

Hoare Logic for Parallel Programs

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Abstract

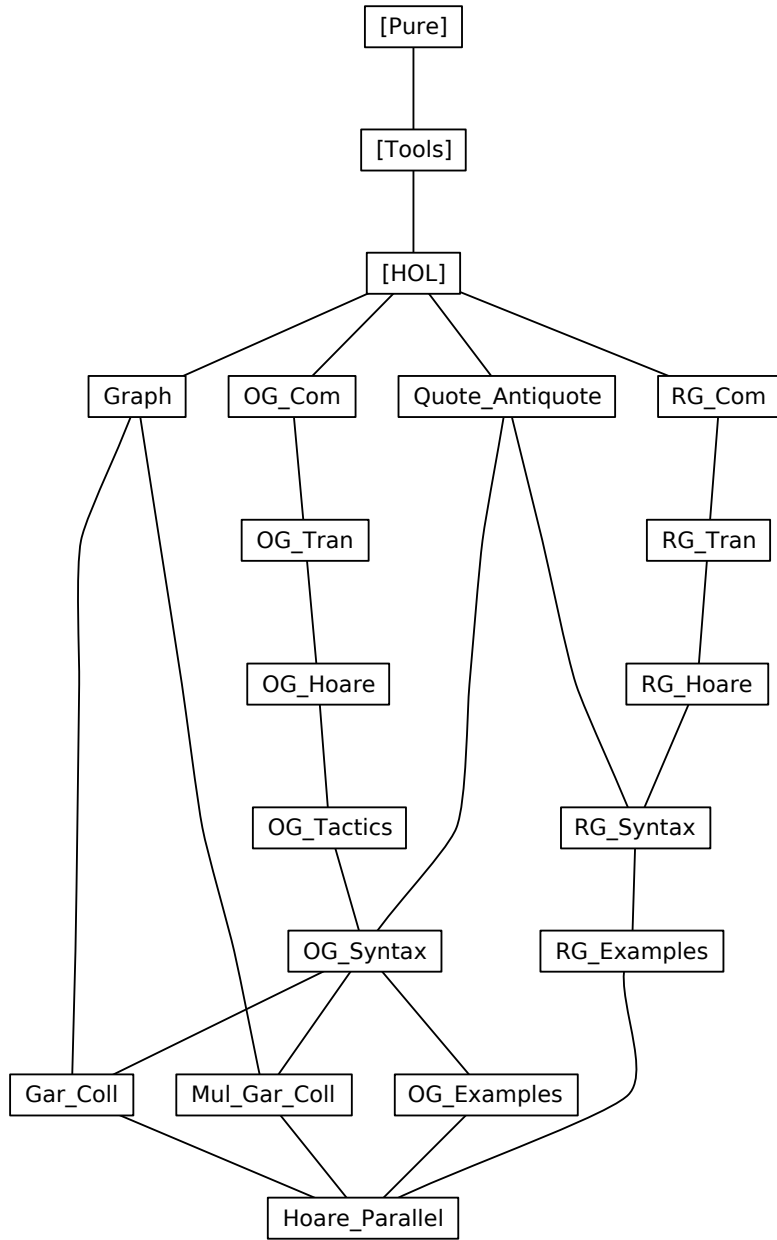
In the following theories a formalization of the Owicki-Gries and the rely-guarantee methods is presented. These methods are widely used for correctness proofs of parallel imperative programs with shared variables. We define syntax, semantics and proof rules in Isabelle/HOL. The proof rules also provide for programs parameterized in the number of parallel components. Their correctness w.r.t. the semantics is proven. Completeness proofs for both methods are extended to the new case of parameterized programs. (These proofs have not been formalized in Isabelle. They can be found in [1].) Using this formalizations we verify several non-trivial examples for parameterized and non-parameterized programs. For the automatic generation of verification conditions with the Owicki-Gries method we define a tactic based on the proof rules. The most involved examples are the verification of two garbage-collection algorithms, the second one parameterized in the number of mutators.

For excellent descriptions of this work see [2, 4, 1, 3].

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Chapter 1

The Owicki-Gries Method

1.1 Abstract Syntax

theory *OG-Com* **imports** *Main* **begin**

Type abbreviations for boolean expressions and assertions:

type-synonym *'a bexp* = *'a set*

type-synonym *'a assn* = *'a set*

The syntax of commands is defined by two mutually recursive datatypes: *'a ann-com* for annotated commands and *'a com* for non-annotated commands.

datatype *'a ann-com* =
 AnnBasic (*'a assn*) (*'a $\Rightarrow$ 'a*)
 | *AnnSeq* (*'a ann-com*) (*'a ann-com*)
 | *AnnCond1* (*'a assn*) (*'a bexp*) (*'a ann-com*) (*'a ann-com*)
 | *AnnCond2* (*'a assn*) (*'a bexp*) (*'a ann-com*)
 | *AnnWhile* (*'a assn*) (*'a bexp*) (*'a assn*) (*'a ann-com*)
 | *AnnAwait* (*'a assn*) (*'a bexp*) (*'a com*)
and *'a com* =
 Parallel (*'a ann-com option* $\times$ *'a assn*) *list*
 | *Basic* (*'a $\Rightarrow$ 'a*)
 | *Seq* (*'a com*) (*'a com*)
 | *Cond* (*'a bexp*) (*'a com*) (*'a com*)
 | *While* (*'a bexp*) (*'a assn*) (*'a com*)

The function *pre* extracts the precondition of an annotated command:

primrec *pre* :: *'a ann-com $\Rightarrow$ 'a assn* **where**
 pre (*AnnBasic* *r f*) = *r*
 | *pre* (*AnnSeq* *c1 c2*) = *pre c1*
 | *pre* (*AnnCond1* *r b c1 c2*) = *r*
 | *pre* (*AnnCond2* *r b c*) = *r*
 | *pre* (*AnnWhile* *r b i c*) = *r*
 | *pre* (*AnnAwait* *r b c*) = *r*

Well-formedness predicate for atomic programs:

```

primrec atom-com :: 'a com  $\Rightarrow$  bool where
  atom-com (Parallel Ts) = False
| atom-com (Basic f) = True
| atom-com (Seq c1 c2) = (atom-com c1  $\wedge$  atom-com c2)
| atom-com (Cond b c1 c2) = (atom-com c1  $\wedge$  atom-com c2)
| atom-com (While b i c) = atom-com c

end

```

1.2 Operational Semantics

theory OG-Tran **imports** OG-Com **begin**

```

type-synonym 'a ann-com-op = ('a ann-com) option
type-synonym 'a ann-triple-op = ('a ann-com-op  $\times$  'a assn)

```

```

primrec com :: 'a ann-triple-op  $\Rightarrow$  'a ann-com-op where
  com (c, q) = c

```

```

primrec post :: 'a ann-triple-op  $\Rightarrow$  'a assn where
  post (c, q) = q

```

```

definition All-None :: 'a ann-triple-op list  $\Rightarrow$  bool where
  All-None Ts  $\equiv \forall (c, q) \in \text{set } Ts. c = \text{None}$ 

```

1.2.1 The Transition Relation

inductive-set

```

  ann-transition :: (('a ann-com-op  $\times$  'a)  $\times$  ('a ann-com-op  $\times$  'a)) set
and transition :: (('a com  $\times$  'a)  $\times$  ('a com  $\times$  'a)) set
and ann-transition' :: ('a ann-com-op  $\times$  'a)  $\Rightarrow$  ('a ann-com-op  $\times$  'a)  $\Rightarrow$  bool
  (-  $-1 \rightarrow$  -[81,81] 100)
and transition' :: ('a com  $\times$  'a)  $\Rightarrow$  ('a com  $\times$  'a)  $\Rightarrow$  bool
  (-  $-P1 \rightarrow$  -[81,81] 100)
and transitions :: ('a com  $\times$  'a)  $\Rightarrow$  ('a com  $\times$  'a)  $\Rightarrow$  bool
  (-  $-P* \rightarrow$  -[81,81] 100)

```

where

```

  con-0  $-1 \rightarrow$  con-1  $\equiv (con-0, con-1) \in \text{ann-transition}$ 
| con-0  $-P1 \rightarrow$  con-1  $\equiv (con-0, con-1) \in \text{transition}$ 
| con-0  $-P* \rightarrow$  con-1  $\equiv (con-0, con-1) \in \text{transition}^*$ 

| AnnBasic: (Some (AnnBasic r f), s)  $-1 \rightarrow$  (None, f s)

| AnnSeq1: (Some c0, s)  $-1 \rightarrow$  (None, t)  $\implies$ 
  (Some (AnnSeq c0 c1), s)  $-1 \rightarrow$  (Some c1, t)
| AnnSeq2: (Some c0, s)  $-1 \rightarrow$  (Some c2, t)  $\implies$ 
  (Some (AnnSeq c0 c1), s)  $-1 \rightarrow$  (Some (AnnSeq c2 c1), t)

| AnnCond1T: s  $\in b \implies (Some (AnnCond1 r b c1 c2), s)  $-1 \rightarrow$  (Some c1, s)$ 

```

| *AnnCond1F*: $s \notin b \implies (\text{Some } (\text{AnnCond1 } r \ b \ c1 \ c2), s) -1 \rightarrow (\text{Some } c2, s)$

| *AnnCond2T*: $s \in b \implies (\text{Some } (\text{AnnCond2 } r \ b \ c), s) -1 \rightarrow (\text{Some } c, s)$

| *AnnCond2F*: $s \notin b \implies (\text{Some } (\text{AnnCond2 } r \ b \ c), s) -1 \rightarrow (\text{None}, s)$

| *AnnWhileF*: $s \notin b \implies (\text{Some } (\text{AnnWhile } r \ b \ i \ c), s) -1 \rightarrow (\text{None}, s)$

| *AnnWhileT*: $s \in b \implies (\text{Some } (\text{AnnWhile } r \ b \ i \ c), s) -1 \rightarrow$
 $(\text{Some } (\text{AnnSeq } c \ (\text{AnnWhile } i \ b \ i \ c)), s)$

| *AnnAwait*: $\llbracket s \in b; \text{atom-com } c; (c, s) -P* \rightarrow (\text{Parallel } [], t) \rrbracket \implies$
 $(\text{Some } (\text{AnnAwait } r \ b \ c), s) -1 \rightarrow (\text{None}, t)$

| *Parallel*: $\llbracket i < \text{length } Ts; Ts!i = (\text{Some } c, q); (\text{Some } c, s) -1 \rightarrow (r, t) \rrbracket$
 $\implies (\text{Parallel } Ts, s) -P1 \rightarrow (\text{Parallel } (Ts \ [i := (r, q)]), t)$

| *Basic*: $(\text{Basic } f, s) -P1 \rightarrow (\text{Parallel } [], f \ s)$

| *Seq1*: $\text{All-None } Ts \implies (\text{Seq } (\text{Parallel } Ts) \ c, s) -P1 \rightarrow (c, s)$

| *Seq2*: $(c0, s) -P1 \rightarrow (c2, t) \implies (\text{Seq } c0 \ c1, s) -P1 \rightarrow (\text{Seq } c2 \ c1, t)$

| *CondT*: $s \in b \implies (\text{Cond } b \ c1 \ c2, s) -P1 \rightarrow (c1, s)$

| *CondF*: $s \notin b \implies (\text{Cond } b \ c1 \ c2, s) -P1 \rightarrow (c2, s)$

| *WhileF*: $s \notin b \implies (\text{While } b \ i \ c, s) -P1 \rightarrow (\text{Parallel } [], s)$

| *WhileT*: $s \in b \implies (\text{While } b \ i \ c, s) -P1 \rightarrow (\text{Seq } c \ (\text{While } b \ i \ c), s)$

monos *rtrancl-mono*

The corresponding abbreviations are:

abbreviation

ann-transition-n :: $('a \ \text{ann-com-op} \times 'a) \Rightarrow \text{nat} \Rightarrow ('a \ \text{ann-com-op} \times 'a)$
 $\Rightarrow \text{bool } (- \dashrightarrow -[81,81] \ 100) \ \mathbf{where}$
con-0 $-n \rightarrow$ *con-1* $\equiv (\text{con-0}, \text{con-1}) \in \text{ann-transition} \ \widetilde{\sim} \ n$

abbreviation

ann-transitions :: $('a \ \text{ann-com-op} \times 'a) \Rightarrow ('a \ \text{ann-com-op} \times 'a) \Rightarrow \text{bool}$
 $(- \dashrightarrow -[81,81] \ 100) \ \mathbf{where}$
con-0 $\dashrightarrow$ *con-1* $\equiv (\text{con-0}, \text{con-1}) \in \text{ann-transition}^*$

abbreviation

transition-n :: $('a \ \text{com} \times 'a) \Rightarrow \text{nat} \Rightarrow ('a \ \text{com} \times 'a) \Rightarrow \text{bool}$
 $(- \dashrightarrow -P \dashrightarrow -[81,81,81] \ 100) \ \mathbf{where}$
con-0 $-Pn \rightarrow$ *con-1* $\equiv (\text{con-0}, \text{con-1}) \in \text{transition} \ \widetilde{\sim} \ n$

1.2.2 Definition of Semantics

definition *ann-sem* :: $'a \ \text{ann-com} \Rightarrow 'a \Rightarrow 'a \ \text{set} \ \mathbf{where}$

ann-sem $c \equiv \lambda s. \{t. (\text{Some } c, s) \dashrightarrow (\text{None}, t)\}$

definition $ann\text{-}SEM :: 'a\ ann\text{-}com \Rightarrow 'a\ set \Rightarrow 'a\ set$ **where**
 $ann\text{-}SEM\ c\ S \equiv \bigcup (ann\text{-}sem\ c\ `S)$

definition $sem :: 'a\ com \Rightarrow 'a \Rightarrow 'a\ set$ **where**
 $sem\ c \equiv \lambda s. \{t. \exists Ts. (c, s) -P* \rightarrow (Parallel\ Ts, t) \wedge All\text{-}None\ Ts\}$

definition $SEM :: 'a\ com \Rightarrow 'a\ set \Rightarrow 'a\ set$ **where**
 $SEM\ c\ S \equiv \bigcup (sem\ c\ `S)$

abbreviation $\Omega :: 'a\ com$ ($\Omega\ 63$)
where $\Omega \equiv While\ UNIV\ UNIV\ (Basic\ id)$

primrec $fwhile :: 'a\ bexp \Rightarrow 'a\ com \Rightarrow nat \Rightarrow 'a\ com$ **where**
 $fwhile\ b\ c\ 0 = \Omega$
 $| fwhile\ b\ c\ (Suc\ n) = Cond\ b\ (Seq\ c\ (fwhile\ b\ c\ n))\ (Basic\ id)$

Proofs

declare $ann\text{-}transition\text{-}transition.intros$ [intro]

inductive-cases $transition\text{-}cases$:

$(Parallel\ T, s) -P1 \rightarrow t$
 $(Basic\ f, s) -P1 \rightarrow t$
 $(Seq\ c1\ c2, s) -P1 \rightarrow t$
 $(Cond\ b\ c1\ c2, s) -P1 \rightarrow t$
 $(While\ b\ i\ c, s) -P1 \rightarrow t$

lemma $Parallel\text{-}empty\text{-}lemma$ [rule-format (no-asm)]:
 $(Parallel\ [], s) -Pn \rightarrow (Parallel\ Ts, t) \longrightarrow Ts=[] \wedge n=0 \wedge s=t$
 $\langle proof \rangle$

lemma $Parallel\text{-}AllNone\text{-}lemma$ [rule-format (no-asm)]:
 $All\text{-}None\ Ss \longrightarrow (Parallel\ Ss, s) -Pn \rightarrow (Parallel\ Ts, t) \longrightarrow Ts=Ss \wedge n=0 \wedge s=t$
 $\langle proof \rangle$

lemma $Parallel\text{-}AllNone$: $All\text{-}None\ Ts \Longrightarrow (SEM\ (Parallel\ Ts)\ X) = X$
 $\langle proof \rangle$

lemma $Parallel\text{-}empty$: $Ts=[] \Longrightarrow (SEM\ (Parallel\ Ts)\ X) = X$
 $\langle proof \rangle$

Set of lemmas from Apt and Olderog "Verification of sequential and concurrent programs", page 63.

lemma $L3\text{-}5i$: $X \subseteq Y \Longrightarrow SEM\ c\ X \subseteq SEM\ c\ Y$
 $\langle proof \rangle$

lemma $L3\text{-}5ii\text{-}lemma1$:
 $\llbracket (c1, s1) -P* \rightarrow (Parallel\ Ts, s2); All\text{-}None\ Ts;$
 $(c2, s2) -P* \rightarrow (Parallel\ Ss, s3); All\text{-}None\ Ss \rrbracket$
 $\Longrightarrow (Seq\ c1\ c2, s1) -P* \rightarrow (Parallel\ Ss, s3)$

$\langle proof \rangle$

lemma *L3-5ii-lemma2* [rule-format (no-asm)]:

$\forall c1\ c2\ s\ t. (Seq\ c1\ c2,\ s) -Pn \rightarrow (Parallel\ Ts,\ t) \rightarrow$
 $(All\ None\ Ts) \rightarrow (\exists y\ m\ Rs. (c1,\ s) -P* \rightarrow (Parallel\ Rs,\ y) \wedge$
 $(All\ None\ Rs) \wedge (c2,\ y) -Pm \rightarrow (Parallel\ Ts,\ t) \wedge m \leq n)$
 $\langle proof \rangle$

lemma *L3-5ii-lemma3*:

$\llbracket (Seq\ c1\ c2,\ s) -P* \rightarrow (Parallel\ Ts,\ t); All\ None\ Ts \rrbracket \implies$
 $(\exists y\ Rs. (c1,\ s) -P* \rightarrow (Parallel\ Rs,\ y) \wedge All\ None\ Rs$
 $\wedge (c2,\ y) -P* \rightarrow (Parallel\ Ts,\ t))$
 $\langle proof \rangle$

lemma *L3-5ii*: $SEM\ (Seq\ c1\ c2)\ X = SEM\ c2\ (SEM\ c1\ X)$

$\langle proof \rangle$

lemma *L3-5iii*: $SEM\ (Seq\ (Seq\ c1\ c2)\ c3)\ X = SEM\ (Seq\ c1\ (Seq\ c2\ c3))\ X$

$\langle proof \rangle$

lemma *L3-5iv*:

$SEM\ (Cond\ b\ c1\ c2)\ X = (SEM\ c1\ (X \cap b))\ Un\ (SEM\ c2\ (X \cap (-b)))$
 $\langle proof \rangle$

lemma *L3-5v-lemma1* [rule-format]:

$(S,\ s) -Pn \rightarrow (T,\ t) \rightarrow S = \Omega \rightarrow (\neg(\exists Rs. T = (Parallel\ Rs) \wedge All\ None\ Rs))$
 $\langle proof \rangle$

lemma *L3-5v-lemma2*: $\llbracket (\Omega,\ s) -P* \rightarrow (Parallel\ Ts,\ t); All\ None\ Ts \rrbracket \implies False$

$\langle proof \rangle$

lemma *L3-5v-lemma3*: $SEM\ (\Omega)\ S = \{\}$

$\langle proof \rangle$

lemma *L3-5v-lemma4* [rule-format]:

$\forall s. (While\ b\ i\ c,\ s) -Pn \rightarrow (Parallel\ Ts,\ t) \rightarrow All\ None\ Ts \rightarrow$
 $(\exists k. (fwhile\ b\ c\ k,\ s) -P* \rightarrow (Parallel\ Ts,\ t))$
 $\langle proof \rangle$

lemma *L3-5v-lemma5* [rule-format]:

$\forall s. (fwhile\ b\ c\ k,\ s) -P* \rightarrow (Parallel\ Ts,\ t) \rightarrow All\ None\ Ts \rightarrow$
 $(While\ b\ i\ c,\ s) -P* \rightarrow (Parallel\ Ts,\ t)$
 $\langle proof \rangle$

lemma *L3-5v*: $SEM\ (While\ b\ i\ c) = (\lambda x. (\bigcup k. SEM\ (fwhile\ b\ c\ k)\ x))$

$\langle proof \rangle$

1.3 Validity of Correctness Formulas

definition *com-validity* :: 'a assn $\Rightarrow$ 'a com $\Rightarrow$ 'a assn $\Rightarrow$ bool (($\exists$ $\models$ -// -//-) [90,55,90] 50) **where**
 $\models p \ c \ q \equiv SEM \ c \ p \subseteq q$

definition *ann-com-validity* :: 'a ann-com $\Rightarrow$ 'a assn $\Rightarrow$ bool ($\models$ - - [60,90] 45) **where**
 $\models c \ q \equiv ann-SEM \ c \ (pre \ c) \subseteq q$

end

1.4 The Proof System

theory *OG-Hoare* **imports** *OG-Tran* **begin**

primrec *assertions* :: 'a ann-com $\Rightarrow$ ('a assn) set **where**
 $assertions \ (AnnBasic \ r \ f) = \{r\}$
 $| \ assertions \ (AnnSeq \ c1 \ c2) = assertions \ c1 \cup assertions \ c2$
 $| \ assertions \ (AnnCond1 \ r \ b \ c1 \ c2) = \{r\} \cup assertions \ c1 \cup assertions \ c2$
 $| \ assertions \ (AnnCond2 \ r \ b \ c) = \{r\} \cup assertions \ c$
 $| \ assertions \ (AnnWhile \ r \ b \ i \ c) = \{r, i\} \cup assertions \ c$
 $| \ assertions \ (AnnAwait \ r \ b \ c) = \{r\}$

primrec *atomics* :: 'a ann-com $\Rightarrow$ ('a assn $\times$ 'a com) set **where**
 $atomics \ (AnnBasic \ r \ f) = \{(r, Basic \ f)\}$
 $| \ atomics \ (AnnSeq \ c1 \ c2) = atomics \ c1 \cup atomics \ c2$
 $| \ atomics \ (AnnCond1 \ r \ b \ c1 \ c2) = atomics \ c1 \cup atomics \ c2$
 $| \ atomics \ (AnnCond2 \ r \ b \ c) = atomics \ c$
 $| \ atomics \ (AnnWhile \ r \ b \ i \ c) = atomics \ c$
 $| \ atomics \ (AnnAwait \ r \ b \ c) = \{(r \cap b, c)\}$

primrec *com* :: 'a ann-triple-op $\Rightarrow$ 'a ann-com-op **where**
 $com \ (c, q) = c$

primrec *post* :: 'a ann-triple-op $\Rightarrow$ 'a assn **where**
 $post \ (c, q) = q$

definition *interfree-aux* :: ('a ann-com-op $\times$ 'a assn $\times$ 'a ann-com-op) $\Rightarrow$ bool **where**
 $interfree-aux \equiv \lambda(co, q, co'). \ co' = None \vee$
 $\quad (\forall (r, a) \in atomics \ (the \ co'). \ \models (q \cap r) \ a \ q \wedge$
 $\quad (co = None \vee (\forall p \in assertions \ (the \ co). \ \models (p \cap r) \ a \ p)))$

definition *interfree* :: (('a ann-triple-op) list) $\Rightarrow$ bool **where**
 $interfree \ Ts \equiv \forall i \ j. \ i < length \ Ts \wedge j < length \ Ts \wedge i \neq j \longrightarrow$
 $\quad interfree-aux \ (com \ (Ts!i), post \ (Ts!i), com \ (Ts!j))$

inductive

$oghoare :: 'a \text{ assn} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ assn} \Rightarrow \text{bool} \ ((\exists ||- \text{ -} // \text{ -} // \text{ -}) [90,55,90] \ 50)$
and $ann\text{-}hoare :: 'a \text{ ann-com} \Rightarrow 'a \text{ assn} \Rightarrow \text{bool} \ ((2\vdash \text{ -} // \text{ -}) [60,90] \ 45)$
where
 $AnnBasic: r \subseteq \{s. f \ s \in q\} \Longrightarrow \vdash (AnnBasic \ r \ f) \ q$
 $| \text{ AnnSeq: } \llbracket \vdash \ c0 \ pre \ c1; \vdash \ c1 \ q \rrbracket \Longrightarrow \vdash (AnnSeq \ c0 \ c1) \ q$
 $| \text{ AnnCond1: } \llbracket r \cap b \subseteq pre \ c1; \vdash \ c1 \ q; r \cap \neg b \subseteq pre \ c2; \vdash \ c2 \ q \rrbracket$
 $\quad \Longrightarrow \vdash (AnnCond1 \ r \ b \ c1 \ c2) \ q$
 $| \text{ AnnCond2: } \llbracket r \cap b \subseteq pre \ c; \vdash \ c \ q; r \cap \neg b \subseteq q \rrbracket \Longrightarrow \vdash (AnnCond2 \ r \ b \ c) \ q$
 $| \text{ AnnWhile: } \llbracket r \subseteq i; i \cap b \subseteq pre \ c; \vdash \ c \ i; i \cap \neg b \subseteq q \rrbracket$
 $\quad \Longrightarrow \vdash (AnnWhile \ r \ b \ i \ c) \ q$
 $| \text{ AnnAwait: } \llbracket atom\text{-}com \ c; ||- \ (r \cap b) \ c \ q \rrbracket \Longrightarrow \vdash (AnnAwait \ r \ b \ c) \ q$
 $| \text{ AnnConseq: } \llbracket \vdash \ c \ q; q \subseteq q' \rrbracket \Longrightarrow \vdash \ c \ q'$
 $| \text{ Parallel: } \llbracket \forall i < length \ Ts. \exists c \ q. Ts!i = (Some \ c, \ q) \wedge \vdash \ c \ q; \text{interfree} \ Ts \rrbracket$
 $\quad \Longrightarrow ||- \ (\bigcap_{i \in \{i. i < length \ Ts\}. pre(the(com(Ts!i)))}$
 $\quad \quad \text{Parallel } Ts$
 $\quad \quad (\bigcap_{i \in \{i. i < length \ Ts\}. post(Ts!i)})$
 $| \text{ Basic: } ||- \ \{s. f \ s \in q\} \ (Basic \ f) \ q$
 $| \text{ Seq: } \llbracket ||- \ p \ c1 \ r; ||- \ r \ c2 \ q \rrbracket \Longrightarrow ||- \ p \ (Seq \ c1 \ c2) \ q$
 $| \text{ Cond: } \llbracket ||- \ (p \cap b) \ c1 \ q; ||- \ (p \cap \neg b) \ c2 \ q \rrbracket \Longrightarrow ||- \ p \ (Cond \ b \ c1 \ c2) \ q$
 $| \text{ While: } \llbracket ||- \ (p \cap b) \ c \ p \rrbracket \Longrightarrow ||- \ p \ (While \ b \ i \ c) \ (p \cap \neg b)$
 $| \text{ Conseq: } \llbracket p' \subseteq p; ||- \ p \ c \ q; q \subseteq q' \rrbracket \Longrightarrow ||- \ p' \ c \ q'$

