The Supplemental Isabelle/HOL Library

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Contents

1 Implementation of Association Lists **21**
- 1.1 *update* and *updates* . . . 21
- 1.2 *delete* . . . 24
- 1.3 *update-with-aux* and *delete-aux* . . . 25
- 1.4 *restrict* . . . 27
- 1.5 *clearjunk* . . . 28
- 1.6 *map-ran* . . . 30
- 1.7 *merge* . . . 31
- 1.8 *compose* . . . 32
- 1.9 *map-entry* . . . 36
- 1.10 *map-default* . . . 36

2 Adhoc overloading of constants based on their types **37**

3 Pointwise instantiation of functions to algebra type classes **37**

4 Algebraic operations on sets **41**

5 Big O notation **50**
- 5.1 Definitions . . . 50
- 5.2 Sum . . . 61
- 5.3 Misc useful stuff . . . 63
- 5.4 Less than or equal to . . . 64

6 Axiomatic Declaration of Bounded Natural Functors **67**

7 Generalized Corecursor Sugar (corec and friends) **68**
- 7.1 Coinduction . . . 69

8 A general "while" combinator **72**
- 8.1 Partial version . . . 72
- 8.2 Total version . . . 76

9 The Bourbaki-Witt tower construction for transfinite iteration 81
9.1 Connect with the while combinator for executability on chain-finite lattices. 87

10 Order on characters 90

11 A generic phantom type 91

12 Cardinality of types 92
12.1 Preliminary lemmas 92
12.2 Cardinalities of types 92
12.3 Classes with at least 1 and 2 95
12.4 A type class for deciding finiteness of types 95
12.5 A type class for computing the cardinality of types 96
12.6 Instantiations for *card-UNIV* 96
12.7 Code setup for sets with cardinality type information 100
12.8 Eliminating pattern matches 103

13 Lazy types in generated code 103
13.1 The type *lazy* 104
13.2 Implementation 108
13.3 YXML encoding for *term* 108
13.4 Test engine and drivers 111

14 A combinator to build partial equivalence relations from a predicate and an equivalence relation 111

15 Formalisation of chain-complete partial orders, continuity and admissibility 112
15.1 Continuity 119
15.1.1 Theorem collection *cont-intro* 120
15.2 Admissibility 130
15.3 (=) as order 134
15.4 ccpo for products 135
15.5 Complete lattices as ccpo 141
15.6 Parallel fixpoint induction 146

16 Confluence 151

17 Old Datatype package: constructing datatypes from Cartesian Products and Disjoint Sums 157
17.1 The datatype universe 157
17.2 Freeness: Distinctness of Constructors 159
17.3 Set Constructions 163

18 Bijections between natural numbers and other types **167**
18.1 Type *nat* × *nat* . . . 167
18.2 Type *nat* + *nat* . . . 169
18.3 Type *int* . . . 170
18.4 Type *nat list* . . . 171
18.5 Finite sets of naturals . . . 172
18.5.1 Preliminaries . . . 172
18.5.2 From sets to naturals . . . 172
18.5.3 From naturals to sets . . . 173
18.5.4 Proof of isomorphism . . . 174

19 Encoding (almost) everything into natural numbers **175**
19.1 The class of countable types . . . 175
19.2 Conversion functions . . . 176
19.3 Finite types are countable . . . 176
19.4 Automatically proving countability of old-style datatypes . . . 176
19.5 Automatically proving countability of datatypes . . . 179
19.6 More Countable types . . . 179
19.7 The rationals are countably infinite . . . 181

20 Infinite Sets and Related Concepts **182**
20.1 The set of natural numbers is infinite . . . 182
20.2 The set of integers is also infinite . . . 183
20.3 Infinitely Many and Almost All . . . 184
20.4 Enumeration of an Infinite Set . . . 186
20.5 Properties of *wellorder-class.enumerate* on finite sets . . . 190

21 Countable sets **194**
21.1 Predicate for countable sets . . . 195
21.2 Enumerate a countable set . . . 195
21.3 Closure properties of countability . . . 199
21.4 Misc lemmas . . . 202
21.5 Uncountable . . . 204

22 Countable Complete Lattices **205**
22.0.1 Instances of countable complete lattices . . . 210

23 Type of (at Most) Countable Sets **211**
23.1 Cardinal stuff . . . 211
23.2 The type of countable sets . . . 212
23.3 Additional lemmas . . . 218
23.3.1 *cempty* . . . 218
23.3.2 *cinsert* . . . 218
23.3.3 *cimage* . . . 219

23.3.4 bounded quantification . . . 219
23.3.5 *cUnion* . . . 219
23.4 Setup for Lifting/Transfer . . . 219
23.4.1 Relator and predicator properties . . . 219
23.4.2 Transfer rules for the Transfer package . . . 220
23.5 Registration as BNF . . . 221

24 Debugging facilities for code generated towards Isabelle/ML 224

25 Sequence of Properties on Subsequences 225

26 Common discrete functions 227
26.1 Discrete logarithm . . . 227
26.2 Discrete square root . . . 230

27 Pi and Function Sets 234
27.1 Basic Properties of *Pi* . . . 235
27.2 Composition With a Restricted Domain: *compose* . . . 237
27.3 Bounded Abstraction: *restrict* . . . 238
27.4 Bijections Between Sets . . . 239
27.5 Extensionality . . . 240
27.6 Cardinality . . . 241
27.7 Extensional Function Spaces . . . 241
27.7.1 Injective Extensional Function Spaces . . . 247
27.7.2 Misc properties of functions, composition and restriction from HOL Light . . . 247
27.7.3 Cardinality . . . 248
27.8 The pigeonhole principle . . . 249

28 Partitions and Disjoint Sets 249
28.1 Set of Disjoint Sets . . . 249
28.1.1 Family of Disjoint Sets . . . 251
28.2 Construct Disjoint Sequences . . . 255
28.3 Partitions . . . 255
28.4 Constructions of partitions . . . 256
28.5 Finiteness of partitions . . . 257
28.6 Equivalence of partitions and equivalence classes . . . 257
28.7 Refinement of partitions . . . 258
28.8 The coarsest common refinement of a set of partitions . . . 259

29 Type of finite sets defined as a subtype of sets 262
29.1 Definition of the type . . . 262
29.2 Basic operations and type class instantiations . . . 262
29.3 Other operations . . . 265
29.4 Transferred lemmas from Set.thy . . . 266

29.5 Additional lemmas . . . 271
29.5.1 *ffUnion* . . . 271
29.5.2 *fbind* . . . 271
29.5.3 *fsingleton* . . . 271
29.5.4 *femepty* . . . 271
29.5.5 *fset* . . . 272
29.5.6 *ffilter* . . . 272
29.5.7 *fset-of-list* . . . 272
29.5.8 *finsert* . . . 273
29.5.9 *fimage* . . . 273
29.5.10 bounded quantification . . . 273
29.5.11 *fcard* . . . 274
29.5.12 *sorted-list-of-fset* . . . 275
29.5.13 *ffold* . . . 276
29.5.14 Group operations . . . 277
29.5.15 Semilattice operations . . . 277
29.6 Choice in fsets . . . 280
29.7 Induction and Cases rules for fsets . . . 280
29.8 Setup for Lifting/Transfer . . . 281
29.8.1 Relator and predicator properties . . . 281
29.8.2 Transfer rules for the Transfer package . . . 282
29.9 BNF setup . . . 284
29.10 Size setup . . . 286
29.11 Advanced relator customization . . . 286
29.11.1 Countability . . . 288
29.12 Quickcheck setup . . . 288

30 Type of finite maps defined as a subtype of maps 290
30.1 Auxiliary constants and lemmas over *map* . . . 290
30.2 Abstract characterisation . . . 293
30.3 Operations . . . 293
30.4 BNF setup . . . 308
30.5 *size* setup . . . 313
30.6 Additional operations . . . 314
30.7 Additional properties . . . 315
30.8 Lifting/transfer setup . . . 315
30.9 View as datatype . . . 316
30.10 Code setup . . . 318
30.11 Instances . . . 320
30.12 Tests . . . 321

31 Disjoint FSets 322

32 Lists with elements distinct as canonical example for datatype invariants **323**
32.1 The type of distinct lists . . . 323
32.2 Executable version obeying invariant . . . 325
32.3 Induction principle and case distinction . . . 326
32.4 Functorial structure . . . 327
32.5 Quickcheck generators . . . 327
32.6 BNF instance . . . 328

33 Type of dual ordered lattices **330**
33.1 Pointwise ordering . . . 331
33.2 Binary infimum and supremum . . . 332
33.3 Top and bottom elements . . . 333
33.4 Complement . . . 334
33.5 Complete lattice operations . . . 335

34 Equipollence and Other Relations Connected with Cardinality **337**
34.1 Eqpoll . . . 338
34.2 The strict relation . . . 341
34.3 Mapping by an injection . . . 341
34.4 Inserting elements into sets . . . 342
34.5 Binary sums and unions . . . 344
34.6 Binary Cartesian products . . . 344
34.7 General Unions . . . 346
34.8 General Cartesian products (Pi) . . . 347

35 Continuity and iterations **356**
35.1 Continuity for complete lattices . . . 357
35.1.1 Least fixed points in countable complete lattices . . . 365

36 Extended natural numbers (i.e. with infinity) **366**
36.1 Type definition . . . 366
36.2 Constructors and numbers . . . 367
36.3 Addition . . . 369
36.4 Multiplication . . . 370
36.5 Numerals . . . 371
36.6 Subtraction . . . 371
36.7 Ordering . . . 372
36.8 Cancellation simprocs . . . 376
36.9 Well-ordering . . . 378
36.10 Complete Lattice . . . 379
36.11 Traditional theorem names . . . 379

37 Liminf and Limsup on conditionally complete lattices **380**
37.0.1 *Liminf* and *Limsup* . . . 381
37.1 More Limits . . . 392

38 Extended real number line **393**
38.1 Definition and basic properties . . . 398
38.1.1 Addition . . . 401
38.1.2 Linear order on *ereal* . . . 403
38.1.3 Multiplication . . . 411
38.1.4 Power . . . 419
38.1.5 Subtraction . . . 419
38.1.6 Division . . . 423
38.2 Complete lattice . . . 428
38.3 Extended real intervals . . . 431
38.4 Topological space . . . 434
38.5 Relation to *enat* . . . 445
38.6 Limits on *ereal* . . . 447
38.6.1 Convergent sequences . . . 451
38.6.2 Sums . . . 461
38.6.3 Continuity . . . 474
38.6.4 liminf and limsup . . . 477
38.6.5 Tests for code generator . . . 481

39 Indicator Function **481**
39.1 The type of non-negative extended real numbers . . . 486
39.2 Defining the extended non-negative reals . . . 490
39.3 Cancellation simprocs . . . 494
39.4 Order with top . . . 495
39.5 Arithmetic . . . 498
39.6 Coercion from *real* to *ennreal* . . . 503
39.7 Coercion from *ennreal* to *real* . . . 508
39.8 Coercion from *enat* to *ennreal* . . . 509
39.9 Topology on *ennreal* . . . 511
39.10 Approximation lemmas . . . 523
39.11 *ennreal* theorems . . . 526

40 Logarithm of Natural Numbers **531**
40.1 Preliminaries . . . 531
40.2 Floorlog . . . 531
40.3 Bitlen . . . 535

41 Various algebraic structures combined with a lattice **537**
41.1 Positive Part, Negative Part, Absolute Value . . . 539

42 Floating-Point Numbers **548**
42.1 Real operations preserving the representation as floating point number . . . 549
42.2 Arithmetic operations on floating point numbers . . . 552
42.3 Quickcheck . . . 555
42.4 Represent floats as unique mantissa and exponent . . . 556
42.5 Compute arithmetic operations . . . 559
42.6 Lemmas for types *real*, *nat*, *int* . . . 562
42.7 Rounding Real Numbers . . . 562
42.8 Rounding Floats . . . 565
42.9 Truncating Real Numbers . . . 569
42.10 Truncating Floats . . . 571
42.11 Approximation of positive rationals . . . 576
42.12 Division . . . 580
42.13 Approximate Addition . . . 581
42.14 Approximate Multiplication . . . 589
42.15 Approximate Power . . . 590
42.16 Lemmas needed by Approximate . . . 595

43 Pointwise instantiation of functions to division **600**
43.1 Syntactic with division . . . 601

44 Lexicographic order on functions **602**

45 The *going-to* filter **604**

46 Big sum and product over function bodies **606**
46.1 Abstract product . . . 606
46.2 Concrete sum . . . 610
46.3 Concrete product . . . 612

47 Interval Type **613**
47.1 Membership . . . 617
47.2 Quickcheck . . . 630

48 Approximate Operations on Intervals of Floating Point Numbers **631**
48.1 Intervals with Floating Point Bounds . . . 632
48.2 intros for *real-interval* . . . 634
48.3 bounds for lists . . . 634
48.4 constants for code generation . . . 639

49 Immutable Arrays with Code Generation **639**
49.1 Fundamental operations . . . 639
49.2 Generic code equations . . . 640
49.3 Auxiliary operations for code generation . . . 641
49.4 Code Generation for SML . . . 643
49.5 Code Generation for Haskell . . . 643

50 Definition of Landau symbols **644**
50.1 Definition of Landau symbols . . . 644
50.2 Landau symbols and limits . . . 666
50.3 Flatness of real functions . . . 677
50.4 Asymptotic Equivalence . . . 679
50.5 Values extended by a bottom element . . . 691
50.6 Values extended by a top element . . . 693
50.7 Values extended by a top and a bottom element . . . 695

51 Infinite Streams **701**
51.1 prepend list to stream . . . 701
51.2 set of streams with elements in some fixed set . . . 702
51.3 nth, take, drop for streams . . . 703
51.4 unary predicates lifted to streams . . . 706
51.5 recurring stream out of a list . . . 707
51.6 iterated application of a function . . . 708
51.7 stream repeating a single element . . . 708
51.8 stream of natural numbers . . . 709
51.9 flatten a stream of lists . . . 710
51.10 merge a stream of streams . . . 711
51.11 product of two streams . . . 711
51.12 interleave two streams . . . 711
51.13 zip . . . 712
51.14 zip via function . . . 713

52 List prefixes, suffixes, and homeomorphic embedding **714**
52.1 Prefix order on lists . . . 714
52.2 Basic properties of prefixes . . . 715
52.3 Prefixes . . . 719
52.4 Longest Common Prefix . . . 721
52.5 Parallel lists . . . 723
52.6 Suffix order on lists . . . 724
52.7 Suffixes . . . 730
52.8 Homeomorphic embedding on lists . . . 731
52.9 Subsequences (special case of homeomorphic embedding) . . . 734
52.10 Appending elements . . . 736
52.11 Relation to standard list operations . . . 737

52.12Contiguous sublists . . . 738
52.12.1 *sublist* . . . 738
52.12.2 *sublists* . . . 743
52.13Parametricity . . . 743

53 Linear Temporal Logic on Streams **745**

54 Preliminaries **745**

55 Linear temporal logic **745**

56 Weak vs. strong until (contributed by Michael Foster, University of Sheffield) **761**

57 Lists as vectors **763**
57.1 + and − . . . 763
57.2 Inner product . . . 765

58 Definitions of Least Upper Bounds and Greatest Lower Bounds **766**
58.1 Rules for the Relations ∗<= and <=∗ . . . 766
58.2 Rules about the Operators *leastP*, *ub* and *lub* . . . 767
58.3 Rules about the Operators *greatestP*, *isLb* and *isGlb* . . . 768

59 An abstract view on maps for code generation. **771**
59.1 Parametricity transfer rules . . . 771
59.2 Type definition and primitive operations . . . 773
59.3 Functorial structure . . . 775
59.4 Derived operations . . . 775
59.5 Properties . . . 776
59.5.1 *entries*, *ordered-entries*, and *fold* . . . 786
59.6 Code generator setup . . . 792

60 Monad notation for arbitrary types **792**

61 Less common functions on lists **793**
61.1 Simproc Set-Up . . . 803

62 (Finite) Multisets **803**
62.1 The type of multisets . . . 803
62.2 Representing multisets . . . 804
62.3 Basic operations . . . 806
62.3.1 Conversion to set and membership . . . 806
62.3.2 Union . . . 808
62.3.3 Difference . . . 809
62.3.4 Min and Max . . . 811

62.3.5 Equality of multisets . . . 811
62.3.6 Pointwise ordering induced by count . . . 813
62.3.7 Intersection and bounded union . . . 817
62.3.8 Additional intersection facts . . . 818
62.3.9 Additional bounded union facts . . . 821
62.4 Replicate and repeat operations . . . 822
62.4.1 Simprocs . . . 823
62.4.2 Conditionally complete lattice . . . 824
62.4.3 Filter (with comprehension syntax) . . . 829
62.4.4 Size . . . 831
62.5 Induction and case splits . . . 833
62.5.1 Strong induction and subset induction for multisets . . . 835
62.6 The fold combinator . . . 836
62.7 Image . . . 838
62.8 Further conversions . . . 841
62.9 More properties of the replicate and repeat operations . . . 848
62.10 Big operators . . . 850
62.11 Multiset as order-ignorant lists . . . 857
62.12 The multiset order . . . 861
62.12.1 Well-foundedness . . . 861
62.12.2 Closure-free presentation . . . 864
62.13 The multiset extension is cancellative for multiset union . . . 866
62.14 Quasi-executable version of the multiset extension . . . 867
62.14.1 Partial-order properties . . . 868
62.14.2 Monotonicity of multiset union . . . 869
62.14.3 Termination proofs with multiset orders . . . 870
62.15 Legacy theorem bindings . . . 873
62.16 Naive implementation using lists . . . 875
62.17 BNF setup . . . 879
62.18 Size setup . . . 885
62.19 Lemmas about Size . . . 886

63 More Theorems about the Multiset Order **887**
63.1 Alternative Characterizations . . . 887
63.2 Simprocs . . . 893
63.3 Additional facts and instantiations . . . 894

64 Non-negative, non-positive integers and reals **896**
64.1 Non-positive integers . . . 896
64.2 Non-negative reals . . . 899
64.3 Non-positive reals . . . 900

65 Numeral Syntax for Types **902**
65.1 Numeral Types . . . 902
65.2 *num1* . . . 903
65.3 Locales for for modular arithmetic subtypes . . . 904
65.4 Ring class instances . . . 907
65.5 Order instances . . . 909
65.6 Code setup and type classes for code generation . . . 909
65.7 Syntax . . . 913
65.8 Examples . . . 914

66 ω-words **914**
66.1 Type declaration and elementary operations . . . 915
66.2 Subsequence, Prefix, and Suffix . . . 916
66.3 Prepending . . . 920
66.4 The limit set of an ω-word . . . 922
66.5 Index sequences and piecewise definitions . . . 929

67 Combinator syntax for generic, open state monads (single-threaded monads) **933**
67.1 Motivation . . . 933
67.2 State transformations and combinators . . . 933
67.3 Monad laws . . . 934
67.4 Do-syntax . . . 935

68 Canonical order on option type **935**

69 Futures and parallel lists for code generated towards Isabelle/ML **945**
69.1 Futures . . . 946
69.2 Parallel lists . . . 946

70 Input syntax for pattern aliases (or "as-patterns" in Haskell) **947**
70.1 Definition . . . 948
70.2 Usage . . . 950

71 Periodic Functions **951**

72 Polynomial mapping: combination of almost everywhere zero functions with an algebraic view **954**
72.1 Preliminary: auxiliary operations for *almost everywhere zero* . 954
72.2 Type definition . . . 958
72.3 Additive structure . . . 960
72.4 Multiplicative structure . . . 962
72.5 Single-point mappings . . . 968
72.6 Integral domains . . . 970

72.7 Mapping order . . . 972
72.8 Fundamental mapping notions . . . 974
72.9 Degree . . . 976
72.10Inductive structure . . . 978
72.11Quasi-functorial structure . . . 979
72.12Canonical dense representation of $nat \Rightarrow_0 'a$ . . . 981
72.13Canonical sparse representation of $'a \Rightarrow_0 'b$ . . . 983
72.14Size estimation . . . 985
72.15Further mapping operations and properties . . . 986
72.16Free Abelian Groups Over a Type . . . 987

73 Exponentiation by Squaring **992**

74 Preorders with explicit equivalence relation **994**

75 Additive group operations on product types **995**
75.1 Operations . . . 996
75.2 Class instances . . . 997

76 Roots of real quadratics **998**

77 Pretty syntax for Quotient operations **1002**

78 Quotient infrastructure for the set type **1002**
78.1 Contravariant set map (vimage) and set relator, rules for the Quotient package . . . 1003

79 Quotient infrastructure for the product type **1005**
79.1 Rules for the Quotient package . . . 1005

80 Quotient infrastructure for the option type **1007**
80.1 Rules for the Quotient package . . . 1007

81 Quotient infrastructure for the list type **1008**
81.1 Rules for the Quotient package . . . 1008

82 Quotient infrastructure for the sum type **1013**
82.1 Rules for the Quotient package . . . 1013

83 Quotient types **1014**
83.1 Equivalence relations and quotient types . . . 1014
83.2 Equality on quotients . . . 1016
83.3 Picking representing elements . . . 1017

84 Ramsey's Theorem **1018**
84.1 Preliminary definitions 1018
84.1.1 The n-element subsets of a set A 1018
84.1.2 Partition predicates 1022
84.2 Finite versions of Ramsey's theorem 1023
84.2.1 Trivial cases 1023
84.2.2 Ramsey's theorem with two colours and arbitrary exponents (hypergraph version) 1024
84.2.3 Full Ramsey's theorem with multiple colours and arbitrary exponents 1030
84.2.4 Simple graph version 1032
84.3 Preliminaries 1033
84.3.1 "Axiom" of Dependent Choice 1033
84.3.2 Partition functions 1034
84.4 Ramsey's Theorem: Infinitary Version 1034
84.5 Disjunctive Well-Foundedness 1037

85 Generic reflection and reification **1039**

86 Assigning lengths to types by type classes **1040**

87 Saturated arithmetic **1043**
87.1 The type of saturated naturals 1043

88 Set Idioms **1048**
88.1 Idioms for being a suitable union/intersection of something . 1048
88.2 The "Relative to" operator 1054
88.3 Signed division: negative results rounded towards zero rather than minus infinity. 1060

89 State monad **1064**

90 Comparators on linear quasi-orders **1069**
90.1 Basic properties 1069
90.2 Fundamental comparator combinators 1073
90.3 Direct implementations for linear orders on selected types . . 1074

91 Stably sorted lists **1075**

92 Alternative sorting algorithms **1082**
92.1 Quicksort 1082
92.2 Mergesort 1084

93 A decision procedure for universal multivariate real arithmetic with addition, multiplication and ordering using semidefinite programming 1089

94 A table-based implementation of the reflexive transitive closure 1089

95 Binary Tree 1094
95.1 *map-tree* . . . 1096
95.2 *size* . . . 1096
95.3 *set-tree* . . . 1096
95.4 *subtrees* . . . 1096
95.5 *height* and *min-height* . . . 1097
95.6 *complete* . . . 1098
95.7 *acomplete* . . . 1099
95.8 *wbalanced* . . . 1100
95.9 *ipl* . . . 1100
95.10 List of entries . . . 1101
95.11 Binary Search Tree . . . 1101
95.12 *heap* . . . 1102
95.13 *mirror* . . . 1102

96 Multiset of Elements of Binary Tree 1103

97 Unordered pairs 1106

98 A type of finite bit strings 1111
98.1 Preliminaries . . . 1111
98.2 Fundamentals . . . 1111
98.2.1 Type definition . . . 1111
98.2.2 Basic arithmetic . . . 1111
98.2.3 Basic tool setup . . . 1113
98.2.4 Basic code generation setup . . . 1113
98.2.5 Basic conversions . . . 1115
98.2.6 Basic ordering . . . 1121
98.3 Enumeration . . . 1123
98.4 Bit-wise operations . . . 1124
98.5 Conversions including casts . . . 1134
98.5.1 Generic unsigned conversion . . . 1134
98.5.2 Generic signed conversion . . . 1137
98.5.3 More . . . 1138
98.6 Arithmetic operations . . . 1143
98.7 Ordering . . . 1146
98.8 Bit-wise operations . . . 1148

98.9 More shift operations . . . 1151
98.10 Rotation . . . 1152
98.11 Split and cat operations . . . 1155
98.12 More on conversions . . . 1156
98.13 Testing bits . . . 1159
98.14 Word Arithmetic . . . 1163
98.15 Transferring goals from words to ints . . . 1167
98.16 Order on fixed-length words . . . 1169
98.17 Conditions for the addition (etc) of two words to overflow . . . 1172
98.18 Some proof tool support . . . 1174
98.19 More on overflows and monotonicity . . . 1176
98.20 Arithmetic type class instantiations . . . 1180
98.21 Word and nat . . . 1180
98.22 Cardinality, finiteness of set of words . . . 1184
98.23 Bitwise Operations on Words . . . 1185
98.23.1 shift functions in terms of lists of bools . . . 1189
98.23.2 Mask . . . 1192
98.23.3 Slices . . . 1194
98.23.4 Revcast . . . 1195
98.24 Split and cat . . . 1196
98.24.1 Split and slice . . . 1196
98.25 Rotation . . . 1198
98.25.1 "Word rotation commutes with bit-wise operations . . . 1200
98.26 Maximum machine word . . . 1200
98.27 Recursion combinator for words . . . 1204
98.28 Tool support . . . 1206

99 The Field of Integers mod 2 1206

100 Pointwise order on product types 1211
100.1 Pointwise ordering . . . 1211
100.2 Binary infimum and supremum . . . 1212
100.3 Top and bottom elements . . . 1213
100.4 Complete lattice operations . . . 1214
100.5 Complete distributive lattices . . . 1215
100.6 Bekic's Theorem . . . 1215
100.7 Finite Distributive Lattices . . . 1220
100.8 Linear Orders . . . 1221
100.9 Finite Linear Orders . . . 1222

101 Lexicographic order on lists 1222

102 Lexicographic order on lists 1224

103 Prefix order on lists as order class instance 1226

104 Lexicographic order on product types 1227

105 Subsequence Ordering 1230
105.1 Definitions and basic lemmas 1230

106 Records based on BNF/datatype machinery 1231

107 Implementation of mappings with Association Lists 1233

108 Avoidance of pattern matching on natural numbers 1236
108.1 Case analysis 1236
108.2 Preprocessors 1236
108.3 One candidate which needs special treatment 1238

109 Implementation of natural numbers as binary numerals 1238
109.1 Representation 1239
109.2 Basic arithmetic 1239
109.3 Conversions 1241

110 Code generation of prolog programs 1241

111 Setup for Numerals 1242

112 Implementation of integer numbers by target-language integers 1242

113 Implementation of natural numbers by target-language integers 1249
113.1 Implementation for *nat* 1249

114 Implementation of natural and integer numbers by target-language integers 1253

115 Preprocessor setup for floats implemented by target language numerals 1254

116 Abstract type of association lists with unique keys 1255
116.1 Preliminaries 1255
116.2 Type (*'key*, *'value*) *alist* 1255
116.3 Primitive operations 1255
116.4 Abstract operation properties 1256
116.5 Further operations 1257
116.5.1 Equality 1257
116.5.2 Size 1257

116.6 Quickcheck generators . . . 1257

117 alist is a BNF **1259**

118 Multisets partially implemented by association lists **1259**

119 Implementation of Red-Black Trees **1268**
119.1 Datatype of RB trees . . . 1269
119.2 Tree properties . . . 1269
119.2.1 Content of a tree . . . 1269
119.2.2 Search tree properties . . . 1270
119.2.3 Tree lookup . . . 1271
119.2.4 Red-black properties . . . 1274
119.3 Insertion . . . 1275
119.4 Deletion . . . 1280
119.5 Modifying existing entries . . . 1290
119.6 Mapping all entries . . . 1291
119.7 Folding over entries . . . 1291
119.8 Bulkloading a tree . . . 1292
119.9 Building a RBT from a sorted list . . . 1293
119.10 Union and intersection of sorted associative lists . . . 1306
119.11 Code generator setup . . . 1334

120 Abstract type of RBT trees **1336**
120.1 Type definition . . . 1336
120.2 Primitive operations . . . 1336
120.3 Derived operations . . . 1337
120.4 Abstract lookup properties . . . 1338
120.5 Quickcheck generators . . . 1341
120.6 Hide implementation details . . . 1341

121 Implementation of mappings with Red-Black Trees **1341**
121.1 Data type and invariant . . . 1341
121.2 Operations . . . 1341
121.3 Invariant preservation . . . 1342
121.4 Map Semantics . . . 1343

122 Implementation of sets using RBT trees **1343**

123 Definition of code datatype constructors **1343**

124 Deletion of already existing code equations **1343**

125 Lemmas **1344**
125.1 Auxiliary lemmas . . . 1344
125.2 fold and filter . . . 1344
125.3 foldi and Ball . . . 1345
125.4 foldi and Bex . . . 1345
125.5 folding over non empty trees and selecting the minimal and maximal element . . . 1346
125.5.1 concrete . . . 1346
125.5.2 abstract . . . 1350

126 Code equations **1352**

127 Common constants **1360**

128 Pairs **1360**

129 Filters **1360**

130 Bounded quantifiers **1360**

131 Operations on Predicates **1360**

132 Setup for Numerals **1361**

133 Arithmetic operations **1361**
133.1 Arithmetic on naturals and integers . . . 1361
133.2 Inductive definitions for ordering on naturals . . . 1363

134 Alternative list definitions **1364**
134.1 Alternative rules for *length* . . . 1364
134.2 Alternative rules for *list-all2* . . . 1364
134.3 Alternative rules for membership in lists . . . 1364

135 Setup for String.literal **1365**

136 Simplification rules for optimisation **1365**

137 A Prototype of Quickcheck based on the Predicate Compiler **1365**

138 TFL: recursive function definitions **1365**
138.1 Lemmas for TFL . . . 1366
138.2 Rule setup . . . 1367

139 Program extraction from proofs involving datatypes and inductive predicates **1367**

140 Refute **1367**

1 Implementation of Association Lists

theory *AList*
imports *Main*
begin

context
begin

The operations preserve distinctness of keys and function *clearjunk* distributes over them. Since *clearjunk* enforces distinctness of keys it can be used to establish the invariant, e.g. for inductive proofs.

1.1 *update* and *updates*

qualified primrec *update* :: $'key \Rightarrow 'val \Rightarrow ('key \times 'val)\ list \Rightarrow ('key \times 'val)\ list$
where
update k v [] = [(*k*, *v*)]
| *update k v* (*p* # *ps*) = (*if fst p* = *k then* (*k*, *v*) # *ps else p* # *update k v ps*)

lemma *update-conv'*: *map-of* (*update k v al*) = (*map-of al*)(*k*↦*v*)
by (*induct al*) (*auto simp add*: *fun-eq-iff*)

corollary *update-conv*: *map-of* (*update k v al*) *k'* = ((*map-of al*)(*k*↦*v*)) *k'*
by (*simp add*: *update-conv'*)

lemma *dom-update*: *fst* ' *set* (*update k v al*) = {*k*} ∪ *fst* ' *set al*
by (*induct al*) *auto*

lemma *update-keys*:
map fst (*update k v al*) =
(*if k* ∈ *set* (*map fst al*) *then map fst al else map fst al* @ [*k*])
by (*induct al*) *simp-all*

lemma *distinct-update*:
assumes *distinct* (*map fst al*)
shows *distinct* (*map fst* (*update k v al*))
using *assms* **by** (*simp add*: *update-keys*)

lemma *update-filter*:
a ≠ *k* ⟹ *update k v* [*q*←*ps*. *fst q* ≠ *a*] = [*q*←*update k v ps*. *fst q* ≠ *a*]
by (*induct ps*) *auto*

lemma *update-triv*: *map-of al k* = *Some v* ⟹ *update k v al* = *al*
by (*induct al*) *auto*

lemma *update-nonempty* [*simp*]: *update k v al* ≠ []
by (*induct al*) *auto*

lemma *update-eqD*: *update k v al = update k v′ al′ ⟹ v = v′*
proof (*induct al arbitrary*: *al′*)
  case *Nil*
  then show *?case*
    by (*cases al′*) (*auto split*: *if-split-asm*)
next
  case *Cons*
  then show *?case*
    by (*cases al′*) (*auto split*: *if-split-asm*)
qed

lemma *update-last* [*simp*]: *update k v* (*update k v′ al*) = *update k v al*
  by (*induct al*) *auto*

Note that the lists are not necessarily the same: *update k v* (*update k′ v′* []) = [(*k′*, *v′*), (*k*, *v*)] and *update k′ v′* (*update k v* []) = [(*k*, *v*), (*k′*, *v′*)].

lemma *update-swap*:
  k ≠ k′ ⟹ map-of (*update k v* (*update k′ v′ al*)) = *map-of* (*update k′ v′* (*update k v al*))
  by (*simp add*: *update-conv′ fun-eq-iff*)

lemma *update-Some-unfold*:
  map-of (*update k v al*) *x* = *Some y* ⟷
    x = *k* ∧ *v* = *y* ∨ *x ≠ k* ∧ *map-of al x* = *Some y*
  by (*simp add*: *update-conv′ map-upd-Some-unfold*)

lemma *image-update* [*simp*]: *x ∉ A ⟹ map-of* (*update x y al*) ‘ *A* = *map-of al* ‘ *A*
  by (*auto simp add*: *update-conv′*)

qualified definition *updates* ::
    ′key list ⇒ ′val list ⇒ (*′key × ′val*) *list ⇒* (*′key × ′val*) *list*
  where *updates ks vs* = *fold* (*case-prod update*) (*zip ks vs*)

lemma *updates-simps* [*simp*]:
  updates [] *vs ps* = *ps*
  updates ks [] *ps* = *ps*
  updates (*k*#*ks*) (*v*#*vs*) *ps* = *updates ks vs* (*update k v ps*)
  by (*simp-all add*: *updates-def*)

lemma *updates-key-simp* [*simp*]:
  updates (*k # ks*) *vs ps* =
    (*case vs of* [] ⇒ *ps* | *v # vs* ⇒ *updates ks vs* (*update k v ps*))
  by (*cases vs*) *simp-all*

lemma *updates-conv′*: *map-of* (*updates ks vs al*) = (*map-of al*)(*ks*[↦]*vs*)
proof −
  have *map-of* ∘ *fold* (*case-prod update*) (*zip ks vs*) =
      fold (λ(*k*, *v*) *f*. *f*(*k* ↦ *v*)) (*zip ks vs*) ∘ *map-of*

by (*rule fold-commute*) (*auto simp add*: *fun-eq-iff update-conv'*)
then show *?thesis*
by (*auto simp add*: *updates-def fun-eq-iff map-upds-fold-map-upd foldl-conv-fold split-def*)
qed

lemma *updates-conv*: *map-of* (*updates ks vs al*) *k* = ((*map-of al*)(*ks*[↦]*vs*)) *k*
by (*simp add*: *updates-conv'*)

lemma *distinct-updates*:
assumes *distinct* (*map fst al*)
shows *distinct* (*map fst* (*updates ks vs al*))
proof −
have *distinct* (*fold*
(λ(*k*, *v*) *al*. *if k* ∈ *set al then al else al* @ [*k*])
(*zip ks vs*) (*map fst al*))
by (*rule fold-invariant* [*of zip ks vs* λ-. *True*]) (*auto intro*: *assms*)
moreover have *map fst* ∘ *fold* (*case-prod update*) (*zip ks vs*) =
fold (λ(*k*, *v*) *al*. *if k* ∈ *set al then al else al* @ [*k*]) (*zip ks vs*) ∘ *map fst*
by (*rule fold-commute*) (*simp add*: *update-keys split-def case-prod-beta comp-def*)
ultimately show *?thesis*
by (*simp add*: *updates-def fun-eq-iff*)
qed

lemma *updates-append1*[*simp*]: *size ks* < *size vs* ⟹
updates (*ks*@[*k*]) *vs al* = *update k* (*vs*!*size ks*) (*updates ks vs al*)
by (*induct ks arbitrary*: *vs al*) (*auto split*: *list.splits*)

lemma *updates-list-update-drop*[*simp*]:
size ks ≤ *i* ⟹ *i* < *size vs* ⟹
updates ks (*vs*[*i*:=*v*]) *al* = *updates ks vs al*
by (*induct ks arbitrary*: *al vs i*) (*auto split*: *list.splits nat.splits*)

lemma *update-updates-conv-if*:
map-of (*updates xs ys* (*update x y al*)) =
map-of
(*if x* ∈ *set* (*take* (*length ys*) *xs*)
then updates xs ys al
else (*update x y* (*updates xs ys al*)))
by (*simp add*: *updates-conv' update-conv' map-upd-upds-conv-if*)

lemma *updates-twist* [*simp*]:
k ∉ *set ks* ⟹
map-of (*updates ks vs* (*update k v al*)) = *map-of* (*update k v* (*updates ks vs al*))
by (*simp add*: *updates-conv' update-conv'*)

lemma *updates-apply-notin* [*simp*]:
k ∉ *set ks* ⟹ *map-of* (*updates ks vs al*) *k* = *map-of al k*
by (*simp add*: *updates-conv*)

lemma *updates-append-drop* [*simp*]:
size xs = size ys ⟹ updates (xs @ zs) ys al = updates xs ys al
by (*induct xs arbitrary*: *ys al*) (*auto split*: *list.splits*)

lemma *updates-append2-drop* [*simp*]:
size xs = size ys ⟹ updates xs (ys @ zs) al = updates xs ys al
by (*induct xs arbitrary*: *ys al*) (*auto split*: *list.splits*)

1.2 *delete*

qualified definition *delete* :: *'key ⇒ ('key × 'val) list ⇒ ('key × 'val) list*
where *delete-eq*: *delete k = filter (λ(k', -). k ≠ k')*

lemma *delete-simps* [*simp*]:
delete k [] = []
delete k (p # ps) = (if fst p = k then delete k ps else p # delete k ps)
by (*auto simp add*: *delete-eq*)

lemma *delete-conv'*: *map-of (delete k al) = (map-of al)(k := None)*
by (*induct al*) (*auto simp add*: *fun-eq-iff*)

corollary *delete-conv*: *map-of (delete k al) k' = ((map-of al)(k := None)) k'*
by (*simp add*: *delete-conv'*)

lemma *delete-keys*: *map fst (delete k al) = removeAll k (map fst al)*
by (*simp add*: *delete-eq removeAll-filter-not-eq filter-map split-def comp-def*)

lemma *distinct-delete*:
assumes *distinct (map fst al)*
shows *distinct (map fst (delete k al))*
using *assms* **by** (*simp add*: *delete-keys distinct-removeAll*)

lemma *delete-id* [*simp*]: *k ∉ fst ` set al ⟹ delete k al = al*
by (*auto simp add*: *image-iff delete-eq filter-id-conv*)

lemma *delete-idem*: *delete k (delete k al) = delete k al*
by (*simp add*: *delete-eq*)

lemma *map-of-delete* [*simp*]: *k' ≠ k ⟹ map-of (delete k al) k' = map-of al k'*
by (*simp add*: *delete-conv'*)

lemma *delete-notin-dom*: *k ∉ fst ` set (delete k al)*
by (*auto simp add*: *delete-eq*)

lemma *dom-delete-subset*: *fst ` set (delete k al) ⊆ fst ` set al*
by (*auto simp add*: *delete-eq*)

lemma *delete-update-same*: *delete k (update k v al) = delete k al*

by (*induct al*) *simp-all*

lemma *delete-update*: $k \neq l \Longrightarrow$ *delete l* (*update k v al*) = *update k v* (*delete l al*)
by (*induct al*) *simp-all*

lemma *delete-twist*: *delete x* (*delete y al*) = *delete y* (*delete x al*)
by (*simp add*: *delete-eq conj-commute*)

lemma *length-delete-le*: *length* (*delete k al*) $\leq$ *length al*
by (*simp add*: *delete-eq*)

1.3 *update-with-aux* and *delete-aux*

qualified primrec *update-with-aux* ::
$'val \Rightarrow 'key \Rightarrow ('val \Rightarrow 'val) \Rightarrow ('key \times 'val)\ list \Rightarrow ('key \times 'val)\ list$
where
update-with-aux v k f [] = [(*k*, *f v*)]
| *update-with-aux v k f* (*p* # *ps*) =
(*if* (*fst p* = *k*) *then* (*k*, *f* (*snd p*)) # *ps else p* # *update-with-aux v k f ps*)

The above *delete* traverses all the list even if it has found the key. This one does not have to keep going because is assumes the invariant that keys are distinct.

qualified fun *delete-aux* :: $'key \Rightarrow ('key \times 'val)\ list \Rightarrow ('key \times 'val)\ list$
where
delete-aux k [] = []
| *delete-aux k* ((*k'*, *v*) # *xs*) = (*if k* = *k' then xs else* (*k'*, *v*) # *delete-aux k xs*)

lemma *map-of-update-with-aux'*:
map-of (*update-with-aux v k f ps*) *k'* =
((*map-of ps*)(*k* $\mapsto$ (*case map-of ps k of None* $\Rightarrow$ *f v* | *Some v* $\Rightarrow$ *f v*))) *k'*
by (*induct ps*) *auto*

lemma *map-of-update-with-aux*:
map-of (*update-with-aux v k f ps*) =
(*map-of ps*)(*k* $\mapsto$ (*case map-of ps k of None* $\Rightarrow$ *f v* | *Some v* $\Rightarrow$ *f v*))
by (*simp add*: *fun-eq-iff map-of-update-with-aux'*)

lemma *dom-update-with-aux*: *fst* ' *set* (*update-with-aux v k f ps*) = {*k*} $\cup$ *fst* ' *set ps*
by (*induct ps*) *auto*

lemma *distinct-update-with-aux* [*simp*]:
distinct (*map fst* (*update-with-aux v k f ps*)) = *distinct* (*map fst ps*)
by (*induct ps*) (*auto simp add*: *dom-update-with-aux*)

lemma *set-update-with-aux*:
distinct (*map fst xs*) $\Longrightarrow$
set (*update-with-aux v k f xs*) =

(*set xs* − {*k*} × *UNIV* ∪ {(*k*, *f* (*case map-of xs k of None* ⇒ *v* | *Some v* ⇒ *v*))})
by (*induct xs*) (*auto intro*: *rev-image-eqI*)

lemma *set-delete-aux*: *distinct* (*map fst xs*) ⟹ *set* (*delete-aux k xs*) = *set xs* − {*k*} × *UNIV*
apply (*induct xs*)
apply *simp-all*
apply *clarsimp*
apply (*fastforce intro*: *rev-image-eqI*)
done

lemma *dom-delete-aux*: *distinct* (*map fst ps*) ⟹ *fst* ‘ *set* (*delete-aux k ps*) = *fst* ‘ *set ps* − {*k*}
by (*auto simp add*: *set-delete-aux*)

lemma *distinct-delete-aux* [*simp*]: *distinct* (*map fst ps*) ⟹ *distinct* (*map fst* (*delete-aux k ps*))
proof (*induct ps*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a ps*)
obtain *k′ v* **where** *a*: *a* = (*k′*, *v*)
by (*cases a*)
show *?case*
proof (*cases k′* = *k*)
case *True*
with *Cons a* **show** *?thesis* **by** *simp*
next
case *False*
with *Cons a* **have** *k′* ∉ *fst* ‘ *set ps distinct* (*map fst ps*)
by *simp-all*
with *False a* **have** *k′* ∉ *fst* ‘ *set* (*delete-aux k ps*)
by (*auto dest*!: *dom-delete-aux*[**where** *k*=*k*])
with *Cons a* **show** *?thesis*
by *simp*
qed
qed

lemma *map-of-delete-aux′*:
distinct (*map fst xs*) ⟹ *map-of* (*delete-aux k xs*) = (*map-of xs*)(*k* := *None*)
apply (*induct xs*)
apply (*fastforce simp add*: *map-of-eq-None-iff fun-upd-twist*)
apply (*auto intro*!: *ext*)
apply (*simp add*: *map-of-eq-None-iff*)
done

lemma *map-of-delete-aux*:

distinct (*map fst xs*) $\Longrightarrow$ *map-of* (*delete-aux k xs*) $k' = ((\textit{map-of}\ xs)(k := None))\ k'$
by (*simp add*: *map-of-delete-aux'*)

lemma *delete-aux-eq-Nil-conv*: *delete-aux k ts* $= [] \longleftrightarrow ts = [] \vee (\exists v.\ ts = [(k, v)])$
by (*cases ts*) (*auto split*: *if-split-asm*)

1.4 *restrict*

qualified definition *restrict* :: *'key set* $\Rightarrow$ (*'key* $\times$ *'val*) *list* $\Rightarrow$ (*'key* $\times$ *'val*) *list*
where *restrict-eq*: *restrict A* = *filter* ($\lambda(k, v).\ k \in A$)

lemma *restr-simps* [*simp*]:
restrict A [] = []
restrict A (*p*#*ps*) = (*if fst p* $\in$ *A then p* # *restrict A ps else restrict A ps*)
by (*auto simp add*: *restrict-eq*)

lemma *restr-conv'*: *map-of* (*restrict A al*) = ((*map-of al*)|‘ *A*)
proof
show *map-of* (*restrict A al*) *k* = ((*map-of al*)|‘ *A*) *k* **for** *k*
apply (*induct al*)
apply *simp*
apply (*cases k* $\in$ *A*)
apply *auto*
done
qed

corollary *restr-conv*: *map-of* (*restrict A al*) *k* = ((*map-of al*)|‘ *A*) *k*
by (*simp add*: *restr-conv'*)

lemma *distinct-restr*: *distinct* (*map fst al*) $\Longrightarrow$ *distinct* (*map fst* (*restrict A al*))
by (*induct al*) (*auto simp add*: *restrict-eq*)

lemma *restr-empty* [*simp*]:
restrict {} *al* = []
restrict A [] = []
by (*induct al*) (*auto simp add*: *restrict-eq*)

lemma *restr-in* [*simp*]: *x* $\in$ *A* $\Longrightarrow$ *map-of* (*restrict A al*) *x* = *map-of al x*
by (*simp add*: *restr-conv'*)

lemma *restr-out* [*simp*]: *x* $\notin$ *A* $\Longrightarrow$ *map-of* (*restrict A al*) *x* = *None*
by (*simp add*: *restr-conv'*)

lemma *dom-restr* [*simp*]: *fst* ‘ *set* (*restrict A al*) = *fst* ‘ *set al* $\cap$ *A*
by (*induct al*) (*auto simp add*: *restrict-eq*)

lemma *restr-upd-same* [*simp*]: *restrict* ($-\{x\}$) (*update x y al*) = *restrict* ($-\{x\}$) *al*

by (*induct al*) (*auto simp add*: *restrict-eq*)

lemma *restr-restr* [*simp*]: *restrict A* (*restrict B al*) = *restrict* (*A*∩*B*) *al*
by (*induct al*) (*auto simp add*: *restrict-eq*)

lemma *restr-update*[*simp*]:
map-of (*restrict D* (*update x y al*)) =
map-of ((*if x* ∈ *D then* (*update x y* (*restrict* (*D*−{*x*}) *al*)) *else restrict D al*))
by (*simp add*: *restr-conv*′ *update-conv*′)

lemma *restr-delete* [*simp*]:
delete x (*restrict D al*) = (*if x* ∈ *D then restrict* (*D* − {*x*}) *al else restrict D al*)
apply (*simp add*: *delete-eq restrict-eq*)
apply (*auto simp add*: *split-def*)
proof −
have *y* ≠ *x* ⟷ *x* ≠ *y* **for** *y*
by *auto*
then show [*p* ← *al*. *fst p* ∈ *D* ∧ *x* ≠ *fst p*] = [*p* ← *al*. *fst p* ∈ *D* ∧ *fst p* ≠ *x*]
by *simp*
assume *x* ∉ *D*
then have *y* ∈ *D* ⟷ *y* ∈ *D* ∧ *x* ≠ *y* **for** *y*
by *auto*
then show [*p* ← *al* . *fst p* ∈ *D* ∧ *x* ≠ *fst p*] = [*p* ← *al* . *fst p* ∈ *D*]
by *simp*
qed

lemma *update-restr*:
map-of (*update x y* (*restrict D al*)) = *map-of* (*update x y* (*restrict* (*D* − {*x*}) *al*))
by (*simp add*: *update-conv*′ *restr-conv*′) (*rule fun-upd-restrict*)

lemma *update-restr-conv* [*simp*]:
x ∈ *D* ⟹
map-of (*update x y* (*restrict D al*)) = *map-of* (*update x y* (*restrict* (*D* − {*x*}) *al*))
by (*simp add*: *update-conv*′ *restr-conv*′)

lemma *restr-updates* [*simp*]:
length xs = *length ys* ⟹ *set xs* ⊆ *D* ⟹
map-of (*restrict D* (*updates xs ys al*)) =
map-of (*updates xs ys* (*restrict* (*D* − *set xs*) *al*))
by (*simp add*: *updates-conv*′ *restr-conv*′)

lemma *restr-delete-twist*: (*restrict A* (*delete a ps*)) = *delete a* (*restrict A ps*)
by (*induct ps*) *auto*

1.5 *clearjunk*

qualified function *clearjunk* :: ('*key* × '*val*) *list* ⇒ ('*key* × '*val*) *list*

```
  where
    clearjunk [] = []
  | clearjunk (p#ps) = p # clearjunk (delete (fst p) ps)
  by pat-completeness auto
termination
  by (relation measure length) (simp-all add: less-Suc-eq-le length-delete-le)

lemma map-of-clearjunk: map-of (clearjunk al) = map-of al
  by (induct al rule: clearjunk.induct) (simp-all add: fun-eq-iff)

lemma clearjunk-keys-set: set (map fst (clearjunk al)) = set (map fst al)
  by (induct al rule: clearjunk.induct) (simp-all add: delete-keys)

lemma dom-clearjunk: fst ‘ set (clearjunk al) = fst ‘ set al
  using clearjunk-keys-set by simp

lemma distinct-clearjunk [simp]: distinct (map fst (clearjunk al))
  by (induct al rule: clearjunk.induct) (simp-all del: set-map add: clearjunk-keys-set
delete-keys)

lemma ran-clearjunk: ran (map-of (clearjunk al)) = ran (map-of al)
  by (simp add: map-of-clearjunk)

lemma ran-map-of: ran (map-of al) = snd ‘ set (clearjunk al)
proof −
  have ran (map-of al) = ran (map-of (clearjunk al))
    by (simp add: ran-clearjunk)
  also have ... = snd ‘ set (clearjunk al)
    by (simp add: ran-distinct)
  finally show ?thesis .
qed

lemma graph-map-of: Map.graph (map-of al) = set (clearjunk al)
  by (metis distinct-clearjunk graph-map-of-if-distinct-dom map-of-clearjunk)

lemma clearjunk-update: clearjunk (update k v al) = update k v (clearjunk al)
  by (induct al rule: clearjunk.induct) (simp-all add: delete-update)

lemma clearjunk-updates: clearjunk (updates ks vs al) = updates ks vs (clearjunk
al)
proof −
  have clearjunk ∘ fold (case-prod update) (zip ks vs) =
      fold (case-prod update) (zip ks vs) ∘ clearjunk
    by (rule fold-commute) (simp add: clearjunk-update case-prod-beta o-def)
  then show ?thesis
    by (simp add: updates-def fun-eq-iff)
qed

lemma clearjunk-delete: clearjunk (delete x al) = delete x (clearjunk al)
```

by (*induct al rule*: *clearjunk.induct*) (*auto simp add*: *delete-idem delete-twist*)

lemma *clearjunk-restrict*: *clearjunk* (*restrict A al*) = *restrict A* (*clearjunk al*)
by (*induct al rule*: *clearjunk.induct*) (*auto simp add*: *restr-delete-twist*)

lemma *distinct-clearjunk-id* [*simp*]: *distinct* (*map fst al*) ⟹ *clearjunk al* = *al*
by (*induct al rule*: *clearjunk.induct*) *auto*

lemma *clearjunk-idem*: *clearjunk* (*clearjunk al*) = *clearjunk al*
by *simp*

lemma *length-clearjunk*: *length* (*clearjunk al*) ≤ *length al*
proof (*induct al rule*: *clearjunk.induct* [*case-names Nil Cons*])
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons kv al*)
moreover have *length* (*delete* (*fst kv*) *al*) ≤ *length al*
by (*fact length-delete-le*)
ultimately have *length* (*clearjunk* (*delete* (*fst kv*) *al*)) ≤ *length al*
by (*rule order-trans*)
then show *?case*
by *simp*
qed

lemma *delete-map*:
assumes ⋀*kv*. *fst* (*f kv*) = *fst kv*
shows *delete k* (*map f ps*) = *map f* (*delete k ps*)
by (*simp add*: *delete-eq filter-map comp-def split-def assms*)

lemma *clearjunk-map*:
assumes ⋀*kv*. *fst* (*f kv*) = *fst kv*
shows *clearjunk* (*map f ps*) = *map f* (*clearjunk ps*)
by (*induct ps rule*: *clearjunk.induct* [*case-names Nil Cons*])
(*simp-all add*: *clearjunk-delete delete-map assms*)

1.6 *map-ran*

definition *map-ran* :: ('*key* ⇒ '*val1* ⇒ '*val2*) ⇒ ('*key* × '*val1*) *list* ⇒ ('*key* × '*val2*) *list*
where *map-ran f* = *map* (λ(*k*, *v*). (*k*, *f k v*))

lemma *map-ran-simps* [*simp*]:
map-ran f [] = []
map-ran f ((*k*, *v*) # *ps*) = (*k*, *f k v*) # *map-ran f ps*
by (*simp-all add*: *map-ran-def*)

lemma *map-ran-Cons-sel*: *map-ran f* (*p* # *ps*) = (*fst p*, *f* (*fst p*) (*snd p*)) # *map-ran f ps*

by (*simp add*: *map-ran-def case-prod-beta*)

lemma *length-map-ran*[*simp*]: *length* (*map-ran f al*) = *length al*
by (*simp add*: *map-ran-def*)

lemma *map-fst-map-ran*[*simp*]: *map fst* (*map-ran f al*) = *map fst al*
by (*simp add*: *map-ran-def case-prod-beta*)

lemma *dom-map-ran*: *fst ‘ set* (*map-ran f al*) = *fst ‘ set al*
by (*simp add*: *map-ran-def image-image split-def*)

lemma *map-ran-conv*: *map-of* (*map-ran f al*) *k* = *map-option* (*f k*) (*map-of al k*)
by (*induct al*) *auto*

lemma *distinct-map-ran*: *distinct* (*map fst al*) ⟹ *distinct* (*map fst* (*map-ran f al*))
by *simp*

lemma *map-ran-filter*: *map-ran f* [*p*←*ps*. *fst p* ≠ *a*] = [*p*←*map-ran f ps*. *fst p* ≠ *a*]
by (*simp add*: *map-ran-def filter-map split-def comp-def*)

lemma *clearjunk-map-ran*: *clearjunk* (*map-ran f al*) = *map-ran f* (*clearjunk al*)
by (*simp add*: *map-ran-def split-def clearjunk-map*)

1.7 *merge*

qualified definition *merge* :: (*'key* × *'val*) *list* ⇒ (*'key* × *'val*) *list* ⇒ (*'key* × *'val*) *list*
where *merge qs ps* = *foldr* (λ(*k*, *v*). *update k v*) *ps qs*

lemma *merge-simps* [*simp*]:
merge qs [] = *qs*
merge qs (*p*#*ps*) = *update* (*fst p*) (*snd p*) (*merge qs ps*)
by (*simp-all add*: *merge-def split-def*)

lemma *merge-updates*: *merge qs ps* = *updates* (*rev* (*map fst ps*)) (*rev* (*map snd ps*)) *qs*
by (*simp add*: *merge-def updates-def foldr-conv-fold zip-rev zip-map-fst-snd*)

lemma *dom-merge*: *fst ‘ set* (*merge xs ys*) = *fst ‘ set xs* ∪ *fst ‘ set ys*
by (*induct ys arbitrary*: *xs*) (*auto simp add*: *dom-update*)

lemma *distinct-merge*: *distinct* (*map fst xs*) ⟹ *distinct* (*map fst* (*merge xs ys*))
by (*simp add*: *merge-updates distinct-updates*)

lemma *clearjunk-merge*: *clearjunk* (*merge xs ys*) = *merge* (*clearjunk xs*) *ys*
by (*simp add*: *merge-updates clearjunk-updates*)

lemma *merge-conv'*: *map-of* (*merge xs ys*) = *map-of xs* ++ *map-of ys*
proof −
have *map-of* ∘ *fold* (*case-prod update*) (*rev ys*) =
fold (λ(*k*, *v*) *m*. *m*(*k* ↦ *v*)) (*rev ys*) ∘ *map-of*
by (*rule fold-commute*) (*simp add*: *update-conv'* *case-prod-beta split-def fun-eq-iff*)
then show *?thesis*
by (*simp add*: *merge-def map-add-map-of-foldr foldr-conv-fold fun-eq-iff*)
qed

corollary *merge-conv*: *map-of* (*merge xs ys*) *k* = (*map-of xs* ++ *map-of ys*) *k*
by (*simp add*: *merge-conv'*)

lemma *merge-empty*: *map-of* (*merge* [] *ys*) = *map-of ys*
by (*simp add*: *merge-conv'*)

lemma *merge-assoc* [*simp*]: *map-of* (*merge m1* (*merge m2 m3*)) = *map-of* (*merge* (*merge m1 m2*) *m3*)
by (*simp add*: *merge-conv'*)

lemma *merge-Some-iff*:
map-of (*merge m n*) *k* = *Some x* ⟷
map-of n k = *Some x* ∨ *map-of n k* = *None* ∧ *map-of m k* = *Some x*
by (*simp add*: *merge-conv'* *map-add-Some-iff*)

lemmas *merge-SomeD* [*dest*!] = *merge-Some-iff* [*THEN iffD1*]

lemma *merge-find-right* [*simp*]: *map-of n k* = *Some v* ⟹ *map-of* (*merge m n*) *k* = *Some v*
by (*simp add*: *merge-conv'*)

lemma *merge-None* [*iff*]: (*map-of* (*merge m n*) *k* = *None*) = (*map-of n k* = *None* ∧ *map-of m k* = *None*)
by (*simp add*: *merge-conv'*)

lemma *merge-upd* [*simp*]: *map-of* (*merge m* (*update k v n*)) = *map-of* (*update k v* (*merge m n*))
by (*simp add*: *update-conv'* *merge-conv'*)

lemma *merge-updatess* [*simp*]:
map-of (*merge m* (*updates xs ys n*)) = *map-of* (*updates xs ys* (*merge m n*))
by (*simp add*: *updates-conv'* *merge-conv'*)

lemma *merge-append*: *map-of* (*xs* @ *ys*) = *map-of* (*merge ys xs*)
by (*simp add*: *merge-conv'*)

1.8 *compose*

qualified function *compose* :: ('*key* × '*a*) *list* ⇒ ('*a* × '*b*) *list* ⇒ ('*key* × '*b*) *list*
where

```
    compose [] ys = []
  | compose (x # xs) ys =
      (case map-of ys (snd x) of
        None ⇒ compose (delete (fst x) xs) ys
      | Some v ⇒ (fst x, v) # compose xs ys)
  by pat-completeness auto
termination
  by (relation measure (length ∘ fst)) (simp-all add: less-Suc-eq-le length-delete-le)

lemma compose-first-None [simp]: map-of xs k = None ⟹ map-of (compose xs
ys) k = None
  by (induct xs ys rule: compose.induct) (auto split: option.splits if-split-asm)

lemma compose-conv: map-of (compose xs ys) k = (map-of ys ∘m map-of xs) k
proof (induct xs ys rule: compose.induct)
  case 1
  then show ?case by simp
next
  case (2 x xs ys)
  show ?case
  proof (cases map-of ys (snd x))
    case None
    with 2 have hyp: map-of (compose (delete (fst x) xs) ys) k =
        (map-of ys ∘m map-of (delete (fst x) xs)) k
      by simp
    show ?thesis
    proof (cases fst x = k)
      case True
      from True delete-notin-dom [of k xs]
      have map-of (delete (fst x) xs) k = None
        by (simp add: map-of-eq-None-iff)
      with hyp show ?thesis
        using True None
        by simp
    next
      case False
      from False have map-of (delete (fst x) xs) k = map-of xs k
        by simp
      with hyp show ?thesis
        using False None by (simp add: map-comp-def)
    qed
  next
    case (Some v)
    with 2
    have map-of (compose xs ys) k = (map-of ys ∘m map-of xs) k
      by simp
    with Some show ?thesis
      by (auto simp add: map-comp-def)
  qed
```

qed

lemma *compose-conv'*: *map-of* (*compose xs ys*) = (*map-of ys* $\circ_m$ *map-of xs*)
by (*rule ext*) (*rule compose-conv*)

lemma *compose-first-Some* [*simp*]: *map-of xs k* = *Some v* $\Longrightarrow$ *map-of* (*compose xs ys*) *k* = *map-of ys v*
by (*simp add*: *compose-conv*)

lemma *dom-compose*: *fst* ' *set* (*compose xs ys*) $\subseteq$ *fst* ' *set xs*
proof (*induct xs ys rule*: *compose.induct*)
case *1*
then show *?case* **by** *simp*
next
case (*2 x xs ys*)
show *?case*
proof (*cases map-of ys* (*snd x*))
case *None*
with *2.hyps* **have** *fst* ' *set* (*compose* (*delete* (*fst x*) *xs*) *ys*) $\subseteq$ *fst* ' *set* (*delete* (*fst x*) *xs*)
by *simp*
also have ... $\subseteq$ *fst* ' *set xs*
by (*rule dom-delete-subset*)
finally show *?thesis*
using *None* **by** *auto*
next
case (*Some v*)
with *2.hyps* **have** *fst* ' *set* (*compose xs ys*) $\subseteq$ *fst* ' *set xs*
by *simp*
with *Some* **show** *?thesis*
by *auto*
qed
qed

lemma *distinct-compose*:
assumes *distinct* (*map fst xs*)
shows *distinct* (*map fst* (*compose xs ys*))
using *assms*
proof (*induct xs ys rule*: *compose.induct*)
case *1*
then show *?case* **by** *simp*
next
case (*2 x xs ys*)
show *?case*
proof (*cases map-of ys* (*snd x*))
case *None*
with *2* **show** *?thesis* **by** *simp*
next
case (*Some v*)

```
    with 2 dom-compose [of xs ys] show ?thesis
      by auto
  qed
qed

lemma compose-delete-twist: compose (delete k xs) ys = delete k (compose xs ys)
proof (induct xs ys rule: compose.induct)
  case 1
  then show ?case by simp
next
  case (2 x xs ys)
  show ?case
  proof (cases map-of ys (snd x))
    case None
    with 2 have hyp: compose (delete k (delete (fst x) xs)) ys =
        delete k (compose (delete (fst x) xs) ys)
      by simp
    show ?thesis
    proof (cases fst x = k)
      case True
      with None hyp show ?thesis
        by (simp add: delete-idem)
    next
      case False
      from None False hyp show ?thesis
        by (simp add: delete-twist)
    qed
  next
    case (Some v)
    with 2 have hyp: compose (delete k xs) ys = delete k (compose xs ys)
      by simp
    with Some show ?thesis
      by simp
  qed
qed

lemma compose-clearjunk: compose xs (clearjunk ys) = compose xs ys
  by (induct xs ys rule: compose.induct)
    (auto simp add: map-of-clearjunk split: option.splits)

lemma clearjunk-compose: clearjunk (compose xs ys) = compose (clearjunk xs) ys
  by (induct xs rule: clearjunk.induct)
   (auto split: option.splits simp add: clearjunk-delete delete-idem compose-delete-twist)

lemma compose-empty [simp]: compose xs [] = []
  by (induct xs) (auto simp add: compose-delete-twist)

lemma compose-Some-iff:
  (map-of (compose xs ys) k = Some v) ⟷
```

