_expansion F*
shows *residue* (λz. *f z* / *z* ^ *Suc n*) *0* = *fps_nth F n*

5.10 Poles and residues of some well-known functions

end

6 The Residue Theorem, the Argument Principle and Rouché's Theorem

theory *Residue_Theorem*
imports *Complex_Residues HOL−Library.Landau_Symbols*
begin

6.1 Cauchy's residue theorem

theorem *Residue_theorem*:
fixes *s pts*::*complex set* **and** *f*::*complex* ⇒ *complex*
and *g*::*real* ⇒ *complex*
assumes *open s connected s finite pts* **and**
holo:*f holomorphic_on s−pts* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_image g ⊆ *s−pts* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number g z* = *0*
shows *contour_integral g f* = *2* ∗ *pi* ∗ i ∗(∑ *p*∈*pts*. *winding_number g p* ∗ *residue f p*)

6.2 The argument principle

theorem *argument_principle*:
fixes *f*::*complex* ⇒ *complex* **and** *poles s*:: *complex set*
defines *pz* ≡ {*w*. *f w* = *0* ∨ *w* ∈ *poles*} — *pz* is the set of poles and zeros
assumes *open s* **and**
connected s **and**
f_holo:*f holomorphic_on s−poles* **and**
h_holo:*h holomorphic_on s* **and**
valid_path g **and**
loop:*pathfinish g* = *pathstart g* **and**
path_img:*path_image g* ⊆ *s* − *pz* **and**
homo:∀ *z*. (*z* ∉ *s*) ⟶ *winding_number g z* = *0* **and**
finite:*finite pz* **and**

poles:∀ p∈poles. is_pole f p
shows contour_integral g (λx. deriv f x ∗ h x / f x) = 2 ∗ pi ∗ i ∗
(∑ p∈pz. winding_number g p ∗ h p ∗ zorder f p)
(**is** ?L=?R)

6.3 Coefficient asymptotics for generating functions

theorem
fixes f :: complex ⇒ complex **and** n :: nat **and** r :: real
defines g ≡ (λw. f w / w ^ Suc n) **and** γ ≡ circlepath 0 r
assumes open A connected A cball 0 r ⊆ A r > 0
assumes f holomorphic_on A − S S ⊆ ball 0 r finite S 0 ∉ S
shows fps_coeff_conv_residues:
(deriv ^^ n) f 0 / fact n =
contour_integral γ g / (2 ∗ pi ∗ i) − (∑ z∈S. residue g z) (**is** ?thesis1)
and fps_coeff_residues_bound:
(⋀z. norm z = r ⟹ z ∉ k ⟹ norm (f z) ≤ C) ⟹ C ≥ 0 ⟹ finite k ⟹
norm ((deriv ^^ n) f 0 / fact n + (∑ z∈S. residue g z)) ≤ C / r ^ n
corollary fps_coeff_residues_bigo:
fixes f :: complex ⇒ complex **and** r :: real
assumes open A connected A cball 0 r ⊆ A r > 0
assumes f holomorphic_on A − S S ⊆ ball 0 r finite S 0 ∉ S
assumes g: eventually (λn. g n = −(∑ z∈S. residue (λz. f z / z ^ Suc n) z)) sequentially
(**is** eventually (λn. _ = −?g′ n) _)
shows (λn. (deriv ^^ n) f 0 / fact n − g n) ∈ O(λn. 1 / r ^ n) (**is** (λn. ?c n − _) ∈ O(_))

corollary fps_coeff_residues_bigo′:
fixes f :: complex ⇒ complex **and** r :: real
assumes exp: f has_fps_expansion F
assumes open A connected A cball 0 r ⊆ A r > 0
assumes f holomorphic_on A − S S ⊆ ball 0 r finite S 0 ∉ S
assumes eventually (λn. g n = −(∑ z∈S. residue (λz. f z / z ^ Suc n) z)) sequentially
(**is** eventually (λn. _ = −?g′ n) _)
shows (λn. fps_nth F n − g n) ∈ O(λn. 1 / r ^ n) (**is** (λn. ?c n − _) ∈ O(_))

6.4 Rouche's theorem

theorem Rouche_theorem:
fixes f g::complex ⇒ complex **and** s:: complex set
defines fg≡(λp. f p + g p)
defines zeros_fg≡{p. fg p = 0} **and** zeros_f≡{p. f p = 0}
assumes
open s **and** connected s **and**