1.5 Soundness

lemmas $[cong \ del] = \text{if-weak-cong}$

lemmas $ann\text{-}hoare\text{-}induct = oghoare\text{-}ann\text{-}hoare.induct \ [THEN \ conjunct2]$

lemmas $oghoare\text{-}induct = oghoare\text{-}ann\text{-}hoare.induct \ [THEN \ conjunct1]$

lemmas $AnnBasic = oghoare\text{-}ann\text{-}hoare.AnnBasic$

lemmas $AnnSeq = oghoare\text{-}ann\text{-}hoare.AnnSeq$

lemmas $AnnCond1 = oghoare\text{-}ann\text{-}hoare.AnnCond1$

lemmas $AnnCond2 = oghoare\text{-}ann\text{-}hoare.AnnCond2$

lemmas $AnnWhile = oghoare\text{-}ann\text{-}hoare.AnnWhile$

lemmas $AnnAwait = oghoare\text{-}ann\text{-}hoare.AnnAwait$

lemmas $AnnConseq = oghoare\text{-}ann\text{-}hoare.AnnConseq$

lemmas *Parallel* = *oghoare-ann-hoare.Parallel*
lemmas *Basic* = *oghoare-ann-hoare.Basic*
lemmas *Seq* = *oghoare-ann-hoare.Seq*
lemmas *Cond* = *oghoare-ann-hoare.Cond*
lemmas *While* = *oghoare-ann-hoare.While*
lemmas *Conseq* = *oghoare-ann-hoare.Conseq*

1.5.1 Soundness of the System for Atomic Programs

lemma *Basic-ntran* [rule-format]:
 $(\text{Basic } f, s) - Pn \rightarrow (\text{Parallel } Ts, t) \longrightarrow \text{All-None } Ts \longrightarrow t = f s$
 <proof>

lemma *SEM-fwhile*: $SEM S (p \cap b) \subseteq p \implies SEM (fwhile b S k) p \subseteq (p \cap -b)$
 <proof>

lemma *atom-hoare-sound* [rule-format]:
 $\| - p \ c \ q \longrightarrow \text{atom-com}(c) \longrightarrow \| = p \ c \ q$
 <proof>

1.5.2 Soundness of the System for Component Programs

inductive-cases *ann-transition-cases*:

$(\text{None}, s) - 1 \rightarrow (c', s')$
 $(\text{Some } (\text{AnnBasic } r \ f), s) - 1 \rightarrow (c', s')$
 $(\text{Some } (\text{AnnSeq } c1 \ c2), s) - 1 \rightarrow (c', s')$
 $(\text{Some } (\text{AnnCond1 } r \ b \ c1 \ c2), s) - 1 \rightarrow (c', s')$
 $(\text{Some } (\text{AnnCond2 } r \ b \ c), s) - 1 \rightarrow (c', s')$
 $(\text{Some } (\text{AnnWhile } r \ b \ I \ c), s) - 1 \rightarrow (c', s')$
 $(\text{Some } (\text{AnnAwait } r \ b \ c), s) - 1 \rightarrow (c', s')$

Strong Soundness for Component Programs:

lemma *ann-hoare-case-analysis* [rule-format]: $\vdash C \ q' \longrightarrow$
 $((\forall r \ f. C = \text{AnnBasic } r \ f \longrightarrow (\exists q. r \subseteq \{s. f \ s \in q\} \wedge q \subseteq q')) \wedge$
 $(\forall c0 \ c1. C = \text{AnnSeq } c0 \ c1 \longrightarrow (\exists q. q \subseteq q' \wedge \vdash c0 \ pre \ c1 \wedge \vdash c1 \ q)) \wedge$
 $(\forall r \ b \ c1 \ c2. C = \text{AnnCond1 } r \ b \ c1 \ c2 \longrightarrow (\exists q. q \subseteq q' \wedge$
 $r \cap b \subseteq pre \ c1 \wedge \vdash c1 \ q \wedge r \cap -b \subseteq pre \ c2 \wedge \vdash c2 \ q)) \wedge$
 $(\forall r \ b \ c. C = \text{AnnCond2 } r \ b \ c \longrightarrow$
 $(\exists q. q \subseteq q' \wedge r \cap b \subseteq pre \ c \wedge \vdash c \ q \wedge r \cap -b \subseteq q)) \wedge$
 $(\forall r \ i \ b \ c. C = \text{AnnWhile } r \ b \ i \ c \longrightarrow$
 $(\exists q. q \subseteq q' \wedge r \subseteq i \wedge i \cap b \subseteq pre \ c \wedge \vdash c \ i \wedge i \cap -b \subseteq q)) \wedge$
 $(\forall r \ b \ c. C = \text{AnnAwait } r \ b \ c \longrightarrow (\exists q. q \subseteq q' \wedge \| - (r \cap b) \ c \ q)))$
 <proof>

lemma *Help*: $(\text{transition} \cap \{(x, y). \text{True}\}) = (\text{transition})$
 <proof>

lemma *Strong-Soundness-aux-aux* [rule-format]:
 $(co, s) - 1 \rightarrow (co', t) \longrightarrow (\forall c. co = \text{Some } c \longrightarrow s \in pre \ c \longrightarrow$

$(\forall q. \vdash c \ q \longrightarrow (\text{if } co' = \text{None then } t \in q \text{ else } t \in \text{pre}(\text{the } co') \wedge \vdash (\text{the } co') \ q)))$
 $\langle \text{proof} \rangle$

lemma *Strong-Soundness-aux*: $\llbracket (\text{Some } c, s) \text{--}^* \rightarrow (co, t); s \in \text{pre } c; \vdash c \ q \rrbracket$
 $\implies \text{if } co = \text{None then } t \in q \text{ else } t \in \text{pre}(\text{the } co) \wedge \vdash (\text{the } co) \ q$
 $\langle \text{proof} \rangle$

lemma *Strong-Soundness*: $\llbracket (\text{Some } c, s) \text{--}^* \rightarrow (co, t); s \in \text{pre } c; \vdash c \ q \rrbracket$
 $\implies \text{if } co = \text{None then } t \in q \text{ else } t \in \text{pre}(\text{the } co)$
 $\langle \text{proof} \rangle$

lemma *ann-hoare-sound*: $\vdash c \ q \implies \models c \ q$
 $\langle \text{proof} \rangle$

1.5.3 Soundness of the System for Parallel Programs

lemma *Parallel-length-post-P1*: $(\text{Parallel } Ts, s) \text{--}P1 \rightarrow (R', t) \implies$
 $(\exists Rs. R' = (\text{Parallel } Rs) \wedge (\text{length } Rs) = (\text{length } Ts) \wedge$
 $(\forall i. i < \text{length } Ts \longrightarrow \text{post}(Rs \ ! \ i) = \text{post}(Ts \ ! \ i)))$
 $\langle \text{proof} \rangle$

lemma *Parallel-length-post-PStar*: $(\text{Parallel } Ts, s) \text{--}P^* \rightarrow (R', t) \implies$
 $(\exists Rs. R' = (\text{Parallel } Rs) \wedge (\text{length } Rs) = (\text{length } Ts) \wedge$
 $(\forall i. i < \text{length } Ts \longrightarrow \text{post}(Ts \ ! \ i) = \text{post}(Rs \ ! \ i)))$
 $\langle \text{proof} \rangle$

lemma *assertions-lemma*: $\text{pre } c \in \text{assertions } c$
 $\langle \text{proof} \rangle$

lemma *interfree-aux1* [rule-format]:
 $(c, s) \text{--}1 \rightarrow (r, t) \longrightarrow (\text{interfree-aux}(c1, q1, c) \longrightarrow \text{interfree-aux}(c1, q1, r))$
 $\langle \text{proof} \rangle$

lemma *interfree-aux2* [rule-format]:
 $(c, s) \text{--}1 \rightarrow (r, t) \longrightarrow (\text{interfree-aux}(c, q, a) \longrightarrow \text{interfree-aux}(r, q, a))$
 $\langle \text{proof} \rangle$

lemma *interfree-lemma*: $\llbracket (\text{Some } c, s) \text{--}1 \rightarrow (r, t); \text{interfree } Ts ; i < \text{length } Ts;$
 $Ts[i] = (\text{Some } c, q) \rrbracket \implies \text{interfree } (Ts[i] := (r, q))$
 $\langle \text{proof} \rangle$

Strong Soundness Theorem for Parallel Programs:

lemma *Parallel-Strong-Soundness-Seq-aux*:
 $\llbracket \text{interfree } Ts; i < \text{length } Ts; \text{com}(Ts \ ! \ i) = \text{Some}(\text{AnnSeq } c0 \ c1) \rrbracket$
 $\implies \text{interfree } (Ts[i] := (\text{Some } c0, \text{pre } c1))$
 $\langle \text{proof} \rangle$

lemma *Parallel-Strong-Soundness-Seq* [rule-format (no-asm)]:
 $\llbracket \forall i < \text{length } Ts. (\text{if } \text{com}(Ts[i]) = \text{None then } b \in \text{post}(Ts[i])) \rrbracket$

$\text{else } b \in \text{pre}(\text{the}(\text{com}(Ts!i))) \wedge \vdash \text{the}(\text{com}(Ts!i)) \text{ post}(Ts!i);$
 $\text{com}(Ts!i) = \text{Some}(\text{AnnSeq } c0 \ c1); i < \text{length } Ts; \text{interfree } Ts \] \implies$
 $(\forall ia < \text{length } Ts. (\text{if } \text{com}(Ts[i := (\text{Some } c0, \text{pre } c1)]! ia) = \text{None}$
 $\text{then } b \in \text{post}(Ts[i := (\text{Some } c0, \text{pre } c1)]! ia)$
 $\text{else } b \in \text{pre}(\text{the}(\text{com}(Ts[i := (\text{Some } c0, \text{pre } c1)]! ia)))) \wedge$
 $\vdash \text{the}(\text{com}(Ts[i := (\text{Some } c0, \text{pre } c1)]! ia)) \text{ post}(Ts[i := (\text{Some } c0, \text{pre } c1)]! ia)))$
 $\wedge \text{interfree } (Ts[i := (\text{Some } c0, \text{pre } c1)])$
 $\langle \text{proof} \rangle$

lemma *Parallel-Strong-Soundness-aux-aux* [rule-format]:

$(\text{Some } c, b) -1 \rightarrow (co, t) \longrightarrow$
 $(\forall Ts. i < \text{length } Ts \longrightarrow \text{com}(Ts!i) = \text{Some } c \longrightarrow$
 $(\forall i < \text{length } Ts. (\text{if } \text{com}(Ts!i) = \text{None} \text{ then } b \in \text{post}(Ts!i)$
 $\text{else } b \in \text{pre}(\text{the}(\text{com}(Ts!i))) \wedge \vdash \text{the}(\text{com}(Ts!i)) \text{ post}(Ts!i))) \longrightarrow$
 $\text{interfree } Ts \longrightarrow$
 $(\forall j. j < \text{length } Ts \wedge i \neq j \longrightarrow (\text{if } \text{com}(Ts!j) = \text{None} \text{ then } t \in \text{post}(Ts!j)$
 $\text{else } t \in \text{pre}(\text{the}(\text{com}(Ts!j))) \wedge \vdash \text{the}(\text{com}(Ts!j)) \text{ post}(Ts!j))))$
 $\langle \text{proof} \rangle$

lemma *Parallel-Strong-Soundness-aux* [rule-format]:

$\llbracket (Ts', s) - P^* \rightarrow (Rs', t); Ts' = (\text{Parallel } Ts); \text{interfree } Ts;$
 $\forall i. i < \text{length } Ts \longrightarrow (\exists c \ q. (Ts!i) = (\text{Some } c, q) \wedge s \in (\text{pre } c) \wedge \vdash c \ q) \rrbracket \implies$
 $\forall Rs. Rs' = (\text{Parallel } Rs) \longrightarrow (\forall j. j < \text{length } Rs \longrightarrow$
 $(\text{if } \text{com}(Rs!j) = \text{None} \text{ then } t \in \text{post}(Ts!j)$
 $\text{else } t \in \text{pre}(\text{the}(\text{com}(Rs!j))) \wedge \vdash \text{the}(\text{com}(Rs!j)) \text{ post}(Ts!j))) \wedge \text{interfree } Rs$
 $\langle \text{proof} \rangle$

lemma *Parallel-Strong-Soundness*:

$\llbracket (\text{Parallel } Ts, s) - P^* \rightarrow (\text{Parallel } Rs, t); \text{interfree } Ts; j < \text{length } Rs;$
 $\forall i. i < \text{length } Ts \longrightarrow (\exists c \ q. Ts!i = (\text{Some } c, q) \wedge s \in \text{pre } c \wedge \vdash c \ q) \rrbracket \implies$
 $\text{if } \text{com}(Rs!j) = \text{None} \text{ then } t \in \text{post}(Ts!j) \text{ else } t \in \text{pre}(\text{the}(\text{com}(Rs!j)))$
 $\langle \text{proof} \rangle$

lemma *oghoare-sound* [rule-format]: $\| - p \ c \ q \longrightarrow \| = p \ c \ q$

$\langle \text{proof} \rangle$

end

1.6 Generation of Verification Conditions

theory *OG-Tactics*

imports *OG-Hoare*

begin

lemmas *ann-hoare-intros* = *AnnBasic AnnSeq AnnCond1 AnnCond2 AnnWhile AnnAwait AnnConseq*

lemmas *oghoare-intros* = *Parallel Basic Seq Cond While Conseq*

lemma *ParallelConseqRule*:

$$\begin{aligned} & \llbracket p \subseteq (\bigcap_{i \in \{i. i < \text{length } Ts\}}. \text{pre}(\text{the}(\text{com}(Ts ! i))))); \\ & \quad \parallel - (\bigcap_{i \in \{i. i < \text{length } Ts\}}. \text{pre}(\text{the}(\text{com}(Ts ! i)))) \\ & \quad (\text{Parallel } Ts) \\ & \quad (\bigcap_{i \in \{i. i < \text{length } Ts\}}. \text{post}(Ts ! i)); \\ & \quad (\bigcap_{i \in \{i. i < \text{length } Ts\}}. \text{post}(Ts ! i)) \subseteq q \rrbracket \\ & \implies \parallel - p \text{ (Parallel } Ts) q \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *SkipRule*: $p \subseteq q \implies \parallel - p \text{ (Basic id)} q$
 $\langle \text{proof} \rangle$

lemma *BasicRule*: $p \subseteq \{s. (f s) \in q\} \implies \parallel - p \text{ (Basic } f) q$
 $\langle \text{proof} \rangle$

lemma *SeqRule*: $\llbracket \parallel - p \text{ c1 } r; \parallel - r \text{ c2 } q \rrbracket \implies \parallel - p \text{ (Seq c1 c2)} q$
 $\langle \text{proof} \rangle$

lemma *CondRule*:

$$\begin{aligned} & \llbracket p \subseteq \{s. (s \in b \longrightarrow s \in w) \wedge (s \notin b \longrightarrow s \in w')\}; \parallel - w \text{ c1 } q; \parallel - w' \text{ c2 } q \rrbracket \\ & \implies \parallel - p \text{ (Cond } b \text{ c1 c2)} q \\ & \langle \text{proof} \rangle \end{aligned}$$

lemma *WhileRule*: $\llbracket p \subseteq i; \parallel - (i \cap b) \text{ c } i; (i \cap (-b)) \subseteq q \rrbracket$
 $\implies \parallel - p \text{ (While } b \text{ i c)} q$
 $\langle \text{proof} \rangle$

Three new proof rules for special instances of the *AnnBasic* and the *AnnAwait* commands when the transformation performed on the state is the identity, and for an *AnnAwait* command where the boolean condition is $\{s. \text{True}\}$:

lemma *AnnatomRule*:

$$\llbracket \text{atom-com}(c); \parallel - r \text{ c } q \rrbracket \implies \vdash (\text{AnnAwait } r \{s. \text{True}\} c) q$$
 $\langle \text{proof} \rangle$

lemma *AnnskipRule*:
 $r \subseteq q \implies \vdash (\text{AnnBasic } r \text{ id}) q$
 $\langle \text{proof} \rangle$

lemma *AnnwaitRule*:

$$\llbracket (r \cap b) \subseteq q \rrbracket \implies \vdash (\text{AnnAwait } r b \text{ (Basic id)}) q$$
 $\langle \text{proof} \rangle$

Lemmata to avoid using the definition of *map-ann-hoare*, *interfree-aux*, *interfree-swap* and *interfree* by splitting it into different cases:

lemma *interfree-aux-rule1*: $\text{interfree-aux}(co, q, \text{None})$
 $\langle \text{proof} \rangle$

lemma *interfree-aux-rule2*:
 $\forall (R, r) \in (\text{atomics } a). \parallel - (q \cap R) r q \implies \text{interfree-aux}(\text{None}, q, \text{Some } a)$

$\langle \text{proof} \rangle$

lemma *interfree-aux-rule3*:

$$(\forall (R, r) \in (\text{atomics } a). \Vdash (q \cap R) \ r \ q \wedge (\forall p \in (\text{assertions } c). \Vdash (p \cap R) \ r \ p)) \\ \implies \text{interfree-aux}(\text{Some } c, q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnBasic-assertions*:

$$\llbracket \text{interfree-aux}(\text{None}, r, \text{Some } a); \text{interfree-aux}(\text{None}, q, \text{Some } a) \rrbracket \implies \\ \text{interfree-aux}(\text{Some } (\text{AnnBasic } r \ f), q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnSeq-assertions*:

$$\llbracket \text{interfree-aux}(\text{Some } c1, q, \text{Some } a); \text{interfree-aux}(\text{Some } c2, q, \text{Some } a) \rrbracket \implies \\ \text{interfree-aux}(\text{Some } (\text{AnnSeq } c1 \ c2), q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnCond1-assertions*:

$$\llbracket \text{interfree-aux}(\text{None}, r, \text{Some } a); \text{interfree-aux}(\text{Some } c1, q, \text{Some } a); \\ \text{interfree-aux}(\text{Some } c2, q, \text{Some } a) \rrbracket \implies \\ \text{interfree-aux}(\text{Some } (\text{AnnCond1 } r \ b \ c1 \ c2), q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnCond2-assertions*:

$$\llbracket \text{interfree-aux}(\text{None}, r, \text{Some } a); \text{interfree-aux}(\text{Some } c, q, \text{Some } a) \rrbracket \implies \\ \text{interfree-aux}(\text{Some } (\text{AnnCond2 } r \ b \ c), q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnWhile-assertions*:

$$\llbracket \text{interfree-aux}(\text{None}, r, \text{Some } a); \text{interfree-aux}(\text{None}, i, \text{Some } a); \\ \text{interfree-aux}(\text{Some } c, q, \text{Some } a) \rrbracket \implies \\ \text{interfree-aux}(\text{Some } (\text{AnnWhile } r \ b \ i \ c), q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnAwait-assertions*:

$$\llbracket \text{interfree-aux}(\text{None}, r, \text{Some } a); \text{interfree-aux}(\text{None}, q, \text{Some } a) \rrbracket \implies \\ \text{interfree-aux}(\text{Some } (\text{AnnAwait } r \ b \ c), q, \text{Some } a)$$

$\langle \text{proof} \rangle$

lemma *AnnBasic-atomics*:

$$\Vdash (q \cap r) \ (\text{Basic } f) \ q \implies \text{interfree-aux}(\text{None}, q, \text{Some } (\text{AnnBasic } r \ f))$$

$\langle \text{proof} \rangle$

lemma *AnnSeq-atomics*:

$$\llbracket \text{interfree-aux}(\text{Any}, q, \text{Some } a1); \text{interfree-aux}(\text{Any}, q, \text{Some } a2) \rrbracket \implies \\ \text{interfree-aux}(\text{Any}, q, \text{Some } (\text{AnnSeq } a1 \ a2))$$

$\langle \text{proof} \rangle$

lemma *AnnCond1-atomics*:

$$\llbracket \text{interfree-aux}(\text{Any}, q, \text{Some } a1); \text{interfree-aux}(\text{Any}, q, \text{Some } a2) \rrbracket \Longrightarrow$$

$$\text{interfree-aux}(\text{Any}, q, \text{Some } (\text{AnnCond1 } r \ b \ a1 \ a2))$$

$$\langle \text{proof} \rangle$$

lemma *AnnCond2-atomics:*

$$\text{interfree-aux}(\text{Any}, q, \text{Some } a) \Longrightarrow \text{interfree-aux}(\text{Any}, q, \text{Some } (\text{AnnCond2 } r \ b \ a))$$

$$\langle \text{proof} \rangle$$

lemma *AnnWhile-atomics:* $\text{interfree-aux}(\text{Any}, q, \text{Some } a)$

$$\Longrightarrow \text{interfree-aux}(\text{Any}, q, \text{Some } (\text{AnnWhile } r \ b \ i \ a))$$

$$\langle \text{proof} \rangle$$

lemma *Annatom-atomics:*

$$\llbracket - \ (q \cap r) \ a \ q \rrbracket \Longrightarrow \text{interfree-aux}(\text{None}, q, \text{Some } (\text{AnnAwait } r \ \{x. \text{True}\} \ a))$$

$$\langle \text{proof} \rangle$$

lemma *AnnAwait-atomics:*

$$\llbracket - \ (q \cap (r \cap b)) \ a \ q \rrbracket \Longrightarrow \text{interfree-aux}(\text{None}, q, \text{Some } (\text{AnnAwait } r \ b \ a))$$

$$\langle \text{proof} \rangle$$

definition *interfree-swap* :: $('a \text{ ann-triple-op} * ('a \text{ ann-triple-op}) \text{ list}) \Rightarrow \text{bool}$ **where**

$$\text{interfree-swap} == \lambda(x, xs). \forall y \in \text{set } xs. \text{interfree-aux}(\text{com } x, \text{post } x, \text{com } y)$$

$$\wedge \text{interfree-aux}(\text{com } y, \text{post } y, \text{com } x)$$

lemma *interfree-swap-Empty:* $\text{interfree-swap}(x, [])$

$\langle \text{proof} \rangle$

lemma *interfree-swap-List:*

$$\llbracket \text{interfree-aux}(\text{com } x, \text{post } x, \text{com } y);$$

$$\text{interfree-aux}(\text{com } y, \text{post } y, \text{com } x); \text{interfree-swap}(x, xs) \rrbracket$$

$$\Longrightarrow \text{interfree-swap}(x, y \# xs)$$

$$\langle \text{proof} \rangle$$

lemma *interfree-swap-Map:* $\forall k. i \leq k \wedge k < j \longrightarrow \text{interfree-aux}(\text{com } x, \text{post } x, c \ k)$

$$\wedge \text{interfree-aux}(c \ k, Q \ k, \text{com } x)$$

$$\Longrightarrow \text{interfree-swap}(x, \text{map}(\lambda k. (c \ k, Q \ k)) [i..<j])$$

$$\langle \text{proof} \rangle$$

lemma *interfree-Empty:* $\text{interfree } []$

$\langle \text{proof} \rangle$

lemma *interfree-List:*

$$\llbracket \text{interfree-swap}(x, xs); \text{interfree } xs \rrbracket \Longrightarrow \text{interfree}(x \# xs)$$

$$\langle \text{proof} \rangle$$

lemma *interfree-Map:*

$$(\forall i \ j. a \leq i \wedge i < b \wedge a \leq j \wedge j < b \wedge i \neq j \longrightarrow \text{interfree-aux}(c \ i, Q \ i, c \ j))$$

$$\Longrightarrow \text{interfree}(\text{map}(\lambda k. (c \ k, Q \ k)) [a..<b])$$

$\langle proof \rangle$

definition *map-ann-hoare* :: $(('a \text{ ann-com-op} * 'a \text{ assn}) \text{ list}) \Rightarrow \text{bool}$ ($\vdash$ - $[0]$ 45)
where
 $\vdash Ts == (\forall i < \text{length } Ts. \exists c \ q. Ts!i = (\text{Some } c, q) \wedge \vdash c \ q)$

lemma *MapAnnEmpty*: $\vdash []$
 $\langle proof \rangle$

lemma *MapAnnList*: $\llbracket \vdash c \ q ; \vdash xs \rrbracket \Longrightarrow \vdash (\text{Some } c, q) \# xs$
 $\langle proof \rangle$

lemma *MapAnnMap*:
 $\forall k. i \leq k \wedge k < j \longrightarrow \vdash (c \ k) (Q \ k) \Longrightarrow \vdash \text{map } (\lambda k. (\text{Some } (c \ k), Q \ k)) [i..<j]$
 $\langle proof \rangle$

lemma *ParallelRule*: $\llbracket \vdash Ts ; \text{interfree } Ts \rrbracket$
 $\Longrightarrow \llbracket - (\bigcap_{i \in \{i. i < \text{length } Ts\}. \text{pre}(\text{the}(\text{com}(Ts!i)))}$
 $\text{Parallel } Ts$
 $(\bigcap_{i \in \{i. i < \text{length } Ts\}. \text{post}(Ts!i)})$
 $\langle proof \rangle$

The following are some useful lemmas and simplification tactics to control which theorems are used to simplify at each moment, so that the original input does not suffer any unexpected transformation.

lemma *Compl-Collect*: $\neg(\text{Collect } b) = \{x. \neg(b \ x)\}$
 $\langle proof \rangle$

lemma *list-length*: $\text{length } [] = 0$ $\text{length } (x \# xs) = \text{Suc}(\text{length } xs)$
 $\langle proof \rangle$

lemma *list-lemmas*: $\text{length } [] = 0$ $\text{length } (x \# xs) = \text{Suc}(\text{length } xs)$
 $(x \# xs) ! 0 = x$ $(x \# xs) ! \text{Suc } n = xs ! n$
 $\langle proof \rangle$

lemma *le-Suc-eq-insert*: $\{i. i < \text{Suc } n\} = \text{insert } n \ \{i. i < n\}$
 $\langle proof \rangle$

lemmas *primrecdef-list* = *pre.simps assertions.simps atomics.simps atom-com.simps*

lemmas *my-simp-list* = *list-lemmas fst-conv snd-conv*
not-less0 refl le-Suc-eq-insert Suc-not-Zero Zero-not-Suc nat.inject
Collect-mem-eq ball-simps option.simps primrecdef-list

lemmas *ParallelConseq-list* = *INTER-eq Collect-conj-eq length-map length-upt length-append*

$\langle ML \rangle$

The following tactic applies *tac* to each conjunct in a subgoal of the form $A \Longrightarrow a1 \wedge a2 \wedge .. \wedge an$ returning n subgoals, one for each conjunct:

$\langle ML \rangle$

Tactic for the generation of the verification conditions

The tactic basically uses two subtactics:

HoareRuleTac is called at the level of parallel programs, it uses the **ParallelTac** to solve parallel composition of programs. This verification has two parts, namely, (1) all component programs are correct and (2) they are interference free. *HoareRuleTac* is also called at the level of atomic regions, i.e. $\langle \rangle$ and *AWAIT b THEN - END*, and at each interference freedom test.