$(\exists k'.\ map\text{-}of\ xs\ k = Some\ k' \wedge map\text{-}of\ ys\ k' = Some\ v)$
by (*simp add*: *compose-conv map-comp-Some-iff*)

lemma *map-comp-None-iff*:
$map\text{-}of\ (compose\ xs\ ys)\ k = None \longleftrightarrow$
$(map\text{-}of\ xs\ k = None \vee (\exists k'.\ map\text{-}of\ xs\ k = Some\ k' \wedge map\text{-}of\ ys\ k' = None))$
by (*simp add*: *compose-conv map-comp-None-iff*)

1.9 *map-entry*

qualified fun *map-entry* :: $'key \Rightarrow ('val \Rightarrow 'val) \Rightarrow ('key \times 'val)\ list \Rightarrow ('key \times 'val)\ list$
where
$map\text{-}entry\ k\ f\ [] = []$
$|\ map\text{-}entry\ k\ f\ (p\ \#\ ps) =$
$(if\ fst\ p = k\ then\ (k,\ f\ (snd\ p))\ \#\ ps\ else\ p\ \#\ map\text{-}entry\ k\ f\ ps)$

lemma *map-of-map-entry*:
$map\text{-}of\ (map\text{-}entry\ k\ f\ xs) =$
$(map\text{-}of\ xs)(k := case\ map\text{-}of\ xs\ k\ of\ None \Rightarrow None \mid Some\ v' \Rightarrow Some\ (f\ v'))$
by (*induct xs*) *auto*

lemma *dom-map-entry*: $fst\ `\ set\ (map\text{-}entry\ k\ f\ xs) = fst\ `\ set\ xs$
by (*induct xs*) *auto*

lemma *distinct-map-entry*:
assumes *distinct* (*map fst xs*)
shows *distinct* (*map fst* (*map-entry k f xs*))
using *assms* **by** (*induct xs*) (*auto simp add*: *dom-map-entry*)

1.10 *map-default*

fun *map-default* :: $'key \Rightarrow 'val \Rightarrow ('val \Rightarrow 'val) \Rightarrow ('key \times 'val)\ list \Rightarrow ('key \times 'val)\ list$
where
$map\text{-}default\ k\ v\ f\ [] = [(k,\ v)]$
$|\ map\text{-}default\ k\ v\ f\ (p\ \#\ ps) =$
$(if\ fst\ p = k\ then\ (k,\ f\ (snd\ p))\ \#\ ps\ else\ p\ \#\ map\text{-}default\ k\ v\ f\ ps)$

lemma *map-of-map-default*:
$map\text{-}of\ (map\text{-}default\ k\ v\ f\ xs) =$
$(map\text{-}of\ xs)(k := case\ map\text{-}of\ xs\ k\ of\ None \Rightarrow Some\ v \mid Some\ v' \Rightarrow Some\ (f\ v'))$
by (*induct xs*) *auto*

lemma *dom-map-default*: $fst\ `\ set\ (map\text{-}default\ k\ v\ f\ xs) = insert\ k\ (fst\ `\ set\ xs)$
by (*induct xs*) *auto*

lemma *distinct-map-default*:
assumes *distinct* (*map fst xs*)
shows *distinct* (*map fst* (*map-default k v f xs*))

using *assms* **by** (*induct xs*) (*auto simp add*: *dom-map-default*)

end

end

2 Adhoc overloading of constants based on their types

theory *Adhoc-Overloading*
imports *Main*
keywords *adhoc-overloading no-adhoc-overloading* :: *thy-decl*
begin

ML-file ‹*adhoc-overloading.ML*›

end

3 Pointwise instantiation of functions to algebra type classes

theory *Function-Algebras*
imports *Main*
begin

Pointwise operations

instantiation *fun* :: (*type*, *plus*) *plus*
begin

definition $f + g = (\lambda x.\ f\ x + g\ x)$
instance ..

end

lemma *plus-fun-apply* [*simp*]:
$(f + g)\ x = f\ x + g\ x$
by (*simp add*: *plus-fun-def*)

instantiation *fun* :: (*type*, *zero*) *zero*
begin

definition $0 = (\lambda x.\ 0)$
instance ..

end

lemma *zero-fun-apply* [*simp*]:

0 x = 0
by (*simp add*: *zero-fun-def*)

instantiation *fun* :: (*type*, *times*) *times*
begin

definition $f * g = (\lambda x.\ f\ x * g\ x)$
instance ..

end

lemma *times-fun-apply* [*simp*]:
$(f * g)\ x = f\ x * g\ x$
by (*simp add*: *times-fun-def*)

instantiation *fun* :: (*type*, *one*) *one*
begin

definition *1* = $(\lambda x.\ 1)$
instance ..

end

lemma *one-fun-apply* [*simp*]:
1 x = 1
by (*simp add*: *one-fun-def*)

Additive structures

instance *fun* :: (*type*, *semigroup-add*) *semigroup-add*
by *standard* (*simp add*: *fun-eq-iff add.assoc*)

instance *fun* :: (*type*, *cancel-semigroup-add*) *cancel-semigroup-add*
by *standard* (*simp-all add*: *fun-eq-iff*)

instance *fun* :: (*type*, *ab-semigroup-add*) *ab-semigroup-add*
by *standard* (*simp add*: *fun-eq-iff add.commute*)

instance *fun* :: (*type*, *cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
by *standard* (*simp-all add*: *fun-eq-iff diff-diff-eq*)

instance *fun* :: (*type*, *monoid-add*) *monoid-add*
by *standard* (*simp-all add*: *fun-eq-iff*)

instance *fun* :: (*type*, *comm-monoid-add*) *comm-monoid-add*
by *standard simp*

instance *fun* :: (*type*, *cancel-comm-monoid-add*) *cancel-comm-monoid-add* **..**

instance *fun* :: (*type*, *group-add*) *group-add*

by *standard* (*simp-all add*: *fun-eq-iff*)

instance *fun* :: (*type*, *ab-group-add*) *ab-group-add*
by *standard simp-all*

Multiplicative structures

instance *fun* :: (*type*, *semigroup-mult*) *semigroup-mult*
by *standard* (*simp add*: *fun-eq-iff mult.assoc*)

instance *fun* :: (*type*, *ab-semigroup-mult*) *ab-semigroup-mult*
by *standard* (*simp add*: *fun-eq-iff mult.commute*)

instance *fun* :: (*type*, *monoid-mult*) *monoid-mult*
by *standard* (*simp-all add*: *fun-eq-iff*)

instance *fun* :: (*type*, *comm-monoid-mult*) *comm-monoid-mult*
by *standard simp*

Misc

instance *fun* :: (*type*, *Rings.dvd*) *Rings.dvd* **..**

instance *fun* :: (*type*, *mult-zero*) *mult-zero*
by *standard* (*simp-all add*: *fun-eq-iff*)

instance *fun* :: (*type*, *zero-neq-one*) *zero-neq-one*
by *standard* (*simp add*: *fun-eq-iff*)

Ring structures

instance *fun* :: (*type*, *semiring*) *semiring*
by *standard* (*simp-all add*: *fun-eq-iff algebra-simps*)

instance *fun* :: (*type*, *comm-semiring*) *comm-semiring*
by *standard* (*simp add*: *fun-eq-iff algebra-simps*)

instance *fun* :: (*type*, *semiring-0*) *semiring-0* **..**

instance *fun* :: (*type*, *comm-semiring-0*) *comm-semiring-0* **..**

instance *fun* :: (*type*, *semiring-0-cancel*) *semiring-0-cancel* **..**

instance *fun* :: (*type*, *comm-semiring-0-cancel*) *comm-semiring-0-cancel* **..**

instance *fun* :: (*type*, *semiring-1*) *semiring-1* **..**

lemma *numeral-fun*:
‹*numeral n* = (λ*x*::'*a*. *numeral n*)›
by (*induction n*) (*simp-all only*: *numeral.simps plus-fun-def*, *simp-all*)

lemma *numeral-fun-apply* [*simp*]:

‹*numeral n x = numeral n*›
by (*simp add*: *numeral-fun*)

lemma *of-nat-fun*: *of-nat n* = (λ*x*::′*a*. *of-nat n*)
proof −
have *comp*: *comp* = (λ*f g x*. *f* (*g x*))
by (*rule ext*)+ *simp*
have *plus-fun*: *plus* = (λ*f g x*. *f x* + *g x*)
by (*rule ext*, *rule ext*) (*fact plus-fun-def*)
have *of-nat n* = (*comp* (*plus* (*1*::′*b*)) ^^ *n*) (λ*x*::′*a*. *0*)
by (*simp add*: *of-nat-def plus-fun zero-fun-def one-fun-def comp*)
also have ... = *comp* ((*plus 1*) ^^ *n*) (λ*x*::′*a*. *0*)
by (*simp only*: *comp-funpow*)
finally show *?thesis* **by** (*simp add*: *of-nat-def comp*)
qed

lemma *of-nat-fun-apply* [*simp*]:
of-nat n x = *of-nat n*
by (*simp add*: *of-nat-fun*)

instance *fun* :: (*type*, *comm-semiring-1*) *comm-semiring-1* **..**

instance *fun* :: (*type*, *semiring-1-cancel*) *semiring-1-cancel* **..**

instance *fun* :: (*type*, *comm-semiring-1-cancel*) *comm-semiring-1-cancel*
by *standard* (*auto simp add*: *times-fun-def algebra-simps*)

instance *fun* :: (*type*, *semiring-char-0*) *semiring-char-0*
proof
from *inj-of-nat* **have** *inj* (λ*n* (*x*::′*a*). *of-nat n* :: ′*b*)
by (*rule inj-fun*)
then have *inj* (λ*n*. *of-nat n* :: ′*a* ⇒ ′*b*)
by (*simp add*: *of-nat-fun*)
then show *inj* (*of-nat* :: *nat* ⇒ ′*a* ⇒ ′*b*) **.**
qed

instance *fun* :: (*type*, *ring*) *ring* **..**

instance *fun* :: (*type*, *comm-ring*) *comm-ring* **..**

instance *fun* :: (*type*, *ring-1*) *ring-1* **..**

instance *fun* :: (*type*, *comm-ring-1*) *comm-ring-1* **..**

instance *fun* :: (*type*, *ring-char-0*) *ring-char-0* **..**

Ordered structures

instance *fun* :: (*type*, *ordered-ab-semigroup-add*) *ordered-ab-semigroup-add*
by *standard* (*auto simp add*: *le-fun-def intro*: *add-left-mono*)

instance *fun* :: (*type*, *ordered-cancel-ab-semigroup-add*) *ordered-cancel-ab-semigroup-add*
..

instance *fun* :: (*type*, *ordered-ab-semigroup-add-imp-le*) *ordered-ab-semigroup-add-imp-le*
by *standard* (*simp add*: *le-fun-def*)

instance *fun* :: (*type*, *ordered-comm-monoid-add*) *ordered-comm-monoid-add* **..**

instance *fun* :: (*type*, *ordered-cancel-comm-monoid-add*) *ordered-cancel-comm-monoid-add*
..

instance *fun* :: (*type*, *ordered-ab-group-add*) *ordered-ab-group-add* **..**

instance *fun* :: (*type*, *ordered-semiring*) *ordered-semiring*
by *standard* (*auto simp add*: *le-fun-def intro*: *mult-left-mono mult-right-mono*)

instance *fun* :: (*type*, *dioid*) *dioid*
proof *standard*
fix *a b* :: *'a* ⇒ *'b*
show *a* ≤ *b* ⟷ (∃ *c*. *b* = *a* + *c*)
unfolding *le-fun-def plus-fun-def fun-eq-iff choice-iff*[*symmetric*, *of* λ*x c*. *b x* = *a x* + *c*]
by (*intro arg-cong*[**where** *f*=*All*] *ext canonically-ordered-monoid-add-class.le-iff-add*)
qed

instance *fun* :: (*type*, *ordered-comm-semiring*) *ordered-comm-semiring*
by *standard* (*fact mult-left-mono*)

instance *fun* :: (*type*, *ordered-cancel-semiring*) *ordered-cancel-semiring* **..**

instance *fun* :: (*type*, *ordered-cancel-comm-semiring*) *ordered-cancel-comm-semiring*
..

instance *fun* :: (*type*, *ordered-ring*) *ordered-ring* **..**

instance *fun* :: (*type*, *ordered-comm-ring*) *ordered-comm-ring* **..**

lemmas *func-plus* = *plus-fun-def*
lemmas *func-zero* = *zero-fun-def*
lemmas *func-times* = *times-fun-def*
lemmas *func-one* = *one-fun-def*

end

4 Algebraic operations on sets

theory *Set-Algebras*

imports *Main*
begin

This library lifts operations like addition and multiplication to sets. It was designed to support asymptotic calculations. See the comments at the top of `BigO.thy`.

instantiation *set* :: (*plus*) *plus*
begin

definition *plus-set* :: *'a*::*plus set* ⇒ *'a set* ⇒ *'a set*
where *set-plus-def*: $A + B = \{c.\ \exists a{\in}A.\ \exists b{\in}B.\ c = a + b\}$

instance ..

end

instantiation *set* :: (*times*) *times*
begin

definition *times-set* :: *'a*::*times set* ⇒ *'a set* ⇒ *'a set*
where *set-times-def*: $A * B = \{c.\ \exists a{\in}A.\ \exists b{\in}B.\ c = a * b\}$

instance ..

end

instantiation *set* :: (*zero*) *zero*
begin

definition *set-zero*[*simp*]: (*0*::*'a*::*zero set*) = {*0*}

instance ..

end

instantiation *set* :: (*one*) *one*
begin

definition *set-one*[*simp*]: (*1*::*'a*::*one set*) = {*1*}

instance ..

end

definition *elt-set-plus* :: *'a*::*plus* ⇒ *'a set* ⇒ *'a set* (**infixl** *+o 70*)
where $a +o B = \{c.\ \exists b{\in}B.\ c = a + b\}$

definition *elt-set-times* :: *'a*::*times* ⇒ *'a set* ⇒ *'a set* (**infixl** *∗o 80*)
where $a *o B = \{c.\ \exists b{\in}B.\ c = a * b\}$

abbreviation (*input*) *elt-set-eq* :: *'a* ⇒ *'a set* ⇒ *bool* (**infix** =*o* *50*)
where *x* =*o* *A* ≡ *x* ∈ *A*

instance *set* :: (*semigroup-add*) *semigroup-add*
by *standard* (*force simp add*: *set-plus-def add.assoc*)

instance *set* :: (*ab-semigroup-add*) *ab-semigroup-add*
by *standard* (*force simp add*: *set-plus-def add.commute*)

instance *set* :: (*monoid-add*) *monoid-add*
by *standard* (*simp-all add*: *set-plus-def*)

instance *set* :: (*comm-monoid-add*) *comm-monoid-add*
by *standard* (*simp-all add*: *set-plus-def*)

instance *set* :: (*semigroup-mult*) *semigroup-mult*
by *standard* (*force simp add*: *set-times-def mult.assoc*)

instance *set* :: (*ab-semigroup-mult*) *ab-semigroup-mult*
by *standard* (*force simp add*: *set-times-def mult.commute*)

instance *set* :: (*monoid-mult*) *monoid-mult*
by *standard* (*simp-all add*: *set-times-def*)

instance *set* :: (*comm-monoid-mult*) *comm-monoid-mult*
by *standard* (*simp-all add*: *set-times-def*)

lemma *set-plus-intro* [*intro*]: *a* ∈ *C* ⟹ *b* ∈ *D* ⟹ *a* + *b* ∈ *C* + *D*
by (*auto simp add*: *set-plus-def*)

lemma *set-plus-elim*:
assumes *x* ∈ *A* + *B*
obtains *a b* **where** *x* = *a* + *b* **and** *a* ∈ *A* **and** *b* ∈ *B*
using *assms* **unfolding** *set-plus-def* **by** *fast*

lemma *set-plus-intro2* [*intro*]: *b* ∈ *C* ⟹ *a* + *b* ∈ *a* +*o* *C*
by (*auto simp add*: *elt-set-plus-def*)

lemma *set-plus-rearrange*: (*a* +*o* *C*) + (*b* +*o* *D*) = (*a* + *b*) +*o* (*C* + *D*)
for *a b* :: *'a*::*comm-monoid-add*
apply (*auto simp add*: *elt-set-plus-def set-plus-def ac-simps*)
apply (*rule-tac x* = *ba* + *bb* **in** *exI*)
apply (*auto simp add*: *ac-simps*)
apply (*rule-tac x* = *aa* + *a* **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-plus-rearrange2*: *a* +*o* (*b* +*o* *C*) = (*a* + *b*) +*o* *C*

for $a\ b :: 'a$::*semigroup-add*
by (*auto simp add*: *elt-set-plus-def add.assoc*)

lemma *set-plus-rearrange3*: $(a +o B) + C = a +o (B + C)$
for $a :: 'a$::*semigroup-add*
apply (*auto simp add*: *elt-set-plus-def set-plus-def*)
apply (*blast intro*: *ac-simps*)
apply (*rule-tac* $x = a + aa$ **in** *exI*)
apply (*rule conjI*)
apply (*rule-tac* $x = aa$ **in** *bexI*)
apply *auto*
apply (*rule-tac* $x = ba$ **in** *bexI*)
apply (*auto simp add*: *ac-simps*)
done

theorem *set-plus-rearrange4*: $C + (a +o D) = a +o (C + D)$
for $a :: 'a$::*comm-monoid-add*
apply (*auto simp add*: *elt-set-plus-def set-plus-def ac-simps*)
apply (*rule-tac* $x = aa + ba$ **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemmas *set-plus-rearranges* = *set-plus-rearrange set-plus-rearrange2 set-plus-rearrange3 set-plus-rearrange4*

lemma *set-plus-mono* [*intro*!]: $C \subseteq D \Longrightarrow a +o C \subseteq a +o D$
by (*auto simp add*: *elt-set-plus-def*)

lemma *set-plus-mono2* [*intro*]: $C \subseteq D \Longrightarrow E \subseteq F \Longrightarrow C + E \subseteq D + F$
for $C\ D\ E\ F :: 'a$::*plus set*
by (*auto simp add*: *set-plus-def*)

lemma *set-plus-mono3* [*intro*]: $a \in C \Longrightarrow a +o D \subseteq C + D$
by (*auto simp add*: *elt-set-plus-def set-plus-def*)

lemma *set-plus-mono4* [*intro*]: $a \in C \Longrightarrow a +o D \subseteq D + C$
for $a :: 'a$::*comm-monoid-add*
by (*auto simp add*: *elt-set-plus-def set-plus-def ac-simps*)

lemma *set-plus-mono5*: $a \in C \Longrightarrow B \subseteq D \Longrightarrow a +o B \subseteq C + D$
apply (*subgoal-tac* $a +o B \subseteq a +o D$)
apply (*erule order-trans*)
apply (*erule set-plus-mono3*)
apply (*erule set-plus-mono*)
done

lemma *set-plus-mono-b*: $C \subseteq D \Longrightarrow x \in a +o C \Longrightarrow x \in a +o D$
apply (*frule set-plus-mono*)
apply *auto*

done

lemma *set-plus-mono2-b*: $C \subseteq D \Longrightarrow E \subseteq F \Longrightarrow x \in C + E \Longrightarrow x \in D + F$
apply (*frule set-plus-mono2*)
prefer *2*
apply *force*
apply *assumption*
done

lemma *set-plus-mono3-b*: $a \in C \Longrightarrow x \in a +o D \Longrightarrow x \in C + D$
apply (*frule set-plus-mono3*)
apply *auto*
done

lemma *set-plus-mono4-b*: $a \in C \Longrightarrow x \in a +o D \Longrightarrow x \in D + C$
for *a x* :: *'a*::*comm-monoid-add*
apply (*frule set-plus-mono4*)
apply *auto*
done

lemma *set-zero-plus* [*simp*]: $0 +o C = C$
for *C* :: *'a*::*comm-monoid-add set*
by (*auto simp add*: *elt-set-plus-def*)

lemma *set-zero-plus2*: $0 \in A \Longrightarrow B \subseteq A + B$
for *A B* :: *'a*::*comm-monoid-add set*
apply (*auto simp add*: *set-plus-def*)
apply (*rule-tac x* = *0* **in** *bexI*)
apply (*rule-tac x* = *x* **in** *bexI*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-plus-imp-minus*: $a \in b +o C \Longrightarrow a - b \in C$
for *a b* :: *'a*::*ab-group-add*
by (*auto simp add*: *elt-set-plus-def ac-simps*)

lemma *set-minus-imp-plus*: $a - b \in C \Longrightarrow a \in b +o C$
for *a b* :: *'a*::*ab-group-add*
apply (*auto simp add*: *elt-set-plus-def ac-simps*)
apply (*subgoal-tac a* = (*a* + − *b*) + *b*)
apply (*rule bexI*)
apply *assumption*
apply (*auto simp add*: *ac-simps*)
done

lemma *set-minus-plus*: $a - b \in C \longleftrightarrow a \in b +o C$
for *a b* :: *'a*::*ab-group-add*
apply (*rule iffI*)
apply (*rule set-minus-imp-plus*)

apply *assumption*
apply (*rule set-plus-imp-minus*)
apply *assumption*
done

lemma *set-times-intro* [*intro*]: $a \in C \Longrightarrow b \in D \Longrightarrow a * b \in C * D$
by (*auto simp add*: *set-times-def*)

lemma *set-times-elim*:
assumes $x \in A * B$
obtains *a b* **where** $x = a * b$ **and** $a \in A$ **and** $b \in B$
using *assms* **unfolding** *set-times-def* **by** *fast*

lemma *set-times-intro2* [*intro*!]: $b \in C \Longrightarrow a * b \in a *o C$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-times-rearrange*: $(a *o C) * (b *o D) = (a * b) *o (C * D)$
for *a b* :: *'a*::*comm-monoid-mult*
apply (*auto simp add*: *elt-set-times-def set-times-def*)
apply (*rule-tac x* = *ba* ∗ *bb* **in** *exI*)
apply (*auto simp add*: *ac-simps*)
apply (*rule-tac x* = *aa* ∗ *a* **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemma *set-times-rearrange2*: $a *o (b *o C) = (a * b) *o C$
for *a b* :: *'a*::*semigroup-mult*
by (*auto simp add*: *elt-set-times-def mult.assoc*)

lemma *set-times-rearrange3*: $(a *o B) * C = a *o (B * C)$
for *a* :: *'a*::*semigroup-mult*
apply (*auto simp add*: *elt-set-times-def set-times-def*)
apply (*blast intro*: *ac-simps*)
apply (*rule-tac x* = *a* ∗ *aa* **in** *exI*)
apply (*rule conjI*)
apply (*rule-tac x* = *aa* **in** *bexI*)
apply *auto*
apply (*rule-tac x* = *ba* **in** *bexI*)
apply (*auto simp add*: *ac-simps*)
done

theorem *set-times-rearrange4*: $C * (a *o D) = a *o (C * D)$
for *a* :: *'a*::*comm-monoid-mult*
apply (*auto simp add*: *elt-set-times-def set-times-def ac-simps*)
apply (*rule-tac x* = *aa* ∗ *ba* **in** *exI*)
apply (*auto simp add*: *ac-simps*)
done

lemmas *set-times-rearranges* = *set-times-rearrange set-times-rearrange2*

set-times-rearrange3 set-times-rearrange4

lemma *set-times-mono* [*intro*]: $C \subseteq D \Longrightarrow a *o C \subseteq a *o D$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-times-mono2* [*intro*]: $C \subseteq D \Longrightarrow E \subseteq F \Longrightarrow C * E \subseteq D * F$
for *C D E F* :: *'a*::*times set*
by (*auto simp add*: *set-times-def*)

lemma *set-times-mono3* [*intro*]: $a \in C \Longrightarrow a *o D \subseteq C * D$
by (*auto simp add*: *elt-set-times-def set-times-def*)

lemma *set-times-mono4* [*intro*]: $a \in C \Longrightarrow a *o D \subseteq D * C$
for *a* :: *'a*::*comm-monoid-mult*
by (*auto simp add*: *elt-set-times-def set-times-def ac-simps*)

lemma *set-times-mono5*: $a \in C \Longrightarrow B \subseteq D \Longrightarrow a *o B \subseteq C * D$
apply (*subgoal-tac* $a *o B \subseteq a *o D$)
apply (*erule order-trans*)
apply (*erule set-times-mono3*)
apply (*erule set-times-mono*)
done

lemma *set-times-mono-b*: $C \subseteq D \Longrightarrow x \in a *o C \Longrightarrow x \in a *o D$
apply (*frule set-times-mono*)
apply *auto*
done

lemma *set-times-mono2-b*: $C \subseteq D \Longrightarrow E \subseteq F \Longrightarrow x \in C * E \Longrightarrow x \in D * F$
apply (*frule set-times-mono2*)
prefer *2*
apply *force*
apply *assumption*
done

lemma *set-times-mono3-b*: $a \in C \Longrightarrow x \in a *o D \Longrightarrow x \in C * D$
apply (*frule set-times-mono3*)
apply *auto*
done

lemma *set-times-mono4-b*: $a \in C \Longrightarrow x \in a *o D \Longrightarrow x \in D * C$
for *a x* :: *'a*::*comm-monoid-mult*
apply (*frule set-times-mono4*)
apply *auto*
done

lemma *set-one-times* [*simp*]: $1 *o C = C$
for *C* :: *'a*::*comm-monoid-mult set*
by (*auto simp add*: *elt-set-times-def*)

lemma *set-times-plus-distrib*: $a *o (b +o C) = (a * b) +o (a *o C)$
for $a\ b :: 'a::semiring$
by (*auto simp add*: *elt-set-plus-def elt-set-times-def ring-distribs*)

lemma *set-times-plus-distrib2*: $a *o (B + C) = (a *o B) + (a *o C)$
for $a :: 'a::semiring$
apply (*auto simp add*: *set-plus-def elt-set-times-def ring-distribs*)
apply *blast*
apply (*rule-tac* $x = b + bb$ **in** *exI*)
apply (*auto simp add*: *ring-distribs*)
done

lemma *set-times-plus-distrib3*: $(a +o C) * D \subseteq a *o D + C * D$
for $a :: 'a::semiring$
apply (*auto simp*: *elt-set-plus-def elt-set-times-def set-times-def set-plus-def ring-distribs*)
apply *auto*
done

lemmas *set-times-plus-distribs* =
set-times-plus-distrib
set-times-plus-distrib2

lemma *set-neg-intro*: $a \in (-1) *o C \Longrightarrow -a \in C$
for $a :: 'a::ring-1$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-neg-intro2*: $a \in C \Longrightarrow -a \in (-1) *o C$
for $a :: 'a::ring-1$
by (*auto simp add*: *elt-set-times-def*)

lemma *set-plus-image*: $S + T = (\lambda(x, y).\ x + y)\ `\ (S \times T)$
by (*fastforce simp*: *set-plus-def image-iff*)

lemma *set-times-image*: $S * T = (\lambda(x, y).\ x * y)\ `\ (S \times T)$
by (*fastforce simp*: *set-times-def image-iff*)

lemma *finite-set-plus*: *finite* $s \Longrightarrow$ *finite* $t \Longrightarrow$ *finite* $(s + t)$
by (*simp add*: *set-plus-image*)

lemma *finite-set-times*: *finite* $s \Longrightarrow$ *finite* $t \Longrightarrow$ *finite* $(s * t)$
by (*simp add*: *set-times-image*)

lemma *set-sum-alt*:
assumes *fin*: *finite* I
shows *sum* $S\ I = \{sum\ s\ I\ |s.\ \forall i{\in}I.\ s\ i \in S\ i\}$
(**is** - = *?sum* I)
using *fin*
proof *induct*

```
  case empty
  then show ?case by simp
next
  case (insert x F)
  have sum S (insert x F) = S x + ?sum F
    using insert.hyps by auto
  also have ... = {s x + sum s F |s. ∀ i∈insert x F. s i ∈ S i}
    unfolding set-plus-def
  proof safe
    fix y s
    assume y ∈ S x ∀ i∈F. s i ∈ S i
    then show ∃ s'. y + sum s F = s' x + sum s' F ∧ (∀ i∈insert x F. s' i ∈ S i)
      using insert.hyps
      by (intro exI[of - λi. if i ∈ F then s i else y]) (auto simp add: set-plus-def)
  qed auto
  finally show ?case
    using insert.hyps by auto
qed

lemma sum-set-cond-linear:
  fixes f :: 'a::comm-monoid-add set ⇒ 'b::comm-monoid-add set
  assumes [intro!]: ⋀A B. P A ⟹ P B ⟹ P (A + B) P {0}
    and f: ⋀A B. P A ⟹ P B ⟹ f (A + B) = f A + f B f {0} = {0}
  assumes all: ⋀i. i ∈ I ⟹ P (S i)
  shows f (sum S I) = sum (f ∘ S) I
proof (cases finite I)
  case True
  from this all show ?thesis
  proof induct
    case empty
    then show ?case by (auto intro!: f)
  next
    case (insert x F)
    from ‹finite F› ‹⋀i. i ∈ insert x F ⟹ P (S i)› have P (sum S F)
      by induct auto
    with insert show ?case
      by (simp, subst f) auto
  qed
next
  case False
  then show ?thesis by (auto intro!: f)
qed

lemma sum-set-linear:
  fixes f :: 'a::comm-monoid-add set ⇒ 'b::comm-monoid-add set
  assumes ⋀A B. f(A) + f(B) = f(A + B) f {0} = {0}
  shows f (sum S I) = sum (f ∘ S) I
  using sum-set-cond-linear[of λx. True f I S] assms by auto
```

lemma *set-times-Un-distrib*:
$A * (B \cup C) = A * B \cup A * C$
$(A \cup B) * C = A * C \cup B * C$
by (*auto simp*: *set-times-def*)

lemma *set-times-UNION-distrib*:
$A * \bigcup(M \text{ ` } I) = (\bigcup i{\in}I.\ A * M\ i)$
$\bigcup(M \text{ ` } I) * A = (\bigcup i{\in}I.\ M\ i * A)$
by (*auto simp*: *set-times-def*)

end

5 Big O notation

theory *BigO*
imports
Complex-Main
Function-Algebras
Set-Algebras
begin

This library is designed to support asymptotic "big O" calculations, i.e. reasoning with expressions of the form $f = O(g)$ and $f = g + O(h)$. An earlier version of this library is described in detail in [1].

The main changes in this version are as follows:

- We have eliminated the O operator on sets. (Most uses of this seem to be inessential.)
- We no longer use + as output syntax for $+o$
- Lemmas involving *sumr* have been replaced by more general lemmas involving '*sum*.
- The library has been expanded, with e.g. support for expressions of the form $f < g + O(h)$.

Note also since the Big O library includes rules that demonstrate set inclusion, to use the automated reasoners effectively with the library one should redeclare the theorem *subsetI* as an intro rule, rather than as an *intro*! rule, for example, using **declare** *subsetI* [*del*, *intro*].

5.1 Definitions

definition *bigo* :: $('a \Rightarrow 'b{::}$*linordered-idom*$) \Rightarrow ('a \Rightarrow 'b)$ *set* (($1O'(\text{-}')$))
where $O(f{::}\ 'a \Rightarrow 'b) = \{h.\ \exists c.\ \forall x.\ |h\ x| \leq c * |f\ x|\}$

lemma *bigo-pos-const*:

```
  (∃ c::'a::linordered-idom. ∀ x. |h x| ≤ c * |f x|) ⟷
    (∃ c. 0 < c ∧ (∀ x. |h x| ≤ c * |f x|))
  apply auto
  apply (case-tac c = 0)
   apply simp
   apply (rule-tac x = 1 in exI)
   apply simp
  apply (rule-tac x = |c| in exI)
  apply auto
  apply (subgoal-tac c * |f x| ≤ |c| * |f x|)
   apply (erule-tac x = x in allE)
   apply force
  apply (rule mult-right-mono)
   apply (rule abs-ge-self)
  apply (rule abs-ge-zero)
  done

lemma bigo-alt-def: O(f) = {h. ∃ c. 0 < c ∧ (∀ x. |h x| ≤ c * |f x|)}
  by (auto simp add: bigo-def bigo-pos-const)

lemma bigo-elt-subset [intro]: f ∈ O(g) ⟹ O(f) ≤ O(g)
  apply (auto simp add: bigo-alt-def)
  apply (rule-tac x = ca * c in exI)
  apply (rule conjI)
   apply simp
  apply (rule allI)
  apply (drule-tac x = xa in spec)+
  apply (subgoal-tac ca * |f xa| ≤ ca * (c * |g xa|))
   apply (erule order-trans)
   apply (simp add: ac-simps)
  apply (rule mult-left-mono, assumption)
  apply (rule order-less-imp-le, assumption)
  done

lemma bigo-refl [intro]: f ∈ O(f)
  apply (auto simp add: bigo-def)
  apply (rule-tac x = 1 in exI)
  apply simp
  done

lemma bigo-zero: 0 ∈ O(g)
  apply (auto simp add: bigo-def func-zero)
  apply (rule-tac x = 0 in exI)
  apply auto
  done

lemma bigo-zero2: O(λx. 0) = {λx. 0}
  by (auto simp add: bigo-def)
```

lemma *bigo-plus-self-subset* [*intro*]: $O(f) + O(f) \subseteq O(f)$
apply (*auto simp add*: *bigo-alt-def set-plus-def*)
apply (*rule-tac* $x = c + ca$ **in** *exI*)
apply *auto*
apply (*simp add*: *ring-distribs func-plus*)
apply (*rule order-trans*)
apply (*rule abs-triangle-ineq*)
apply (*rule add-mono*)
apply *force*
apply *force*
done

lemma *bigo-plus-idemp* [*simp*]: $O(f) + O(f) = O(f)$
apply (*rule equalityI*)
apply (*rule bigo-plus-self-subset*)
apply (*rule set-zero-plus2*)
apply (*rule bigo-zero*)
done

lemma *bigo-plus-subset* [*intro*]: $O(f + g) \subseteq O(f) + O(g)$
apply (*rule subsetI*)
apply (*auto simp add*: *bigo-def bigo-pos-const func-plus set-plus-def*)
apply (*subst bigo-pos-const* [*symmetric*])+
apply (*rule-tac* $x = \lambda n.$ *if* $|g\ n| \leq |f\ n|$ *then* $x\ n$ *else* 0 **in** *exI*)
apply (*rule conjI*)
apply (*rule-tac* $x = c + c$ **in** *exI*)
apply (*clarsimp*)
apply (*subgoal-tac* $c * |f\ xa + g\ xa| \leq (c + c) * |f\ xa|$)
apply (*erule-tac* $x = xa$ **in** *allE*)
apply (*erule order-trans*)
apply (*simp*)
apply (*subgoal-tac* $c * |f\ xa + g\ xa| \leq c * (|f\ xa| + |g\ xa|)$)
apply (*erule order-trans*)
apply (*simp add*: *ring-distribs*)
apply (*rule mult-left-mono*)
apply (*simp add*: *abs-triangle-ineq*)
apply (*simp add*: *order-less-le*)
apply (*rule-tac* $x = \lambda n.$ *if* $|f\ n| < |g\ n|$ *then* $x\ n$ *else* 0 **in** *exI*)
apply (*rule conjI*)
apply (*rule-tac* $x = c + c$ **in** *exI*)
apply *auto*
apply (*subgoal-tac* $c * |f\ xa + g\ xa| \leq (c + c) * |g\ xa|$)
apply (*erule-tac* $x = xa$ **in** *allE*)
apply (*erule order-trans*)
apply *simp*
apply (*subgoal-tac* $c * |f\ xa + g\ xa| \leq c * (|f\ xa| + |g\ xa|)$)
apply (*erule order-trans*)
apply (*simp add*: *ring-distribs*)
apply (*rule mult-left-mono*)

apply (*rule abs-triangle-ineq*)
apply (*simp add*: *order-less-le*)
done

lemma *bigo-plus-subset2* [*intro*]: $A \subseteq O(f) \Longrightarrow B \subseteq O(f) \Longrightarrow A + B \subseteq O(f)$
apply (*subgoal-tac* $A + B \subseteq O(f) + O(f)$)
apply (*erule order-trans*)
apply *simp*
apply (*auto del*: *subsetI simp del*: *bigo-plus-idemp*)
done

lemma *bigo-plus-eq*: $\forall x.\ 0 \leq f\ x \Longrightarrow \forall x.\ 0 \leq g\ x \Longrightarrow O(f + g) = O(f) + O(g)$
apply (*rule equalityI*)
apply (*rule bigo-plus-subset*)
apply (*simp add*: *bigo-alt-def set-plus-def func-plus*)
apply *clarify*
apply (*rule-tac* $x = max\ c\ ca$ **in** *exI*)
apply (*rule conjI*)
apply (*subgoal-tac* $c \leq max\ c\ ca$)
apply (*erule order-less-le-trans*)
apply *assumption*
apply (*rule max.cobounded1*)
apply *clarify*
apply (*drule-tac* $x = xa$ **in** *spec*)+
apply (*subgoal-tac* $0 \leq f\ xa + g\ xa$)
apply (*simp add*: *ring-distribs*)
apply (*subgoal-tac* $|a\ xa + b\ xa| \leq |a\ xa| + |b\ xa|$)
apply (*subgoal-tac* $|a\ xa| + |b\ xa| \leq max\ c\ ca * f\ xa + max\ c\ ca * g\ xa$)
apply *force*
apply (*rule add-mono*)
apply (*subgoal-tac* $c * f\ xa \leq max\ c\ ca * f\ xa$)
apply *force*
apply (*rule mult-right-mono*)
apply (*rule max.cobounded1*)
apply *assumption*
apply (*subgoal-tac* $ca * g\ xa \leq max\ c\ ca * g\ xa$)
apply *force*
apply (*rule mult-right-mono*)
apply (*rule max.cobounded2*)
apply *assumption*
apply (*rule abs-triangle-ineq*)
apply (*rule add-nonneg-nonneg*)
apply *assumption*+
done

lemma *bigo-bounded-alt*: $\forall x.\ 0 \leq f\ x \Longrightarrow \forall x.\ f\ x \leq c * g\ x \Longrightarrow f \in O(g)$
apply (*auto simp add*: *bigo-def*)
apply (*rule-tac* $x = |c|$ **in** *exI*)
apply *auto*

apply (*drule-tac* $x = x$ **in** *spec*)+
apply (*simp flip*: *abs-mult*)
done

lemma *bigo-bounded*: $\forall x.\ 0 \leq f\ x \Longrightarrow \forall x.\ f\ x \leq g\ x \Longrightarrow f \in O(g)$
apply (*erule bigo-bounded-alt* [*of f 1 g*])
apply *simp*
done

lemma *bigo-bounded2*: $\forall x.\ lb\ x \leq f\ x \Longrightarrow \forall x.\ f\ x \leq lb\ x + g\ x \Longrightarrow f \in lb +o O(g)$
apply (*rule set-minus-imp-plus*)
apply (*rule bigo-bounded*)
apply (*auto simp add*: *fun-Compl-def func-plus*)
apply (*drule-tac* $x = x$ **in** *spec*)+
apply *force*
done

lemma *bigo-abs*: $(\lambda x.\ |f\ x|) =o\ O(f)$
apply (*unfold bigo-def*)
apply *auto*
apply (*rule-tac* $x = 1$ **in** *exI*)
apply *auto*
done

lemma *bigo-abs2*: $f =o\ O(\lambda x.\ |f\ x|)$
apply (*unfold bigo-def*)
apply *auto*
apply (*rule-tac* $x = 1$ **in** *exI*)
apply *auto*
done

lemma *bigo-abs3*: $O(f) = O(\lambda x.\ |f\ x|)$
apply (*rule equalityI*)
apply (*rule bigo-elt-subset*)
apply (*rule bigo-abs2*)
apply (*rule bigo-elt-subset*)
apply (*rule bigo-abs*)
done

lemma *bigo-abs4*: $f =o\ g +o\ O(h) \Longrightarrow (\lambda x.\ |f\ x|) =o\ (\lambda x.\ |g\ x|) +o\ O(h)$
apply (*drule set-plus-imp-minus*)
apply (*rule set-minus-imp-plus*)
apply (*subst fun-diff-def*)
proof −
assume ∗: $f - g \in O(h)$
have $(\lambda x.\ |f\ x| - |g\ x|) =o\ O(\lambda x.\ ||f\ x| - |g\ x||)$
by (*rule bigo-abs2*)
also have $\ldots \subseteq O(\lambda x.\ |f\ x - g\ x|)$

```
    apply (rule bigo-elt-subset)
    apply (rule bigo-bounded)
     apply force
    apply (rule allI)
    apply (rule abs-triangle-ineq3)
    done
  also have ... ⊆ O(f − g)
    apply (rule bigo-elt-subset)
    apply (subst fun-diff-def)
    apply (rule bigo-abs)
    done
  also from * have ... ⊆ O(h)
    by (rule bigo-elt-subset)
  finally show (λx. |f x| − |g x|) ∈ O(h).
qed

lemma bigo-abs5: f =o O(g) ⟹ (λx. |f x|) =o O(g)
  by (auto simp: bigo-def)

lemma bigo-elt-subset2 [intro]:
  assumes *: f ∈ g +o O(h)
  shows O(f) ⊆ O(g) + O(h)
proof −
  note *
  also have g +o O(h) ⊆ O(g) + O(h)
    by (auto del: subsetI)
  also have ... = O(λx. |g x|) + O(λx. |h x|)
    by (subst bigo-abs3 [symmetric])+ (rule refl)
  also have ... = O((λx. |g x|) + (λx. |h x|))
    by (rule bigo-plus-eq [symmetric]) auto
  finally have f ∈ ... .
  then have O(f) ⊆ ...
    by (elim bigo-elt-subset)
  also have ... = O(λx. |g x|) + O(λx. |h x|)
    by (rule bigo-plus-eq, auto)
  finally show ?thesis
    by (simp flip: bigo-abs3)
qed

lemma bigo-mult [intro]: O(f)*O(g) ⊆ O(f * g)
  apply (rule subsetI)
  apply (subst bigo-def)
  apply (auto simp add: bigo-alt-def set-times-def func-times)
  apply (rule-tac x = c * ca in exI)
  apply (rule allI)
  apply (erule-tac x = x in allE)+
  apply (subgoal-tac c * ca * |f x * g x| = (c * |f x|) * (ca * |g x|))
   apply (erule ssubst)
   apply (subst abs-mult)
```

```
   apply (rule mult-mono)
     apply assumption+
    apply auto
  apply (simp add: ac-simps abs-mult)
  done

lemma bigo-mult2 [intro]: f *o O(g) ⊆ O(f * g)
  apply (auto simp add: bigo-def elt-set-times-def func-times abs-mult)
  apply (rule-tac x = c in exI)
  apply auto
  apply (drule-tac x = x in spec)
  apply (subgoal-tac |f x| * |b x| ≤ |f x| * (c * |g x|))
   apply (force simp add: ac-simps)
  apply (rule mult-left-mono, assumption)
  apply (rule abs-ge-zero)
  done

lemma bigo-mult3: f ∈ O(h) ⟹ g ∈ O(j) ⟹ f * g ∈ O(h * j)
  apply (rule subsetD)
   apply (rule bigo-mult)
  apply (erule set-times-intro, assumption)
  done

lemma bigo-mult4 [intro]: f ∈ k +o O(h) ⟹ g * f ∈ (g * k) +o O(g * h)
  apply (drule set-plus-imp-minus)
  apply (rule set-minus-imp-plus)
  apply (drule bigo-mult3 [where g = g and j = g])
   apply (auto simp add: algebra-simps)
  done

lemma bigo-mult5:
  fixes f :: 'a ⇒ 'b::linordered-field
  assumes ∀ x. f x ≠ 0
  shows O(f * g) ⊆ f *o O(g)
proof
  fix h
  assume h ∈ O(f * g)
  then have (λx. 1 / (f x)) * h ∈ (λx. 1 / f x) *o O(f * g)
    by auto
  also have ... ⊆ O((λx. 1 / f x) * (f * g))
    by (rule bigo-mult2)
  also have (λx. 1 / f x) * (f * g) = g
    apply (simp add: func-times)
    apply (rule ext)
    apply (simp add: assms nonzero-divide-eq-eq ac-simps)
    done
  finally have (λx. (1::'b) / f x) * h ∈ O(g) .
  then have f * ((λx. (1::'b) / f x) * h) ∈ f *o O(g)
    by auto
```

also have $f * ((\lambda x.\ (1::'b) \ /\ f\ x) * h) = h$
apply (*simp add*: *func-times*)
apply (*rule ext*)
apply (*simp add*: *assms nonzero-divide-eq-eq ac-simps*)
done
finally show $h \in f *o O(g)$ **.**
qed

lemma *bigo-mult6*: $\forall x.\ f\ x \neq 0 \Longrightarrow O(f * g) = f *o O(g)$
for $f :: 'a \Rightarrow 'b$::*linordered-field*
apply (*rule equalityI*)
apply (*erule bigo-mult5*)
apply (*rule bigo-mult2*)
done

lemma *bigo-mult7*: $\forall x.\ f\ x \neq 0 \Longrightarrow O(f * g) \subseteq O(f) * O(g)$
for $f :: 'a \Rightarrow 'b$::*linordered-field*
apply (*subst bigo-mult6*)
apply *assumption*
apply (*rule set-times-mono3*)
apply (*rule bigo-refl*)
done

lemma *bigo-mult8*: $\forall x.\ f\ x \neq 0 \Longrightarrow O(f * g) = O(f) * O(g)$
for $f :: 'a \Rightarrow 'b$::*linordered-field*
apply (*rule equalityI*)
apply (*erule bigo-mult7*)
apply (*rule bigo-mult*)
done

lemma *bigo-minus* [*intro*]: $f \in O(g) \Longrightarrow - f \in O(g)$
by (*auto simp add*: *bigo-def fun-Compl-def*)

lemma *bigo-minus2*: $f \in g +o O(h) \Longrightarrow - f \in -g +o O(h)$
apply (*rule set-minus-imp-plus*)
apply (*drule set-plus-imp-minus*)
apply (*drule bigo-minus*)
apply *simp*
done

lemma *bigo-minus3*: $O(- f) = O(f)$
by (*auto simp add*: *bigo-def fun-Compl-def*)

lemma *bigo-plus-absorb-lemma1*:
assumes *: $f \in O(g)$
shows $f +o O(g) \subseteq O(g)$
proof −
have $f \in O(f)$ **by** *auto*
then have $f +o O(g) \subseteq O(f) + O(g)$

```
    by (auto del: subsetI)
  also have ... ⊆ O(g) + O(g)
  proof −
    from ∗ have O(f) ⊆ O(g)
      by (auto del: subsetI)
    then show ?thesis
      by (auto del: subsetI)
  qed
  also have ... ⊆ O(g) by simp
  finally show ?thesis .
qed

lemma bigo-plus-absorb-lemma2:
  assumes ∗: f ∈ O(g)
  shows O(g) ⊆ f +o O(g)
proof −
  from ∗ have − f ∈ O(g)
    by auto
  then have − f +o O(g) ⊆ O(g)
    by (elim bigo-plus-absorb-lemma1)
  then have f +o (− f +o O(g)) ⊆ f +o O(g)
    by auto
  also have f +o (− f +o O(g)) = O(g)
    by (simp add: set-plus-rearranges)
  finally show ?thesis .
qed

lemma bigo-plus-absorb [simp]: f ∈ O(g) ⟹ f +o O(g) = O(g)
  apply (rule equalityI)
   apply (erule bigo-plus-absorb-lemma1)
  apply (erule bigo-plus-absorb-lemma2)
  done

lemma bigo-plus-absorb2 [intro]: f ∈ O(g) ⟹ A ⊆ O(g) ⟹ f +o A ⊆ O(g)
  apply (subgoal-tac f +o A ⊆ f +o O(g))
   apply force+
  done

lemma bigo-add-commute-imp: f ∈ g +o O(h) ⟹ g ∈ f +o O(h)
  apply (subst set-minus-plus [symmetric])
  apply (subgoal-tac g − f = − (f − g))
   apply (erule ssubst)
   apply (rule bigo-minus)
   apply (subst set-minus-plus)
   apply assumption
  apply (simp add: ac-simps)
  done

lemma bigo-add-commute: f ∈ g +o O(h) ⟷ g ∈ f +o O(h)
```

apply (*rule iffI*)
apply (*erule bigo-add-commute-imp*)+
done

lemma *bigo-const1*: $(\lambda x.\ c) \in O(\lambda x.\ 1)$
by (*auto simp add*: *bigo-def ac-simps*)

lemma *bigo-const2* [*intro*]: $O(\lambda x.\ c) \subseteq O(\lambda x.\ 1)$
apply (*rule bigo-elt-subset*)
apply (*rule bigo-const1*)
done

lemma *bigo-const3*: $c \neq 0 \Longrightarrow (\lambda x.\ 1) \in O(\lambda x.\ c)$
for $c :: 'a$::*linordered-field*
apply (*simp add*: *bigo-def*)
apply (*rule-tac* $x = |inverse\ c|$ **in** *exI*)
apply (*simp flip*: *abs-mult*)
done

lemma *bigo-const4*: $c \neq 0 \Longrightarrow O(\lambda x.\ 1) \subseteq O(\lambda x.\ c)$
for $c :: 'a$::*linordered-field*
apply (*rule bigo-elt-subset*)
apply (*rule bigo-const3*)
apply *assumption*
done

lemma *bigo-const* [*simp*]: $c \neq 0 \Longrightarrow O(\lambda x.\ c) = O(\lambda x.\ 1)$
for $c :: 'a$::*linordered-field*
apply (*rule equalityI*)
apply (*rule bigo-const2*)
apply (*rule bigo-const4*)
apply *assumption*
done

lemma *bigo-const-mult1*: $(\lambda x.\ c * f\ x) \in O(f)$
apply (*simp add*: *bigo-def*)
apply (*rule-tac* $x = |c|$ **in** *exI*)
apply (*auto simp flip*: *abs-mult*)
done

lemma *bigo-const-mult2*: $O(\lambda x.\ c * f\ x) \subseteq O(f)$
apply (*rule bigo-elt-subset*)
apply (*rule bigo-const-mult1*)
done

lemma *bigo-const-mult3*: $c \neq 0 \Longrightarrow f \in O(\lambda x.\ c * f\ x)$
for $c :: 'a$::*linordered-field*
apply (*simp add*: *bigo-def*)
apply (*rule-tac* $x = |inverse\ c|$ **in** *exI*)

apply (*simp add*: *abs-mult mult.assoc* [*symmetric*])
done

lemma *bigo-const-mult4*: $c \neq 0 \Longrightarrow O(f) \subseteq O(\lambda x.\ c * f\ x)$
for $c :: 'a$::*linordered-field*
apply (*rule bigo-elt-subset*)
apply (*rule bigo-const-mult3*)
apply *assumption*
done

lemma *bigo-const-mult* [*simp*]: $c \neq 0 \Longrightarrow O(\lambda x.\ c * f\ x) = O(f)$
for $c :: 'a$::*linordered-field*
apply (*rule equalityI*)
apply (*rule bigo-const-mult2*)
apply (*erule bigo-const-mult4*)
done

lemma *bigo-const-mult5* [*simp*]: $c \neq 0 \Longrightarrow (\lambda x.\ c) *o\ O(f) = O(f)$
for $c :: 'a$::*linordered-field*
apply (*auto del*: *subsetI*)
apply (*rule order-trans*)
apply (*rule bigo-mult2*)
apply (*simp add*: *func-times*)
apply (*auto intro*!: *simp add*: *bigo-def elt-set-times-def func-times*)
apply (*rule-tac* $x = \lambda y.\ inverse\ c * x\ y$ **in** *exI*)
apply (*simp add*: *mult.assoc* [*symmetric*] *abs-mult*)
apply (*rule-tac* $x = |inverse\ c| * ca$ **in** *exI*)
apply *auto*
done

lemma *bigo-const-mult6* [*intro*]: $(\lambda x.\ c) *o\ O(f) \subseteq O(f)$
apply (*auto intro*!: *simp add*: *bigo-def elt-set-times-def func-times*)
apply (*rule-tac* $x = ca * |c|$ **in** *exI*)
apply (*rule allI*)
apply (*subgoal-tac* $ca * |c| * |f\ x| = |c| * (ca * |f\ x|)$)
apply (*erule ssubst*)
apply (*subst abs-mult*)
apply (*rule mult-left-mono*)
apply (*erule spec*)
apply *simp*
apply (*simp add*: *ac-simps*)
done

lemma *bigo-const-mult7* [*intro*]:
assumes *: $f =o\ O(g)$
shows $(\lambda x.\ c * f\ x) =o\ O(g)$
proof −
from * **have** $(\lambda x.\ c) * f =o\ (\lambda x.\ c) *o\ O(g)$
by *auto*

also have $(\lambda x.\ c) * f = (\lambda x.\ c * f\ x)$
by (*simp add*: *func-times*)
also have $(\lambda x.\ c) *o\ O(g) \subseteq O(g)$
by (*auto del*: *subsetI*)
finally show *?thesis* **.**
qed

lemma *bigo-compose1*: $f =o\ O(g) \Longrightarrow (\lambda x.\ f\ (k\ x)) =o\ O(\lambda x.\ g\ (k\ x))$
by (*auto simp*: *bigo-def*)

lemma *bigo-compose2*: $f =o\ g +o\ O(h) \Longrightarrow (\lambda x.\ f\ (k\ x)) =o\ (\lambda x.\ g\ (k\ x)) +o\ O(\lambda x.\ h(k\ x))$
apply (*simp only*: *set-minus-plus* [*symmetric*] *fun-Compl-def func-plus*)
apply (*drule bigo-compose1*)
apply (*simp add*: *fun-diff-def*)
done

5.2 Sum

lemma *bigo-sum-main*: $\forall x.\ \forall y \in A\ x.\ 0 \leq h\ x\ y \Longrightarrow$
$\exists c.\ \forall x.\ \forall y \in A\ x.\ |f\ x\ y| \leq c * h\ x\ y \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ f\ x\ y) =o\ O(\lambda x.\ \sum y \in A\ x.\ h\ x\ y)$
apply (*auto simp add*: *bigo-def*)
apply (*rule-tac* $x = |c|$ **in** *exI*)
apply (*subst abs-of-nonneg*) **back back**
apply (*rule sum-nonneg*)
apply *force*
apply (*subst sum-distrib-left*)
apply (*rule allI*)
apply (*rule order-trans*)
apply (*rule sum-abs*)
apply (*rule sum-mono*)
apply (*rule order-trans*)
apply (*drule spec*)+
apply (*drule bspec*)+
apply *assumption*+
apply (*drule bspec*)
apply *assumption*+
apply (*rule mult-right-mono*)
apply (*rule abs-ge-self*)
apply *force*
done

lemma *bigo-sum1*: $\forall x\ y.\ 0 \leq h\ x\ y \Longrightarrow$
$\exists c.\ \forall x\ y.\ |f\ x\ y| \leq c * h\ x\ y \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ f\ x\ y) =o\ O(\lambda x.\ \sum y \in A\ x.\ h\ x\ y)$
apply (*rule bigo-sum-main*)
apply *force*
apply *clarsimp*

apply (*rule-tac* $x = c$ **in** *exI*)
apply *force*
done

lemma *bigo-sum2*: $\forall y.\ 0 \leq h\ y \Longrightarrow$
$\exists c.\ \forall y.\ |f\ y| \leq c * (h\ y) \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ f\ y) =o\ O(\lambda x.\ \sum y \in A\ x.\ h\ y)$
by (*rule bigo-sum1*) *auto*

lemma *bigo-sum3*: $f =o\ O(h) \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * f\ (k\ x\ y)) =o\ O(\lambda x.\ \sum y \in A\ x.\ |l\ x\ y * h\ (k\ x\ y)|)$
apply (*rule bigo-sum1*)
apply (*rule allI*)+
apply (*rule abs-ge-zero*)
apply (*unfold bigo-def*)
apply *auto*
apply (*rule-tac* $x = c$ **in** *exI*)
apply (*rule allI*)+
apply (*subst abs-mult*)+
apply (*subst mult.left-commute*)
apply (*rule mult-left-mono*)
apply (*erule spec*)
apply (*rule abs-ge-zero*)
done

lemma *bigo-sum4*: $f =o\ g +o\ O(h) \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * f\ (k\ x\ y)) =o$
$(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * g\ (k\ x\ y)) +o$
$O(\lambda x.\ \sum y \in A\ x.\ |l\ x\ y * h\ (k\ x\ y)|)$
apply (*rule set-minus-imp-plus*)
apply (*subst fun-diff-def*)
apply (*subst sum-subtractf* [*symmetric*])
apply (*subst right-diff-distrib* [*symmetric*])
apply (*rule bigo-sum3*)
apply (*subst fun-diff-def* [*symmetric*])
apply (*erule set-plus-imp-minus*)
done

lemma *bigo-sum5*: $f =o\ O(h) \Longrightarrow \forall x\ y.\ 0 \leq l\ x\ y \Longrightarrow$
$\forall x.\ 0 \leq h\ x \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * f\ (k\ x\ y)) =o$
$O(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * h\ (k\ x\ y))$
apply (*subgoal-tac* $(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * h\ (k\ x\ y)) =$
$(\lambda x.\ \sum y \in A\ x.\ |l\ x\ y * h\ (k\ x\ y)|)$)
apply (*erule ssubst*)
apply (*erule bigo-sum3*)
apply (*rule ext*)
apply (*rule sum.cong*)
apply (*rule refl*)

apply (*subst abs-of-nonneg*)
apply *auto*
done

lemma *bigo-sum6*: $f =o\ g +o\ O(h) \Longrightarrow \forall x\ y.\ 0 \leq l\ x\ y \Longrightarrow$
$\forall x.\ 0 \leq h\ x \Longrightarrow$
$(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * f\ (k\ x\ y)) =o$
$(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * g\ (k\ x\ y)) +o$
$O(\lambda x.\ \sum y \in A\ x.\ l\ x\ y * h\ (k\ x\ y))$
apply (*rule set-minus-imp-plus*)
apply (*subst fun-diff-def*)
apply (*subst sum-subtractf* [*symmetric*])
apply (*subst right-diff-distrib* [*symmetric*])
apply (*rule bigo-sum5*)
apply (*subst fun-diff-def* [*symmetric*])
apply (*drule set-plus-imp-minus*)
apply *auto*
done

5.3 Misc useful stuff

lemma *bigo-useful-intro*: $A \subseteq O(f) \Longrightarrow B \subseteq O(f) \Longrightarrow A + B \subseteq O(f)$
apply (*subst bigo-plus-idemp* [*symmetric*])
apply (*rule set-plus-mono2*)
apply *assumption+*
done

lemma *bigo-useful-add*: $f =o\ O(h) \Longrightarrow g =o\ O(h) \Longrightarrow f + g =o\ O(h)$
apply (*subst bigo-plus-idemp* [*symmetric*])
apply (*rule set-plus-intro*)
apply *assumption+*
done

lemma *bigo-useful-const-mult*: $c \neq 0 \Longrightarrow (\lambda x.\ c) * f =o\ O(h) \Longrightarrow f =o\ O(h)$
for $c :: 'a$::*linordered-field*
apply (*rule subsetD*)
apply (*subgoal-tac* $(\lambda x.\ 1\ /\ c) *o\ O(h) \subseteq O(h)$)
apply *assumption*
apply (*rule bigo-const-mult6*)
apply (*subgoal-tac* $f = (\lambda x.\ 1\ /\ c) * ((\lambda x.\ c) * f)$)
apply (*erule ssubst*)
apply (*erule set-times-intro2*)
apply (*simp add*: *func-times*)
done

lemma *bigo-fix*: $(\lambda x$::*nat*. $f\ (x + 1)) =o\ O(\lambda x.\ h\ (x + 1)) \Longrightarrow f\ 0 = 0 \Longrightarrow f =o$
$O(h)$
apply (*simp add*: *bigo-alt-def*)
apply *auto*

apply (*rule-tac* $x = c$ **in** *exI*)
apply *auto*
apply (*case-tac* $x = 0$)
apply *simp*
apply (*subgoal-tac* $x = Suc\ (x - 1)$)
apply (*erule ssubst*) **back**
apply (*erule spec*)
apply *simp*
done

lemma *bigo-fix2*:
$(\lambda x.\ f\ ((x::nat) + 1)) =o\ (\lambda x.\ g(x + 1)) +o\ O(\lambda x.\ h(x + 1)) \Longrightarrow$
$f\ 0 = g\ 0 \Longrightarrow f =o\ g +o\ O(h)$
apply (*rule set-minus-imp-plus*)
apply (*rule bigo-fix*)
apply (*subst fun-diff-def*)
apply (*subst fun-diff-def* [*symmetric*])
apply (*rule set-plus-imp-minus*)
apply *simp*
apply (*simp add*: *fun-diff-def*)
done