$\textit{finite zeros\_fg}$ **and**
$\textit{finite zeros\_f}$ **and**
$\textit{f\_holo}{:}\textit{f holomorphic\_on } s$ **and**
$\textit{g\_holo}{:}\textit{g holomorphic\_on } s$ **and**
$\textit{valid\_path } \gamma$ **and**
$\textit{loop}{:}\textit{pathfinish } \gamma = \textit{pathstart } \gamma$ **and**
$\textit{path\_img}{:}\textit{path\_image } \gamma \subseteq s$ **and**
$\textit{path\_less}{:}\forall z{\in}\textit{path\_image } \gamma.\ \textit{cmod}(f\ z) > \textit{cmod}(g\ z)$ **and**
$\textit{homo}{:}\forall z.\ (z \notin s) \longrightarrow \textit{winding\_number } \gamma\ z = 0$
shows $(\sum p{\in}\textit{zeros\_fg}.\ \textit{winding\_number } \gamma\ p * \textit{zorder fg } p)$
$= (\sum p{\in}\textit{zeros\_f}.\ \textit{winding\_number } \gamma\ p * \textit{zorder } f\ p)$

end

7 The Great Picard Theorem and its Applications

theory *Great_Picard*
imports *Conformal_Mappings*
begin

7.1 Schottky's theorem

theorem *Schottky*:
assumes $\textit{holf}$: $f \textit{ holomorphic\_on cball } 0\ 1$
and $\textit{nof0}$: $\textit{norm}(f\ 0) \le r$
and $\textit{not01}$: $\bigwedge z.\ z \in \textit{cball } 0\ 1 \Longrightarrow \neg(f\ z = 0 \lor f\ z = 1)$
and $0 < t\ t < 1\ \textit{norm } z \le t$
shows $\textit{norm}(f\ z) \le \textit{exp}(\textit{pi} * \textit{exp}(\textit{pi} * (2 + 2 * r + 12 * t / (1 - t))))$

7.2 The Little Picard Theorem

theorem *Landau_Picard*:
obtains R
where $\bigwedge z.\ 0 < R\ z$
$\bigwedge f.\ [\![f \textit{ holomorphic\_on cball } 0\ (R(f\ 0));$
$\bigwedge z.\ \textit{norm } z \le R(f\ 0) \Longrightarrow f\ z \ne 0 \land f\ z \ne 1]\!] \Longrightarrow \textit{norm}(\textit{deriv } f\ 0)$
< 1

theorem *little_Picard*:
assumes $\textit{holf}$: $f \textit{ holomorphic\_on UNIV}$
and $a \ne b\ \textit{range } f \cap \{a,b\} = \{\}$
obtains c **where** $f = (\lambda x.\ c)$

7.3 The Arzelà–Ascoli theorem

theorem *Arzela_Ascoli*:
fixes $\mathcal{F}$ :: [*nat*,'*a*::*euclidean_space*] $\Rightarrow$ '*b*::{*real_normed_vector*,*heine_borel*}
assumes *compact S*
and *M*: $\bigwedge n\ x.\ x \in S \Longrightarrow norm(\mathcal{F}\ n\ x) \le M$
and *equicont*:
$\bigwedge x\ e.\ [\![x \in S;\ 0 < e]\!]$
$\Longrightarrow \exists\, d.\ 0 < d \wedge (\forall\, n\ y.\ y \in S \wedge norm(x - y) < d \longrightarrow norm(\mathcal{F}\ n\ x - \mathcal{F}\ n\ y) < e)$
obtains *g k* **where** *continuous_on S g strict_mono* $(k :: nat \Rightarrow nat)$
$\bigwedge e.\ 0 < e \Longrightarrow \exists\, N.\ \forall\, n\ x.\ n \ge N \wedge x \in S \longrightarrow norm(\mathcal{F}(k\ n)\ x - g\ x) < e$

7.3.1 Montel's theorem

theorem *Montel*:
fixes $\mathcal{F}$ :: [*nat*,*complex*] $\Rightarrow$ *complex*
assumes *open S*
and $\mathcal{H}$: $\bigwedge h.\ h \in \mathcal{H} \Longrightarrow h$ *holomorphic_on S*
and *bounded*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow \exists\, B.\ \forall\, h \in \mathcal{H}.\ \forall\ z \in K.\ norm(h\ z) \le B$
and *rng_f*: *range* $\mathcal{F} \subseteq \mathcal{H}$
obtains *g r*
where *g holomorphic_on S strict_mono* $(r :: nat \Rightarrow nat)$
$\bigwedge x.\ x \in S \Longrightarrow ((\lambda n.\ \mathcal{F}\ (r\ n)\ x) \longrightarrow g\ x)$ *sequentially*
$\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $(\mathcal{F} \circ r)$ *g sequentially*