AnnHoareRuleTac is for component programs which are annotated programs and so, there are not unknown assertions (no need to use the parameter *precond*, see NOTE).

NOTE: *precond*(::bool) informs if the subgoal has the form $\parallel - ?p \ c \ q$, in this case we have *precond*=False and the generated verification condition would have the form $?p \subseteq \dots$ which can be solved by *rtac subset-refl*, if True we proceed to simplify it using the simplification tactics above.

$\langle ML \rangle$

The final tactic is given the name *oghoare*:

$\langle ML \rangle$

Notice that the tactic for parallel programs *oghoare-tac* is initially invoked with the value *true* for the parameter *precond*.

Parts of the tactic can be also individually used to generate the verification conditions for annotated sequential programs and to generate verification conditions out of interference freedom tests:

$\langle ML \rangle$

The so defined ML tactics are then “exported” to be used in Isabelle proofs.

$\langle ML \rangle$

Tactics useful for dealing with the generated verification conditions:

$\langle ML \rangle$

end

1.7 Concrete Syntax

theory *Quote-Antiquote* **imports** *Main* **begin**

syntax

-quote :: 'b $\Rightarrow$ ('a $\Rightarrow$ 'b) ((«-») [0] 1000)
 -antiquote :: ('a $\Rightarrow$ 'b) $\Rightarrow$ 'b (' - [1000] 1000)

-Assert :: 'a $\Rightarrow$ 'a set (($\{\}$) [0] 1000)

translations

$\{\!|b|\!\} \rightarrow \text{CONST Collect } \langle b \rangle$

$\langle ML \rangle$

end

theory *OG-Syntax*

imports *OG-Tactics Quote-Antiquote*

begin

Syntax for commands and for assertions and boolean expressions in commands *com* and annotated commands *ann-com*.

abbreviation *Skip* :: 'a com (*SKIP* 63)

where *SKIP* $\equiv$ *Basic id*

abbreviation *AnnSkip* :: 'a assn $\Rightarrow$ 'a ann-com (-//*SKIP* [90] 63)

where *r SKIP* $\equiv$ *AnnBasic r id*

notation

Seq (-, / - [55, 56] 55) **and**

AnnSeq (-; / - [60, 61] 60)

syntax

-Assign :: *idt* $\Rightarrow$ 'b $\Rightarrow$ 'a com (($\prime$ - := / -) [70, 65] 61)

-AnnAssign :: 'a assn $\Rightarrow$ *idt* $\Rightarrow$ 'b $\Rightarrow$ 'a com (($\prime$ - := / -) [90, 70, 65] 61)

translations

$\prime x := a \rightarrow \text{CONST Basic } \langle \prime(-\text{update-name } x (\lambda-. a)) \rangle$

r $\prime x := a \rightarrow \text{CONST AnnBasic } r \langle \prime(-\text{update-name } x (\lambda-. a)) \rangle$

syntax

-AnnCond1 :: 'a assn $\Rightarrow$ 'a bexp $\Rightarrow$ 'a ann-com $\Rightarrow$ 'a ann-com $\Rightarrow$ 'a ann-com
(- // *IF* - / *THEN* - / *ELSE* - / *FI* [90, 0, 0, 0] 61)

-AnnCond2 :: 'a assn $\Rightarrow$ 'a bexp $\Rightarrow$ 'a ann-com $\Rightarrow$ 'a ann-com
(- // *IF* - / *THEN* - / *FI* [90, 0, 0] 61)

-AnnWhile :: 'a assn $\Rightarrow$ 'a bexp $\Rightarrow$ 'a assn $\Rightarrow$ 'a ann-com $\Rightarrow$ 'a ann-com
(- // *WHILE* - / *INV* - // *DO* - // *OD* [90, 0, 0, 0] 61)

-AnnAwait :: 'a assn $\Rightarrow$ 'a bexp $\Rightarrow$ 'a com $\Rightarrow$ 'a ann-com
(- // *AWAIT* - / *THEN* / - / *END* [90, 0, 0] 61)

-AnnAtom :: 'a assn $\Rightarrow$ 'a com $\Rightarrow$ 'a ann-com (-// $\langle$ - $\rangle$ [90, 0] 61)

-AnnWait :: 'a assn $\Rightarrow$ 'a bexp $\Rightarrow$ 'a ann-com (-// *WAIT* - *END* [90, 0] 61)

-Cond :: 'a bexp $\Rightarrow$ 'a com $\Rightarrow$ 'a com $\Rightarrow$ 'a com
((*0IF* - / *THEN* - / *ELSE* - / *FI*) [0, 0, 0] 61)

-Cond2 :: 'a bexp $\Rightarrow$ 'a com $\Rightarrow$ 'a com (*IF* - *THEN* - *FI* [0, 0] 56)

-While-inv :: 'a bexp $\Rightarrow$ 'a assn $\Rightarrow$ 'a com $\Rightarrow$ 'a com
((*0WHILE* - / *INV* - // *DO* - / *OD*) [0, 0, 0] 61)

-While :: 'a bexp $\Rightarrow$ 'a com $\Rightarrow$ 'a com
 ((0WHILE - //DO - /OD) [0, 0] 61)

translations

IF b THEN c1 ELSE c2 FI $\rightarrow$ CONST Cond $\{b\}$ c1 c2
 IF b THEN c FI $\Rightarrow$ IF b THEN c ELSE SKIP FI
 WHILE b INV i DO c OD $\rightarrow$ CONST While $\{b\}$ i c
 WHILE b DO c OD $\Rightarrow$ WHILE b INV CONST undefined DO c OD

r IF b THEN c1 ELSE c2 FI $\rightarrow$ CONST AnnCond1 r $\{b\}$ c1 c2
 r IF b THEN c FI $\rightarrow$ CONST AnnCond2 r $\{b\}$ c
 r WHILE b INV i DO c OD $\rightarrow$ CONST AnnWhile r $\{b\}$ i c
 r AWAIT b THEN c END $\rightarrow$ CONST AnnAwait r $\{b\}$ c
 r $\langle c \rangle \Rightarrow$ r AWAIT CONST True THEN c END
 r WAIT b END $\Rightarrow$ r AWAIT b THEN SKIP END

nonterminal prgs

syntax

-PAR :: prgs $\Rightarrow$ 'a (COBEGIN//-/COEND [57] 56)
 -prg :: ['a, 'a] $\Rightarrow$ prgs (-// - [60, 90] 57)
 -prgs :: ['a, 'a, prgs] $\Rightarrow$ prgs (-//-/||/- [60,90,57] 57)
 -prg-scheme :: ['a, 'a, 'a, 'a, 'a] $\Rightarrow$ prgs
 (SCHEME [- $\leq$ - < -] -// - [0,0,0,60, 90] 57)

translations

-prg c q $\Rightarrow$ [(CONST Some c, q)]
 -prgs c q ps $\Rightarrow$ (CONST Some c, q) # ps
 -PAR ps $\Rightarrow$ CONST Parallel ps
 -prg-scheme j i k c q $\Rightarrow$ CONST map ($\lambda i.$ (CONST Some c, q)) [j..<k]

$\langle ML \rangle$

end

1.8 Examples

theory OG-Examples imports OG-Syntax begin

1.8.1 Mutual Exclusion

Peterson's Algorithm I

Eike Best. "Semantics of Sequential and Parallel Programs", page 217.

record Petersons-mutex-1 =

pr1 :: nat

pr2 :: nat

$in1 :: bool$
 $in2 :: bool$
 $hold :: nat$

lemma *Petersons-mutex-1:*

$\parallel - \{ 'pr1=0 \wedge \neg 'in1 \wedge 'pr2=0 \wedge \neg 'in2 \}$
 $COBEGIN \{ 'pr1=0 \wedge \neg 'in1 \}$
 $WHILE \text{True} \text{ INV } \{ 'pr1=0 \wedge \neg 'in1 \}$
 DO
 $\{ 'pr1=0 \wedge \neg 'in1 \} \langle 'in1:=\text{True}, 'pr1:=1 \rangle;;$
 $\{ 'pr1=1 \wedge 'in1 \} \langle 'hold:=1, 'pr1:=2 \rangle;;$
 $\{ 'pr1=2 \wedge 'in1 \wedge ('hold=1 \vee 'hold=2 \wedge 'pr2=2) \}$
 $AWAIT (\neg 'in2 \vee \neg ('hold=1)) \text{ THEN } 'pr1:=3 \text{ END};;$
 $\{ 'pr1=3 \wedge 'in1 \wedge ('hold=1 \vee 'hold=2 \wedge 'pr2=2) \}$
 $\langle 'in1:=\text{False}, 'pr1:=0 \rangle$
 $OD \{ 'pr1=0 \wedge \neg 'in1 \}$
 $\parallel$
 $\{ 'pr2=0 \wedge \neg 'in2 \}$
 $WHILE \text{True} \text{ INV } \{ 'pr2=0 \wedge \neg 'in2 \}$
 DO
 $\{ 'pr2=0 \wedge \neg 'in2 \} \langle 'in2:=\text{True}, 'pr2:=1 \rangle;;$
 $\{ 'pr2=1 \wedge 'in2 \} \langle 'hold:=2, 'pr2:=2 \rangle;;$
 $\{ 'pr2=2 \wedge 'in2 \wedge ('hold=2 \vee ('hold=1 \wedge 'pr1=2)) \}$
 $AWAIT (\neg 'in1 \vee \neg ('hold=2)) \text{ THEN } 'pr2:=3 \text{ END};;$
 $\{ 'pr2=3 \wedge 'in2 \wedge ('hold=2 \vee ('hold=1 \wedge 'pr1=2)) \}$
 $\langle 'in2:=\text{False}, 'pr2:=0 \rangle$
 $OD \{ 'pr2=0 \wedge \neg 'in2 \}$
 $COEND$
 $\{ 'pr1=0 \wedge \neg 'in1 \wedge 'pr2=0 \wedge \neg 'in2 \}$
 $\langle \text{proof} \rangle$

Peterson's Algorithm II: A Busy Wait Solution

Apt and Olderog. "Verification of sequential and concurrent Programs", page 282.

record *Busy-wait-mutex* =

$flag1 :: bool$
 $flag2 :: bool$
 $turn :: nat$
 $after1 :: bool$
 $after2 :: bool$

lemma *Busy-wait-mutex:*

$\parallel - \{ \text{True} \}$
 $'flag1:=\text{False}, 'flag2:=\text{False},$
 $COBEGIN \{ \neg 'flag1 \}$
 $WHILE \text{True}$
 $INV \{ \neg 'flag1 \}$
 $DO \{ \neg 'flag1 \} \langle 'flag1:=\text{True}, 'after1:=\text{False} \rangle;;$

```

    { 'flag1 ∧ ¬ 'after1 } < 'turn:=1,, 'after1:=True >;
    { 'flag1 ∧ 'after1 ∧ ( 'turn=1 ∨ 'turn=2 ) }
    WHILE ¬( 'flag2 → 'turn=2 )
    INV { 'flag1 ∧ 'after1 ∧ ( 'turn=1 ∨ 'turn=2 ) }
    DO { 'flag1 ∧ 'after1 ∧ ( 'turn=1 ∨ 'turn=2 ) } SKIP OD;;
    { 'flag1 ∧ 'after1 ∧ ( 'flag2 ∧ 'after2 → 'turn=2 ) }
    'flag1:=False
  OD
  { False }
||
  { ¬ 'flag2 }
  WHILE True
  INV { ¬ 'flag2 }
  DO { ¬ 'flag2 } < 'flag2:=True,, 'after2:=False >;
    { 'flag2 ∧ ¬ 'after2 } < 'turn:=2,, 'after2:=True >;
    { 'flag2 ∧ 'after2 ∧ ( 'turn=1 ∨ 'turn=2 ) }
    WHILE ¬( 'flag1 → 'turn=1 )
    INV { 'flag2 ∧ 'after2 ∧ ( 'turn=1 ∨ 'turn=2 ) }
    DO { 'flag2 ∧ 'after2 ∧ ( 'turn=1 ∨ 'turn=2 ) } SKIP OD;;
    { 'flag2 ∧ 'after2 ∧ ( 'flag1 ∧ 'after1 → 'turn=1 ) }
    'flag2:=False
  OD
  { False }
COEND
{ False }
<proof>

```

Peterson's Algorithm III: A Solution using Semaphores

record *Semaphores-mutex* =

out :: bool

who :: nat

lemma *Semaphores-mutex*:

```

||- { i≠j }
  'out:=True ,,
  COBEGIN { i≠j }
    WHILE True INV { i≠j }
    DO { i≠j } AWAIT 'out THEN 'out:=False,, 'who:=i END;;
    { ¬ 'out ∧ 'who=i ∧ i≠j } 'out:=True OD
    { False }
  ||
    { i≠j }
    WHILE True INV { i≠j }
    DO { i≠j } AWAIT 'out THEN 'out:=False,, 'who:=j END;;
    { ¬ 'out ∧ 'who=j ∧ i≠j } 'out:=True OD
    { False }
  COEND
  { False }

```


$\langle proof \rangle$

Peterson's Algorithm III: Parameterized version:

lemma *Semaphores-parameterized-mutex:*

$0 < n \implies \parallel - \{ \{ True \} \}$
 $\text{'out} := True \text{ , ,}$
COBEGIN
SCHEME $[0 \leq i < n]$
 $\{ \{ True \} \}$
WHILE $True$ **INV** $\{ \{ True \} \}$
DO $\{ \{ True \} \}$ **AWAIT** $\text{'out} = False \text{ , , 'who} := i$ **END**;;
 $\{ \neg \text{'out} \wedge \text{'who} = i \} \text{'out} := True$ **OD**
 $\{ \{ False \} \}$
COEND
 $\{ \{ False \} \}$
 $\langle proof \rangle$

The Ticket Algorithm

record *Ticket-mutex* =

$num :: nat$
 $nextv :: nat$
 $turn :: nat \text{ list}$
 $index :: nat$

lemma *Ticket-mutex:*

$\parallel [0 < n; I = \langle n = length \text{'turn} \wedge 0 < \text{'nextv} \wedge (\forall k \ l. k < n \wedge l < n \wedge k \neq l$
 $\implies \text{'turn}!k < \text{'num} \wedge (\text{'turn}!k = 0 \vee \text{'turn}!k \neq \text{'turn}!l)) \rangle]$
 $\implies \parallel - \{ \{ n = length \text{'turn} \} \}$
 $\text{'index} := 0 \text{ , ,}$
WHILE $\text{'index} < n$ **INV** $\{ \{ n = length \text{'turn} \wedge (\forall i < \text{'index}. \text{'turn}!i = 0) \} \}$
DO $\text{'turn} := \text{'turn}[\text{'index} := 0] \text{ , , 'index} := \text{'index} + 1$ **OD** ,,
 $\text{'num} := 1 \text{ , , 'nextv} := 1 \text{ , ,}$
COBEGIN
SCHEME $[0 \leq i < n]$
 $\{ \{ I \} \}$
WHILE $True$ **INV** $\{ \{ I \} \}$
DO $\{ \{ I \} \}$ $\langle \text{'turn} := \text{'turn}[i := \text{'num}] \text{ , , 'num} := \text{'num} + 1 \rangle$;;
 $\{ \{ I \} \}$ **WAIT** $\text{'turn}!i = \text{'nextv}$ **END**;;
 $\{ \{ I \wedge \text{'turn}!i = \text{'nextv} \} \text{'nextv} := \text{'nextv} + 1$
OD
 $\{ \{ False \} \}$
COEND
 $\{ \{ False \} \}$
 $\langle proof \rangle$

1.8.2 Parallel Zero Search

Synchronized Zero Search. Zero-6

record *Zero-search* =
turn :: nat
found :: bool
x :: nat
y :: nat

lemma *Zero-search*:

$$\begin{aligned} & \llbracket I1 = \langle a \leq 'x \wedge ('found \longrightarrow (a < 'x \wedge f('x)=0) \vee ('y \leq a \wedge f('y)=0)) \\ & \quad \wedge (\neg 'found \wedge a < 'x \longrightarrow f('x) \neq 0) \rangle \rrbracket ; \\ & \quad I2 = \langle 'y \leq a+1 \wedge ('found \longrightarrow (a < 'x \wedge f('x)=0) \vee ('y \leq a \wedge f('y)=0)) \\ & \quad \wedge (\neg 'found \wedge 'y \leq a \longrightarrow f('y) \neq 0) \rangle \rrbracket \implies \\ & \parallel - \llbracket \exists u. f(u)=0 \rrbracket \\ & \quad 'turn:=1, 'found:=False, \\ & \quad 'x:=a, 'y:=a+1, \\ & \quad COBEGIN \llbracket 'I1 \rrbracket \\ & \quad \quad WHILE \neg 'found \\ & \quad \quad INV \llbracket 'I1 \rrbracket \\ & \quad \quad DO \llbracket a \leq 'x \wedge ('found \longrightarrow 'y \leq a \wedge f('y)=0) \wedge (a < 'x \longrightarrow f('x) \neq 0) \rrbracket \\ & \quad \quad \quad WAIT 'turn=1 END;; \\ & \quad \quad \quad \llbracket a \leq 'x \wedge ('found \longrightarrow 'y \leq a \wedge f('y)=0) \wedge (a < 'x \longrightarrow f('x) \neq 0) \rrbracket \\ & \quad \quad \quad 'turn:=2;; \\ & \quad \quad \quad \llbracket a \leq 'x \wedge ('found \longrightarrow 'y \leq a \wedge f('y)=0) \wedge (a < 'x \longrightarrow f('x) \neq 0) \rrbracket \\ & \quad \quad \quad \langle 'x:='x+1, \\ & \quad \quad \quad IF f('x)=0 THEN 'found:=True ELSE SKIP FI \rangle \\ & \quad \quad OD;; \\ & \quad \quad \llbracket 'I1 \wedge 'found \rrbracket \\ & \quad \quad 'turn:=2 \\ & \quad \quad \llbracket 'I1 \wedge 'found \rrbracket \\ & \parallel \\ & \quad \llbracket 'I2 \rrbracket \\ & \quad \quad WHILE \neg 'found \\ & \quad \quad INV \llbracket 'I2 \rrbracket \\ & \quad \quad DO \llbracket 'y \leq a+1 \wedge ('found \longrightarrow a < 'x \wedge f('x)=0) \wedge ('y \leq a \longrightarrow f('y) \neq 0) \rrbracket \\ & \quad \quad \quad WAIT 'turn=2 END;; \\ & \quad \quad \quad \llbracket 'y \leq a+1 \wedge ('found \longrightarrow a < 'x \wedge f('x)=0) \wedge ('y \leq a \longrightarrow f('y) \neq 0) \rrbracket \\ & \quad \quad \quad 'turn:=1;; \\ & \quad \quad \quad \llbracket 'y \leq a+1 \wedge ('found \longrightarrow a < 'x \wedge f('x)=0) \wedge ('y \leq a \longrightarrow f('y) \neq 0) \rrbracket \\ & \quad \quad \quad \langle 'y:='y-1, \\ & \quad \quad \quad IF f('y)=0 THEN 'found:=True ELSE SKIP FI \rangle \\ & \quad \quad OD;; \\ & \quad \quad \llbracket 'I2 \wedge 'found \rrbracket \\ & \quad \quad 'turn:=1 \\ & \quad \quad \llbracket 'I2 \wedge 'found \rrbracket \\ & \quad COEND \\ & \quad \llbracket f('x)=0 \vee f('y)=0 \rrbracket \\ & \langle proof \rangle \end{aligned}$$

Easier Version: without AWAIT. Apt and Olderog. page 256:

lemma *Zero-Search-2*:

$$\begin{aligned}
& \llbracket I1 = \langle a \leq 'x \wedge ('found \longrightarrow (a < 'x \wedge f('x) = 0) \vee ('y \leq a \wedge f('y) = 0)) \\
& \quad \wedge (\neg 'found \wedge a < 'x \longrightarrow f('x) \neq 0) \rangle \rrbracket; \\
& I2 = \langle 'y \leq a+1 \wedge ('found \longrightarrow (a < 'x \wedge f('x) = 0) \vee ('y \leq a \wedge f('y) = 0)) \\
& \quad \wedge (\neg 'found \wedge 'y \leq a \longrightarrow f('y) \neq 0) \rangle \rrbracket \implies \\
& \parallel - \{ \exists u. f(u) = 0 \} \\
& 'found := False, \\
& 'x := a, 'y := a+1, \\
& COBEGIN \{ 'I1 \} \\
& \quad WHILE \neg 'found \\
& \quad INV \{ 'I1 \} \\
& \quad DO \{ a \leq 'x \wedge ('found \longrightarrow 'y \leq a \wedge f('y) = 0) \wedge (a < 'x \longrightarrow f('x) \neq 0) \} \\
& \quad \quad \langle 'x := 'x+1, IF f('x) = 0 THEN 'found := True ELSE SKIP FI \rangle \\
& \quad OD \\
& \quad \{ 'I1 \wedge 'found \} \\
& \parallel \\
& \{ 'I2 \} \\
& \quad WHILE \neg 'found \\
& \quad INV \{ 'I2 \} \\
& \quad DO \{ 'y \leq a+1 \wedge ('found \longrightarrow a < 'x \wedge f('x) = 0) \wedge ('y \leq a \longrightarrow f('y) \neq 0) \} \\
& \quad \quad \langle 'y := ('y - 1), IF f('y) = 0 THEN 'found := True ELSE SKIP FI \rangle \\
& \quad OD \\
& \quad \{ 'I2 \wedge 'found \} \\
& COEND \\
& \{ f('x) = 0 \vee f('y) = 0 \} \\
& \langle proof \rangle
\end{aligned}$$

1.8.3 Producer/Consumer

Previous lemmas

lemma *nat-lemma2*: $\llbracket b = m * (n :: nat) + t; a = s * n + u; t = u; b - a < n \rrbracket \implies m \leq s$
 $\langle proof \rangle$

lemma *mod-lemma*: $\llbracket (c :: nat) \leq a; a < b; b - c < n \rrbracket \implies b \bmod n \neq a \bmod n$
 $\langle proof \rangle$

Producer/Consumer Algorithm

record *Producer-consumer* =

$ins :: nat$
 $outs :: nat$
 $li :: nat$
 $lj :: nat$
 $vx :: nat$
 $vy :: nat$
 $buffer :: nat \text{ list}$
 $b :: nat \text{ list}$

The whole proof takes aprox. 4 minutes.

lemma *Producer-consumer*:

```

[[INIT = «0 < length a ∧ 0 < length 'buffer ∧ length 'b = length a» ;
  I = «(∀ k < 'ins. 'outs ≤ k → (a ! k) = 'buffer ! (k mod (length 'buffer))) ∧
    'outs ≤ 'ins ∧ 'ins - 'outs ≤ length 'buffer» ;
  I1 = «'I ∧ 'li ≤ length a» ;
  p1 = «'I1 ∧ 'li = 'ins» ;
  I2 = «'I ∧ (∀ k < 'lj. (a ! k) = ('b ! k)) ∧ 'lj ≤ length a» ;
  p2 = «'I2 ∧ 'lj = 'outs» ]] ⇒
||- { 'INIT }
'ins := 0,, 'outs := 0,, 'li := 0,, 'lj := 0,,
COBEGIN { 'p1 ∧ 'INIT }
  WHILE 'li < length a
    INV { 'p1 ∧ 'INIT }
    DO { 'p1 ∧ 'INIT ∧ 'li < length a }
      'vx := (a ! 'li);
      { 'p1 ∧ 'INIT ∧ 'li < length a ∧ 'vx = (a ! 'li) }
      WAIT 'ins - 'outs < length 'buffer END;;
      { 'p1 ∧ 'INIT ∧ 'li < length a ∧ 'vx = (a ! 'li)
        ∧ 'ins - 'outs < length 'buffer }
      'buffer := (list-update 'buffer ('ins mod (length 'buffer)) 'vx);
      { 'p1 ∧ 'INIT ∧ 'li < length a
        ∧ (a ! 'li) = ('buffer ! ('ins mod (length 'buffer)))
        ∧ 'ins - 'outs < length 'buffer }
      'ins := 'ins + 1;;
      { 'I1 ∧ 'INIT ∧ ('li + 1) = 'ins ∧ 'li < length a }
      'li := 'li + 1
    OD
  { 'p1 ∧ 'INIT ∧ 'li = length a }
||
{ 'p2 ∧ 'INIT }
  WHILE 'lj < length a
    INV { 'p2 ∧ 'INIT }
    DO { 'p2 ∧ 'lj < length a ∧ 'INIT }
      WAIT 'outs < 'ins END;;
      { 'p2 ∧ 'lj < length a ∧ 'outs < 'ins ∧ 'INIT }
      'vy := ('buffer ! ('outs mod (length 'buffer)));
      { 'p2 ∧ 'lj < length a ∧ 'outs < 'ins ∧ 'vy = (a ! 'lj) ∧ 'INIT }
      'outs := 'outs + 1;;
      { 'I2 ∧ ('lj + 1) = 'outs ∧ 'lj < length a ∧ 'vy = (a ! 'lj) ∧ 'INIT }
      'b := (list-update 'b 'lj 'vy);
      { 'I2 ∧ ('lj + 1) = 'outs ∧ 'lj < length a ∧ (a ! 'lj) = ('b ! 'lj) ∧ 'INIT }
      'lj := 'lj + 1
    OD
  { 'p2 ∧ 'lj = length a ∧ 'INIT }
COEND
{ ∀ k < length a. (a ! k) = ('b ! k) }
⟨proof⟩