5.4 Less than or equal to

definition *lesso* :: $('a \Rightarrow 'b::linordered\text{-}idom) \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ (**infixl** $<o$ *70*)
where $f <o\ g = (\lambda x.\ max\ (f\ x - g\ x)\ 0)$

lemma *bigo-lesseq1*: $f =o\ O(h) \Longrightarrow \forall x.\ |g\ x| \leq |f\ x| \Longrightarrow g =o\ O(h)$
apply (*unfold bigo-def*)
apply *clarsimp*
apply (*rule-tac* $x = c$ **in** *exI*)
apply (*rule allI*)
apply (*rule order-trans*)
apply (*erule spec*)+
done

lemma *bigo-lesseq2*: $f =o\ O(h) \Longrightarrow \forall x.\ |g\ x| \leq f\ x \Longrightarrow g =o\ O(h)$
apply (*erule bigo-lesseq1*)
apply (*rule allI*)
apply (*drule-tac* $x = x$ **in** *spec*)
apply (*rule order-trans*)
apply *assumption*
apply (*rule abs-ge-self*)
done

lemma *bigo-lesseq3*: $f =o\ O(h) \Longrightarrow \forall x.\ 0 \leq g\ x \Longrightarrow \forall x.\ g\ x \leq f\ x \Longrightarrow g =o\ O(h)$
apply (*erule bigo-lesseq2*)

apply (*rule allI*)
apply (*subst abs-of-nonneg*)
apply (*erule spec*)+
done

lemma *bigo-lesseq4*: $f =o O(h) \Longrightarrow$
$\forall x.\ 0 \leq g\ x \Longrightarrow \forall x.\ g\ x \leq |f\ x| \Longrightarrow g =o O(h)$
apply (*erule bigo-lesseq1*)
apply (*rule allI*)
apply (*subst abs-of-nonneg*)
apply (*erule spec*)+
done

lemma *bigo-lesso1*: $\forall x.\ f\ x \leq g\ x \Longrightarrow f <o g =o O(h)$
apply (*unfold lesso-def*)
apply (*subgoal-tac* ($\lambda x.\ max\ (f\ x - g\ x)\ 0) = 0$)
apply (*erule ssubst*)
apply (*rule bigo-zero*)
apply (*unfold func-zero*)
apply (*rule ext*)
apply (*simp split*: *split-max*)
done

lemma *bigo-lesso2*: $f =o g +o O(h) \Longrightarrow \forall x.\ 0 \leq k\ x \Longrightarrow \forall x.\ k\ x \leq f\ x \Longrightarrow k <o g =o O(h)$
apply (*unfold lesso-def*)
apply (*rule bigo-lesseq4*)
apply (*erule set-plus-imp-minus*)
apply (*rule allI*)
apply (*rule max.cobounded2*)
apply (*rule allI*)
apply (*subst fun-diff-def*)
apply (*case-tac* $0 \leq k\ x - g\ x$)
apply *simp*
apply (*subst abs-of-nonneg*)
apply (*drule-tac* $x = x$ **in** *spec*) **back**
apply (*simp add*: *algebra-simps*)
apply (*subst diff-conv-add-uminus*)+
apply (*rule add-right-mono*)
apply (*erule spec*)
apply (*rule order-trans*)
prefer *2*
apply (*rule abs-ge-zero*)
apply (*simp add*: *algebra-simps*)
done

lemma *bigo-lesso3*: $f =o g +o O(h) \Longrightarrow \forall x.\ 0 \leq k\ x \Longrightarrow \forall x.\ g\ x \leq k\ x \Longrightarrow f <o k =o O(h)$
apply (*unfold lesso-def*)

apply (*rule bigo-lesseq4*)
apply (*erule set-plus-imp-minus*)
apply (*rule allI*)
apply (*rule max.cobounded2*)
apply (*rule allI*)
apply (*subst fun-diff-def*)
apply (*case-tac* $0 \leq f\ x - k\ x$)
apply *simp*
apply (*subst abs-of-nonneg*)
apply (*drule-tac* $x = x$ **in** *spec*) **back**
apply (*simp add*: *algebra-simps*)
apply (*subst diff-conv-add-uminus*)+
apply (*rule add-left-mono*)
apply (*rule le-imp-neg-le*)
apply (*erule spec*)
apply (*rule order-trans*)
prefer *2*
apply (*rule abs-ge-zero*)
apply (*simp add*: *algebra-simps*)
done

lemma *bigo-lesso4*: $f <o g =o O(k) \Longrightarrow g =o h +o O(k) \Longrightarrow f <o h =o O(k)$
for $k :: 'a \Rightarrow 'b$::*linordered-field*
apply (*unfold lesso-def*)
apply (*drule set-plus-imp-minus*)
apply (*drule bigo-abs5*) **back**
apply (*simp add*: *fun-diff-def*)
apply (*drule bigo-useful-add*)
apply *assumption*
apply (*erule bigo-lesseq2*) **back**
apply (*rule allI*)
apply (*auto simp add*: *func-plus fun-diff-def algebra-simps split*: *split-max abs-split*)
done

lemma *bigo-lesso5*: $f <o g =o O(h) \Longrightarrow \exists C.\ \forall x.\ f\ x \leq g\ x + C * |h\ x|$
apply (*simp only*: *lesso-def bigo-alt-def*)
apply *clarsimp*
apply (*rule-tac* $x = c$ **in** *exI*)
apply (*rule allI*)
apply (*drule-tac* $x = x$ **in** *spec*)
apply (*subgoal-tac* $|max\ (f\ x - g\ x)\ 0| = max\ (f\ x - g\ x)\ 0$)
apply (*clarsimp simp add*: *algebra-simps*)
apply (*rule abs-of-nonneg*)
apply (*rule max.cobounded2*)
done

lemma *lesso-add*: $f <o g =o O(h) \Longrightarrow k <o l =o O(h) \Longrightarrow (f + k) <o (g + l) =o O(h)$
apply (*unfold lesso-def*)

apply (*rule bigo-lesseq3*)
apply (*erule bigo-useful-add*)
apply *assumption*
apply (*force split*: *split-max*)
apply (*auto split*: *split-max simp add*: *func-plus*)
done

lemma *bigo-LIMSEQ1*: $f =o O(g) \Longrightarrow g \longrightarrow 0 \Longrightarrow f \longrightarrow 0$
for *f g* :: *nat* ⇒ *real*
apply (*simp add*: *LIMSEQ-iff bigo-alt-def*)
apply *clarify*
apply (*drule-tac x* = *r* / *c* **in** *spec*)
apply (*drule mp*)
apply *simp*
apply *clarify*
apply (*rule-tac x* = *no* **in** *exI*)
apply (*rule allI*)
apply (*drule-tac x* = *n* **in** *spec*)+
apply (*rule impI*)
apply (*drule mp*)
apply *assumption*
apply (*rule order-le-less-trans*)
apply *assumption*
apply (*rule order-less-le-trans*)
apply (*subgoal-tac c* ∗ |*g n*| < *c* ∗ (*r* / *c*))
apply *assumption*
apply (*erule mult-strict-left-mono*)
apply *assumption*
apply *simp*
done

lemma *bigo-LIMSEQ2*: $f =o g +o O(h) \Longrightarrow h \longrightarrow 0 \Longrightarrow f \longrightarrow a \Longrightarrow g \longrightarrow a$
for *f g h* :: *nat* ⇒ *real*
apply (*drule set-plus-imp-minus*)
apply (*drule bigo-LIMSEQ1*)
apply *assumption*
apply (*simp only*: *fun-diff-def*)
apply (*erule Lim-transform2*)
apply *assumption*
done

end

6 Axiomatic Declaration of Bounded Natural Functors

theory *BNF-Axiomatization*

imports *Main*
keywords
bnf-axiomatization :: *thy-decl*
begin

ML-file ‹../*Tools*/*BNF*/*bnf-axiomatization.ML*›

end

7 Generalized Corecursor Sugar (corec and friends)

theory *BNF-Corec*
imports *Main*
keywords
corec :: *thy-defn* **and**
corecursive :: *thy-goal-defn* **and**
friend-of-corec :: *thy-goal-defn* **and**
coinduction-upto :: *thy-decl*
begin

lemma *obj-distinct-prems*: $P \longrightarrow P \longrightarrow Q \Longrightarrow P \Longrightarrow Q$
by *auto*

lemma *inject-refine*: $g\ (f\ x) = x \Longrightarrow g\ (f\ y) = y \Longrightarrow f\ x = f\ y \longleftrightarrow x = y$
by (*metis* (*no-types*))

lemma *convol-apply*: *BNF-Def.convol* $f\ g\ x = (f\ x,\ g\ x)$
unfolding *convol-def* **..**

lemma *Grp-UNIV-id*: *BNF-Def.Grp UNIV id* $= (=)$
unfolding *BNF-Def.Grp-def* **by** *auto*

lemma *sum-comp-cases*:
assumes $f \circ$ *Inl* $= g \circ$ *Inl* **and** $f \circ$ *Inr* $= g \circ$ *Inr*
shows $f = g$
proof (*rule ext*)
fix *a* **show** $f\ a = g\ a$
using *assms* **unfolding** *comp-def fun-eq-iff* **by** (*cases a*) *auto*
qed

lemma *case-sum-Inl-Inr-L*: *case-sum* $(f \circ$ *Inl*$)\ (f \circ$ *Inr*$) = f$
by (*metis case-sum-expand-Inr'*)

lemma *eq-o-InrI*: $[\![g \circ$ *Inl* $= h$; *case-sum* $h\ f = g]\!] \Longrightarrow f = g \circ$ *Inr*
by (*auto simp*: *fun-eq-iff split*: *sum.splits*)

lemma *id-bnf-o*: *BNF-Composition.id-bnf* $\circ\ f = f$
unfolding *BNF-Composition.id-bnf-def* **by** (*rule o-def*)

lemma *o-id-bnf*: $f \circ$ *BNF-Composition.id-bnf* $= f$
unfolding *BNF-Composition.id-bnf-def* **by** (*rule o-def*)

lemma *if-True-False*:
(*if P then True else Q*) $\longleftrightarrow P \vee Q$
(*if P then False else Q*) $\longleftrightarrow \neg P \wedge Q$
(*if P then Q else True*) $\longleftrightarrow \neg P \vee Q$
(*if P then Q else False*) $\longleftrightarrow P \wedge Q$
by *auto*

lemma *if-distrib-fun*: (*if c then f else g*) $x =$ (*if c then f x else g x*)
by *simp*

7.1 Coinduction

lemma *eq-comp-compI*: $a \circ b = f \circ x \Longrightarrow x \circ c = id \Longrightarrow f = a \circ (b \circ c)$
unfolding *fun-eq-iff* **by** *simp*

lemma *self-bounded-weaken-left*: $(a :: 'a ::$ *semilattice-inf*$) \leq$ *inf a b* $\Longrightarrow a \leq b$
by (*erule le-infE*)

lemma *self-bounded-weaken-right*: $(a :: 'a ::$ *semilattice-inf*$) \leq$ *inf b a* $\Longrightarrow a \leq b$
by (*erule le-infE*)

lemma *symp-iff*: *symp* $R \longleftrightarrow R = R^{-1-1}$
by (*metis antisym conversep.cases conversep-le-swap predicate2I symp-def*)

lemma *equivp-inf*: $[\![$*equivp R*; *equivp S*$]\!] \Longrightarrow$ *equivp* (*inf R S*)
unfolding *equivp-def inf-fun-def inf-bool-def* **by** *metis*

lemma *vimage2p-rel-prod*:
($\lambda x\ y.$ *rel-prod R S* (*BNF-Def.convol f1 g1 x*) (*BNF-Def.convol f2 g2 y*)) =
(*inf* (*BNF-Def.vimage2p f1 f2 R*) (*BNF-Def.vimage2p g1 g2 S*))
unfolding *vimage2p-def rel-prod.simps convol-def* **by** *auto*

lemma *predicate2I-obj*: $(\forall x\ y.\ P\ x\ y \longrightarrow Q\ x\ y) \Longrightarrow P \leq Q$
by *auto*

lemma *predicate2D-obj*: $P \leq Q \Longrightarrow P\ x\ y \longrightarrow Q\ x\ y$
by *auto*

locale *cong* =
fixes *rel* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('b \Rightarrow 'b \Rightarrow bool)$
and *eval* :: $'b \Rightarrow 'a$
and *retr* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool)$
assumes *rel-mono*: $\bigwedge R\ S.\ R \leq S \Longrightarrow$ *rel* $R \leq$ *rel* S
and *equivp-retr*: $\bigwedge R.$ *equivp* $R \Longrightarrow$ *equivp* (*retr R*)
and *retr-eval*: $\bigwedge R\ x\ y.$ $[\![$(*rel-fun* (*rel R*) *R*) *eval eval*; *rel* (*inf R* (*retr R*)) *x y*$]\!]$
$\Longrightarrow$

retr R (*eval x*) (*eval y*)
begin

definition *cong* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *bool* **where**
cong R ≡ *equivp R* ∧ (*rel-fun* (*rel R*) *R*) *eval eval*

lemma *cong-retr*: *cong R* ⟹ *cong* (*inf R* (*retr R*))
unfolding *cong-def*
by (*auto simp*: *rel-fun-def dest*: *predicate2D*[*OF rel-mono*, *rotated*]
intro: *equivp-inf equivp-retr retr-eval*)

lemma *cong-equivp*: *cong R* ⟹ *equivp R*
unfolding *cong-def* **by** *simp*

definition *gen-cong* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a* ⇒ *'a* ⇒ *bool* **where**
gen-cong R j1 j2 ≡ ∀ *R'*. *R* ≤ *R'* ∧ *cong R'* ⟶ *R' j1 j2*

lemma *gen-cong-reflp*[*intro*, *simp*]: *x* = *y* ⟹ *gen-cong R x y*
unfolding *gen-cong-def* **by** (*auto dest*: *cong-equivp equivp-reflp*)

lemma *gen-cong-symp*[*intro*]: *gen-cong R x y* ⟹ *gen-cong R y x*
unfolding *gen-cong-def* **by** (*auto dest*: *cong-equivp equivp-symp*)

lemma *gen-cong-transp*[*intro*]: *gen-cong R x y* ⟹ *gen-cong R y z* ⟹ *gen-cong R x z*
unfolding *gen-cong-def* **by** (*auto dest*: *cong-equivp equivp-transp*)

lemma *equivp-gen-cong*: *equivp* (*gen-cong R*)
by (*intro equivpI reflpI sympI transpI*) *auto*

lemma *leq-gen-cong*: *R* ≤ *gen-cong R*
unfolding *gen-cong-def*[*abs-def*] **by** *auto*

lemmas *imp-gen-cong*[*intro*] = *predicate2D*[*OF leq-gen-cong*]

lemma *gen-cong-minimal*: ⟦*R* ≤ *R'*; *cong R'*⟧ ⟹ *gen-cong R* ≤ *R'*
unfolding *gen-cong-def*[*abs-def*] **by** (*rule predicate2I*) *metis*

lemma *congdd-base-gen-congdd-base-aux*:
rel (*gen-cong R*) *x y* ⟹ *R* ≤ *R'* ⟹ *cong R'* ⟹ *R'* (*eval x*) (*eval y*)
by (*force simp*: *rel-fun-def gen-cong-def cong-def dest*: *spec*[*of - R'*] *predicate2D*[*OF rel-mono*, *rotated* −1, *of - - - R'*])

lemma *cong-gen-cong*: *cong* (*gen-cong R*)
proof −
{ fix *R' x y*
have *rel* (*gen-cong R*) *x y* ⟹ *R* ≤ *R'* ⟹ *cong R'* ⟹ *R'* (*eval x*) (*eval y*)
by (*force simp*: *rel-fun-def gen-cong-def cong-def dest*: *spec*[*of - R'*]
predicate2D[*OF rel-mono*, *rotated* −1, *of - - - R'*])

}
then show *cong* (*gen-cong R*) **by** (*auto simp*: *equivp-gen-cong rel-fun-def gen-cong-def cong-def*)
qed

lemma *gen-cong-eval-rel-fun*:
(*rel-fun* (*rel* (*gen-cong R*)) (*gen-cong R*)) *eval eval*
using *cong-gen-cong*[*of R*] **unfolding** *cong-def* **by** *simp*

lemma *gen-cong-eval*:
rel (*gen-cong R*) *x y* ⟹ *gen-cong R* (*eval x*) (*eval y*)
by (*erule rel-funD*[*OF gen-cong-eval-rel-fun*])

lemma *gen-cong-idem*: *gen-cong* (*gen-cong R*) = *gen-cong R*
by (*simp add*: *antisym cong-gen-cong gen-cong-minimal leq-gen-cong*)

lemma *gen-cong-rho*:
ϱ = *eval* ∘ *f* ⟹ *rel* (*gen-cong R*) (*f x*) (*f y*) ⟹ *gen-cong R* (ϱ *x*) (ϱ *y*)
by (*simp add*: *gen-cong-eval*)
lemma *coinduction*:
assumes *coind*: ∀ *R*. *R* ≤ *retr R* ⟶ *R* ≤ (=)
assumes *cih*: *R* ≤ *retr* (*gen-cong R*)
shows *R* ≤ (=)
apply (*rule order-trans*[*OF leq-gen-cong mp*[*OF spec*[*OF coind*]]])
apply (*rule self-bounded-weaken-left*[*OF gen-cong-minimal*])
apply (*rule inf-greatest*[*OF leq-gen-cong cih*])
apply (*rule cong-retr*[*OF cong-gen-cong*])
done

end

lemma *rel-sum-case-sum*:
rel-fun (*rel-sum R S*) *T* (*case-sum f1 g1*) (*case-sum f2 g2*) = (*rel-fun R T f1 f2* ∧ *rel-fun S T g1 g2*)
by (*auto simp*: *rel-fun-def rel-sum.simps split*: *sum.splits*)

context
fixes *rel eval rel′ eval′ retr emb*
assumes *base*: *cong rel eval retr*
and *step*: *cong rel′ eval′ retr*
and *emb*: *eval′* ∘ *emb* = *eval*
and *emb-transfer*: *rel-fun* (*rel R*) (*rel′ R*) *emb emb*
begin

interpretation *base*: *cong rel eval retr* **by** (*rule base*)
interpretation *step*: *cong rel′ eval′ retr* **by** (*rule step*)

lemma *gen-cong-emb*: *base.gen-cong R* ≤ *step.gen-cong R*
proof (*rule base.gen-cong-minimal*[*OF step.leq-gen-cong*])

note *step.gen-cong-eval-rel-fun*[*transfer-rule*] *emb-transfer*[*transfer-rule*]
have (*rel-fun* (*rel* (*step.gen-cong R*)) (*step.gen-cong R*)) *eval eval*
unfolding *emb*[*symmetric*] **by** *transfer-prover*
then show *base.cong* (*step.gen-cong R*)
by (*auto simp*: *base.cong-def step.equivp-gen-cong*)
qed

end

named-theorems *friend-of-corec-simps*

ML-file ‹../*Tools*/*BNF*/*bnf-gfp-grec-tactics.ML*›
ML-file ‹../*Tools*/*BNF*/*bnf-gfp-grec.ML*›
ML-file ‹../*Tools*/*BNF*/*bnf-gfp-grec-sugar-util.ML*›
ML-file ‹../*Tools*/*BNF*/*bnf-gfp-grec-sugar-tactics.ML*›
ML-file ‹../*Tools*/*BNF*/*bnf-gfp-grec-sugar.ML*›
ML-file ‹../*Tools*/*BNF*/*bnf-gfp-grec-unique-sugar.ML*›

method-setup *transfer-prover-eq* = ‹
Scan.succeed (*SIMPLE-METHOD'* o *BNF-GFP-Grec-Tactics.transfer-prover-eq-tac*)
› *apply transfer-prover after folding relator-eq*

method-setup *corec-unique* = ‹
Scan.succeed (*SIMPLE-METHOD'* o *BNF-GFP-Grec-Unique-Sugar.corec-unique-tac*)
› *prove uniqueness of corecursive equation*

end

8 A general "while" combinator

theory *While-Combinator*
imports *Main*
begin

8.1 Partial version

definition *while-option* :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a*) ⇒ '*a* ⇒ '*a option* **where**
while-option b c s = (*if* (∃ *k*. ¬ *b* ((*c* ^^ *k*) *s*))
then Some ((*c* ^^ (*LEAST k*. ¬ *b* ((*c* ^^ *k*) *s*))) *s*)
else None)

theorem *while-option-unfold*[*code*]:
while-option b c s = (*if b s then while-option b c* (*c s*) *else Some s*)
proof *cases*
assume *b s*
show *?thesis*
proof (*cases* ∃ *k*. ¬ *b* ((*c* ^^ *k*) *s*))
case *True*
then obtain *k* **where** *1*: ¬ *b* ((*c* ^^ *k*) *s*) **..**

```
    with ‹b s› obtain l where k = Suc l by (cases k) auto
    with 1 have ¬ b ((c ^^ l) (c s)) by (auto simp: funpow-swap1)
    then have 2: ∃ l. ¬ b ((c ^^ l) (c s)) ..
    from 1
    have (LEAST k. ¬ b ((c ^^ k) s)) = Suc (LEAST l. ¬ b ((c ^^ Suc l) s))
      by (rule Least-Suc) (simp add: ‹b s›)
    also have ... = Suc (LEAST l. ¬ b ((c ^^ l) (c s)))
      by (simp add: funpow-swap1)
    finally
    show ?thesis
      using True 2 ‹b s› by (simp add: funpow-swap1 while-option-def)
  next
    case False
    then have ¬ (∃ l. ¬ b ((c ^^ Suc l) s)) by blast
    then have ¬ (∃ l. ¬ b ((c ^^ l) (c s)))
      by (simp add: funpow-swap1)
    with False ‹b s› show ?thesis by (simp add: while-option-def)
  qed
next
  assume [simp]: ¬ b s
  have least: (LEAST k. ¬ b ((c ^^ k) s)) = 0
    by (rule Least-equality) auto
  moreover
  have ∃ k. ¬ b ((c ^^ k) s) by (rule exI[of - 0::nat]) auto
  ultimately show ?thesis unfolding while-option-def by auto
qed

lemma while-option-stop2:
 while-option b c s = Some t ⟹ ∃ k. t = (c^^k) s ∧ ¬ b t
apply(simp add: while-option-def split: if-splits)
by (metis (lifting) LeastI-ex)

lemma while-option-stop: while-option b c s = Some t ⟹ ¬ b t
by(metis while-option-stop2)

theorem while-option-rule:
  assumes step: !!s. P s ==> b s ==> P (c s)
    and result: while-option b c s = Some t
    and init: P s
  shows P t
proof −
  define k where k = (LEAST k. ¬ b ((c ^^ k) s))
  from assms have t: t = (c ^^ k) s
    by (simp add: while-option-def k-def split: if-splits)
  have 1: ∀ i<k. b ((c ^^ i) s)
    by (auto simp: k-def dest: not-less-Least)

  { fix i assume i ≤ k then have P ((c ^^ i) s)
      by (induct i) (auto simp: init step 1) }
```

```
  thus P t by (auto simp: t)
qed

lemma funpow-commute:
  ⟦∀k'<k. f (c ((c^^k') s)) = c' (f ((c^^k') s))⟧ ⟹ f ((c^^k) s) = (c'^^k) (f s)
by (induct k arbitrary: s) auto

lemma while-option-commute-invariant:
assumes Invariant: ⋀s. P s ⟹ b s ⟹ P (c s)
assumes TestCommute: ⋀s. P s ⟹ b s = b' (f s)
assumes BodyCommute: ⋀s. P s ⟹ b s ⟹ f (c s) = c' (f s)
assumes Initial: P s
shows map-option f (while-option b c s) = while-option b' c' (f s)
unfolding while-option-def
proof (rule trans[OF if-distrib if-cong], safe, unfold option.inject)
  fix k
  assume ¬ b ((c ^^ k) s)
  with Initial show ∃k. ¬ b' ((c' ^^ k) (f s))
  proof (induction k arbitrary: s)
    case 0 thus ?case by (auto simp: TestCommute intro: exI[of - 0])
  next
    case (Suc k) thus ?case
    proof (cases b s)
      assume b s
      with Suc.IH[of c s] Suc.prems show ?thesis
      by (metis BodyCommute Invariant comp-apply funpow.simps(2) funpow-swap1)
    next
      assume ¬ b s
      with Suc show ?thesis by (auto simp: TestCommute intro: exI [of - 0])
    qed
  qed
next
  fix k
  assume ¬ b' ((c' ^^ k) (f s))
  with Initial show ∃k. ¬ b ((c ^^ k) s)
  proof (induction k arbitrary: s)
    case 0 thus ?case by (auto simp: TestCommute intro: exI[of - 0])
  next
    case (Suc k) thus ?case
    proof (cases b s)
       assume b s
      with Suc.IH[of c s] Suc.prems show ?thesis
      by (metis BodyCommute Invariant comp-apply funpow.simps(2) funpow-swap1)
    next
      assume ¬ b s
      with Suc show ?thesis by (auto simp: TestCommute intro: exI [of - 0])
    qed
  qed
next
```

fix k
assume k: $\neg\ b'\ ((c'\ ^^\ k)\ (f\ s))$
have $*$: $(LEAST\ k.\ \neg\ b'\ ((c'\ ^^\ k)\ (f\ s))) = (LEAST\ k.\ \neg\ b\ ((c\ ^^\ k)\ s))$
(**is** $?k' = ?k$)
proof (*cases* $?k'$)
case *0*
have $\neg\ b'\ ((c'\ ^^\ 0)\ (f\ s))$
unfolding *0*[*symmetric*] **by** (*rule LeastI*[*of* - k]) (*rule* k)
hence $\neg\ b\ s$ **by** (*auto simp*: *TestCommute Initial*)
hence $?k = 0$ **by** (*intro Least-equality*) *auto*
with *0* **show** *?thesis* **by** *auto*
next
case (*Suc* k')
have $\neg\ b'\ ((c'\ ^^\ Suc\ k')\ (f\ s))$
unfolding *Suc*[*symmetric*] **by** (*rule LeastI*) (*rule* k)
moreover
{ fix k **assume** $k \leq k'$
hence $k < ?k'$ **unfolding** *Suc* **by** *simp*
hence $b'\ ((c'\ ^^\ k)\ (f\ s))$ **by** (*rule iffD1*[*OF not-not*, *OF not-less-Least*])
}
note $b' = this$
{ fix k **assume** $k \leq k'$
hence $f\ ((c\ ^^\ k)\ s) = (c'\ ^^\ k)\ (f\ s)$
and $b\ ((c\ ^^\ k)\ s) = b'\ ((c'\ ^^\ k)\ (f\ s))$
and $P\ ((c\ ^^\ k)\ s)$
by (*induct* k) (*auto simp*: b' *assms*)
with $\langle k \leq k' \rangle$
have $b\ ((c\ ^^\ k)\ s)$
and $f\ ((c\ ^^\ k)\ s) = (c'\ ^^\ k)\ (f\ s)$
and $P\ ((c\ ^^\ k)\ s)$
by (*auto simp*: b')
}
note $b = this(1)$ **and** $body = this(2)$ **and** $inv = this(3)$
hence k': $f\ ((c\ ^^\ k')\ s) = (c'\ ^^\ k')\ (f\ s)$ **by** *auto*
ultimately show *?thesis* **unfolding** *Suc* **using** b
proof (*intro Least-equality*[*symmetric*], *goal-cases*)
case *1*
hence *Test*: $\neg\ b'\ (f\ ((c\ ^^\ Suc\ k')\ s))$
by (*auto simp*: *BodyCommute inv b*)
have $P\ ((c\ ^^\ Suc\ k')\ s)$ **by** (*auto simp*: *Invariant inv b*)
with *Test* **show** *?case* **by** (*auto simp*: *TestCommute*)
next
case *2*
thus *?case* **by** (*metis not-less-eq-eq*)
qed
qed
have $f\ ((c\ ^^\ ?k)\ s) = (c'\ ^^\ ?k')\ (f\ s)$ **unfolding** $*$
proof (*rule funpow-commute*, *clarify*)
fix k **assume** $k < ?k$

hence *TestTrue*: $b\ ((c\ ^^\ k)\ s)$ **by** (*auto dest*: *not-less-Least*)
from ‹$k < ?k$› **have** $P\ ((c\ ^^\ k)\ s)$
proof (*induct k*)
case *0* **thus** *?case* **by** (*auto simp*: *assms*)
next
case (*Suc h*)
hence $P\ ((c\ ^^\ h)\ s)$ **by** *auto*
with *Suc* **show** *?case*
by (*auto*, *metis* (*lifting*, *no-types*) *Invariant Suc-lessD not-less-Least*)
qed
with *TestTrue* **show** $f\ (c\ ((c\ ^^\ k)\ s)) = c'\ (f\ ((c\ ^^\ k)\ s))$
by (*metis BodyCommute*)
qed
thus $\exists z.\ (c\ ^^\ ?k)\ s = z \wedge f\ z = (c'\ ^^\ ?k')\ (f\ s)$ **by** *blast*
qed

lemma *while-option-commute*:
assumes $\bigwedge s.\ b\ s = b'\ (f\ s)$ $\bigwedge s.\ [\![b\ s]\!] \Longrightarrow f\ (c\ s) = c'\ (f\ s)$
shows *map-option f* (*while-option b c s*) = *while-option b' c'* (*f s*)
by(*rule while-option-commute-invariant*[**where** $P = \lambda$-. *True*])
(*auto simp add*: *assms*)

8.2 Total version

definition *while* :: $('a \Rightarrow bool) \Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a \Rightarrow 'a$
where *while b c s* = *the* (*while-option b c s*)

lemma *while-unfold* [*code*]:
while b c s = (*if b s then while b c* (*c s*) *else s*)
unfolding *while-def* **by** (*subst while-option-unfold*) *simp*

lemma *def-while-unfold*:
assumes *fdef*: *f* == *while test do*
shows *f x* = (*if test x then f*(*do x*) *else x*)
unfolding *fdef* **by** (*fact while-unfold*)

The proof rule for *while*, where P is the invariant.

theorem *while-rule-lemma*:
assumes *invariant*: $!!s.\ P\ s \Longrightarrow b\ s \Longrightarrow P\ (c\ s)$
and *terminate*: $!!s.\ P\ s \Longrightarrow \neg\ b\ s \Longrightarrow Q\ s$
and *wf*: *wf* $\{(t, s).\ P\ s \wedge b\ s \wedge t = c\ s\}$
shows $P\ s \Longrightarrow Q$ (*while b c s*)
using *wf*
apply (*induct s*)
apply *simp*
apply (*subst while-unfold*)
apply (*simp add*: *invariant terminate*)
done

theorem *while-rule*:

```
[| P s;
    !!s. [| P s; b s  |] ==> P (c s);
    !!s. [| P s; ¬ b s  |] ==> Q s;
    wf r;
    !!s. [| P s; b s  |] ==> (c s, s) ∈ r |] ==>
 Q (while b c s)
apply (rule while-rule-lemma)
  prefer 4 apply assumption
 apply blast
apply blast
apply (erule wf-subset)
apply blast
done
```

Proving termination:

```
theorem wf-while-option-Some:
 assumes wf {(t, s). (P s ∧ b s) ∧ t = c s}
 and ⋀s. P s ⟹ b s ⟹ P(c s) and P s
 shows ∃t. while-option b c s = Some t
using assms(1,3)
proof (induction s)
 case less thus ?case using assms(2)
  by (subst while-option-unfold) simp
qed

lemma wf-rel-while-option-Some:
assumes wf: wf R
assumes smaller: ⋀s. P s ∧ b s ⟹ (c s, s) ∈ R
assumes inv: ⋀s. P s ∧ b s ⟹ P(c s)
assumes init: P s
shows ∃t. while-option b c s = Some t
proof −
 from smaller have {(t,s). P s ∧ b s ∧ t = c s} ⊆ R by auto
 with wf have wf {(t,s). P s ∧ b s ∧ t = c s} by (auto simp: wf-subset)
 with inv init show ?thesis by (auto simp: wf-while-option-Some)
qed

theorem measure-while-option-Some: fixes f :: 's ⇒ nat
shows (⋀s. P s ⟹ b s ⟹ P(c s) ∧ f(c s) < f s)
 ⟹ P s ⟹ ∃t. while-option b c s = Some t
by(blast intro: wf-while-option-Some[OF wf-if-measure, of P b f])
```

Kleene iteration starting from the empty set and assuming some finite bounding set:

```
lemma while-option-finite-subset-Some: fixes C :: 'a set
 assumes mono f and !!X. X ⊆ C ⟹ f X ⊆ C and finite C
 shows ∃P. while-option (λA. f A ≠ A) f {} = Some P
proof(rule measure-while-option-Some[where
   f= %A::'a set. card C − card A and P= %A. A ⊆ C ∧ A ⊆ f A and s= {}])
```

fix A **assume** A: $A \subseteq C \wedge A \subseteq f\ A\ f\ A \neq A$
show $(f\ A \subseteq C \wedge f\ A \subseteq f\ (f\ A)) \wedge card\ C - card\ (f\ A) < card\ C - card\ A$
(**is** $?L \wedge ?R$)
proof
show $?L$ **by**(*metis A*(*1*) *assms*(*2*) *monoD*[*OF* ‹*mono f*›])
show $?R$ **by** (*metis A assms*(*2*,*3*) *card-seteq diff-less-mono2 equalityI linorder-le-less-linear rev-finite-subset*)
qed
qed *simp*

lemma *lfp-the-while-option*:
assumes *mono f* **and** $!!X.\ X \subseteq C \Longrightarrow f\ X \subseteq C$ **and** *finite C*
shows *lfp f* = *the*(*while-option* ($\lambda A.\ f\ A \neq A$) *f* {})
proof−
obtain *P* **where** *while-option* ($\lambda A.\ f\ A \neq A$) *f* {} = *Some P*
using *while-option-finite-subset-Some*[*OF assms*] **by** *blast*
with *while-option-stop2*[*OF this*] *lfp-Kleene-iter*[*OF assms*(*1*)]
show *?thesis* **by** *auto*
qed

lemma *lfp-while*:
assumes *mono f* **and** $!!X.\ X \subseteq C \Longrightarrow f\ X \subseteq C$ **and** *finite C*
shows *lfp f* = *while* ($\lambda A.\ f\ A \neq A$) *f* {}
unfolding *while-def* **using** *assms* **by** (*rule lfp-the-while-option*) *blast*

lemma *wf-finite-less*:
assumes *finite* (*C* :: *'a*::*order set*)
shows *wf* {(*x*, *y*). {*x*, *y*} ⊆ *C* ∧ *x* < *y*}
by (*rule wf-measure*[**where** *f*=λ*b*. *card* {*a*. *a* ∈ *C* ∧ *a* < *b*}, *THEN wf-subset*])
(*fastforce simp*: *less-eq assms intro*: *psubset-card-mono*)

lemma *wf-finite-greater*:
assumes *finite* (*C* :: *'a*::*order set*)
shows *wf* {(*x*, *y*). {*x*, *y*} ⊆ *C* ∧ *y* < *x*}
by (*rule wf-measure*[**where** *f*=λ*b*. *card* {*a*. *a* ∈ *C* ∧ *b* < *a*}, *THEN wf-subset*])
(*fastforce simp*: *less-eq assms intro*: *psubset-card-mono*)

lemma *while-option-finite-increasing-Some*:
fixes *f* :: *'a*::*order* ⇒ *'a*
assumes *mono f* **and** *finite* (*UNIV* :: *'a set*) **and** *s* ≤ *f s*
shows ∃ *P*. *while-option* (λ*A*. *f A* ≠ *A*) *f s* = *Some P*
by (*rule wf-rel-while-option-Some*[**where** *R*={(*x*, *y*). *y* < *x*} **and** *P*=λ*A*. *A* ≤ *f A* **and** *s*=*s*])
(*auto simp*: *assms monoD intro*: *wf-finite-greater*[**where** *C*=*UNIV*::*'a set*, *simplified*])

lemma *lfp-the-while-option-lattice*:
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*
assumes *mono f* **and** *finite* (*UNIV* :: *'a set*)

shows *lfp f = the (while-option (λA. f A ≠ A) f bot)*
proof −
obtain *P* **where** *while-option (λA. f A ≠ A) f bot = Some P*
using *while-option-finite-increasing-Some*[*OF assms*, **where** *s=bot*] **by** *simp blast*
with *while-option-stop2*[*OF this*] *lfp-Kleene-iter*[*OF assms(1)*]
show *?thesis* **by** *auto*
qed

lemma *lfp-while-lattice*:
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*
assumes *mono f* **and** *finite* (*UNIV* :: *'a set*)
shows *lfp f = while (λA. f A ≠ A) f bot*
unfolding *while-def* **using** *assms* **by** (*rule lfp-the-while-option-lattice*)

lemma *while-option-finite-decreasing-Some*:
fixes *f* :: *'a*::*order* ⇒ *'a*
assumes *mono f* **and** *finite* (*UNIV* :: *'a set*) **and** *f s* ≤ *s*
shows ∃ *P*. *while-option (λA. f A ≠ A) f s = Some P*
by (*rule wf-rel-while-option-Some*[**where** *R*={(*x*, *y*). *x* < *y*} **and** *P*=λA. *f A* ≤ *A* **and** *s*=*s*])
(*auto simp add*: *assms monoD intro*: *wf-finite-less*[**where** *C*=*UNIV*::*'a set*, *simplified*])

lemma *gfp-the-while-option-lattice*:
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*
assumes *mono f* **and** *finite* (*UNIV* :: *'a set*)
shows *gfp f = the*(*while-option (λA. f A ≠ A) f top*)
proof −
obtain *P* **where** *while-option (λA. f A ≠ A) f top = Some P*
using *while-option-finite-decreasing-Some*[*OF assms*, **where** *s=top*] **by** *simp blast*
with *while-option-stop2*[*OF this*] *gfp-Kleene-iter*[*OF assms(1)*]
show *?thesis* **by** *auto*
qed

lemma *gfp-while-lattice*:
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*
assumes *mono f* **and** *finite* (*UNIV* :: *'a set*)
shows *gfp f = while (λA. f A ≠ A) f top*
unfolding *while-def* **using** *assms* **by** (*rule gfp-the-while-option-lattice*)

Computing the reflexive, transitive closure by iterating a successor function. Stops when an element is found that dos not satisfy the test.

More refined (and hence more efficient) versions can be found in ITP 2011 paper by Nipkow (the theories are in the AFP entry Flyspeck by Nipkow) and the AFP article Executable Transitive Closures by René Thiemann.

context
fixes *p* :: *'a* ⇒ *bool*

and $f :: 'a \Rightarrow 'a\ list$
and $x :: 'a$
begin

qualified fun *rtrancl-while-test* :: $'a\ list \times 'a\ set \Rightarrow bool$
where *rtrancl-while-test* $(ws,\text{-}) = (ws \neq [] \wedge p(hd\ ws))$

qualified fun *rtrancl-while-step* :: $'a\ list \times 'a\ set \Rightarrow 'a\ list \times 'a\ set$
where *rtrancl-while-step* $(ws,\ Z) =$
$(let\ x = hd\ ws;\ new = remdups\ (filter\ (\lambda y.\ y \notin Z)\ (f\ x))$
$in\ (new\ @\ tl\ ws,\ set\ new \cup Z))$

definition *rtrancl-while* :: $('a\ list * 'a\ set)\ option$
where *rtrancl-while* = *while-option rtrancl-while-test rtrancl-while-step* $([x],\{x\})$

qualified fun *rtrancl-while-invariant* :: $'a\ list \times 'a\ set \Rightarrow bool$
where *rtrancl-while-invariant* $(ws,\ Z) =$
$(x \in Z \wedge set\ ws \subseteq Z \wedge distinct\ ws \wedge \{(x,y).\ y \in set(f\ x)\}\ ``\ (Z - set\ ws) \subseteq Z$
$\wedge$
$Z \subseteq \{(x,y).\ y \in set(f\ x)\}^*\ ``\ \{x\} \wedge (\forall z{\in}Z - set\ ws.\ p\ z))$

qualified lemma *rtrancl-while-invariant*:
assumes *inv*: *rtrancl-while-invariant st* **and** *test*: *rtrancl-while-test st*
shows *rtrancl-while-invariant* (*rtrancl-while-step st*)
proof (*cases st*)
fix *ws Z* **assume** *st*: $st = (ws,\ Z)$
with *test* **obtain** *h t* **where** $ws = h\ \#\ t\ p\ h$ **by** (*cases ws*) *auto*
with *inv st* **show** *?thesis* **by** (*auto intro*: *rtrancl.rtrancl-into-rtrancl*)
qed

lemma *rtrancl-while-Some*: **assumes** *rtrancl-while* = *Some*(ws,Z)
shows *if* $ws = []$
then $Z = \{(x,y).\ y \in set(f\ x)\}^*\ ``\ \{x\} \wedge (\forall z{\in}Z.\ p\ z)$
else $\neg p(hd\ ws) \wedge hd\ ws \in \{(x,y).\ y \in set(f\ x)\}^*\ ``\ \{x\}$
proof −
have *rtrancl-while-invariant* $([x],\{x\})$ **by** *simp*
with *rtrancl-while-invariant* **have** *I*: *rtrancl-while-invariant* (ws,Z)
by (*rule while-option-rule*[*OF - assms*[*unfolded rtrancl-while-def*]])
{ assume $ws = []$
hence *?thesis* **using** *I*
by (*auto simp del*:*Image-Collect-case-prod dest*: *Image-closed-trancl*)
} moreover
{ assume $ws \neq []$
hence *?thesis* **using** *I while-option-stop*[*OF assms*[*unfolded rtrancl-while-def*]]
by (*simp add*: *subset-iff*)
}
ultimately show *?thesis* **by** *simp*
qed

lemma *rtrancl-while-finite-Some*:
assumes *finite* ({(*x*, *y*). *y* ∈ *set* (*f x*)}* `` {*x*}) (**is** *finite ?Cl*)
shows ∃ *y*. *rtrancl-while* = *Some y*
proof −
let *?R* = (λ(-, *Z*). *card* (*?Cl* − *Z*)) <∗*mlex*∗> (λ(*ws*, -). *length ws*) <∗*mlex*∗> {}
have *wf ?R* **by** (*blast intro*: *wf-mlex*)
then show *?thesis* **unfolding** *rtrancl-while-def*
proof (*rule wf-rel-while-option-Some*[*of ?R rtrancl-while-invariant*])
fix *st* **assume** ∗: *rtrancl-while-invariant st* ∧ *rtrancl-while-test st*
hence *I*: *rtrancl-while-invariant* (*rtrancl-while-step st*)
by (*blast intro*: *rtrancl-while-invariant*)
show (*rtrancl-while-step st*, *st*) ∈ *?R*
proof (*cases st*)
fix *ws Z* **let** *?ws* = *fst* (*rtrancl-while-step st*) **and** *?Z* = *snd* (*rtrancl-while-step st*)
assume *st*: *st* = (*ws*, *Z*)
with ∗ **obtain** *h t* **where** *ws*: *ws* = *h* # *t p h* **by** (*cases ws*) *auto*
{ **assume** *remdups* (*filter* (λ*y*. *y* ∉ *Z*) (*f h*)) ≠ []
then obtain *z* **where** *z* ∈ *set* (*remdups* (*filter* (λ*y*. *y* ∉ *Z*) (*f h*))) **by** *fastforce*
with *st ws I* **have** *Z* ⊂ *?Z Z* ⊆ *?Cl ?Z* ⊆ *?Cl* **by** *auto*
with *assms* **have** *card* (*?Cl* − *?Z*) < *card* (*?Cl* − *Z*) **by** (*blast intro*: *psubset-card-mono*)
with *st ws* **have** *?thesis* **unfolding** *mlex-prod-def* **by** *simp*
}
moreover
{ **assume** *remdups* (*filter* (λ*y*. *y* ∉ *Z*) (*f h*)) = []
with *st ws* **have** *?Z* = *Z ?ws* = *t* **by** (*auto simp*: *filter-empty-conv*)
with *st ws* **have** *?thesis* **unfolding** *mlex-prod-def* **by** *simp*
}
ultimately show *?thesis* **by** *blast*
qed
qed (*simp-all add*: *rtrancl-while-invariant*)
qed

end

end

9 The Bourbaki-Witt tower construction for transfinite iteration

theory *Bourbaki-Witt-Fixpoint*
imports *While-Combinator*
begin

lemma *ChainsI* [*intro?*]:

$(\bigwedge a\ b.\ [\![\ a \in Y;\ b \in Y\ ]\!] \Longrightarrow (a,\ b) \in r \vee (b,\ a) \in r) \Longrightarrow Y \in Chains\ r$
unfolding *Chains-def* **by** *blast*

lemma *in-Chains-subset*: $[\![\ M \in Chains\ r;\ M' \subseteq M\ ]\!] \Longrightarrow M' \in Chains\ r$
by(*auto simp add*: *Chains-def*)

lemma *in-ChainsD*: $[\![\ M \in Chains\ r;\ x \in M;\ y \in M\ ]\!] \Longrightarrow (x,\ y) \in r \vee (y,\ x) \in r$
unfolding *Chains-def* **by** *fast*

lemma *Chains-FieldD*: $[\![\ M \in Chains\ r;\ x \in M\ ]\!] \Longrightarrow x \in Field\ r$
by(*auto simp add*: *Chains-def intro*: *FieldI1 FieldI2*)

lemma *in-Chains-conv-chain*: $M \in Chains\ r \longleftrightarrow Complete\text{-}Partial\text{-}Order.chain\ (\lambda x\ y.\ (x,\ y) \in r)\ M$
by(*simp add*: *Chains-def chain-def*)

lemma *partial-order-on-trans*:
$[\![\ partial\text{-}order\text{-}on\ A\ r;\ (x,\ y) \in r;\ (y,\ z) \in r\ ]\!] \Longrightarrow (x,\ z) \in r$
by(*auto simp add*: *order-on-defs dest*: *transD*)

locale *bourbaki-witt-fixpoint* =
fixes *lub* :: $'a\ set \Rightarrow 'a$
and *leq* :: $('a \times 'a)\ set$
and *f* :: $'a \Rightarrow 'a$
assumes *po*: *Partial-order leq*
and *lub-least*: $[\![\ M \in Chains\ leq;\ M \neq \{\};\ \bigwedge x.\ x \in M \Longrightarrow (x,\ z) \in leq\ ]\!] \Longrightarrow (lub\ M,\ z) \in leq$
and *lub-upper*: $[\![\ M \in Chains\ leq;\ x \in M\ ]\!] \Longrightarrow (x,\ lub\ M) \in leq$
and *lub-in-Field*: $[\![\ M \in Chains\ leq;\ M \neq \{\}\ ]\!] \Longrightarrow lub\ M \in Field\ leq$
and *increasing*: $\bigwedge x.\ x \in Field\ leq \Longrightarrow (x,\ f\ x) \in leq$
begin

lemma *leq-trans*: $[\![\ (x,\ y) \in leq;\ (y,\ z) \in leq\ ]\!] \Longrightarrow (x,\ z) \in leq$
by(*rule partial-order-on-trans*[*OF po*])

lemma *leq-refl*: $x \in Field\ leq \Longrightarrow (x,\ x) \in leq$
using *po* **by**(*simp add*: *order-on-defs refl-on-def*)

lemma *leq-antisym*: $[\![\ (x,\ y) \in leq;\ (y,\ x) \in leq\ ]\!] \Longrightarrow x = y$
using *po* **by**(*simp add*: *order-on-defs antisym-def*)

inductive-set *iterates-above* :: $'a \Rightarrow 'a\ set$
for *a*
where
base: $a \in iterates\text{-}above\ a$
| *step*: $x \in iterates\text{-}above\ a \Longrightarrow f\ x \in iterates\text{-}above\ a$
| *Sup*: $[\![\ M \in Chains\ leq;\ M \neq \{\};\ \bigwedge x.\ x \in M \Longrightarrow x \in iterates\text{-}above\ a\ ]\!] \Longrightarrow lub\ M \in iterates\text{-}above\ a$

definition *fixp-above* :: $'a \Rightarrow 'a$
where *fixp-above a* = (*if a* ∈ *Field leq then lub* (*iterates-above a*) *else a*)

lemma *fixp-above-outside*: *a* ∉ *Field leq* ⟹ *fixp-above a* = *a*
by(*simp add*: *fixp-above-def*)

lemma *fixp-above-inside*: *a* ∈ *Field leq* ⟹ *fixp-above a* = *lub* (*iterates-above a*)
by(*simp add*: *fixp-above-def*)

context
  notes *leq-refl* [*intro*!, *simp*]
  and *base* [*intro*]
  and *step* [*intro*]
  and *Sup* [*intro*]
  and *leq-trans* [*trans*]
begin

lemma *iterates-above-le-f*: ⟦ *x* ∈ *iterates-above a*; *a* ∈ *Field leq* ⟧ ⟹ (*x*, *f x*) ∈ *leq*
by(*induction x rule*: *iterates-above.induct*)(*blast intro*: *increasing FieldI2 lub-in-Field*)+

lemma *iterates-above-Field*: ⟦ *x* ∈ *iterates-above a*; *a* ∈ *Field leq* ⟧ ⟹ *x* ∈ *Field leq*
by(*drule* (*1*) *iterates-above-le-f*)(*rule FieldI1*)

lemma *iterates-above-ge*:
  assumes *y*: *y* ∈ *iterates-above a*
  and *a*: *a* ∈ *Field leq*
  shows (*a*, *y*) ∈ *leq*
using *y* **by**(*induction*)(*auto intro*: *a increasing iterates-above-le-f leq-trans leq-trans*[*OF - lub-upper*])

lemma *iterates-above-lub*:
  assumes *M*: *M* ∈ *Chains leq*
  and *nempty*: *M* ≠ {}
  and *upper*: ⋀*y*. *y* ∈ *M* ⟹ ∃ *z* ∈ *M*. (*y*, *z*) ∈ *leq* ∧ *z* ∈ *iterates-above a*
  shows *lub M* ∈ *iterates-above a*
proof −
  let *?M* = *M* ∩ *iterates-above a*
  from *M* **have** *M'*: *?M* ∈ *Chains leq* **by**(*rule in-Chains-subset*)*simp*
  have *?M* ≠ {} **using** *nempty* **by**(*auto dest*: *upper*)
  with *M'* **have** *lub ?M* ∈ *iterates-above a* **by**(*rule Sup*) *blast*
  also have *lub ?M* = *lub M* **using** *nempty*
   by(*intro leq-antisym*)(*blast intro*!: *lub-least*[*OF M*] *lub-least*[*OF M'*] *intro*: *lub-upper*[*OF M'*] *lub-upper*[*OF M*] *leq-trans dest*: *upper*)+
  finally show *?thesis* **.**
qed

lemma *iterates-above-successor*:
  assumes *y*: *y* ∈ *iterates-above a*

and *a*: $a \in Field\ leq$
shows $y = a \vee y \in iterates\text{-}above\ (f\ a)$
using *y*
proof *induction*
case *base* **thus** *?case* **by** *simp*
next
case (*step x*) **thus** *?case* **by** *auto*
next
case (*Sup M*)
show *?case*
proof(*cases* $\exists x.\ M \subseteq \{x\}$)
case *True*
with ‹$M \neq \{\}$› **obtain** *y* **where** *M*: $M = \{y\}$ **by** *auto*
have *lub M* = *y*
by(*rule leq-antisym*)(*auto intro*!: *lub-upper Sup lub-least ChainsI simp add*: *a M Sup.hyps*(*3*)[*of y*, *THEN iterates-above-Field*] *dest*: *iterates-above-Field*)
with *Sup.IH*[*of y*] *M* **show** *?thesis* **by** *simp*
next
case *False*
from *Sup*(*1*−*2*) **have** *lub M* ∈ *iterates-above* (*f a*)
proof(*rule iterates-above-lub*)
fix *y*
assume *y*: $y \in M$
from *Sup.IH*[*OF this*] **show** $\exists z{\in}M.\ (y,\ z) \in leq \wedge z \in iterates\text{-}above\ (f\ a)$
proof
assume $y = a$
from *y False* **obtain** *z* **where** *z*: $z \in M$ **and** *neq*: $y \neq z$ **by** (*metis insertI1 subsetI*)
with *Sup.IH*[*OF z*] ‹$y = a$› *Sup.hyps*(*3*)[*OF z*]
show *?thesis* **by**(*auto dest*: *iterates-above-ge intro*: *a*)
next
assume ∗: $y \in iterates\text{-}above\ (f\ a)$
with *increasing*[*OF a*] **have** $y \in Field\ leq$
by(*auto dest*!: *iterates-above-Field intro*: *FieldI2*)
with ∗ **show** *?thesis* **using** *y* **by** *auto*
qed
qed
thus *?thesis* **by** *simp*
qed
qed

lemma *iterates-above-Sup-aux*:
assumes *M*: $M \in Chains\ leq\ M \neq \{\}$
and *M′*: $M' \in Chains\ leq\ M' \neq \{\}$
and *comp*: $\bigwedge x.\ x \in M \Longrightarrow x \in iterates\text{-}above\ (lub\ M') \vee lub\ M' \in iterates\text{-}above\ x$
shows $(lub\ M,\ lub\ M') \in leq \vee lub\ M \in iterates\text{-}above\ (lub\ M')$
proof(*cases* $\exists x \in M.\ x \in iterates\text{-}above\ (lub\ M')$)
case *True*

then obtain *x* **where** *x*: $x \in M$ $x \in$ *iterates-above* (*lub M'*) **by** *blast*
have *lub-M'*: *lub M'* $\in$ *Field leq* **using** *M'* **by**(*rule lub-in-Field*)
have *lub M* $\in$ *iterates-above* (*lub M'*) **using** *M*
proof(*rule iterates-above-lub*)
fix *y*
assume *y*: $y \in M$
from *comp*[*OF y*] **show** $\exists z{\in}M.\ (y,\ z) \in$ *leq* $\wedge$ $z \in$ *iterates-above* (*lub M'*)
proof
assume $y \in$ *iterates-above* (*lub M'*)
from *this iterates-above-Field*[*OF this*] *y lub-M'* **show** *?thesis* **by** *blast*
next
assume *lub M'* $\in$ *iterates-above y*
hence (*y*, *lub M'*) $\in$ *leq* **using** *Chains-FieldD*[*OF M*(*1*) *y*] **by**(*rule iterates-above-ge*)
also have (*lub M'*, *x*) $\in$ *leq* **using** *x*(*2*) *lub-M'* **by**(*rule iterates-above-ge*)
finally show *?thesis* **using** *x* **by** *blast*
qed
qed
thus *?thesis* **..**
next
case *False*
have (*lub M*, *lub M'*) $\in$ *leq* **using** *M*
proof(*rule lub-least*)
fix *x*
assume *x*: $x \in M$
from *comp*[*OF x*] *x False* **have** *lub M'* $\in$ *iterates-above x* **by** *auto*
moreover from *M*(*1*) *x* **have** $x \in$ *Field leq* **by**(*rule Chains-FieldD*)
ultimately show (*x*, *lub M'*) $\in$ *leq* **by**(*rule iterates-above-ge*)
qed
thus *?thesis* **..**
qed

lemma *iterates-above-triangle*:
assumes *x*: $x \in$ *iterates-above a*
and *y*: $y \in$ *iterates-above a*
and *a*: $a \in$ *Field leq*
shows $x \in$ *iterates-above y* $\vee$ $y \in$ *iterates-above x*
using *x y*
proof(*induction arbitrary*: *y*)
case *base* **then show** *?case* **by** *simp*
next
case (*step x*) **thus** *?case* **using** *a*
by(*auto dest*: *iterates-above-successor intro*: *iterates-above-Field*)
next
case *x*: (*Sup M*)
hence *lub*: *lub M* $\in$ *iterates-above a* **by** *blast*
from ‹*y* $\in$ *iterates-above a*› **show** *?case*
proof(*induction*)
case *base* **show** *?case* **using** *lub* **by** *simp*

next
case (*step y*) **thus** *?case* **using** *a*
by(*auto dest*: *iterates-above-successor intro*: *iterates-above-Field*)
next
case *y*: (*Sup M'*)
hence *lub'*: *lub M'* ∈ *iterates-above a* **by** *blast*
have ∗: *x* ∈ *iterates-above* (*lub M'*) ∨ *lub M'* ∈ *iterates-above x* **if** *x* ∈ *M* **for** *x*
using *that lub'* **by**(*rule x.IH*)
with *x*(*1*−*2*) *y*(*1*−*2*) **have** (*lub M*, *lub M'*) ∈ *leq* ∨ *lub M* ∈ *iterates-above* (*lub M'*)
by(*rule iterates-above-Sup-aux*)
moreover from *y*(*1*−*2*) *x*(*1*−*2*) **have** (*lub M'*, *lub M*) ∈ *leq* ∨ *lub M'* ∈ *iterates-above* (*lub M*)
by(*rule iterates-above-Sup-aux*)(*blast dest*: *y.IH*)
ultimately show *?case* **by**(*auto 4 3 dest*: *leq-antisym*)
qed
qed

lemma *chain-iterates-above*:
assumes *a*: *a* ∈ *Field leq*
shows *iterates-above a* ∈ *Chains leq* (**is** *?C* ∈ -)
proof (*rule ChainsI*)
fix *x y*
assume *x* ∈ *?C y* ∈ *?C*
hence *x* ∈ *iterates-above y* ∨ *y* ∈ *iterates-above x* **using** *a* **by**(*rule iterates-above-triangle*)
moreover from ‹*x* ∈ *?C*› *a* **have** *x* ∈ *Field leq* **by**(*rule iterates-above-Field*)
moreover from ‹*y* ∈ *?C*› *a* **have** *y* ∈ *Field leq* **by**(*rule iterates-above-Field*)
ultimately show (*x*, *y*) ∈ *leq* ∨ (*y*, *x*) ∈ *leq* **by**(*auto dest*: *iterates-above-ge*)
qed

lemma *fixp-iterates-above*: *fixp-above a* ∈ *iterates-above a*
by(*auto intro*: *chain-iterates-above simp add*: *fixp-above-def*)

lemma *fixp-above-Field*: *a* ∈ *Field leq* ⟹ *fixp-above a* ∈ *Field leq*
using *fixp-iterates-above* **by**(*rule iterates-above-Field*)

lemma *fixp-above-unfold*:
assumes *a*: *a* ∈ *Field leq*
shows *fixp-above a* = *f* (*fixp-above a*) (**is** *?a* = *f ?a*)
proof(*rule leq-antisym*)
show (*?a*, *f ?a*) ∈ *leq* **using** *fixp-above-Field*[*OF a*] **by**(*rule increasing*)

have *f ?a* ∈ *iterates-above a* **using** *fixp-iterates-above* **by**(*rule iterates-above.step*)
with *chain-iterates-above*[*OF a*] **show** (*f ?a*, *?a*) ∈ *leq*
by(*simp add*: *fixp-above-inside assms lub-upper*)
qed

end

lemma *fixp-above-induct* [*case-names adm base step*]:
assumes *adm*: *ccpo.admissible lub* (λ*x y*. (*x*, *y*) ∈ *leq*) *P*
and *base*: *P a*
and *step*: ⋀*x*. *P x* ⟹ *P* (*f x*)
shows *P* (*fixp-above a*)
proof(*cases a* ∈ *Field leq*)
case *True*
from *adm chain-iterates-above*[*OF True*]
show *?thesis* **unfolding** *fixp-above-inside*[*OF True*] *in-Chains-conv-chain*
proof(*rule ccpo.admissibleD*)
have *a* ∈ *iterates-above a* **..**
then show *iterates-above a* ≠ {} **by**(*auto*)
show *P x* **if** *x* ∈ *iterates-above a* **for** *x* **using** *that*
by *induction*(*auto intro*: *base step simp add*: *in-Chains-conv-chain dest*: *ccpo.admissibleD*[*OF adm*])
qed
qed(*simp add*: *fixp-above-outside base*)

end

9.1 Connect with the while combinator for executability on chain-finite lattices.

context *bourbaki-witt-fixpoint* **begin**

lemma *in-Chains-finite*: — Translation from ⟦*Complete-Partial-Order.chain* (≤) *?A*; *finite ?A*; *?A* ≠ {}⟧ ⟹ *Sup ?A* ∈ *?A*.
assumes *M* ∈ *Chains leq*
and *M* ≠ {}
and *finite M*
shows *lub M* ∈ *M*
using *assms*(*3*,*1*,*2*)
proof *induction*
case *empty* **thus** *?case* **by** *simp*
next
case (*insert x M*)
note *chain* = ‹*insert x M* ∈ *Chains leq*›
show *?case*
proof(*cases M* = {})
case *True* **thus** *?thesis*
using *chain in-ChainsD leq-antisym lub-least lub-upper* **by** *fastforce*
next
case *False*
from *chain* **have** *chain'*: *M* ∈ *Chains leq*
using *in-Chains-subset subset-insertI* **by** *blast*
hence *lub M* ∈ *M* **using** *False* **by**(*rule insert.IH*)
show *?thesis*
proof(*cases* (*x*, *lub M*) ∈ *leq*)
case *True*

```
    have (lub (insert x M), lub M) ∈ leq using chain
      by (rule lub-least) (auto simp: True intro: lub-upper[OF chain'])
    with False have lub (insert x M) = lub M
     using lub-upper[OF chain] lub-least[OF chain'] by (blast intro: leq-antisym)
    with ‹lub M ∈ M› show ?thesis by simp
   next
    case False
    with in-ChainsD[OF chain, of x lub M] ‹lub M ∈ M›
    have lub (insert x M) = x
      by − (rule leq-antisym, (blast intro: FieldI2 chain chain' insert.prems(2)
leq-refl leq-trans lub-least lub-upper)+)
    thus ?thesis by simp
   qed
  qed
qed

lemma fun-pow-iterates-above: (f ^^ k) a ∈ iterates-above a
using iterates-above.base iterates-above.step by (induct k) simp-all

lemma chfin-iterates-above-fun-pow:
  assumes x ∈ iterates-above a
  assumes ∀ M ∈ Chains leq. finite M
  shows ∃ j. x = (f ^^ j) a
using assms(1)
proof induct
  case base then show ?case by (simp add: exI[where x=0])
next
  case (step x) then obtain j where x = (f ^^ j) a by blast
  with step(1) show ?case by (simp add: exI[where x=Suc j])
next
  case (Sup M) with in-Chains-finite assms(2) show ?case by blast
qed

lemma Chain-finite-iterates-above-fun-pow-iff:
  assumes ∀ M ∈ Chains leq. finite M
  shows x ∈ iterates-above a ⟷ (∃ j. x = (f ^^ j) a)
using chfin-iterates-above-fun-pow fun-pow-iterates-above assms by blast

lemma fixp-above-Kleene-iter-ex:
  assumes (∀ M ∈ Chains leq. finite M)
  obtains k where fixp-above a = (f ^^ k) a
using assms by atomize-elim (simp add: chfin-iterates-above-fun-pow fixp-iterates-above)

context fixes a assumes a: a ∈ Field leq begin

lemma funpow-Field-leq: (f ^^ k) a ∈ Field leq
using a by (induct k) (auto intro: increasing FieldI2)

lemma funpow-prefix: j < k ⟹ ((f ^^ j) a, (f ^^ k) a) ∈ leq
```

proof(*induct k*)
case (*Suc k*)
with *leq-trans*[*OF - increasing*[*OF funpow-Field-leq*]] *funpow-Field-leq increasing a*
show *?case* **by** *simp* (*metis less-antisym*)
qed *simp*

lemma *funpow-suffix*: $(f \frown\frown \mathit{Suc}\ k)\ a = (f \frown\frown k)\ a \Longrightarrow ((f \frown\frown (j + k))\ a, (f \frown\frown k)\ a) \in \mathit{leq}$
using *funpow-Field-leq*
by (*induct j*) (*simp-all del*: *funpow.simps add*: *funpow-Suc-right funpow-add leq-refl*)

lemma *funpow-stability*: $(f \frown\frown \mathit{Suc}\ k)\ a = (f \frown\frown k)\ a \Longrightarrow ((f \frown\frown j)\ a, (f \frown\frown k)\ a) \in \mathit{leq}$
using *funpow-prefix funpow-suffix*[**where** $j{=}j - k$ **and** $k{=}k$] **by** (*cases* $j < k$) *simp-all*

lemma *funpow-in-Chains*: $\{(f \frown\frown k)\ a \mid k.\ \mathit{True}\} \in \mathit{Chains}\ \mathit{leq}$
using *chain-iterates-above*[*OF a*] *fun-pow-iterates-above*
by (*blast intro*: *ChainsI dest*: *in-ChainsD*)

lemma *fixp-above-Kleene-iter*:
assumes $\forall M \in \mathit{Chains}\ \mathit{leq}.\ \mathit{finite}\ M$ — convenient but surely not necessary
assumes $(f \frown\frown \mathit{Suc}\ k)\ a = (f \frown\frown k)\ a$
shows $\mathit{fixp\text{-}above}\ a = (f \frown\frown k)\ a$
proof(*rule leq-antisym*)
show (*fixp-above a*, $(f \frown\frown k)\ a) \in \mathit{leq}$ **using** *assms a*
by(*auto simp add*: *fixp-above-def chain-iterates-above Chain-finite-iterates-above-fun-pow-iff funpow-stability*[*OF assms*(*2*)] *intro*!: *lub-least intro*: *iterates-above.base*)
show $((f \frown\frown k)\ a, \mathit{fixp\text{-}above}\ a) \in \mathit{leq}$ **using** *a*
by(*auto simp add*: *fixp-above-def chain-iterates-above fun-pow-iterates-above intro*!: *lub-upper*)
qed

context assumes *chfin*: $\forall M \in \mathit{Chains}\ \mathit{leq}.\ \mathit{finite}\ M$ **begin**

lemma *Chain-finite-wf*: *wf* $\{(f\ ((f \frown\frown k)\ a), (f \frown\frown k)\ a) \mid k.\ f\ ((f \frown\frown k)\ a) \neq (f \frown\frown k)\ a\}$
apply(*rule wf-measure*[**where** $f{=}\lambda b.\ \mathit{card}\ \{(f \frown\frown j)\ a \mid j.\ (b, (f \frown\frown j)\ a) \in \mathit{leq}\}$, *THEN wf-subset*])
apply(*auto simp*: *set-eq-iff intro*!: *psubset-card-mono*[*OF finite-subset*[*OF - bspec*[*OF chfin funpow-in-Chains*]]])
apply(*metis funpow-Field-leq increasing leq-antisym leq-trans leq-refl*)+
done

lemma *while-option-finite-increasing*: $\exists P.\ \mathit{while\text{-}option}\ (\lambda A.\ f\ A \neq A)\ f\ a = \mathit{Some}\ P$
by(*rule wf-rel-while-option-Some*[*OF Chain-finite-wf*, **where** $P{=}\lambda A.\ (\exists k.\ A = (f \frown\frown k)\ a) \wedge (A, f\ A) \in \mathit{leq}$ **and** $s{=}a$])