7.4 Some simple but useful cases of Hurwitz's theorem

proposition *Hurwitz_no_zeros*:
assumes *S*: *open S connected S*
and *holf*: $\bigwedge n{::}nat.\ \mathcal{F}\ n$ *holomorphic_on S*
and *holg*: *g holomorphic_on S*
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit K* $\mathcal{F}$ *g sequentially*
and *nonconst*: $\neg$ *g constant_on S*
and *nz*: $\bigwedge n\ z.\ z \in S \Longrightarrow \mathcal{F}\ n\ z \ne 0$
and $z0 \in S$
shows $g\ z0 \ne 0$

corollary *Hurwitz_injective*:
assumes *S*: *open S connected S*

and *holf*: $\bigwedge n{::}nat.\ \mathcal{F}\ n$ *holomorphic_on* S
and *holg*: g *holomorphic_on* S
and *ul_g*: $\bigwedge K.\ [\![compact\ K;\ K \subseteq S]\!] \Longrightarrow$ *uniform_limit* $K\ \mathcal{F}\ g$ *sequentially*
and *nonconst*: $\neg\ g$ *constant_on* S
and *inj*: $\bigwedge n.$ *inj_on* $(\mathcal{F}\ n)\ S$
shows *inj_on* $g\ S$

7.5 The Great Picard theorem

theorem *great_Picard*:
assumes *open* $M\ z \in M\ a \neq b$ **and** *holf*: f *holomorphic_on* $(M - \{z\})$
and *fab*: $\bigwedge w.\ w \in M - \{z\} \Longrightarrow f\ w \neq a \wedge f\ w \neq b$
obtains l **where** $(f \longrightarrow l)\ (at\ z) \vee ((inverse \circ f) \longrightarrow l)\ (at\ z)$

corollary *great_Picard_alt*:
assumes M: *open* $M\ z \in M$ **and** *holf*: f *holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains a **where** $- \{a\} \subseteq f\ `\ (M - \{z\})$

corollary *great_Picard_infinite*:
assumes M: *open* $M\ z \in M$ **and** *holf*: f *holomorphic_on* $(M - \{z\})$
and *non*: $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
obtains a **where** $\bigwedge w.\ w \neq a \Longrightarrow infinite\ \{x.\ x \in M - \{z\} \wedge f\ x = w\}$

theorem *Casorati_Weierstrass*:
assumes *open* $M\ z \in M\ f$ *holomorphic_on* $(M - \{z\})$
and $\bigwedge l.\ \neg\ (f \longrightarrow l)\ (at\ z)\ \bigwedge l.\ \neg\ ((inverse \circ f) \longrightarrow l)\ (at\ z)$
shows $closure(f\ `\ (M - \{z\})) = UNIV$

end

8 Moebius functions, Equivalents of Simply Connected Sets, Riemann Mapping Theorem

theory *Riemann_Mapping*
imports *Great_Picard*
begin

8.1 Moebius functions are biholomorphisms of the unit disc

definition *Moebius_function* :: [*real*,*complex*,*complex*] ⇒ *complex* **where**
Moebius_function ≡ λ*t w z*. *exp*(i ∗ *of_real t*) ∗ (*z* − *w*) / (*1* − *cnj w* ∗ *z*)

8.2 A big chain of equivalents of simple connectedness for an open set

proposition
assumes *open S*
shows *simply_connected_eq_winding_number_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g z*. *path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ (*z* ∉ *S*)
⟶ *winding_number g z* = *0*) (**is** *?wn0*)
and *simply_connected_eq_contour_integral_zero*:
simply_connected S ⟷
connected S ∧
(∀ *g f*. *valid_path g* ∧ *path_image g* ⊆ *S* ∧
pathfinish g = *pathstart g* ∧ *f holomorphic_on S*
⟶ (*f has_contour_integral 0*) *g*) (**is** *?ci0*)
and *simply_connected_eq_global_primitive*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ⟶
(∃ *h*. ∀ *z*. *z* ∈ *S* ⟶ (*h has_field_derivative f z*) (*at z*))) (**is** *?gp*)
and *simply_connected_eq_holomorphic_log*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *g holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* = *exp*(*g z*)))) (**is** *?log*)
and *simply_connected_eq_holomorphic_sqrt*:
simply_connected S ⟷
connected S ∧
(∀ *f*. *f holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* ≠ *0*)
⟶ (∃ *g*. *g holomorphic_on S* ∧ (∀ *z* ∈ *S*. *f z* = $(g\ z)^2$))) (**is** *?sqrt*)
and *simply_connected_eq_biholomorphic_to_disc*:
simply_connected S ⟷
S = {} ∨ *S* = *UNIV* ∨
(∃ *f g*. *f holomorphic_on S* ∧ *g holomorphic_on ball 0 1* ∧
(∀ *z* ∈ *S*. *f z* ∈ *ball 0 1* ∧ *g*(*f z*) = *z*) ∧
(∀ *z* ∈ *ball 0 1*. *g z* ∈ *S* ∧ *f*(*g z*) = *z*)) (**is** *?bih*)
and *simply_connected_eq_homeomorphic_to_disc*:
simply_connected S ⟷ *S* = {} ∨ *S homeomorphic ball* (*0*::*complex*) *1*
(**is** *?disc*)