```

1.8.4 Parameterized Examples

Set Elements of an Array to Zero

record *Example1* =
 $a :: \text{nat} \Rightarrow \text{nat}$

lemma *Example1*:
 $\| - \{ \text{True} \}$
 $\text{COBEGIN SCHEME } [0 \leq i < n] \{ \text{True} \} \text{ 'a} := \text{'a } (i := 0) \{ \text{'a } i = 0 \} \text{ COEND}$
 $\{ \forall i < n. \text{'a } i = 0 \}$
 $\langle \text{proof} \rangle$

Same example with lists as auxiliary variables.

record *Example1-list* =
 $A :: \text{nat list}$

lemma *Example1-list*:
 $\| - \{ n < \text{length 'A} \}$
 COBEGIN
 $\text{SCHEME } [0 \leq i < n] \{ n < \text{length 'A} \} \text{ 'A} := \text{'A}[i := 0] \{ \text{'A}[i] = 0 \}$
 COEND
 $\{ \forall i < n. \text{'A}[i] = 0 \}$
 $\langle \text{proof} \rangle$

Increment a Variable in Parallel

First some lemmas about summation properties.

lemma *Example2-lemma2-aux*: $!!b. j < n \Rightarrow$
 $(\sum i=0..<n. (b \text{ i}::\text{nat})) =$
 $(\sum i=0..<j. b \text{ i}) + b \text{ j} + (\sum i=0..<n-(\text{Suc } j). b (\text{Suc } j + i))$
 $\langle \text{proof} \rangle$

lemma *Example2-lemma2-aux2*:
 $!!b. j \leq s \Rightarrow (\sum i::\text{nat}=0..<j. (b (s:=t)) \text{ i}) = (\sum i=0..<j. b \text{ i})$
 $\langle \text{proof} \rangle$

lemma *Example2-lemma2*:
 $!!b. [j < n; b \text{ j} = 0] \Rightarrow \text{Suc } (\sum i::\text{nat}=0..<n. b \text{ i}) = (\sum i=0..<n. (b (j := \text{Suc } 0)) \text{ i})$
 $\langle \text{proof} \rangle$

record *Example2* =
 $c :: \text{nat} \Rightarrow \text{nat}$
 $x :: \text{nat}$

lemma *Example-2*: $0 < n \Rightarrow$
 $\| - \{ x = 0 \wedge (\sum i=0..<n. \text{'c } i) = 0 \}$
 COBEGIN

```

    SCHEME [0 ≤ i < n]
    { 'x = (∑ i=0..<n. 'c i) ∧ 'c i = 0 }
    ⟨ 'x := 'x + (Suc 0),, 'c := 'c (i := (Suc 0)) ⟩
    { 'x = (∑ i=0..<n. 'c i) ∧ 'c i = (Suc 0) }
    COEND
    { 'x = n }
    ⟨ proof ⟩

end

```

Chapter 2

Case Study: Single and Multi-Mutator Garbage Collection Algorithms

2.1 Formalization of the Memory

theory *Graph* imports *Main* begin

datatype *node* = *Black* | *White*

type-synonym *nodes* = *node list*

type-synonym *edge* = *nat* $\times$ *nat*

type-synonym *edges* = *edge list*

consts *Roots* :: *nat set*

definition *Proper-Roots* :: *nodes* $\Rightarrow$ *bool* **where**
 $Proper-Roots\ M \equiv Roots \neq \{\} \wedge Roots \subseteq \{i. i < length\ M\}$

definition *Proper-Edges* :: (*nodes* $\times$ *edges*) $\Rightarrow$ *bool* **where**
 $Proper-Edges \equiv (\lambda(M, E). \forall i < length\ E. fst(E!i) < length\ M \wedge snd(E!i) < length\ M)$

definition *BtoW* :: (*edge* $\times$ *nodes*) $\Rightarrow$ *bool* **where**
 $BtoW \equiv (\lambda(e, M). (M!fst\ e) = Black \wedge (M!snd\ e) \neq Black)$

definition *Blacks* :: *nodes* $\Rightarrow$ *nat set* **where**
 $Blacks\ M \equiv \{i. i < length\ M \wedge M!i = Black\}$

definition *Reach* :: *edges* $\Rightarrow$ *nat set* **where**
 $Reach\ E \equiv \{x. (\exists path. 1 < length\ path \wedge path!(length\ path - 1) \in Roots \wedge x = path!0 \wedge (\forall i < length\ path - 1. (\exists j < length\ E. E!j = (path!(i+1), path!i)))) \vee x \in Roots\}$

Reach: the set of reachable nodes is the set of Roots together with the nodes reachable from some Root by a path represented by a list of nodes (at least two since we traverse at least one edge), where two consecutive nodes correspond to an edge in E.

2.1.1 Proofs about Graphs

lemmas *Graph-defs* = *Blacks-def Proper-Roots-def Proper-Edges-def BtoW-def*
declare *Graph-defs* [simp]

Graph 1

lemma *Graph1-aux* [rule-format]:

$$\begin{aligned} & \llbracket \text{Roots} \subseteq \text{Blacks } M; \forall i < \text{length } E. \neg \text{BtoW}(E!i, M) \rrbracket \\ & \implies 1 < \text{length path} \longrightarrow (\text{path}!(\text{length path} - 1)) \in \text{Roots} \longrightarrow \\ & (\forall i < \text{length path} - 1. (\exists j. j < \text{length } E \wedge E!j = (\text{path}!(\text{Suc } i), \text{path}!i))) \\ & \longrightarrow M!(\text{path}!0) = \text{Black} \end{aligned}$$

 <proof>

lemma *Graph1*:

$$\begin{aligned} & \llbracket \text{Roots} \subseteq \text{Blacks } M; \text{Proper-Edges}(M, E); \forall i < \text{length } E. \neg \text{BtoW}(E!i, M) \rrbracket \\ & \implies \text{Reach } E \subseteq \text{Blacks } M \end{aligned}$$

 <proof>

Graph 2

lemma *Ex-first-occurrence* [rule-format]:

$$P (n::\text{nat}) \longrightarrow (\exists m. P m \wedge (\forall i. i < m \longrightarrow \neg P i))$$

 <proof>

lemma *Compl-lemma*: $(n::\text{nat}) \leq l \implies (\exists m. m \leq l \wedge n = l - m)$
 <proof>

lemma *Ex-last-occurrence*:

$$\llbracket P (n::\text{nat}); n \leq l \rrbracket \implies (\exists m. P (l - m) \wedge (\forall i. i < m \longrightarrow \neg P (l - i)))$$

 <proof>

lemma *Graph2*:

$$\llbracket T \in \text{Reach } E; R < \text{length } E \rrbracket \implies T \in \text{Reach } (E[R := (\text{fst}(E!R), T)])$$

 <proof>

Graph 3

declare *min.absorb1* [simp] *min.absorb2* [simp]

lemma *Graph3*:

$$\llbracket T \in \text{Reach } E; R < \text{length } E \rrbracket \implies \text{Reach}(E[R := (\text{fst}(E!R), T)]) \subseteq \text{Reach } E$$

 <proof>

Graph 4

lemma *Graph4*:

$$\begin{aligned} & \llbracket T \in \text{Reach } E; \text{Roots} \subseteq \text{Blacks } M; I \leq \text{length } E; T < \text{length } M; R < \text{length } E; \\ & \forall i < I. \neg \text{BtoW}(E!i, M); R < I; M!fst(E!R) = \text{Black}; M!T \neq \text{Black} \rrbracket \implies \\ & (\exists r. I \leq r \wedge r < \text{length } E \wedge \text{BtoW}(E[R := (fst(E!R), T)]!r, M)) \\ & \langle \text{proof} \rangle \end{aligned}$$

declare *min.absorb1* [*simp del*] *min.absorb2* [*simp del*]

Graph 5

lemma *Graph5*:

$$\begin{aligned} & \llbracket T \in \text{Reach } E; \text{Roots} \subseteq \text{Blacks } M; \forall i < R. \neg \text{BtoW}(E!i, M); T < \text{length } M; \\ & R < \text{length } E; M!fst(E!R) = \text{Black}; M!snd(E!R) = \text{Black}; M!T \neq \text{Black} \rrbracket \\ & \implies (\exists r. R < r \wedge r < \text{length } E \wedge \text{BtoW}(E[R := (fst(E!R), T)]!r, M)) \\ & \langle \text{proof} \rangle \end{aligned}$$

Other lemmas about graphs

lemma *Graph6*:

$$\llbracket \text{Proper-Edges}(M, E); R < \text{length } E; T < \text{length } M \rrbracket \implies \text{Proper-Edges}(M, E[R := (fst(E!R), T)])$$

 $\langle \text{proof} \rangle$

lemma *Graph7*:

$$\llbracket \text{Proper-Edges}(M, E) \rrbracket \implies \text{Proper-Edges}(M[T := a], E)$$

 $\langle \text{proof} \rangle$

lemma *Graph8*:

$$\llbracket \text{Proper-Roots}(M) \rrbracket \implies \text{Proper-Roots}(M[T := a])$$

 $\langle \text{proof} \rangle$

Some specific lemmata for the verification of garbage collection algorithms.

lemma *Graph9*: $j < \text{length } M \implies \text{Blacks } M \subseteq \text{Blacks } (M[j := \text{Black}])$
 $\langle \text{proof} \rangle$

lemma *Graph10* [*rule-format (no-asm)*]: $\forall i. M!i = a \longrightarrow M[i := a] = M$
 $\langle \text{proof} \rangle$

lemma *Graph11* [*rule-format (no-asm)*]:

$$\llbracket M!j \neq \text{Black}; j < \text{length } M \rrbracket \implies \text{Blacks } M \subset \text{Blacks } (M[j := \text{Black}])$$

 $\langle \text{proof} \rangle$

lemma *Graph12*: $\llbracket a \subseteq \text{Blacks } M; j < \text{length } M \rrbracket \implies a \subseteq \text{Blacks } (M[j := \text{Black}])$
 $\langle \text{proof} \rangle$

lemma *Graph13*: $\llbracket a \subset \text{Blacks } M; j < \text{length } M \rrbracket \implies a \subset \text{Blacks } (M[j := \text{Black}])$
 $\langle \text{proof} \rangle$

declare *Graph-defs* [*simp del*]

end

2.2 The Single Mutator Case

theory *Gar-Coll* **imports** *Graph OG-Syntax* **begin**

declare *psubsetE* [*rule del*]

Declaration of variables:

record *gar-coll-state* =

M :: *nodes*
E :: *edges*
bc :: *nat set*
obc :: *nat set*
Ma :: *nodes*
ind :: *nat*
k :: *nat*
z :: *bool*

2.2.1 The Mutator

The mutator first redirects an arbitrary edge R from an arbitrary accessible node towards an arbitrary accessible node T . It then colors the new target T black.

We declare the arbitrarily selected node and edge as constants:

consts $R :: nat$ $T :: nat$

The following predicate states, given a list of nodes m and a list of edges e , the conditions under which the selected edge R and node T are valid:

definition *Mut-init* :: *gar-coll-state* $\Rightarrow$ *bool* **where**

$Mut-init \equiv \langle T \in Reach \ 'E \wedge R < length \ 'E \wedge T < length \ 'M \rangle$

For the mutator we consider two modules, one for each action. An auxiliary variable z is set to false if the mutator has already redirected an edge but has not yet colored the new target.

definition *Redirect-Edge* :: *gar-coll-state ann-com* **where**

$Redirect-Edge \equiv \{ \ 'Mut-init \wedge \ 'z \} \langle \ 'E := \ 'E[R := (fst(\ 'E!R), T)], \ 'z := (\neg \ 'z) \rangle$

definition *Color-Target* :: *gar-coll-state ann-com* **where**

$Color-Target \equiv \{ \ 'Mut-init \wedge \neg \ 'z \} \langle \ 'M := \ 'M[T := Black], \ 'z := (\neg \ 'z) \rangle$

definition *Mutator* :: *gar-coll-state ann-com* **where**

$Mutator \equiv$
 $\{ \ 'Mut-init \wedge \ 'z \}$
 $WHILE \ True \ INV \ \{ \ 'Mut-init \wedge \ 'z \}$
 $DO \ Redirect-Edge \ ; \ Color-Target \ OD$

Correctness of the mutator

lemmas *mutator-defs* = *Mut-init-def Redirect-Edge-def Color-Target-def*

lemma *Redirect-Edge*:

$\vdash \text{Redirect-Edge } \text{pre}(\text{Color-Target})$

$\langle \text{proof} \rangle$

lemma *Color-Target*:

$\vdash \text{Color-Target } \{\text{'Mut-init} \wedge \text{'z}\}$

$\langle \text{proof} \rangle$

lemma *Mutator*:

$\vdash \text{Mutator } \{\text{False}\}$

$\langle \text{proof} \rangle$

2.2.2 The Collector

A constant *M-init* is used to give *'Ma* a suitable first value, defined as a list of nodes where only the *Roots* are black.

consts *M-init* :: *nodes*

definition *Proper-M-init* :: *gar-coll-state* $\Rightarrow$ *bool* **where**

Proper-M-init $\equiv$ « *Blacks M-init=Roots* $\wedge$ *length M-init=length 'M* »

definition *Proper* :: *gar-coll-state* $\Rightarrow$ *bool* **where**

Proper $\equiv$ « *Proper-Roots 'M* $\wedge$ *Proper-Edges('M, 'E)* $\wedge$ *'Proper-M-init* »

definition *Safe* :: *gar-coll-state* $\Rightarrow$ *bool* **where**

Safe $\equiv$ « *Reach 'E* $\subseteq$ *Blacks 'M* »

lemmas *collector-defs* = *Proper-M-init-def Proper-def Safe-def*

Blackening the roots

definition *Blacken-Roots* :: *gar-coll-state ann-com* **where**

Blacken-Roots $\equiv$

$\{\text{'Proper}\}$

'ind:=0;;

$\{\text{'Proper} \wedge \text{'ind}=0\}$

WHILE *'ind<length 'M*

INV $\{\text{'Proper} \wedge (\forall i < \text{'ind}. i \in \text{Roots} \longrightarrow \text{'M}[i] = \text{Black}) \wedge \text{'ind} \leq \text{length 'M}\}$

DO $\{\text{'Proper} \wedge (\forall i < \text{'ind}. i \in \text{Roots} \longrightarrow \text{'M}[i] = \text{Black}) \wedge \text{'ind} < \text{length 'M}\}$

IF *'ind* $\in$ *Roots* **THEN**

$\{\text{'Proper} \wedge (\forall i < \text{'ind}. i \in \text{Roots} \longrightarrow \text{'M}[i] = \text{Black}) \wedge \text{'ind} < \text{length 'M} \wedge$

'ind $\in$ *Roots}\}*

'M:= 'M['ind:=Black] **FI**;;

$\{\text{'Proper} \wedge (\forall i < \text{'ind}+1. i \in \text{Roots} \longrightarrow \text{'M}[i] = \text{Black}) \wedge \text{'ind} < \text{length 'M}\}$

'ind:= 'ind+1

OD

lemma *Blacken-Roots*:

$\vdash \text{Blacken-Roots } \{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \} \}$
 $\langle \text{proof} \rangle$

Propagating black

definition *PBInv* :: *gar-coll-state* $\Rightarrow$ *nat* $\Rightarrow$ *bool* **where**

$\text{PBInv} \equiv \ll \lambda \text{ind. } 'obc < \text{Blacks } 'M \vee (\forall i < \text{ind. } \neg \text{BtoW } ('E!i, 'M) \vee$
 $(\neg 'z \wedge i = R \wedge (\text{snd}('E!R)) = T \wedge (\exists r. \text{ind} \leq r \wedge r < \text{length } 'E \wedge \text{BtoW}('E!r, 'M)))) \gg$

definition *Propagate-Black-aux* :: *gar-coll-state* *ann-com* **where**

$\text{Propagate-Black-aux} \equiv$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \}$
 $\text{'ind} := 0;;$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \wedge \text{'ind} = 0 \}$
 $\text{WHILE } 'ind < \text{length } 'E$
 $\text{INV } \{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge \text{'PBInv } 'ind \wedge 'ind \leq \text{length } 'E \}$
 $\text{DO } \{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge \text{'PBInv } 'ind \wedge 'ind < \text{length } 'E \}$
 $\text{IF } 'M!(\text{fst } ('E! 'ind)) = \text{Black THEN}$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge \text{'PBInv } 'ind \wedge 'ind < \text{length } 'E \wedge 'M!(\text{fst } ('E! 'ind)) = \text{Black} \}$
 $\text{'M} := 'M[\text{snd}('E! 'ind) := \text{Black}];;$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge \text{'PBInv } ('ind + 1) \wedge 'ind < \text{length } 'E \}$
 $\text{'ind} := 'ind + 1$
 FI
 OD

lemma *Propagate-Black-aux*:

$\vdash \text{Propagate-Black-aux}$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge ('obc < \text{Blacks } 'M \vee 'Safe) \}$
 $\langle \text{proof} \rangle$

Refining propagating black

definition *Auxk* :: *gar-coll-state* $\Rightarrow$ *bool* **where**

$\text{Auxk} \equiv \ll 'k < \text{length } 'M \wedge ('M!'k \neq \text{Black} \vee \neg \text{BtoW}('E!'ind, 'M) \vee$
 $'obc < \text{Blacks } 'M \vee (\neg 'z \wedge 'ind = R \wedge \text{snd}('E!R) = T$
 $\wedge (\exists r. 'ind < r \wedge r < \text{length } 'E \wedge \text{BtoW}('E!r, 'M)))) \gg$

definition *Propagate-Black* :: *gar-coll-state* *ann-com* **where**

$\text{Propagate-Black} \equiv$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \}$
 $\text{'ind} := 0;;$
 $\{\{ \text{'Proper} \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \wedge \text{'ind} = 0 \}$

```

WHILE 'ind < length 'E
  INV { 'Proper ∧ Roots ⊆ Blacks 'M ∧ 'obc ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
        ∧ 'PBIInv 'ind ∧ 'ind ≤ length 'E }
  DO { 'Proper ∧ Roots ⊆ Blacks 'M ∧ 'obc ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
        ∧ 'PBIInv 'ind ∧ 'ind < length 'E }
  IF ( 'M!(fst ( 'E! 'ind )) = Black THEN
    { 'Proper ∧ Roots ⊆ Blacks 'M ∧ 'obc ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
      ∧ 'PBIInv 'ind ∧ 'ind < length 'E ∧ ( 'M!(fst ( 'E! 'ind )) = Black }
    'k := (snd ( 'E! 'ind ));
    { 'Proper ∧ Roots ⊆ Blacks 'M ∧ 'obc ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
      ∧ 'PBIInv 'ind ∧ 'ind < length 'E ∧ ( 'M!(fst ( 'E! 'ind )) = Black
      ∧ 'Auxk }
    ⟨ 'M := 'M[ 'k := Black ],, 'ind := 'ind + 1 ⟩
  ELSE { 'Proper ∧ Roots ⊆ Blacks 'M ∧ 'obc ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
        ∧ 'PBIInv 'ind ∧ 'ind < length 'E }
    ⟨ IF ( 'M!(fst ( 'E! 'ind )) ≠ Black THEN 'ind := 'ind + 1 FI ⟩
  FI
OD

```

lemma *Propagate-Black*:

```

⊢ Propagate-Black
{ 'Proper ∧ Roots ⊆ Blacks 'M ∧ 'obc ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
  ∧ ( 'obc < Blacks 'M ∨ 'Safe ) }
⟨ proof ⟩

```

Counting black nodes

definition *CountInv* :: *gar-coll-state* ⇒ *nat* ⇒ *bool* **where**

CountInv ≡ « λ ind. { i. i < ind ∧ 'M! i = Black } ⊆ 'bc »

definition *Count* :: *gar-coll-state* *ann-com* **where**

```

Count ≡
{ 'Proper ∧ Roots ⊆ Blacks 'M
  ∧ 'obc ⊆ Blacks 'Ma ∧ Blacks 'Ma ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
  ∧ length 'Ma = length 'M ∧ ( 'obc < Blacks 'Ma ∨ 'Safe ) ∧ 'bc = {} }
'ind := 0;;
{ 'Proper ∧ Roots ⊆ Blacks 'M
  ∧ 'obc ⊆ Blacks 'Ma ∧ Blacks 'Ma ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
  ∧ length 'Ma = length 'M ∧ ( 'obc < Blacks 'Ma ∨ 'Safe ) ∧ 'bc = {}
  ∧ 'ind = 0 }
WHILE 'ind < length 'M
  INV { 'Proper ∧ Roots ⊆ Blacks 'M
        ∧ 'obc ⊆ Blacks 'Ma ∧ Blacks 'Ma ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
        ∧ length 'Ma = length 'M ∧ 'CountInv 'ind
        ∧ ( 'obc < Blacks 'Ma ∨ 'Safe ) ∧ 'ind ≤ length 'M }
  DO { 'Proper ∧ Roots ⊆ Blacks 'M
        ∧ 'obc ⊆ Blacks 'Ma ∧ Blacks 'Ma ⊆ Blacks 'M ∧ 'bc ⊆ Blacks 'M
        ∧ length 'Ma = length 'M ∧ 'CountInv 'ind
        ∧ ( 'obc < Blacks 'Ma ∨ 'Safe ) ∧ 'ind < length 'M }

```

IF 'M!'ind=Black
 THEN { 'Proper $\wedge$ Roots $\subseteq$ Blacks 'M
 $\wedge$ 'obc $\subseteq$ Blacks 'Ma $\wedge$ Blacks 'Ma $\subseteq$ Blacks 'M $\wedge$ 'bc $\subseteq$ Blacks 'M
 $\wedge$ length 'Ma=length 'M $\wedge$ 'CountInv 'ind
 $\wedge$ ('obc < Blacks 'Ma $\vee$ 'Safe) $\wedge$ 'ind<length 'M $\wedge$ 'M!'ind=Black }
 'bc:=insert 'ind 'bc
 FI;;
 { 'Proper $\wedge$ Roots $\subseteq$ Blacks 'M
 $\wedge$ 'obc $\subseteq$ Blacks 'Ma $\wedge$ Blacks 'Ma $\subseteq$ Blacks 'M $\wedge$ 'bc $\subseteq$ Blacks 'M
 $\wedge$ length 'Ma=length 'M $\wedge$ 'CountInv ('ind+1)
 $\wedge$ ('obc < Blacks 'Ma $\vee$ 'Safe) $\wedge$ 'ind<length 'M }
 'ind:='ind+1
 OD

lemma Count:

$\vdash$ Count
 { 'Proper $\wedge$ Roots $\subseteq$ Blacks 'M
 $\wedge$ 'obc $\subseteq$ Blacks 'Ma $\wedge$ Blacks 'Ma $\subseteq$ 'bc $\wedge$ 'bc $\subseteq$ Blacks 'M $\wedge$ length 'Ma=length
 'M
 $\wedge$ ('obc < Blacks 'Ma $\vee$ 'Safe) }
 <proof>

Appending garbage nodes to the free list

axiomatization Append-to-free :: nat $\times$ edges $\Rightarrow$ edges

where

Append-to-free0: length (Append-to-free (i, e)) = length e **and**
 Append-to-free1: Proper-Edges (m, e)
 $\implies$ Proper-Edges (m, Append-to-free(i, e)) **and**
 Append-to-free2: i $\notin$ Reach e
 $\implies n \in$ Reach (Append-to-free(i, e)) = (n = i $\vee$ n $\in$ Reach e)

definition AppendInv :: gar-coll-state $\Rightarrow$ nat $\Rightarrow$ bool **where**

AppendInv $\equiv \llbracket \lambda ind. \forall i < \text{length 'M}. ind \leq i \longrightarrow i \in \text{Reach 'E} \longrightarrow \text{'M!}i = \text{Black} \rrbracket$

definition Append :: gar-coll-state ann-com **where**

Append $\equiv$
 { 'Proper $\wedge$ Roots $\subseteq$ Blacks 'M $\wedge$ 'Safe }
 'ind:=0;;
 { 'Proper $\wedge$ Roots $\subseteq$ Blacks 'M $\wedge$ 'Safe $\wedge$ 'ind=0 }
 WHILE 'ind<length 'M
 INV { 'Proper $\wedge$ 'AppendInv 'ind $\wedge$ 'ind $\leq$ length 'M }
 DO { 'Proper $\wedge$ 'AppendInv 'ind $\wedge$ 'ind<length 'M }
 IF 'M!'ind=Black THEN
 { 'Proper $\wedge$ 'AppendInv 'ind $\wedge$ 'ind<length 'M $\wedge$ 'M!'ind=Black }
 'M:='M['ind:=White]
 ELSE { 'Proper $\wedge$ 'AppendInv 'ind $\wedge$ 'ind<length 'M $\wedge$ 'ind $\notin$ Reach 'E }
 'E:=Append-to-free('ind, 'E)
 FI;;