(*auto simp*: *a increasing chfin FieldI2 chfin-iterates-above-fun-pow fun-pow-iterates-above iterates-above.step intro*: *exI*[**where** *x=0*])

lemma *fixp-above-the-while-option*: *fixp-above a = the* (*while-option* (λ*A*. *f A* ≠ *A*) *f a*)
proof −
obtain *P* **where** *while-option* (λ*A*. *f A* ≠ *A*) *f a* = *Some P*
using *while-option-finite-increasing* **by** *blast*
with *while-option-stop2*[*OF this*] *fixp-above-Kleene-iter*[*OF chfin*]
show *?thesis* **by** *fastforce*
qed

lemma *fixp-above-conv-while*: *fixp-above a = while* (λ*A*. *f A* ≠ *A*) *f a*
unfolding *while-def* **by** (*rule fixp-above-the-while-option*)

end

end

end

lemma *bourbaki-witt-fixpoint-complete-latticeI*:
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*
assumes ⋀*x*. *x* ≤ *f x*
shows *bourbaki-witt-fixpoint Sup* {(*x*, *y*). *x* ≤ *y*} *f*
by *unfold-locales* (*auto simp*: *assms Sup-upper order-on-defs Field-def intro*: *refl-onI transI antisymI Sup-least*)

end

10 Order on characters

theory *Char-ord*
imports *Main*
begin

instantiation *char* :: *linorder*
begin

definition *less-eq-char* :: *char* ⇒ *char* ⇒ *bool*
where *c1* ≤ *c2* ⟷ *of-char c1* ≤ (*of-char c2* :: *nat*)

definition *less-char* :: *char* ⇒ *char* ⇒ *bool*
where *c1* < *c2* ⟷ *of-char c1* < (*of-char c2* :: *nat*)

instance
by *standard* (*auto simp add*: *less-eq-char-def less-char-def*)

end

lemma *less-eq-char-simp* [*simp*]:
Char b0 b1 b2 b3 b4 b5 b6 b7 ≤ *Char c0 c1 c2 c3 c4 c5 c6 c7*
⟷ *foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*, *b7*] *0*
≤ *foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*c0*, *c1*, *c2*, *c3*, *c4*, *c5*, *c6*, *c7*] (*0*::*nat*)
by (*simp add*: *less-eq-char-def*)

lemma *less-char-simp* [*simp*]:
Char b0 b1 b2 b3 b4 b5 b6 b7 < *Char c0 c1 c2 c3 c4 c5 c6 c7*
⟷ *foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*b0*, *b1*, *b2*, *b3*, *b4*, *b5*, *b6*, *b7*] *0*
< *foldr* (λ*b k*. *of-bool b* + *k* ∗ *2*) [*c0*, *c1*, *c2*, *c3*, *c4*, *c5*, *c6*, *c7*] (*0*::*nat*)
by (*simp add*: *less-char-def*)

instantiation *char* :: *distrib-lattice*
begin

definition (*inf* :: *char* ⇒ -) = *min*
definition (*sup* :: *char* ⇒ -) = *max*

instance
by *standard* (*auto simp add*: *inf-char-def sup-char-def max-min-distrib2*)

end

end

11 A generic phantom type

theory *Phantom-Type*
imports *Main*
begin

datatype ('*a*, '*b*) *phantom* = *phantom* (*of-phantom*: '*b*)

lemma *type-definition-phantom'*: *type-definition of-phantom phantom UNIV*
by(*unfold-locales*) *simp-all*

lemma *phantom-comp-of-phantom* [*simp*]: *phantom* ∘ *of-phantom* = *id*
and *of-phantom-comp-phantom* [*simp*]: *of-phantom* ∘ *phantom* = *id*
by(*simp-all add*: *o-def id-def*)

syntax -*Phantom* :: *type* ⇒ *logic* ((*1Phantom*/(*1* '(-'))))
translations
Phantom('*t*) => *CONST phantom* :: - ⇒ ('*t*, -) *phantom*

typed-print-translation ‹
let
fun phantom-tr' ctxt (*Type* (***type-name*** ‹*fun*›, [-, *Type* (***type-name*** ‹*phantom*›,

```
[T, -])])) ts =
        list-comb
        (Syntax.const syntax-const ‹-Phantom› $ Syntax-Phases.term-of-typ ctxt
T, ts)
    | phantom-tr' - - - = raise Match;
  in [(const-syntax ‹phantom›, phantom-tr')] end
›

lemma of-phantom-inject [simp]:
  of-phantom x = of-phantom y ⟷ x = y
by(cases x y rule: phantom.exhaust[case-product phantom.exhaust]) simp

end
```

12 Cardinality of types

```
theory Cardinality
imports Phantom-Type
begin
```

12.1 Preliminary lemmas

```
lemma (in type-definition) univ:
  UNIV = Abs ' A
proof
  show Abs ' A ⊆ UNIV by (rule subset-UNIV)
  show UNIV ⊆ Abs ' A
  proof
    fix x :: 'b
    have x = Abs (Rep x) by (rule Rep-inverse [symmetric])
    moreover have Rep x ∈ A by (rule Rep)
    ultimately show x ∈ Abs ' A by (rule image-eqI)
  qed
qed

lemma (in type-definition) card: card (UNIV :: 'b set) = card A
  by (simp add: univ card-image inj-on-def Abs-inject)
```

12.2 Cardinalities of types

```
syntax -type-card :: type => nat ((1CARD/(1'(-'))))

translations CARD('t) => CONST card (CONST UNIV :: 't set)

print-translation ‹
  let
    fun card-univ-tr' ctxt [Const (const-syntax ‹UNIV›, Type (-, [T]))] =
      Syntax.const syntax-const ‹-type-card› $ Syntax-Phases.term-of-typ ctxt T
  in [(const-syntax ‹card›, card-univ-tr')] end
```

›

lemma *card-prod* [*simp*]: $CARD('a \times 'b) = CARD('a) * CARD('b)$
unfolding *UNIV-Times-UNIV* [*symmetric*] **by** (*simp only*: *card-cartesian-product*)

lemma *card-UNIV-sum*: $CARD('a + 'b) = (\mathit{if}\ CARD('a) \neq 0 \wedge CARD('b) \neq 0\ \mathit{then}\ CARD('a) + CARD('b)\ \mathit{else}\ 0)$
unfolding *UNIV-Plus-UNIV*[*symmetric*]
by(*auto simp add*: *card-eq-0-iff card-Plus simp del*: *UNIV-Plus-UNIV*)

lemma *card-sum* [*simp*]: $CARD('a + 'b) = CARD('a::finite) + CARD('b::finite)$
by(*simp add*: *card-UNIV-sum*)

lemma *card-UNIV-option*: $CARD('a\ option) = (\mathit{if}\ CARD('a) = 0\ \mathit{then}\ 0\ \mathit{else}\ CARD('a) + 1)$
proof −
have $(None :: 'a\ option) \notin range\ Some$ **by** *clarsimp*
thus *?thesis*
by (*simp add*: *UNIV-option-conv card-eq-0-iff finite-range-Some card-image*)
qed

lemma *card-option* [*simp*]: $CARD('a\ option) = Suc\ CARD('a::finite)$
by(*simp add*: *card-UNIV-option*)

lemma *card-UNIV-set*: $CARD('a\ set) = (\mathit{if}\ CARD('a) = 0\ \mathit{then}\ 0\ \mathit{else}\ 2\ \hat{}\ CARD('a))$
by(*simp add*: *card-eq-0-iff card-Pow flip*: *Pow-UNIV*)

lemma *card-set* [*simp*]: $CARD('a\ set) = 2\ \hat{}\ CARD('a::finite)$
by(*simp add*: *card-UNIV-set*)

lemma *card-nat* [*simp*]: $CARD(nat) = 0$
by (*simp add*: *card-eq-0-iff*)

lemma *card-fun*: $CARD('a \Rightarrow 'b) = (\mathit{if}\ CARD('a) \neq 0 \wedge CARD('b) \neq 0 \vee CARD('b) = 1\ \mathit{then}\ CARD('b)\ \hat{}\ CARD('a)\ \mathit{else}\ 0)$
proof −
{ **assume** $0 < CARD('a)$ **and** $0 < CARD('b)$
hence *fina*: *finite* $(UNIV :: 'a\ set)$ **and** *finb*: *finite* $(UNIV :: 'b\ set)$
by(*simp-all only*: *card-ge-0-finite*)
from *finite-distinct-list*[*OF finb*] **obtain** *bs*
where *bs*: $set\ bs = (UNIV :: 'b\ set)$ **and** *distb*: *distinct bs* **by** *blast*
from *finite-distinct-list*[*OF fina*] **obtain** *as*
where *as*: $set\ as = (UNIV :: 'a\ set)$ **and** *dista*: *distinct as* **by** *blast*
have *cb*: $CARD('b) = length\ bs$
unfolding *bs*[*symmetric*] *distinct-card*[*OF distb*] **..**
have *ca*: $CARD('a) = length\ as$
unfolding *as*[*symmetric*] *distinct-card*[*OF dista*] **..**
let *?xs* = $map\ (\lambda ys.\ the \circ map\text{-}of\ (zip\ as\ ys))\ (List.n\text{-}lists\ (length\ as)\ bs)$
have $UNIV = set\ ?xs$

```
  proof(rule UNIV-eq-I)
    fix f :: 'a ⇒ 'b
    from as have f = the ∘ map-of (zip as (map f as))
      by(auto simp add: map-of-zip-map)
    thus f ∈ set ?xs using bs by(auto simp add: set-n-lists)
  qed
  moreover have distinct ?xs unfolding distinct-map
  proof(intro conjI distinct-n-lists distb inj-onI)
    fix xs ys :: 'b list
    assume xs: xs ∈ set (List.n-lists (length as) bs)
      and ys: ys ∈ set (List.n-lists (length as) bs)
      and eq: the ∘ map-of (zip as xs) = the ∘ map-of (zip as ys)
    from xs ys have [simp]: length xs = length as length ys = length as
      by(simp-all add: length-n-lists-elem)
    have map-of (zip as xs) = map-of (zip as ys)
    proof
      fix x
       from as bs have ∃ y. map-of (zip as xs) x = Some y ∃ y. map-of (zip as
ys) x = Some y
        by(simp-all add: map-of-zip-is-Some[symmetric])
      with eq show map-of (zip as xs) x = map-of (zip as ys) x
        by(auto dest: fun-cong[where x=x])
    qed
    with dista show xs = ys by(simp add: map-of-zip-inject)
  qed
  hence card (set ?xs) = length ?xs by(simp only: distinct-card)
  moreover have length ?xs = length bs ^ length as by(simp add: length-n-lists)
   ultimately have CARD('a ⇒ 'b) = CARD('b) ^ CARD('a) using cb ca by
simp }
 moreover {
  assume cb: CARD('b) = 1
  then obtain b where b: UNIV = {b :: 'b} by(auto simp add: card-Suc-eq)
  have eq: UNIV = {λx :: 'a. b ::'b}
  proof(rule UNIV-eq-I)
    fix x :: 'a ⇒ 'b
    { fix y
      have x y ∈ UNIV ..
      hence x y = b unfolding b by simp }
    thus x ∈ {λx. b} by(auto)
  qed
  have CARD('a ⇒ 'b) = 1 unfolding eq by simp }
 ultimately show ?thesis
  by(auto simp del: One-nat-def)(auto simp add: card-eq-0-iff dest: finite-fun-UNIVD2
finite-fun-UNIVD1)
qed

corollary finite-UNIV-fun:
 finite (UNIV :: ('a ⇒ 'b) set) ⟷
  finite (UNIV :: 'a set) ∧ finite (UNIV :: 'b set) ∨ CARD('b) = 1
```

(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof −
have *?lhs* $\longleftrightarrow$ *CARD*('*a* $\Rightarrow$ '*b*) > *0* **by**(*simp add*: *card-gt-0-iff*)
also have ... $\longleftrightarrow$ *CARD*('*a*) > *0* $\wedge$ *CARD*('*b*) > *0* $\vee$ *CARD*('*b*) = *1*
by(*simp add*: *card-fun*)
also have ... = *?rhs* **by**(*simp add*: *card-gt-0-iff*)
finally show *?thesis* **.**
qed

lemma *card-literal*: *CARD*(*String.literal*) = *0*
by(*simp add*: *card-eq-0-iff infinite-literal*)

12.3 Classes with at least 1 and 2

Class finite already captures "at least 1"

lemma *zero-less-card-finite* [*simp*]: *0* < *CARD*('*a*::*finite*)
unfolding *neq0-conv* [*symmetric*] **by** *simp*

lemma *one-le-card-finite* [*simp*]: *Suc 0* ≤ *CARD*('*a*::*finite*)
by (*simp add*: *less-Suc-eq-le* [*symmetric*])

class *CARD-1* =
assumes *CARD-1*: *CARD* ('*a*) = *1*
begin

subclass *finite*
proof
from *CARD-1* **show** *finite* (*UNIV* :: '*a set*)
using *finite-UNIV-fun* **by** *fastforce*
qed

end

Class for cardinality "at least 2"

class *card2* = *finite* +
assumes *two-le-card*: *2* ≤ *CARD*('*a*)

lemma *one-less-card*: *Suc 0* < *CARD*('*a*::*card2*)
using *two-le-card* [**where** '*a*='*a*] **by** *simp*

lemma *one-less-int-card*: *1* < *int CARD*('*a*::*card2*)
using *one-less-card* [**where** '*a*='*a*] **by** *simp*

12.4 A type class for deciding finiteness of types

type-synonym '*a finite-UNIV* = ('*a*, *bool*) *phantom*

class *finite-UNIV* =

fixes *finite-UNIV* :: (*'a*, *bool*) *phantom*
assumes *finite-UNIV*: *finite-UNIV* = *Phantom*(*'a*) (*finite* (*UNIV* :: *'a set*))

lemma *finite-UNIV-code* [*code-unfold*]:
finite (*UNIV* :: *'a* :: *finite-UNIV set*)
⟷ *of-phantom* (*finite-UNIV* :: *'a finite-UNIV*)
by(*simp add*: *finite-UNIV*)

12.5 A type class for computing the cardinality of types

definition *is-list-UNIV* :: *'a list* ⇒ *bool*
where *is-list-UNIV xs* = (*let c* = *CARD*(*'a*) *in if c* = *0 then False else size* (*remdups xs*) = *c*)

lemma *is-list-UNIV-iff*: *is-list-UNIV xs* ⟷ *set xs* = *UNIV*
by(*auto simp add*: *is-list-UNIV-def Let-def card-eq-0-iff List.card-set*[*symmetric*]
dest: *subst*[**where** *P*=*finite*, *OF - finite-set*] *card-eq-UNIV-imp-eq-UNIV*)

type-synonym *'a card-UNIV* = (*'a*, *nat*) *phantom*

class *card-UNIV* = *finite-UNIV* +
fixes *card-UNIV* :: *'a card-UNIV*
assumes *card-UNIV*: *card-UNIV* = *Phantom*(*'a*) *CARD*(*'a*)

12.6 Instantiations for *card-UNIV*

instantiation *nat* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*nat*) *False*
definition *card-UNIV* = *Phantom*(*nat*) *0*
instance by *intro-classes* (*simp-all add*: *finite-UNIV-nat-def card-UNIV-nat-def*)
end

instantiation *int* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*int*) *False*
definition *card-UNIV* = *Phantom*(*int*) *0*
instance by *intro-classes* (*simp-all add*: *card-UNIV-int-def finite-UNIV-int-def*)
end

instantiation *natural* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*natural*) *False*
definition *card-UNIV* = *Phantom*(*natural*) *0*
instance
by *standard*
(*auto simp add*: *finite-UNIV-natural-def card-UNIV-natural-def card-eq-0-iff*
type-definition.univ [*OF type-definition-natural*] *natural-eq-iff*
dest!: *finite-imageD intro*: *inj-onI*)
end

instantiation *integer* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*integer*) *False*

definition *card-UNIV* = *Phantom*(*integer*) *0*
instance
by *standard*
(*auto simp add*: *finite-UNIV-integer-def card-UNIV-integer-def card-eq-0-iff*
type-definition.univ [*OF type-definition-integer*]
dest!: *finite-imageD intro*: *inj-onI*)
end

instantiation *list* :: (*type*) *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a list*) *False*
definition *card-UNIV* = *Phantom*('*a list*) *0*
instance by *intro-classes* (*simp-all add*: *card-UNIV-list-def finite-UNIV-list-def infinite-UNIV-listI*)
end

instantiation *unit* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*unit*) *True*
definition *card-UNIV* = *Phantom*(*unit*) *1*
instance by *intro-classes* (*simp-all add*: *card-UNIV-unit-def finite-UNIV-unit-def*)
end

instantiation *bool* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*bool*) *True*
definition *card-UNIV* = *Phantom*(*bool*) *2*
instance by(*intro-classes*)(*simp-all add*: *card-UNIV-bool-def finite-UNIV-bool-def*)
end

instantiation *char* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*char*) *True*
definition *card-UNIV* = *Phantom*(*char*) *256*
instance by *intro-classes* (*simp-all add*: *card-UNIV-char-def card-UNIV-char finite-UNIV-char-def*)
end

instantiation *prod* :: (*finite-UNIV*, *finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* × '*b*)
(*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*) ∧ *of-phantom* (*finite-UNIV* :: '*b finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-prod-def finite-UNIV finite-prod*)
end

instantiation *prod* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a* × '*b*)
(*of-phantom* (*card-UNIV* :: '*a card-UNIV*) ∗ *of-phantom* (*card-UNIV* :: '*b card-UNIV*))
instance by *intro-classes* (*simp add*: *card-UNIV-prod-def card-UNIV*)
end

instantiation *sum* :: (*finite-UNIV*, *finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a* + '*b*)

(*of-phantom* (*finite-UNIV* :: *'a finite-UNIV*) ∧ *of-phantom* (*finite-UNIV* :: *'b finite-UNIV*))
instance
by *intro-classes* (*simp add*: *finite-UNIV-sum-def finite-UNIV*)
end

instantiation *sum* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*(*'a* + *'b*)
(*let ca* = *of-phantom* (*card-UNIV* :: *'a card-UNIV*);
cb = *of-phantom* (*card-UNIV* :: *'b card-UNIV*)
in if ca ≠ *0* ∧ *cb* ≠ *0 then ca* + *cb else 0*)
instance by *intro-classes* (*auto simp add*: *card-UNIV-sum-def card-UNIV card-UNIV-sum*)
end

instantiation *fun* :: (*finite-UNIV*, *card-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*'a* ⇒ *'b*)
(*let cb* = *of-phantom* (*card-UNIV* :: *'b card-UNIV*)
in cb = *1* ∨ *of-phantom* (*finite-UNIV* :: *'a finite-UNIV*) ∧ *cb* ≠ *0*)
instance
by *intro-classes* (*auto simp add*: *finite-UNIV-fun-def Let-def card-UNIV finite-UNIV finite-UNIV-fun card-gt-0-iff*)
end

instantiation *fun* :: (*card-UNIV*, *card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*(*'a* ⇒ *'b*)
(*let ca* = *of-phantom* (*card-UNIV* :: *'a card-UNIV*);
cb = *of-phantom* (*card-UNIV* :: *'b card-UNIV*)
in if ca ≠ *0* ∧ *cb* ≠ *0* ∨ *cb* = *1 then cb* ^ *ca else 0*)
instance by *intro-classes* (*simp add*: *card-UNIV-fun-def card-UNIV Let-def card-fun*)
end

instantiation *option* :: (*finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*'a option*) (*of-phantom* (*finite-UNIV* :: *'a finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-option-def finite-UNIV*)
end

instantiation *option* :: (*card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*(*'a option*)
(*let c* = *of-phantom* (*card-UNIV* :: *'a card-UNIV*) *in if c* ≠ *0 then Suc c else 0*)
instance by *intro-classes* (*simp add*: *card-UNIV-option-def card-UNIV card-UNIV-option*)
end

instantiation *String.literal* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*String.literal*) *False*
definition *card-UNIV* = *Phantom*(*String.literal*) *0*
instance
by *intro-classes* (*simp-all add*: *card-UNIV-literal-def finite-UNIV-literal-def infinite-literal card-literal*)

end

instantiation *set* :: (*finite-UNIV*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a set*) (*of-phantom* (*finite-UNIV* :: '*a finite-UNIV*))
instance by *intro-classes* (*simp add*: *finite-UNIV-set-def finite-UNIV Finite-Set.finite-set*)
end

instantiation *set* :: (*card-UNIV*) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a set*)
(*let c* = *of-phantom* (*card-UNIV* :: '*a card-UNIV*) *in if c* = *0 then 0 else 2* ^ *c*)
instance by *intro-classes* (*simp add*: *card-UNIV-set-def card-UNIV-set card-UNIV*)
end

lemma *UNIV-finite-1*: *UNIV* = *set* [*finite-1*.a_1]
by(*auto intro*: *finite-1.exhaust*)

lemma *UNIV-finite-2*: *UNIV* = *set* [*finite-2*.a_1, *finite-2*.a_2]
by(*auto intro*: *finite-2.exhaust*)

lemma *UNIV-finite-3*: *UNIV* = *set* [*finite-3*.a_1, *finite-3*.a_2, *finite-3*.a_3]
by(*auto intro*: *finite-3.exhaust*)

lemma *UNIV-finite-4*: *UNIV* = *set* [*finite-4*.a_1, *finite-4*.a_2, *finite-4*.a_3, *finite-4*.a_4]
by(*auto intro*: *finite-4.exhaust*)

lemma *UNIV-finite-5*:
UNIV = *set* [*finite-5*.a_1, *finite-5*.a_2, *finite-5*.a_3, *finite-5*.a_4, *finite-5*.a_5]
by(*auto intro*: *finite-5.exhaust*)

instantiation *Enum.finite-1* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-1*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-1*) *1*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-1 card-UNIV-finite-1-def finite-UNIV-finite-1-def*)
end

instantiation *Enum.finite-2* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-2*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-2*) *2*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-2 card-UNIV-finite-2-def finite-UNIV-finite-2-def*)
end

instantiation *Enum.finite-3* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-3*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-3*) *3*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-3 card-UNIV-finite-3-def finite-UNIV-finite-3-def*)

end

instantiation *Enum.finite-4* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-4*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-4*) *4*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-4 card-UNIV-finite-4-def finite-UNIV-finite-4-def*)
end

instantiation *Enum.finite-5* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*Enum.finite-5*) *True*
definition *card-UNIV* = *Phantom*(*Enum.finite-5*) *5*
instance
by *intro-classes* (*simp-all add*: *UNIV-finite-5 card-UNIV-finite-5-def finite-UNIV-finite-5-def*)
end

end

12.7 Code setup for sets with cardinality type information

theory *Code-Cardinality* **imports** *Cardinality* **begin**

Implement *CARD*('*a*) via *card-UNIV-class.card-UNIV* and provide implementations for *finite*, *card*, (⊆), and (=)if the calling context already provides *finite-UNIV* and *card-UNIV* instances. If we implemented the latter always via *card-UNIV-class.card-UNIV*, we would require instances of essentially all element types, i.e., a lot of instantiation proofs and – at run time – possibly slow dictionary constructions.

context
begin

qualified definition *card-UNIV'* :: '*a card-UNIV*
where [*code del*]: *card-UNIV'* = *Phantom*('*a*) *CARD*('*a*)

lemma *CARD-code* [*code-unfold*]:
CARD('*a*) = *of-phantom* (*card-UNIV'* :: '*a card-UNIV*)
by(*simp add*: *card-UNIV'-def*)

lemma *card-UNIV'-code* [*code*]:
card-UNIV' = *card-UNIV*
by(*simp add*: *card-UNIV card-UNIV'-def*)

end

lemma *card-Compl*:
finite A ⟹ *card* (− *A*) = *card* (*UNIV* :: '*a set*) − *card* (*A* :: '*a set*)
by (*metis Compl-eq-Diff-UNIV card-Diff-subset top-greatest*)

context fixes *xs* :: '*a* :: *finite-UNIV list*

begin

qualified definition *finite'* :: *'a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *finite'* = *finite*

lemma *finite'-code* [*code*]:
finite' (*set xs*) ⟷ *True*
finite' (*List.coset xs*) ⟷ *of-phantom* (*finite-UNIV* :: *'a finite-UNIV*)
by(*simp-all add*: *card-gt-0-iff finite-UNIV*)

end

context fixes *xs* :: *'a* :: *card-UNIV list*
begin

qualified definition *card'* :: *'a set* ⇒ *nat*
where [*simp*, *code del*, *code-abbrev*]: *card'* = *card*

lemma *card'-code* [*code*]:
card' (*set xs*) = *length* (*remdups xs*)
card' (*List.coset xs*) = *of-phantom* (*card-UNIV* :: *'a card-UNIV*) − *length* (*remdups xs*)
by(*simp-all add*: *List.card-set card-Compl card-UNIV*)

qualified definition *subset'* :: *'a set* ⇒ *'a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *subset'* = (⊆)

lemma *subset'-code* [*code*]:
subset' *A* (*List.coset ys*) ⟷ (∀ *y* ∈ *set ys*. *y* ∉ *A*)
subset' (*set ys*) *B* ⟷ (∀ *y* ∈ *set ys*. *y* ∈ *B*)
subset' (*List.coset xs*) (*set ys*) ⟷ (*let n* = *CARD*(*'a*) *in n* > *0* ∧ *card*(*set* (*xs* @ *ys*)) = *n*)
by(*auto simp add*: *Let-def card-gt-0-iff dest*: *card-eq-UNIV-imp-eq-UNIV intro*: *arg-cong*[**where** *f*=*card*])
(*metis finite-compl finite-set rev-finite-subset*)

qualified definition *eq-set* :: *'a set* ⇒ *'a set* ⇒ *bool*
where [*simp*, *code del*, *code-abbrev*]: *eq-set* = (=)

lemma *eq-set-code* [*code*]:
fixes *ys*
defines *rhs* ≡
let n = *CARD*(*'a*)
in if n = *0 then False else*
let xs' = *remdups xs*; *ys'* = *remdups ys*
in length xs' + *length ys'* = *n* ∧ (∀ *x* ∈ *set xs'*. *x* ∉ *set ys'*) ∧ (∀ *y* ∈ *set ys'*. *y* ∉ *set xs'*)
shows *eq-set* (*List.coset xs*) (*set ys*) ⟷ *rhs*

```
  and eq-set (set ys) (List.coset xs) ⟷ rhs
  and eq-set (set xs) (set ys) ⟷ (∀ x ∈ set xs. x ∈ set ys) ∧ (∀ y ∈ set ys. y ∈
set xs)
  and eq-set (List.coset xs) (List.coset ys) ⟷ (∀ x ∈ set xs. x ∈ set ys) ∧ (∀ y ∈
set ys. y ∈ set xs)
proof goal-cases
  {
    case 1
    show ?case (is ?lhs ⟷ ?rhs)
    proof
      show ?rhs if ?lhs
        using that
        by (auto simp add: rhs-def Let-def List.card-set[symmetric]
          card-Un-Int[where A=set xs and B=− set xs] card-UNIV
          Compl-partition card-gt-0-iff dest: sym)(metis finite-compl finite-set)
      show ?lhs if ?rhs
      proof −
        have ⟦ ∀ y∈set xs. y ∉ set ys; ∀ x∈set ys. x ∉ set xs ⟧ ⟹ set xs ∩ set ys =
{} by blast
        with that show ?thesis
          by (auto simp add: rhs-def Let-def List.card-set[symmetric]
            card-UNIV card-gt-0-iff card-Un-Int[where A=set xs and B=set ys]
            dest: card-eq-UNIV-imp-eq-UNIV split: if-split-asm)
      qed
    qed
  }
  moreover
  case 2
  ultimately show ?case unfolding eq-set-def by blast
next
  case 3
  show ?case unfolding eq-set-def List.coset-def by blast
next
  case 4
  show ?case unfolding eq-set-def List.coset-def by blast
qed

end
```

Provide more informative exceptions than Match for non-rewritten cases. If generated code raises one these exceptions, then a code equation calls the mentioned operator for an element type that is not an instance of *card-UNIV* and is therefore not implemented via *card-UNIV-class.card-UNIV*. Constrain the element type with sort *card-UNIV* to change this.

```
lemma card-coset-error [code]:
  card (List.coset xs) =
   Code.abort (STR ''card (List.coset -) requires type class instance card-UNIV'')
     (λ-. card (List.coset xs))
by(simp)
```

lemma *coset-subseteq-set-code* [*code*]:
List.coset xs ⊆ *set ys* ⟷
(*if xs* = [] ∧ *ys* = [] *then False*
else Code.abort
(*STR ''subset-eq* (*List.coset* -) (*List.set* -) *requires type class instance card-UNIV ''*)
(λ-. *List.coset xs* ⊆ *set ys*))
by *simp*

notepad begin — test code setup
have *List.coset* [*True*] = *set* [*False*] ∧
List.coset [] ⊆ *List.set* [*True*, *False*] ∧
finite (*List.coset* [*True*])
by *eval*

end

end

theory *Case-Converter*
imports *Main*
begin

12.8 Eliminating pattern matches

definition *missing-pattern-match* :: *String.literal* ⇒ (*unit* ⇒ *'a*) ⇒ *'a* **where**
[*code del*]: *missing-pattern-match m f* = *f* ()

lemma *missing-pattern-match-cong* [*cong*]:
m = *m'* ⟹ *missing-pattern-match m f* = *missing-pattern-match m' f*
by(*rule arg-cong*)

lemma *missing-pattern-match-code* [*code-unfold*]:
missing-pattern-match = *Code.abort*
unfolding *missing-pattern-match-def Code.abort-def* **..**

ML-file ‹*case-converter.ML*›

end

13 Lazy types in generated code

theory *Code-Lazy*
imports *Case-Converter*
keywords
code-lazy-type
activate-lazy-type
deactivate-lazy-type

activate-lazy-types
deactivate-lazy-types
print-lazy-types :: *thy-decl*
begin

This theory and the CodeLazy tool described in [2].

It hooks into Isabelle's code generator such that the generated code evaluates a user-specified set of type constructors lazily, even in target languages with eager evaluation. The lazy type must be algebraic, i.e., values must be built from constructors and a corresponding case operator decomposes them. Every datatype and codatatype is algebraic and thus eligible for lazification.

13.1 The type *lazy*

typedef *'a lazy = UNIV* :: *'a set* **..**
setup-lifting *type-definition-lazy*
lift-definition *delay* :: *(unit ⇒ 'a) ⇒ 'a lazy* **is** *λf. f ()* **.**
lift-definition *force* :: *'a lazy ⇒ 'a* **is** *λx. x* **.**

code-datatype *delay*
lemma *force-delay* [*code*]: *force (delay f) = f ()* **by** *transfer (rule refl)*
lemma *delay-force*: *delay (λ-. force s) = s* **by** *transfer (rule refl)*

definition *termify-lazy2* :: *'a* :: *typerep lazy ⇒ term*
where *termify-lazy2 x =*
Code-Evaluation.App (Code-Evaluation.Const (STR ''Code-Lazy.delay'') (TYPEREP((unit ⇒ 'a) ⇒ 'a lazy)))
(Code-Evaluation.Const (STR ''Pure.dummy-pattern'') (TYPEREP((unit ⇒ 'a))))

definition *termify-lazy* ::
(String.literal ⇒ 'typerep ⇒ 'term) ⇒
('term ⇒ 'term ⇒ 'term) ⇒
(String.literal ⇒ 'typerep ⇒ 'term ⇒ 'term) ⇒
'typerep ⇒ ('typerep ⇒ 'typerep ⇒ 'typerep) ⇒ ('typerep ⇒ 'typerep) ⇒
('a ⇒ 'term) ⇒ 'typerep ⇒ 'a :: *typerep lazy ⇒ 'term ⇒ term*
where *termify-lazy - - - - - - - - x - = termify-lazy2 x*

declare [[*code drop*: *Code-Evaluation.term-of* :: *- lazy ⇒ -*]]

lemma *term-of-lazy-code* [*code*]:
Code-Evaluation.term-of x ≡
termify-lazy
Code-Evaluation.Const Code-Evaluation.App Code-Evaluation.Abs
TYPEREP(unit) (λT U. typerep.Typerep (STR ''fun'') [T, U]) (λT. typerep.Typerep (STR ''Code-Lazy.lazy'') [T])
Code-Evaluation.term-of TYPEREP('a) x (Code-Evaluation.Const (STR '''') (TYPEREP(unit)))
for *x* :: *'a* :: *{typerep, term-of} lazy*

by (*rule term-of-anything*)

The implementations of - *lazy* using language primitives cache forced values.

Term reconstruction for lazy looks into the lazy value and reconstructs it to the depth it has been evaluated. This is not done for Haskell as we do not know of any portable way to inspect whether a lazy value has been evaluated to or not.

code-printing code-module *Lazy* ⇀ (*SML*)
‹*signature LAZY* =
sig
  type 'a lazy;
  val lazy : (*unit* −> *'a*) −> *'a lazy*;
  val force : *'a lazy* −> *'a*;
  val peek : *'a lazy* −> *'a option*
  val termify-lazy :
   (*string* −> *'typerep* −> *'term*) −>
   (*'term* −> *'term* −> *'term*) −>
   (*string* −> *'typerep* −> *'term* −> *'term*) −>
   'typerep −> (*'typerep* −> *'typerep* −> *'typerep*) −> (*'typerep* −> *'typerep*) −>
   (*'a* −> *'term*) −> *'typerep* −> *'a lazy* −> *'term* −> *'term*;
end;

structure Lazy : *LAZY* =
struct

datatype 'a content =
   Delay of unit −> *'a*
 | *Value of 'a*
 | *Exn of exn*;

datatype 'a lazy = *Lazy of 'a content ref*;

fun lazy f = *Lazy* (*ref* (*Delay f*));

fun force (*Lazy x*) = *case* !*x of*
   Delay f => (
     let val res = *f* (); *val* - = *x* := *Value res*; *in res end*
     handle exn => (*x* := *Exn exn*; *raise exn*))
  | *Value x* => *x*
  | *Exn exn* => *raise exn*;

fun peek (*Lazy x*) = *case* !*x of*
    Value x => *SOME x*
  | - => *NONE*;

fun termify-lazy const app abs unitT funT lazyT term-of T x - =
  app (*const Code-Lazy.delay* (*funT* (*funT unitT T*) (*lazyT T*)))
    (*case peek x of SOME y* => *abs* - *unitT* (*term-of y*)

```
    | - => const Pure.dummy-pattern (funT unitT T));

end;› for type-constructor lazy constant delay force termify-lazy
| type-constructor lazy ⇀ (SML) - Lazy.lazy
| constant delay ⇀ (SML) Lazy.lazy
| constant force ⇀ (SML) Lazy.force
| constant termify-lazy ⇀ (SML) Lazy.termify'-lazy

code-reserved SML Lazy

code-printing — For code generation within the Isabelle environment, we reuse
the thread-safe implementation of lazy from ~~/src/Pure/Concurrent/lazy.ML
  code-module Lazy ⇀ (Eval) ‹› for constant undefined
| type-constructor lazy ⇀ (Eval) - Lazy.lazy
| constant delay ⇀ (Eval) Lazy.lazy
| constant force ⇀ (Eval) Lazy.force
| code-module Termify-Lazy ⇀ (Eval)
‹structure Termify-Lazy = struct
fun termify-lazy
  (-: string -> typ -> term) (-: term -> term -> term)  (-: string -> typ ->
term -> term)
  (-: typ) (-: typ -> typ -> typ) (-: typ -> typ)
  (term-of: 'a -> term) (T: typ) (x: 'a Lazy.lazy) (-: term) =
  Const (Code-Lazy.delay, (HOLogic.unitT --> T) --> Type (Code-Lazy.lazy,
[T])) $
  (case Lazy.peek x of
    SOME (Exn.Res x) => absdummy HOLogic.unitT (term-of x)
  | - => Const (Pure.dummy-pattern, HOLogic.unitT --> T));
end;› for constant termify-lazy
| constant termify-lazy ⇀ (Eval) Termify'-Lazy.termify'-lazy

code-reserved Eval Termify-Lazy

code-printing
  type-constructor lazy ⇀ (OCaml) - Lazy.t
| constant delay ⇀ (OCaml) Lazy.from'-fun
| constant force ⇀ (OCaml) Lazy.force
| code-module Termify-Lazy ⇀ (OCaml)
‹module Termify-Lazy : sig
  val termify-lazy :
    (string -> 'typerep -> 'term) ->
    ('term -> 'term -> 'term) ->
    (string -> 'typerep -> 'term -> 'term) ->
    'typerep -> ('typerep -> 'typerep -> 'typerep) -> ('typerep -> 'typerep) ->
    ('a -> 'term) -> 'typerep -> 'a Lazy.t -> 'term -> 'term
end = struct

let termify-lazy const app abs unitT funT lazyT term-of ty x - =
  app (const Code-Lazy.delay (funT (funT unitT ty) (lazyT ty)))
```

```
    (if Lazy.is-val x then abs - unitT (term-of (Lazy.force x))
     else const Pure.dummy-pattern (funT unitT ty));;

end;;› for constant termify-lazy
| constant termify-lazy ⇀ (OCaml) Termify'-Lazy.termify'-lazy

code-reserved OCaml Lazy Termify-Lazy


code-printing
  code-module Lazy ⇀ (Haskell) ‹
module Lazy(Lazy, delay, force) where

newtype Lazy a = Lazy a
delay f = Lazy (f ())
force (Lazy x) = x› for type-constructor lazy constant delay force
| type-constructor lazy ⇀ (Haskell) Lazy.Lazy -
| constant delay ⇀ (Haskell) Lazy.delay
| constant force ⇀ (Haskell) Lazy.force

code-reserved Haskell Lazy

code-printing
  code-module Lazy ⇀ (Scala)
‹object Lazy {
  final class Lazy[A] (f: Unit => A) {
    var evaluated = false;
    lazy val x: A = f(())

    def get() : A = {
      evaluated = true;
      return x
    }
  }

  def force[A] (x: Lazy[A]) : A = {
    return x.get()
  }

  def delay[A] (f: Unit => A) : Lazy[A] = {
    return new Lazy[A] (f)
  }

  def termify-lazy[Typerep, Term, A] (
    const: String => Typerep => Term,
    app: Term => Term => Term,
    abs: String => Typerep => Term => Term,
    unitT: Typerep,
    funT: Typerep => Typerep => Typerep,
```

```
    lazyT: Typerep => Typerep,
    term-of: A => Term,
    ty: Typerep,
    x: Lazy[A],
    dummy: Term) : Term = {
    if (x.evaluated)
    app(const(Code-Lazy.delay)(funT(funT(unitT)(ty))(lazyT(ty))))(abs(-)(unitT)(term-of(x.get())))
    else
    app(const(Code-Lazy.delay)(funT(funT(unitT)(ty))(lazyT(ty))))(const(Pure.dummy-pattern)(funT(unitT
  }
}› for type-constructor lazy constant delay force termify-lazy
| type-constructor lazy ⇀ (Scala) Lazy.Lazy[-]
| constant delay ⇀ (Scala) Lazy.delay
| constant force ⇀ (Scala) Lazy.force
| constant termify-lazy ⇀ (Scala) Lazy.termify'-lazy

code-reserved Scala Lazy
```

Make evaluation with the simplifier respect *delay*s.

```
lemma delay-lazy-cong: delay f = delay f by simp
setup ‹Code-Simp.map-ss (Simplifier.add-cong @{thm delay-lazy-cong})›
```

13.2 Implementation

```
ML-file ‹code-lazy.ML›

setup ‹
  Code-Preproc.add-functrans (lazy-datatype, Code-Lazy.transform-code-eqs)
›

end


theory Code-Test
imports Main
keywords test-code :: diag
begin
```

13.3 YXML encoding for *term*

```
datatype (plugins del: code size quickcheck) yxml-of-term = YXML

lemma yot-anything: x = (y :: yxml-of-term)
by(cases x y rule: yxml-of-term.exhaust[case-product yxml-of-term.exhaust])(simp)

definition yot-empty :: yxml-of-term where [code del]: yot-empty = YXML
definition yot-literal :: String.literal ⇒ yxml-of-term
  where [code del]: yot-literal - = YXML
definition yot-append :: yxml-of-term ⇒ yxml-of-term ⇒ yxml-of-term
  where [code del]: yot-append - - = YXML
```

definition *yot-concat* :: *yxml-of-term list* ⇒ *yxml-of-term*
where [*code del*]: *yot-concat - = YXML*

Serialise *yxml-of-term* to native string of target language

code-printing type-constructor *yxml-of-term*
⇀ (*SML*) *string*
and (*OCaml*) *string*
and (*Haskell*) *String*
and (*Scala*) *String*
| **constant** *yot-empty*
⇀ (*SML*)
and (*OCaml*)
and (*Haskell*)
and (*Scala*)
| **constant** *yot-literal*
⇀ (*SML*) -
and (*OCaml*) -
and (*Haskell*) -
and (*Scala*) -
| **constant** *yot-append*
⇀ (*SML*) *String.concat* [(-), (-)]
and (*OCaml*) *String.concat* [(-); (-)]
and (*Haskell*) **infixr** *5* ++
and (*Scala*) **infixl** *5* +
| **constant** *yot-concat*
⇀ (*SML*) *String.concat*
and (*OCaml*) *String.concat*
and (*Haskell*) *Prelude.concat*
and (*Scala*) -.*mkString*()

Stripped-down implementations of Isabelle's XML tree with YXML encoding as defined in `~~/src/Pure/PIDE/xml.ML`, `~~/src/Pure/PIDE/yxml.ML` sufficient to encode *term* as in `~~/src/Pure/term_xml.ML`.

datatype (*plugins del*: *code size quickcheck*) *xml-tree* = *XML-Tree*

lemma *xml-tree-anything*: *x* = (*y* :: *xml-tree*)
by(*cases x y rule*: *xml-tree.exhaust*[*case-product xml-tree.exhaust*])(*simp*)

context begin
local-setup ‹*Local-Theory.map-background-naming* (*Name-Space.mandatory-path xml*)›

type-synonym *attributes* = (*String.literal* × *String.literal*) *list*
type-synonym *body* = *xml-tree list*

definition *Elem* :: *String.literal* ⇒ *attributes* ⇒ *xml-tree list* ⇒ *xml-tree*
where [*code del*]: *Elem - - - = XML-Tree*

definition *Text* :: *String.literal* ⇒ *xml-tree*

where [*code del*]: *Text - = XML-Tree*

definition *node* :: *xml-tree list* ⇒ *xml-tree*
where *node ts* = *Elem* (*STR ''*:*''*) [] *ts*

definition *tagged* :: *String.literal* ⇒ *String.literal option* ⇒ *xml-tree list* ⇒ *xml-tree*
where *tagged tag x ts* = *Elem tag* (*case x of None* ⇒ [] | *Some x'* ⇒ [(*STR ''0''*, *x'*)]) *ts*

definition *list* **where** *list f xs* = *map* (*node* ∘ *f*) *xs*

definition *X* :: *yxml-of-term* **where** *X* = *yot-literal* (*STR 0x05*)
definition *Y* :: *yxml-of-term* **where** *Y* = *yot-literal* (*STR 0x06*)
definition *XY* :: *yxml-of-term* **where** *XY* = *yot-append X Y*
definition *XYX* :: *yxml-of-term* **where** *XYX* = *yot-append XY X*

end

code-datatype *xml.Elem xml.Text*

definition *yxml-string-of-xml-tree* :: *xml-tree* ⇒ *yxml-of-term* ⇒ *yxml-of-term*
where [*code del*]: *yxml-string-of-xml-tree - - = YXML*

lemma *yxml-string-of-xml-tree-code* [*code*]:
yxml-string-of-xml-tree (*xml.Elem name atts ts*) *rest* =
yot-append xml.XY (
yot-append (*yot-literal name*) (
foldr (λ(*a*, *x*) *rest*.
yot-append xml.Y (
yot-append (*yot-literal a*) (
yot-append (*yot-literal* (*STR ''=''*)) (
yot-append (*yot-literal x*) *rest*)))) *atts* (
foldr yxml-string-of-xml-tree ts (
yot-append xml.XYX rest))))
yxml-string-of-xml-tree (*xml.Text s*) *rest* = *yot-append* (*yot-literal s*) *rest*
by(*rule yot-anything*)+

definition *yxml-string-of-body* :: *xml.body* ⇒ *yxml-of-term*
where *yxml-string-of-body ts* = *foldr yxml-string-of-xml-tree ts yot-empty*

Encoding *term* into XML trees as defined in `~~/src/Pure/term_xml.ML`.

definition *xml-of-typ* :: *Typerep.typerep* ⇒ *xml.body*
where [*code del*]: *xml-of-typ - =* [*XML-Tree*]

definition *xml-of-term* :: *Code-Evaluation.term* ⇒ *xml.body*
where [*code del*]: *xml-of-term - =* [*XML-Tree*]

lemma *xml-of-typ-code* [*code*]:
xml-of-typ (*typerep.Typerep t args*) = [*xml.tagged* (*STR ''0''*) (*Some t*) (*xml.list*

xml-of-typ args)]
by(*simp add*: *xml-of-typ-def xml-tree-anything*)

lemma *xml-of-term-code* [*code*]:
xml-of-term (*Code-Evaluation.Const x ty*) = [*xml.tagged* (*STR ''0''*) (*Some x*) (*xml-of-typ ty*)]
xml-of-term (*Code-Evaluation.App t1 t2*) = [*xml.tagged* (*STR ''5''*) *None* [*xml.node* (*xml-of-term t1*), *xml.node* (*xml-of-term t2*)]]
xml-of-term (*Code-Evaluation.Abs x ty t*) = [*xml.tagged* (*STR ''4''*) (*Some x*) [*xml.node* (*xml-of-typ ty*), *xml.node* (*xml-of-term t*)]]
— FIXME: *Code-Evaluation.Free* is used only in *HOL.Quickcheck-Narrowing* to represent uninstantiated parameters in constructors. Here, we always translate them to `Free` variables.
xml-of-term (*Code-Evaluation.Free x ty*) = [*xml.tagged* (*STR ''1''*) (*Some x*) (*xml-of-typ ty*)]
by(*simp-all add*: *xml-of-term-def xml-tree-anything*)

definition *yxml-string-of-term* :: *Code-Evaluation.term* ⇒ *yxml-of-term*
where *yxml-string-of-term* = *yxml-string-of-body* ∘ *xml-of-term*

13.4 Test engine and drivers

ML-file ‹*code-test.ML*›

end

14 A combinator to build partial equivalence relations from a predicate and an equivalence relation

theory *Combine-PER*
imports *Main*
begin

unbundle *lattice-syntax*

definition *combine-per* :: ('*a* ⇒ *bool*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a* ⇒ '*a* ⇒ *bool*
where *combine-per P R* = (λ*x y*. *P x* ∧ *P y*) ⊓ *R*

lemma *combine-per-simp* [*simp*]:
combine-per P R x y ⟷ *P x* ∧ *P y* ∧ *x* ≈ *y* **for** *R* (**infixl** ≈ *50*)
by (*simp add*: *combine-per-def*)

lemma *combine-per-top* [*simp*]: *combine-per* ⊤ *R* = *R*
by (*simp add*: *fun-eq-iff*)

lemma *combine-per-eq* [*simp*]: *combine-per P HOL.eq* = *HOL.eq* ⊓ (λ*x y*. *P x*)
by (*auto simp add*: *fun-eq-iff*)

lemma *symp-combine-per*: *symp R* $\Longrightarrow$ *symp* (*combine-per P R*)
by (*auto simp add*: *symp-def sym-def combine-per-def*)

lemma *transp-combine-per*: *transp R* $\Longrightarrow$ *transp* (*combine-per P R*)
by (*auto simp add*: *transp-def trans-def combine-per-def*)

lemma *combine-perI*: *P x* $\Longrightarrow$ *P y* $\Longrightarrow$ *x* $\approx$ *y* $\Longrightarrow$ *combine-per P R x y* **for** *R*
(**infixl** $\approx$ *50*)
by (*simp add*: *combine-per-def*)

lemma *symp-combine-per-symp*: *symp R* $\Longrightarrow$ *symp* (*combine-per P R*)
by (*auto intro*!: *sympI elim*: *sympE*)

lemma *transp-combine-per-transp*: *transp R* $\Longrightarrow$ *transp* (*combine-per P R*)
by (*auto intro*!: *transpI elim*: *transpE*)

lemma *equivp-combine-per-part-equivp* [*intro?*]:
fixes *R* (**infixl** $\approx$ *50*)
assumes $\exists x.$ *P x* **and** *equivp R*
shows *part-equivp* (*combine-per P R*)
proof −
from ‹$\exists x.$ *P x*› **obtain** *x* **where** *P x* **..**
moreover from ‹*equivp R*› **have** *x* $\approx$ *x*
by (*rule equivp-reflp*)
ultimately have $\exists x.$ *P x* $\wedge$ *x* $\approx$ *x*
by *blast*
with ‹*equivp R*› **show** *?thesis*
by (*auto intro*!: *part-equivpI symp-combine-per-symp transp-combine-per-transp*
elim: *equivpE*)
qed

end

15 Formalisation of chain-complete partial orders, continuity and admissibility

theory *Complete-Partial-Order2* **imports**
Main
begin

unbundle *lattice-syntax*

lemma *chain-transfer* [*transfer-rule*]:
includes *lifting-syntax*
shows ((*A* ===> *A* ===> (=)) ===> *rel-set A* ===> (=)) *Complete-Partial-Order.chain*
Complete-Partial-Order.chain
unfolding *chain-def*[*abs-def*] **by** *transfer-prover*

lemma *linorder-chain* [*simp*, *intro*!]:
  fixes *Y* :: - :: *linorder set*
  shows *Complete-Partial-Order.chain* (≤) *Y*
by(*auto intro*: *chainI*)

lemma *fun-lub-apply*: ⋀*Sup*. *fun-lub Sup Y x* = *Sup* ((λ*f*. *f x*) ‘ *Y*)
by(*simp add*: *fun-lub-def image-def*)

lemma *fun-lub-empty* [*simp*]: *fun-lub lub* {} = (λ-. *lub* {})
by(*rule ext*)(*simp add*: *fun-lub-apply*)

lemma *chain-fun-ordD*:
  assumes *Complete-Partial-Order.chain* (*fun-ord le*) *Y*
  shows *Complete-Partial-Order.chain le* ((λ*f*. *f x*) ‘ *Y*)
by(*rule chainI*)(*auto dest*: *chainD*[*OF assms*] *simp add*: *fun-ord-def*)

lemma *chain-Diff*:
  Complete-Partial-Order.chain ord A
  ⟹ *Complete-Partial-Order.chain ord* (*A* − *B*)
by(*erule chain-subset*) *blast*

lemma *chain-rel-prodD1*:
  Complete-Partial-Order.chain (*rel-prod orda ordb*) *Y*
  ⟹ *Complete-Partial-Order.chain orda* (*fst* ‘ *Y*)
by(*auto 4 3 simp add*: *chain-def*)

lemma *chain-rel-prodD2*:
  Complete-Partial-Order.chain (*rel-prod orda ordb*) *Y*
  ⟹ *Complete-Partial-Order.chain ordb* (*snd* ‘ *Y*)
by(*auto 4 3 simp add*: *chain-def*)

context *ccpo* **begin**

lemma *ccpo-fun*: *class.ccpo* (*fun-lub Sup*) (*fun-ord* (≤)) (*mk-less* (*fun-ord* (≤)))
  by *standard* (*auto 4 3 simp add*: *mk-less-def fun-ord-def fun-lub-apply*
    intro: *order.trans order.antisym chain-imageI ccpo-Sup-upper ccpo-Sup-least*)

lemma *ccpo-Sup-below-iff*: *Complete-Partial-Order.chain* (≤) *Y* ⟹ *Sup Y* ≤ *x*
⟷ (∀ *y*∈*Y*. *y* ≤ *x*)
by(*fast intro*: *order-trans*[*OF ccpo-Sup-upper*] *ccpo-Sup-least*)

lemma *Sup-minus-bot*:
  assumes *chain*: *Complete-Partial-Order.chain* (≤) *A*
  shows ⨆(*A* − {⨆{}}) = ⨆*A*
    (**is** *?lhs* = *?rhs*)
proof (*rule order.antisym*)
  show *?lhs* ≤ *?rhs*

```
    by (blast intro: ccpo-Sup-least chain-Diff[OF chain] ccpo-Sup-upper[OF chain])
  show ?rhs ≤ ?lhs
  proof (rule ccpo-Sup-least [OF chain])
    show x ∈ A ⟹ x ≤ ?lhs for x
      by (cases x = ⨆{})
         (blast intro: ccpo-Sup-least chain-empty ccpo-Sup-upper[OF chain-Diff[OF
chain]])+
  qed
qed

lemma mono-lub:
  fixes le-b (infix ⊑ 60)
  assumes chain: Complete-Partial-Order.chain (fun-ord (≤)) Y
  and mono: ⋀f. f ∈ Y ⟹ monotone le-b (≤) f
  shows monotone (⊑) (≤) (fun-lub Sup Y)
proof(rule monotoneI)
  fix x y
  assume x ⊑ y

  have chain'': ⋀x. Complete-Partial-Order.chain (≤) ((λf. f x) ` Y)
    using chain by(rule chain-imageI)(simp add: fun-ord-def)
  then show fun-lub Sup Y x ≤ fun-lub Sup Y y unfolding fun-lub-apply
  proof(rule ccpo-Sup-least)
    fix x'
    assume x' ∈ (λf. f x) ` Y
    then obtain f where f ∈ Y x' = f x by blast
    note ‹x' = f x› also
    from ‹f ∈ Y› ‹x ⊑ y› have f x ≤ f y by(blast dest: mono monotoneD)
    also have ... ≤ ⨆((λf. f y) ` Y) using chain''
      by(rule ccpo-Sup-upper)(simp add: ‹f ∈ Y›)
    finally show x' ≤ ⨆((λf. f y) ` Y) .
  qed
qed

context
  fixes le-b (infix ⊑ 60) and Y f
  assumes chain: Complete-Partial-Order.chain le-b Y
  and mono1: ⋀y. y ∈ Y ⟹ monotone le-b (≤) (λx. f x y)
  and mono2: ⋀x a b. ⟦ x ∈ Y; a ⊑ b; a ∈ Y; b ∈ Y ⟧ ⟹ f x a ≤ f x b
begin

lemma Sup-mono:
  assumes le: x ⊑ y and x: x ∈ Y and y: y ∈ Y
  shows ⨆(f x ` Y) ≤ ⨆(f y ` Y) (is - ≤ ?rhs)
proof(rule ccpo-Sup-least)
  from chain show chain': Complete-Partial-Order.chain (≤) (f x ` Y) when x ∈
Y for x
    by(rule chain-imageI) (insert that, auto dest: mono2)
```

fix x'
assume $x' \in f\ x\ `\ Y$
then obtain y' **where** $y' \in Y\ x' = f\ x\ y'$ **by** *blast* **note** *this*(*2*)
also from *mono1*[*OF* ‹$y' \in Y$›] *le* **have** $\ldots \leq f\ y\ y'$ **by**(*rule monotoneD*)
also have $\ldots \leq$ *?rhs* **using** *chain'*[*OF y*]
by (*auto intro*!: *ccpo-Sup-upper simp add*: ‹$y' \in Y$›)
finally show $x' \leq$ *?rhs* **.**
qed(*rule x*)

lemma *diag-Sup*: $\bigsqcup((\lambda x.\ \bigsqcup(f\ x\ `\ Y))\ `\ Y) = \bigsqcup((\lambda x.\ f\ x\ x)\ `\ Y)$ (**is** *?lhs* = *?rhs*)
proof(*rule order.antisym*)
have *chain1*: *Complete-Partial-Order.chain* $(\leq)$ $((\lambda x.\ \bigsqcup(f\ x\ `\ Y))\ `\ Y)$
using *chain* **by**(*rule chain-imageI*)(*rule Sup-mono*)
have *chain2*: $\bigwedge y'.\ y' \in Y \Longrightarrow$ *Complete-Partial-Order.chain* $(\leq)$ $(f\ y'\ `\ Y)$ **using** *chain*
by(*rule chain-imageI*)(*auto dest*: *mono2*)
have *chain3*: *Complete-Partial-Order.chain* $(\leq)$ $((\lambda x.\ f\ x\ x)\ `\ Y)$
using *chain* **by**(*rule chain-imageI*)(*auto intro*: *monotoneD*[*OF mono1*] *mono2 order.trans*)

show *?lhs* $\leq$ *?rhs* **using** *chain1*
proof(*rule ccpo-Sup-least*)
fix x'
assume $x' \in (\lambda x.\ \bigsqcup(f\ x\ `\ Y))\ `\ Y$
then obtain y' **where** $y' \in Y\ x' = \bigsqcup(f\ y'\ `\ Y)$ **by** *blast* **note** *this*(*2*)
also have $\ldots \leq$ *?rhs* **using** *chain2*[*OF* ‹$y' \in Y$›]
proof(*rule ccpo-Sup-least*)
fix x
assume $x \in f\ y'\ `\ Y$
then obtain y **where** $y \in Y$ **and** x: $x = f\ y'\ y$ **by** *blast*
define y'' **where** $y'' = (\mathit{if}\ y \sqsubseteq y'\ \mathit{then}\ y'\ \mathit{else}\ y)$
from *chain* ‹$y \in Y$› ‹$y' \in Y$› **have** $y \sqsubseteq y' \vee y' \sqsubseteq y$ **by**(*rule chainD*)
hence $f\ y'\ y \leq f\ y''\ y''$ **using** ‹$y \in Y$› ‹$y' \in Y$›
by(*auto simp add*: *y''-def intro*: *mono2 monotoneD*[*OF mono1*])
also from ‹$y \in Y$› ‹$y' \in Y$› **have** $y'' \in Y$ **by**(*simp add*: *y''-def*)
from *chain3* **have** $f\ y''\ y'' \leq$ *?rhs* **by**(*rule ccpo-Sup-upper*)(*simp add*: ‹$y'' \in Y$›)
finally show $x \leq$ *?rhs* **by**(*simp add*: *x*)
qed
finally show $x' \leq$ *?rhs* **.**
qed

show *?rhs* $\leq$ *?lhs* **using** *chain3*
proof(*rule ccpo-Sup-least*)
fix y
assume $y \in (\lambda x.\ f\ x\ x)\ `\ Y$
then obtain x **where** $x \in Y$ **and** $y = f\ x\ x$ **by** *blast* **note** *this*(*2*)
also from *chain2*[*OF* ‹$x \in Y$›] **have** $\ldots \leq \bigsqcup(f\ x\ `\ Y)$
by(*rule ccpo-Sup-upper*)(*simp add*: ‹$x \in Y$›)

```
    also have ... ≤ ?lhs by(rule ccpo-Sup-upper[OF chain1])(simp add: ‹x ∈ Y›)
    finally show y ≤ ?lhs .
  qed
qed

end

lemma Sup-image-mono-le:
  fixes le-b (infix ⊑ 60) and Sup-b (⋁)
  assumes ccpo: class.ccpo Sup-b (⊑) lt-b
  assumes chain: Complete-Partial-Order.chain (⊑) Y
  and mono: ⋀x y. ⟦ x ⊑ y; x ∈ Y ⟧ ⟹ f x ≤ f y
  shows Sup (f ‘ Y) ≤ f (⋁ Y)
proof(rule ccpo-Sup-least)
  show Complete-Partial-Order.chain (≤) (f ‘ Y)
    using chain by(rule chain-imageI)(rule mono)

  fix x
  assume x ∈ f ‘ Y
  then obtain y where y ∈ Y and x = f y by blast note this(2)
  also have y ⊑ ⋁ Y using ccpo chain ‹y ∈ Y› by(rule ccpo.ccpo-Sup-upper)
  hence f y ≤ f (⋁ Y) using ‹y ∈ Y› by(rule mono)
  finally show x ≤ ... .
qed

lemma swap-Sup:
  fixes le-b (infix ⊑ 60)
  assumes Y: Complete-Partial-Order.chain (⊑) Y
  and Z: Complete-Partial-Order.chain (fun-ord (≤)) Z
  and mono: ⋀f. f ∈ Z ⟹ monotone (⊑) (≤) f
  shows ⨆((λx. ⨆(x ‘ Y)) ‘ Z) = ⨆((λx. ⨆((λf. f x) ‘ Z)) ‘ Y)
  (is ?lhs = ?rhs)
proof(cases Y = {})
  case True
  then show ?thesis
    by (simp add: image-constant-conv cong del: SUP-cong-simp)
next
  case False
  have chain1: ⋀f. f ∈ Z ⟹ Complete-Partial-Order.chain (≤) (f ‘ Y)
    by(rule chain-imageI[OF Y])(rule monotoneD[OF mono])
  have chain2: Complete-Partial-Order.chain (≤) ((λx. ⨆(x ‘ Y)) ‘ Z) using Z
  proof(rule chain-imageI)
    fix f g
    assume f ∈ Z g ∈ Z
      and fun-ord (≤) f g
    from chain1[OF ‹f ∈ Z›] show ⨆(f ‘ Y) ≤ ⨆(g ‘ Y)
    proof(rule ccpo-Sup-least)
      fix x
      assume x ∈ f ‘ Y
```

```
    then obtain y where y ∈ Y x = f y by blast note this(2)
    also have ... ≤ g y using ‹fun-ord (≤) f g› by(simp add: fun-ord-def)
    also have ... ≤ ⨆(g ‘ Y) using chain1[OF ‹g ∈ Z›]
      by(rule ccpo-Sup-upper)(simp add: ‹y ∈ Y›)
    finally show x ≤ ⨆(g ‘ Y) .
  qed
 qed
 have chain3: ⋀x. Complete-Partial-Order.chain (≤) ((λf. f x) ‘ Z)
  using Z by(rule chain-imageI)(simp add: fun-ord-def)
 have chain4: Complete-Partial-Order.chain (≤) ((λx. ⨆((λf. f x) ‘ Z)) ‘ Y)
  using Y
 proof(rule chain-imageI)
  fix f x y
  assume x ⊑ y
  show ⨆((λf. f x) ‘ Z) ≤ ⨆((λf. f y) ‘ Z) (is - ≤ ?rhs) using chain3
  proof(rule ccpo-Sup-least)
    fix x'
    assume x' ∈ (λf. f x) ‘ Z
    then obtain f where f ∈ Z x' = f x by blast note this(2)
    also have f x ≤ f y using ‹f ∈ Z› ‹x ⊑ y› by(rule monotoneD[OF mono])
    also have f y ≤ ?rhs using chain3
      by(rule ccpo-Sup-upper)(simp add: ‹f ∈ Z›)
    finally show x' ≤ ?rhs .
  qed
 qed

 from chain2 have ?lhs ≤ ?rhs
 proof(rule ccpo-Sup-least)
  fix x
  assume x ∈ (λx. ⨆(x ‘ Y)) ‘ Z
  then obtain f where f ∈ Z x = ⨆(f ‘ Y) by blast note this(2)
  also have ... ≤ ?rhs using chain1[OF ‹f ∈ Z›]
  proof(rule ccpo-Sup-least)
    fix x'
    assume x' ∈ f ‘ Y
    then obtain y where y ∈ Y x' = f y by blast note this(2)
    also have f y ≤ ⨆((λf. f y) ‘ Z) using chain3
      by(rule ccpo-Sup-upper)(simp add: ‹f ∈ Z›)
    also have ... ≤ ?rhs using chain4 by(rule ccpo-Sup-upper)(simp add: ‹y ∈
Y›)
    finally show x' ≤ ?rhs .
  qed
  finally show x ≤ ?rhs .
 qed
 moreover
 have ?rhs ≤ ?lhs using chain4
 proof(rule ccpo-Sup-least)
  fix x
  assume x ∈ (λx. ⨆((λf. f x) ‘ Z)) ‘ Y
```

then obtain *y* **where** $y \in Y\ x = \bigsqcup((\lambda f.\ f\ y)\ `\ Z)$ **by** *blast* **note** *this*(*2*)
also have $\ldots \leq$ *?lhs* **using** *chain3*
proof(*rule ccpo-Sup-least*)
fix x'
assume $x' \in (\lambda f.\ f\ y)\ `\ Z$
then obtain *f* **where** $f \in Z\ x' = f\ y$ **by** *blast* **note** *this*(*2*)
also have $f\ y \leq \bigsqcup(f\ `\ Y)$ **using** *chain1*[*OF* ‹$f \in Z$›]
by(*rule ccpo-Sup-upper*)(*simp add*: ‹$y \in Y$›)
also have $\ldots \leq$ *?lhs* **using** *chain2*
by(*rule ccpo-Sup-upper*)(*simp add*: ‹$f \in Z$›)
finally show $x' \leq$ *?lhs* **.**
qed
finally show $x \leq$ *?lhs* **.**
qed
ultimately show *?lhs* = *?rhs*
by (*rule order.antisym*)
qed

lemma *fixp-mono*:
assumes *fg*: *fun-ord* (≤) *f g*
and *f*: *monotone* (≤) (≤) *f*
and *g*: *monotone* (≤) (≤) *g*
shows *ccpo-class.fixp f* ≤ *ccpo-class.fixp g*
unfolding *fixp-def*
proof(*rule ccpo-Sup-least*)
fix *x*
assume *x* ∈ *ccpo-class.iterates f*
thus *x* ≤ ⨆ *ccpo-class.iterates g*
proof *induction*
case (*step x*)
from *f step.IH* **have** *f x* ≤ *f* (⨆ *ccpo-class.iterates g*) **by**(*rule monotoneD*)
also have $\ldots \leq g$ (⨆ *ccpo-class.iterates g*) **using** *fg* **by**(*simp add*: *fun-ord-def*)
also have $\ldots =$ ⨆ *ccpo-class.iterates g* **by**(*fold fixp-def fixp-unfold*[*OF g*]) *simp*
finally show *?case* **.**
qed(*blast intro*: *ccpo-Sup-least*)
qed(*rule chain-iterates*[*OF f*])

context fixes *ordb* :: *'b* ⇒ *'b* ⇒ *bool* (**infix** ⊑ *60*) **begin**

lemma *iterates-mono*:
assumes *f*: *f* ∈ *ccpo.iterates* (*fun-lub Sup*) (*fun-ord* (≤)) *F*
and *mono*: ⋀*f*. *monotone* (⊑) (≤) *f* ⟹ *monotone* (⊑) (≤) (*F f*)
shows *monotone* (⊑) (≤) *f*
using *f*
by(*induction rule*: *ccpo.iterates.induct*[*OF ccpo-fun*, *consumes 1*, *case-names step Sup*])(*blast intro*: *mono mono-lub*)+

lemma *fixp-preserves-mono*:
assumes *mono*: ⋀*x*. *monotone* (*fun-ord* (≤)) (≤) (λ*f*. *F f x*)

and *mono2*: $\bigwedge f.$ *monotone* ($\sqsubseteq$) ($\leq$) $f \Longrightarrow$ *monotone* ($\sqsubseteq$) ($\leq$) ($F\ f$)
shows *monotone* ($\sqsubseteq$) ($\leq$) (*ccpo.fixp* (*fun-lub Sup*) (*fun-ord* ($\leq$)) F)
(**is** *monotone - - ?fixp*)
proof(*rule monotoneI*)
have *mono*: *monotone* (*fun-ord* ($\leq$)) (*fun-ord* ($\leq$)) F
by(*rule monotoneI*)(*auto simp add*: *fun-ord-def intro*: *monotoneD*[*OF mono*])
let *?iter* = *ccpo.iterates* (*fun-lub Sup*) (*fun-ord* ($\leq$)) F
have *chain*: $\bigwedge x.$ *Complete-Partial-Order.chain* ($\leq$) ((λ$f.$ $f\ x$) ‘ *?iter*)
by(*rule chain-imageI*[*OF ccpo.chain-iterates*[*OF ccpo-fun mono*]])(*simp add*: *fun-ord-def*)

fix x y
assume $x \sqsubseteq y$
show *?fixp* $x \leq$ *?fixp* y
apply (*simp only*: *ccpo.fixp-def*[*OF ccpo-fun*] *fun-lub-apply*)
using *chain*
proof(*rule ccpo-Sup-least*)
fix x'
assume $x' \in$ (λ$f.$ $f\ x$) ‘ *?iter*
then obtain f **where** $f \in$ *?iter* $x' = f\ x$ **by** *blast* **note** *this*(*2*)
also have $f\ x \leq f\ y$
by(*rule monotoneD*[*OF iterates-mono*[*OF* ‹$f \in$ *?iter*› *mono2*]])(*blast intro*: ‹$x \sqsubseteq y$›)+
also have $f\ y \leq \bigsqcup$ ((λ$f.$ $f\ y$) ‘ *?iter*) **using** *chain*
by(*rule ccpo-Sup-upper*)(*simp add*: ‹$f \in$ *?iter*›)
finally show $x' \leq \ldots$ **.**
qed
qed

end

end

lemma *monotone2monotone*:
assumes *2*: $\bigwedge x.$ *monotone ordb ordc* (λ$y.$ $f\ x\ y$)
and *t*: *monotone orda ordb* (λ$x.$ $t\ x$)
and *1*: $\bigwedge y.$ *monotone orda ordc* (λ$x.$ $f\ x\ y$)
and *trans*: *transp ordc*
shows *monotone orda ordc* (λ$x.$ $f\ x\ (t\ x)$)
by(*blast intro*: *monotoneI transpD*[*OF trans*] *monotoneD*[*OF t*] *monotoneD*[*OF 2*] *monotoneD*[*OF 1*])