corollary *contractible_eq_simply_connected_2d*:
  fixes S :: *complex set*
  shows *open* $S \Longrightarrow$ (*contractible* $S \longleftrightarrow$ *simply_connected* S)

8.3 A further chain of equivalences about components of the complement of a simply connected set

proposition
  fixes S :: *complex set*
  assumes *open* S
  shows *simply_connected_eq_frontier_properties*:
    simply_connected $S \longleftrightarrow$
     connected $S \wedge$
      (*if bounded* S *then connected*(*frontier* S)
      else ($\forall$ $C \in$ *components*(*frontier* S). $\neg$*bounded* C)) (**is** *?fp*)
   and *simply_connected_eq_unbounded_complement_components*:
    simply_connected $S \longleftrightarrow$
     connected $S \wedge$ ($\forall$ $C \in$ *components*($-$ S). $\neg$*bounded* C) (**is** *?ucc*)
   and *simply_connected_eq_empty_inside*:
    simply_connected $S \longleftrightarrow$
     connected $S \wedge$ *inside* S = {} (**is** *?ei*)

8.4 Further equivalences based on continuous logs and sqrts

proposition
  fixes S :: *complex set*
  assumes *open* S
  shows *simply_connected_eq_continuous_log*:
    simply_connected $S \longleftrightarrow$
     connected $S \wedge$
     ($\forall f$::*complex*$\Rightarrow$*complex*. *continuous_on* S f $\wedge$ ($\forall z \in S$. f $z \neq 0$)
      $\longrightarrow$ ($\exists g$. *continuous_on* S g $\wedge$ ($\forall z \in S$. f z = *exp* (g z)))) (**is** *?log*)
   and *simply_connected_eq_continuous_sqrt*:
    simply_connected $S \longleftrightarrow$
     connected $S \wedge$
     ($\forall f$::*complex*$\Rightarrow$*complex*. *continuous_on* S f $\wedge$ ($\forall z \in S$. f $z \neq 0$)
      $\longrightarrow$ ($\exists g$. *continuous_on* S g $\wedge$ ($\forall z \in S$. f z = $(g\ z)^2$))) (**is** *?sqrt*)

8.5 Finally, the Riemann Mapping Theorem

theorem *Riemann_mapping_theorem*:
   open $S \wedge$ *simply_connected* $S \longleftrightarrow$
    S = {} $\vee$ S = *UNIV* $\vee$
    ($\exists f\ g$. f *holomorphic_on* S $\wedge$ g *holomorphic_on ball 0 1* $\wedge$

$(\forall z \in S.\ f\ z \in ball\ 0\ 1 \wedge g(f\ z) = z) \wedge$
$(\forall z \in ball\ 0\ 1.\ g\ z \in S \wedge f(g\ z) = z))$
(**is** _ = *?rhs*)

8.6 Applications to Winding Numbers

8.7 Winding number equality is the same as path/loop homotopy in C - 0

proposition *winding_number_homotopic_paths_eq*:
assumes *path p* **and** ζp: $\zeta \notin$ *path_image p*
and *path q* **and** ζq: $\zeta \notin$ *path_image q*
and *qp*: *pathstart q* = *pathstart p pathfinish q* = *pathfinish p*
shows *winding_number p* ζ = *winding_number q* $\zeta \longleftrightarrow$ *homotopic_paths* $(-\{\zeta\})$ *p q*
(**is** *?lhs* = *?rhs*)

end
theory *Complex_Analysis*
imports
Residue_Theorem
Riemann_Mapping
begin

end

References

[1]