$\{\text{' } Proper \wedge \text{' } AppendInv (\text{' } ind+1) \wedge \text{' } ind < length \text{' } M\}$
 $\text{' } ind := \text{' } ind + 1$
OD

lemma *Append*:
 $\vdash Append \{\text{' } Proper\}$
<proof>

Correctness of the Collector

definition *Collector* :: *gar-coll-state ann-com where*
Collector $\equiv$
 $\{\text{' } Proper\}$
WHILE True INV $\{\text{' } Proper\}$
DO
Blacken-Roots;;
 $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M\}$
 $\text{' } obc := \{\}$;;
 $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M \wedge \text{' } obc = \{\}\}$
 $\text{' } bc := Roots$;;
 $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M \wedge \text{' } obc = \{\} \wedge \text{' } bc = Roots\}$
 $\text{' } Ma := M\text{-}init$;;
 $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M \wedge \text{' } obc = \{\} \wedge \text{' } bc = Roots \wedge \text{' } Ma = M\text{-}init\}$
WHILE $\text{' } obc \neq \text{' } bc$
INV $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M$
 $\wedge \text{' } obc \subseteq Blacks \text{' } Ma \wedge Blacks \text{' } Ma \subseteq \text{' } bc \wedge \text{' } bc \subseteq Blacks \text{' } M$
 $\wedge length \text{' } Ma = length \text{' } M \wedge (\text{' } obc < Blacks \text{' } Ma \vee \text{' } Safe)\}$
DO $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M \wedge \text{' } bc \subseteq Blacks \text{' } M\}$
 $\text{' } obc := \text{' } bc$;;
Propagate-Black;;
 $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M \wedge \text{' } obc \subseteq Blacks \text{' } M \wedge \text{' } bc \subseteq Blacks \text{' } M$
 $\wedge (\text{' } obc < Blacks \text{' } M \vee \text{' } Safe)\}$
 $\text{' } Ma := \text{' } M$;;
 $\{\text{' } Proper \wedge Roots \subseteq Blacks \text{' } M \wedge \text{' } obc \subseteq Blacks \text{' } Ma$
 $\wedge Blacks \text{' } Ma \subseteq Blacks \text{' } M \wedge \text{' } bc \subseteq Blacks \text{' } M \wedge length \text{' } Ma = length \text{' } M$
 $\wedge (\text{' } obc < Blacks \text{' } Ma \vee \text{' } Safe)\}$
 $\text{' } bc := \{\}$;;
Count
OD;;
Append
OD

lemma *Collector*:
 $\vdash Collector \{\text{' } False\}$
<proof>

2.2.3 Interference Freedom

lemmas *modules = Redirect-Edge-def Color-Target-def Blacken-Roots-def*
Propagate-Black-def Count-def Append-def

lemmas *Invariants* = *PBInv-def Auxk-def CountInv-def AppendInv-def*
lemmas *abbrev* = *collector-defs mutator-defs Invariants*

lemma *interfree-Blacken-Roots-Redirect-Edge*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Redirect-Edge*)
 ⟨*proof*⟩

lemma *interfree-Redirect-Edge-Blacken-Roots*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Blacken-Roots*)
 ⟨*proof*⟩

lemma *interfree-Blacken-Roots-Color-Target*:
interfree-aux (*Some Blacken-Roots*, {}, *Some Color-Target*)
 ⟨*proof*⟩

lemma *interfree-Color-Target-Blacken-Roots*:
interfree-aux (*Some Color-Target*, {}, *Some Blacken-Roots*)
 ⟨*proof*⟩

lemma *interfree-Propagate-Black-Redirect-Edge*:
interfree-aux (*Some Propagate-Black*, {}, *Some Redirect-Edge*)
 ⟨*proof*⟩

lemma *interfree-Redirect-Edge-Propagate-Black*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Propagate-Black*)
 ⟨*proof*⟩

lemma *interfree-Propagate-Black-Color-Target*:
interfree-aux (*Some Propagate-Black*, {}, *Some Color-Target*)
 ⟨*proof*⟩

lemma *interfree-Color-Target-Propagate-Black*:
interfree-aux (*Some Color-Target*, {}, *Some Propagate-Black*)
 ⟨*proof*⟩

lemma *interfree-Count-Redirect-Edge*:
interfree-aux (*Some Count*, {}, *Some Redirect-Edge*)
 ⟨*proof*⟩

lemma *interfree-Redirect-Edge-Count*:
interfree-aux (*Some Redirect-Edge*, {}, *Some Count*)
 ⟨*proof*⟩

lemma *interfree-Count-Color-Target*:
interfree-aux (*Some Count*, {}, *Some Color-Target*)
 ⟨*proof*⟩

lemma *interfree-Color-Target-Count*:
interfree-aux (*Some Color-Target*, {}, *Some Count*)

<proof>

lemma *interfree-Append-Redirect-Edge:*

interfree-aux (Some Append, {}, Some Redirect-Edge)

<proof>

lemma *interfree-Redirect-Edge-Append:*

interfree-aux (Some Redirect-Edge, {}, Some Append)

<proof>

lemma *interfree-Append-Color-Target:*

interfree-aux (Some Append, {}, Some Color-Target)

<proof>

lemma *interfree-Color-Target-Append:*

interfree-aux (Some Color-Target, {}, Some Append)

<proof>

lemmas *collector-mutator-interfree =*

interfree-Blacken-Roots-Redirect-Edge interfree-Blacken-Roots-Color-Target
interfree-Propagate-Black-Redirect-Edge interfree-Propagate-Black-Color-Target
interfree-Count-Redirect-Edge interfree-Count-Color-Target
interfree-Append-Redirect-Edge interfree-Append-Color-Target
interfree-Redirect-Edge-Blacken-Roots interfree-Color-Target-Blacken-Roots
interfree-Redirect-Edge-Propagate-Black interfree-Color-Target-Propagate-Black
interfree-Redirect-Edge-Count interfree-Color-Target-Count
interfree-Redirect-Edge-Append interfree-Color-Target-Append

Interference freedom Collector-Mutator

lemma *interfree-Collector-Mutator:*

interfree-aux (Some Collector, {}, Some Mutator)

<proof>

Interference freedom Mutator-Collector

lemma *interfree-Mutator-Collector:*

interfree-aux (Some Mutator, {}, Some Collector)

<proof>

The Garbage Collection algorithm

In total there are 289 verification conditions.

lemma *Gar-Coll:*

$\| - \{ \text{'Proper} \wedge \text{'Mut-init} \wedge \text{'z} \}$

COBEGIN

Collector

$\{ \text{False} \}$

$\|$

```

    Mutator
    {False}
  COEND
    {False}
<proof>

```

end

2.3 The Multi-Mutator Case

theory *Mul-Gar-Coll* **imports** *Graph OG-Syntax* **begin**

The full theory takes aprox. 18 minutes.

```

record mut =
  Z :: bool
  R :: nat
  T :: nat

```

Declaration of variables:

```

record mul-gar-coll-state =
  M :: nodes
  E :: edges
  bc :: nat set
  obc :: nat set
  Ma :: nodes
  ind :: nat
  k :: nat
  q :: nat
  l :: nat
  Muts :: mut list

```

2.3.1 The Mutators

definition *Mul-mut-init* :: *mul-gar-coll-state* $\Rightarrow$ *nat* $\Rightarrow$ *bool* **where**

$$\text{Mul-mut-init} \equiv \llbracket \lambda n. n = \text{length } 'Muts \wedge (\forall i < n. R ('Muts!i) < \text{length } 'E \wedge T ('Muts!i) < \text{length } 'M) \rrbracket$$

definition *Mul-Redirect-Edge* *j* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *mul-gar-coll-state* *ann-com* **where**

$$\text{Mul-Redirect-Edge } j \ n \equiv \llbracket 'Mul\text{-mut-init } n \wedge Z ('Muts!j) \rrbracket$$

$$\llbracket \text{IF } T ('Muts!j) \in \text{Reach } 'E \text{ THEN } 'E := 'E[R ('Muts!j) := (fst ('E!R ('Muts!j)), T ('Muts!j))] \text{ FI}, 'Muts := 'Muts[j := ('Muts!j) (Z := False)] \rrbracket$$

definition *Mul-Color-Target* *j* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *mul-gar-coll-state* *ann-com* **where**

$$\text{Mul-Color-Target } j \ n \equiv \llbracket 'Mul\text{-mut-init } n \wedge \neg Z ('Muts!j) \rrbracket$$

$$\llbracket 'M := 'M[T ('Muts!j) := \text{Black}], 'Muts := 'Muts[j := ('Muts!j) (Z := True)] \rrbracket$$

definition *Mul-Mutator* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *mul-gar-coll-state ann-com* **where**

Mul-Mutator j n $\equiv$
 $\{\text{'Mul-mut-init } n \wedge Z (\text{'Muts!}j)\}$
WHILE *True*
INV $\{\text{'Mul-mut-init } n \wedge Z (\text{'Muts!}j)\}$
DO *Mul-Redirect-Edge j n* ;;
Mul-Color-Target j n
OD

lemmas *mul-mutator-defs* = *Mul-mut-init-def Mul-Redirect-Edge-def Mul-Color-Target-def*

Correctness of the proof outline of one mutator

lemma *Mul-Redirect-Edge*: $0 \leq j \wedge j < n \implies$

$\vdash$ *Mul-Redirect-Edge j n*
 $\text{pre}(\text{Mul-Color-Target } j \text{ } n)$
 $\langle \text{proof} \rangle$

lemma *Mul-Color-Target*: $0 \leq j \wedge j < n \implies$

$\vdash$ *Mul-Color-Target j n*
 $\{\text{'Mul-mut-init } n \wedge Z (\text{'Muts!}j)\}$
 $\langle \text{proof} \rangle$

lemma *Mul-Mutator*: $0 \leq j \wedge j < n \implies$

$\vdash$ *Mul-Mutator j n* $\{\text{False}\}$
 $\langle \text{proof} \rangle$

Interference freedom between mutators

lemma *Mul-interfree-Redirect-Edge-Redirect-Edge*:

$\llbracket 0 \leq i; i < n; 0 \leq j; j < n; i \neq j \rrbracket \implies$
 $\text{interfree-aux } (\text{Some } (\text{Mul-Redirect-Edge } i \text{ } n), \{\}, \text{Some}(\text{Mul-Redirect-Edge } j \text{ } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Redirect-Edge-Color-Target*:

$\llbracket 0 \leq i; i < n; 0 \leq j; j < n; i \neq j \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Redirect-Edge } i \text{ } n), \{\}, \text{Some}(\text{Mul-Color-Target } j \text{ } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Color-Target-Redirect-Edge*:

$\llbracket 0 \leq i; i < n; 0 \leq j; j < n; i \neq j \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Color-Target } i \text{ } n), \{\}, \text{Some}(\text{Mul-Redirect-Edge } j \text{ } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Color-Target-Color-Target*:

$\llbracket 0 \leq i; i < n; 0 \leq j; j < n; i \neq j \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Color-Target } i \text{ } n), \{\}, \text{Some}(\text{Mul-Color-Target } j \text{ } n))$
 $\langle \text{proof} \rangle$

lemmas *mul-mutator-interfree* =
Mul-interfree-Redirect-Edge-Redirect-Edge *Mul-interfree-Redirect-Edge-Color-Target*
Mul-interfree-Color-Target-Redirect-Edge *Mul-interfree-Color-Target-Color-Target*

lemma *Mul-interfree-Mutator-Mutator*: $\llbracket i < n; j < n; i \neq j \rrbracket \implies$
interfree-aux (*Some* (*Mul-Mutator* *i n*), {}, *Some* (*Mul-Mutator* *j n*))
 <proof>

Modular Parameterized Mutators

lemma *Mul-Parameterized-Mutators*: $0 < n \implies$
 $\llbracket - \rrbracket \{ \text{'Mul-mut-init } n \wedge (\forall i < n. Z \text{'Muts!i}) \}$
 COBEGIN
SCHEME [$0 \leq j < n$]
Mul-Mutator *j n*
 $\{ \text{False} \}$
 COEND
 $\{ \text{False} \}$
 <proof>

2.3.2 The Collector

definition *Queue* :: *mul-gar-coll-state* $\Rightarrow$ *nat* **where**
Queue $\equiv$ « *length* (*filter* ($\lambda i. \neg Z \text{'Muts!i} \wedge \text{'M!}(T \text{'Muts!i}) \neq \text{Black}$) *'Muts*) »

consts *M-init* :: *nodes*

definition *Proper-M-init* :: *mul-gar-coll-state* $\Rightarrow$ *bool* **where**
Proper-M-init $\equiv$ « *Blacks* *M-init=Roots* $\wedge$ *length* *M-init=length* *'M* »

definition *Mul-Proper* :: *mul-gar-coll-state* $\Rightarrow$ *nat* $\Rightarrow$ *bool* **where**
Mul-Proper $\equiv$ « $\lambda n. \text{Proper-Roots } 'M \wedge \text{Proper-Edges } ('M, 'E) \wedge ' \text{Proper-M-init}$
 $\wedge n = \text{length } ' \text{Muts}$ »

definition *Safe* :: *mul-gar-coll-state* $\Rightarrow$ *bool* **where**
Safe $\equiv$ « *Reach* *'E* $\subseteq$ *Blacks* *'M* »

lemmas *mul-collector-defs* = *Proper-M-init-def* *Mul-Proper-def* *Safe-def*

Blackening Roots

definition *Mul-Blacken-Roots* :: *nat* $\Rightarrow$ *mul-gar-coll-state* *ann-com* **where**
Mul-Blacken-Roots *n* $\equiv$
 $\{ \text{'Mul-Proper } n \}$
 $\text{'ind} := 0;$
 $\{ \text{'Mul-Proper } n \wedge \text{'ind} = 0 \}$
 WHILE $\text{'ind} < \text{length } 'M$
 INV $\{ \text{'Mul-Proper } n \wedge (\forall i < \text{'ind}. i \in \text{Roots} \longrightarrow \text{'M!}i = \text{Black}) \wedge \text{'ind} \leq \text{length}$
 $\text{'M} \}$
 DO $\{ \text{'Mul-Proper } n \wedge (\forall i < \text{'ind}. i \in \text{Roots} \longrightarrow \text{'M!}i = \text{Black}) \wedge \text{'ind} < \text{length } 'M \}$

$$\begin{array}{l}
\text{IF } 'ind \in \text{Roots} \text{ THEN} \\
\{ 'Mul\text{-}Proper\ n \wedge (\forall i < 'ind. i \in \text{Roots} \longrightarrow 'M!i = \text{Black}) \wedge 'ind < \text{length } 'M \wedge \\
'ind \in \text{Roots} \} \\
'M := 'M['ind := \text{Black}] \text{ FI}; \\
\{ 'Mul\text{-}Proper\ n \wedge (\forall i < 'ind + 1. i \in \text{Roots} \longrightarrow 'M!i = \text{Black}) \wedge 'ind < \text{length} \\
'M \} \\
'ind := 'ind + 1 \\
\text{OD}
\end{array}$$

lemma *Mul-Blacken-Roots*:

$$\begin{array}{l}
\vdash \text{Mul-Blacken-Roots } n \\
\{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \} \\
\langle \text{proof} \rangle
\end{array}$$

Propagating Black

definition *Mul-PBInv* :: *mul-gar-coll-state* $\Rightarrow$ *bool* **where**

$$\begin{array}{l}
\text{Mul-PBInv} \equiv \ll 'Safe \vee 'obc \subseteq \text{Blacks } 'M \vee 'l < 'Queue \\
\vee (\forall i < 'ind. \neg \text{BtoW}('E!i, 'M)) \wedge 'l \leq 'Queue \gg
\end{array}$$

definition *Mul-Auxk* :: *mul-gar-coll-state* $\Rightarrow$ *bool* **where**

$$\text{Mul-Auxk} \equiv \ll 'l < 'Queue \vee 'M!k \neq \text{Black} \vee \neg \text{BtoW}('E!ind, 'M) \vee 'obc \subseteq \text{Blacks } 'M \gg$$

definition *Mul-Propagate-Black* :: *nat* $\Rightarrow$ *mul-gar-coll-state ann-com* **where**

$$\begin{array}{l}
\text{Mul-Propagate-Black } n \equiv \\
\{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \\
\wedge ('Safe \vee 'l \leq 'Queue \vee 'obc \subseteq \text{Blacks } 'M) \} \\
'ind := 0; \\
\{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \\
\wedge 'obc \subseteq \text{Blacks } 'M \wedge \text{Blacks } 'M \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \\
\wedge ('Safe \vee 'l \leq 'Queue \vee 'obc \subseteq \text{Blacks } 'M) \wedge 'ind = 0 \} \\
\text{WHILE } 'ind < \text{length } 'E \\
\text{INV } \{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \\
\wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \\
\wedge 'Mul\text{-}PBInv \wedge 'ind \leq \text{length } 'E \} \\
\text{DO } \{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \\
\wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \\
\wedge 'Mul\text{-}PBInv \wedge 'ind < \text{length } 'E \} \\
\text{IF } 'M!(fst('E!ind)) = \text{Black} \text{ THEN} \\
\{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \\
\wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \\
\wedge 'Mul\text{-}PBInv \wedge ('M!fst('E!ind)) = \text{Black} \wedge 'ind < \text{length } 'E \} \\
'k := snd('E!ind); \\
\{ 'Mul\text{-}Proper\ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \\
\wedge 'obc \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M \\
\wedge ('Safe \vee 'obc \subseteq \text{Blacks } 'M \vee 'l < 'Queue \vee (\forall i < 'ind. \neg \text{BtoW}('E!i, 'M)) \\
\wedge 'l \leq 'Queue \wedge 'Mul\text{-}Auxk) \wedge 'k < \text{length } 'M \wedge 'M!fst('E!ind) = \text{Black} \\
\wedge 'ind < \text{length } 'E \}
\end{array}$$

$\langle 'M := 'M['k := Black], 'ind := 'ind + 1 \rangle$
 $ELSE \{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M$
 $\wedge 'obc \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge 'Mul-PBInv \wedge 'ind < length \text{ } 'E \}$
 $\langle IF 'M!(fst('E! 'ind)) \neq Black THEN 'ind := 'ind + 1 FI \rangle FI$
 OD

lemma *Mul-Propagate-Black*:

$\vdash Mul-Propagate-Black \text{ } n$
 $\{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M \wedge 'obc \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge ('Safe \vee 'obc \subseteq Blacks \text{ } 'M \vee 'l < 'Queue \wedge ('l \leq 'Queue \vee 'obc \subseteq Blacks$
 $\text{ } 'M)) \}$
 $\langle proof \rangle$

Counting Black Nodes

definition *Mul-CountInv* :: *mul-gar-coll-state* $\Rightarrow$ *nat* $\Rightarrow$ *bool* **where**

$Mul-CountInv \equiv \ll \lambda ind. \{ i. i < ind \wedge 'Ma!i = Black \} \subseteq 'bc \gg$

definition *Mul-Count* :: *nat* $\Rightarrow$ *mul-gar-coll-state ann-com* **where**

$Mul-Count \text{ } n \equiv$
 $\{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M$
 $\wedge 'obc \subseteq Blacks \text{ } 'Ma \wedge Blacks \text{ } 'Ma \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge length \text{ } 'Ma = length \text{ } 'M$
 $\wedge ('Safe \vee 'obc \subseteq Blacks \text{ } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq Blacks \text{ } 'M))$
 $\wedge 'q < n + 1 \wedge 'bc = \{ \} \}$
 $'ind := 0;;$
 $\{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M$
 $\wedge 'obc \subseteq Blacks \text{ } 'Ma \wedge Blacks \text{ } 'Ma \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge length \text{ } 'Ma = length \text{ } 'M$
 $\wedge ('Safe \vee 'obc \subseteq Blacks \text{ } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq Blacks \text{ } 'M))$
 $\wedge 'q < n + 1 \wedge 'bc = \{ \} \wedge 'ind = 0 \}$
 $WHILE 'ind < length \text{ } 'M$
 $INV \{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M$
 $\wedge 'obc \subseteq Blacks \text{ } 'Ma \wedge Blacks \text{ } 'Ma \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge length \text{ } 'Ma = length \text{ } 'M \wedge 'Mul-CountInv \text{ } 'ind$
 $\wedge ('Safe \vee 'obc \subseteq Blacks \text{ } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq Blacks \text{ } 'M))$
 $\wedge 'q < n + 1 \wedge 'ind \leq length \text{ } 'M \}$
 $DO \{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M$
 $\wedge 'obc \subseteq Blacks \text{ } 'Ma \wedge Blacks \text{ } 'Ma \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge length \text{ } 'Ma = length \text{ } 'M \wedge 'Mul-CountInv \text{ } 'ind$
 $\wedge ('Safe \vee 'obc \subseteq Blacks \text{ } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq Blacks \text{ } 'M))$
 $\wedge 'q < n + 1 \wedge 'ind < length \text{ } 'M \}$
 $IF 'M! 'ind = Black$
 $THEN \{ 'Mul-Prop\text{-}n \wedge Roots \subseteq Blacks \text{ } 'M$
 $\wedge 'obc \subseteq Blacks \text{ } 'Ma \wedge Blacks \text{ } 'Ma \subseteq Blacks \text{ } 'M \wedge 'bc \subseteq Blacks \text{ } 'M$
 $\wedge length \text{ } 'Ma = length \text{ } 'M \wedge 'Mul-CountInv \text{ } 'ind$
 $\wedge ('Safe \vee 'obc \subseteq Blacks \text{ } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq Blacks$
 $\text{ } 'M))$

$\wedge 'q < n+1 \wedge 'ind < \text{length } 'M \wedge 'M! 'ind = \text{Black}\}$
 $'bc := \text{insert } 'ind \ 'bc$
FI;;
 $\{\{ 'Mul\text{-}Proper \ n \wedge \text{Roots} \subseteq \text{Blacks } 'M$
 $\wedge 'obc \subseteq \text{Blacks } 'Ma \wedge \text{Blacks } 'Ma \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge \text{length } 'Ma = \text{length } 'M \wedge 'Mul\text{-}CountInv ('ind+1)$
 $\wedge ('Safe \vee 'obc \subseteq \text{Blacks } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq \text{Blacks } 'M))$
 $\wedge 'q < n+1 \wedge 'ind < \text{length } 'M\}$
 $'ind := 'ind+1$
OD

lemma *Mul-Count*:

$\vdash \text{Mul-Count } n$
 $\{\{ 'Mul\text{-}Proper \ n \wedge \text{Roots} \subseteq \text{Blacks } 'M$
 $\wedge 'obc \subseteq \text{Blacks } 'Ma \wedge \text{Blacks } 'Ma \subseteq \text{Blacks } 'M \wedge 'bc \subseteq \text{Blacks } 'M$
 $\wedge \text{length } 'Ma = \text{length } 'M \wedge \text{Blacks } 'Ma \subseteq 'bc$
 $\wedge ('Safe \vee 'obc \subseteq \text{Blacks } 'Ma \vee 'l < 'q \wedge ('q \leq 'Queue \vee 'obc \subseteq \text{Blacks } 'M))$
 $\wedge 'q < n+1\}$
 $\langle \text{proof} \rangle$

Appending garbage nodes to the free list

axiomatization *Append-to-free* :: $\text{nat} \times \text{edges} \Rightarrow \text{edges}$

where

Append-to-free0: $\text{length } (\text{Append-to-free } (i, e)) = \text{length } e$ **and**
Append-to-free1: $\text{Proper-Edges } (m, e)$
 $\implies \text{Proper-Edges } (m, \text{Append-to-free}(i, e))$ **and**
Append-to-free2: $i \notin \text{Reach } e$
 $\implies n \in \text{Reach } (\text{Append-to-free}(i, e)) = (n = i \vee n \in \text{Reach } e)$

definition *Mul-AppendInv* :: $\text{mul-gar-coll-state} \Rightarrow \text{nat} \Rightarrow \text{bool}$ **where**

Mul-AppendInv $\equiv \ll \lambda ind. (\forall i. ind \leq i \longrightarrow i < \text{length } 'M \longrightarrow i \in \text{Reach } 'E \longrightarrow 'M! i = \text{Black}) \gg$

definition *Mul-Append* :: $\text{nat} \Rightarrow \text{mul-gar-coll-state ann-com}$ **where**

Mul-Append $n \equiv$
 $\{\{ 'Mul\text{-}Proper \ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'Safe\}$
 $'ind := 0;$
 $\{\{ 'Mul\text{-}Proper \ n \wedge \text{Roots} \subseteq \text{Blacks } 'M \wedge 'Safe \wedge 'ind = 0\}$
 $\text{WHILE } 'ind < \text{length } 'M$
 $\text{INV } \{\{ 'Mul\text{-}Proper \ n \wedge 'Mul\text{-}AppendInv \ 'ind \wedge 'ind \leq \text{length } 'M\}$
 $\text{DO } \{\{ 'Mul\text{-}Proper \ n \wedge 'Mul\text{-}AppendInv \ 'ind \wedge 'ind < \text{length } 'M\}$
 $\text{IF } 'M! 'ind = \text{Black} \text{ THEN}$
 $\{\{ 'Mul\text{-}Proper \ n \wedge 'Mul\text{-}AppendInv \ 'ind \wedge 'ind < \text{length } 'M \wedge 'M! 'ind = \text{Black}\}$
 $'M := 'M['ind := \text{White}]$
 ELSE
 $\{\{ 'Mul\text{-}Proper \ n \wedge 'Mul\text{-}AppendInv \ 'ind \wedge 'ind < \text{length } 'M \wedge 'ind \notin \text{Reach}$
 $'E\}$
 $'E := \text{Append-to-free}('ind, 'E)$

$FI;;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{'Mul-AppendInv}\ (\text{'ind}+1) \wedge \text{'ind} < \text{length}\ \text{'M}\}$
 $\text{'ind} := \text{'ind} + 1$
 OD

lemma *Mul-Append:*

$\vdash \text{Mul-Append}\ n$
 $\{\text{'Mul-Prop}er\ n\}$
 $\langle \text{proof} \rangle$

Collector

definition *Mul-Collector* :: $\text{nat} \Rightarrow \text{mul-gar-coll-state ann-com}$ **where**