15.1 Continuity

definition *cont* :: ('*a set* ⇒ '*a*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ ('*b set* ⇒ '*b*) ⇒ ('*b* ⇒ '*b* ⇒ *bool*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*
where
cont luba orda lubb ordb f ⟷
(∀ *Y*. *Complete-Partial-Order.chain orda Y* ⟶ *Y* ≠ {} ⟶ *f* (*luba Y*) = *lubb*

$(f \text{ ` } Y))$

definition *mcont* :: $('a\ set \Rightarrow 'a) \Rightarrow ('a \Rightarrow 'a \Rightarrow bool) \Rightarrow ('b\ set \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
where
mcont luba orda lubb ordb f $\longleftrightarrow$
monotone orda ordb f $\wedge$ *cont luba orda lubb ordb f*

15.1.1 Theorem collection *cont-intro*

named-theorems *cont-intro continuity and admissibility intro rules*
ML ‹
(∗ *apply cont-intro rules as intro and try to solve*
the remaining of the emerging subgoals with simp ∗)
fun cont-intro-tac ctxt =
REPEAT-ALL-NEW (*resolve-tac ctxt* (*rev* (*Named-Theorems.get ctxt* ***named-theorems*** ‹*cont-intro*›)))
THEN-ALL-NEW (*SOLVED'* (*simp-tac ctxt*))

fun cont-intro-simproc ctxt ct =
let
fun mk-stmt t = t
|> *HOLogic.mk-Trueprop*
|> *Thm.cterm-of ctxt*
|> *Goal.init*
fun mk-thm t =
case SINGLE (*cont-intro-tac ctxt 1*) (*mk-stmt t*) *of*
SOME thm => SOME (*Goal.finish ctxt thm RS* @{*thm Eq-TrueI*})
| *NONE => NONE*
in
case Thm.term-of ct of
t as Const (***const-name*** ‹*ccpo.admissible*›, -) $ - $ - $ - => *mk-thm t*
| *t as Const* (***const-name*** ‹*mcont*›, -) $ - $ - $ - $ - $ - => *mk-thm t*
| *t as Const* (***const-name*** ‹*monotone*›, -) $ - $ - $ - => *mk-thm t*
| - => *NONE*
end
handle THM - => NONE
| *TYPE - => NONE*
›

simproc-setup *cont-intro*
(*ccpo.admissible lub ord P*
| *mcont lub ord lub' ord' f*
| *monotone ord ord' f*
) = ‹*K cont-intro-simproc*›

lemmas [*cont-intro*] =
call-mono
let-mono

if-mono
option.const-mono
tailrec.const-mono
bind-mono

declare *if-mono*[*simp*]

lemma *monotone-id'* [*cont-intro*]: *monotone ord ord* (λ*x*. *x*)
by(*simp add*: *monotone-def*)

lemma *monotone-applyI*:
monotone orda ordb F ⟹ *monotone* (*fun-ord orda*) *ordb* (λ*f*. *F* (*f x*))
by(*rule monotoneI*)(*auto simp add*: *fun-ord-def dest*: *monotoneD*)

lemma *monotone-if-fun* [*partial-function-mono*]:
⟦ *monotone* (*fun-ord orda*) (*fun-ord ordb*) *F*; *monotone* (*fun-ord orda*) (*fun-ord ordb*) *G* ⟧
⟹ *monotone* (*fun-ord orda*) (*fun-ord ordb*) (λ*f n*. *if c n then F f n else G f n*)
by(*simp add*: *monotone-def fun-ord-def*)

lemma *monotone-fun-apply-fun* [*partial-function-mono*]:
monotone (*fun-ord* (*fun-ord ord*)) (*fun-ord ord*) (λ*f n*. *f t* (*g n*))
by(*rule monotoneI*)(*simp add*: *fun-ord-def*)

lemma *monotone-fun-ord-apply*:
monotone orda (*fun-ord ordb*) *f* ⟷ (∀ *x*. *monotone orda ordb* (λ*y*. *f y x*))
by(*auto simp add*: *monotone-def fun-ord-def*)

context *preorder* **begin**

declare *transp-le*[*cont-intro*]

lemma *monotone-const* [*simp*, *cont-intro*]: *monotone ord* (≤) (λ-. *c*)
by(*rule monotoneI*) *simp*

end

lemma *transp-le* [*cont-intro*, *simp*]:
class.preorder ord (*mk-less ord*) ⟹ *transp ord*
by(*rule preorder.transp-le*)

context *partial-function-definitions* **begin**

declare *const-mono* [*cont-intro*, *simp*]

lemma *transp-le* [*cont-intro*, *simp*]: *transp leq*
by(*rule transpI*)(*rule leq-trans*)

lemma *preorder* [*cont-intro*, *simp*]: *class.preorder leq* (*mk-less leq*)

by(*unfold-locales*)(*auto simp add*: *mk-less-def intro*: *leq-refl leq-trans*)

declare *ccpo*[*cont-intro*, *simp*]

end

lemma *contI* [*intro?*]:
(⋀*Y*. ⟦ *Complete-Partial-Order.chain orda Y*; *Y* ≠ {} ⟧ ⟹ *f* (*luba Y*) = *lubb* (*f* ‘ *Y*))
⟹ *cont luba orda lubb ordb f*
unfolding *cont-def* **by** *blast*

lemma *contD*:
⟦ *cont luba orda lubb ordb f*; *Complete-Partial-Order.chain orda Y*; *Y* ≠ {} ⟧
⟹ *f* (*luba Y*) = *lubb* (*f* ‘ *Y*)
unfolding *cont-def* **by** *blast*

lemma *cont-id* [*simp*, *cont-intro*]: ⋀*Sup*. *cont Sup ord Sup ord id*
by(*rule contI*) *simp*

lemma *cont-id′* [*simp*, *cont-intro*]: ⋀*Sup*. *cont Sup ord Sup ord* (λ*x*. *x*)
using *cont-id*[*unfolded id-def*] **.**

lemma *cont-applyI* [*cont-intro*]:
assumes *cont*: *cont luba orda lubb ordb g*
shows *cont* (*fun-lub luba*) (*fun-ord orda*) *lubb ordb* (λ*f*. *g* (*f x*))
by(*rule contI*)(*drule chain-fun-ordD*[**where** *x*=*x*], *simp add*: *fun-lub-apply image-image contD*[*OF cont*])

lemma *call-cont*: *cont* (*fun-lub lub*) (*fun-ord ord*) *lub ord* (λ*f*. *f t*)
by(*simp add*: *cont-def fun-lub-apply*)

lemma *cont-if* [*cont-intro*]:
⟦ *cont luba orda lubb ordb f*; *cont luba orda lubb ordb g* ⟧
⟹ *cont luba orda lubb ordb* (λ*x*. *if c then f x else g x*)
by(*cases c*) *simp-all*

lemma *mcontI* [*intro?*]:
⟦ *monotone orda ordb f*; *cont luba orda lubb ordb f* ⟧ ⟹ *mcont luba orda lubb ordb f*
by(*simp add*: *mcont-def*)

lemma *mcont-mono*: *mcont luba orda lubb ordb f* ⟹ *monotone orda ordb f*
by(*simp add*: *mcont-def*)

lemma *mcont-cont* [*simp*]: *mcont luba orda lubb ordb f* ⟹ *cont luba orda lubb ordb f*
by(*simp add*: *mcont-def*)

lemma *mcont-monoD*:
⟦ *mcont luba orda lubb ordb f*; *orda x y* ⟧ ⟹ *ordb* (*f x*) (*f y*)
by(*auto simp add*: *mcont-def dest*: *monotoneD*)

lemma *mcont-contD*:
⟦ *mcont luba orda lubb ordb f*; *Complete-Partial-Order.chain orda Y*; *Y* ≠ {} ⟧
⟹ *f* (*luba Y*) = *lubb* (*f* ‘ *Y*)
by(*auto simp add*: *mcont-def dest*: *contD*)

lemma *mcont-call* [*cont-intro*, *simp*]:
mcont (*fun-lub lub*) (*fun-ord ord*) *lub ord* (λ*f*. *f t*)
by(*simp add*: *mcont-def call-mono call-cont*)

lemma *mcont-id′* [*cont-intro*, *simp*]: *mcont lub ord lub ord* (λ*x*. *x*)
by(*simp add*: *mcont-def monotone-id′*)

lemma *mcont-applyI*:
mcont luba orda lubb ordb (λ*x*. *F x*) ⟹ *mcont* (*fun-lub luba*) (*fun-ord orda*) *lubb ordb* (λ*f*. *F* (*f x*))
by(*simp add*: *mcont-def monotone-applyI cont-applyI*)

lemma *mcont-if* [*cont-intro*, *simp*]:
⟦ *mcont luba orda lubb ordb* (λ*x*. *f x*); *mcont luba orda lubb ordb* (λ*x*. *g x*) ⟧
⟹ *mcont luba orda lubb ordb* (λ*x*. *if c then f x else g x*)
by(*simp add*: *mcont-def cont-if*)

lemma *cont-fun-lub-apply*:
cont luba orda (*fun-lub lubb*) (*fun-ord ordb*) *f* ⟷ (∀ *x*. *cont luba orda lubb ordb* (λ*y*. *f y x*))
by(*simp add*: *cont-def fun-lub-def fun-eq-iff*)(*auto simp add*: *image-def*)

lemma *mcont-fun-lub-apply*:
mcont luba orda (*fun-lub lubb*) (*fun-ord ordb*) *f* ⟷ (∀ *x*. *mcont luba orda lubb ordb* (λ*y*. *f y x*))
by(*auto simp add*: *monotone-fun-ord-apply cont-fun-lub-apply mcont-def*)

context *ccpo* **begin**

lemma *cont-const* [*simp*, *cont-intro*]: *cont luba orda Sup* (≤) (λ*x*. *c*)
by (*rule contI*) (*simp add*: *image-constant-conv cong del*: *SUP-cong-simp*)

lemma *mcont-const* [*cont-intro*, *simp*]:
mcont luba orda Sup (≤) (λ*x*. *c*)
by(*simp add*: *mcont-def*)

lemma *cont-apply*:
assumes *2*: ⋀*x*. *cont lubb ordb Sup* (≤) (λ*y*. *f x y*)
and *t*: *cont luba orda lubb ordb* (λ*x*. *t x*)
and *1*: ⋀*y*. *cont luba orda Sup* (≤) (λ*x*. *f x y*)

and *mono*: *monotone orda ordb* ($\lambda x.\ t\ x$)
and *mono2*: $\bigwedge x.$ *monotone ordb* ($\leq$) ($\lambda y.\ f\ x\ y$)
and *mono1*: $\bigwedge y.$ *monotone orda* ($\leq$) ($\lambda x.\ f\ x\ y$)
shows *cont luba orda Sup* ($\leq$) ($\lambda x.\ f\ x\ (t\ x)$)
proof
fix *Y*
assume *chain*: *Complete-Partial-Order.chain orda Y* **and** $Y \neq \{\}$
moreover from *chain* **have** *chain'*: *Complete-Partial-Order.chain ordb* (*t* ‘ *Y*)
by(*rule chain-imageI*)(*rule monotoneD*[*OF mono*])
ultimately show *f* (*luba Y*) (*t* (*luba Y*)) = $\bigsqcup$((λ*x*. *f x* (*t x*)) ‘ *Y*)
by(*simp add*: *contD*[*OF 1*] *contD*[*OF t*] *contD*[*OF 2*] *image-image*)
(*rule diag-Sup*[*OF chain*], *auto intro*: *monotone2monotone*[*OF mono2 mono monotone-const transpI*] *monotoneD*[*OF mono1*])
qed

lemma *mcont2mcont'*:
⟦ $\bigwedge x.$ *mcont lub' ord' Sup* ($\leq$) ($\lambda y.\ f\ x\ y$);
$\bigwedge y.$ *mcont lub ord Sup* ($\leq$) ($\lambda x.\ f\ x\ y$);
mcont lub ord lub' ord' ($\lambda y.\ t\ y$) ⟧
⟹ *mcont lub ord Sup* ($\leq$) ($\lambda x.\ f\ x\ (t\ x)$)
unfolding *mcont-def* **by**(*blast intro*: *transp-le monotone2monotone cont-apply*)

lemma *mcont2mcont*:
⟦*mcont lub' ord' Sup* ($\leq$) ($\lambda x.\ f\ x$); *mcont lub ord lub' ord'* ($\lambda x.\ t\ x$)⟧
⟹ *mcont lub ord Sup* ($\leq$) ($\lambda x.\ f\ (t\ x)$)
by(*rule mcont2mcont'*[*OF - mcont-const*])

context
fixes *ord* :: *'b* ⇒ *'b* ⇒ *bool* (**infix** $\sqsubseteq$ *60*)
and *lub* :: *'b set* ⇒ *'b* ($\bigvee$)
begin

lemma *cont-fun-lub-Sup*:
assumes *chainM*: *Complete-Partial-Order.chain* (*fun-ord* ($\leq$)) *M*
and *mcont* [*rule-format*]: $\forall f{\in}M.$ *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) *f*
shows *cont lub* ($\sqsubseteq$) *Sup* ($\leq$) (*fun-lub Sup M*)
proof(*rule contI*)
fix *Y*
assume *chain*: *Complete-Partial-Order.chain* ($\sqsubseteq$) *Y*
and *Y*: $Y \neq \{\}$
from *swap-Sup*[*OF chain chainM mcont*[*THEN mcont-mono*]]
show *fun-lub Sup M* ($\bigvee Y$) = $\bigsqcup$(*fun-lub Sup M* ‘ *Y*)
by(*simp add*: *mcont-contD*[*OF mcont chain Y*] *fun-lub-apply cong*: *image-cong*)
qed

lemma *mcont-fun-lub-Sup*:
⟦ *Complete-Partial-Order.chain* (*fun-ord* ($\leq$)) *M*;
$\forall f{\in}M.$ *mcont lub ord Sup* ($\leq$) *f* ⟧
⟹ *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) (*fun-lub Sup M*)

by(*simp add*: *mcont-def cont-fun-lub-Sup mono-lub*)

lemma *iterates-mcont*:
assumes *f*: $f \in$ *ccpo.iterates* (*fun-lub Sup*) (*fun-ord* ($\leq$)) *F*
and *mono*: $\bigwedge f$. *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) *f* $\Longrightarrow$ *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) (*F f*)
shows *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) *f*
using *f*
by(*induction rule*: *ccpo.iterates.induct*[*OF ccpo-fun*, *consumes 1*, *case-names step Sup*])(*blast intro*: *mono mcont-fun-lub-Sup*)+

lemma *fixp-preserves-mcont*:
assumes *mono*: $\bigwedge x$. *monotone* (*fun-ord* ($\leq$)) ($\leq$) (λf. *F f x*)
and *mcont*: $\bigwedge f$. *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) *f* $\Longrightarrow$ *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) (*F f*)
shows *mcont lub* ($\sqsubseteq$) *Sup* ($\leq$) (*ccpo.fixp* (*fun-lub Sup*) (*fun-ord* ($\leq$)) *F*)
(**is** *mcont - - - - ?fixp*)
unfolding *mcont-def*
proof(*intro conjI monotoneI contI*)
have *mono*: *monotone* (*fun-ord* ($\leq$)) (*fun-ord* ($\leq$)) *F*
by(*rule monotoneI*)(*auto simp add*: *fun-ord-def intro*: *monotoneD*[*OF mono*])
let *?iter* = *ccpo.iterates* (*fun-lub Sup*) (*fun-ord* ($\leq$)) *F*
have *chain*: $\bigwedge x$. *Complete-Partial-Order.chain* ($\leq$) ((λ*f*. *f x*) ‘ *?iter*)
by(*rule chain-imageI*[*OF ccpo.chain-iterates*[*OF ccpo-fun mono*]])(*simp add*: *fun-ord-def*)

{
fix *x y*
assume $x \sqsubseteq y$
show *?fixp x* $\leq$ *?fixp y*
apply (*simp only*: *ccpo.fixp-def*[*OF ccpo-fun*] *fun-lub-apply*)
using *chain*
proof(*rule ccpo-Sup-least*)
fix x'
assume $x' \in$ (λ*f*. *f x*) ‘ *?iter*
then obtain *f* **where** *f* $\in$ *?iter* $x' = f\ x$ **by** *blast* **note** *this*(*2*)
also from - ‹$x \sqsubseteq y$› **have** $f\ x \leq f\ y$
by(*rule mcont-monoD*[*OF iterates-mcont*[*OF* ‹*f* $\in$ *?iter*› *mcont*]])
also have $f\ y \leq \bigsqcup$((λ*f*. *f y*) ‘ *?iter*) **using** *chain*
by(*rule ccpo-Sup-upper*)(*simp add*: ‹*f* $\in$ *?iter*›)
finally show $x' \leq \ldots$ **.**
qed
next
fix *Y*
assume *chain*: *Complete-Partial-Order.chain* ($\sqsubseteq$) *Y*
and *Y*: $Y \neq \{\}$
{ fix *f*
assume *f* $\in$ *?iter*
hence *f* ($\bigvee Y$) = $\bigsqcup$(*f* ‘ *Y*)
using *mcont chain Y* **by**(*rule mcont-contD*[*OF iterates-mcont*]) **}**
moreover have $\bigsqcup$((λ*f*. $\bigsqcup$(*f* ‘ *Y*)) ‘ *?iter*) = $\bigsqcup$((λ*x*. $\bigsqcup$((λ*f*. *f x*) ‘ *?iter*)) ‘

Y)
using *chain ccpo.chain-iterates*[*OF ccpo-fun mono*]
by(*rule swap-Sup*)(*rule mcont-mono*[*OF iterates-mcont*[*OF - mcont*]])
ultimately show *?fixp* (⋁ *Y*) = ⨆(*?fixp* ‘ *Y*) **unfolding** *ccpo.fixp-def*[*OF ccpo-fun*]
by(*simp add*: *fun-lub-apply cong*: *image-cong*)
}
qed

end

context
fixes *F* :: *'c* ⇒ *'c* **and** *U* :: *'c* ⇒ *'b* ⇒ *'a* **and** *C* :: (*'b* ⇒ *'a*) ⇒ *'c* **and** *f*
assumes *mono*: ⋀*x*. *monotone* (*fun-ord* (≤)) (≤) (λ*f*. *U* (*F* (*C f*)) *x*)
and *eq*: *f* ≡ *C* (*ccpo.fixp* (*fun-lub Sup*) (*fun-ord* (≤)) (λ*f*. *U* (*F* (*C f*))))
and *inverse*: ⋀*f*. *U* (*C f*) = *f*
begin

lemma *fixp-preserves-mono-uc*:
assumes *mono2*: ⋀*f*. *monotone ord* (≤) (*U f*) ⟹ *monotone ord* (≤) (*U* (*F f*))
shows *monotone ord* (≤) (*U f*)
using *fixp-preserves-mono*[*OF mono mono2*] **by**(*subst eq*)(*simp add*: *inverse*)

lemma *fixp-preserves-mcont-uc*:
assumes *mcont*: ⋀*f*. *mcont lubb ordb Sup* (≤) (*U f*) ⟹ *mcont lubb ordb Sup* (≤) (*U* (*F f*))
shows *mcont lubb ordb Sup* (≤) (*U f*)
using *fixp-preserves-mcont*[*OF mono mcont*] **by**(*subst eq*)(*simp add*: *inverse*)

end

lemmas *fixp-preserves-mono1* = *fixp-preserves-mono-uc*[*of* λ*x*. *x* - λ*x*. *x*, *OF* - - *refl*]
lemmas *fixp-preserves-mono2* =
fixp-preserves-mono-uc[*of case-prod* - *curry*, *unfolded case-prod-curry curry-case-prod*, *OF* - - *refl*]
lemmas *fixp-preserves-mono3* =
fixp-preserves-mono-uc[*of* λ*f*. *case-prod* (*case-prod f*) - λ*f*. *curry* (*curry f*), *unfolded case-prod-curry curry-case-prod*, *OF* - - *refl*]
lemmas *fixp-preserves-mono4* =
fixp-preserves-mono-uc[*of* λ*f*. *case-prod* (*case-prod* (*case-prod f*)) - λ*f*. *curry* (*curry* (*curry f*)), *unfolded case-prod-curry curry-case-prod*, *OF* - - *refl*]

lemmas *fixp-preserves-mcont1* = *fixp-preserves-mcont-uc*[*of* λ*x*. *x* - λ*x*. *x*, *OF* - - *refl*]
lemmas *fixp-preserves-mcont2* =
fixp-preserves-mcont-uc[*of case-prod* - *curry*, *unfolded case-prod-curry curry-case-prod*, *OF* - - *refl*]
lemmas *fixp-preserves-mcont3* =

fixp-preserves-mcont-uc[*of* $\lambda f.$ *case-prod* (*case-prod f*) - $\lambda f.$ *curry* (*curry f*), *unfolded case-prod-curry curry-case-prod*, *OF* - - *refl*]
lemmas *fixp-preserves-mcont4* =
fixp-preserves-mcont-uc[*of* $\lambda f.$ *case-prod* (*case-prod* (*case-prod f*)) - $\lambda f.$ *curry* (*curry* (*curry f*)), *unfolded case-prod-curry curry-case-prod*, *OF* - - *refl*]

end

lemma (**in** *preorder*) *monotone-if-bot*:
fixes *bot*
assumes *mono*: $\bigwedge x\ y.\ [\![\ x \le y;\ \neg\ (x \le bound)\ ]\!] \Longrightarrow ord\ (f\ x)\ (f\ y)$
and *bot*: $\bigwedge x.\ \neg\ x \le bound \Longrightarrow ord\ bot\ (f\ x)\ ord\ bot\ bot$
shows *monotone* ($\le$) *ord* ($\lambda x.$ *if* $x \le bound$ *then bot else f x*)
by(*rule monotoneI*)(*auto intro*: *bot intro*: *mono order-trans*)

lemma (**in** *ccpo*) *mcont-if-bot*:
fixes *bot* **and** *lub* ($\bigvee$) **and** *ord* (**infix** $\sqsubseteq$ *60*)
assumes *ccpo*: *class.ccpo lub* ($\sqsubseteq$) *lt*
and *mono*: $\bigwedge x\ y.\ [\![\ x \le y;\ \neg\ x \le bound\ ]\!] \Longrightarrow f\ x \sqsubseteq f\ y$
and *cont*: $\bigwedge Y.\ [\![$ *Complete-Partial-Order.chain* ($\le$) Y; $Y \neq \{\}$; $\bigwedge x.\ x \in Y \Longrightarrow \neg\ x \le bound\ ]\!] \Longrightarrow f\ (\bigsqcup Y) = \bigvee(f\ `\ Y)$
and *bot*: $\bigwedge x.\ \neg\ x \le bound \Longrightarrow bot \sqsubseteq f\ x$
shows *mcont Sup* ($\le$) *lub* ($\sqsubseteq$) ($\lambda x.$ *if* $x \le bound$ *then bot else f x*) (**is** *mcont - - - - ?g*)
proof(*intro mcontI contI*)
interpret *c*: *ccpo lub* ($\sqsubseteq$) *lt* **by**(*fact ccpo*)
show *monotone* ($\le$) ($\sqsubseteq$) *?g* **by**(*rule monotone-if-bot*)(*simp-all add*: *mono bot*)

fix Y
assume *chain*: *Complete-Partial-Order.chain* ($\le$) Y **and** Y: $Y \neq \{\}$
show *?g* ($\bigsqcup Y$) = $\bigvee$(*?g* ` Y)
proof(*cases* $Y \subseteq \{x.\ x \le bound\}$)
case *True*
hence $\bigsqcup Y \le bound$ **using** *chain* **by**(*auto intro*: *ccpo-Sup-least*)
moreover have $Y \cap \{x.\ \neg\ x \le bound\} = \{\}$ **using** *True* **by** *auto*
ultimately show *?thesis* **using** *True Y*
by (*auto simp add*: *image-constant-conv cong del*: *c.SUP-cong-simp*)
next
case *False*
let *?Y* = $Y \cap \{x.\ \neg\ x \le bound\}$
have *chain'*: *Complete-Partial-Order.chain* ($\le$) *?Y*
using *chain* **by**(*rule chain-subset*) *simp*

from *False* **obtain** y **where** *ybound*: $\neg\ y \le bound$ **and** y: $y \in Y$ **by** *blast*
hence $\neg\ \bigsqcup Y \le bound$ **by** (*metis ccpo-Sup-upper chain order.trans*)
hence *?g* ($\bigsqcup Y$) = $f\ (\bigsqcup Y)$ **by** *simp*
also have $\bigsqcup Y \le \bigsqcup$ *?Y* **using** *chain*
proof(*rule ccpo-Sup-least*)
fix x

assume x: $x \in Y$
show $x \leq \bigsqcup ?Y$
proof(*cases* $x \leq$ *bound*)
case *True*
with *chainD*[*OF chain x y*] **have** $x \leq y$ **using** *ybound* **by**(*auto intro*: *order-trans*)
thus *?thesis* **by**(*rule order-trans*)(*auto intro*: *ccpo-Sup-upper*[*OF chain'*] *simp add*: *y ybound*)
qed(*auto intro*: *ccpo-Sup-upper*[*OF chain'*] *simp add*: *x*)
qed
hence $\bigsqcup Y = \bigsqcup ?Y$ **by**(*rule order.antisym*)(*blast intro*: *ccpo-Sup-least*[*OF chain'*] *ccpo-Sup-upper*[*OF chain*])
hence $f\ (\bigsqcup Y) = f\ (\bigsqcup ?Y)$ **by** *simp*
also have $f\ (\bigsqcup ?Y) = \bigsqcup (f\ `\ ?Y)$ **using** *chain'* **by**(*rule cont*)(*insert y ybound*, *auto*)
also have $\bigsqcup (f\ `\ ?Y) = \bigsqcup (?g\ `\ Y)$
proof(*cases* $Y \cap \{x.\ x \leq bound\} = \{\}$)
case *True*
hence $f\ `\ ?Y = ?g\ `\ Y$ **by** *auto*
thus *?thesis* **by**(*rule arg-cong*)
next
case *False*
have *chain''*: *Complete-Partial-Order.chain* ($\sqsubseteq$) (*insert bot* ($f\ `\ ?Y$))
using *chain* **by**(*auto intro*!: *chainI bot dest*: *chainD intro*: *mono*)
hence *chain'''*: *Complete-Partial-Order.chain* ($\sqsubseteq$) ($f\ `\ ?Y$) **by**(*rule chain-subset*) *blast*
have *bot* $\sqsubseteq \bigsqcup (f\ `\ ?Y)$ **using** *y ybound* **by**(*blast intro*: *c.order-trans*[*OF bot*] *c.ccpo-Sup-upper*[*OF chain'''*])
hence $\bigsqcup$(*insert bot* ($f\ `\ ?Y$)) $\sqsubseteq \bigsqcup (f\ `\ ?Y)$ **using** *chain''*
by(*auto intro*: *c.ccpo-Sup-least c.ccpo-Sup-upper*[*OF chain'''*])
with - **have** ... $= \bigsqcup$(*insert bot* ($f\ `\ ?Y$))
by(*rule c.order.antisym*)(*blast intro*: *c.ccpo-Sup-least*[*OF chain'''*] *c.ccpo-Sup-upper*[*OF chain'''*])
also have *insert bot* ($f\ `\ ?Y$) = *?g* ` *Y* **using** *False* **by** *auto*
finally show *?thesis* **.**
qed
finally show *?thesis* **.**
qed
qed

context *partial-function-definitions* **begin**

lemma *mcont-const* [*cont-intro*, *simp*]:
mcont luba orda lub leq ($\lambda x.\ c$)
by(*rule ccpo.mcont-const*)(*rule Partial-Function.ccpo*[*OF partial-function-definitions-axioms*])

lemmas [*cont-intro*, *simp*] =
ccpo.cont-const[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]

lemma *mono2mono*:
  assumes *monotone ordb leq* (λ*y*. *f y*) *monotone orda ordb* (λ*x*. *t x*)
  shows *monotone orda leq* (λ*x*. *f* (*t x*))
using *assms* **by**(*rule monotone2monotone*) *simp-all*

lemmas *mcont2mcont′* = *ccpo.mcont2mcont′*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *mcont2mcont* = *ccpo.mcont2mcont*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]

lemmas *fixp-preserves-mono1* = *ccpo.fixp-preserves-mono1*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mono2* = *ccpo.fixp-preserves-mono2*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mono3* = *ccpo.fixp-preserves-mono3*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mono4* = *ccpo.fixp-preserves-mono4*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mcont1* = *ccpo.fixp-preserves-mcont1*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mcont2* = *ccpo.fixp-preserves-mcont2*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mcont3* = *ccpo.fixp-preserves-mcont3*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
lemmas *fixp-preserves-mcont4* = *ccpo.fixp-preserves-mcont4*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]

lemma *monotone-if-bot*:
  fixes *bot*
  assumes *g*: ⋀*x*. *g x* = (*if leq x bound then bot else f x*)
  and *mono*: ⋀*x y*. ⟦ *leq x y*; ¬ *leq x bound* ⟧ ⟹ *ord* (*f x*) (*f y*)
  and *bot*: ⋀*x*. ¬ *leq x bound* ⟹ *ord bot* (*f x*) *ord bot bot*
  shows *monotone leq ord g*
unfolding *g*[*abs-def*] **using** *preorder mono bot* **by**(*rule preorder.monotone-if-bot*)

lemma *mcont-if-bot*:
  fixes *bot*
  assumes *ccpo*: *class.ccpo lub′ ord* (*mk-less ord*)
  and *bot*: ⋀*x*. ¬ *leq x bound* ⟹ *ord bot* (*f x*)
  and *g*: ⋀*x*. *g x* = (*if leq x bound then bot else f x*)
  and *mono*: ⋀*x y*. ⟦ *leq x y*; ¬ *leq x bound* ⟧ ⟹ *ord* (*f x*) (*f y*)
  and *cont*: ⋀*Y*. ⟦ *Complete-Partial-Order.chain leq Y*; *Y* ≠ {}; ⋀*x*. *x* ∈ *Y* ⟹ ¬ *leq x bound* ⟧ ⟹ *f* (*lub Y*) = *lub′* (*f* ‘ *Y*)
  shows *mcont lub leq lub′ ord g*
unfolding *g*[*abs-def*] **using** *ccpo mono cont bot* **by**(*rule ccpo.mcont-if-bot*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]])

end

15.2 Admissibility

lemma *admissible-subst*:
  assumes *adm*: *ccpo.admissible luba orda* (λ*x*. *P x*)
  and *mcont*: *mcont lubb ordb luba orda f*
  shows *ccpo.admissible lubb ordb* (λ*x*. *P* (*f x*))
apply(*rule ccpo.admissibleI*)
apply(*frule* (*1*) *mcont-contD*[*OF mcont*])
apply(*auto intro*: *ccpo.admissibleD*[*OF adm*] *chain-imageI dest*: *mcont-monoD*[*OF mcont*])
done

lemmas [*simp*, *cont-intro*] =
  admissible-all
  admissible-ball
  admissible-const
  admissible-conj

lemma *admissible-disj'* [*simp*, *cont-intro*]:
  ⟦ *class.ccpo lub ord* (*mk-less ord*); *ccpo.admissible lub ord P*; *ccpo.admissible lub ord Q* ⟧
  ⟹ *ccpo.admissible lub ord* (λ*x*. *P x* ∨ *Q x*)
by(*rule ccpo.admissible-disj*)

lemma *admissible-imp'* [*cont-intro*]:
  ⟦ *class.ccpo lub ord* (*mk-less ord*);
    ccpo.admissible lub ord (λ*x*. ¬ *P x*);
    ccpo.admissible lub ord (λ*x*. *Q x*) ⟧
  ⟹ *ccpo.admissible lub ord* (λ*x*. *P x* ⟶ *Q x*)
unfolding *imp-conv-disj* **by**(*rule ccpo.admissible-disj*)

lemma *admissible-imp* [*cont-intro*]:
  (*Q* ⟹ *ccpo.admissible lub ord* (λ*x*. *P x*))
  ⟹ *ccpo.admissible lub ord* (λ*x*. *Q* ⟶ *P x*)
by(*rule ccpo.admissibleI*)(*auto dest*: *ccpo.admissibleD*)

lemma *admissible-not-mem'* [*THEN admissible-subst*, *cont-intro*, *simp*]:
  shows *admissible-not-mem*: *ccpo.admissible Union* (⊆) (λ*A*. *x* ∉ *A*)
by(*rule ccpo.admissibleI*) *auto*

lemma *admissible-eqI*:
  assumes *f*: *cont luba orda lub ord* (λ*x*. *f x*)
  and *g*: *cont luba orda lub ord* (λ*x*. *g x*)
  shows *ccpo.admissible luba orda* (λ*x*. *f x* = *g x*)
apply(*rule ccpo.admissibleI*)
apply(*simp-all add*: *contD*[*OF f*] *contD*[*OF g*] *cong*: *image-cong*)
done

corollary *admissible-eq-mcontI* [*cont-intro*]:
  ⟦ *mcont luba orda lub ord* (λ*x*. *f x*);

mcont luba orda lub ord $(\lambda x.\ g\ x)$ ⟧
⟹ *ccpo.admissible luba orda* $(\lambda x.\ f\ x = g\ x)$
by(*rule admissible-eqI*)(*auto simp add*: *mcont-def*)

lemma *admissible-iff* [*cont-intro*, *simp*]:
⟦ *ccpo.admissible lub ord* $(\lambda x.\ P\ x \longrightarrow Q\ x)$; *ccpo.admissible lub ord* $(\lambda x.\ Q\ x \longrightarrow P\ x)$ ⟧
⟹ *ccpo.admissible lub ord* $(\lambda x.\ P\ x \longleftrightarrow Q\ x)$
by(*subst iff-conv-conj-imp*)(*rule admissible-conj*)

context *ccpo* **begin**

lemma *admissible-leI*:
assumes *f*: *mcont luba orda Sup* $(\leq)$ $(\lambda x.\ f\ x)$
and *g*: *mcont luba orda Sup* $(\leq)$ $(\lambda x.\ g\ x)$
shows *ccpo.admissible luba orda* $(\lambda x.\ f\ x \leq g\ x)$
proof(*rule ccpo.admissibleI*)
fix *A*
assume *chain*: *Complete-Partial-Order.chain orda A*
and *le*: $\forall x{\in}A.\ f\ x \leq g\ x$
and *False*: $A \neq \{\}$
have *f* (*luba A*) = $\bigsqcup$(*f* ‘ *A*) **by**(*simp add*: *mcont-contD*[*OF f*] *chain False*)
also have ... ≤ $\bigsqcup$(*g* ‘ *A*)
proof(*rule ccpo-Sup-least*)
from *chain* **show** *Complete-Partial-Order.chain* $(\leq)$ (*f* ‘ *A*)
by(*rule chain-imageI*)(*rule mcont-monoD*[*OF f*])

fix *x*
assume *x* ∈ *f* ‘ *A*
then obtain *y* **where** *y* ∈ *A x* = *f y* **by** *blast* **note** *this*(*2*)
also have *f y* ≤ *g y* **using** *le* ‹*y* ∈ *A*› **by** *simp*
also have *Complete-Partial-Order.chain* $(\leq)$ (*g* ‘ *A*)
using *chain* **by**(*rule chain-imageI*)(*rule mcont-monoD*[*OF g*])
hence *g y* ≤ $\bigsqcup$(*g* ‘ *A*) **by**(*rule ccpo-Sup-upper*)(*simp add*: ‹*y* ∈ *A*›)
finally show *x* ≤ ... **.**
qed
also have ... = *g* (*luba A*) **by**(*simp add*: *mcont-contD*[*OF g*] *chain False*)
finally show *f* (*luba A*) ≤ *g* (*luba A*) **.**
qed

end

lemma *admissible-leI*:
fixes *ord* (**infix** ⊑ *60*) **and** *lub* (⋁)
assumes *class.ccpo lub* (⊑) (*mk-less* (⊑))
and *mcont luba orda lub* (⊑) $(\lambda x.\ f\ x)$
and *mcont luba orda lub* (⊑) $(\lambda x.\ g\ x)$
shows *ccpo.admissible luba orda* $(\lambda x.\ f\ x \sqsubseteq g\ x)$
using *assms* **by**(*rule ccpo.admissible-leI*)

declare *ccpo-class.admissible-leI*[*cont-intro*]

context *ccpo* **begin**

lemma *admissible-not-below*: *ccpo.admissible Sup* (≤) (λ*x*. ¬ (≤) *x y*)
by(*rule ccpo.admissibleI*)(*simp add*: *ccpo-Sup-below-iff*)

end

lemma (**in** *preorder*) *preorder* [*cont-intro*, *simp*]: *class.preorder* (≤) (*mk-less* (≤))
by(*unfold-locales*)(*auto simp add*: *mk-less-def intro*: *order-trans*)

context *partial-function-definitions* **begin**

lemmas [*cont-intro*, *simp*] =
admissible-leI[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]
ccpo.admissible-not-below[*THEN admissible-subst*, *OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]

end

setup ‹*Sign.map-naming* (*Name-Space.mandatory-path ccpo*)›

inductive *compact* :: ('*a set* ⇒ '*a*) ⇒ ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a* ⇒ *bool*
for *lub ord x*
where *compact*:
⟦ *ccpo.admissible lub ord* (λ*y*. ¬ *ord x y*);
ccpo.admissible lub ord (λ*y*. *x* ≠ *y*) ⟧
⟹ *compact lub ord x*

setup ‹*Sign.map-naming Name-Space.parent-path*›

context *ccpo* **begin**

lemma *compactI*:
assumes *ccpo.admissible Sup* (≤) (λ*y*. ¬ *x* ≤ *y*)
shows *ccpo.compact Sup* (≤) *x*
using *assms*
proof(*rule ccpo.compact.intros*)
have *neq*: (λ*y*. *x* ≠ *y*) = (λ*y*. ¬ *x* ≤ *y* ∨ ¬ *y* ≤ *x*) **by**(*auto*)
show *ccpo.admissible Sup* (≤) (λ*y*. *x* ≠ *y*)
by(*subst neq*)(*rule admissible-disj admissible-not-below assms*)+
qed

lemma *compact-bot*:
assumes *x* = *Sup* {}
shows *ccpo.compact Sup* (≤) *x*
proof(*rule compactI*)

show *ccpo.admissible Sup* (≤) (λ*y*. ¬ *x* ≤ *y*) **using** *assms*
by(*auto intro*!: *ccpo.admissibleI intro*: *ccpo-Sup-least chain-empty*)
qed

end

lemma *admissible-compact-neq'* [*THEN admissible-subst*, *cont-intro*, *simp*]:
shows *admissible-compact-neq*: *ccpo.compact lub ord k* ⟹ *ccpo.admissible lub ord* (λ*x*. *k* ≠ *x*)
by(*simp add*: *ccpo.compact.simps*)

lemma *admissible-neq-compact'* [*THEN admissible-subst*, *cont-intro*, *simp*]:
shows *admissible-neq-compact*: *ccpo.compact lub ord k* ⟹ *ccpo.admissible lub ord* (λ*x*. *x* ≠ *k*)
by(*subst eq-commute*)(*rule admissible-compact-neq*)

context *partial-function-definitions* **begin**

lemmas [*cont-intro*, *simp*] = *ccpo.compact-bot*[*OF Partial-Function.ccpo*[*OF partial-function-definitions-axioms*]]

end

context *ccpo* **begin**

lemma *fixp-strong-induct*:
assumes [*cont-intro*]: *ccpo.admissible Sup* (≤) *P*
and *mono*: *monotone* (≤) (≤) *f*
and *bot*: *P* (⨆{})
and *step*: ⋀*x*. ⟦ *x* ≤ *ccpo-class.fixp f*; *P x* ⟧ ⟹ *P* (*f x*)
shows *P* (*ccpo-class.fixp f*)
proof(*rule fixp-induct*[**where** *P*=λ*x*. *x* ≤ *ccpo-class.fixp f* ∧ *P x*, *THEN conjunct2*])
note [*cont-intro*] = *admissible-leI*
show *ccpo.admissible Sup* (≤) (λ*x*. *x* ≤ *ccpo-class.fixp f* ∧ *P x*) **by** *simp*
next
show ⨆{} ≤ *ccpo-class.fixp f* ∧ *P* (⨆{})
by(*auto simp add*: *bot intro*: *ccpo-Sup-least chain-empty*)
next
fix *x*
assume *x* ≤ *ccpo-class.fixp f* ∧ *P x*
thus *f x* ≤ *ccpo-class.fixp f* ∧ *P* (*f x*)
by(*subst fixp-unfold*[*OF mono*])(*auto dest*: *monotoneD*[*OF mono*] *intro*: *step*)
qed(*rule mono*)

end

context *partial-function-definitions* **begin**

lemma *fixp-strong-induct-uc*:
  fixes $F :: 'c \Rightarrow 'c$
    and $U :: 'c \Rightarrow 'b \Rightarrow 'a$
    and $C :: ('b \Rightarrow 'a) \Rightarrow 'c$
    and $P :: ('b \Rightarrow 'a) \Rightarrow bool$
  assumes *mono*: $\bigwedge x.\ mono\text{-}body\ (\lambda f.\ U\ (F\ (C\ f))\ x)$
    and *eq*: $f \equiv C\ (fixp\text{-}fun\ (\lambda f.\ U\ (F\ (C\ f))))$
    and *inverse*: $\bigwedge f.\ U\ (C\ f) = f$
    and *adm*: *ccpo.admissible lub-fun le-fun P*
    and *bot*: $P\ (\lambda\text{-}.\ lub\ \{\})$
    and *step*: $\bigwedge f'.\ [\![\ P\ (U\ f');\ le\text{-}fun\ (U\ f')\ (U\ f)\ ]\!] \Longrightarrow P\ (U\ (F\ f'))$
  shows $P\ (U\ f)$
unfolding *eq inverse*
apply (*rule ccpo.fixp-strong-induct*[*OF ccpo adm*])
apply (*insert mono*, *auto simp*: *monotone-def fun-ord-def bot fun-lub-def*)[*2*]
apply (*rule-tac f'5*=*C x* **in** *step*)
apply (*simp-all add*: *inverse eq*)
done

end

15.3 (=) as order

definition *lub-singleton* :: $('a\ set \Rightarrow 'a) \Rightarrow bool$
where *lub-singleton lub* $\longleftrightarrow$ $(\forall a.\ lub\ \{a\} = a)$

definition *the-Sup* :: $'a\ set \Rightarrow 'a$
where *the-Sup A* = $(THE\ a.\ a \in A)$

lemma *lub-singleton-the-Sup* [*cont-intro*, *simp*]: *lub-singleton the-Sup*
by(*simp add*: *lub-singleton-def the-Sup-def*)

lemma (**in** *ccpo*) *lub-singleton*: *lub-singleton Sup*
by(*simp add*: *lub-singleton-def*)

lemma (**in** *partial-function-definitions*) *lub-singleton* [*cont-intro*, *simp*]: *lub-singleton lub*
by(*rule ccpo.lub-singleton*)(*rule Partial-Function.ccpo*[*OF partial-function-definitions-axioms*])

lemma *preorder-eq* [*cont-intro*, *simp*]:
  class.preorder (=) (*mk-less* (=))
by(*unfold-locales*)(*simp-all add*: *mk-less-def*)

lemma *monotone-eqI* [*cont-intro*]:
  assumes *class.preorder ord* (*mk-less ord*)
  shows *monotone* (=) *ord f*
proof −
  interpret *preorder ord mk-less ord* **by** *fact*
  show *?thesis* **by**(*simp add*: *monotone-def*)

qed

lemma *cont-eqI* [*cont-intro*]:
fixes *f* :: *'a* ⇒ *'b*
assumes *lub-singleton lub*
shows *cont the-Sup* (=) *lub ord f*
proof(*rule contI*)
fix *Y* :: *'a set*
assume *Complete-Partial-Order.chain* (=) *Y Y* ≠ {}
then obtain *a* **where** *Y* = {*a*} **by**(*auto simp add*: *chain-def*)
thus *f* (*the-Sup Y*) = *lub* (*f* ‘ *Y*) **using** *assms*
by(*simp add*: *the-Sup-def lub-singleton-def*)
qed

lemma *mcont-eqI* [*cont-intro*, *simp*]:
⟦ *class.preorder ord* (*mk-less ord*); *lub-singleton lub* ⟧
⟹ *mcont the-Sup* (=) *lub ord f*
by(*simp add*: *mcont-def cont-eqI monotone-eqI*)

15.4 ccpo for products

definition *prod-lub* :: (*'a set* ⇒ *'a*) ⇒ (*'b set* ⇒ *'b*) ⇒ (*'a* × *'b*) *set* ⇒ *'a* × *'b*
where *prod-lub Sup-a Sup-b Y* = (*Sup-a* (*fst* ‘ *Y*), *Sup-b* (*snd* ‘ *Y*))

lemma *lub-singleton-prod-lub* [*cont-intro*, *simp*]:
⟦ *lub-singleton luba*; *lub-singleton lubb* ⟧ ⟹ *lub-singleton* (*prod-lub luba lubb*)
by(*simp add*: *lub-singleton-def prod-lub-def*)

lemma *prod-lub-empty* [*simp*]: *prod-lub luba lubb* {} = (*luba* {}, *lubb* {})
by(*simp add*: *prod-lub-def*)

lemma *preorder-rel-prodI* [*cont-intro*, *simp*]:
assumes *class.preorder orda* (*mk-less orda*)
and *class.preorder ordb* (*mk-less ordb*)
shows *class.preorder* (*rel-prod orda ordb*) (*mk-less* (*rel-prod orda ordb*))
proof −
interpret *a*: *preorder orda mk-less orda* **by** *fact*
interpret *b*: *preorder ordb mk-less ordb* **by** *fact*
show *?thesis* **by**(*unfold-locales*)(*auto simp add*: *mk-less-def intro*: *a.order-trans b.order-trans*)
qed

lemma *order-rel-prodI*:
assumes *a*: *class.order orda* (*mk-less orda*)
and *b*: *class.order ordb* (*mk-less ordb*)
shows *class.order* (*rel-prod orda ordb*) (*mk-less* (*rel-prod orda ordb*))
(**is** *class.order ?ord ?ord'*)
proof(*intro class.order.intro class.order-axioms.intro*)
interpret *a*: *order orda mk-less orda* **by**(*fact a*)

interpret *b*: *order ordb mk-less ordb* **by**(*fact b*)
show *class.preorder ?ord ?ord'* **by**(*rule preorder-rel-prodI*) *unfold-locales*

fix *x y*
assume *?ord x y ?ord y x*
thus *x = y* **by**(*cases x y rule*: *prod.exhaust*[*case-product prod.exhaust*]) *auto*
qed

lemma *monotone-rel-prodI*:
assumes *mono2*: ⋀*a*. *monotone ordb ordc* (λ*b*. *f* (*a*, *b*))
and *mono1*: ⋀*b*. *monotone orda ordc* (λ*a*. *f* (*a*, *b*))
and *a*: *class.preorder orda* (*mk-less orda*)
and *b*: *class.preorder ordb* (*mk-less ordb*)
and *c*: *class.preorder ordc* (*mk-less ordc*)
shows *monotone* (*rel-prod orda ordb*) *ordc f*
proof −
interpret *a*: *preorder orda mk-less orda* **by**(*rule a*)
interpret *b*: *preorder ordb mk-less ordb* **by**(*rule b*)
interpret *c*: *preorder ordc mk-less ordc* **by**(*rule c*)
show *?thesis* **using** *mono2 mono1*
by(*auto 7 2 simp add*: *monotone-def intro*: *c.order-trans*)
qed

lemma *monotone-rel-prodD1*:
assumes *mono*: *monotone* (*rel-prod orda ordb*) *ordc f*
and *preorder*: *class.preorder ordb* (*mk-less ordb*)
shows *monotone orda ordc* (λ*a*. *f* (*a*, *b*))
proof −
interpret *preorder ordb mk-less ordb* **by**(*rule preorder*)
show *?thesis* **using** *mono* **by**(*simp add*: *monotone-def*)
qed

lemma *monotone-rel-prodD2*:
assumes *mono*: *monotone* (*rel-prod orda ordb*) *ordc f*
and *preorder*: *class.preorder orda* (*mk-less orda*)
shows *monotone ordb ordc* (λ*b*. *f* (*a*, *b*))
proof −
interpret *preorder orda mk-less orda* **by**(*rule preorder*)
show *?thesis* **using** *mono* **by**(*simp add*: *monotone-def*)
qed

lemma *monotone-case-prodI*:
⟦ ⋀*a*. *monotone ordb ordc* (*f a*); ⋀*b*. *monotone orda ordc* (λ*a*. *f a b*);
class.preorder orda (*mk-less orda*); *class.preorder ordb* (*mk-less ordb*);
class.preorder ordc (*mk-less ordc*) ⟧
⟹ *monotone* (*rel-prod orda ordb*) *ordc* (*case-prod f*)
by(*rule monotone-rel-prodI*) *simp-all*

lemma *monotone-case-prodD1*:

assumes *mono*: *monotone* (*rel-prod orda ordb*) *ordc* (*case-prod f*)
and *preorder*: *class.preorder ordb* (*mk-less ordb*)
shows *monotone orda ordc* (λ*a*. *f a b*)
using *monotone-rel-prodD1*[*OF assms*] **by** *simp*

lemma *monotone-case-prodD2*:
assumes *mono*: *monotone* (*rel-prod orda ordb*) *ordc* (*case-prod f*)
and *preorder*: *class.preorder orda* (*mk-less orda*)
shows *monotone ordb ordc* (*f a*)
using *monotone-rel-prodD2*[*OF assms*] **by** *simp*

context
fixes *orda ordb ordc*
assumes *a*: *class.preorder orda* (*mk-less orda*)
and *b*: *class.preorder ordb* (*mk-less ordb*)
and *c*: *class.preorder ordc* (*mk-less ordc*)
begin

lemma *monotone-rel-prod-iff*:
monotone (*rel-prod orda ordb*) *ordc f* ⟷
(∀ *a*. *monotone ordb ordc* (λ*b*. *f* (*a*, *b*))) ∧
(∀ *b*. *monotone orda ordc* (λ*a*. *f* (*a*, *b*)))
using *a b c* **by**(*blast intro*: *monotone-rel-prodI dest*: *monotone-rel-prodD1 monotone-rel-prodD2*)

lemma *monotone-case-prod-iff* [*simp*]:
monotone (*rel-prod orda ordb*) *ordc* (*case-prod f*) ⟷
(∀ *a*. *monotone ordb ordc* (*f a*)) ∧ (∀ *b*. *monotone orda ordc* (λ*a*. *f a b*))
by(*simp add*: *monotone-rel-prod-iff*)

end

lemma *monotone-case-prod-apply-iff*:
monotone orda ordb (λ*x*. (*case-prod f x*) *y*) ⟷ *monotone orda ordb* (*case-prod* (λ*a b*. *f a b y*))
by(*simp add*: *monotone-def*)

lemma *monotone-case-prod-applyD*:
monotone orda ordb (λ*x*. (*case-prod f x*) *y*)
⟹ *monotone orda ordb* (*case-prod* (λ*a b*. *f a b y*))
by(*simp add*: *monotone-case-prod-apply-iff*)

lemma *monotone-case-prod-applyI*:
monotone orda ordb (*case-prod* (λ*a b*. *f a b y*))
⟹ *monotone orda ordb* (λ*x*. (*case-prod f x*) *y*)
by(*simp add*: *monotone-case-prod-apply-iff*)

lemma *cont-case-prod-apply-iff*:

cont luba orda lubb ordb (λ*x*. (*case-prod f x*) *y*) ⟷ *cont luba orda lubb ordb* (*case-prod* (λ*a b*. *f a b y*))
by(*simp add*: *cont-def split-def*)

lemma *cont-case-prod-applyI*:
cont luba orda lubb ordb (*case-prod* (λ*a b*. *f a b y*))
⟹ *cont luba orda lubb ordb* (λ*x*. (*case-prod f x*) *y*)
by(*simp add*: *cont-case-prod-apply-iff*)

lemma *cont-case-prod-applyD*:
cont luba orda lubb ordb (λ*x*. (*case-prod f x*) *y*)
⟹ *cont luba orda lubb ordb* (*case-prod* (λ*a b*. *f a b y*))
by(*simp add*: *cont-case-prod-apply-iff*)

lemma *mcont-case-prod-apply-iff* [*simp*]:
mcont luba orda lubb ordb (λ*x*. (*case-prod f x*) *y*) ⟷
mcont luba orda lubb ordb (*case-prod* (λ*a b*. *f a b y*))
by(*simp add*: *mcont-def monotone-case-prod-apply-iff cont-case-prod-apply-iff*)

lemma *cont-prodD1*:
assumes *cont*: *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lubc ordc f*
and *class.preorder orda* (*mk-less orda*)
and *luba*: *lub-singleton luba*
shows *cont lubb ordb lubc ordc* (λ*y*. *f* (*x*, *y*))
proof(*rule contI*)
interpret *preorder orda mk-less orda* **by** *fact*

fix *Y* :: *'b set*
let *?Y* = {*x*} × *Y*
assume *Complete-Partial-Order.chain ordb Y Y* ≠ {}
hence *Complete-Partial-Order.chain* (*rel-prod orda ordb*) *?Y ?Y* ≠ {}
by(*simp-all add*: *chain-def*)
with *cont* **have** *f* (*prod-lub luba lubb ?Y*) = *lubc* (*f* ‘ *?Y*) **by**(*rule contD*)
moreover have *f* ‘ *?Y* = (λ*y*. *f* (*x*, *y*)) ‘ *Y* **by** *auto*
ultimately show *f* (*x*, *lubb Y*) = *lubc* ((λ*y*. *f* (*x*, *y*)) ‘ *Y*) **using** *luba*
by(*simp add*: *prod-lub-def* ‹*Y* ≠ {}› *lub-singleton-def*)
qed

lemma *cont-prodD2*:
assumes *cont*: *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lubc ordc f*
and *class.preorder ordb* (*mk-less ordb*)
and *lubb*: *lub-singleton lubb*
shows *cont luba orda lubc ordc* (λ*x*. *f* (*x*, *y*))
proof(*rule contI*)
interpret *preorder ordb mk-less ordb* **by** *fact*

fix *Y*
assume *Y*: *Complete-Partial-Order.chain orda Y Y* ≠ {}
let *?Y* = *Y* × {*y*}

have *f* (*luba Y*, *y*) = *f* (*prod-lub luba lubb ?Y*)
using *lubb* **by**(*simp add*: *prod-lub-def Y lub-singleton-def*)
also from *Y* **have** *Complete-Partial-Order.chain* (*rel-prod orda ordb*) *?Y ?Y* ≠ {}
by(*simp-all add*: *chain-def*)
with *cont* **have** *f* (*prod-lub luba lubb ?Y*) = *lubc* (*f* ‘ *?Y*) **by**(*rule contD*)
also have *f* ‘ *?Y* = (λ*x*. *f* (*x*, *y*)) ‘ *Y* **by** *auto*
finally show *f* (*luba Y*, *y*) = *lubc* ... **.**
qed

lemma *cont-case-prodD1*:
assumes *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lubc ordc* (*case-prod f*)
and *class.preorder orda* (*mk-less orda*)
and *lub-singleton luba*
shows *cont lubb ordb lubc ordc* (*f x*)
using *cont-prodD1*[*OF assms*] **by** *simp*

lemma *cont-case-prodD2*:
assumes *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lubc ordc* (*case-prod f*)
and *class.preorder ordb* (*mk-less ordb*)
and *lub-singleton lubb*
shows *cont luba orda lubc ordc* (λ*x*. *f x y*)
using *cont-prodD2*[*OF assms*] **by** *simp*

context *ccpo* **begin**

lemma *cont-prodI*:
assumes *mono*: *monotone* (*rel-prod orda ordb*) (≤) *f*
and *cont1*: ⋀*x*. *cont lubb ordb Sup* (≤) (λ*y*. *f* (*x*, *y*))
and *cont2*: ⋀*y*. *cont luba orda Sup* (≤) (λ*x*. *f* (*x*, *y*))
and *class.preorder orda* (*mk-less orda*)
and *class.preorder ordb* (*mk-less ordb*)
shows *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *Sup* (≤) *f*
proof(*rule contI*)
interpret *a*: *preorder orda mk-less orda* **by** *fact*
interpret *b*: *preorder ordb mk-less ordb* **by** *fact*

fix *Y*
assume *chain*: *Complete-Partial-Order.chain* (*rel-prod orda ordb*) *Y*
and *Y* ≠ {}
have *f* (*prod-lub luba lubb Y*) = *f* (*luba* (*fst* ‘ *Y*), *lubb* (*snd* ‘ *Y*))
by(*simp add*: *prod-lub-def*)
also from *cont2* **have** *f* (*luba* (*fst* ‘ *Y*), *lubb* (*snd* ‘ *Y*)) = ⨆((λ*x*. *f* (*x*, *lubb* (*snd* ‘ *Y*))) ‘ *fst* ‘ *Y*)
by(*rule contD*)(*simp-all add*: *chain-rel-prodD1*[*OF chain*] ‹*Y* ≠ {}›)
also from *cont1* **have** ⋀*x*. *f* (*x*, *lubb* (*snd* ‘ *Y*)) = ⨆((λ*y*. *f* (*x*, *y*)) ‘ *snd* ‘ *Y*)
by(*rule contD*)(*simp-all add*: *chain-rel-prodD2*[*OF chain*] ‹*Y* ≠ {}›)
hence ⨆((λ*x*. *f* (*x*, *lubb* (*snd* ‘ *Y*))) ‘ *fst* ‘ *Y*) = ⨆((λ*x*. ... *x*) ‘ *fst* ‘ *Y*) **by** *simp*

also have ... = $\bigsqcup((\lambda x.\ f\ (fst\ x,\ snd\ x))\ `\ Y)$
unfolding *image-image split-def* **using** *chain*
apply(*rule diag-Sup*)
using *monotoneD*[*OF mono*]
by(*auto intro*: *monotoneI*)
finally show $f\ (prod\text{-}lub\ luba\ lubb\ Y) = \bigsqcup(f\ `\ Y)$ **by** *simp*
qed

lemma *cont-case-prodI*:
assumes *monotone* (*rel-prod orda ordb*) (≤) (*case-prod f*)
and $\bigwedge x.$ *cont lubb ordb Sup* (≤) ($\lambda y.$ *f x y*)
and $\bigwedge y.$ *cont luba orda Sup* (≤) ($\lambda x.$ *f x y*)
and *class.preorder orda* (*mk-less orda*)
and *class.preorder ordb* (*mk-less ordb*)
shows *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *Sup* (≤) (*case-prod f*)
by(*rule cont-prodI*)(*simp-all add*: *assms*)

lemma *cont-case-prod-iff*:
⟦ *monotone* (*rel-prod orda ordb*) (≤) (*case-prod f*);
class.preorder orda (*mk-less orda*); *lub-singleton luba*;
class.preorder ordb (*mk-less ordb*); *lub-singleton lubb* ⟧
⟹ *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *Sup* (≤) (*case-prod f*) ⟷
($\forall x.$ *cont lubb ordb Sup* (≤) ($\lambda y.$ *f x y*)) ∧ ($\forall y.$ *cont luba orda Sup* (≤) ($\lambda x.$ *f x y*))
by(*blast dest*: *cont-case-prodD1 cont-case-prodD2 intro*: *cont-case-prodI*)

end

context *partial-function-definitions* **begin**

lemma *mono2mono2*:
assumes *f*: *monotone* (*rel-prod ordb ordc*) *leq* ($\lambda(x,\ y).$ *f x y*)
and *t*: *monotone orda ordb* ($\lambda x.$ *t x*)
and *t'*: *monotone orda ordc* ($\lambda x.$ *t' x*)
shows *monotone orda leq* ($\lambda x.$ *f* (*t x*) (*t' x*))
proof(*rule monotoneI*)
fix *x y*
assume *orda x y*
hence *rel-prod ordb ordc* (*t x*, *t' x*) (*t y*, *t' y*)
using *t t'* **by**(*auto dest*: *monotoneD*)
from *monotoneD*[*OF f this*] **show** *leq* (*f* (*t x*) (*t' x*)) (*f* (*t y*) (*t' y*)) **by** *simp*
qed

lemma *cont-case-prodI* [*cont-intro*]:
⟦ *monotone* (*rel-prod orda ordb*) *leq* (*case-prod f*);
$\bigwedge x.$ *cont lubb ordb lub leq* ($\lambda y.$ *f x y*);
$\bigwedge y.$ *cont luba orda lub leq* ($\lambda x.$ *f x y*);
class.preorder orda (*mk-less orda*);
class.preorder ordb (*mk-less ordb*) ⟧

$\Longrightarrow$ *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lub leq* (*case-prod f*)
by(*rule ccpo.cont-case-prodI*)(*rule Partial-Function.ccpo*[*OF partial-function-definitions-axioms*])

lemma *cont-case-prod-iff*:
⟦ *monotone* (*rel-prod orda ordb*) *leq* (*case-prod f*);
class.preorder orda (*mk-less orda*); *lub-singleton luba*;
class.preorder ordb (*mk-less ordb*); *lub-singleton lubb* ⟧
$\Longrightarrow$ *cont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lub leq* (*case-prod f*) $\longleftrightarrow$
($\forall$ *x*. *cont lubb ordb lub leq* (λ*y*. *f x y*)) $\wedge$ ($\forall$ *y*. *cont luba orda lub leq* (λ*x*. *f x y*))
by(*blast dest*: *cont-case-prodD1 cont-case-prodD2 intro*: *cont-case-prodI*)

lemma *mcont-case-prod-iff* [*simp*]:
⟦ *class.preorder orda* (*mk-less orda*); *lub-singleton luba*;
class.preorder ordb (*mk-less ordb*); *lub-singleton lubb* ⟧
$\Longrightarrow$ *mcont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lub leq* (*case-prod f*) $\longleftrightarrow$
($\forall$ *x*. *mcont lubb ordb lub leq* (λ*y*. *f x y*)) $\wedge$ ($\forall$ *y*. *mcont luba orda lub leq* (λ*x*. *f x y*))
unfolding *mcont-def* **by**(*auto simp add*: *cont-case-prod-iff*)

end

lemma *mono2mono-case-prod* [*cont-intro*]:
assumes $\bigwedge$*x y*. *monotone orda ordb* (λ*f*. *pair f x y*)
shows *monotone orda ordb* (λ*f*. *case-prod* (*pair f*) *x*)
by(*rule monotoneI*)(*auto split*: *prod.split dest*: *monotoneD*[*OF assms*])

15.5 Complete lattices as ccpo

context *complete-lattice* **begin**

lemma *complete-lattice-ccpo*: *class.ccpo Sup* ($\leq$) ($<$)
by(*unfold-locales*)(*fast intro*: *Sup-upper Sup-least*)+

lemma *complete-lattice-ccpo'*: *class.ccpo Sup* ($\leq$) (*mk-less* ($\leq$))
by(*unfold-locales*)(*auto simp add*: *mk-less-def intro*: *Sup-upper Sup-least*)

lemma *complete-lattice-partial-function-definitions*:
partial-function-definitions ($\leq$) *Sup*
by(*unfold-locales*)(*auto intro*: *Sup-least Sup-upper*)

lemma *complete-lattice-partial-function-definitions-dual*:
partial-function-definitions ($\geq$) *Inf*
by(*unfold-locales*)(*auto intro*: *Inf-lower Inf-greatest*)

lemmas [*cont-intro*, *simp*] =
Partial-Function.ccpo[*OF complete-lattice-partial-function-definitions*]
Partial-Function.ccpo[*OF complete-lattice-partial-function-definitions-dual*]

lemma *mono2mono-inf*:

assumes f: *monotone ord* $(\leq)$ $(\lambda x.\ f\ x)$
and g: *monotone ord* $(\leq)$ $(\lambda x.\ g\ x)$
shows *monotone ord* $(\leq)$ $(\lambda x.\ f\ x \sqcap g\ x)$
by(*auto 4 3 dest*: *monotoneD*[*OF f*] *monotoneD*[*OF g*] *intro*: *le-infI1 le-infI2 intro*!: *monotoneI*)

lemma *mcont-const* [*simp*]: *mcont lub ord Sup* $(\leq)$ $(\lambda\text{-}.\ c)$
by(*rule ccpo.mcont-const*[*OF complete-lattice-ccpo*])

lemma *mono2mono-sup*:
assumes f: *monotone ord* $(\leq)$ $(\lambda x.\ f\ x)$
and g: *monotone ord* $(\leq)$ $(\lambda x.\ g\ x)$
shows *monotone ord* $(\leq)$ $(\lambda x.\ f\ x \sqcup g\ x)$
by(*auto 4 3 intro*!: *monotoneI intro*: *sup.coboundedI1 sup.coboundedI2 dest*: *monotoneD*[*OF f*] *monotoneD*[*OF g*])

lemma *Sup-image-sup*:
assumes $Y \neq \{\}$
shows $\bigsqcup((\sqcup)\ x\ `\ Y) = x \sqcup \bigsqcup Y$
proof(*rule Sup-eqI*)
fix y
assume $y \in (\sqcup)\ x\ `\ Y$
then obtain z **where** $y = x \sqcup z$ **and** $z \in Y$ **by** *blast*
from ‹$z \in Y$› **have** $z \leq \bigsqcup Y$ **by**(*rule Sup-upper*)
with - **show** $y \leq x \sqcup \bigsqcup Y$ **unfolding** ‹$y = x \sqcup z$› **by**(*rule sup-mono*) *simp*
next
fix y
assume *upper*: $\bigwedge z.\ z \in (\sqcup)\ x\ `\ Y \Longrightarrow z \leq y$
show $x \sqcup \bigsqcup Y \leq y$ **unfolding** *Sup-insert*[*symmetric*]
proof(*rule Sup-least*)
fix z
assume $z \in$ *insert x Y*
from *assms* **obtain** z' **where** $z' \in Y$ **by** *blast*
let *?z* = *if* $z \in Y$ *then* $x \sqcup z$ *else* $x \sqcup z'$
have $z \leq x \sqcup$ *?z* **using** ‹$z' \in Y$› ‹$z \in$ *insert x Y*› **by** *auto*
also have $\ldots \leq y$ **by**(*rule upper*)(*auto split*: *if-split-asm intro*: ‹$z' \in Y$›)
finally show $z \leq y$ **.**
qed
qed

lemma *mcont-sup1*: *mcont Sup* $(\leq)$ *Sup* $(\leq)$ $(\lambda y.\ x \sqcup y)$
by(*auto 4 3 simp add*: *mcont-def sup.coboundedI1 sup.coboundedI2 intro*!: *monotoneI contI intro*: *Sup-image-sup*[*symmetric*])

lemma *mcont-sup2*: *mcont Sup* $(\leq)$ *Sup* $(\leq)$ $(\lambda x.\ x \sqcup y)$
by(*subst sup-commute*)(*rule mcont-sup1*)

lemma *mcont2mcont-sup* [*cont-intro*, *simp*]:
⟦ *mcont lub ord Sup* $(\leq)$ $(\lambda x.\ f\ x)$;

mcont lub ord Sup $(\leq)$ $(\lambda x.\ g\ x)$ ⟧
⟹ *mcont lub ord Sup* $(\leq)$ $(\lambda x.\ f\ x \sqcup g\ x)$
by(*best intro*: *ccpo.mcont2mcont'*[*OF complete-lattice-ccpo*] *mcont-sup1 mcont-sup2 ccpo.mcont-const*[*OF complete-lattice-ccpo*])

end

lemmas [*cont-intro*] = *admissible-leI*[*OF complete-lattice-ccpo'*]

context *complete-distrib-lattice* **begin**

lemma *mcont-inf1*: *mcont Sup* $(\leq)$ *Sup* $(\leq)$ $(\lambda y.\ x \sqcap y)$
by(*auto intro*: *monotoneI contI simp add*: *le-infI2 inf-Sup mcont-def*)

lemma *mcont-inf2*: *mcont Sup* $(\leq)$ *Sup* $(\leq)$ $(\lambda x.\ x \sqcap y)$
by(*auto intro*: *monotoneI contI simp add*: *le-infI1 Sup-inf mcont-def*)

lemma *mcont2mcont-inf* [*cont-intro*, *simp*]:
⟦ *mcont lub ord Sup* $(\leq)$ $(\lambda x.\ f\ x)$;
mcont lub ord Sup $(\leq)$ $(\lambda x.\ g\ x)$ ⟧
⟹ *mcont lub ord Sup* $(\leq)$ $(\lambda x.\ f\ x \sqcap g\ x)$
by(*best intro*: *ccpo.mcont2mcont'*[*OF complete-lattice-ccpo*] *mcont-inf1 mcont-inf2 ccpo.mcont-const*[*OF complete-lattice-ccpo*])

end

interpretation *lfp*: *partial-function-definitions* $(\leq)$ :: - :: *complete-lattice* ⇒ - *Sup*
by(*rule complete-lattice-partial-function-definitions*)

declaration ‹*Partial-Function.init lfp* ***term*** ‹*lfp.fixp-fun*› ***term*** ‹*lfp.mono-body*›
@{*thm lfp.fixp-rule-uc*} @{*thm lfp.fixp-induct-uc*} *NONE*›

interpretation *gfp*: *partial-function-definitions* $(\geq)$ :: - :: *complete-lattice* ⇒ - *Inf*
by(*rule complete-lattice-partial-function-definitions-dual*)

declaration ‹*Partial-Function.init gfp* ***term*** ‹*gfp.fixp-fun*› ***term*** ‹*gfp.mono-body*›
@{*thm gfp.fixp-rule-uc*} @{*thm gfp.fixp-induct-uc*} *NONE*›

lemma *insert-mono* [*partial-function-mono*]:
monotone (*fun-ord* $(\subseteq)$) $(\subseteq)$ *A* ⟹ *monotone* (*fun-ord* $(\subseteq)$) $(\subseteq)$ $(\lambda y.\ insert\ x\ (A\ y))$
by(*rule monotoneI*)(*auto simp add*: *fun-ord-def dest*: *monotoneD*)

lemma *mono2mono-insert* [*THEN lfp.mono2mono*, *cont-intro*, *simp*]:
shows *monotone-insert*: *monotone* $(\subseteq)$ $(\subseteq)$ (*insert x*)
by(*rule monotoneI*) *blast*

lemma *mcont2mcont-insert*[*THEN lfp.mcont2mcont*, *cont-intro*, *simp*]:
shows *mcont-insert*: *mcont Union* $(\subseteq)$ *Union* $(\subseteq)$ (*insert x*)

by(*blast intro*: *mcontI contI monotone-insert*)

lemma *mono2mono-image* [*THEN lfp.mono2mono*, *cont-intro*, *simp*]:
  shows *monotone-image*: *monotone* (⊆) (⊆) ((`) *f*)
by(*rule monotoneI*) *blast*

lemma *cont-image*: *cont Union* (⊆) *Union* (⊆) ((`) *f*)
by(*rule contI*)(*auto*)

lemma *mcont2mcont-image* [*THEN lfp.mcont2mcont*, *cont-intro*, *simp*]:
  shows *mcont-image*: *mcont Union* (⊆) *Union* (⊆) ((`) *f*)
by(*blast intro*: *mcontI monotone-image cont-image*)

context *complete-lattice* **begin**

lemma *monotone-Sup* [*cont-intro*, *simp*]:
  monotone ord (⊆) *f* ⟹ *monotone ord* (≤) (λ*x*. ⨆*f x*)
by(*blast intro*: *monotoneI Sup-least Sup-upper dest*: *monotoneD*)

lemma *cont-Sup*:
  assumes *cont lub ord Union* (⊆) *f*
  shows *cont lub ord Sup* (≤) (λ*x*. ⨆*f x*)
apply(*rule contI*)
apply(*simp add*: *contD*[*OF assms*])
apply(*blast intro*: *Sup-least Sup-upper order-trans order.antisym*)
done

lemma *mcont-Sup*: *mcont lub ord Union* (⊆) *f* ⟹ *mcont lub ord Sup* (≤) (λ*x*. ⨆*f x*)
unfolding *mcont-def* **by**(*blast intro*: *monotone-Sup cont-Sup*)

lemma *monotone-SUP*:
  ⟦ *monotone ord* (⊆) *f*; ⋀*y*. *monotone ord* (≤) (λ*x*. *g x y*) ⟧ ⟹ *monotone ord* (≤) (λ*x*. ⨆*y*∈*f x*. *g x y*)
by(*rule monotoneI*)(*blast dest*: *monotoneD intro*: *Sup-upper order-trans intro*!: *Sup-least*)

lemma *monotone-SUP2*:
  (⋀*y*. *y* ∈ *A* ⟹ *monotone ord* (≤) (λ*x*. *g x y*)) ⟹ *monotone ord* (≤) (λ*x*. ⨆*y*∈*A*. *g x y*)
by(*rule monotoneI*)(*blast intro*: *Sup-upper order-trans dest*: *monotoneD intro*!: *Sup-least*)

lemma *cont-SUP*:
  assumes *f*: *mcont lub ord Union* (⊆) *f*
  and *g*: ⋀*y*. *mcont lub ord Sup* (≤) (λ*x*. *g x y*)
  shows *cont lub ord Sup* (≤) (λ*x*. ⨆*y*∈*f x*. *g x y*)
proof(*rule contI*)
  fix *Y*
  assume *chain*: *Complete-Partial-Order.chain ord Y*
    and *Y*: *Y* ≠ {}

show $\bigsqcup(g\ (lub\ Y)\ `\ f\ (lub\ Y)) = \bigsqcup((\lambda x.\ \bigsqcup(g\ x\ `\ f\ x))\ `\ Y)$ (**is** *?lhs* = *?rhs*)
proof(*rule order.antisym*)
  show *?lhs* ≤ *?rhs*
  proof(*rule Sup-least*)
    fix x
    assume $x \in g\ (lub\ Y)\ `\ f\ (lub\ Y)$
    with *mcont-contD*[*OF f chain Y*] *mcont-contD*[*OF g chain Y*]
    obtain $y\ z$ **where** $y \in Y\ z \in f\ y$
      and x: $x = \bigsqcup((\lambda x.\ g\ x\ z)\ `\ Y)$ **by** *auto*
    show $x \leq$ *?rhs* **unfolding** x
    proof(*rule Sup-least*)
      fix u
      assume $u \in (\lambda x.\ g\ x\ z)\ `\ Y$
      then obtain y' **where** $u = g\ y'\ z\ y' \in Y$ **by** *auto*
      from *chain* ‹$y \in Y$› ‹$y' \in Y$› **have** *ord y y'* ∨ *ord y' y* **by**(*rule chainD*)
      thus $u \leq$ *?rhs*
      proof
        note ‹$u = g\ y'\ z$› **also**
        assume *ord y y'*
        with f **have** $f\ y \subseteq f\ y'$ **by**(*rule mcont-monoD*)
        with ‹$z \in f\ y$›
        have $g\ y'\ z \leq \bigsqcup(g\ y'\ `\ f\ y')$ **by**(*auto intro*: *Sup-upper*)
        also have ... ≤ *?rhs* **using** ‹$y' \in Y$› **by**(*auto intro*: *Sup-upper*)
        finally show *?thesis* **.**
      next
        note ‹$u = g\ y'\ z$› **also**
        assume *ord y' y*
        with g **have** $g\ y'\ z \leq g\ y\ z$ **by**(*rule mcont-monoD*)
        also have ... ≤ $\bigsqcup(g\ y\ `\ f\ y)$ **using** ‹$z \in f\ y$›
          by(*auto intro*: *Sup-upper*)
        also have ... ≤ *?rhs* **using** ‹$y \in Y$› **by**(*auto intro*: *Sup-upper*)
        finally show *?thesis* **.**
      qed
    qed
  qed
next
  show *?rhs* ≤ *?lhs*
  proof(*rule Sup-least*)
    fix x
    assume $x \in (\lambda x.\ \bigsqcup(g\ x\ `\ f\ x))\ `\ Y$
    then obtain y **where** x: $x = \bigsqcup(g\ y\ `\ f\ y)$ **and** $y \in Y$ **by** *auto*
    show $x \leq$ *?lhs* **unfolding** x
    proof(*rule Sup-least*)
      fix u
      assume $u \in g\ y\ `\ f\ y$
      then obtain z **where** $u = g\ y\ z\ z \in f\ y$ **by** *auto*
      note ‹$u = g\ y\ z$›
      also have $g\ y\ z \leq \bigsqcup((\lambda x.\ g\ x\ z)\ `\ Y)$
        using ‹$y \in Y$› **by**(*auto intro*: *Sup-upper*)

also have ... = *g* (*lub Y*) *z* **by**(*simp add*: *mcont-contD*[*OF g chain Y*])
also have ... ≤ *?lhs* **using** ‹*z* ∈ *f y*› ‹*y* ∈ *Y*›
by(*auto intro*: *Sup-upper simp add*: *mcont-contD*[*OF f chain Y*])
finally show *u* ≤ *?lhs* **.**
qed
qed
qed
qed

lemma *mcont-SUP* [*cont-intro*, *simp*]:
⟦ *mcont lub ord Union* (⊆) *f*; ⋀*y*. *mcont lub ord Sup* (≤) (λ*x*. *g x y*) ⟧
⟹ *mcont lub ord Sup* (≤) (λ*x*. ⨆ *y*∈*f x*. *g x y*)
by(*blast intro*: *mcontI cont-SUP monotone-SUP mcont-mono*)

end

lemma *admissible-Ball* [*cont-intro*, *simp*]:
⟦ ⋀*x*. *ccpo.admissible lub ord* (λ*A*. *P A x*);
mcont lub ord Union (⊆) *f*;
class.ccpo lub ord (*mk-less ord*) ⟧
⟹ *ccpo.admissible lub ord* (λ*A*. ∀ *x*∈*f A*. *P A x*)
unfolding *Ball-def* **by** *simp*

lemma *admissible-Bex'*[*THEN admissible-subst*, *cont-intro*, *simp*]:
shows *admissible-Bex*: *ccpo.admissible Union* (⊆) (λ*A*. ∃ *x*∈*A*. *P x*)
by(*rule ccpo.admissibleI*)(*auto*)

15.6 Parallel fixpoint induction

context
fixes *luba* :: *'a set* ⇒ *'a*
and *orda* :: *'a* ⇒ *'a* ⇒ *bool*
and *lubb* :: *'b set* ⇒ *'b*
and *ordb* :: *'b* ⇒ *'b* ⇒ *bool*
assumes *a*: *class.ccpo luba orda* (*mk-less orda*)
and *b*: *class.ccpo lubb ordb* (*mk-less ordb*)
begin

interpretation *a*: *ccpo luba orda mk-less orda* **by**(*rule a*)
interpretation *b*: *ccpo lubb ordb mk-less ordb* **by**(*rule b*)

lemma *ccpo-rel-prodI*:
class.ccpo (*prod-lub luba lubb*) (*rel-prod orda ordb*) (*mk-less* (*rel-prod orda ordb*))
(**is** *class.ccpo ?lub ?ord ?ord'*)
proof(*intro class.ccpo.intro class.ccpo-axioms.intro*)
show *class.order ?ord ?ord'* **by**(*rule order-rel-prodI*) *intro-locales*
qed(*auto 4 4 simp add*: *prod-lub-def intro*: *a.ccpo-Sup-upper b.ccpo-Sup-upper a.ccpo-Sup-least b.ccpo-Sup-least rev-image-eqI dest*: *chain-rel-prodD1 chain-rel-prodD2*)

interpretation *ab*: *ccpo prod-lub luba lubb rel-prod orda ordb mk-less* (*rel-prod orda ordb*)
by(*rule ccpo-rel-prodI*)

lemma *monotone-map-prod* [*simp*]:
  monotone (*rel-prod orda ordb*) (*rel-prod ordc ordd*) (*map-prod f g*) ⟷
   monotone orda ordc f ∧ *monotone ordb ordd g*
by(*auto simp add*: *monotone-def*)

lemma *parallel-fixp-induct*:
  assumes *adm*: *ccpo.admissible* (*prod-lub luba lubb*) (*rel-prod orda ordb*) (λ*x*. *P* (*fst x*) (*snd x*))
  and *f*: *monotone orda orda f*
  and *g*: *monotone ordb ordb g*
  and *bot*: *P* (*luba* {}) (*lubb* {})
  and *step*: ⋀*x y*. *P x y* ⟹ *P* (*f x*) (*g y*)
  shows *P* (*ccpo.fixp luba orda f*) (*ccpo.fixp lubb ordb g*)
proof −
  let *?lub* = *prod-lub luba lubb*
    and *?ord* = *rel-prod orda ordb*
    and *?P* = λ(*x*, *y*). *P x y*
  from *adm* **have** *adm'*: *ccpo.admissible ?lub ?ord ?P* **by**(*simp add*: *split-def*)
  hence *?P* (*ccpo.fixp* (*prod-lub luba lubb*) (*rel-prod orda ordb*) (*map-prod f g*))
    by(*rule ab.fixp-induct*)(*auto simp add*: *f g step bot*)
  also have *ccpo.fixp* (*prod-lub luba lubb*) (*rel-prod orda ordb*) (*map-prod f g*) =
          (*ccpo.fixp luba orda f*, *ccpo.fixp lubb ordb g*) (**is** *?lhs* = (*?rhs1*, *?rhs2*))
  proof(*rule ab.order.antisym*)
    have *ccpo.admissible ?lub ?ord* (λ*xy*. *?ord xy* (*?rhs1*, *?rhs2*))
    by(*rule admissible-leI*[*OF ccpo-rel-prodI*])(*auto simp add*: *prod-lub-def chain-empty intro*: *a.ccpo-Sup-least b.ccpo-Sup-least*)
    thus *?ord ?lhs* (*?rhs1*, *?rhs2*)
      by(*rule ab.fixp-induct*)(*auto 4 3 dest*: *monotoneD*[*OF f*] *monotoneD*[*OF g*] *simp add*: *b.fixp-unfold*[*OF g*, *symmetric*] *a.fixp-unfold*[*OF f*, *symmetric*] *f g intro*: *a.ccpo-Sup-least b.ccpo-Sup-least chain-empty*)
  next
    have *ccpo.admissible luba orda* (λ*x*. *orda x* (*fst ?lhs*))
    by(*rule admissible-leI*[*OF a*])(*auto intro*: *a.ccpo-Sup-least simp add*: *chain-empty*)
    hence *orda ?rhs1* (*fst ?lhs*) **using** *f*
    proof(*rule a.fixp-induct*)
      fix *x*
      assume *orda x* (*fst ?lhs*)
      thus *orda* (*f x*) (*fst ?lhs*)
        by(*subst ab.fixp-unfold*)(*auto simp add*: *f g dest*: *monotoneD*[*OF f*])
    qed(*auto intro*: *a.ccpo-Sup-least chain-empty*)
    moreover
    have *ccpo.admissible lubb ordb* (λ*y*. *ordb y* (*snd ?lhs*))
    by(*rule admissible-leI*[*OF b*])(*auto intro*: *b.ccpo-Sup-least simp add*: *chain-empty*)
    hence *ordb ?rhs2* (*snd ?lhs*) **using** *g*
    proof(*rule b.fixp-induct*)

fix *y*
assume *ordb y* (*snd ?lhs*)
thus *ordb* (*g y*) (*snd ?lhs*)
by(*subst ab.fixp-unfold*)(*auto simp add*: *f g dest*: *monotoneD*[*OF g*])
qed(*auto intro*: *b.ccpo-Sup-least chain-empty*)
ultimately show *?ord* (*?rhs1*, *?rhs2*) *?lhs*
by(*simp add*: *rel-prod-conv split-beta*)
qed
finally show *?thesis* **by** *simp*
qed

end

lemma *parallel-fixp-induct-uc*:
assumes *a*: *partial-function-definitions orda luba*
and *b*: *partial-function-definitions ordb lubb*
and *F*: $\bigwedge x.$ *monotone* (*fun-ord orda*) *orda* ($\lambda f.$ *U1* (*F* (*C1 f*)) *x*)
and *G*: $\bigwedge y.$ *monotone* (*fun-ord ordb*) *ordb* ($\lambda g.$ *U2* (*G* (*C2 g*)) *y*)
and *eq1*: $f \equiv$ *C1* (*ccpo.fixp* (*fun-lub luba*) (*fun-ord orda*) ($\lambda f.$ *U1* (*F* (*C1 f*))))
and *eq2*: $g \equiv$ *C2* (*ccpo.fixp* (*fun-lub lubb*) (*fun-ord ordb*) ($\lambda g.$ *U2* (*G* (*C2 g*))))
and *inverse*: $\bigwedge f.$ *U1* (*C1 f*) = *f*
and *inverse2*: $\bigwedge g.$ *U2* (*C2 g*) = *g*
and *adm*: *ccpo.admissible* (*prod-lub* (*fun-lub luba*) (*fun-lub lubb*)) (*rel-prod* (*fun-ord orda*) (*fun-ord ordb*)) ($\lambda x.$ *P* (*fst x*) (*snd x*))
and *bot*: *P* (λ-. *luba* {}) (λ-. *lubb* {})
and *step*: $\bigwedge f\ g.$ *P* (*U1 f*) (*U2 g*) $\Longrightarrow$ *P* (*U1* (*F f*)) (*U2* (*G g*))
shows *P* (*U1 f*) (*U2 g*)
apply(*unfold eq1 eq2 inverse inverse2*)
apply(*rule parallel-fixp-induct*[*OF partial-function-definitions.ccpo*[*OF a*] *partial-function-definitions.ccpo*[*OF b*] *adm*])
using *F* **apply**(*simp add*: *monotone-def fun-ord-def*)
using *G* **apply**(*simp add*: *monotone-def fun-ord-def*)
apply(*simp add*: *fun-lub-def bot*)
apply(*rule step*, *simp add*: *inverse inverse2*)
done

lemmas *parallel-fixp-induct-1-1* = *parallel-fixp-induct-uc*[
of - - - - $\lambda x.$ *x* - $\lambda x.$ *x* $\lambda x.$ *x* - $\lambda x.$ *x*,
OF - - - - - - *refl refl*]

lemmas *parallel-fixp-induct-2-2* = *parallel-fixp-induct-uc*[
of - - - - *case-prod* - *curry case-prod* - *curry*,
where *P*=$\lambda f\ g.$ *P* (*curry f*) (*curry g*),
unfolded case-prod-curry curry-case-prod curry-K,
OF - - - - - - *refl refl*]
for *P*

lemma *monotone-fst*: *monotone* (*rel-prod orda ordb*) *orda fst*
by(*auto intro*: *monotoneI*)

lemma *mcont-fst*: *mcont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *luba orda fst*
by(*auto intro*!: *mcontI monotoneI contI simp add*: *prod-lub-def*)

lemma *mcont2mcont-fst* [*cont-intro*, *simp*]:
mcont lub ord (*prod-lub luba lubb*) (*rel-prod orda ordb*) *t*
⟹ *mcont lub ord luba orda* (λ*x*. *fst* (*t x*))
by(*auto intro*!: *mcontI monotoneI contI dest*: *mcont-monoD mcont-contD simp add*: *rel-prod-sel split-beta prod-lub-def image-image*)

lemma *monotone-snd*: *monotone* (*rel-prod orda ordb*) *ordb snd*
by(*auto intro*: *monotoneI*)

lemma *mcont-snd*: *mcont* (*prod-lub luba lubb*) (*rel-prod orda ordb*) *lubb ordb snd*
by(*auto intro*!: *mcontI monotoneI contI simp add*: *prod-lub-def*)

lemma *mcont2mcont-snd* [*cont-intro*, *simp*]:
mcont lub ord (*prod-lub luba lubb*) (*rel-prod orda ordb*) *t*
⟹ *mcont lub ord lubb ordb* (λ*x*. *snd* (*t x*))
by(*auto intro*!: *mcontI monotoneI contI dest*: *mcont-monoD mcont-contD simp add*: *rel-prod-sel split-beta prod-lub-def image-image*)

lemma *monotone-Pair*:
⟦ *monotone ord orda f*; *monotone ord ordb g* ⟧
⟹ *monotone ord* (*rel-prod orda ordb*) (λ*x*. (*f x*, *g x*))
by(*simp add*: *monotone-def*)

lemma *cont-Pair*:
⟦ *cont lub ord luba orda f*; *cont lub ord lubb ordb g* ⟧
⟹ *cont lub ord* (*prod-lub luba lubb*) (*rel-prod orda ordb*) (λ*x*. (*f x*, *g x*))
by(*rule contI*)(*auto simp add*: *prod-lub-def image-image dest*!: *contD*)

lemma *mcont-Pair*:
⟦ *mcont lub ord luba orda f*; *mcont lub ord lubb ordb g* ⟧
⟹ *mcont lub ord* (*prod-lub luba lubb*) (*rel-prod orda ordb*) (λ*x*. (*f x*, *g x*))
by(*rule mcontI*)(*simp-all add*: *monotone-Pair mcont-mono cont-Pair*)

context *partial-function-definitions* **begin**

Specialised versions of *mcont-call* for admissibility proofs for parallel fixpoint inductions

lemmas *mcont-call-fst* [*cont-intro*] = *mcont-call*[*THEN mcont2mcont*, *OF mcont-fst*]
lemmas *mcont-call-snd* [*cont-intro*] = *mcont-call*[*THEN mcont2mcont*, *OF mcont-snd*]
end

lemma *map-option-mono* [*partial-function-mono*]:
mono-option B ⟹ *mono-option* (λ*f*. *map-option g* (*B f*))
unfolding *map-conv-bind-option* **by**(*rule bind-mono*) *simp-all*

lemma *compact-flat-lub* [*cont-intro*]: *ccpo.compact* (*flat-lub x*) (*flat-ord x*) *y*
using *flat-interpretation*[*THEN ccpo*]
proof(*rule ccpo.compactI*[*OF - ccpo.admissibleI*])
fix *A*
assume *chain*: *Complete-Partial-Order.chain* (*flat-ord x*) *A*
and *A*: $A \neq \{\}$
and $*$: $\forall z{\in}A.\ \neg$ *flat-ord x y z*
from *A* **obtain** *z* **where** $z \in A$ **by** *blast*
with $*$ **have** *z*: $\neg$ *flat-ord x y z* **..**
hence *y*: $x \neq y\ y \neq z$ **by**(*auto simp add*: *flat-ord-def*)
{ assume $\neg\ A \subseteq \{x\}$
then obtain z' **where** $z' \in A\ z' \neq x$ **by** *auto*
then have (*THE z*. $z \in A - \{x\}) = z'$
by(*intro the-equality*)(*auto dest*: *chainD*[*OF chain*] *simp add*: *flat-ord-def*)
moreover have $z' \neq y$ **using** ‹$z' \in A$› $*$ **by**(*auto simp add*: *flat-ord-def*)
ultimately have $y \neq$ (*THE z*. $z \in A - \{x\}$) **by** *simp* **}**
with *z* **show** $\neg$ *flat-ord x y* (*flat-lub x A*) **by**(*simp add*: *flat-ord-def flat-lub-def*)
qed

end

theory *Conditional-Parametricity*
imports *Main*
keywords *parametric-constant* :: *thy-decl*
begin

context includes *lifting-syntax* **begin**

qualified definition *Rel-match* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a* ⇒ '*b* ⇒ *bool* **where**
Rel-match R x y = *R x y*

named-theorems *parametricity-preprocess*

lemma *bi-unique-Rel-match* [*parametricity-preprocess*]:
bi-unique A = *Rel-match* (*A* ===> *A* ===> (=)) (=) (=)
unfolding *bi-unique-alt-def2 Rel-match-def* **..**

lemma *bi-total-Rel-match* [*parametricity-preprocess*]:
bi-total A = *Rel-match* ((*A* ===> (=)) ===> (=)) *All All*
unfolding *bi-total-alt-def2 Rel-match-def* **..**

lemma *is-equality-Rel*: *is-equality A* ⟹ *Transfer.Rel A t t*
by (*fact transfer-raw*)

lemma *Rel-Rel-match*: *Transfer.Rel R x y* ⟹ *Rel-match R x y*
unfolding *Rel-match-def Rel-def* **.**

lemma *Rel-match-Rel*: *Rel-match R x y* ⟹ *Transfer.Rel R x y*

unfolding *Rel-match-def Rel-def* **.**

lemma *Rel-Rel-match-eq*: *Transfer.Rel R x y = Rel-match R x y*
using *Rel-Rel-match Rel-match-Rel* **by** *fast*

lemma *Rel-match-app*:
assumes *Rel-match* $(A ===> B)$ *f g* **and** *Transfer.Rel A x y*
shows *Rel-match B* $(f\ x)$ $(g\ y)$
using *assms Rel-match-Rel Rel-app Rel-Rel-match* **by** *fast*

end

ML-file ‹*conditional-parametricity.ML*›

end
theory *Confluence* **imports**
Main
begin

16 Confluence

definition *semiconfluentp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ **where**
semiconfluentp r $\longleftrightarrow$ $r^{-1-1}\ OO\ r^{**} \leq r^{**}\ OO\ r^{-1-1**}$

definition *confluentp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ **where**
confluentp r $\longleftrightarrow$ $r^{-1-1**}\ OO\ r^{**} \leq r^{**}\ OO\ r^{-1-1**}$

definition *strong-confluentp* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ **where**
strong-confluentp r $\longleftrightarrow$ $r^{-1-1}\ OO\ r \leq r^{**}\ OO\ (r^{-1-1})^{==}$

lemma *semiconfluentpI* [*intro?*]:
semiconfluentp r **if** $\bigwedge x\ y\ z.\ [\![\ r\ x\ y;\ r^{**}\ x\ z\ ]\!] \Longrightarrow \exists u.\ r^{**}\ y\ u \wedge r^{**}\ z\ u$
using *that* **unfolding** *semiconfluentp-def rtranclp-conversep* **by** *blast*

lemma *semiconfluentpD*: $\exists u.\ r^{**}\ y\ u \wedge r^{**}\ z\ u$ **if** *semiconfluentp r r x y* $r^{**}\ x\ z$
using *that* **unfolding** *semiconfluentp-def rtranclp-conversep* **by** *blast*

lemma *confluentpI*:
confluentp r **if** $\bigwedge x\ y\ z.\ [\![\ r^{**}\ x\ y;\ r^{**}\ x\ z\ ]\!] \Longrightarrow \exists u.\ r^{**}\ y\ u \wedge r^{**}\ z\ u$
using *that* **unfolding** *confluentp-def rtranclp-conversep* **by** *blast*

lemma *confluentpD*: $\exists u.\ r^{**}\ y\ u \wedge r^{**}\ z\ u$ **if** *confluentp r* $r^{**}\ x\ y\ r^{**}\ x\ z$
using *that* **unfolding** *confluentp-def rtranclp-conversep* **by** *blast*

lemma *strong-confluentpI* [*intro?*]:
strong-confluentp r **if** $\bigwedge x\ y\ z.\ [\![\ r\ x\ y;\ r\ x\ z\ ]\!] \Longrightarrow \exists u.\ r^{**}\ y\ u \wedge r^{==}\ z\ u$
using *that* **unfolding** *strong-confluentp-def* **by** *blast*

lemma *strong-confluentpD*: $\exists u.\ r^{**}\ y\ u \wedge r^{==}\ z\ u$ **if** *strong-confluentp r r x y r x*

z
using *that* **unfolding** *strong-confluentp-def* **by** *blast*

lemma *semiconfluentp-imp-confluentp*: *confluentp r* **if** *r*: *semiconfluentp r*
proof(*rule confluentpI*)
show ∃ *u*. r^{**} *y u* ∧ r^{**} *z u* **if** r^{**} *x y* r^{**} *x z* **for** *x y z*
using *that*(*2*,*1*)
by(*induction arbitrary*: *y rule*: *converse-rtranclp-induct*)
(*blast intro*: *rtranclp-trans dest*: *r*[*THEN semiconfluentpD*])+
qed

lemma *confluentp-imp-semiconfluentp*: *semiconfluentp r* **if** *confluentp r*
using *that* **by**(*auto intro*!: *semiconfluentpI dest*: *confluentpD*[*OF that*])

lemma *confluentp-eq-semiconfluentp*: *confluentp r* ⟷ *semiconfluentp r*
by(*blast intro*: *semiconfluentp-imp-confluentp confluentp-imp-semiconfluentp*)

lemma *confluentp-conv-strong-confluentp-rtranclp*:
confluentp r ⟷ *strong-confluentp* (r^{**})
by(*auto simp add*: *confluentp-def strong-confluentp-def rtranclp-conversep*)

lemma *strong-confluentp-into-semiconfluentp*:
semiconfluentp r **if** *r*: *strong-confluentp r*
proof
show ∃ *u*. r^{**} *y u* ∧ r^{**} *z u* **if** *r x y* r^{**} *x z* **for** *x y z*
using *that*(*2*,*1*)
apply(*induction arbitrary*: *y rule*: *converse-rtranclp-induct*)
subgoal by *blast*
subgoal for *a b c*
by (*drule* (*1*) *strong-confluentpD*[*OF r*, *of a c*])(*auto 10 0 intro*: *rtranclp-trans*)
done
qed

lemma *strong-confluentp-imp-confluentp*: *confluentp r* **if** *strong-confluentp r*
unfolding *confluentp-eq-semiconfluentp* **using** *that* **by**(*rule strong-confluentp-into-semiconfluentp*)

lemma *semiconfluentp-equivclp*: *equivclp r* = r^{**} *OO* r^{-1-1**} **if** *r*: *semiconfluentp r*
proof(*rule antisym*[*rotated*] *r-OO-conversep-into-equivclp predicate2I*)+
show (r^{**} *OO* r^{-1-1**}) *x y* **if** *equivclp r x y* **for** *x y* **using** *that* **unfolding** *equivclp-def rtranclp-conversep*
by(*induction rule*: *converse-rtranclp-induct*)
(*blast elim*!: *symclpE intro*: *converse-rtranclp-into-rtranclp rtranclp-trans dest*: *semiconfluentpD*[*OF r*])+
qed

end
theory *Confluent-Quotient* **imports**
Confluence

begin

Functors with finite setters preserve wide intersection for any equivalence relation that respects the mapper.

lemma *Inter-finite-subset*:
assumes $\forall A \in \mathcal{A}$. *finite* A
shows $\exists \mathcal{B} \subseteq \mathcal{A}$. *finite* $\mathcal{B} \wedge (\bigcap \mathcal{B}) = (\bigcap \mathcal{A})$
proof(*cases* $\mathcal{A} = \{\}$)
case *False*
then obtain A **where** A: $A \in \mathcal{A}$ **by** *auto*
then have *finA*: *finite* A **using** *assms* **by** *auto*
hence *fin*: *finite* $(A - \bigcap \mathcal{A})$ **by**(*rule finite-subset*[*rotated*]) *auto*
let *?P* $= \lambda x\ A.\ A \in \mathcal{A} \wedge x \notin A$
define f **where** $f\ x = Eps\ (?P\ x)$ **for** x
let *?B* $= insert\ A\ (f\ `\ (A - \bigcap \mathcal{A}))$
have *?P* $x\ (f\ x)$ **if** $x \in A - \bigcap \mathcal{A}$ **for** x **unfolding** *f-def* **by**(*rule someI-ex*)(*use that A* **in** *auto*)
hence $(\bigcap ?\mathcal{B}) = (\bigcap \mathcal{A})$ *?B* $\subseteq \mathcal{A}$ **using** A **by** *auto*
moreover have *finite ?B* **using** *fin* **by** *simp*
ultimately show *?thesis* **by** *blast*
qed *simp*

locale *wide-intersection-finite* =
fixes $E :: 'Fa \Rightarrow 'Fa \Rightarrow bool$
and $mapFa :: ('a \Rightarrow 'a) \Rightarrow 'Fa \Rightarrow 'Fa$
and $setFa :: 'Fa \Rightarrow 'a\ set$
assumes *equiv*: *equivp E*
and *map-E*: $E\ x\ y \Longrightarrow E\ (mapFa\ f\ x)\ (mapFa\ f\ y)$
and *map-id*: $mapFa\ id\ x = x$
and *map-cong*: $\forall a \in setFa\ x.\ f\ a = g\ a \Longrightarrow mapFa\ f\ x = mapFa\ g\ x$
and *set-map*: $setFa\ (mapFa\ f\ x) = f\ `\ setFa\ x$
and *finite*: *finite* $(setFa\ x)$
begin

lemma *binary-intersection*:
assumes $E\ y\ z$ **and** y: $setFa\ y \subseteq Y$ **and** z: $setFa\ z \subseteq Z$ **and** a: $a \in Y\ a \in Z$
shows $\exists x.\ E\ x\ y \wedge setFa\ x \subseteq Y \wedge setFa\ x \subseteq Z$
proof −
let *?f* $= \lambda b.$ *if* $b \in Z$ *then* b *else* a
let *?u* $= mapFa$ *?f* y
from ‹$E\ y\ z$› **have** E *?u* $(mapFa$ *?f* $z)$ **by**(*rule map-E*)
also have $mapFa$ *?f* $z = mapFa\ id\ z$ **by**(*rule map-cong*)(*use z* **in** *auto*)
also have $\ldots = z$ **by**(*rule map-id*)
finally have E *?u* y **using** ‹$E\ y\ z$› *equivp-symp*[*OF equiv*] *equivp-transp*[*OF equiv*]
by *blast*
moreover have $setFa$ *?u* $\subseteq Y$ **using** $a\ y$ **by**(*subst set-map*) *auto*
moreover have $setFa$ *?u* $\subseteq Z$ **using** a **by**(*subst set-map*) *auto*
ultimately show *?thesis* **by** *blast*
qed

lemma *finite-intersection*:
assumes *E*: $\forall\, y{\in}A.\ E\ y\ z$
and *fin*: *finite A*
and *sub*: $\forall\, y{\in}A.\ setFa\ y \subseteq Y\ y \wedge a \in Y\ y$
shows $\exists\, x.\ E\ x\ z \wedge (\forall\, y{\in}A.\ setFa\ x \subseteq Y\ y)$
using *fin E sub*
proof(*induction*)
case *empty*
then show *?case* **using** *equivp-reflp*[*OF equiv*, *of z*] **by**(*auto*)
next
case (*insert y A*)
then obtain *x* **where** *x*: $E\ x\ z\ \forall\, y{\in}A.\ setFa\ x \subseteq Y\ y \wedge a \in Y\ y$ **by** *auto*
hence *set-x*: $setFa\ x \subseteq (\bigcap y{\in}A.\ Y\ y)\ a \in (\bigcap y{\in}A.\ Y\ y)$ **by** *auto*
from *insert.prems* **have** $E\ y\ z$ **and** *set-y*: $setFa\ y \subseteq Y\ y\ a \in Y\ y$ **by** *auto*
from ‹$E\ y\ z$› ‹$E\ x\ z$› **have** $E\ x\ y$ **using** *equivp-symp*[*OF equiv*] *equivp-transp*[*OF equiv*] **by** *blast*
from *binary-intersection*[*OF this set-x*(*1*) *set-y*(*1*) *set-x*(*2*) *set-y*(*2*)]
obtain x' **where** $E\ x'\ x\ setFa\ x' \subseteq \bigcap\ (Y\ `\ A)\ setFa\ x' \subseteq Y\ y$ **by** *blast*
then show *?case* **using** ‹$E\ x\ z$› *equivp-transp*[*OF equiv*] **by** *blast*
qed

lemma *wide-intersection*:
assumes *inter-nonempty*: $\bigcap\ Ss \neq \{\}$
shows $(\bigcap As \in Ss.\ \{(x, x').\ E\ x\ x'\}\ ``\ \{x.\ setFa\ x \subseteq As\}) \subseteq \{(x, x').\ E\ x\ x'\}\ ``\ \{x.\ setFa\ x \subseteq \bigcap\ Ss\}$ (**is** *?lhs* $\subseteq$ *?rhs*)
proof
fix *x*
assume *lhs*: $x \in$ *?lhs*
from *inter-nonempty* **obtain** *a* **where** *a*: $\forall\, As \in Ss.\ a \in As$ **by** *auto*
from *lhs* **obtain** *y* **where** *y*: $\bigwedge As.\ As \in Ss \Longrightarrow E\ (y\ As)\ x \wedge setFa\ (y\ As) \subseteq As$
by *atomize-elim*(*rule choice*, *auto*)
define *Ts* **where** $Ts = (\lambda As.\ insert\ a\ (setFa\ (y\ As)))\ `\ Ss$
have *Ts-subset*: $(\bigcap Ts) \subseteq (\bigcap Ss)$ **using** *a* **unfolding** *Ts-def* **by**(*auto dest*: *y*)
have *Ts-finite*: $\forall\, Bs \in Ts.\ finite\ Bs$ **unfolding** *Ts-def* **by**(*auto dest*: *y intro*: *finite*)
from *Inter-finite-subset*[*OF this*] **obtain** *Us*
where *Us*: $Us \subseteq Ts$ **and** *finite-Us*: *finite Us* **and** *Int-Us*: $(\bigcap Us) \subseteq (\bigcap Ts)$ **by** *force*
let $?P = \lambda U\ As.\ As \in Ss \wedge U = insert\ a\ (setFa\ (y\ As))$
define *Y* **where** $Y\ U = Eps\ (?P\ U)$ **for** *U*
have *Y*: $?P\ U\ (Y\ U)$ **if** $U \in Us$ **for** *U* **unfolding** *Y-def*
by(*rule someI-ex*)(*use that Us* **in** ‹*auto simp add*: *Ts-def*›)
let $?f = \lambda U.\ y\ (Y\ U)$
have $*$: $\forall\, z{\in}(?f\ `\ Us).\ E\ z\ x$ **by**(*auto dest*!: *Y y*)
have $**$: $\forall\, z{\in}(?f\ `\ Us).\ setFa\ z \subseteq insert\ a\ (setFa\ z) \wedge a \in insert\ a\ (setFa\ z)$ **by** *auto*
from *finite-intersection*[*OF* $*$ - $**$] *finite-Us* **obtain** *u*
where *u*: $E\ u\ x$ **and** *set-u*: $\forall\, z{\in}(?f\ `\ Us).\ setFa\ u \subseteq insert\ a\ (setFa\ z)$ **by** *auto*

from *set-u* **have** *setFa u* ⊆ (⋂ *Us*) **by**(*auto dest*: *Y*)
with *Int-Us Ts-subset* **have** *setFa u* ⊆ (⋂ *Ss*) **by** *auto*
with *u* **show** *x* ∈ *?rhs* **by** *auto*
qed

end

Subdistributivity for quotients via confluence

lemma *rtranclp-transp-reflp*: $R^{**} = R$ **if** *transp R reflp R*
apply(*rule ext iffI*)+
subgoal premises *prems* **for** *x y* **using** *prems* **by**(*induction*)(*use that* **in** ‹*auto intro*: *reflpD transpD*›)
subgoal by(*rule r-into-rtranclp*)
done

lemma *rtranclp-equivp*: $R^{**} = R$ **if** *equivp R*
using *that* **by**(*simp add*: *rtranclp-transp-reflp equivp-reflp-symp-transp*)

locale *confluent-quotient* =
fixes *Rb* :: *'Fb* ⇒ *'Fb* ⇒ *bool*
and *Ea* :: *'Fa* ⇒ *'Fa* ⇒ *bool*
and *Eb* :: *'Fb* ⇒ *'Fb* ⇒ *bool*
and *Ec* :: *'Fc* ⇒ *'Fc* ⇒ *bool*
and *Eab* :: *'Fab* ⇒ *'Fab* ⇒ *bool*
and *Ebc* :: *'Fbc* ⇒ *'Fbc* ⇒ *bool*
and *π-Faba* :: *'Fab* ⇒ *'Fa*
and *π-Fabb* :: *'Fab* ⇒ *'Fb*
and *π-Fbcb* :: *'Fbc* ⇒ *'Fb*
and *π-Fbcc* :: *'Fbc* ⇒ *'Fc*
and *rel-Fab* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ *'Fa* ⇒ *'Fb* ⇒ *bool*
and *rel-Fbc* :: (*'b* ⇒ *'c* ⇒ *bool*) ⇒ *'Fb* ⇒ *'Fc* ⇒ *bool*
and *rel-Fac* :: (*'a* ⇒ *'c* ⇒ *bool*) ⇒ *'Fa* ⇒ *'Fc* ⇒ *bool*
and *set-Fab* :: *'Fab* ⇒ (*'a* × *'b*) *set*
and *set-Fbc* :: *'Fbc* ⇒ (*'b* × *'c*) *set*
assumes *confluent*: *confluentp Rb*
and *retract1-ab*: ⋀*x y*. *Rb* (*π-Fabb x*) *y* ⟹ ∃ *z*. *Eab x z* ∧ *y* = *π-Fabb z* ∧ *set-Fab z* ⊆ *set-Fab x*
and *retract1-bc*: ⋀*x y*. *Rb* (*π-Fbcb x*) *y* ⟹ ∃ *z*. *Ebc x z* ∧ *y* = *π-Fbcb z* ∧ *set-Fbc z* ⊆ *set-Fbc x*
and *generated-b*: *Eb* ≤ *equivclp Rb*
and *transp-a*: *transp Ea*
and *transp-c*: *transp Ec*
and *equivp-ab*: *equivp Eab*
and *equivp-bc*: *equivp Ebc*
and *in-rel-Fab*: ⋀*A x y*. *rel-Fab A x y* ⟷ (∃ *z*. *z* ∈ {*x*. *set-Fab x* ⊆ {(*x*, *y*). *A x y*}} ∧ *π-Faba z* = *x* ∧ *π-Fabb z* = *y*)
and *in-rel-Fbc*: ⋀*B x y*. *rel-Fbc B x y* ⟷ (∃ *z*. *z* ∈ {*x*. *set-Fbc x* ⊆ {(*x*, *y*). *B x y*}} ∧ *π-Fbcb z* = *x* ∧ *π-Fbcc z* = *y*)
and *rel-compp*: ⋀*A B*. *rel-Fac* (*A OO B*) = *rel-Fab A OO rel-Fbc B*

and *π-Faba-respect*: *rel-fun Eab Ea π-Faba π-Faba*
and *π-Fbcc-respect*: *rel-fun Ebc Ec π-Fbcc π-Fbcc*
begin

lemma *retract-ab*: Rb^{**} (*π-Fabb x*) *y* $\Longrightarrow$ $\exists$ *z*. *Eab x z* $\wedge$ *y* = *π-Fabb z* $\wedge$ *set-Fab z* $\subseteq$ *set-Fab x*
by(*induction rule*: *rtranclp-induct*)(*blast dest*: *retract1-ab intro*: *equivp-transp*[*OF equivp-ab*] *equivp-reflp*[*OF equivp-ab*])+

lemma *retract-bc*: Rb^{**} (*π-Fbcb x*) *y* $\Longrightarrow$ $\exists$ *z*. *Ebc x z* $\wedge$ *y* = *π-Fbcb z* $\wedge$ *set-Fbc z* $\subseteq$ *set-Fbc x*
by(*induction rule*: *rtranclp-induct*)(*blast dest*: *retract1-bc intro*: *equivp-transp*[*OF equivp-bc*] *equivp-reflp*[*OF equivp-bc*])+

lemma *subdistributivity*: *rel-Fab A OO Eb OO rel-Fbc B* $\leq$ *Ea OO rel-Fac* (*A OO B*) *OO Ec*
proof(*rule predicate2I*; *elim relcomppE*)
fix *x y y′ z*
assume *rel-Fab A x y* **and** *Eb y y′* **and** *rel-Fbc B y′ z*
then obtain *xy y′z*
where *xy*: *set-Fab xy* $\subseteq$ {(*a*, *b*). *A a b*} *x* = *π-Faba xy y* = *π-Fabb xy*
and *y′z*: *set-Fbc y′z* $\subseteq$ {(*a*, *b*). *B a b*} *y′* = *π-Fbcb y′z z* = *π-Fbcc y′z*
by(*auto simp add*: *in-rel-Fab in-rel-Fbc*)
from ‹*Eb y y′*› **have** *equivclp Rb y y′* **using** *generated-b* **by** *blast*
then obtain *u* **where** *u*: Rb^{**} *y u* Rb^{**} *y′ u*
unfolding *semiconfluentp-equivclp*[*OF confluent*[*THEN confluentp-imp-semiconfluentp*]]
by(*auto simp add*: *rtranclp-conversep*)
with *xy y′z* **obtain** *xy′ y′z′*
where *retract1*: *Eab xy xy′ π-Fabb xy′* = *u set-Fab xy′* $\subseteq$ *set-Fab xy*
and *retract2*: *Ebc y′z y′z′ π-Fbcb y′z′* = *u set-Fbc y′z′* $\subseteq$ *set-Fbc y′z*
by(*auto dest*!: *retract-ab retract-bc*)
from *retract1*(*1*) *xy* **have** *Ea x* (*π-Faba xy′*) **by**(*auto dest*: *π-Faba-respect*[*THEN rel-funD*])
moreover have *rel-Fab A* (*π-Faba xy′*) *u* **using** *xy retract1* **by**(*auto simp add*: *in-rel-Fab*)
moreover have *rel-Fbc B u* (*π-Fbcc y′z′*) **using** *y′z retract2* **by**(*auto simp add*: *in-rel-Fbc*)
moreover have *Ec* (*π-Fbcc y′z′*) *z* **using** *retract2 y′z equivp-symp*[*OF equivp-bc*]
by(*auto intro*: *π-Fbcc-respect*[*THEN rel-funD*])
ultimately show (*Ea OO rel-Fac* (*A OO B*) *OO Ec*) *x z* **unfolding** *rel-compp*
by *blast*
qed

end

end

17 Old Datatype package: constructing datatypes from Cartesian Products and Disjoint Sums

theory *Old-Datatype*
imports *Main*
begin

17.1 The datatype universe

definition *Node* = $\{p.\ \exists f\ x\ k.\ p = (f :: nat => 'b + nat,\ x ::'a + nat) \wedge f\ k = Inr\ 0\}$

typedef $('a, 'b)$ *node* = *Node* :: $((nat => 'b + nat) * ('a + nat))$ *set*
morphisms *Rep-Node Abs-Node*
unfolding *Node-def* **by** *auto*

Datatypes will be represented by sets of type *node*

type-synonym $'a$ *item* = $('a,$ *unit*$)$ *node set*
type-synonym $('a, 'b)$ *dtree* = $('a, 'b)$ *node set*

definition *Push* :: $[('b + nat),\ nat => ('b + nat)] => (nat => ('b + nat))$

where *Push* == $(\%b\ h.$ *case-nat b h*$)$

definition *Push-Node* :: $[('b + nat),\ ('a, 'b)$ *node*$] => ('a, 'b)$ *node*
where *Push-Node* == $(\%n\ x.$ *Abs-Node* $($*apfst* $($*Push n*$)$ $($*Rep-Node x*$)))$

definition *Atom* :: $('a + nat) => ('a, 'b)$ *dtree*
where *Atom* == $(\%x.\ \{$*Abs-Node*$((\%k.$ *Inr 0*$,\ x))\})$
definition *Scons* :: $[('a, 'b)$ *dtree*$, ('a, 'b)$ *dtree*$] => ('a, 'b)$ *dtree*
where *Scons M N* == $($*Push-Node* $($*Inr 1*$)$ ‘ *M*$)$ *Un* $($*Push-Node* $($*Inr* $($*Suc 1*$))$ ‘ *N*$)$

definition *Leaf* :: $'a => ('a, 'b)$ *dtree*
where *Leaf* == *Atom* ∘ *Inl*
definition *Numb* :: *nat* => $('a, 'b)$ *dtree*
where *Numb* == *Atom* ∘ *Inr*

definition *In0* :: $('a, 'b)$ *dtree* => $('a, 'b)$ *dtree*
where *In0*(M) == *Scons* $($*Numb 0*$)$ *M*
definition *In1* :: $('a, 'b)$ *dtree* => $('a, 'b)$ *dtree*
where *In1*(M) == *Scons* $($*Numb 1*$)$ *M*

definition *Lim* :: ('*b* => ('*a*, '*b*) *dtree*) => ('*a*, '*b*) *dtree*
  where *Lim f* == $\bigcup${*z*. ∃ *x*. *z* = *Push-Node* (*Inl x*) ' (*f x*)}

definition *ndepth* :: ('*a*, '*b*) *node* => *nat*
  where *ndepth*(*n*) == (%(*f*,*x*). *LEAST k*. *f k* = *Inr 0*) (*Rep-Node n*)
definition *ntrunc* :: [*nat*, ('*a*, '*b*) *dtree*] => ('*a*, '*b*) *dtree*
  where *ntrunc k N* == {*n*. *n*∈*N* ∧ *ndepth*(*n*)<*k*}

definition *uprod* :: [('*a*, '*b*) *dtree set*, ('*a*, '*b*) *dtree set*]=> ('*a*, '*b*) *dtree set*
  where *uprod A B* == *UN x*:*A*. *UN y*:*B*. { *Scons x y* }
definition *usum* :: [('*a*, '*b*) *dtree set*, ('*a*, '*b*) *dtree set*]=> ('*a*, '*b*) *dtree set*
  where *usum A B* == *In0*'*A Un In1*'*B*

definition *Split* :: [[('*a*, '*b*) *dtree*, ('*a*, '*b*) *dtree*]=>'*c*, ('*a*, '*b*) *dtree*] => '*c*
  where *Split c M* == *THE u*. ∃ *x y*. *M* = *Scons x y* ∧ *u* = *c x y*

definition *Case* :: [[('*a*, '*b*) *dtree*]=>'*c*, [('*a*, '*b*) *dtree*]=>'*c*, ('*a*, '*b*) *dtree*] => '*c*
  where *Case c d M* == *THE u*. (∃ *x* . *M* = *In0*(*x*) ∧ *u* = *c*(*x*)) ∨ (∃ *y* . *M* = *In1*(*y*) ∧ *u* = *d*(*y*))

definition *dprod* :: [(('*a*, '*b*) *dtree* * ('*a*, '*b*) *dtree*)*set*, (('*a*, '*b*) *dtree* * ('*a*, '*b*) *dtree*)*set*]
    => (('*a*, '*b*) *dtree* * ('*a*, '*b*) *dtree*)*set*
  where *dprod r s* == *UN* (*x*,*x*'):*r*. *UN* (*y*,*y*'):*s*. {(*Scons x y*, *Scons x*' *y*')}

definition *dsum* :: [(('*a*, '*b*) *dtree* * ('*a*, '*b*) *dtree*)*set*, (('*a*, '*b*) *dtree* * ('*a*, '*b*) *dtree*)*set*]
    => (('*a*, '*b*) *dtree* * ('*a*, '*b*) *dtree*)*set*
 where *dsum r s* == (*UN* (*x*,*x*'):*r*. {(*In0*(*x*),*In0*(*x*'))}) *Un* (*UN* (*y*,*y*'):*s*. {(*In1*(*y*),*In1*(*y*'))})

lemma *apfst-convE*:
    [| *q* = *apfst f p*; !!*x y*. [| *p* = (*x*,*y*); *q* = (*f*(*x*),*y*) |] ==> *R*
    |] ==> *R*
by (*force simp add*: *apfst-def*)

lemma *Push-inject1*: *Push i f* = *Push j g* ==> *i*=*j*
apply (*simp add*: *Push-def fun-eq-iff*)
apply (*drule-tac x*=*0* **in** *spec*, *simp*)
done

lemma *Push-inject2*: *Push i f = Push j g ==> f=g*
apply (*auto simp add*: *Push-def fun-eq-iff*)
apply (*drule-tac x=Suc x* **in** *spec*, *simp*)
done

lemma *Push-inject*:
  [| *Push i f =Push j g*; [| *i=j*; *f=g* |] ==> *P* |] ==> *P*
by (*blast dest*: *Push-inject1 Push-inject2*)

lemma *Push-neq-K0*: *Push (Inr (Suc k)) f = (%z. Inr 0) ==> P*
by (*auto simp add*: *Push-def fun-eq-iff split*: *nat.split-asm*)

lemmas *Abs-Node-inj = Abs-Node-inject* [*THEN* [*2*] *rev-iffD1*]

lemma *Node-K0-I*: (*λk. Inr 0*, *a*) ∈ *Node*
by (*simp add*: *Node-def*)

lemma *Node-Push-I*: *p* ∈ *Node* ⟹ *apfst* (*Push i*) *p* ∈ *Node*
apply (*simp add*: *Node-def Push-def*)
apply (*fast intro*!: *apfst-conv nat.case*(*2*)[*THEN trans*])
done