$\text{Mul-Collector}\ n \equiv$
 $\{\text{'Mul-Prop}er\ n\}$
 $WHILE\ True\ INV\ \{\text{'Mul-Prop}er\ n\}$
 DO
 $\text{Mul-Blacken-Roots}\ n;;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M}\}$
 $\text{'obc} := \{\};;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'obc} = \{\}\}$
 $\text{'bc} := \text{Roots};;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'obc} = \{\} \wedge \text{'bc} = \text{Roots}\}$
 $\text{'l} := 0;;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'obc} = \{\} \wedge \text{'bc} = \text{Roots} \wedge \text{'l} = 0\}$
 $WHILE\ \text{'l} < n + 1$
 $INV\ \{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'bc} \subseteq \text{Blacks}\ \text{'M} \wedge$
 $(\text{'Safe} \vee (\text{'l} \leq \text{'Queue} \vee \text{'bc} \subseteq \text{Blacks}\ \text{'M})) \wedge \text{'l} < n + 1\}$
 $DO\ \{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'bc} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge (\text{'Safe} \vee \text{'l} \leq \text{'Queue} \vee \text{'bc} \subseteq \text{Blacks}\ \text{'M}))\}$
 $\text{'obc} := \text{'bc};;$
 $\text{Mul-Propagate-Black}\ n;;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge \text{'obc} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'bc} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge (\text{'Safe} \vee \text{'obc} \subseteq \text{Blacks}\ \text{'M} \vee \text{'l} < \text{'Queue}$
 $\wedge (\text{'l} \leq \text{'Queue} \vee \text{'obc} \subseteq \text{Blacks}\ \text{'M}))\}$
 $\text{'bc} := \{\};;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge \text{'obc} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'bc} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge (\text{'Safe} \vee \text{'obc} \subseteq \text{Blacks}\ \text{'M} \vee \text{'l} < \text{'Queue}$
 $\wedge (\text{'l} \leq \text{'Queue} \vee \text{'obc} \subseteq \text{Blacks}\ \text{'M})) \wedge \text{'bc} = \{\}\}$
 $\langle \text{'Ma} := \text{'M},, \text{'q} := \text{'Queue} \rangle;;$
 $\text{Mul-Count}\ n;;$
 $\{\text{'Mul-Prop}er\ n \wedge \text{Roots} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge \text{'obc} \subseteq \text{Blacks}\ \text{'Ma} \wedge \text{Blacks}\ \text{'Ma} \subseteq \text{Blacks}\ \text{'M} \wedge \text{'bc} \subseteq \text{Blacks}\ \text{'M}$
 $\wedge \text{length}\ \text{'Ma} = \text{length}\ \text{'M} \wedge \text{Blacks}\ \text{'Ma} \subseteq \text{'bc}$
 $\wedge (\text{'Safe} \vee \text{'obc} \subseteq \text{Blacks}\ \text{'Ma} \vee \text{'l} < \text{'q} \wedge (\text{'q} \leq \text{'Queue} \vee \text{'obc} \subseteq \text{Blacks}\ \text{'M}))$
 $\wedge \text{'q} < n + 1\}$
 $IF\ \text{'obc} = \text{'bc}\ THEN$

$\{ \text{'Mul-} \textit{Proper } n \wedge \textit{Roots} \subseteq \textit{Blacks } 'M$
 $\wedge \text{'obc} \subseteq \textit{Blacks } 'Ma \wedge \textit{Blacks } 'Ma \subseteq \textit{Blacks } 'M \wedge \text{'bc} \subseteq \textit{Blacks } 'M$
 $\wedge \textit{length } 'Ma = \textit{length } 'M \wedge \textit{Blacks } 'Ma \subseteq \text{'bc}$
 $\wedge (\text{'Safe} \vee \text{'obc} \subseteq \textit{Blacks } 'Ma \vee \text{'l} < \text{'q} \wedge (\text{'q} \leq \text{'Queue} \vee \text{'obc} \subseteq \textit{Blacks } 'M))$
 $\wedge \text{'q} < n+1 \wedge \text{'obc} = \text{'bc} \}$
 $\text{'l} := \text{'l} + 1$
 $\textit{ELSE } \{ \text{'Mul-} \textit{Proper } n \wedge \textit{Roots} \subseteq \textit{Blacks } 'M$
 $\wedge \text{'obc} \subseteq \textit{Blacks } 'Ma \wedge \textit{Blacks } 'Ma \subseteq \textit{Blacks } 'M \wedge \text{'bc} \subseteq \textit{Blacks } 'M$
 $\wedge \textit{length } 'Ma = \textit{length } 'M \wedge \textit{Blacks } 'Ma \subseteq \text{'bc}$
 $\wedge (\text{'Safe} \vee \text{'obc} \subseteq \textit{Blacks } 'Ma \vee \text{'l} < \text{'q} \wedge (\text{'q} \leq \text{'Queue} \vee \text{'obc} \subseteq \textit{Blacks } 'M))$
 $\wedge \text{'q} < n+1 \wedge \text{'obc} \neq \text{'bc} \}$
 $\text{'l} := 0 \textit{ FI}$
 $\textit{OD};;$
 $\textit{Mul-Append } n$
 $\textit{OD}$

lemmas $\textit{mul-modules} = \textit{Mul-Redirect-Edge-def } \textit{Mul-Color-Target-def}$
 $\textit{Mul-Blacken-Roots-def } \textit{Mul-Propagate-Black-def}$
 $\textit{Mul-Count-def } \textit{Mul-Append-def}$

lemma $\textit{Mul-Collector}:$

$\vdash \textit{Mul-Collector } n$
 $\{ \textit{False} \}$
 $\langle \textit{proof} \rangle$

2.3.3 Interference Freedom

lemma $\textit{le-length-filter-update}[\textit{rule-format}]:$

$\forall i. (\neg P(\textit{list}!i) \vee P j) \wedge i < \textit{length list}$
 $\longrightarrow \textit{length}(\textit{filter } P \textit{ list}) \leq \textit{length}(\textit{filter } P (\textit{list}[i:=j]))$
 $\langle \textit{proof} \rangle$

lemma $\textit{less-length-filter-update} [\textit{rule-format}]:$

$\forall i. P j \wedge \neg(P(\textit{list}!i)) \wedge i < \textit{length list}$
 $\longrightarrow \textit{length}(\textit{filter } P \textit{ list}) < \textit{length}(\textit{filter } P (\textit{list}[i:=j]))$
 $\langle \textit{proof} \rangle$

lemma $\textit{Mul-interfree-Blacken-Roots-Redirect-Edge}: \llbracket 0 \leq j; j < n \rrbracket \Longrightarrow$
 $\textit{interfree-aux } (\textit{Some}(\textit{Mul-Blacken-Roots } n), \{\}, \textit{Some}(\textit{Mul-Redirect-Edge } j \textit{ n}))$
 $\langle \textit{proof} \rangle$

lemma $\textit{Mul-interfree-Redirect-Edge-Blacken-Roots}: \llbracket 0 \leq j; j < n \rrbracket \Longrightarrow$
 $\textit{interfree-aux } (\textit{Some}(\textit{Mul-Redirect-Edge } j \textit{ n}), \{\}, \textit{Some}(\textit{Mul-Blacken-Roots } n))$
 $\langle \textit{proof} \rangle$

lemma $\textit{Mul-interfree-Blacken-Roots-Color-Target}: \llbracket 0 \leq j; j < n \rrbracket \Longrightarrow$
 $\textit{interfree-aux } (\textit{Some}(\textit{Mul-Blacken-Roots } n), \{\}, \textit{Some}(\textit{Mul-Color-Target } j \textit{ n}))$
 $\langle \textit{proof} \rangle$

lemma *Mul-interfree-Color-Target-Blacken-Roots*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Color-Target } j \ n), \{\}, \text{Some } (\text{Mul-Blacken-Roots } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Propagate-Black-Redirect-Edge*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Propagate-Black } n), \{\}, \text{Some } (\text{Mul-Redirect-Edge } j \ n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Redirect-Edge-Propagate-Black*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Redirect-Edge } j \ n), \{\}, \text{Some } (\text{Mul-Propagate-Black } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Propagate-Black-Color-Target*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Propagate-Black } n), \{\}, \text{Some } (\text{Mul-Color-Target } j \ n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Color-Target-Propagate-Black*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Color-Target } j \ n), \{\}, \text{Some}(\text{Mul-Propagate-Black } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Count-Redirect-Edge*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Count } n), \{\}, \text{Some}(\text{Mul-Redirect-Edge } j \ n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Redirect-Edge-Count*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Redirect-Edge } j \ n), \{\}, \text{Some}(\text{Mul-Count } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Count-Color-Target*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Count } n), \{\}, \text{Some}(\text{Mul-Color-Target } j \ n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Color-Target-Count*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Color-Target } j \ n), \{\}, \text{Some}(\text{Mul-Count } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Append-Redirect-Edge*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Append } n), \{\}, \text{Some}(\text{Mul-Redirect-Edge } j \ n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Redirect-Edge-Append*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Redirect-Edge } j \ n), \{\}, \text{Some}(\text{Mul-Append } n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Append-Color-Target*: $\llbracket 0 \leq j; j < n \rrbracket \implies$
 $\text{interfree-aux } (\text{Some}(\text{Mul-Append } n), \{\}, \text{Some}(\text{Mul-Color-Target } j \ n))$
 $\langle \text{proof} \rangle$

lemma *Mul-interfree-Color-Target-Append*: $\llbracket 0 \leq j; j < n \rrbracket \implies$

interfree-aux (*Some*(*Mul-Color-Target* *j n*), {}, *Some*(*Mul-Append* *n*))
 <proof>

Interference freedom Collector-Mutator

lemmas *mul-collector-mutator-interfree* =
Mul-interfree-Blacken-Roots-Redirect-Edge *Mul-interfree-Blacken-Roots-Color-Target*
Mul-interfree-Propagate-Black-Redirect-Edge *Mul-interfree-Propagate-Black-Color-Target*
Mul-interfree-Count-Redirect-Edge *Mul-interfree-Count-Color-Target*
Mul-interfree-Append-Redirect-Edge *Mul-interfree-Append-Color-Target*
Mul-interfree-Redirect-Edge-Blacken-Roots *Mul-interfree-Color-Target-Blacken-Roots*
Mul-interfree-Redirect-Edge-Propagate-Black *Mul-interfree-Color-Target-Propagate-Black*
Mul-interfree-Redirect-Edge-Count *Mul-interfree-Color-Target-Count*
Mul-interfree-Redirect-Edge-Append *Mul-interfree-Color-Target-Append*

lemma *Mul-interfree-Collector-Mutator*: $j < n \implies$
interfree-aux (*Some* (*Mul-Collector* *n*), {}, *Some* (*Mul-Mutator* *j n*))
 <proof>

Interference freedom Mutator-Collector

lemma *Mul-interfree-Mutator-Collector*: $j < n \implies$
interfree-aux (*Some* (*Mul-Mutator* *j n*), {}, *Some* (*Mul-Collector* *n*))
 <proof>

The Multi-Mutator Garbage Collection Algorithm

The total number of verification conditions is 328

lemma *Mul-Gar-Coll*:
 || – {*Mul-Proper* *n* $\wedge$ *Mul-mut-init* *n* $\wedge$ ($\forall i < n. Z$ (*Muts!**i*))}
 COBEGIN
 Mul-Collector *n*
 {*False*}
 ||
 SCHEME [$0 \leq j < n$]
 Mul-Mutator *j n*
 {*False*}
 COEND
 {*False*}
 <proof>

end

Chapter 3

The Rely-Guarantee Method

3.1 Abstract Syntax

theory *RG-Com* **imports** *Main* **begin**

Semantics of assertions and boolean expressions (*bexp*) as sets of states.
Syntax of commands *com* and parallel commands *par-com*.

type-synonym *'a bexp* = *'a set*

datatype *'a com* =
 Basic 'a $\Rightarrow$ 'a
 | *Seq 'a com 'a com*
 | *Cond 'a bexp 'a com 'a com*
 | *While 'a bexp 'a com*
 | *Await 'a bexp 'a com*

type-synonym *'a par-com* = *'a com option list*

end

3.2 Operational Semantics

theory *RG-Tran*
imports *RG-Com*
begin

3.2.1 Semantics of Component Programs

Environment transitions

type-synonym *'a conf* = *(('a com) option) $\times$ 'a*

inductive-set

$etran :: ('a\ conf \times 'a\ conf)\ set$
 and $etran' :: 'a\ conf \Rightarrow 'a\ conf \Rightarrow bool \ (- \ -e \rightarrow - \ [81,81] \ 80)$

where

$P -e\rightarrow Q \equiv (P, Q) \in etran$
 $| Env: (P, s) -e\rightarrow (P, t)$

lemma *etranE*: $c -e\rightarrow c' \implies (\bigwedge P \text{ s.t. } c = (P, s) \implies c' = (P, t) \implies Q) \implies Q$
 $\langle proof \rangle$

Component transitions

inductive-set

$ctran :: ('a \text{ conf} \times 'a \text{ conf}) \text{ set}$
and $ctran' :: 'a \text{ conf} \Rightarrow 'a \text{ conf} \Rightarrow \text{bool} \quad (- -c\rightarrow - [81,81] \ 80)$
and $ctrans :: 'a \text{ conf} \Rightarrow 'a \text{ conf} \Rightarrow \text{bool} \quad (- -c*\rightarrow - [81,81] \ 80)$

where

$P -c\rightarrow Q \equiv (P, Q) \in ctran$
 $| P -c*\rightarrow Q \equiv (P, Q) \in ctran^*$

$| Basic: (Some(Basic \ f), s) -c\rightarrow (None, f \ s)$

$| Seq1: (Some \ P0, s) -c\rightarrow (None, t) \implies (Some(Seq \ P0 \ P1), s) -c\rightarrow (Some \ P1, t)$

$| Seq2: (Some \ P0, s) -c\rightarrow (Some \ P2, t) \implies (Some(Seq \ P0 \ P1), s) -c\rightarrow (Some(Seq \ P2 \ P1), t)$

$| CondT: s \in b \implies (Some(Cond \ b \ P1 \ P2), s) -c\rightarrow (Some \ P1, s)$

$| CondF: s \notin b \implies (Some(Cond \ b \ P1 \ P2), s) -c\rightarrow (Some \ P2, s)$

$| WhileF: s \notin b \implies (Some(While \ b \ P), s) -c\rightarrow (None, s)$

$| WhileT: s \in b \implies (Some(While \ b \ P), s) -c\rightarrow (Some(Seq \ P \ (While \ b \ P)), s)$

$| Await: \llbracket s \in b; (Some \ P, s) -c*\rightarrow (None, t) \rrbracket \implies (Some(Await \ b \ P), s) -c\rightarrow (None, t)$

monos *rtrancl-mono*

3.2.2 Semantics of Parallel Programs

type-synonym $'a \text{ par-conf} = ('a \text{ par-com}) \times 'a$

inductive-set

$par-etran :: ('a \text{ par-conf} \times 'a \text{ par-conf}) \text{ set}$
and $par-etran' :: ['a \text{ par-conf}, 'a \text{ par-conf}] \Rightarrow \text{bool} \quad (- -pe\rightarrow - [81,81] \ 80)$

where

$P -pe\rightarrow Q \equiv (P, Q) \in par-etran$
 $| ParEnv: (Ps, s) -pe\rightarrow (Ps, t)$

inductive-set

$par-ctran :: ('a \text{ par-conf} \times 'a \text{ par-conf}) \text{ set}$
and $par-ctran' :: ['a \text{ par-conf}, 'a \text{ par-conf}] \Rightarrow \text{bool} \quad (- -pc\rightarrow - [81,81] \ 80)$

where

$P -pc\rightarrow Q \equiv (P, Q) \in \text{par-ctran}$
 $| \text{ParComp}: \llbracket i < \text{length } Ps; (Ps!i, s) -c\rightarrow (r, t) \rrbracket \implies (Ps, s) -pc\rightarrow (Ps[i:=r], t)$

lemma $\text{par-ctranE}: c -pc\rightarrow c' \implies$

$(\bigwedge i \text{ Ps } s \ r \ t. c = (Ps, s) \implies c' = (Ps[i := r], t) \implies i < \text{length } Ps \implies$

$(Ps ! i, s) -c\rightarrow (r, t) \implies P) \implies P$

$\langle \text{proof} \rangle$

3.2.3 Computations

Sequential computations

type-synonym $'a \text{ confs} = 'a \text{ conf list}$

inductive-set $\text{cptn} :: 'a \text{ confs set}$

where

$\text{CptnOne}: [(P, s)] \in \text{cptn}$

$| \text{CptnEnv}: (P, t) \# xs \in \text{cptn} \implies (P, s) \# (P, t) \# xs \in \text{cptn}$

$| \text{CptnComp}: \llbracket (P, s) -c\rightarrow (Q, t); (Q, t) \# xs \in \text{cptn} \rrbracket \implies (P, s) \# (Q, t) \# xs \in \text{cptn}$

definition $\text{cp} :: ('a \text{ com}) \text{ option} \Rightarrow 'a \Rightarrow ('a \text{ confs}) \text{ set}$ **where**

$\text{cp } P \ s \equiv \{l. !l = (P, s) \wedge l \in \text{cptn}\}$

Parallel computations

type-synonym $'a \text{ par-confs} = 'a \text{ par-conf list}$

inductive-set $\text{par-cptn} :: 'a \text{ par-confs set}$

where

$\text{ParCptnOne}: [(P, s)] \in \text{par-cptn}$

$| \text{ParCptnEnv}: (P, t) \# xs \in \text{par-cptn} \implies (P, s) \# (P, t) \# xs \in \text{par-cptn}$

$| \text{ParCptnComp}: \llbracket (P, s) -pc\rightarrow (Q, t); (Q, t) \# xs \in \text{par-cptn} \rrbracket \implies (P, s) \# (Q, t) \# xs \in \text{par-cptn}$

definition $\text{par-cp} :: 'a \text{ par-com} \Rightarrow 'a \Rightarrow ('a \text{ par-confs}) \text{ set}$ **where**

$\text{par-cp } P \ s \equiv \{l. !l = (P, s) \wedge l \in \text{par-cptn}\}$

3.2.4 Modular Definition of Computation

definition $\text{lift} :: 'a \text{ com} \Rightarrow 'a \text{ conf} \Rightarrow 'a \text{ conf}$ **where**

$\text{lift } Q \equiv \lambda(P, s). (\text{if } P = \text{None then } (\text{Some } Q, s) \text{ else } (\text{Some } (\text{Seq } (\text{the } P) \ Q), s))$

inductive-set $\text{cptn-mod} :: ('a \text{ confs}) \text{ set}$

where

$\text{CptnModOne}: [(P, s)] \in \text{cptn-mod}$

$| \text{CptnModEnv}: (P, t) \# xs \in \text{cptn-mod} \implies (P, s) \# (P, t) \# xs \in \text{cptn-mod}$

$| \text{CptnModNone}: \llbracket (\text{Some } P, s) -c\rightarrow (\text{None}, t); (\text{None}, t) \# xs \in \text{cptn-mod} \rrbracket \implies (\text{Some } P, s) \# (\text{None}, t) \# xs \in \text{cptn-mod}$

$| \text{CptnModCondT}: \llbracket (\text{Some } P0, s) \# ys \in \text{cptn-mod}; s \in b \rrbracket \implies (\text{Some}(\text{Cond } b \ P0 \ P1), s) \# (\text{Some } P0, s) \# ys \in \text{cptn-mod}$
 $| \text{CptnModCondF}: \llbracket (\text{Some } P1, s) \# ys \in \text{cptn-mod}; s \notin b \rrbracket \implies (\text{Some}(\text{Cond } b \ P0 \ P1), s) \# (\text{Some } P1, s) \# ys \in \text{cptn-mod}$
 $| \text{CptnModSeq1}: \llbracket (\text{Some } P0, s) \# xs \in \text{cptn-mod}; zs = \text{map } (\text{lift } P1) \ xs \rrbracket \implies (\text{Some}(\text{Seq } P0 \ P1), s) \# zs \in \text{cptn-mod}$
 $| \text{CptnModSeq2}: \llbracket (\text{Some } P0, s) \# xs \in \text{cptn-mod}; \text{fst}(\text{last } ((\text{Some } P0, s) \# xs)) = \text{None};$
 $(\text{Some } P1, \text{snd}(\text{last } ((\text{Some } P0, s) \# xs))) \# ys \in \text{cptn-mod};$
 $zs = (\text{map } (\text{lift } P1) \ xs) @ ys \rrbracket \implies (\text{Some}(\text{Seq } P0 \ P1), s) \# zs \in \text{cptn-mod}$
 $| \text{CptnModWhile1}: \llbracket (\text{Some } P, s) \# xs \in \text{cptn-mod}; s \in b; zs = \text{map } (\text{lift } (\text{While } b \ P)) \ xs \rrbracket \implies (\text{Some}(\text{While } b \ P), s) \# (\text{Some}(\text{Seq } P \ (\text{While } b \ P)), s) \# zs \in \text{cptn-mod}$
 $| \text{CptnModWhile2}: \llbracket (\text{Some } P, s) \# xs \in \text{cptn-mod}; \text{fst}(\text{last } ((\text{Some } P, s) \# xs)) = \text{None}; s \in b;$
 $zs = (\text{map } (\text{lift } (\text{While } b \ P)) \ xs) @ ys;$
 $(\text{Some}(\text{While } b \ P), \text{snd}(\text{last } ((\text{Some } P, s) \# xs))) \# ys \in \text{cptn-mod} \rrbracket \implies (\text{Some}(\text{While } b \ P), s) \# (\text{Some}(\text{Seq } P \ (\text{While } b \ P)), s) \# zs \in \text{cptn-mod}$

3.2.5 Equivalence of Both Definitions.

lemma *last-length*: $((a \# xs)!(\text{length } xs)) = \text{last } (a \# xs)$
 $\langle \text{proof} \rangle$

lemma *div-seq [rule-format]*: $\text{list} \in \text{cptn-mod} \implies$
 $(\forall s \ P \ Q \ zs. \text{list} = (\text{Some } (\text{Seq } P \ Q), s) \# zs \longrightarrow$
 $(\exists xs. (\text{Some } P, s) \# xs \in \text{cptn-mod} \wedge (zs = (\text{map } (\text{lift } Q) \ xs) \vee$
 $(\text{fst}(((\text{Some } P, s) \# xs)!\text{length } xs) = \text{None} \wedge$
 $(\exists ys. (\text{Some } Q, \text{snd}(((\text{Some } P, s) \# xs)!\text{length } xs))) \# ys \in \text{cptn-mod}$
 $\wedge zs = (\text{map } (\text{lift } (Q)) \ xs) @ ys))))$
 $\langle \text{proof} \rangle$

lemma *cptn-onlyif-cptn-mod-aux [rule-format]*:
 $\forall s \ Q \ t \ xs. ((\text{Some } a, s), Q, t) \in \text{ctran} \longrightarrow (Q, t) \# xs \in \text{cptn-mod}$
 $\longrightarrow (\text{Some } a, s) \# (Q, t) \# xs \in \text{cptn-mod}$
 $\langle \text{proof} \rangle$

lemma *cptn-onlyif-cptn-mod [rule-format]*: $c \in \text{cptn} \implies c \in \text{cptn-mod}$
 $\langle \text{proof} \rangle$

lemma *lift-is-cptn*: $c \in \text{cptn} \implies \text{map } (\text{lift } P) \ c \in \text{cptn}$
 $\langle \text{proof} \rangle$

lemma *cptn-append-is-cptn [rule-format]*:
 $\forall b \ a. b \# c1 \in \text{cptn} \longrightarrow a \# c2 \in \text{cptn} \longrightarrow (b \# c1)!\text{length } c1 = a \longrightarrow b \# c1 @ c2 \in \text{cptn}$
 $\langle \text{proof} \rangle$

lemma *last-lift*: $\llbracket xs \neq []; \text{fst}(xs!(\text{length } xs - (\text{Suc } 0))) = \text{None} \rrbracket$

$\implies \text{fst}((\text{map } (\text{lift } P) \text{ } xs)!(\text{length } (\text{map } (\text{lift } P) \text{ } xs) - (\text{Suc } 0))) = (\text{Some } P)$
 $\langle \text{proof} \rangle$

lemma *last-fst* [rule-format]: $P((a \# x)!\text{length } x) \longrightarrow \neg P \text{ } a \longrightarrow P \text{ } (x!(\text{length } x - (\text{Suc } 0)))$
 $\langle \text{proof} \rangle$

lemma *last-fst-esp*:
 $\text{snd}(((\text{Some } a, s) \# xs)!(\text{length } xs)) = \text{None} \implies \text{fst}(xs!(\text{length } xs - (\text{Suc } 0))) = \text{None}$
 $\langle \text{proof} \rangle$

lemma *last-snd*: $xs \neq [] \implies$
 $\text{snd}(((\text{map } (\text{lift } P) \text{ } xs)!(\text{length } (\text{map } (\text{lift } P) \text{ } xs) - (\text{Suc } 0)))) = \text{snd}(xs!(\text{length } xs - (\text{Suc } 0)))$
 $\langle \text{proof} \rangle$

lemma *Cons-lift*: $(\text{Some } (\text{Seq } P \text{ } Q), s) \# (\text{map } (\text{lift } Q) \text{ } xs) = \text{map } (\text{lift } Q) ((\text{Some } P, s) \# xs)$
 $\langle \text{proof} \rangle$

lemma *Cons-lift-append*:
 $(\text{Some } (\text{Seq } P \text{ } Q), s) \# (\text{map } (\text{lift } Q) \text{ } xs) @ ys = \text{map } (\text{lift } Q) ((\text{Some } P, s) \# xs) @ ys$
 $\langle \text{proof} \rangle$

lemma *lift-nth*: $i < \text{length } xs \implies \text{map } (\text{lift } Q) \text{ } xs ! i = \text{lift } Q \text{ } (xs ! i)$
 $\langle \text{proof} \rangle$

lemma *snd-lift*: $i < \text{length } xs \implies \text{snd}(\text{lift } Q \text{ } (xs ! i)) = \text{snd } (xs ! i)$
 $\langle \text{proof} \rangle$

lemma *cptn-if-cptn-mod*: $c \in \text{cptn-mod} \implies c \in \text{cptn}$
 $\langle \text{proof} \rangle$

theorem *cptn-iff-cptn-mod*: $(c \in \text{cptn}) = (c \in \text{cptn-mod})$
 $\langle \text{proof} \rangle$

3.3 Validity of Correctness Formulas

3.3.1 Validity for Component Programs.

type-synonym $'a \text{ } \text{rgformula} =$
 $'a \text{ } \text{com} \times 'a \text{ } \text{set} \times ('a \times 'a) \text{ } \text{set} \times ('a \times 'a) \text{ } \text{set} \times 'a \text{ } \text{set}$

definition *assum* :: $('a \text{ } \text{set} \times ('a \times 'a) \text{ } \text{set}) \Rightarrow ('a \text{ } \text{confs}) \text{ } \text{set}$ **where**
 $\text{assum} \equiv \lambda(\text{pre}, \text{rely}). \{c. \text{snd}(c!0) \in \text{pre} \wedge (\forall i. \text{Suc } i < \text{length } c \longrightarrow$
 $c!i - e \longrightarrow c!(\text{Suc } i) \longrightarrow (\text{snd}(c!i), \text{snd}(c!\text{Suc } i)) \in \text{rely}\}$

definition *comm* :: $(('a \times 'a) \text{ } \text{set} \times 'a \text{ } \text{set}) \Rightarrow ('a \text{ } \text{confs}) \text{ } \text{set}$ **where**

$comm \equiv \lambda(guar, post). \{c. (\forall i. Suc\ i < length\ c \longrightarrow$
 $c!i - c \rightarrow c!(Suc\ i) \longrightarrow (snd(c!i), snd(c!Suc\ i)) \in guar) \wedge$
 $(fst\ (last\ c) = None \longrightarrow snd\ (last\ c) \in post)\}$

definition *com-validity* :: 'a com $\Rightarrow$ 'a set $\Rightarrow$ ('a $\times$ 'a) set $\Rightarrow$ ('a $\times$ 'a) set $\Rightarrow$ 'a set $\Rightarrow$ bool

($\models$ - sat [-, -, -, -] [60,0,0,0,0] 45) **where**
 $\models P\ sat\ [pre, rely, guar, post] \equiv$
 $\forall s. cp\ (Some\ P)\ s \cap assum(pre, rely) \subseteq comm(guar, post)$

3.3.2 Validity for Parallel Programs.

definition *All-None* :: ('a com) option list $\Rightarrow$ bool **where**

All-None xs $\equiv \forall c \in set\ xs. c = None$

definition *par-assum* :: ('a set $\times$ ('a $\times$ 'a) set) $\Rightarrow$ ('a par-confs) set **where**

par-assum $\equiv \lambda(pre, rely). \{c. snd(c!0) \in pre \wedge (\forall i. Suc\ i < length\ c \longrightarrow$
 $c!i - pc \rightarrow c!Suc\ i \longrightarrow (snd(c!i), snd(c!Suc\ i)) \in rely)\}$

definition *par-comm* :: (('a $\times$ 'a) set $\times$ 'a set) $\Rightarrow$ ('a par-confs) set **where**

par-comm $\equiv \lambda(guar, post). \{c. (\forall i. Suc\ i < length\ c \longrightarrow$
 $c!i - pc \rightarrow c!Suc\ i \longrightarrow (snd(c!i), snd(c!Suc\ i)) \in guar) \wedge$
 $(All-None\ (fst\ (last\ c)) \longrightarrow snd\ (last\ c) \in post)\}$

definition *par-com-validity* :: 'a par-com $\Rightarrow$ 'a set $\Rightarrow$ ('a $\times$ 'a) set $\Rightarrow$ ('a $\times$ 'a) set $\Rightarrow$ bool

($\models$ - SAT [-, -, -, -] [60,0,0,0,0] 45) **where**
 $\models Ps\ SAT\ [pre, rely, guar, post] \equiv$
 $\forall s. par-cp\ Ps\ s \cap par-assum(pre, rely) \subseteq par-comm(guar, post)$

3.3.3 Compositionality of the Semantics

Definition of the conjoin operator

definition *same-length* :: 'a par-confs $\Rightarrow$ ('a confs) list $\Rightarrow$ bool **where**

same-length c clist $\equiv (\forall i < length\ clist. length(c!i) = length\ c)$

definition *same-state* :: 'a par-confs $\Rightarrow$ ('a confs) list $\Rightarrow$ bool **where**

same-state c clist $\equiv (\forall i < length\ clist. \forall j < length\ c. snd(c!j) = snd((clist!i)!j))$

definition *same-program* :: 'a par-confs $\Rightarrow$ ('a confs) list $\Rightarrow$ bool **where**

same-program c clist $\equiv (\forall j < length\ c. fst(c!j) = map\ (\lambda x. fst(nth\ x\ j))\ clist)$

definition *compat-label* :: 'a par-confs $\Rightarrow$ ('a confs) list $\Rightarrow$ bool **where**

compat-label c clist $\equiv (\forall j. Suc\ j < length\ c \longrightarrow$
 $(c!j - pc \rightarrow c!Suc\ j \wedge (\exists i < length\ clist. (clist!i)!j - c \rightarrow (clist!i)! Suc\ j \wedge$
 $(\forall l < length\ clist. l \neq i \longrightarrow (clist!l)!j - e \rightarrow (clist!l)! Suc\ j))) \vee$
 $(c!j - pe \rightarrow c!Suc\ j \wedge (\forall i < length\ clist. (clist!i)!j - e \rightarrow (clist!i)! Suc\ j)))$

definition *conjoin* :: 'a par-confs $\Rightarrow$ ('a confs) list $\Rightarrow$ bool (- $\propto$ - [65,65] 64)
where
 $c \propto clist \equiv (\text{same-length } c \text{ clist}) \wedge (\text{same-state } c \text{ clist}) \wedge (\text{same-program } c \text{ clist})$
 $\wedge (\text{compat-label } c \text{ clist})$

Some previous lemmas

lemma *list-eq-if* [rule-format]:
 $\forall ys. xs=ys \longrightarrow (\text{length } xs = \text{length } ys) \longrightarrow (\forall i < \text{length } xs. xs!i=ys!i)$
 $\langle \text{proof} \rangle$

lemma *list-eq*: $(\text{length } xs = \text{length } ys \wedge (\forall i < \text{length } xs. xs!i=ys!i)) = (xs=ys)$
 $\langle \text{proof} \rangle$

lemma *nth-tl*: $\llbracket ys!0=a; ys \neq [] \rrbracket \Longrightarrow ys=(a\#(\text{tl } ys))$
 $\langle \text{proof} \rangle$

lemma *nth-tl-if* [rule-format]: $ys \neq [] \longrightarrow ys!0=a \longrightarrow P \text{ } ys \longrightarrow P (a\#(\text{tl } ys))$
 $\langle \text{proof} \rangle$

lemma *nth-tl-onlyif* [rule-format]: $ys \neq [] \longrightarrow ys!0=a \longrightarrow P (a\#(\text{tl } ys)) \longrightarrow P \text{ } ys$
 $\langle \text{proof} \rangle$

lemma *seq-not-eq1*: $\text{Seq } c1 \text{ } c2 \neq c1$
 $\langle \text{proof} \rangle$

lemma *seq-not-eq2*: $\text{Seq } c1 \text{ } c2 \neq c2$
 $\langle \text{proof} \rangle$

lemma *if-not-eq1*: $\text{Cond } b \text{ } c1 \text{ } c2 \neq c1$
 $\langle \text{proof} \rangle$

lemma *if-not-eq2*: $\text{Cond } b \text{ } c1 \text{ } c2 \neq c2$
 $\langle \text{proof} \rangle$

lemmas *seq-and-if-not-eq* [simp] = *seq-not-eq1 seq-not-eq2*
seq-not-eq1 [THEN not-sym] *seq-not-eq2* [THEN not-sym]
if-not-eq1 if-not-eq2 if-not-eq1 [THEN not-sym] *if-not-eq2* [THEN not-sym]

lemma *prog-not-eq-in-ctran-aux*:
assumes $c: (P,s) -c \rightarrow (Q,t)$
shows $P \neq Q$ $\langle \text{proof} \rangle$

lemma *prog-not-eq-in-ctran* [simp]: $\neg (P,s) -c \rightarrow (P,t)$
 $\langle \text{proof} \rangle$

lemma *prog-not-eq-in-par-ctran-aux* [rule-format]: $(P,s) -pc \rightarrow (Q,t) \Longrightarrow (P \neq Q)$
 $\langle \text{proof} \rangle$

lemma *prog-not-eq-in-par-ctran* [*simp*]: $\neg (P, s) -pc \rightarrow (P, t)$
 $\langle proof \rangle$

lemma *tl-in-cptn*: $\llbracket a \# xs \in cptn; xs \neq [] \rrbracket \implies xs \in cptn$
 $\langle proof \rangle$

lemma *tl-zero*[*rule-format*]:
 $P (ys!Suc\ j) \longrightarrow Suc\ j < length\ ys \longrightarrow ys \neq [] \longrightarrow P (tl(ys)!j)$
 $\langle proof \rangle$

3.3.4 The Semantics is Compositional

lemma *aux-if* [*rule-format*]:
 $\forall xs\ s\ clist. (length\ clist = length\ xs \wedge (\forall i < length\ xs. (xs!i, s) \# clist!i \in cptn))$
 $\wedge ((xs, s) \# ys \propto map\ (\lambda i. (fst\ i, s) \# snd\ i)\ (zip\ xs\ clist))$
 $\longrightarrow (xs, s) \# ys \in par-cptn$
 $\langle proof \rangle$

lemma *aux-onlyif* [*rule-format*]: $\forall xs\ s. (xs, s) \# ys \in par-cptn \longrightarrow$
 $(\exists clist. (length\ clist = length\ xs) \wedge$
 $(xs, s) \# ys \propto map\ (\lambda i. (fst\ i, s) \# (snd\ i))\ (zip\ xs\ clist) \wedge$
 $(\forall i < length\ xs. (xs!i, s) \# (clist!i) \in cptn))$
 $\langle proof \rangle$

lemma *one-iff-aux*: $xs \neq [] \implies (\forall ys. ((xs, s) \# ys \in par-cptn) =$
 $(\exists clist. length\ clist = length\ xs \wedge$
 $((xs, s) \# ys \propto map\ (\lambda i. (fst\ i, s) \# (snd\ i))\ (zip\ xs\ clist)) \wedge$
 $(\forall i < length\ xs. (xs!i, s) \# (clist!i) \in cptn))) =$
 $(par-cp\ (xs)\ s = \{c. \exists clist. (length\ clist) = (length\ xs) \wedge$
 $(\forall i < length\ clist. (clist!i) \in cp(xs!i)\ s) \wedge c \propto clist\})$
 $\langle proof \rangle$

theorem *one*: $xs \neq [] \implies$
 $par-cp\ xs\ s = \{c. \exists clist. (length\ clist) = (length\ xs) \wedge$
 $(\forall i < length\ clist. (clist!i) \in cp(xs!i)\ s) \wedge c \propto clist\}$
 $\langle proof \rangle$

end

3.4 The Proof System

theory *RG-Hoare* **imports** *RG-Tran* **begin**

3.4.1 Proof System for Component Programs

declare *Un-subset-iff* [*simp del*] *sup.bounded-iff* [*simp del*]

definition *stable* :: $'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow bool$ **where**
 $stable \equiv \lambda f\ g. (\forall x\ y. x \in f \longrightarrow (x, y) \in g \longrightarrow y \in f)$

inductive

$rghoare :: ['a\ com, 'a\ set, ('a \times 'a)\ set, ('a \times 'a)\ set, 'a\ set] \Rightarrow bool$
 $(\vdash -\ sat\ [-, -, -, -]\ [60,0,0,0,0]\ 45)$

where

$Basic: \llbracket pre \subseteq \{s. f\ s \in post\}; \{(s,t). s \in pre \wedge (t=f\ s \vee t=s)\} \subseteq guar; \\ stable\ pre\ rely; stable\ post\ rely \rrbracket \\ \Longrightarrow \vdash Basic\ f\ sat\ [pre, rely, guar, post]$

$| Seq: \llbracket \vdash P\ sat\ [pre, rely, guar, mid]; \vdash Q\ sat\ [mid, rely, guar, post] \rrbracket \\ \Longrightarrow \vdash Seq\ P\ Q\ sat\ [pre, rely, guar, post]$

$| Cond: \llbracket stable\ pre\ rely; \vdash P1\ sat\ [pre \cap b, rely, guar, post]; \\ \vdash P2\ sat\ [pre \cap \neg b, rely, guar, post]; \forall s. (s,s) \in guar \rrbracket \\ \Longrightarrow \vdash Cond\ b\ P1\ P2\ sat\ [pre, rely, guar, post]$

$| While: \llbracket stable\ pre\ rely; (pre \cap \neg b) \subseteq post; stable\ post\ rely; \\ \vdash P\ sat\ [pre \cap b, rely, guar, pre]; \forall s. (s,s) \in guar \rrbracket \\ \Longrightarrow \vdash While\ b\ P\ sat\ [pre, rely, guar, post]$

$| Await: \llbracket stable\ pre\ rely; stable\ post\ rely; \\ \forall V. \vdash P\ sat\ [pre \cap b \cap \{V\}, \{(s, t). s = t\}, \\ UNIV, \{s. (V, s) \in guar\} \cap post] \rrbracket \\ \Longrightarrow \vdash Await\ b\ P\ sat\ [pre, rely, guar, post]$

$| Conseq: \llbracket pre \subseteq pre'; rely \subseteq rely'; guar' \subseteq guar; post' \subseteq post; \\ \vdash P\ sat\ [pre', rely', guar', post'] \rrbracket \\ \Longrightarrow \vdash P\ sat\ [pre, rely, guar, post]$

definition $Pre :: 'a\ rgformula \Rightarrow 'a\ set$ **where**

$Pre\ x \equiv fst(snd\ x)$

definition $Post :: 'a\ rgformula \Rightarrow 'a\ set$ **where**

$Post\ x \equiv snd(snd(snd(snd\ x)))$

definition $Rely :: 'a\ rgformula \Rightarrow ('a \times 'a)\ set$ **where**

$Rely\ x \equiv fst(snd(snd\ x))$

definition $Guar :: 'a\ rgformula \Rightarrow ('a \times 'a)\ set$ **where**

$Guar\ x \equiv fst(snd(snd(snd\ x)))$

definition $Com :: 'a\ rgformula \Rightarrow 'a\ com$ **where**

$Com\ x \equiv fst\ x$

3.4.2 Proof System for Parallel Programs

type-synonym $'a\ par\ rgformula =$

$('a\ rgformula)\ list \times 'a\ set \times ('a \times 'a)\ set \times ('a \times 'a)\ set \times 'a\ set$

inductive

$par\text{-}rg\text{-}hoare :: ('a\ rgformula)\ list \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set \Rightarrow ('a \times 'a)\ set \Rightarrow 'a\ set \Rightarrow bool$

$(\vdash - SAT [-, -, -, -] [60,0,0,0,0] 45)$

where

Parallel:

$$\begin{aligned} & \llbracket \forall i < \text{length } xs. \text{ rely} \cup (\bigcup j \in \{j. j < \text{length } xs \wedge j \neq i\}. \text{ Guar}(xs!j)) \subseteq \text{ Rely}(xs!i); \\ & (\bigcup j \in \{j. j < \text{length } xs\}. \text{ Guar}(xs!j)) \subseteq \text{ guar}; \\ & \text{ pre} \subseteq (\bigcap i \in \{i. i < \text{length } xs\}. \text{ Pre}(xs!i)); \\ & (\bigcap i \in \{i. i < \text{length } xs\}. \text{ Post}(xs!i)) \subseteq \text{ post}; \\ & \forall i < \text{length } xs. \vdash \text{ Com}(xs!i) \text{ sat } [\text{ Pre}(xs!i), \text{ Rely}(xs!i), \text{ Guar}(xs!i), \text{ Post}(xs!i)] \rrbracket \\ & \implies \vdash xs\ SAT [\text{ pre}, \text{ rely}, \text{ guar}, \text{ post}] \end{aligned}$$

3.5 Soundness

Some previous lemmas

lemma *tl-of-assum-in-assum:*

$(P, s) \# (P, t) \# xs \in \text{assum}(\text{pre}, \text{rely}) \implies \text{stable pre rely} \\ \implies (P, t) \# xs \in \text{assum}(\text{pre}, \text{rely})$
 $\langle \text{proof} \rangle$

lemma *etran-in-comm:*

$(P, t) \# xs \in \text{comm}(\text{guar}, \text{post}) \implies (P, s) \# (P, t) \# xs \in \text{comm}(\text{guar}, \text{post})$
 $\langle \text{proof} \rangle$

lemma *ctran-in-comm:*

$\llbracket (s, s) \in \text{guar}; (Q, s) \# xs \in \text{comm}(\text{guar}, \text{post}) \rrbracket \\ \implies (P, s) \# (Q, s) \# xs \in \text{comm}(\text{guar}, \text{post})$
 $\langle \text{proof} \rangle$

lemma *takecptn-is-cptn [rule-format, elim]:*

$\forall j. c \in \text{cptn} \longrightarrow \text{take}(\text{Suc } j) c \in \text{cptn}$
 $\langle \text{proof} \rangle$

lemma *dropcptn-is-cptn [rule-format, elim]:*

$\forall j < \text{length } c. c \in \text{cptn} \longrightarrow \text{drop } j c \in \text{cptn}$
 $\langle \text{proof} \rangle$

lemma *takepar-cptn-is-par-cptn [rule-format, elim]:*

$\forall j. c \in \text{par-cptn} \longrightarrow \text{take}(\text{Suc } j) c \in \text{par-cptn}$
 $\langle \text{proof} \rangle$

lemma *droppar-cptn-is-par-cptn [rule-format]:*

$\forall j < \text{length } c. c \in \text{par-cptn} \longrightarrow \text{drop } j c \in \text{par-cptn}$
 $\langle \text{proof} \rangle$

lemma *tl-of-cptn-is-cptn:* $\llbracket x \# xs \in \text{cptn}; xs \neq [] \rrbracket \implies xs \in \text{cptn}$

$\langle \text{proof} \rangle$

lemma *not-ctran-None* [rule-format]:

$\forall s. (None, s) \# xs \in cptn \longrightarrow (\forall i < \text{length } xs. ((None, s) \# xs)!i -e\rightarrow xs!i)$
 $\langle \text{proof} \rangle$

lemma *cptn-not-empty* [simp]: $[] \notin cptn$

$\langle \text{proof} \rangle$

lemma *etran-or-ctran* [rule-format]:

$\forall m i. x \in cptn \longrightarrow m \leq \text{length } x$
 $\longrightarrow (\forall i. \text{Suc } i < m \longrightarrow \neg x!i -c\rightarrow x!\text{Suc } i) \longrightarrow \text{Suc } i < m$
 $\longrightarrow x!i -e\rightarrow x!\text{Suc } i$
 $\langle \text{proof} \rangle$

lemma *etran-or-ctran2* [rule-format]:

$\forall i. \text{Suc } i < \text{length } x \longrightarrow x \in cptn \longrightarrow (x!i -c\rightarrow x!\text{Suc } i \longrightarrow \neg x!i -e\rightarrow x!\text{Suc } i)$
 $\vee (x!i -e\rightarrow x!\text{Suc } i \longrightarrow \neg x!i -c\rightarrow x!\text{Suc } i)$
 $\langle \text{proof} \rangle$

lemma *etran-or-ctran2-disjI1*:

$\llbracket x \in cptn; \text{Suc } i < \text{length } x; x!i -c\rightarrow x!\text{Suc } i \rrbracket \Longrightarrow \neg x!i -e\rightarrow x!\text{Suc } i$
 $\langle \text{proof} \rangle$

lemma *etran-or-ctran2-disjI2*:

$\llbracket x \in cptn; \text{Suc } i < \text{length } x; x!i -e\rightarrow x!\text{Suc } i \rrbracket \Longrightarrow \neg x!i -c\rightarrow x!\text{Suc } i$
 $\langle \text{proof} \rangle$

lemma *not-ctran-None2* [rule-format]:

$\llbracket (None, s) \# xs \in cptn; i < \text{length } xs \rrbracket \Longrightarrow \neg ((None, s) \# xs)!i -c\rightarrow xs!i$
 $\langle \text{proof} \rangle$

lemma *Ex-first-occurrence* [rule-format]: $P (n::nat) \longrightarrow (\exists m. P m \wedge (\forall i < m. \neg P i))$

$\langle \text{proof} \rangle$

lemma *stability* [rule-format]:

$\forall j k. x \in cptn \longrightarrow \text{stable } p \text{ rely} \longrightarrow j \leq k \longrightarrow k < \text{length } x \longrightarrow \text{snd}(x!j) \in p \longrightarrow$
 $(\forall i. (\text{Suc } i) < \text{length } x \longrightarrow$
 $(x!i -e\rightarrow x!(\text{Suc } i)) \longrightarrow (\text{snd}(x!i), \text{snd}(x!(\text{Suc } i))) \in \text{rely}) \longrightarrow$
 $(\forall i. j \leq i \wedge i < k \longrightarrow x!i -e\rightarrow x!\text{Suc } i) \longrightarrow \text{snd}(x!k) \in p \wedge \text{fst}(x!j) = \text{fst}(x!k)$
 $\langle \text{proof} \rangle$

3.5.1 Soundness of the System for Component Programs

Soundness of the Basic rule

lemma *unique-ctran-Basic* [rule-format]:

$\forall s i. x \in cptn \longrightarrow x!0 = (\text{Some } (\text{Basic } f), s) \longrightarrow$
 $\text{Suc } i < \text{length } x \longrightarrow x!i -c\rightarrow x!\text{Suc } i \longrightarrow$
 $(\forall j. \text{Suc } j < \text{length } x \longrightarrow i \neq j \longrightarrow x!j -e\rightarrow x!\text{Suc } j)$

$\langle proof \rangle$

lemma *exists-ctran-Basic-None* [rule-format]:

$\forall s \ i. \ x \in \text{cptn} \longrightarrow x ! 0 = (\text{Some } (\text{Basic } f), s)$
 $\longrightarrow i < \text{length } x \longrightarrow \text{fst}(x!i) = \text{None} \longrightarrow (\exists j < i. x!j -c\rightarrow x! \text{Suc } j)$
 $\langle proof \rangle$

lemma *Basic-sound*:

$\llbracket \text{pre} \subseteq \{s. f \ s \in \text{post}\}; \{(s, t). s \in \text{pre} \wedge t = f \ s\} \subseteq \text{guar};$
 $\text{stable pre rely}; \text{stable post rely} \rrbracket$
 $\implies \models \text{Basic } f \text{ sat } [\text{pre}, \text{rely}, \text{guar}, \text{post}]$
 $\langle proof \rangle$

Soundness of the Await rule

lemma *unique-ctran-Await* [rule-format]:

$\forall s \ i. \ x \in \text{cptn} \longrightarrow x ! 0 = (\text{Some } (\text{Await } b \ c), s) \longrightarrow$
 $\text{Suc } i < \text{length } x \longrightarrow x!i -c\rightarrow x! \text{Suc } i \longrightarrow$
 $(\forall j. \text{Suc } j < \text{length } x \longrightarrow i \neq j \longrightarrow x!j -e\rightarrow x! \text{Suc } j)$
 $\langle proof \rangle$

lemma *exists-ctran-Await-None* [rule-format]:

$\forall s \ i. \ x \in \text{cptn} \longrightarrow x ! 0 = (\text{Some } (\text{Await } b \ c), s)$
 $\longrightarrow i < \text{length } x \longrightarrow \text{fst}(x!i) = \text{None} \longrightarrow (\exists j < i. x!j -c\rightarrow x! \text{Suc } j)$
 $\langle proof \rangle$

lemma *Star-imp-cptn*:

$(P, s) -c*\rightarrow (R, t) \implies \exists l \in \text{cp } P \ s. (\text{last } l) = (R, t)$
 $\wedge (\forall i. \text{Suc } i < \text{length } l \longrightarrow l!i -c\rightarrow l! \text{Suc } i)$
 $\langle proof \rangle$

lemma *Await-sound*:

$\llbracket \text{stable pre rely}; \text{stable post rely};$
 $\forall V. \vdash P \text{ sat } [\text{pre} \cap b \cap \{s. s = V\}, \{(s, t). s = t\},$
 $\text{UNIV}, \{s. (V, s) \in \text{guar}\} \cap \text{post}] \wedge$
 $\models P \text{ sat } [\text{pre} \cap b \cap \{s. s = V\}, \{(s, t). s = t\},$
 $\text{UNIV}, \{s. (V, s) \in \text{guar}\} \cap \text{post}] \rrbracket$
 $\implies \models \text{Await } b \ P \text{ sat } [\text{pre}, \text{rely}, \text{guar}, \text{post}]$
 $\langle proof \rangle$

Soundness of the Conditional rule

lemma *Cond-sound*:

$\llbracket \text{stable pre rely}; \models P1 \text{ sat } [\text{pre} \cap b, \text{rely}, \text{guar}, \text{post}];$
 $\models P2 \text{ sat } [\text{pre} \cap \neg b, \text{rely}, \text{guar}, \text{post}]; \forall s. (s, s) \in \text{guar} \rrbracket$
 $\implies \models (\text{Cond } b \ P1 \ P2) \text{ sat } [\text{pre}, \text{rely}, \text{guar}, \text{post}]$
 $\langle proof \rangle$

Soundness of the Sequential rule

inductive-cases *Seq-cases* [*elim!*]: $(\text{Some } (\text{Seq } P \ Q), s) -c \rightarrow t$

lemma *last-lift-not-None*: $\text{fst } ((\text{lift } Q) ((x\#xs)!(\text{length } xs))) \neq \text{None}$
 $\langle \text{proof} \rangle$

lemma *Seq-sound1* [*rule-format*]:
 $x \in \text{cptn-mod} \implies \forall s \ P. \ x \neq (\text{Some } (\text{Seq } P \ Q), s) \longrightarrow$
 $(\forall i < \text{length } x. \ \text{fst}(x!i) \neq \text{Some } Q) \longrightarrow$
 $(\exists xs \in \text{cp } (\text{Some } P) \ s. \ x = \text{map } (\text{lift } Q) \ xs)$
 $\langle \text{proof} \rangle$

lemma *Seq-sound2* [*rule-format*]:
 $x \in \text{cptn} \implies \forall s \ P \ i. \ x!i \neq (\text{Some } (\text{Seq } P \ Q), s) \longrightarrow i < \text{length } x$
 $\longrightarrow \text{fst}(x!i) = \text{Some } Q \longrightarrow$
 $(\forall j < i. \ \text{fst}(x!j) \neq (\text{Some } Q)) \longrightarrow$
 $(\exists xs \ ys. \ xs \in \text{cp } (\text{Some } P) \ s \wedge \text{length } xs = \text{Suc } i$
 $\wedge \ ys \in \text{cp } (\text{Some } Q) \ (\text{snd}(xs \ !i)) \wedge x = (\text{map } (\text{lift } Q) \ xs) @ \text{tl } ys)$
 $\langle \text{proof} \rangle$

lemma *last-lift-not-None2*: $\text{fst } ((\text{lift } Q) (\text{last } (x\#xs))) \neq \text{None}$
 $\langle \text{proof} \rangle$

lemma *Seq-sound*:
 $\llbracket \models P \text{ sat } [\text{pre}, \text{rely}, \text{guar}, \text{mid}]; \models Q \text{ sat } [\text{mid}, \text{rely}, \text{guar}, \text{post}] \rrbracket$
 $\implies \models \text{Seq } P \ Q \text{ sat } [\text{pre}, \text{rely}, \text{guar}, \text{post}]$
 $\langle \text{proof} \rangle$