17.2 Freeness: Distinctness of Constructors

lemma *Scons-not-Atom* [*iff*]: *Scons M N* ≠ *Atom*(*a*)
unfolding *Atom-def Scons-def Push-Node-def One-nat-def*
by (*blast intro*: *Node-K0-I Rep-Node* [*THEN Node-Push-I*]
    dest!: *Abs-Node-inj*
    elim!: *apfst-convE sym* [*THEN Push-neq-K0*])

lemmas *Atom-not-Scons* [*iff*] = *Scons-not-Atom* [*THEN not-sym*]

lemma *inj-Atom*: *inj*(*Atom*)
apply (*simp add*: *Atom-def*)
apply (*blast intro*!: *inj-onI Node-K0-I dest*!: *Abs-Node-inj*)
done
lemmas *Atom-inject = inj-Atom* [*THEN injD*]

lemma *Atom-Atom-eq* [*iff*]: (*Atom*(*a*)=*Atom*(*b*)) = (*a=b*)
by (*blast dest*!: *Atom-inject*)

lemma *inj-Leaf*: *inj*(*Leaf*)
apply (*simp add*: *Leaf-def o-def*)
apply (*rule inj-onI*)
apply (*erule Atom-inject* [*THEN Inl-inject*])
done

lemmas *Leaf-inject* [*dest*!] = *inj-Leaf* [*THEN injD*]

lemma *inj-Numb*: *inj*(*Numb*)
apply (*simp add*: *Numb-def o-def*)
apply (*rule inj-onI*)
apply (*erule Atom-inject* [*THEN Inr-inject*])
done

lemmas *Numb-inject* [*dest*!] = *inj-Numb* [*THEN injD*]

lemma *Push-Node-inject*:
[| *Push-Node i m* =*Push-Node j n*; [| *i*=*j*; *m*=*n* |] ==> *P*
|] ==> *P*
apply (*simp add*: *Push-Node-def*)
apply (*erule Abs-Node-inj* [*THEN apfst-convE*])
apply (*rule Rep-Node* [*THEN Node-Push-I*])+
apply (*erule sym* [*THEN apfst-convE*])
apply (*blast intro*: *Rep-Node-inject* [*THEN iffD1*] *trans sym elim*!: *Push-inject*)
done

lemma *Scons-inject-lemma1*: *Scons M N* <= *Scons M' N'* ==> *M*<=*M'*
unfolding *Scons-def One-nat-def*
by (*blast dest*!: *Push-Node-inject*)

lemma *Scons-inject-lemma2*: *Scons M N* <= *Scons M' N'* ==> *N*<=*N'*
unfolding *Scons-def One-nat-def*
by (*blast dest*!: *Push-Node-inject*)

lemma *Scons-inject1*: *Scons M N* = *Scons M' N'* ==> *M*=*M'*
apply (*erule equalityE*)
apply (*iprover intro*: *equalityI Scons-inject-lemma1*)
done

lemma *Scons-inject2*: *Scons M N* = *Scons M' N'* ==> *N*=*N'*
apply (*erule equalityE*)
apply (*iprover intro*: *equalityI Scons-inject-lemma2*)

done

lemma *Scons-inject*:
[| *Scons M N = Scons M′ N′*; [| *M=M′*; *N=N′* |] ==> *P* |] ==> *P*
by (*iprover dest*: *Scons-inject1 Scons-inject2*)

lemma *Scons-Scons-eq* [*iff*]: (*Scons M N = Scons M′ N′*) = (*M=M′* ∧ *N=N′*)
by (*blast elim*!: *Scons-inject*)

lemma *Scons-not-Leaf* [*iff*]: *Scons M N* ≠ *Leaf*(*a*)
unfolding *Leaf-def o-def* **by** (*rule Scons-not-Atom*)

lemmas *Leaf-not-Scons* [*iff*] = *Scons-not-Leaf* [*THEN not-sym*]

lemma *Scons-not-Numb* [*iff*]: *Scons M N* ≠ *Numb*(*k*)
unfolding *Numb-def o-def* **by** (*rule Scons-not-Atom*)

lemmas *Numb-not-Scons* [*iff*] = *Scons-not-Numb* [*THEN not-sym*]

lemma *Leaf-not-Numb* [*iff*]: *Leaf*(*a*) ≠ *Numb*(*k*)
by (*simp add*: *Leaf-def Numb-def*)

lemmas *Numb-not-Leaf* [*iff*] = *Leaf-not-Numb* [*THEN not-sym*]

lemma *ndepth-K0*: *ndepth* (*Abs-Node*(%*k*. *Inr 0*, *x*)) = *0*
by (*simp add*: *ndepth-def Node-K0-I* [*THEN Abs-Node-inverse*] *Least-equality*)

lemma *ndepth-Push-Node-aux*:
case-nat (*Inr* (*Suc i*)) *f k* = *Inr 0* ⟶ *Suc*(*LEAST x*. *f x* = *Inr 0*) ≤ *k*
apply (*induct-tac k*, *auto*)
apply (*erule Least-le*)
done

lemma *ndepth-Push-Node*:
ndepth (*Push-Node* (*Inr* (*Suc i*)) *n*) = *Suc*(*ndepth*(*n*))
apply (*insert Rep-Node* [*of n*, *unfolded Node-def*])
apply (*auto simp add*: *ndepth-def Push-Node-def*

Rep-Node [*THEN Node-Push-I*, *THEN Abs-Node-inverse*])
apply (*rule Least-equality*)
apply (*auto simp add*: *Push-def ndepth-Push-Node-aux*)
apply (*erule LeastI*)
done

lemma *ntrunc-0* [*simp*]: *ntrunc 0 M* = {}
by (*simp add*: *ntrunc-def*)

lemma *ntrunc-Atom* [*simp*]: *ntrunc* (*Suc k*) (*Atom a*) = *Atom*(*a*)
by (*auto simp add*: *Atom-def ntrunc-def ndepth-K0*)

lemma *ntrunc-Leaf* [*simp*]: *ntrunc* (*Suc k*) (*Leaf a*) = *Leaf*(*a*)
unfolding *Leaf-def o-def* **by** (*rule ntrunc-Atom*)

lemma *ntrunc-Numb* [*simp*]: *ntrunc* (*Suc k*) (*Numb i*) = *Numb*(*i*)
unfolding *Numb-def o-def* **by** (*rule ntrunc-Atom*)

lemma *ntrunc-Scons* [*simp*]:
ntrunc (*Suc k*) (*Scons M N*) = *Scons* (*ntrunc k M*) (*ntrunc k N*)
unfolding *Scons-def ntrunc-def One-nat-def*
by (*auto simp add*: *ndepth-Push-Node*)

lemma *ntrunc-one-In0* [*simp*]: *ntrunc* (*Suc 0*) (*In0 M*) = {}
apply (*simp add*: *In0-def*)
apply (*simp add*: *Scons-def*)
done

lemma *ntrunc-In0* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In0 M*) = *In0* (*ntrunc* (*Suc k*) *M*)
by (*simp add*: *In0-def*)

lemma *ntrunc-one-In1* [*simp*]: *ntrunc* (*Suc 0*) (*In1 M*) = {}
apply (*simp add*: *In1-def*)
apply (*simp add*: *Scons-def*)
done

lemma *ntrunc-In1* [*simp*]: *ntrunc* (*Suc*(*Suc k*)) (*In1 M*) = *In1* (*ntrunc* (*Suc k*) *M*)
by (*simp add*: *In1-def*)

17.3 Set Constructions

lemma *uprodI* [*intro*!]: $[\![M{\in}A;\ N{\in}B]\!] \Longrightarrow Scons\ M\ N \in uprod\ A\ B$
by (*simp add*: *uprod-def*)

lemma *uprodE* [*elim*!]:
$[\![c \in uprod\ A\ B;$
$\bigwedge x\ y.\ [\![x \in A;\ y \in B;\ c = Scons\ x\ y]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*auto simp add*: *uprod-def*)

lemma *uprodE2*: $[\![Scons\ M\ N \in uprod\ A\ B;\ [\![M \in A;\ N \in B]\!] \Longrightarrow P]\!] \Longrightarrow P$
by (*auto simp add*: *uprod-def*)

lemma *usum-In0I* [*intro*]: $M \in A \Longrightarrow In0(M) \in usum\ A\ B$
by (*simp add*: *usum-def*)

lemma *usum-In1I* [*intro*]: $N \in B \Longrightarrow In1(N) \in usum\ A\ B$
by (*simp add*: *usum-def*)

lemma *usumE* [*elim*!]:
$[\![u \in usum\ A\ B;$
$\bigwedge x.\ [\![x \in A;\ u{=}In0(x)]\!] \Longrightarrow P;$
$\bigwedge y.\ [\![y \in B;\ u{=}In1(y)]\!] \Longrightarrow P$
$]\!] \Longrightarrow P$
by (*auto simp add*: *usum-def*)

lemma *In0-not-In1* [*iff*]: $In0(M) \neq In1(N)$
unfolding *In0-def In1-def One-nat-def* **by** *auto*

lemmas *In1-not-In0* [*iff*] = *In0-not-In1* [*THEN not-sym*]

lemma *In0-inject*: $In0(M) = In0(N) \Longrightarrow M{=}N$
by (*simp add*: *In0-def*)

lemma *In1-inject*: $In1(M) = In1(N) \Longrightarrow M{=}N$
by (*simp add*: *In1-def*)

lemma *In0-eq* [*iff*]: $(In0\ M = In0\ N) = (M{=}N)$
by (*blast dest*!: *In0-inject*)

lemma *In1-eq* [*iff*]: (*In1 M = In1 N*) = (*M=N*)
by (*blast dest*!: *In1-inject*)

lemma *inj-In0*: *inj In0*
by (*blast intro*!: *inj-onI*)

lemma *inj-In1*: *inj In1*
by (*blast intro*!: *inj-onI*)

lemma *Lim-inject*: *Lim f = Lim g ==> f = g*
apply (*simp add*: *Lim-def*)
apply (*rule ext*)
apply (*blast elim*!: *Push-Node-inject*)
done

lemma *ntrunc-subsetI*: *ntrunc k M <= M*
by (*auto simp add*: *ntrunc-def*)

lemma *ntrunc-subsetD*: (!!*k*. *ntrunc k M <= N*) *==> M<=N*
by (*auto simp add*: *ntrunc-def*)

lemma *ntrunc-equality*: (!!*k*. *ntrunc k M = ntrunc k N*) *==> M=N*
apply (*rule equalityI*)
apply (*rule-tac* [!] *ntrunc-subsetD*)
apply (*rule-tac* [!] *ntrunc-subsetI* [*THEN* [*2*] *subset-trans*], *auto*)
done

lemma *ntrunc-o-equality*:
[| !!*k*. (*ntrunc*(*k*) ∘ *h1*) = (*ntrunc*(*k*) ∘ *h2*) |] *==> h1=h2*
apply (*rule ntrunc-equality* [*THEN ext*])
apply (*simp add*: *fun-eq-iff*)
done

lemma *uprod-mono*: [| *A<=A'*; *B<=B'* |] *==> uprod A B <= uprod A' B'*
by (*simp add*: *uprod-def*, *blast*)

lemma *usum-mono*: [| *A<=A'*; *B<=B'* |] *==> usum A B <= usum A' B'*
by (*simp add*: *usum-def*, *blast*)

lemma *Scons-mono*: [| *M*<=*M'*; *N*<=*N'* |] ==> *Scons M N* <= *Scons M' N'*
by (*simp add*: *Scons-def*, *blast*)

lemma *In0-mono*: *M*<=*N* ==> *In0*(*M*) <= *In0*(*N*)
by (*simp add*: *In0-def Scons-mono*)

lemma *In1-mono*: *M*<=*N* ==> *In1*(*M*) <= *In1*(*N*)
by (*simp add*: *In1-def Scons-mono*)

lemma *Split* [*simp*]: *Split c* (*Scons M N*) = *c M N*
by (*simp add*: *Split-def*)

lemma *Case-In0* [*simp*]: *Case c d* (*In0 M*) = *c*(*M*)
by (*simp add*: *Case-def*)

lemma *Case-In1* [*simp*]: *Case c d* (*In1 N*) = *d*(*N*)
by (*simp add*: *Case-def*)

lemma *ntrunc-UN1*: *ntrunc k* (*UN x. f*(*x*)) = (*UN x. ntrunc k* (*f x*))
by (*simp add*: *ntrunc-def*, *blast*)

lemma *Scons-UN1-x*: *Scons* (*UN x. f x*) *M* = (*UN x. Scons* (*f x*) *M*)
by (*simp add*: *Scons-def*, *blast*)

lemma *Scons-UN1-y*: *Scons M* (*UN x. f x*) = (*UN x. Scons M* (*f x*))
by (*simp add*: *Scons-def*, *blast*)

lemma *In0-UN1*: *In0*(*UN x. f*(*x*)) = (*UN x. In0*(*f*(*x*)))
by (*simp add*: *In0-def Scons-UN1-y*)

lemma *In1-UN1*: *In1*(*UN x. f*(*x*)) = (*UN x. In1*(*f*(*x*)))
by (*simp add*: *In1-def Scons-UN1-y*)

lemma *dprodI* [*intro*!]:
⟦(*M*,*M'*) ∈ *r*; (*N*,*N'*) ∈ *s*⟧ ⟹ (*Scons M N*, *Scons M' N'*) ∈ *dprod r s*
by (*auto simp add*: *dprod-def*)

lemma *dprodE* [*elim*!]:

⟦$c \in dprod\ r\ s$;
⋀$x\ y\ x'\ y'$. ⟦$(x,x') \in r$; $(y,y') \in s$;
$c = (Scons\ x\ y,\ Scons\ x'\ y')$⟧ ⟹ P
⟧ ⟹ P
by (*auto simp add*: *dprod-def*)

lemma *dsum-In0I* [*intro*]: $(M,M') \in r \Longrightarrow (In0(M),\ In0(M')) \in dsum\ r\ s$
by (*auto simp add*: *dsum-def*)

lemma *dsum-In1I* [*intro*]: $(N,N') \in s \Longrightarrow (In1(N),\ In1(N')) \in dsum\ r\ s$
by (*auto simp add*: *dsum-def*)

lemma *dsumE* [*elim*!]:
⟦$w \in dsum\ r\ s$;
⋀$x\ x'$. ⟦ $(x,x') \in r$; $w = (In0(x),\ In0(x'))$ ⟧ ⟹ P;
⋀$y\ y'$. ⟦ $(y,y') \in s$; $w = (In1(y),\ In1(y'))$ ⟧ ⟹ P
⟧ ⟹ P
by (*auto simp add*: *dsum-def*)

lemma *dprod-mono*: [| $r<=r'$; $s<=s'$ |] ==> $dprod\ r\ s <= dprod\ r'\ s'$
by *blast*

lemma *dsum-mono*: [| $r<=r'$; $s<=s'$ |] ==> $dsum\ r\ s <= dsum\ r'\ s'$
by *blast*

lemma *dprod-Sigma*: $(dprod\ (A \times B)\ (C \times D)) <= (uprod\ A\ C) \times (uprod\ B\ D)$
by *blast*

lemmas *dprod-subset-Sigma* = *subset-trans* [*OF dprod-mono dprod-Sigma*]

lemma *dprod-subset-Sigma2*:
$(dprod\ (Sigma\ A\ B)\ (Sigma\ C\ D)) <= Sigma\ (uprod\ A\ C)\ (Split\ (\%x\ y.\ uprod\ (B\ x)\ (D\ y)))$
by *auto*

lemma *dsum-Sigma*: $(dsum\ (A \times B)\ (C \times D)) <= (usum\ A\ C) \times (usum\ B\ D)$
by *blast*

lemmas *dsum-subset-Sigma* = *subset-trans* [*OF dsum-mono dsum-Sigma*]

lemma *Domain-dprod* [*simp*]: *Domain* (*dprod r s*) = *uprod* (*Domain r*) (*Domain s*)
by *auto*

lemma *Domain-dsum* [*simp*]: *Domain* (*dsum r s*) = *usum* (*Domain r*) (*Domain s*)
by *auto*

hides popular names

hide-type (**open**) *node item*
hide-const (**open**) *Push Node Atom Leaf Numb Lim Split Case*

ML-file ‹~~/*src*/*HOL*/*Tools*/*Old-Datatype*/*old-datatype.ML*›

end

18 Bijections between natural numbers and other types

theory *Nat-Bijection*
imports *Main*
begin

18.1 Type *nat* × *nat*

Triangle numbers: 0, 1, 3, 6, 10, 15, ...

definition *triangle* :: *nat* ⇒ *nat*
where *triangle n* = (*n* ∗ *Suc n*) *div 2*

lemma *triangle-0* [*simp*]: *triangle 0* = *0*
by (*simp add*: *triangle-def*)

lemma *triangle-Suc* [*simp*]: *triangle* (*Suc n*) = *triangle n* + *Suc n*
by (*simp add*: *triangle-def*)

definition *prod-encode* :: *nat* × *nat* ⇒ *nat*
where *prod-encode* = (λ(*m*, *n*). *triangle* (*m* + *n*) + *m*)

In this auxiliary function, *triangle k* + *m* is an invariant.

fun *prod-decode-aux* :: *nat* ⇒ *nat* ⇒ *nat* × *nat*
where *prod-decode-aux k m* =
(*if m* ≤ *k then* (*m*, *k* − *m*) *else prod-decode-aux* (*Suc k*) (*m* − *Suc k*))

declare *prod-decode-aux.simps* [*simp del*]

definition *prod-decode* :: *nat* ⇒ *nat* × *nat*
where *prod-decode* = *prod-decode-aux 0*

lemma *prod-encode-prod-decode-aux*: *prod-encode* (*prod-decode-aux k m*) = *triangle k* + *m*
proof (*induction k m rule*: *prod-decode-aux.induct*)
case (*1 k m*)
then show *?case*
by (*simp add*: *prod-encode-def prod-decode-aux.simps*)
qed

lemma *prod-decode-inverse* [*simp*]: *prod-encode* (*prod-decode n*) = *n*
by (*simp add*: *prod-decode-def prod-encode-prod-decode-aux*)

lemma *prod-decode-triangle-add*: *prod-decode* (*triangle k* + *m*) = *prod-decode-aux k m*
proof (*induct k arbitrary*: *m*)
case *0*
then show *?case*
by (*simp add*: *prod-decode-def*)
next
case (*Suc k*)
then show *?case*
by (*metis ab-semigroup-add-class.add-ac*(*1*) *add-diff-cancel-left' le-add1 not-less-eq-eq prod-decode-aux.simps triangle-Suc*)
qed

lemma *prod-encode-inverse* [*simp*]: *prod-decode* (*prod-encode x*) = *x*
unfolding *prod-encode-def*
proof (*induct x*)
case (*Pair a b*)
then show *?case*
by (*simp add*: *prod-decode-triangle-add prod-decode-aux.simps*)
qed

lemma *inj-prod-encode*: *inj-on prod-encode A*
by (*rule inj-on-inverseI*) (*rule prod-encode-inverse*)

lemma *inj-prod-decode*: *inj-on prod-decode A*
by (*rule inj-on-inverseI*) (*rule prod-decode-inverse*)

lemma *surj-prod-encode*: *surj prod-encode*
by (*rule surjI*) (*rule prod-decode-inverse*)

lemma *surj-prod-decode*: *surj prod-decode*
by (*rule surjI*) (*rule prod-encode-inverse*)

lemma *bij-prod-encode*: *bij prod-encode*
by (*rule bijI* [*OF inj-prod-encode surj-prod-encode*])

lemma *bij-prod-decode*: *bij prod-decode*
by (*rule bijI* [*OF inj-prod-decode surj-prod-decode*])

lemma *prod-encode-eq*: *prod-encode x = prod-encode y ⟷ x = y*
by (*rule inj-prod-encode* [*THEN inj-eq*])

lemma *prod-decode-eq*: *prod-decode x = prod-decode y ⟷ x = y*
by (*rule inj-prod-decode* [*THEN inj-eq*])

Ordering properties

lemma *le-prod-encode-1*: *a ≤ prod-encode* (*a*, *b*)
by (*simp add*: *prod-encode-def*)

lemma *le-prod-encode-2*: *b ≤ prod-encode* (*a*, *b*)
by (*induct b*) (*simp-all add*: *prod-encode-def*)

18.2 Type *nat + nat*

definition *sum-encode* :: *nat + nat ⇒ nat*
where *sum-encode x* = (*case x of Inl a ⇒ 2 * a | Inr b ⇒ Suc* (*2 * b*))

definition *sum-decode* :: *nat ⇒ nat + nat*
where *sum-decode n* = (*if even n then Inl* (*n div 2*) *else Inr* (*n div 2*))

lemma *sum-encode-inverse* [*simp*]: *sum-decode* (*sum-encode x*) = *x*
by (*induct x*) (*simp-all add*: *sum-decode-def sum-encode-def*)

lemma *sum-decode-inverse* [*simp*]: *sum-encode* (*sum-decode n*) = *n*
by (*simp add*: *even-two-times-div-two sum-decode-def sum-encode-def*)

lemma *inj-sum-encode*: *inj-on sum-encode A*
by (*rule inj-on-inverseI*) (*rule sum-encode-inverse*)

lemma *inj-sum-decode*: *inj-on sum-decode A*
by (*rule inj-on-inverseI*) (*rule sum-decode-inverse*)

lemma *surj-sum-encode*: *surj sum-encode*
by (*rule surjI*) (*rule sum-decode-inverse*)

lemma *surj-sum-decode*: *surj sum-decode*
by (*rule surjI*) (*rule sum-encode-inverse*)

lemma *bij-sum-encode*: *bij sum-encode*
by (*rule bijI* [*OF inj-sum-encode surj-sum-encode*])

lemma *bij-sum-decode*: *bij sum-decode*
by (*rule bijI* [*OF inj-sum-decode surj-sum-decode*])

lemma *sum-encode-eq*: *sum-encode x = sum-encode y ⟷ x = y*
by (*rule inj-sum-encode* [*THEN inj-eq*])

lemma *sum-decode-eq*: *sum-decode x = sum-decode y ⟷ x = y*
by (*rule inj-sum-decode* [*THEN inj-eq*])

18.3 Type *int*

definition *int-encode* :: *int ⇒ nat*
where *int-encode i = sum-encode* (*if 0 ≤ i then Inl* (*nat i*) *else Inr* (*nat* (*− i − 1*)))

definition *int-decode* :: *nat ⇒ int*
where *int-decode n* = (*case sum-decode n of Inl a ⇒ int a* | *Inr b ⇒ − int b − 1*)

lemma *int-encode-inverse* [*simp*]: *int-decode* (*int-encode x*) = *x*
by (*simp add*: *int-decode-def int-encode-def*)

lemma *int-decode-inverse* [*simp*]: *int-encode* (*int-decode n*) = *n*
unfolding *int-decode-def int-encode-def*
using *sum-decode-inverse* [*of n*] **by** (*cases sum-decode n*) *simp-all*

lemma *inj-int-encode*: *inj-on int-encode A*
by (*rule inj-on-inverseI*) (*rule int-encode-inverse*)

lemma *inj-int-decode*: *inj-on int-decode A*
by (*rule inj-on-inverseI*) (*rule int-decode-inverse*)

lemma *surj-int-encode*: *surj int-encode*
by (*rule surjI*) (*rule int-decode-inverse*)

lemma *surj-int-decode*: *surj int-decode*
by (*rule surjI*) (*rule int-encode-inverse*)

lemma *bij-int-encode*: *bij int-encode*
by (*rule bijI* [*OF inj-int-encode surj-int-encode*])

lemma *bij-int-decode*: *bij int-decode*
by (*rule bijI* [*OF inj-int-decode surj-int-decode*])

lemma *int-encode-eq*: *int-encode x = int-encode y ⟷ x = y*
by (*rule inj-int-encode* [*THEN inj-eq*])

lemma *int-decode-eq*: *int-decode x = int-decode y ⟷ x = y*
by (*rule inj-int-decode* [*THEN inj-eq*])

18.4 Type *nat list*

fun *list-encode* :: *nat list* ⇒ *nat*
  where
    list-encode [] = *0*
  | *list-encode* (*x* # *xs*) = *Suc* (*prod-encode* (*x*, *list-encode xs*))

function *list-decode* :: *nat* ⇒ *nat list*
  where
    list-decode 0 = []
  | *list-decode* (*Suc n*) = (*case prod-decode n of* (*x*, *y*) ⇒ *x* # *list-decode y*)
  by *pat-completeness auto*

termination *list-decode*
proof −
  have ⋀*n x y*. (*x*, *y*) = *prod-decode n* ⟹ *y* < *Suc n*
    by (*metis le-imp-less-Suc le-prod-encode-2 prod-decode-inverse*)
  then show *?thesis*
    using *termination* **by** *blast*
qed

lemma *list-encode-inverse* [*simp*]: *list-decode* (*list-encode x*) = *x*
  by (*induct x rule*: *list-encode.induct*) *simp-all*

lemma *list-decode-inverse* [*simp*]: *list-encode* (*list-decode n*) = *n*
proof (*induct n rule*: *list-decode.induct*)
  case (*2 n*)
  then show *?case*
    by (*metis list-encode.simps*(*2*) *list-encode-inverse prod-decode-inverse surj-pair*)
qed *auto*

lemma *inj-list-encode*: *inj-on list-encode A*
  by (*rule inj-on-inverseI*) (*rule list-encode-inverse*)

lemma *inj-list-decode*: *inj-on list-decode A*
  by (*rule inj-on-inverseI*) (*rule list-decode-inverse*)

lemma *surj-list-encode*: *surj list-encode*
  by (*rule surjI*) (*rule list-decode-inverse*)

lemma *surj-list-decode*: *surj list-decode*
  by (*rule surjI*) (*rule list-encode-inverse*)

lemma *bij-list-encode*: *bij list-encode*
  by (*rule bijI* [*OF inj-list-encode surj-list-encode*])

lemma *bij-list-decode*: *bij list-decode*
  by (*rule bijI* [*OF inj-list-decode surj-list-decode*])

lemma *list-encode-eq*: *list-encode x* = *list-encode y* ⟷ *x* = *y*

by (*rule inj-list-encode* [*THEN inj-eq*])

lemma *list-decode-eq*: *list-decode x* = *list-decode y* ⟷ *x* = *y*
by (*rule inj-list-decode* [*THEN inj-eq*])

18.5 Finite sets of naturals

18.5.1 Preliminaries

lemma *finite-vimage-Suc-iff*: *finite* (*Suc* −‘ *F*) ⟷ *finite F*
proof
have *F* ⊆ *insert 0* (*Suc* ‘ *Suc* −‘ *F*)
using *nat.nchotomy* **by** *force*
moreover
assume *finite* (*Suc* −‘ *F*)
then have *finite* (*insert 0* (*Suc* ‘ *Suc* −‘ *F*))
by *blast*
ultimately show *finite F*
using *finite-subset* **by** *blast*
qed (*force intro*: *finite-vimageI inj-Suc*)

lemma *vimage-Suc-insert-0*: *Suc* −‘ *insert 0 A* = *Suc* −‘ *A*
by *auto*

lemma *vimage-Suc-insert-Suc*: *Suc* −‘ *insert* (*Suc n*) *A* = *insert n* (*Suc* −‘ *A*)
by *auto*

lemma *div2-even-ext-nat*:
fixes *x y* :: *nat*
assumes *x div 2* = *y div 2*
and *even x* ⟷ *even y*
shows *x* = *y*
proof −
from ‹*even x* ⟷ *even y*› **have** *x mod 2* = *y mod 2*
by (*simp only*: *even-iff-mod-2-eq-zero*) *auto*
with *assms* **have** *x div 2* ∗ *2* + *x mod 2* = *y div 2* ∗ *2* + *y mod 2*
by *simp*
then show *?thesis*
by *simp*
qed

18.5.2 From sets to naturals

definition *set-encode* :: *nat set* ⇒ *nat*
where *set-encode* = *sum* ((^) *2*)

lemma *set-encode-empty* [*simp*]: *set-encode* {} = *0*
by (*simp add*: *set-encode-def*)

lemma *set-encode-inf*: ¬ *finite A* ⟹ *set-encode A* = *0*

by (*simp add*: *set-encode-def*)

lemma *set-encode-insert* [*simp*]: *finite A* $\Longrightarrow$ *n* $\notin$ *A* $\Longrightarrow$ *set-encode* (*insert n A*) = *2^n* + *set-encode A*
by (*simp add*: *set-encode-def*)

lemma *even-set-encode-iff*: *finite A* $\Longrightarrow$ *even* (*set-encode A*) $\longleftrightarrow$ *0* $\notin$ *A*
by (*induct set*: *finite*) (*auto simp*: *set-encode-def*)

lemma *set-encode-vimage-Suc*: *set-encode* (*Suc* `-` *A*) = *set-encode A div 2*
proof (*induction A rule*: *infinite-finite-induct*)
case (*infinite A*)
then show *?case*
by (*simp add*: *finite-vimage-Suc-iff set-encode-inf*)
next
case (*insert x A*)
show *?case*
proof (*cases x*)
case *0*
with *insert* **show** *?thesis*
by (*simp add*: *even-set-encode-iff vimage-Suc-insert-0*)
next
case (*Suc y*)
with *insert* **show** *?thesis*
by (*simp add*: *finite-vimageI add.commute vimage-Suc-insert-Suc*)
qed
qed *auto*

lemmas *set-encode-div-2* = *set-encode-vimage-Suc* [*symmetric*]

18.5.3 From naturals to sets

definition *set-decode* :: *nat* $\Rightarrow$ *nat set*
where *set-decode x* = {*n*. *odd* (*x div 2* ^ *n*)}

lemma *set-decode-0* [*simp*]: *0* $\in$ *set-decode x* $\longleftrightarrow$ *odd x*
by (*simp add*: *set-decode-def*)

lemma *set-decode-Suc* [*simp*]: *Suc n* $\in$ *set-decode x* $\longleftrightarrow$ *n* $\in$ *set-decode* (*x div 2*)
by (*simp add*: *set-decode-def div-mult2-eq*)

lemma *set-decode-zero* [*simp*]: *set-decode 0* = {}
by (*simp add*: *set-decode-def*)

lemma *set-decode-div-2*: *set-decode* (*x div 2*) = *Suc* `-` *set-decode x*
by *auto*

lemma *set-decode-plus-power-2*:
n $\notin$ *set-decode z* $\Longrightarrow$ *set-decode* (*2* ^ *n* + *z*) = *insert n* (*set-decode z*)

proof (*induct n arbitrary*: *z*)
case *0*
show *?case*
proof (*rule set-eqI*)
show *q* ∈ *set-decode* (*2* ^ *0* + *z*) ⟷ *q* ∈ *insert 0* (*set-decode z*) **for** *q*
by (*induct q*) (*use 0* **in** *simp-all*)
qed
next
case (*Suc n*)
show *?case*
proof (*rule set-eqI*)
show *q* ∈ *set-decode* (*2* ^ *Suc n* + *z*) ⟷ *q* ∈ *insert* (*Suc n*) (*set-decode z*) **for** *q*
by (*induct q*) (*use Suc* **in** *simp-all*)
qed
qed

lemma *finite-set-decode* [*simp*]: *finite* (*set-decode n*)
proof (*induction n rule*: *less-induct*)
case (*less n*)
show *?case*
proof (*cases n* = *0*)
case *False*
then show *?thesis*
using *less.IH* [*of n div 2*] *finite-vimage-Suc-iff set-decode-div-2* **by** *auto*
qed *auto*
qed

18.5.4 Proof of isomorphism

lemma *set-decode-inverse* [*simp*]: *set-encode* (*set-decode n*) = *n*
proof (*induction n rule*: *less-induct*)
case (*less n*)
show *?case*
proof (*cases n* = *0*)
case *False*
then have *set-encode* (*set-decode* (*n div 2*)) = *n div 2*
using *less.IH* **by** *auto*
then show *?thesis*
by (*metis div2-even-ext-nat even-set-encode-iff finite-set-decode set-decode-0 set-decode-div-2 set-encode-div-2*)
qed *auto*
qed

lemma *set-encode-inverse* [*simp*]: *finite A* ⟹ *set-decode* (*set-encode A*) = *A*
proof (*induction rule*: *finite-induct*)
case (*insert x A*)
then show *?case*
by (*simp add*: *set-decode-plus-power-2*)

qed *auto*

lemma *inj-on-set-encode*: *inj-on set-encode* (*Collect finite*)
by (*rule inj-on-inverseI* [**where** *g* = *set-decode*]) *simp*

lemma *set-encode-eq*: *finite A* ⟹ *finite B* ⟹ *set-encode A* = *set-encode B* ⟷ *A* = *B*
by (*rule iffI*) (*simp-all add*: *inj-onD* [*OF inj-on-set-encode*])

lemma *subset-decode-imp-le*:
assumes *set-decode m* ⊆ *set-decode n*
shows *m* ≤ *n*
proof −
have *n* = *m* + *set-encode* (*set-decode n* − *set-decode m*)
proof −
obtain *A B* **where**
m = *set-encode A finite A*
n = *set-encode B finite B*
by (*metis finite-set-decode set-decode-inverse*)
with *assms* **show** *?thesis*
by *auto* (*simp add*: *set-encode-def add.commute sum.subset-diff*)
qed
then show *?thesis*
by (*metis le-add1*)
qed

end

19 Encoding (almost) everything into natural numbers

theory *Countable*
imports *Old-Datatype HOL.Rat Nat-Bijection*
begin

19.1 The class of countable types

class *countable* =
assumes *ex-inj*: ∃ *to-nat* :: *'a* ⇒ *nat*. *inj to-nat*

lemma *countable-classI*:
fixes *f* :: *'a* ⇒ *nat*
assumes ⋀*x y*. *f x* = *f y* ⟹ *x* = *y*
shows *OFCLASS*(*'a*, *countable-class*)
proof (*intro-classes*, *rule exI*)
show *inj f*
by (*rule injI* [*OF assms*]) *assumption*
qed

19.2 Conversion functions

definition *to-nat* :: *'a*::*countable* ⇒ *nat* **where**
to-nat = (*SOME f*. *inj f*)

definition *from-nat* :: *nat* ⇒ *'a*::*countable* **where**
from-nat = *inv* (*to-nat* :: *'a* ⇒ *nat*)

lemma *inj-to-nat* [*simp*]: *inj to-nat*
by (*rule exE-some* [*OF ex-inj*]) (*simp add*: *to-nat-def*)

lemma *inj-on-to-nat*[*simp*, *intro*]: *inj-on to-nat S*
using *inj-to-nat* **by** (*auto simp*: *inj-on-def*)

lemma *surj-from-nat* [*simp*]: *surj from-nat*
unfolding *from-nat-def* **by** (*simp add*: *inj-imp-surj-inv*)

lemma *to-nat-split* [*simp*]: *to-nat x* = *to-nat y* ⟷ *x* = *y*
using *injD* [*OF inj-to-nat*] **by** *auto*

lemma *from-nat-to-nat* [*simp*]:
from-nat (*to-nat x*) = *x*
by (*simp add*: *from-nat-def*)

19.3 Finite types are countable

subclass (**in** *finite*) *countable*
proof
have *finite* (*UNIV*::*'a set*) **by** (*rule finite-UNIV*)
with *finite-conv-nat-seg-image* [*of UNIV*::*'a set*]
obtain *n* **and** *f* :: *nat* ⇒ *'a*
where *UNIV* = *f* ' {*i*. *i* < *n*} **by** *auto*
then have *surj f* **unfolding** *surj-def* **by** *auto*
then have *inj* (*inv f*) **by** (*rule surj-imp-inj-inv*)
then show ∃ *to-nat* :: *'a* ⇒ *nat*. *inj to-nat* **by** (*rule exI*[*of inj*])
qed

19.4 Automatically proving countability of old-style datatypes

context
begin

qualified inductive *finite-item* :: *'a Old-Datatype.item* ⇒ *bool* **where**
undefined: *finite-item undefined*
| *In0*: *finite-item x* ⟹ *finite-item* (*Old-Datatype.In0 x*)
| *In1*: *finite-item x* ⟹ *finite-item* (*Old-Datatype.In1 x*)
| *Leaf*: *finite-item* (*Old-Datatype.Leaf a*)
| *Scons*: ⟦*finite-item x*; *finite-item y*⟧ ⟹ *finite-item* (*Old-Datatype.Scons x y*)

qualified function *nth-item* :: *nat* ⇒ (*'a*::*countable*) *Old-Datatype.item*

where
nth-item 0 = undefined
| *nth-item (Suc n) =*
(case sum-decode n of
Inl i ⇒
(case sum-decode i of
Inl j ⇒ Old-Datatype.In0 (nth-item j)
| *Inr j ⇒ Old-Datatype.In1 (nth-item j))*
| *Inr i ⇒*
(case sum-decode i of
Inl j ⇒ Old-Datatype.Leaf (from-nat j)
| *Inr j ⇒*
(case prod-decode j of
(a, b) ⇒ Old-Datatype.Scons (nth-item a) (nth-item b))))
by *pat-completeness auto*

lemma *le-sum-encode-Inl*: $x \leq y \Longrightarrow x \leq$ *sum-encode (Inl y)*
unfolding *sum-encode-def* **by** *simp*

lemma *le-sum-encode-Inr*: $x \leq y \Longrightarrow x \leq$ *sum-encode (Inr y)*
unfolding *sum-encode-def* **by** *simp*

qualified termination
by (*relation measure id*)
(*auto simp flip*: *sum-encode-eq prod-encode-eq*
simp: *le-imp-less-Suc le-sum-encode-Inl le-sum-encode-Inr*
le-prod-encode-1 le-prod-encode-2)

lemma *nth-item-covers*: *finite-item x* $\Longrightarrow \exists n.$ *nth-item n = x*
proof (*induct set*: *finite-item*)
case *undefined*
have *nth-item 0 = undefined* **by** *simp*
thus *?case* **..**
next
case (*In0 x*)
then obtain *n* **where** *nth-item n = x* **by** *fast*
hence *nth-item (Suc (sum-encode (Inl (sum-encode (Inl n))))) = Old-Datatype.In0 x* **by** *simp*
thus *?case* **..**
next
case (*In1 x*)
then obtain *n* **where** *nth-item n = x* **by** *fast*
hence *nth-item (Suc (sum-encode (Inl (sum-encode (Inr n))))) = Old-Datatype.In1 x* **by** *simp*
thus *?case* **..**
next
case (*Leaf a*)
have *nth-item (Suc (sum-encode (Inr (sum-encode (Inl (to-nat a)))))) = Old-Datatype.Leaf a*

```
    by simp
  thus ?case ..
next
  case (Scons x y)
  then obtain i j where nth-item i = x and nth-item j = y by fast
  hence nth-item
   (Suc (sum-encode (Inr (sum-encode (Inr (prod-encode (i, j)))))) = Old-Datatype.Scons
x y
    by simp
  thus ?case ..
qed

theorem countable-datatype:
  fixes Rep :: 'b ⇒ ('a::countable) Old-Datatype.item
  fixes Abs :: ('a::countable) Old-Datatype.item ⇒ 'b
  fixes rep-set :: ('a::countable) Old-Datatype.item ⇒ bool
  assumes type: type-definition Rep Abs (Collect rep-set)
  assumes finite-item: ⋀x. rep-set x ⟹ finite-item x
  shows OFCLASS('b, countable-class)
proof
  define f where f y = (LEAST n. nth-item n = Rep y) for y
  {
    fix y :: 'b
    have rep-set (Rep y)
      using type-definition.Rep [OF type] by simp
    hence finite-item (Rep y)
      by (rule finite-item)
    hence ∃n. nth-item n = Rep y
      by (rule nth-item-covers)
    hence nth-item (f y) = Rep y
      unfolding f-def by (rule LeastI-ex)
    hence Abs (nth-item (f y)) = y
      using type-definition.Rep-inverse [OF type] by simp
  }
  hence inj f
    by (rule inj-on-inverseI)
  thus ∃f::'b ⇒ nat. inj f
    by − (rule exI)
qed

ML ‹
  fun old-countable-datatype-tac ctxt =
    SUBGOAL (fn (goal, -) =>
      let
        val ty-name =
          (case goal of
             (- $ Const (const-name ‹Pure.type›, Type (type-name ‹itself›, [Type
(n, -)]))) => n
           | - => raise Match)
```

```
    val typedef-info = hd (Typedef.get-info ctxt ty-name)
    val typedef-thm = #type-definition (snd typedef-info)
    val pred-name =
      (case HOLogic.dest-Trueprop (Thm.concl-of typedef-thm) of
        (- $ - $ - $ (- $ Const (n, -))) => n
      | - => raise Match)
    val induct-info = Inductive.the-inductive-global ctxt pred-name
    val pred-names = #names (fst induct-info)
    val induct-thms = #inducts (snd induct-info)
    val alist = pred-names ~~ induct-thms
    val induct-thm = the (AList.lookup (op =) alist pred-name)
    val vars = rev (Term.add-vars (Thm.prop-of induct-thm) [])
    val insts = vars |> map (fn (-, T) => try (Thm.cterm-of ctxt)
      (Const (const-name‹Countable.finite-item›, T)))
    val induct-thm' = Thm.instantiate' [] insts induct-thm
    val rules = @{thms finite-item.intros}
  in
    SOLVED' (fn i => EVERY
      [resolve-tac ctxt @{thms countable-datatype} i,
       resolve-tac ctxt [typedef-thm] i,
       eresolve-tac ctxt [induct-thm'] i,
       REPEAT (resolve-tac ctxt rules i ORELSE assume-tac ctxt i)]) 1
  end)
›

end
```

19.5 Automatically proving countability of datatypes

ML-file ‹../Tools/BNF/bnf-lfp-countable.ML›

```
ML ‹
fun countable-datatype-tac ctxt st =
  (case try ‹HEADGOAL (old-countable-datatype-tac ctxt) st› of
    SOME res => res
  | NONE => BNF-LFP-Countable.countable-datatype-tac ctxt st);

(* compatibility *)
fun countable-tac ctxt =
  SELECT-GOAL (countable-datatype-tac ctxt);
›

method-setup countable-datatype = ‹
  Scan.succeed (SIMPLE-METHOD o countable-datatype-tac)
› prove countable class instances for datatypes
```

19.6 More Countable types

Naturals

instance *nat* :: *countable*
by (*rule countable-classI* [*of id*]) *simp*

Pairs

instance *prod* :: (*countable*, *countable*) *countable*
by (*rule countable-classI* [*of* λ(*x*, *y*). *prod-encode* (*to-nat x*, *to-nat y*)])
(*auto simp add*: *prod-encode-eq*)

Sums

instance *sum* :: (*countable*, *countable*) *countable*
by (*rule countable-classI* [*of* (λ*x*. *case x of Inl a* ⇒ *to-nat* (*False*, *to-nat a*)
| *Inr b* ⇒ *to-nat* (*True*, *to-nat b*))])
(*simp split*: *sum.split-asm*)

Integers

instance *int* :: *countable*
by (*rule countable-classI* [*of int-encode*]) (*simp add*: *int-encode-eq*)

Options

instance *option* :: (*countable*) *countable*
by *countable-datatype*

Lists

instance *list* :: (*countable*) *countable*
by *countable-datatype*

String literals

instance *String.literal* :: *countable*
by (*rule countable-classI* [*of to-nat* ∘ *String.explode*]) (*simp add*: *String.explode-inject*)

Functions

instance *fun* :: (*finite*, *countable*) *countable*
proof
obtain *xs* :: *'a list* **where** *xs*: *set xs* = *UNIV*
using *finite-list* [*OF finite-UNIV*] **..**
show ∃ *to-nat*::(*'a* ⇒ *'b*) ⇒ *nat*. *inj to-nat*
proof
show *inj* (λ*f*. *to-nat* (*map f xs*))
by (*rule injI*, *simp add*: *xs fun-eq-iff*)
qed
qed

Typereps

instance *typerep* :: *countable*
by *countable-datatype*

19.7 The rationals are countably infinite

definition *nat-to-rat-surj* :: *nat* ⇒ *rat* **where**
nat-to-rat-surj n = (*let* (*a*, *b*) = *prod-decode n in Fract* (*int-decode a*) (*int-decode b*))

lemma *surj-nat-to-rat-surj*: *surj nat-to-rat-surj*
unfolding *surj-def*
proof
fix *r*::*rat*
show ∃ *n*. *r* = *nat-to-rat-surj n*
proof (*cases r*)
fix *i j* **assume** [*simp*]: *r* = *Fract i j* **and** *j* > *0*
have *r* = (*let m* = *int-encode i*; *n* = *int-encode j in nat-to-rat-surj* (*prod-encode* (*m*, *n*)))
by (*simp add*: *Let-def nat-to-rat-surj-def*)
thus ∃ *n*. *r* = *nat-to-rat-surj n* **by**(*auto simp*: *Let-def*)
qed
qed

lemma *Rats-eq-range-nat-to-rat-surj*: ℚ = *range nat-to-rat-surj*
by (*simp add*: *Rats-def surj-nat-to-rat-surj*)

context *field-char-0*
begin

lemma *Rats-eq-range-of-rat-o-nat-to-rat-surj*:
ℚ = *range* (*of-rat* ∘ *nat-to-rat-surj*)
using *surj-nat-to-rat-surj*
by (*auto simp*: *Rats-def image-def surj-def*) (*blast intro*: *arg-cong*[**where** *f* = *of-rat*])

lemma *surj-of-rat-nat-to-rat-surj*:
r ∈ ℚ ⟹ ∃ *n*. *r* = *of-rat* (*nat-to-rat-surj n*)
by (*simp add*: *Rats-eq-range-of-rat-o-nat-to-rat-surj image-def*)

end

instance *rat* :: *countable*
proof
show ∃ *to-nat*::*rat* ⇒ *nat*. *inj to-nat*
proof
have *surj nat-to-rat-surj*
by (*rule surj-nat-to-rat-surj*)
then show *inj* (*inv nat-to-rat-surj*)
by (*rule surj-imp-inj-inv*)
qed
qed

theorem *rat-denum*: ∃ *f* :: *nat* ⇒ *rat*. *surj f*

using *surj-nat-to-rat-surj* **by** *metis*

end

20 Infinite Sets and Related Concepts

theory *Infinite-Set*
imports *Main*
begin

20.1 The set of natural numbers is infinite

lemma *infinite-nat-iff-unbounded-le*: *infinite* $S \longleftrightarrow (\forall\, m.\ \exists\, n{\geq}m.\ n \in S)$
for S :: *nat set*
using *frequently-cofinite*[*of* $\lambda x.\ x \in S$]
by (*simp add*: *cofinite-eq-sequentially frequently-def eventually-sequentially*)

lemma *infinite-nat-iff-unbounded*: *infinite* $S \longleftrightarrow (\forall\, m.\ \exists\, n{>}m.\ n \in S)$
for S :: *nat set*
using *frequently-cofinite*[*of* $\lambda x.\ x \in S$]
by (*simp add*: *cofinite-eq-sequentially frequently-def eventually-at-top-dense*)

lemma *finite-nat-iff-bounded*: *finite* $S \longleftrightarrow (\exists\, k.\ S \subseteq \{..{<}k\})$
for S :: *nat set*
using *infinite-nat-iff-unbounded-le*[*of* S] **by** (*simp add*: *subset-eq*) (*metis not-le*)

lemma *finite-nat-iff-bounded-le*: *finite* $S \longleftrightarrow (\exists\, k.\ S \subseteq \{..\ k\})$
for S :: *nat set*
using *infinite-nat-iff-unbounded*[*of* S] **by** (*simp add*: *subset-eq*) (*metis not-le*)

lemma *finite-nat-bounded*: *finite* $S \Longrightarrow \exists\, k.\ S \subseteq \{..{<}k\}$
for S :: *nat set*
by (*simp add*: *finite-nat-iff-bounded*)

For a set of natural numbers to be infinite, it is enough to know that for any number larger than some k, there is some larger number that is an element of the set.

lemma *unbounded-k-infinite*: $\forall\, m{>}k.\ \exists\, n{>}m.\ n \in S \Longrightarrow$ *infinite* (S::*nat set*)
apply (*clarsimp simp add*: *finite-nat-set-iff-bounded*)
apply (*drule-tac* x=*Suc* (*max m k*) **in** *spec*)
using *less-Suc-eq* **apply** *fastforce*
done

lemma *nat-not-finite*: *finite* (*UNIV*::*nat set*) $\Longrightarrow R$
by *simp*

lemma *range-inj-infinite*:
fixes f :: *nat* $\Rightarrow$ $'a$
assumes *inj f*

shows *infinite* (*range f*)
proof
assume *finite* (*range f*)
from *this assms* **have** *finite* (*UNIV*::*nat set*)
by (*rule finite-imageD*)
then show *False* **by** *simp*
qed

20.2 The set of integers is also infinite

lemma *infinite-int-iff-infinite-nat-abs*: *infinite* $S \longleftrightarrow$ *infinite* ((*nat* $\circ$ *abs*) ' *S*)
for *S* :: *int set*
proof (*unfold Not-eq-iff*, *rule iffI*)
assume *finite* ((*nat* $\circ$ *abs*) ' *S*)
then have *finite* (*nat* ' (*abs* ' *S*))
by (*simp add*: *image-image cong*: *image-cong*)
moreover have *inj-on nat* (*abs* ' *S*)
by (*rule inj-onI*) *auto*
ultimately have *finite* (*abs* ' *S*)
by (*rule finite-imageD*)
then show *finite S*
by (*rule finite-image-absD*)
qed *simp*

proposition *infinite-int-iff-unbounded-le*: *infinite* $S \longleftrightarrow (\forall\, m.\ \exists\, n.\ |n| \geq m \wedge n \in S)$
for *S* :: *int set*
by (*simp add*: *infinite-int-iff-infinite-nat-abs infinite-nat-iff-unbounded-le o-def image-def*)
(*metis abs-ge-zero nat-le-eq-zle le-nat-iff*)

proposition *infinite-int-iff-unbounded*: *infinite* $S \longleftrightarrow (\forall\, m.\ \exists\, n.\ |n| > m \wedge n \in S)$
for *S* :: *int set*
by (*simp add*: *infinite-int-iff-infinite-nat-abs infinite-nat-iff-unbounded o-def image-def*)
(*metis* (*full-types*) *nat-le-iff nat-mono not-le*)

proposition *finite-int-iff-bounded*: *finite* $S \longleftrightarrow (\exists\, k.$ *abs* ' $S \subseteq \{..<k\})$
for *S* :: *int set*
using *infinite-int-iff-unbounded-le*[*of S*] **by** (*simp add*: *subset-eq*) (*metis not-le*)

proposition *finite-int-iff-bounded-le*: *finite* $S \longleftrightarrow (\exists\, k.$ *abs* ' $S \subseteq \{..\ k\})$
for *S* :: *int set*
using *infinite-int-iff-unbounded*[*of S*] **by** (*simp add*: *subset-eq*) (*metis not-le*)

20.3 Infinitely Many and Almost All

We often need to reason about the existence of infinitely many (resp., all but finitely many) objects satisfying some predicate, so we introduce corresponding binders and their proof rules.

lemma *not-INFM* [*simp*]: $\neg$ (*INFM x. P x*) $\longleftrightarrow$ (*MOST x.* $\neg$ *P x*)
by (*rule not-frequently*)

lemma *not-MOST* [*simp*]: $\neg$ (*MOST x. P x*) $\longleftrightarrow$ (*INFM x.* $\neg$ *P x*)
by (*rule not-eventually*)

lemma *INFM-const* [*simp*]: (*INFM x*::'*a. P*) $\longleftrightarrow$ *P* $\wedge$ *infinite* (*UNIV*::'*a set*)
by (*simp add*: *frequently-const-iff*)

lemma *MOST-const* [*simp*]: (*MOST x*::'*a. P*) $\longleftrightarrow$ *P* $\vee$ *finite* (*UNIV*::'*a set*)
by (*simp add*: *eventually-const-iff*)

lemma *INFM-imp-distrib*: (*INFM x. P x* $\longrightarrow$ *Q x*) $\longleftrightarrow$ ((*MOST x. P x*) $\longrightarrow$ (*INFM x. Q x*))
by (*rule frequently-imp-iff*)

lemma *MOST-imp-iff*: *MOST x. P x* $\Longrightarrow$ (*MOST x. P x* $\longrightarrow$ *Q x*) $\longleftrightarrow$ (*MOST x. Q x*)
by (*auto intro*: *eventually-rev-mp eventually-mono*)

lemma *INFM-conjI*: *INFM x. P x* $\Longrightarrow$ *MOST x. Q x* $\Longrightarrow$ *INFM x. P x* $\wedge$ *Q x*
by (*rule frequently-rev-mp*[*of P*]) (*auto elim*: *eventually-mono*)

Properties of quantifiers with injective functions.

lemma *INFM-inj*: *INFM x. P* (*f x*) $\Longrightarrow$ *inj f* $\Longrightarrow$ *INFM x. P x*
using *finite-vimageI*[*of* {*x. P x*} *f*] **by** (*auto simp*: *frequently-cofinite*)

lemma *MOST-inj*: *MOST x. P x* $\Longrightarrow$ *inj f* $\Longrightarrow$ *MOST x. P* (*f x*)
using *finite-vimageI*[*of* {*x.* $\neg$ *P x*} *f*] **by** (*auto simp*: *eventually-cofinite*)

Properties of quantifiers with singletons.

lemma *not-INFM-eq* [*simp*]:
$\neg$ (*INFM x. x* = *a*)
$\neg$ (*INFM x. a* = *x*)
unfolding *frequently-cofinite* **by** *simp-all*

lemma *MOST-neq* [*simp*]:
MOST x. x $\neq$ *a*
MOST x. a $\neq$ *x*
unfolding *eventually-cofinite* **by** *simp-all*

lemma *INFM-neq* [*simp*]:
(*INFM x*::'*a. x* $\neq$ *a*) $\longleftrightarrow$ *infinite* (*UNIV*::'*a set*)
(*INFM x*::'*a. a* $\neq$ *x*) $\longleftrightarrow$ *infinite* (*UNIV*::'*a set*)

unfolding *frequently-cofinite* **by** *simp-all*

lemma *MOST-eq* [*simp*]:
(*MOST* x::$'a$. $x = a$) $\longleftrightarrow$ *finite* (*UNIV*::$'a$ *set*)
(*MOST* x::$'a$. $a = x$) $\longleftrightarrow$ *finite* (*UNIV*::$'a$ *set*)
unfolding *eventually-cofinite* **by** *simp-all*

lemma *MOST-eq-imp*:
MOST x. $x = a \longrightarrow P\ x$
MOST x. $a = x \longrightarrow P\ x$
unfolding *eventually-cofinite* **by** *simp-all*

Properties of quantifiers over the naturals.

lemma *MOST-nat*: $(\forall_\infty n.\ P\ n) \longleftrightarrow (\exists\, m.\ \forall\, n{>}m.\ P\ n)$
for P :: *nat* $\Rightarrow$ *bool*
by (*auto simp add*: *eventually-cofinite finite-nat-iff-bounded-le subset-eq simp flip*: *not-le*)

lemma *MOST-nat-le*: $(\forall_\infty n.\ P\ n) \longleftrightarrow (\exists\, m.\ \forall\, n{\geq}m.\ P\ n)$
for P :: *nat* $\Rightarrow$ *bool*
by (*auto simp add*: *eventually-cofinite finite-nat-iff-bounded subset-eq simp flip*: *not-le*)

lemma *INFM-nat*: $(\exists_\infty n.\ P\ n) \longleftrightarrow (\forall\, m.\ \exists\, n{>}m.\ P\ n)$
for P :: *nat* $\Rightarrow$ *bool*
by (*simp add*: *frequently-cofinite infinite-nat-iff-unbounded*)

lemma *INFM-nat-le*: $(\exists_\infty n.\ P\ n) \longleftrightarrow (\forall\, m.\ \exists\, n{\geq}m.\ P\ n)$
for P :: *nat* $\Rightarrow$ *bool*
by (*simp add*: *frequently-cofinite infinite-nat-iff-unbounded-le*)

lemma *MOST-INFM*: *infinite* (*UNIV*::$'a$ *set*) $\Longrightarrow$ *MOST* x::$'a$. $P\ x \Longrightarrow$ *INFM* x::$'a$. $P\ x$
by (*simp add*: *eventually-frequently*)

lemma *MOST-Suc-iff*: (*MOST* n. P (*Suc* n)) $\longleftrightarrow$ (*MOST* n. $P\ n$)
by (*simp add*: *cofinite-eq-sequentially*)

lemma *MOST-SucI*: *MOST* n. $P\ n \Longrightarrow$ *MOST* n. P (*Suc* n)
and *MOST-SucD*: *MOST* n. P (*Suc* n) $\Longrightarrow$ *MOST* n. $P\ n$
by (*simp-all add*: *MOST-Suc-iff*)

lemma *MOST-ge-nat*: *MOST* n::*nat*. $m \leq n$
by (*simp add*: *cofinite-eq-sequentially*)

— legacy names
lemma *Inf-many-def*: *Inf-many* $P \longleftrightarrow$ *infinite* $\{x.\ P\ x\}$ **by** (*fact frequently-cofinite*)
lemma *Alm-all-def*: *Alm-all* $P \longleftrightarrow \neg$ (*INFM* x. $\neg\ P\ x$) **by** *simp*
lemma *INFM-iff-infinite*: (*INFM* x. $P\ x$) $\longleftrightarrow$ *infinite* $\{x.\ P\ x\}$ **by** (*fact fre-*

quently-cofinite)
lemma *MOST-iff-cofinite*: (*MOST x. P x*) ⟷ *finite* {*x*. ¬ *P x*} **by** (*fact eventually-cofinite*)
lemma *INFM-EX*: (∃$_\infty$*x. P x*) ⟹ (∃ *x. P x*) **by** (*fact frequently-ex*)
lemma *ALL-MOST*: ∀ *x. P x* ⟹ ∀$_\infty$*x. P x* **by** (*fact always-eventually*)
lemma *INFM-mono*: ∃$_\infty$*x. P x* ⟹ (⋀*x. P x* ⟹ *Q x*) ⟹ ∃$_\infty$*x. Q x* **by** (*fact frequently-elim1*)
lemma *MOST-mono*: ∀$_\infty$*x. P x* ⟹ (⋀*x. P x* ⟹ *Q x*) ⟹ ∀$_\infty$*x. Q x* **by** (*fact eventually-mono*)
lemma *INFM-disj-distrib*: (∃$_\infty$*x. P x* ∨ *Q x*) ⟷ (∃$_\infty$*x. P x*) ∨ (∃$_\infty$*x. Q x*) **by** (*fact frequently-disj-iff*)
lemma *MOST-rev-mp*: ∀$_\infty$*x. P x* ⟹ ∀$_\infty$*x. P x* ⟶ *Q x* ⟹ ∀$_\infty$*x. Q x* **by** (*fact eventually-rev-mp*)
lemma *MOST-conj-distrib*: (∀$_\infty$*x. P x* ∧ *Q x*) ⟷ (∀$_\infty$*x. P x*) ∧ (∀$_\infty$*x. Q x*) **by** (*fact eventually-conj-iff*)
lemma *MOST-conjI*: *MOST x. P x* ⟹ *MOST x. Q x* ⟹ *MOST x. P x* ∧ *Q x* **by** (*fact eventually-conj*)
lemma *INFM-finite-Bex-distrib*: *finite A* ⟹ (*INFM y.* ∃ *x*∈*A. P x y*) ⟷ (∃ *x*∈*A. INFM y. P x y*) **by** (*fact frequently-bex-finite-distrib*)
lemma *MOST-finite-Ball-distrib*: *finite A* ⟹ (*MOST y.* ∀ *x*∈*A. P x y*) ⟷ (∀ *x*∈*A. MOST y. P x y*) **by** (*fact eventually-ball-finite-distrib*)
lemma *INFM-E*: *INFM x. P x* ⟹ (⋀*x. P x* ⟹ *thesis*) ⟹ *thesis* **by** (*fact frequentlyE*)
lemma *MOST-I*: (⋀*x. P x*) ⟹ *MOST x. P x* **by** (*rule eventuallyI*)
lemmas *MOST-iff-finiteNeg* = *MOST-iff-cofinite*

20.4 Enumeration of an Infinite Set

The set's element type must be wellordered (e.g. the natural numbers).