Soundness of the While rule

lemma *last-append* [*rule-format*]:
 $\forall xs. \ ys \neq [] \longrightarrow ((xs @ ys)!(\text{length } (xs @ ys) - (\text{Suc } 0))) = (ys!(\text{length } ys - (\text{Suc } 0)))$
 $\langle \text{proof} \rangle$

lemma *assum-after-body*:
 $\llbracket \models P \text{ sat } [\text{pre} \cap b, \text{rely}, \text{guar}, \text{pre}];$
 $(\text{Some } P, s) \# xs \in \text{cptn-mod}; \text{fst } (\text{last } ((\text{Some } P, s) \# xs)) = \text{None}; s \in b;$
 $(\text{Some } (\text{While } b \ P), s) \# (\text{Some } (\text{Seq } P \ (\text{While } b \ P)), s) \#$
 $\text{map } (\text{lift } (\text{While } b \ P)) \ xs \ @ \ ys \in \text{assum } (\text{pre}, \text{rely}) \rrbracket$
 $\implies (\text{Some } (\text{While } b \ P), \text{snd } (\text{last } ((\text{Some } P, s) \# xs))) \# ys \in \text{assum } (\text{pre}, \text{rely})$
 $\langle \text{proof} \rangle$

lemma *While-sound-aux* [*rule-format*]:
 $\llbracket \text{pre} \cap - \ b \subseteq \text{post}; \models P \text{ sat } [\text{pre} \cap b, \text{rely}, \text{guar}, \text{pre}]; \forall s. (s, s) \in \text{guar};$
 $\text{stable pre rely}; \text{stable post rely}; x \in \text{cptn-mod} \rrbracket$
 $\implies \forall s \ xs. \ x = (\text{Some } (\text{While } b \ P), s) \# xs \longrightarrow x \in \text{assum}(\text{pre}, \text{rely}) \longrightarrow x \in \text{comm}$
 $(\text{guar}, \text{post})$
 $\langle \text{proof} \rangle$

lemma *While-sound:*

$\llbracket \text{stable } pre \text{ rely}; pre \cap -b \subseteq post; \text{stable } post \text{ rely};$
 $\models P \text{ sat } [pre \cap b, rely, guar, pre]; \forall s. (s, s) \in guar \rrbracket$
 $\implies \models \text{While } b \text{ } P \text{ sat } [pre, rely, guar, post]$
 $\langle \text{proof} \rangle$

Soundness of the Rule of Consequence

lemma *Conseq-sound:*

$\llbracket pre \subseteq pre'; rely \subseteq rely'; guar' \subseteq guar; post' \subseteq post;$
 $\models P \text{ sat } [pre', rely', guar', post'] \rrbracket$
 $\implies \models P \text{ sat } [pre, rely, guar, post]$
 $\langle \text{proof} \rangle$

Soundness of the system for sequential component programs

theorem *rgsound:*

$\vdash P \text{ sat } [pre, rely, guar, post] \implies \models P \text{ sat } [pre, rely, guar, post]$
 $\langle \text{proof} \rangle$

3.5.2 Soundness of the System for Parallel Programs

definition *ParallelCom* :: ('a rgformula) list $\Rightarrow$ 'a par-com **where**

ParallelCom Ps $\equiv \text{map } (\text{Some} \circ \text{fst}) \text{ Ps}$

lemma *two:*

$\llbracket \forall i < \text{length } xs. rely \cup (\bigcup j \in \{j. j < \text{length } xs \wedge j \neq i\}. Guar (xs ! j))$
 $\subseteq Rely (xs ! i);$
 $pre \subseteq (\bigcap i \in \{i. i < \text{length } xs\}. Pre (xs ! i));$
 $\forall i < \text{length } xs.$
 $\models Com (xs ! i) \text{ sat } [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)];$
 $\text{length } xs = \text{length } clist; x \in \text{par-cp } (ParallelCom \text{ } xs) \text{ } s; x \in \text{par-assum}(pre, rely);$
 $\forall i < \text{length } clist. clist ! i \in \text{cp } (Some(Com(xs!i))) \text{ } s; x \propto clist \rrbracket$
 $\implies \forall j \text{ } i. i < \text{length } clist \wedge Suc \text{ } j < \text{length } x \longrightarrow (clist ! i ! j) -c \rightarrow (clist ! i ! Suc \text{ } j)$
 $\longrightarrow (snd(clist ! i ! j), snd(clist ! i ! Suc \text{ } j)) \in Guar(xs ! i)$
 $\langle \text{proof} \rangle$

lemma *three [rule-format]:*

$\llbracket xs \neq []; \forall i < \text{length } xs. rely \cup (\bigcup j \in \{j. j < \text{length } xs \wedge j \neq i\}. Guar (xs ! j))$
 $\subseteq Rely (xs ! i);$
 $pre \subseteq (\bigcap i \in \{i. i < \text{length } xs\}. Pre (xs ! i));$
 $\forall i < \text{length } xs.$
 $\models Com (xs ! i) \text{ sat } [Pre (xs ! i), Rely (xs ! i), Guar (xs ! i), Post (xs ! i)];$
 $\text{length } xs = \text{length } clist; x \in \text{par-cp } (ParallelCom \text{ } xs) \text{ } s; x \in \text{par-assum}(pre, rely);$
 $\forall i < \text{length } clist. clist ! i \in \text{cp } (Some(Com(xs!i))) \text{ } s; x \propto clist \rrbracket$
 $\implies \forall j \text{ } i. i < \text{length } clist \wedge Suc \text{ } j < \text{length } x \longrightarrow (clist ! i ! j) -e \rightarrow (clist ! i ! Suc \text{ } j)$
 $\longrightarrow (snd(clist ! i ! j), snd(clist ! i ! Suc \text{ } j)) \in rely \cup (\bigcup j \in \{j. j < \text{length } xs \wedge j \neq i\}. Guar (xs ! j))$

$\langle proof \rangle$

lemma four:

$$\begin{aligned} & \llbracket xs \neq [] \rrbracket; \forall i < \text{length } xs. \text{rely} \cup (\bigcup j \in \{j. j < \text{length } xs \wedge j \neq i\}. \text{Guar } (xs ! j)) \\ & \subseteq \text{Rely } (xs ! i); \\ & (\bigcup j \in \{j. j < \text{length } xs\}. \text{Guar } (xs ! j)) \subseteq \text{guar}; \\ & \text{pre} \subseteq (\bigcap i \in \{i. i < \text{length } xs\}. \text{Pre } (xs ! i)); \\ & \forall i < \text{length } xs. \\ & \quad \models \text{Com } (xs ! i) \text{ sat } [\text{Pre } (xs ! i), \text{Rely } (xs ! i), \text{Guar } (xs ! i), \text{Post } (xs ! i)]; \\ & \quad x \in \text{par-cp } (\text{ParallelCom } xs) \text{ s}; x \in \text{par-assum } (\text{pre}, \text{rely}); \text{Suc } i < \text{length } x; \\ & \quad x ! i -pc \rightarrow x ! \text{Suc } i \\ & \implies (\text{snd } (x ! i), \text{snd } (x ! \text{Suc } i)) \in \text{guar} \end{aligned}$$

$\langle proof \rangle$

lemma parcptn-not-empty [simp]: $[] \notin \text{par-cptn}$

$\langle proof \rangle$

lemma five:

$$\begin{aligned} & \llbracket xs \neq [] \rrbracket; \forall i < \text{length } xs. \text{rely} \cup (\bigcup j \in \{j. j < \text{length } xs \wedge j \neq i\}. \text{Guar } (xs ! j)) \\ & \subseteq \text{Rely } (xs ! i); \\ & \text{pre} \subseteq (\bigcap i \in \{i. i < \text{length } xs\}. \text{Pre } (xs ! i)); \\ & (\bigcap i \in \{i. i < \text{length } xs\}. \text{Post } (xs ! i)) \subseteq \text{post}; \\ & \forall i < \text{length } xs. \\ & \quad \models \text{Com } (xs ! i) \text{ sat } [\text{Pre } (xs ! i), \text{Rely } (xs ! i), \text{Guar } (xs ! i), \text{Post } (xs ! i)]; \\ & \quad x \in \text{par-cp } (\text{ParallelCom } xs) \text{ s}; x \in \text{par-assum } (\text{pre}, \text{rely}); \\ & \quad \text{All-None } (\text{fst } (\text{last } x)) \implies \text{snd } (\text{last } x) \in \text{post} \end{aligned}$$

$\langle proof \rangle$

lemma ParallelEmpty [rule-format]:

$$\begin{aligned} & \forall i \text{ s. } x \in \text{par-cp } (\text{ParallelCom } []) \text{ s} \longrightarrow \\ & \text{Suc } i < \text{length } x \longrightarrow (x ! i, x ! \text{Suc } i) \notin \text{par-ctran} \end{aligned}$$

$\langle proof \rangle$

theorem par-rgsound:

$$\begin{aligned} & \vdash c \text{ SAT } [\text{pre}, \text{rely}, \text{guar}, \text{post}] \implies \\ & \quad \models (\text{ParallelCom } c) \text{ SAT } [\text{pre}, \text{rely}, \text{guar}, \text{post}] \end{aligned}$$

$\langle proof \rangle$

end

3.6 Concrete Syntax

theory *RG-Syntax*

imports *RG-Hoare Quote-Antiquote*

begin

abbreviation *Skip* :: 'a com (*SKIP*)

where *SKIP* $\equiv$ *Basic id*

notation *Seq* ((-;;/-) [60,61] 60)

syntax

-Assign :: $idt \Rightarrow 'b \Rightarrow 'a \text{ com}$ (($'- := / -$) [70, 65] 61)
 -Cond :: $'a \text{ bexp} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ com}$ ((0IF -/ THEN -/ ELSE -/FI) [0, 0, 0] 61)
 -Cond2 :: $'a \text{ bexp} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ com}$ ((0IF - THEN - FI) [0,0] 56)
 -While :: $'a \text{ bexp} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ com}$ ((0WHILE - /DO - /OD) [0, 0] 61)
 -Await :: $'a \text{ bexp} \Rightarrow 'a \text{ com} \Rightarrow 'a \text{ com}$ ((0AWAIT - /THEN - /END) [0,0] 61)
 -Atom :: $'a \text{ com} \Rightarrow 'a \text{ com}$ ((($\langle - \rangle$) 61)
 -Wait :: $'a \text{ bexp} \Rightarrow 'a \text{ com}$ ((0WAIT - END) 61)

translations

$'x := a \rightarrow \text{CONST Basic } \langle '(-\text{update-name } x (\lambda-. a)) \rangle$
 $\text{IF } b \text{ THEN } c1 \text{ ELSE } c2 \text{ FI} \rightarrow \text{CONST Cond } \{b\} \ c1 \ c2$
 $\text{IF } b \text{ THEN } c \text{ FI} \rightleftharpoons \text{IF } b \text{ THEN } c \text{ ELSE SKIP FI}$
 $\text{WHILE } b \text{ DO } c \text{ OD} \rightarrow \text{CONST While } \{b\} \ c$
 $\text{AWAIT } b \text{ THEN } c \text{ END} \rightleftharpoons \text{CONST Await } \{b\} \ c$
 $\langle c \rangle \rightleftharpoons \text{AWAIT CONST True THEN } c \text{ END}$
 $\text{WAIT } b \text{ END} \rightleftharpoons \text{AWAIT } b \text{ THEN SKIP END}$

nonterminal *prgs*

syntax

-PAR :: $\text{prgs} \Rightarrow 'a$ (COBEGIN// -//COEND 60)
 -prg :: $'a \Rightarrow \text{prgs}$ (- 57)
 -prgs :: $['a, \text{prgs}] \Rightarrow \text{prgs}$ (-//||// - [60,57] 57)

translations

-prg $a \rightarrow [a]$
 -prgs $a \text{ ps} \rightarrow a \ \# \ \text{ps}$
 -PAR $\text{ps} \rightarrow \text{ps}$

syntax

-prg-scheme :: $['a, 'a, 'a, 'a] \Rightarrow \text{prgs}$ (SCHEME [- $\leq$ - < -] - [0,0,0,60] 57)

translations

-prg-scheme $j \ i \ k \ c \rightleftharpoons (\text{CONST map } (\lambda i. \ c) \ [j..<k])$

Translations for variables before and after a transition:

syntax

-before :: $id \Rightarrow 'a \ (^{\circ}-)$
 -after :: $id \Rightarrow 'a \ (^{\text{a}}-)$

translations

$^{\circ}x \rightleftharpoons x \ ' \ \text{CONST fst}$
 $^{\text{a}}x \rightleftharpoons x \ ' \ \text{CONST snd}$

$\langle ML \rangle$

end

3.7 Examples

theory *RG-Examples*

imports *RG-Syntax*

begin

lemmas *definitions* [*simp*] = *stable-def Pre-def Rely-def Guar-def Post-def Com-def*

3.7.1 Set Elements of an Array to Zero

lemma *le-less-trans2*: $\llbracket (j::nat) < k; i \leq j \rrbracket \implies i < k$
 $\langle proof \rangle$

lemma *add-le-less-mono*: $\llbracket (a::nat) < c; b \leq d \rrbracket \implies a + b < c + d$
 $\langle proof \rangle$

record *Example1* =
 A :: nat list

lemma *Example1*:

⊢ COBEGIN
 SCHEME $[0 \leq i < n]$
 ($'A := 'A[i := 0]$),
 $\llbracket n < \text{length } 'A \rrbracket$,
 $\llbracket \text{length } ^oA = \text{length } ^aA \wedge ^oA ! i = ^aA ! i \rrbracket$,
 $\llbracket \text{length } ^oA = \text{length } ^aA \wedge (\forall j < n. i \neq j \longrightarrow ^oA ! j = ^aA ! j) \rrbracket$,
 $\llbracket 'A ! i = 0 \rrbracket$)
 COEND
 SAT $\llbracket n < \text{length } 'A \rrbracket, \llbracket ^oA = ^aA \rrbracket, \llbracket \text{True} \rrbracket, \llbracket \forall i < n. 'A ! i = 0 \rrbracket$
 $\langle proof \rangle$

lemma *Example1-parameterized*:

$k < t \implies$
 ⊢ COBEGIN
 SCHEME $[k * n \leq i < (Suc\ k) * n]$ ($'A := 'A[i := 0]$),
 $\llbracket t * n < \text{length } 'A \rrbracket$,
 $\llbracket t * n < \text{length } ^oA \wedge \text{length } ^oA = \text{length } ^aA \wedge ^oA ! i = ^aA ! i \rrbracket$,
 $\llbracket t * n < \text{length } ^oA \wedge \text{length } ^oA = \text{length } ^aA \wedge (\forall j < \text{length } ^oA. i \neq j \longrightarrow ^oA ! j = ^aA ! j) \rrbracket$,
 $\llbracket 'A ! i = 0 \rrbracket$)
 COEND
 SAT $\llbracket t * n < \text{length } 'A \rrbracket$,
 $\llbracket t * n < \text{length } ^oA \wedge \text{length } ^oA = \text{length } ^aA \wedge (\forall i < n. ^oA ! (k * n + i) = ^aA ! (k * n + i)) \rrbracket$,
 $\llbracket t * n < \text{length } ^oA \wedge \text{length } ^oA = \text{length } ^aA \wedge$

$(\forall i < \text{length } {}^o A . (i < k * n \longrightarrow {}^o A!i = {}^a A!i) \wedge ((\text{Suc } k) * n \leq i \longrightarrow {}^o A!i = {}^a A!i))\},$
 $\{\forall i < n. {}^a A!(k * n + i) = 0\}$
 $\langle \text{proof} \rangle$

3.7.2 Increment a Variable in Parallel

Two components

record *Example2* =

$x :: \text{nat}$
 $c-0 :: \text{nat}$
 $c-1 :: \text{nat}$

lemma *Example2*:

$\vdash \text{COBEGIN}$
 $(\langle {}^a x := {}^a x + 1;; {}^a c-0 := {}^a c-0 + 1 \rangle,$
 $\{\{{}^a x = {}^a c-0 + {}^a c-1 \wedge {}^a c-0 = 0\},$
 $\{\{{}^o c-0 = {}^a c-0 \wedge$
 $({}^o x = {}^o c-0 + {}^o c-1$
 $\longrightarrow {}^a x = {}^a c-0 + {}^a c-1)\},$
 $\{\{{}^o c-1 = {}^a c-1 \wedge$
 $({}^o x = {}^o c-0 + {}^o c-1$
 $\longrightarrow {}^a x = {}^a c-0 + {}^a c-1)\},$
 $\{\{{}^a x = {}^a c-0 + {}^a c-1 \wedge {}^a c-0 = 1\}\})$
 $\parallel$
 $(\langle {}^a x := {}^a x + 1;; {}^a c-1 := {}^a c-1 + 1 \rangle,$
 $\{\{{}^a x = {}^a c-0 + {}^a c-1 \wedge {}^a c-1 = 0\},$
 $\{\{{}^o c-1 = {}^a c-1 \wedge$
 $({}^o x = {}^o c-0 + {}^o c-1$
 $\longrightarrow {}^a x = {}^a c-0 + {}^a c-1)\},$
 $\{\{{}^o c-0 = {}^a c-0 \wedge$
 $({}^o x = {}^o c-0 + {}^o c-1$
 $\longrightarrow {}^a x = {}^a c-0 + {}^a c-1)\},$
 $\{\{{}^a x = {}^a c-0 + {}^a c-1 \wedge {}^a c-1 = 1\}\})$
 COEND
 $\text{SAT } [\{\{{}^a x = 0 \wedge {}^a c-0 = 0 \wedge {}^a c-1 = 0\},$
 $\{\{{}^o x = {}^a x \wedge {}^o c-0 = {}^a c-0 \wedge {}^o c-1 = {}^a c-1\},$
 $\{\text{True}\},$
 $\{\{{}^a x = 2\}\}]$
 $\langle \text{proof} \rangle$

Parameterized

lemma *Example2-lemma2-aux*: $j < n \implies$

$(\sum i=0..<n. (b \ i :: \text{nat})) =$
 $(\sum i=0..<j. b \ i) + b \ j + (\sum i=0..<n-(\text{Suc } j) . b \ (\text{Suc } j + i))$
 $\langle \text{proof} \rangle$

lemma *Example2-lemma2-aux2*:

$j \leq s \implies (\sum i :: \text{nat} = 0..<j. (b \ (s := t)) \ i) = (\sum i=0..<j. b \ i)$

$\langle \text{proof} \rangle$

lemma *Example2-lemma2*:

$\llbracket j < n; b \ j = 0 \rrbracket \implies \text{Suc} \ (\sum i :: \text{nat} = 0 .. < n. b \ i) = (\sum i = 0 .. < n. (b \ (j := \text{Suc } 0)) \ i)$
 $\langle \text{proof} \rangle$

lemma *Example2-lemma2-Suc0*: $\llbracket j < n; b \ j = 0 \rrbracket \implies$

$\text{Suc} \ (\sum i :: \text{nat} = 0 .. < n. b \ i) = (\sum i = 0 .. < n. (b \ (j := \text{Suc } 0)) \ i)$
 $\langle \text{proof} \rangle$

record *Example2-parameterized* =

$C :: \text{nat} \Rightarrow \text{nat}$
 $y :: \text{nat}$

lemma *Example2-parameterized*: $0 < n \implies$

$\vdash \text{COBEGIN SCHEME } [0 \leq i < n]$
 $(\langle \ 'y := \ 'y + 1;; \ 'C := \ 'C \ (i := 1) \ \rangle,$
 $\{\ 'y = (\sum i = 0 .. < n. \ 'C \ i) \wedge \ 'C \ i = 0 \},$
 $\{\ ^o C \ i = \ ^a C \ i \wedge$
 $(\ ^o y = (\sum i = 0 .. < n. \ ^o C \ i) \longrightarrow \ ^a y = (\sum i = 0 .. < n. \ ^a C \ i)) \},$
 $\{\ (\forall j < n. \ i \neq j \longrightarrow \ ^o C \ j = \ ^a C \ j) \wedge$
 $(\ ^o y = (\sum i = 0 .. < n. \ ^o C \ i) \longrightarrow \ ^a y = (\sum i = 0 .. < n. \ ^a C \ i)) \},$
 $\{\ 'y = (\sum i = 0 .. < n. \ 'C \ i) \wedge \ 'C \ i = 1 \} \}$
 COEND
 $\text{SAT } [\{\ 'y = 0 \wedge (\sum i = 0 .. < n. \ 'C \ i) = 0 \}, \{\ ^o C = \ ^a C \wedge \ ^o y = \ ^a y \}, \{\ \text{True} \}, \{\ 'y = n \}]$
 $\langle \text{proof} \rangle$

3.7.3 Find Least Element

A previous lemma:

lemma *mod-aux*: $\llbracket i < (n :: \text{nat}); a \bmod n = i; j < a + n; j \bmod n = i; a < j \rrbracket \implies$
 False
 $\langle \text{proof} \rangle$

record *Example3* =

$X :: \text{nat} \Rightarrow \text{nat}$
 $Y :: \text{nat} \Rightarrow \text{nat}$

lemma *Example3*: $m \bmod n = 0 \implies$

$\vdash \text{COBEGIN}$
 $\text{SCHEME } [0 \leq i < n]$
 $(\text{WHILE } (\forall j < n. \ 'X \ i < \ 'Y \ j) \ \text{DO}$
 $\ \ \text{IF } P(B!(\ 'X \ i)) \ \text{THEN } \ 'Y := \ 'Y \ (i := \ 'X \ i)$
 $\ \ \text{ELSE } \ 'X := \ 'X \ (i := (\ 'X \ i) + n) \ \text{FI}$
 $\text{OD},$
 $\{\ (\ 'X \ i) \bmod n = i \wedge (\forall j < \ 'X \ i. \ j \bmod n = i \longrightarrow \neg P(B!j)) \wedge (\ 'Y \ i < m \longrightarrow P(B!(\ 'Y$
 $i)) \wedge \ 'Y \ i \leq m + i) \},$
 $\{\ (\forall j < n. \ i \neq j \longrightarrow \ ^a Y \ j \leq \ ^o Y \ j) \wedge \ ^o X \ i = \ ^a X \ i \wedge$
 $\ \ ^o Y \ i = \ ^a Y \ i \} \},$

$\{\{(\forall j < n. i \neq j \longrightarrow {}^oX\ j = {}^aX\ j \wedge {}^oY\ j = {}^aY\ j) \wedge$
 $\quad {}^aY\ i \leq {}^oY\ i\}\},$
 $\{\{('X\ i) \bmod n = i \wedge (\forall j < 'X\ i. j \bmod n = i \longrightarrow \neg P(B!j)) \wedge ('Y\ i < m \longrightarrow P(B!('Y\ i))) \wedge 'Y\ i \leq m+i) \wedge (\exists j < n. 'Y\ j \leq 'X\ i)\}\}$
 $COEND$
 $SAT\ [\{\{ \forall i < n. 'X\ i = i \wedge 'Y\ i = m+i\}, \{\{ {}^oX = {}^aX \wedge {}^oY = {}^aY\}, \{\{ True\}\},$
 $\quad \{\{ \forall i < n. ('X\ i) \bmod n = i \wedge (\forall j < 'X\ i. j \bmod n = i \longrightarrow \neg P(B!j)) \wedge$
 $\quad ('Y\ i < m \longrightarrow P(B!('Y\ i))) \wedge 'Y\ i \leq m+i) \wedge (\exists j < n. 'Y\ j \leq 'X\ i)\}\}\}$
 $\langle proof \rangle$

Same but with a list as auxiliary variable:

record *Example3-list* =

X :: nat list

Y :: nat list

lemma *Example3-list*: $m \bmod n = 0 \implies \vdash (COBEGIN\ SCHEME\ [0 \leq i < n]$
 $(WHILE\ (\forall j < n. 'X!i < 'Y!j)\ DO$
 $\quad IF\ P(B!('X!i))\ THEN\ 'Y := 'Y[i := 'X!i]\ ELSE\ 'X := 'X[i := ('X!i) + n]\ FI$
 $OD,$
 $\{\{n < length\ 'X \wedge n < length\ 'Y \wedge ('X!i) \bmod n = i \wedge (\forall j < 'X!i. j \bmod n = i \longrightarrow$
 $\neg P(B!j)) \wedge ('Y!i < m \longrightarrow P(B!('Y!i))) \wedge 'Y!i \leq m+i\}\},$
 $\{\{(\forall j < n. i \neq j \longrightarrow {}^aY!j \leq {}^oY!j) \wedge {}^oX!i = {}^aX!i \wedge$
 $\quad {}^oY!i = {}^aY!i \wedge length\ {}^oX = length\ {}^aX \wedge length\ {}^oY = length\ {}^aY\}\},$
 $\{\{(\forall j < n. i \neq j \longrightarrow {}^oX!j = {}^aX!j \wedge {}^oY!j = {}^aY!j) \wedge$
 $\quad {}^aY!i \leq {}^oY!i \wedge length\ {}^oX = length\ {}^aX \wedge length\ {}^oY = length\ {}^aY\}\},$
 $\{\{('X!i) \bmod n = i \wedge (\forall j < 'X!i. j \bmod n = i \longrightarrow \neg P(B!j)) \wedge ('Y!i < m \longrightarrow P(B!('Y!i)))$
 $\wedge 'Y!i \leq m+i) \wedge (\exists j < n. 'Y!j \leq 'X!i)\}\}\} COEND)$
 $SAT\ [\{\{n < length\ 'X \wedge n < length\ 'Y \wedge (\forall i < n. 'X!i = i \wedge 'Y!i = m+i)\},$
 $\quad \{\{ {}^oX = {}^aX \wedge {}^oY = {}^aY\}\},$
 $\quad \{\{ True\}\},$
 $\quad \{\{ \forall i < n. ('X!i) \bmod n = i \wedge (\forall j < 'X!i. j \bmod n = i \longrightarrow \neg P(B!j)) \wedge$
 $\quad ('Y!i < m \longrightarrow P(B!('Y!i))) \wedge 'Y!i \leq m+i) \wedge (\exists j < n. 'Y!j \leq 'X!i)\}\}\}$
 $\langle proof \rangle$

end

theory *Hoare-Parallel*

imports *OG-Examples Gar-Coll Mul-Gar-Coll RG-Examples*

begin

end

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