Could be generalized to *enumerate′ S n* = (*SOME t. t* ∈ *s* ∧ *finite* {*s* ∈ *S. s* < *t*} ∧ *card* {*s* ∈ *S. s* < *t*} = *n*).

primrec (**in** *wellorder*) *enumerate* :: *'a set* ⇒ *nat* ⇒ *'a*
where
enumerate-0: *enumerate S 0* = (*LEAST n. n* ∈ *S*)
| *enumerate-Suc*: *enumerate S* (*Suc n*) = *enumerate* (*S* − {*LEAST n. n* ∈ *S*}) *n*

lemma *enumerate-Suc′*: *enumerate S* (*Suc n*) = *enumerate* (*S* − {*enumerate S 0*}) *n*
by *simp*

lemma *enumerate-in-set*: *infinite S* ⟹ *enumerate S n* ∈ *S*
proof (*induct n arbitrary*: *S*)
case *0*
then show *?case*
by (*fastforce intro*: *LeastI dest*!: *infinite-imp-nonempty*)
next
case (*Suc n*)
then show *?case*

by *simp* (*metis DiffE infinite-remove*)
qed

declare *enumerate-0* [*simp del*] *enumerate-Suc* [*simp del*]

lemma *enumerate-step*: *infinite S* ⟹ *enumerate S n* < *enumerate S* (*Suc n*)
proof (*induction n arbitrary*: *S*)
case *0*
then have *enumerate S 0* ≤ *enumerate S* (*Suc 0*)
by (*simp add*: *enumerate-0 Least-le enumerate-in-set*)
moreover have *enumerate* (*S* − {*enumerate S 0*}) *0* ∈ *S* − {*enumerate S 0*}
by (*meson 0.prems enumerate-in-set infinite-remove*)
then have *enumerate S 0* ≠ *enumerate* (*S* − {*enumerate S 0*}) *0*
by *auto*
ultimately show *?case*
by (*simp add*: *enumerate-Suc′*)
next
case (*Suc n*)
then show *?case*
by (*simp add*: *enumerate-Suc′*)
qed

lemma *enumerate-mono*: *m* < *n* ⟹ *infinite S* ⟹ *enumerate S m* < *enumerate S n*
by (*induct m n rule*: *less-Suc-induct*) (*auto intro*: *enumerate-step*)

lemma *enumerate-mono-iff* [*simp*]:
infinite S ⟹ *enumerate S m* < *enumerate S n* ⟷ *m* < *n*
by (*metis enumerate-mono less-asym less-linear*)

lemma *le-enumerate*:
assumes *S*: *infinite S*
shows *n* ≤ *enumerate S n*
using *S*
proof (*induct n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
then have *n* ≤ *enumerate S n* **by** *simp*
also note *enumerate-mono*[*of n Suc n*, *OF* - ‹*infinite S*›]
finally show *?case* **by** *simp*
qed

lemma *infinite-enumerate*:
assumes *fS*: *infinite S*
shows ∃ *r*::*nat*⇒*nat*. *strict-mono r* ∧ (∀ *n*. *r n* ∈ *S*)
unfolding *strict-mono-def*
using *enumerate-in-set*[*OF fS*] *enumerate-mono*[*of* - - *S*] *fS* **by** *blast*

lemma *enumerate-Suc''*:
fixes S :: *'a::wellorder set*
assumes *infinite S*
shows *enumerate S (Suc n)* = $(LEAST\ s.\ s \in S \wedge enumerate\ S\ n < s)$
using *assms*
proof (*induct n arbitrary*: *S*)
case *0*
then have $\forall\, s \in S.\ enumerate\ S\ 0 \leq s$
by (*auto simp*: *enumerate.simps intro*: *Least-le*)
then show *?case*
unfolding *enumerate-Suc' enumerate-0*[*of S* − {*enumerate S 0*}]
by (*intro arg-cong*[**where** *f* = *Least*] *ext*) *auto*
next
case (*Suc n S*)
show *?case*
using *enumerate-mono*[*OF zero-less-Suc* ‹*infinite S*›, *of n*] ‹*infinite S*›
apply (*subst* (*1 2*) *enumerate-Suc'*)
apply (*subst Suc*)
apply (*use* ‹*infinite S*› **in** *simp*)
apply (*intro arg-cong*[**where** *f* = *Least*] *ext*)
apply (*auto simp flip*: *enumerate-Suc'*)
done
qed

lemma *enumerate-Ex*:
fixes *S* :: *nat set*
assumes *S*: *infinite S*
and *s*: $s \in S$
shows $\exists\, n.\ enumerate\ S\ n = s$
using *s*
proof (*induct s rule*: *less-induct*)
case (*less s*)
show *?case*
proof (*cases* $\exists\, y{\in}S.\ y < s$)
case *True*
let *?y* = *Max* $\{s'{\in}S.\ s' < s\}$
from *True* **have** *y*: $\bigwedge x.\ ?y < x \longleftrightarrow (\forall\, s'{\in}S.\ s' < s \longrightarrow s' < x)$
by (*subst Max-less-iff*) *auto*
then have *y-in*: *?y* ∈ $\{s'{\in}S.\ s' < s\}$
by (*intro Max-in*) *auto*
with *less.hyps*[*of ?y*] **obtain** *n* **where** *enumerate S n* = *?y*
by *auto*
with *S* **have** *enumerate S* (*Suc n*) = *s*
by (*auto simp*: *y less enumerate-Suc'' intro*!: *Least-equality*)
then show *?thesis* **by** *auto*
next
case *False*
then have $\forall\, t{\in}S.\ s \leq t$ **by** *auto*

```
    with ‹s ∈ S› show ?thesis
      by (auto intro!: exI[of - 0] Least-equality simp: enumerate-0)
  qed
qed

lemma inj-enumerate:
  fixes S :: 'a::wellorder set
  assumes S: infinite S
  shows inj (enumerate S)
  unfolding inj-on-def
proof clarsimp
  show ⋀x y. enumerate S x = enumerate S y ⟹ x = y
    by (metis neq-iff enumerate-mono[OF - ‹infinite S›])
qed
```

To generalise this, we'd need a condition that all initial segments were finite

```
lemma bij-enumerate:
  fixes S :: nat set
  assumes S: infinite S
  shows bij-betw (enumerate S) UNIV S
proof −
  have ∀ s ∈ S. ∃ i. enumerate S i = s
    using enumerate-Ex[OF S] by auto
  moreover note ‹infinite S› inj-enumerate
  ultimately show ?thesis
    unfolding bij-betw-def by (auto intro: enumerate-in-set)
qed

lemma
  fixes S :: nat set
  assumes S: infinite S
  shows range-enumerate: range (enumerate S) = S
    and strict-mono-enumerate: strict-mono (enumerate S)
  by (auto simp add: bij-betw-imp-surj-on bij-enumerate assms strict-mono-def)
```

A pair of weird and wonderful lemmas from HOL Light.

```
lemma finite-transitivity-chain:
  assumes finite A
    and R: ⋀x. ¬ R x x ⋀x y z. ⟦R x y; R y z⟧ ⟹ R x z
    and A: ⋀x. x ∈ A ⟹ ∃ y. y ∈ A ∧ R x y
  shows A = {}
  using ‹finite A› A
proof (induct A)
  case empty
  then show ?case by simp
next
  case (insert a A)
  have False
```

using *R(1)*[*of a*] *R(2)*[*of - a*] *insert(3,4)* **by** *blast*
thus *?case* **..**
qed

corollary *Union-maximal-sets*:
assumes *finite* $\mathcal{F}$
shows $\bigcup\{T \in \mathcal{F}.\ \forall U{\in}\mathcal{F}.\ \neg\ T \subset U\} = \bigcup\mathcal{F}$
(**is** *?lhs = ?rhs*)
proof
show *?lhs* $\subseteq$ *?rhs* **by** *force*
show *?rhs* $\subseteq$ *?lhs*
proof (*rule Union-subsetI*)
fix *S*
assume $S \in \mathcal{F}$
have $\{T \in \mathcal{F}.\ S \subseteq T\} = \{\}$
if $\neg\ (\exists y.\ y \in \{T \in \mathcal{F}.\ \forall U{\in}\mathcal{F}.\ \neg\ T \subset U\} \wedge S \subseteq y)$
proof −
have §: $\bigwedge x.\ x \in \mathcal{F} \wedge S \subseteq x \Longrightarrow \exists y.\ y \in \mathcal{F} \wedge S \subseteq y \wedge x \subset y$
using *that* **by** (*blast intro*: *dual-order.trans psubset-imp-subset*)
show *?thesis*
proof (*rule finite-transitivity-chain* [*of* - $\lambda T\ U.\ S \subseteq T \wedge T \subset U$])
qed (*use assms* **in** ‹*auto intro*: §›)
qed
with ‹$S \in \mathcal{F}$› **show** $\exists y.\ y \in \{T \in \mathcal{F}.\ \forall U{\in}\mathcal{F}.\ \neg\ T \subset U\} \wedge S \subseteq y$
by *blast*
qed
qed

20.5 Properties of *wellorder-class.enumerate* on finite sets

lemma *finite-enumerate-in-set*: $[\![$*finite S*; *n* < *card S*$]\!] \Longrightarrow$ *enumerate S n* $\in$ *S*
proof (*induction n arbitrary*: *S*)
case *0*
then show *?case*
by (*metis all-not-in-conv card.empty enumerate.simps*(*1*) *not-less0 wellorder-Least-lemma*(*1*))
next
case (*Suc n*)
show *?case*
using *Suc.prems Suc.IH* [*of S* − {*LEAST n. n* $\in$ *S*}]
apply (*simp add*: *enumerate.simps*)
by (*metis Diff-empty Diff-insert0 Suc-lessD card.remove less-Suc-eq*)
qed

lemma *finite-enumerate-step*: $[\![$*finite S*; *Suc n* < *card S*$]\!] \Longrightarrow$ *enumerate S n* < *enumerate S* (*Suc n*)
proof (*induction n arbitrary*: *S*)
case *0*
then have *enumerate S 0* $\leq$ *enumerate S* (*Suc 0*)
by (*simp add*: *Least-le enumerate.simps*(*1*) *finite-enumerate-in-set*)

moreover have *enumerate* $(S - \{enumerate\ S\ 0\})\ 0 \in S - \{enumerate\ S\ 0\}$
by (*metis 0 Suc-lessD Suc-less-eq card-Suc-Diff1 enumerate-in-set finite-enumerate-in-set*)
then have $enumerate\ S\ 0 \neq enumerate\ (S - \{enumerate\ S\ 0\})\ 0$
by *auto*
ultimately show *?case*
by (*simp add*: *enumerate-Suc'*)
next
case (*Suc n*)
then show *?case*
by (*simp add*: *enumerate-Suc' finite-enumerate-in-set*)
qed

lemma *finite-enumerate-mono*: $[\![m < n; finite\ S; n < card\ S]\!] \Longrightarrow enumerate\ S\ m < enumerate\ S\ n$
by (*induct m n rule*: *less-Suc-induct*) (*auto intro*: *finite-enumerate-step*)

lemma *finite-enumerate-mono-iff* [*simp*]:
$[\![finite\ S; m < card\ S; n < card\ S]\!] \Longrightarrow enumerate\ S\ m < enumerate\ S\ n \longleftrightarrow m < n$
by (*metis finite-enumerate-mono less-asym less-linear*)

lemma *finite-le-enumerate*:
assumes *finite S* $n < card\ S$
shows $n \leq enumerate\ S\ n$
using *assms*
proof (*induction n*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
then have $n \leq enumerate\ S\ n$ **by** *simp*
also note *finite-enumerate-mono*[*of n Suc n*, *OF* - ‹*finite S*›]
finally show *?case*
using *Suc.prems*(*2*) *Suc-leI* **by** *blast*
qed

lemma *finite-enumerate*:
assumes *fS*: *finite S*
shows $\exists r{::}nat \Rightarrow nat.\ strict\text{-}mono\text{-}on\ r\ \{..<card\ S\} \wedge (\forall n{<}card\ S.\ r\ n \in S)$
unfolding *strict-mono-def*
using *finite-enumerate-in-set*[*OF fS*] *finite-enumerate-mono*[*of - - S*] *fS*
by (*metis lessThan-iff strict-mono-on-def*)

lemma *finite-enumerate-Suc''*:
fixes $S :: 'a{::}wellorder\ set$
assumes *finite S Suc n* $< card\ S$
shows $enumerate\ S\ (Suc\ n) = (LEAST\ s.\ s \in S \wedge enumerate\ S\ n < s)$
using *assms*
proof (*induction n arbitrary*: *S*)

```
  case 0
  then have ∀ s ∈ S. enumerate S 0 ≤ s
    by (auto simp: enumerate.simps intro: Least-le)
  then show ?case
    unfolding enumerate-Suc′ enumerate-0[of S − {enumerate S 0}]
    by (metis Diff-iff dual-order.strict-iff-order singletonD singletonI)
next
  case (Suc n S)
  then have Suc n < card (S − {enumerate S 0})
    using Suc.prems(2) finite-enumerate-in-set by force
  then show ?case
    apply (subst (1 2) enumerate-Suc′)
    apply (simp add: Suc)
    apply (intro arg-cong[where f = Least] HOL.ext)
    using finite-enumerate-mono[OF zero-less-Suc ‹finite S›, of n] Suc.prems
    by (auto simp flip: enumerate-Suc′)
qed

lemma finite-enumerate-initial-segment:
  fixes S :: 'a::wellorder set
  assumes finite S and n: n < card (S ∩ {..<s})
  shows enumerate (S ∩ {..<s}) n = enumerate S n
  using n
proof (induction n)
  case 0
  have (LEAST n. n ∈ S ∧ n < s) = (LEAST n. n ∈ S)
  proof (rule Least-equality)
    have ∃ t. t ∈ S ∧ t < s
      by (metis 0 card-gt-0-iff disjoint-iff-not-equal lessThan-iff)
    then show (LEAST n. n ∈ S) ∈ S ∧ (LEAST n. n ∈ S) < s
      by (meson LeastI Least-le le-less-trans)
  qed (simp add: Least-le)
  then show ?case
    by (auto simp: enumerate-0)
next
  case (Suc n)
  then have less-card: Suc n < card S
    by (meson assms(1) card-mono inf-sup-ord(1) leD le-less-linear order.trans)
  obtain T where T: T ∈ {s ∈ S. enumerate S n < s}
    by (metis Infinite-Set.enumerate-step enumerate-in-set finite-enumerate-in-set
finite-enumerate-step less-card mem-Collect-eq)
  have (LEAST x. x ∈ S ∧ x < s ∧ enumerate S n < x) = (LEAST x. x ∈ S ∧
enumerate S n < x)
        (is - = ?r)
  proof (intro Least-equality conjI)
    show ?r ∈ S
      by (metis (mono-tags, lifting) LeastI mem-Collect-eq T)
    have ¬ s ≤ ?r
      using not-less-Least [of - λx. x ∈ S ∧ enumerate S n < x] Suc assms
```

by (*metis* (*mono-tags*, *lifting*) *Int-Collect Suc-lessD finite-Int finite-enumerate-in-set finite-enumerate-step lessThan-def less-le-trans*)
then show *?r* < *s*
by *auto*
show *enumerate S n* < *?r*
by (*metis* (*no-types*, *lifting*) *LeastI mem-Collect-eq T*)
qed (*auto simp*: *Least-le*)
then show *?case*
using *Suc assms* **by** (*simp add*: *finite-enumerate-Suc''* *less-card*)
qed

lemma *finite-enumerate-Ex*:
fixes *S* :: *'a*::*wellorder set*
assumes *S*: *finite S*
and *s*: *s* ∈ *S*
shows ∃ *n*<*card S*. *enumerate S n* = *s*
using *s S*
proof (*induction s arbitrary*: *S rule*: *less-induct*)
case (*less s*)
show *?case*
proof (*cases* ∃ *y*∈*S*. *y* < *s*)
case *True*
let *?T* = *S* ∩ {..<*s*}
have *finite ?T*
using *less.prems*(*2*) **by** *blast*
have *TS*: *card ?T* < *card S*
using *less.prems* **by** (*blast intro*: *psubset-card-mono* [*OF* ‹*finite S*›])
from *True* **have** *y*: ⋀*x*. *Max ?T* < *x* ⟷ (∀ *s'*∈*S*. *s'* < *s* ⟶ *s'* < *x*)
by (*subst Max-less-iff*) (*auto simp*: ‹*finite ?T*›)
then have *y-in*: *Max ?T* ∈ {*s'*∈*S*. *s'* < *s*}
using *Max-in* ‹*finite ?T*› **by** *fastforce*
with *less.IH*[*of Max ?T ?T*] **obtain** *n* **where** *n*: *enumerate ?T n* = *Max ?T n* < *card ?T*
using ‹*finite ?T*› **by** *blast*
then have *Suc n* < *card S*
using *TS less-trans-Suc* **by** *blast*
with *S n* **have** *enumerate S* (*Suc n*) = *s*
by (*subst finite-enumerate-Suc''*) (*auto simp*: *y finite-enumerate-initial-segment less finite-enumerate-Suc'' intro*!: *Least-equality*)
then show *?thesis*
using ‹*Suc n* < *card S*› **by** *blast*
next
case *False*
then have ∀ *t*∈*S*. *s* ≤ *t* **by** *auto*
moreover have *0* < *card S*
using *card-0-eq less.prems* **by** *blast*
ultimately show *?thesis*
using ‹*s* ∈ *S*›
by (*auto intro*!: *exI*[*of - 0*] *Least-equality simp*: *enumerate-0*)

qed
qed

lemma *finite-enum-subset*:
assumes $\bigwedge i.\ i < card\ X \Longrightarrow enumerate\ X\ i = enumerate\ Y\ i$ **and** *finite X finite Y card X* $\leq$ *card Y*
shows $X \subseteq Y$
by (*metis assms finite-enumerate-Ex finite-enumerate-in-set less-le-trans subsetI*)

lemma *finite-enum-ext*:
assumes $\bigwedge i.\ i < card\ X \Longrightarrow enumerate\ X\ i = enumerate\ Y\ i$ **and** *finite X finite Y card X* $=$ *card Y*
shows $X = Y$
by (*intro antisym finite-enum-subset*) (*auto simp*: *assms*)

lemma *finite-bij-enumerate*:
fixes $S :: 'a::wellorder\ set$
assumes *S*: *finite S*
shows *bij-betw* (*enumerate S*) {..<*card S*} *S*
proof −
have $\bigwedge n\ m.\ [\![n \neq m;\ n < card\ S;\ m < card\ S]\!] \Longrightarrow enumerate\ S\ n \neq enumerate\ S\ m$
using *finite-enumerate-mono*[*OF* - ‹*finite S*›] **by** (*auto simp*: *neq-iff*)
then have *inj-on* (*enumerate S*) {..<*card S*}
by (*auto simp*: *inj-on-def*)
moreover have $\forall\, s \in S.\ \exists\, i{<}card\ S.\ enumerate\ S\ i = s$
using *finite-enumerate-Ex*[*OF S*] **by** *auto*
moreover note ‹*finite S*›
ultimately show *?thesis*
unfolding *bij-betw-def* **by** (*auto intro*: *finite-enumerate-in-set*)
qed

lemma *ex-bij-betw-strict-mono-card*:
fixes $M :: 'a::wellorder\ set$
assumes *finite M*
obtains *h* **where** *bij-betw h* {..<*card M*} *M* **and** *strict-mono-on h* {..<*card M*}
proof
show *bij-betw* (*enumerate M*) {..<*card M*} *M*
by (*simp add*: *assms finite-bij-enumerate*)
show *strict-mono-on* (*enumerate M*) {..<*card M*}
by (*simp add*: *assms finite-enumerate-mono strict-mono-on-def*)
qed

end

21 Countable sets

theory *Countable-Set*
imports *Countable Infinite-Set*

begin

21.1 Predicate for countable sets

definition *countable* :: *'a set* ⇒ *bool* **where**
countable S ⟷ (∃*f*::*'a* ⇒ *nat*. *inj-on f S*)

lemma *countableE*:
assumes *S*: *countable S* **obtains** *f* :: *'a* ⇒ *nat* **where** *inj-on f S*
using *S* **by** (*auto simp*: *countable-def*)

lemma *countableI*: *inj-on* (*f*::*'a* ⇒ *nat*) *S* ⟹ *countable S*
by (*auto simp*: *countable-def*)

lemma *countableI'*: *inj-on* (*f*::*'a* ⇒ *'b*::*countable*) *S* ⟹ *countable S*
using *comp-inj-on*[*of f S to-nat*] **by** (*auto intro*: *countableI*)

lemma *countableE-bij*:
assumes *S*: *countable S* **obtains** *f* :: *nat* ⇒ *'a* **and** *C* :: *nat set* **where** *bij-betw f C S*
using *S* **by** (*blast elim*: *countableE dest*: *inj-on-imp-bij-betw bij-betw-inv*)

lemma *countableI-bij*: *bij-betw f* (*C*::*nat set*) *S* ⟹ *countable S*
by (*blast intro*: *countableI bij-betw-inv-into bij-betw-imp-inj-on*)

lemma *countable-finite*: *finite S* ⟹ *countable S*
by (*blast dest*: *finite-imp-inj-to-nat-seg countableI*)

lemma *countableI-bij1*: *bij-betw f A B* ⟹ *countable A* ⟹ *countable B*
by (*blast elim*: *countableE-bij intro*: *bij-betw-trans countableI-bij*)

lemma *countableI-bij2*: *bij-betw f B A* ⟹ *countable A* ⟹ *countable B*
by (*blast elim*: *countableE-bij intro*: *bij-betw-trans bij-betw-inv-into countableI-bij*)

lemma *countable-iff-bij*[*simp*]: *bij-betw f A B* ⟹ *countable A* ⟷ *countable B*
by (*blast intro*: *countableI-bij1 countableI-bij2*)

lemma *countable-subset*: *A* ⊆ *B* ⟹ *countable B* ⟹ *countable A*
by (*auto simp*: *countable-def intro*: *subset-inj-on*)

lemma *countableI-type*[*intro*, *simp*]: *countable* (*A*:: *'a* :: *countable set*)
using *countableI*[*of to-nat A*] **by** *auto*

21.2 Enumerate a countable set

lemma *countableE-infinite*:
assumes *countable S infinite S*
obtains *e* :: *'a* ⇒ *nat* **where** *bij-betw e S UNIV*
proof −
obtain *f* :: *'a* ⇒ *nat* **where** *inj-on f S*

```
    using ‹countable S› by (rule countableE)
  then have bij-betw f S (f‘S)
    unfolding bij-betw-def by simp
  moreover
  from ‹inj-on f S› ‹infinite S› have inf-fS: infinite (f‘S)
    by (auto dest: finite-imageD)
  then have bij-betw (the-inv-into UNIV (enumerate (f‘S))) (f‘S) UNIV
    by (intro bij-betw-the-inv-into bij-enumerate)
  ultimately have bij-betw (the-inv-into UNIV (enumerate (f‘S)) ∘ f) S UNIV
    by (rule bij-betw-trans)
  then show thesis ..
qed

lemma countable-infiniteE':
  assumes countable A infinite A
  obtains g where bij-betw g (UNIV :: nat set) A
  using bij-betw-inv[OF countableE-infinite[OF assms]] that by blast

lemma countable-enum-cases:
  assumes countable S
  obtains (finite) f :: 'a ⇒ nat where finite S bij-betw f S {..<card S}
        | (infinite) f :: 'a ⇒ nat where infinite S bij-betw f S UNIV
  using ex-bij-betw-finite-nat[of S] countableE-infinite ‹countable S›
  by (cases finite S) (auto simp add: atLeast0LessThan)

definition to-nat-on :: 'a set ⇒ 'a ⇒ nat where
  to-nat-on S = (SOME f. if finite S then bij-betw f S {..< card S} else bij-betw f
S UNIV)

definition from-nat-into :: 'a set ⇒ nat ⇒ 'a where
  from-nat-into S n = (if n ∈ to-nat-on S ‘ S then inv-into S (to-nat-on S) n else
SOME s. s∈S)

lemma to-nat-on-finite: finite S ⟹ bij-betw (to-nat-on S) S {..< card S}
  using ex-bij-betw-finite-nat unfolding to-nat-on-def
  by (intro someI2-ex[where Q=λf. bij-betw f S {..<card S}]) (auto simp add:
atLeast0LessThan)

lemma to-nat-on-infinite: countable S ⟹ infinite S ⟹ bij-betw (to-nat-on S) S
UNIV
  using countableE-infinite unfolding to-nat-on-def
  by (intro someI2-ex[where Q=λf. bij-betw f S UNIV]) auto

lemma bij-betw-from-nat-into-finite: finite S ⟹ bij-betw (from-nat-into S) {..<
card S} S
  unfolding from-nat-into-def[abs-def]
  using to-nat-on-finite[of S]
  apply (subst bij-betw-cong)
  apply (split if-split)
```

apply (*simp add*: *bij-betw-def*)
apply (*auto cong*: *bij-betw-cong*
intro: *bij-betw-inv-into to-nat-on-finite*)
done

lemma *bij-betw-from-nat-into*: *countable S* $\Longrightarrow$ *infinite S* $\Longrightarrow$ *bij-betw* (*from-nat-into S*) *UNIV S*
unfolding *from-nat-into-def*[*abs-def*]
using *to-nat-on-infinite*[*of S*, *unfolded bij-betw-def*]
by (*auto cong*: *bij-betw-cong intro*: *bij-betw-inv-into to-nat-on-infinite*)

The sum/product over the enumeration of a finite set equals simply the sum/product over the set

context *comm-monoid-set*
begin

lemma *card-from-nat-into*:
F (*λi*. *h* (*from-nat-into A i*)) {..<*card A*} = *F h A*
proof (*cases finite A*)
case *True*
have *F* (*λi*. *h* (*from-nat-into A i*)) {..<*card A*} = *F h* (*from-nat-into A* ' {..<*card A*})
by (*metis True bij-betw-def bij-betw-from-nat-into-finite reindex-cong*)
also have ... = *F h A*
by (*metis True bij-betw-def bij-betw-from-nat-into-finite*)
finally show *?thesis* **.**
qed *auto*

end

lemma *countable-as-injective-image*:
assumes *countable A infinite A*
obtains *f* :: *nat* ⇒ *'a* **where** *A* = *range f inj f*
by (*metis bij-betw-def bij-betw-from-nat-into* [*OF assms*])

lemma *inj-on-to-nat-on*[*intro*]: *countable A* $\Longrightarrow$ *inj-on* (*to-nat-on A*) *A*
using *to-nat-on-infinite*[*of A*] *to-nat-on-finite*[*of A*]
by (*cases finite A*) (*auto simp*: *bij-betw-def*)

lemma *to-nat-on-inj*[*simp*]:
countable A $\Longrightarrow$ *a* ∈ *A* $\Longrightarrow$ *b* ∈ *A* $\Longrightarrow$ *to-nat-on A a* = *to-nat-on A b* $\longleftrightarrow$ *a* = *b*
using *inj-on-to-nat-on*[*of A*] **by** (*auto dest*: *inj-onD*)

lemma *from-nat-into-to-nat-on*[*simp*]: *countable A* $\Longrightarrow$ *a* ∈ *A* $\Longrightarrow$ *from-nat-into A* (*to-nat-on A a*) = *a*
by (*auto simp*: *from-nat-into-def intro*!: *inv-into-f-f*)

lemma *subset-range-from-nat-into*: *countable A* $\Longrightarrow$ *A* ⊆ *range* (*from-nat-into A*)
by (*auto intro*: *from-nat-into-to-nat-on*[*symmetric*])

lemma *from-nat-into*: $A \neq \{\} \Longrightarrow$ *from-nat-into* $A\ n \in A$
unfolding *from-nat-into-def* **by** (*metis equals0I inv-into-into someI-ex*)

lemma *range-from-nat-into-subset*: $A \neq \{\} \Longrightarrow$ *range* (*from-nat-into* A) $\subseteq A$
using *from-nat-into*[*of A*] **by** *auto*

lemma *range-from-nat-into*[*simp*]: $A \neq \{\} \Longrightarrow$ *countable* $A \Longrightarrow$ *range* (*from-nat-into* A) $= A$
by (*metis equalityI range-from-nat-into-subset subset-range-from-nat-into*)

lemma *image-to-nat-on*: *countable* $A \Longrightarrow$ *infinite* $A \Longrightarrow$ *to-nat-on* A ' A = *UNIV*
using *to-nat-on-infinite*[*of A*] **by** (*simp add*: *bij-betw-def*)

lemma *to-nat-on-surj*: *countable* $A \Longrightarrow$ *infinite* $A \Longrightarrow \exists\, a{\in}A.$ *to-nat-on* $A\ a = n$
by (*metis* (*no-types*) *image-iff iso-tuple-UNIV-I image-to-nat-on*)

lemma *to-nat-on-from-nat-into*[*simp*]: $n \in$ *to-nat-on* A ' $A \Longrightarrow$ *to-nat-on* A (*from-nat-into* $A\ n$) $= n$
by (*simp add*: *f-inv-into-f from-nat-into-def*)

lemma *to-nat-on-from-nat-into-infinite*[*simp*]:
countable $A \Longrightarrow$ *infinite* $A \Longrightarrow$ *to-nat-on* A (*from-nat-into* $A\ n$) $= n$
by (*metis image-iff to-nat-on-surj to-nat-on-from-nat-into*)

lemma *from-nat-into-inj*:
countable $A \Longrightarrow m \in$ *to-nat-on* A ' $A \Longrightarrow n \in$ *to-nat-on* A ' $A \Longrightarrow$
from-nat-into $A\ m$ = *from-nat-into* $A\ n \longleftrightarrow m = n$
by (*subst to-nat-on-inj*[*symmetric*, *of A*]) *auto*

lemma *from-nat-into-inj-infinite*[*simp*]:
countable $A \Longrightarrow$ *infinite* $A \Longrightarrow$ *from-nat-into* $A\ m$ = *from-nat-into* $A\ n \longleftrightarrow m = n$
using *image-to-nat-on*[*of A*] *from-nat-into-inj*[*of A m n*] **by** *simp*

lemma *eq-from-nat-into-iff*:
countable $A \Longrightarrow x \in A \Longrightarrow i \in$ *to-nat-on* A ' $A \Longrightarrow x =$ *from-nat-into* $A\ i \longleftrightarrow i =$ *to-nat-on* $A\ x$
by *auto*

lemma *from-nat-into-surj*: *countable* $A \Longrightarrow a \in A \Longrightarrow \exists\, n.$ *from-nat-into* $A\ n = a$
by (*rule exI*[*of - to-nat-on A a*]) *simp*

lemma *from-nat-into-inject*[*simp*]:
$A \neq \{\} \Longrightarrow$ *countable* $A \Longrightarrow B \neq \{\} \Longrightarrow$ *countable* $B \Longrightarrow$ *from-nat-into* $A =$ *from-nat-into* $B \longleftrightarrow A = B$
by (*metis range-from-nat-into*)

lemma *inj-on-from-nat-into*: *inj-on from-nat-into* ({*A*. *A* ≠ {} ∧ *countable A*})
unfolding *inj-on-def* **by** *auto*

21.3 Closure properties of countability

lemma *countable-SIGMA*[*intro*, *simp*]:
countable I ⟹ (⋀*i*. *i* ∈ *I* ⟹ *countable* (*A i*)) ⟹ *countable* (*SIGMA i* : *I*. *A i*)
by (*intro countableI'*[*of* λ(*i*, *a*). (*to-nat-on I i*, *to-nat-on* (*A i*) *a*)]) (*auto simp*: *inj-on-def*)

lemma *countable-image*[*intro*, *simp*]:
assumes *countable A*
shows *countable* (*f*‘*A*)
proof −
obtain *g* :: *'a* ⇒ *nat* **where** *inj-on g A*
using *assms* **by** (*rule countableE*)
moreover have *inj-on* (*inv-into A f*) (*f*‘*A*) *inv-into A f* ‘ *f* ‘ *A* ⊆ *A*
by (*auto intro*: *inj-on-inv-into inv-into-into*)
ultimately show *?thesis*
by (*blast dest*: *comp-inj-on subset-inj-on intro*: *countableI*)
qed

lemma *countable-image-inj-on*: *countable* (*f* ‘ *A*) ⟹ *inj-on f A* ⟹ *countable A*
by (*metis countable-image the-inv-into-onto*)

lemma *countable-image-inj-Int-vimage*:
⟦*inj-on f S*; *countable A*⟧ ⟹ *countable* (*S* ∩ *f* −‘ *A*)
by (*meson countable-image-inj-on countable-subset image-subset-iff-subset-vimage inf-le2 inj-on-Int*)

lemma *countable-image-inj-gen*:
⟦*inj-on f S*; *countable A*⟧ ⟹ *countable* {*x* ∈ *S*. *f x* ∈ *A*}
using *countable-image-inj-Int-vimage*
by (*auto simp*: *vimage-def Collect-conj-eq*)

lemma *countable-image-inj-eq*:
inj-on f S ⟹ *countable*(*f* ‘ *S*) ⟷ *countable S*
using *countable-image-inj-on* **by** *blast*

lemma *countable-image-inj*:
⟦*countable A*; *inj f*⟧ ⟹ *countable* {*x*. *f x* ∈ *A*}
by (*metis* (*mono-tags*, *lifting*) *countable-image-inj-eq countable-subset image-Collect-subsetI inj-on-inverseI the-inv-f-f*)

lemma *countable-UN*[*intro*, *simp*]:
fixes *I* :: *'i set* **and** *A* :: *'i* => *'a set*
assumes *I*: *countable I*
assumes *A*: ⋀*i*. *i* ∈ *I* ⟹ *countable* (*A i*)

shows *countable* ($\bigcup i{\in}I.\ A\ i$)
proof −
have ($\bigcup i{\in}I.\ A\ i$) = *snd* ‘ (*SIGMA i* : *I*. *A i*) **by** (*auto simp*: *image-iff*)
then show *?thesis* **by** (*simp add*: *assms*)
qed

lemma *countable-Un*[*intro*]: *countable A* ⟹ *countable B* ⟹ *countable* ($A \cup B$)
by (*rule countable-UN*[*of* {*True*, *False*} λ*True* ⇒ *A* | *False* ⇒ *B*, *simplified*])
(*simp split*: *bool.split*)

lemma *countable-Un-iff*[*simp*]: *countable* ($A \cup B$) ⟷ *countable A* ∧ *countable B*
by (*metis countable-Un countable-subset inf-sup-ord*(*3*,*4*))

lemma *countable-Plus*[*intro*, *simp*]:
countable A ⟹ *countable B* ⟹ *countable* (*A* <+> *B*)
by (*simp add*: *Plus-def*)

lemma *countable-empty*[*intro*, *simp*]: *countable* {}
by (*blast intro*: *countable-finite*)

lemma *countable-insert*[*intro*, *simp*]: *countable A* ⟹ *countable* (*insert a A*)
using *countable-Un*[*of* {*a*} *A*] **by** (*auto simp*: *countable-finite*)

lemma *countable-Int1*[*intro*, *simp*]: *countable A* ⟹ *countable* ($A \cap B$)
by (*force intro*: *countable-subset*)

lemma *countable-Int2*[*intro*, *simp*]: *countable B* ⟹ *countable* ($A \cap B$)
by (*blast intro*: *countable-subset*)

lemma *countable-INT*[*intro*, *simp*]: *i* ∈ *I* ⟹ *countable* (*A i*) ⟹ *countable* ($\bigcap i{\in}I.$ *A i*)
by (*blast intro*: *countable-subset*)

lemma *countable-Diff*[*intro*, *simp*]: *countable A* ⟹ *countable* ($A - B$)
by (*blast intro*: *countable-subset*)

lemma *countable-insert-eq* [*simp*]: *countable* (*insert x A*) = *countable A*
by *auto* (*metis Diff-insert-absorb countable-Diff insert-absorb*)

lemma *countable-vimage*: *B* ⊆ *range f* ⟹ *countable* (*f* −‘ *B*) ⟹ *countable B*
by (*metis Int-absorb2 countable-image image-vimage-eq*)

lemma *surj-countable-vimage*: *surj f* ⟹ *countable* (*f* −‘ *B*) ⟹ *countable B*
by (*metis countable-vimage top-greatest*)

lemma *countable-Collect*[*simp*]: *countable A* ⟹ *countable* {*a* ∈ *A*. φ *a*}
by (*metis Collect-conj-eq Int-absorb Int-commute Int-def countable-Int1*)

lemma *countable-Image*:

assumes $\bigwedge y.\ y \in Y \Longrightarrow$ *countable* $(X$ `` $\{y\})$
assumes *countable* Y
shows *countable* $(X$ `` $Y)$
proof −
have *countable* $(X$ `` $(\bigcup y{\in}Y.\ \{y\}))$
unfolding *Image-UN* **by** (*intro countable-UN assms*)
then show *?thesis* **by** *simp*
qed

lemma *countable-relpow*:
fixes X :: *'a rel*
assumes *Image-X*: $\bigwedge Y.$ *countable* $Y \Longrightarrow$ *countable* $(X$ `` $Y)$
assumes Y: *countable* Y
shows *countable* $((X \frown\frown i)$ `` $Y)$
using Y **by** (*induct i arbitrary*: Y) (*auto simp*: *relcomp-Image Image-X*)

lemma *countable-funpow*:
fixes f :: *'a set* ⇒ *'a set*
assumes $\bigwedge A.$ *countable* $A \Longrightarrow$ *countable* $(f\ A)$
and *countable* A
shows *countable* $((f \frown\frown n)\ A)$
by(*induction n*)(*simp-all add*: *assms*)

lemma *countable-rtrancl*:
$(\bigwedge Y.$ *countable* $Y \Longrightarrow$ *countable* $(X$ `` $Y)) \Longrightarrow$ *countable* $Y \Longrightarrow$ *countable* $(X^*$ `` $Y)$
unfolding *rtrancl-is-UN-relpow UN-Image* **by** (*intro countable-UN countableI-type countable-relpow*)

lemma *countable-lists*[*intro*, *simp*]:
assumes A: *countable* A **shows** *countable* (*lists* A)
proof −
have *countable* (*lists* (*range* (*from-nat-into* A)))
by (*auto simp*: *lists-image*)
with A **show** *?thesis*
by (*auto dest*: *subset-range-from-nat-into countable-subset lists-mono*)
qed

lemma *Collect-finite-eq-lists*: *Collect finite* = *set* ` *lists UNIV*
using *finite-list* **by** *auto*

lemma *countable-Collect-finite*: *countable* (*Collect* (*finite*::*'a*::*countable set*⇒*bool*))
by (*simp add*: *Collect-finite-eq-lists*)

lemma *countable-int*: *countable* $\mathbb{Z}$
unfolding *Ints-def* **by** *auto*

lemma *countable-rat*: *countable* $\mathbb{Q}$
unfolding *Rats-def* **by** *auto*

lemma *Collect-finite-subset-eq-lists*: $\{A.\ finite\ A \wedge A \subseteq T\} = set\ `\ lists\ T$
using *finite-list* **by** (*auto simp*: *lists-eq-set*)

lemma *countable-Collect-finite-subset*:
$countable\ T \Longrightarrow countable\ \{A.\ finite\ A \wedge A \subseteq T\}$
unfolding *Collect-finite-subset-eq-lists* **by** *auto*

lemma *countable-Fpow*: $countable\ S \Longrightarrow countable\ (Fpow\ S)$
using *countable-Collect-finite-subset*
by (*force simp add*: *Fpow-def conj-commute*)

lemma *countable-set-option* [*simp*]: $countable\ (set\text{-}option\ x)$
by (*cases x*) *auto*

21.4 Misc lemmas

lemma *countable-subset-image*:
$countable\ B \wedge B \subseteq (f\ `\ A) \longleftrightarrow (\exists A'.\ countable\ A' \wedge A' \subseteq A \wedge (B = f\ `\ A'))$
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
show *?rhs*
by (*rule exI* [**where** *x=inv-into A f ` B*])
(*use ‹?lhs› in ‹auto simp: f-inv-into-f subset-iff image-inv-into-cancel inv-into-into›*)
next
assume *?rhs*
then show *?lhs* **by** *force*
qed

lemma *ex-subset-image-inj*:
$(\exists T.\ T \subseteq f\ `\ S \wedge P\ T) \longleftrightarrow (\exists T.\ T \subseteq S \wedge inj\text{-}on\ f\ T \wedge P\ (f\ `\ T))$
by (*auto simp*: *subset-image-inj*)

lemma *all-subset-image-inj*:
$(\forall T.\ T \subseteq f\ `\ S \longrightarrow P\ T) \longleftrightarrow (\forall T.\ T \subseteq S \wedge inj\text{-}on\ f\ T \longrightarrow P(f\ `\ T))$
by (*metis subset-image-inj*)

lemma *ex-countable-subset-image-inj*:
$(\exists T.\ countable\ T \wedge T \subseteq f\ `\ S \wedge P\ T) \longleftrightarrow$
$(\exists T.\ countable\ T \wedge T \subseteq S \wedge inj\text{-}on\ f\ T \wedge P\ (f\ `\ T))$
by (*metis countable-image-inj-eq subset-image-inj*)

lemma *all-countable-subset-image-inj*:
$(\forall T.\ countable\ T \wedge T \subseteq f\ `\ S \longrightarrow P\ T) \longleftrightarrow (\forall T.\ countable\ T \wedge T \subseteq S \wedge inj\text{-}on\ f\ T \longrightarrow P(f\ `\ T))$
by (*metis countable-image-inj-eq subset-image-inj*)

lemma *ex-countable-subset-image*:

$(\exists\, T.\ countable\ T \wedge T \subseteq f\ `\ S \wedge P\ T) \longleftrightarrow (\exists\, T.\ countable\ T \wedge T \subseteq S \wedge P\ (f\ `\ T))$
by (*metis countable-subset-image*)

lemma *all-countable-subset-image*:
$(\forall\, T.\ countable\ T \wedge T \subseteq f\ `\ S \longrightarrow P\ T) \longleftrightarrow (\forall\, T.\ countable\ T \wedge T \subseteq S \longrightarrow P(f\ `\ T))$
by (*metis countable-subset-image*)

lemma *countable-image-eq*:
$countable(f\ `\ S) \longleftrightarrow (\exists\, T.\ countable\ T \wedge T \subseteq S \wedge f\ `\ S = f\ `\ T)$
by (*metis countable-image countable-image-inj-eq order-refl subset-image-inj*)

lemma *countable-image-eq-inj*:
$countable(f\ `\ S) \longleftrightarrow (\exists\, T.\ countable\ T \wedge T \subseteq S \wedge f\ `\ S = f\ `\ T \wedge inj\text{-}on\ f\ T)$
by (*metis countable-image-inj-eq order-refl subset-image-inj*)

lemma *infinite-countable-subset'*:
assumes X: *infinite X* **shows** $\exists\, C \subseteq X.\ countable\ C \wedge infinite\ C$
proof −
obtain $f :: nat \Rightarrow 'a$ **where** *inj f range f* $\subseteq X$
using *infinite-countable-subset* [*OF X*] **by** *blast*
then show *?thesis*
by (*intro exI*[*of - range f*]) (*auto simp*: *range-inj-infinite*)
qed

lemma *countable-all*:
assumes S: *countable S*
shows $(\forall\, s{\in}S.\ P\ s) \longleftrightarrow (\forall\, n{::}nat.\ from\text{-}nat\text{-}into\ S\ n \in S \longrightarrow P\ (from\text{-}nat\text{-}into\ S\ n))$
using *S*[*THEN subset-range-from-nat-into*] **by** *auto*

lemma *finite-sequence-to-countable-set*:
assumes *countable X*
obtains F **where** $\bigwedge i.\ F\ i \subseteq X\ \bigwedge i.\ F\ i \subseteq F\ (Suc\ i)\ \bigwedge i.\ finite\ (F\ i)\ (\bigcup i.\ F\ i) = X$
proof −
show *thesis*
apply (*rule that*[*of* $\lambda i.\ if\ X = \{\}\ then\ \{\}\ else\ from\text{-}nat\text{-}into\ X\ `\ \{..i\}$])
apply (*auto simp add*: *image-iff intro*: *from-nat-into split*: *if-splits*)
using *assms from-nat-into-surj* **by** (*fastforce cong*: *image-cong*)
qed

lemma *transfer-countable*[*transfer-rule*]:
bi-unique R $\Longrightarrow$ *rel-fun* (*rel-set R*) (=) *countable countable*
by (*rule rel-funI*, *erule* (*1*) *bi-unique-rel-set-lemma*)
(*auto dest*: *countable-image-inj-on*)

21.5 Uncountable

abbreviation *uncountable* **where**
uncountable A ≡ ¬ *countable A*

lemma *uncountable-def*: *uncountable A* ⟷ *A* ≠ {} ∧ ¬ (∃*f*::(*nat* ⇒ *'a*). *range f* = *A*)
by (*auto intro*: *inj-on-inv-into simp*: *countable-def*)
(*metis all-not-in-conv inj-on-iff-surj subset-UNIV*)

lemma *uncountable-bij-betw*: *bij-betw f A B* ⟹ *uncountable B* ⟹ *uncountable A*
unfolding *bij-betw-def* **by** (*metis countable-image*)

lemma *uncountable-infinite*: *uncountable A* ⟹ *infinite A*
by (*metis countable-finite*)

lemma *uncountable-minus-countable*:
uncountable A ⟹ *countable B* ⟹ *uncountable* (*A* − *B*)
using *countable-Un*[*of B A* − *B*] **by** *auto*

lemma *countable-Diff-eq* [*simp*]: *countable* (*A* − {*x*}) = *countable A*
by (*meson countable-Diff countable-empty countable-insert uncountable-minus-countable*)

Every infinite set can be covered by a pairwise disjoint family of infinite sets. This version doesn't achieve equality, as it only covers a countable subset

lemma *infinite-infinite-partition*:
assumes *infinite A*
obtains *C* :: *nat* ⇒ *'a set*
where *pairwise* (λ*i j*. *disjnt* (*C i*) (*C j*)) *UNIV* (⋃ *i*. *C i*) ⊆ *A* ⋀*i*. *infinite* (*C i*)
proof −
obtain *f* :: *nat*⇒*'a* **where** *range f* ⊆ *A inj f*
using *assms infinite-countable-subset* **by** *blast*
let *?C* = λ*i*. *range* (λ*j*. *f* (*prod-encode* (*i*,*j*)))
show *thesis*
proof
show *pairwise* (λ*i j*. *disjnt* (*?C i*) (*?C j*)) *UNIV*
by (*auto simp*: *pairwise-def disjnt-def inj-on-eq-iff* [*OF* ‹*inj f*›] *inj-on-eq-iff* [*OF inj-prod-encode*, *of* - *UNIV*])
show (⋃ *i*. *?C i*) ⊆ *A*
using ‹*range f* ⊆ *A*› **by** *blast*
have *infinite* (*range* (λ*j*. *f* (*prod-encode* (*i*, *j*)))) **for** *i*
by (*rule range-inj-infinite*) (*meson Pair-inject* ‹*inj f*› *inj-def prod-encode-eq*)
then show ⋀*i*. *infinite* (*?C i*)
using *that* **by** *auto*
qed
qed

end

22 Countable Complete Lattices

theory *Countable-Complete-Lattices*
imports *Main Countable-Set*
begin

lemma *UNIV-nat-eq*: $UNIV = insert\ 0\ (range\ Suc)$
by (*metis UNIV-eq-I nat.nchotomy insertCI rangeI*)

class *countable-complete-lattice* = *lattice* + *Inf* + *Sup* + *bot* + *top* +
assumes *ccInf-lower*: $countable\ A \Longrightarrow x \in A \Longrightarrow Inf\ A \leq x$
assumes *ccInf-greatest*: $countable\ A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow z \leq x) \Longrightarrow z \leq Inf\ A$
assumes *ccSup-upper*: $countable\ A \Longrightarrow x \in A \Longrightarrow x \leq Sup\ A$
assumes *ccSup-least*: $countable\ A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow x \leq z) \Longrightarrow Sup\ A \leq z$
assumes *ccInf-empty* [*simp*]: $Inf\ \{\} = top$
assumes *ccSup-empty* [*simp*]: $Sup\ \{\} = bot$
begin

subclass *bounded-lattice*
proof
fix *a*
show $bot \leq a$ **by** (*auto intro*: *ccSup-least simp only*: *ccSup-empty* [*symmetric*])
show $a \leq top$ **by** (*auto intro*: *ccInf-greatest simp only*: *ccInf-empty* [*symmetric*])
qed

lemma *ccINF-lower*: $countable\ A \Longrightarrow i \in A \Longrightarrow (INF\ i \in A.\ f\ i) \leq f\ i$
using *ccInf-lower* [*of f ' A*] **by** *simp*

lemma *ccINF-greatest*: $countable\ A \Longrightarrow (\bigwedge i.\ i \in A \Longrightarrow u \leq f\ i) \Longrightarrow u \leq (INF\ i \in A.\ f\ i)$
using *ccInf-greatest* [*of f ' A*] **by** *auto*

lemma *ccSUP-upper*: $countable\ A \Longrightarrow i \in A \Longrightarrow f\ i \leq (SUP\ i \in A.\ f\ i)$
using *ccSup-upper* [*of f ' A*] **by** *simp*

lemma *ccSUP-least*: $countable\ A \Longrightarrow (\bigwedge i.\ i \in A \Longrightarrow f\ i \leq u) \Longrightarrow (SUP\ i \in A.\ f\ i) \leq u$
using *ccSup-least* [*of f ' A*] **by** *auto*

lemma *ccInf-lower2*: $countable\ A \Longrightarrow u \in A \Longrightarrow u \leq v \Longrightarrow Inf\ A \leq v$
using *ccInf-lower* [*of A u*] **by** *auto*

lemma *ccINF-lower2*: $countable\ A \Longrightarrow i \in A \Longrightarrow f\ i \leq u \Longrightarrow (INF\ i \in A.\ f\ i) \leq u$
using *ccINF-lower* [*of A i f*] **by** *auto*

lemma *ccSup-upper2*: $countable\ A \Longrightarrow u \in A \Longrightarrow v \leq u \Longrightarrow v \leq Sup\ A$
using *ccSup-upper* [*of A u*] **by** *auto*

lemma *ccSUP-upper2*: *countable A* $\Longrightarrow$ $i \in A \Longrightarrow u \leq f\ i \Longrightarrow u \leq (SUP\ i \in A.\ f\ i)$
using *ccSUP-upper* [*of A i f*] **by** *auto*

lemma *le-ccInf-iff*: *countable A* $\Longrightarrow$ $b \leq Inf\ A \longleftrightarrow (\forall\, a{\in}A.\ b \leq a)$
by (*auto intro*: *ccInf-greatest dest*: *ccInf-lower*)

lemma *le-ccINF-iff*: *countable A* $\Longrightarrow$ $u \leq (INF\ i \in A.\ f\ i) \longleftrightarrow (\forall\, i{\in}A.\ u \leq f\ i)$
using *le-ccInf-iff* [*of f ' A*] **by** *simp*

lemma *ccSup-le-iff*: *countable A* $\Longrightarrow$ $Sup\ A \leq b \longleftrightarrow (\forall\, a{\in}A.\ a \leq b)$
by (*auto intro*: *ccSup-least dest*: *ccSup-upper*)

lemma *ccSUP-le-iff*: *countable A* $\Longrightarrow$ $(SUP\ i \in A.\ f\ i) \leq u \longleftrightarrow (\forall\, i{\in}A.\ f\ i \leq u)$
using *ccSup-le-iff* [*of f ' A*] **by** *simp*

lemma *ccInf-insert* [*simp*]: *countable A* $\Longrightarrow$ $Inf\ (insert\ a\ A) = inf\ a\ (Inf\ A)$
by (*force intro*: *le-infI le-infI1 le-infI2 order.antisym ccInf-greatest ccInf-lower*)

lemma *ccINF-insert* [*simp*]: *countable A* $\Longrightarrow$ $(INF\ x{\in}insert\ a\ A.\ f\ x) = inf\ (f\ a)\ (Inf\ (f\ `\ A))$
unfolding *image-insert* **by** *simp*

lemma *ccSup-insert* [*simp*]: *countable A* $\Longrightarrow$ $Sup\ (insert\ a\ A) = sup\ a\ (Sup\ A)$
by (*force intro*: *le-supI le-supI1 le-supI2 order.antisym ccSup-least ccSup-upper*)

lemma *ccSUP-insert* [*simp*]: *countable A* $\Longrightarrow$ $(SUP\ x{\in}insert\ a\ A.\ f\ x) = sup\ (f\ a)\ (Sup\ (f\ `\ A))$
unfolding *image-insert* **by** *simp*

lemma *ccINF-empty* [*simp*]: $(INF\ x{\in}\{\}.\ f\ x) = top$
unfolding *image-empty* **by** *simp*

lemma *ccSUP-empty* [*simp*]: $(SUP\ x{\in}\{\}.\ f\ x) = bot$
unfolding *image-empty* **by** *simp*

lemma *ccInf-superset-mono*: *countable A* $\Longrightarrow$ $B \subseteq A \Longrightarrow Inf\ A \leq Inf\ B$
by (*auto intro*: *ccInf-greatest ccInf-lower countable-subset*)

lemma *ccSup-subset-mono*: *countable B* $\Longrightarrow$ $A \subseteq B \Longrightarrow Sup\ A \leq Sup\ B$
by (*auto intro*: *ccSup-least ccSup-upper countable-subset*)

lemma *ccInf-mono*:
assumes [*intro*]: *countable B countable A*
assumes $\bigwedge b.\ b \in B \Longrightarrow \exists\, a{\in}A.\ a \leq b$
shows $Inf\ A \leq Inf\ B$
proof (*rule ccInf-greatest*)
fix *b* **assume** $b \in B$
with *assms* **obtain** *a* **where** $a \in A$ **and** $a \leq b$ **by** *blast*

from ‹$a \in A$› **have** *Inf A* $\leq a$ **by** (*rule ccInf-lower*[*rotated*]) *auto*
with ‹$a \leq b$› **show** *Inf A* $\leq b$ **by** *auto*
qed *auto*

lemma *ccINF-mono*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ ($\bigwedge m.\ m \in B \Longrightarrow \exists\, n{\in}A.\ f\ n \leq g\ m$) $\Longrightarrow$ (*INF* $n{\in}A.\ f\ n$) $\leq$ (*INF* $n{\in}B.\ g\ n$)
using *ccInf-mono* [*of g ' B f ' A*] **by** *auto*

lemma *ccSup-mono*:
assumes [*intro*]: *countable B countable A*
assumes $\bigwedge a.\ a \in A \Longrightarrow \exists\, b{\in}B.\ a \leq b$
shows *Sup A* $\leq$ *Sup B*
proof (*rule ccSup-least*)
fix *a* **assume** $a \in A$
with *assms* **obtain** *b* **where** $b \in B$ **and** $a \leq b$ **by** *blast*
from ‹$b \in B$› **have** $b \leq$ *Sup B* **by** (*rule ccSup-upper*[*rotated*]) *auto*
with ‹$a \leq b$› **show** $a \leq$ *Sup B* **by** *auto*
qed *auto*

lemma *ccSUP-mono*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ ($\bigwedge n.\ n \in A \Longrightarrow \exists\, m{\in}B.\ f\ n \leq g\ m$) $\Longrightarrow$ (*SUP* $n{\in}A.\ f\ n$) $\leq$ (*SUP* $n{\in}B.\ g\ n$)
using *ccSup-mono* [*of g ' B f ' A*] **by** *auto*

lemma *ccINF-superset-mono*:
countable A $\Longrightarrow B \subseteq A \Longrightarrow$ ($\bigwedge x.\ x \in B \Longrightarrow f\ x \leq g\ x$) $\Longrightarrow$ (*INF* $x{\in}A.\ f\ x$) $\leq$ (*INF* $x{\in}B.\ g\ x$)
by (*blast intro*: *ccINF-mono countable-subset dest*: *subsetD*)

lemma *ccSUP-subset-mono*:
countable B $\Longrightarrow A \subseteq B \Longrightarrow$ ($\bigwedge x.\ x \in A \Longrightarrow f\ x \leq g\ x$) $\Longrightarrow$ (*SUP* $x{\in}A.\ f\ x$) $\leq$ (*SUP* $x{\in}B.\ g\ x$)
by (*blast intro*: *ccSUP-mono countable-subset dest*: *subsetD*)

lemma *less-eq-ccInf-inter*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *sup* (*Inf A*) (*Inf B*) $\leq$ *Inf* ($A \cap B$)
by (*auto intro*: *ccInf-greatest ccInf-lower*)

lemma *ccSup-inter-less-eq*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *Sup* ($A \cap B$) $\leq$ *inf* (*Sup A*) (*Sup B*)
by (*auto intro*: *ccSup-least ccSup-upper*)

lemma *ccInf-union-distrib*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *Inf* ($A \cup B$) = *inf* (*Inf A*) (*Inf B*)
by (*rule order.antisym*) (*auto intro*: *ccInf-greatest ccInf-lower le-infI1 le-infI2*)

lemma *ccINF-union*:

countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ (*INF i*$\in$*A* $\cup$ *B. M i*) = *inf* (*INF i*$\in$*A. M i*) (*INF i*$\in$*B. M i*)
by (*auto intro*!: *order.antisym ccINF-mono intro*: *le-infI1 le-infI2 ccINF-greatest ccINF-lower*)

lemma *ccSup-union-distrib*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ *Sup* (*A* $\cup$ *B*) = *sup* (*Sup A*) (*Sup B*)
by (*rule order.antisym*) (*auto intro*: *ccSup-least ccSup-upper le-supI1 le-supI2*)

lemma *ccSUP-union*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ (*SUP i*$\in$*A* $\cup$ *B. M i*) = *sup* (*SUP i*$\in$*A. M i*) (*SUP i*$\in$*B. M i*)
by (*auto intro*!: *order.antisym ccSUP-mono intro*: *le-supI1 le-supI2 ccSUP-least ccSUP-upper*)

lemma *ccINF-inf-distrib*: *countable A* $\Longrightarrow$ *inf* (*INF a*$\in$*A. f a*) (*INF a*$\in$*A. g a*) = (*INF a*$\in$*A. inf* (*f a*) (*g a*))
by (*rule order.antisym*) (*rule ccINF-greatest*, *auto intro*: *le-infI1 le-infI2 ccINF-lower ccINF-mono*)

lemma *ccSUP-sup-distrib*: *countable A* $\Longrightarrow$ *sup* (*SUP a*$\in$*A. f a*) (*SUP a*$\in$*A. g a*) = (*SUP a*$\in$*A. sup* (*f a*) (*g a*))
by (*rule order.antisym*[*rotated*]) (*rule ccSUP-least*, *auto intro*: *le-supI1 le-supI2 ccSUP-upper ccSUP-mono*)

lemma *ccINF-const* [*simp*]: *A* $\neq$ {} $\Longrightarrow$ (*INF i* $\in$ *A. f*) = *f*
unfolding *image-constant-conv* **by** *auto*

lemma *ccSUP-const* [*simp*]: *A* $\neq$ {} $\Longrightarrow$ (*SUP i* $\in$ *A. f*) = *f*
unfolding *image-constant-conv* **by** *auto*

lemma *ccINF-top* [*simp*]: (*INF x*$\in$*A. top*) = *top*
by (*cases A* = {}) *simp-all*

lemma *ccSUP-bot* [*simp*]: (*SUP x*$\in$*A. bot*) = *bot*
by (*cases A* = {}) *simp-all*

lemma *ccINF-commute*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ (*INF i*$\in$*A. INF j*$\in$*B. f i j*) = (*INF j*$\in$*B. INF i*$\in$*A. f i j*)
by (*iprover intro*: *ccINF-lower ccINF-greatest order-trans order.antisym*)

lemma *ccSUP-commute*: *countable A* $\Longrightarrow$ *countable B* $\Longrightarrow$ (*SUP i*$\in$*A. SUP j*$\in$*B. f i j*) = (*SUP j*$\in$*B. SUP i*$\in$*A. f i j*)
by (*iprover intro*: *ccSUP-upper ccSUP-least order-trans order.antisym*)

end

context
fixes *a* :: *'a*::{*countable-complete-lattice*, *linorder*}

begin

lemma *less-ccSup-iff*: *countable* $S \Longrightarrow a < Sup\ S \longleftrightarrow (\exists\, x{\in}S.\ a < x)$
unfolding *not-le* [*symmetric*] **by** (*subst ccSup-le-iff*) *auto*

lemma *less-ccSUP-iff*: *countable* $A \Longrightarrow a < (SUP\ i{\in}A.\ f\ i) \longleftrightarrow (\exists\, x{\in}A.\ a < f\ x)$
using *less-ccSup-iff* [*of f ' A*] **by** *simp*

lemma *ccInf-less-iff*: *countable* $S \Longrightarrow Inf\ S < a \longleftrightarrow (\exists\, x{\in}S.\ x < a)$
unfolding *not-le* [*symmetric*] **by** (*subst le-ccInf-iff*) *auto*

lemma *ccINF-less-iff*: *countable* $A \Longrightarrow (INF\ i{\in}A.\ f\ i) < a \longleftrightarrow (\exists\, x{\in}A.\ f\ x < a)$
using *ccInf-less-iff* [*of f ' A*] **by** *simp*

end

class *countable-complete-distrib-lattice* = *countable-complete-lattice* +
assumes *sup-ccInf*: *countable* $B \Longrightarrow sup\ a\ (Inf\ B) = (INF\ b{\in}B.\ sup\ a\ b)$
assumes *inf-ccSup*: *countable* $B \Longrightarrow inf\ a\ (Sup\ B) = (SUP\ b{\in}B.\ inf\ a\ b)$
begin

lemma *sup-ccINF*:
countable $B \Longrightarrow sup\ a\ (INF\ b{\in}B.\ f\ b) = (INF\ b{\in}B.\ sup\ a\ (f\ b))$
by (*simp only*: *sup-ccInf image-image countable-image*)

lemma *inf-ccSUP*:
countable $B \Longrightarrow inf\ a\ (SUP\ b{\in}B.\ f\ b) = (SUP\ b{\in}B.\ inf\ a\ (f\ b))$
by (*simp only*: *inf-ccSup image-image countable-image*)

subclass *distrib-lattice*
proof
fix *a b c*
from *sup-ccInf*[*of* {*b*, *c*} *a*] **have** $sup\ a\ (Inf\ \{b, c\}) = (INF\ d{\in}\{b, c\}.\ sup\ a\ d)$
by *simp*
then show $sup\ a\ (inf\ b\ c) = inf\ (sup\ a\ b)\ (sup\ a\ c)$
by *simp*
qed

lemma *ccInf-sup*:
countable $B \Longrightarrow sup\ (Inf\ B)\ a = (INF\ b{\in}B.\ sup\ b\ a)$
by (*simp add*: *sup-ccInf sup-commute*)

lemma *ccSup-inf*:
countable $B \Longrightarrow inf\ (Sup\ B)\ a = (SUP\ b{\in}B.\ inf\ b\ a)$
by (*simp add*: *inf-ccSup inf-commute*)

lemma *ccINF-sup*:
countable $B \Longrightarrow sup\ (INF\ b{\in}B.\ f\ b)\ a = (INF\ b{\in}B.\ sup\ (f\ b)\ a)$
by (*simp add*: *sup-ccINF sup-commute*)

lemma *ccSUP-inf*:
countable B $\Longrightarrow$ *inf* (*SUP b*$\in$*B. f b*) *a* = (*SUP b*$\in$*B. inf* (*f b*) *a*)
by (*simp add*: *inf-ccSUP inf-commute*)

lemma *ccINF-sup-distrib2*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ *sup* (*INF a*$\in$*A. f a*) (*INF b*$\in$*B. g b*) = (*INF a*$\in$*A. INF b*$\in$*B. sup* (*f a*) (*g b*))
by (*subst ccINF-commute*) (*simp-all add*: *sup-ccINF ccINF-sup*)

lemma *ccSUP-inf-distrib2*:
countable A $\Longrightarrow$ *countable B* $\Longrightarrow$ *inf* (*SUP a*$\in$*A. f a*) (*SUP b*$\in$*B. g b*) = (*SUP a*$\in$*A. SUP b*$\in$*B. inf* (*f a*) (*g b*))
by (*subst ccSUP-commute*) (*simp-all add*: *inf-ccSUP ccSUP-inf*)

context
fixes *f* :: *'a* $\Rightarrow$ *'b*::*countable-complete-lattice*
assumes *mono f*
begin

lemma *mono-ccInf*:
countable A $\Longrightarrow$ *f* (*Inf A*) $\leq$ (*INF x*$\in$*A. f x*)
using ‹*mono f*›
by (*auto intro*!: *countable-complete-lattice-class.ccINF-greatest intro*: *ccInf-lower dest*: *monoD*)

lemma *mono-ccSup*:
countable A $\Longrightarrow$ (*SUP x*$\in$*A. f x*) $\leq$ *f* (*Sup A*)
using ‹*mono f*› **by** (*auto intro*: *countable-complete-lattice-class.ccSUP-least ccSup-upper dest*: *monoD*)

lemma *mono-ccINF*:
countable I $\Longrightarrow$ *f* (*INF i* $\in$ *I. A i*) $\leq$ (*INF x* $\in$ *I. f* (*A x*))
by (*intro countable-complete-lattice-class.ccINF-greatest monoD*[*OF* ‹*mono f*›] *ccINF-lower*)

lemma *mono-ccSUP*:
countable I $\Longrightarrow$ (*SUP x* $\in$ *I. f* (*A x*)) $\leq$ *f* (*SUP i* $\in$ *I. A i*)
by (*intro countable-complete-lattice-class.ccSUP-least monoD*[*OF* ‹*mono f*›] *ccSUP-upper*)

end

end

22.0.1 Instances of countable complete lattices

instance *fun* :: (*type*, *countable-complete-lattice*) *countable-complete-lattice*
by *standard*

(*auto simp*: *le-fun-def intro*!: *ccSUP-upper ccSUP-least ccINF-lower ccINF-greatest*)

subclass (**in** *complete-lattice*) *countable-complete-lattice*
by *standard* (*auto intro*: *Sup-upper Sup-least Inf-lower Inf-greatest*)

subclass (**in** *complete-distrib-lattice*) *countable-complete-distrib-lattice*
by *standard* (*auto intro*: *sup-Inf inf-Sup*)

end

23 Type of (at Most) Countable Sets

theory *Countable-Set-Type*
imports *Countable-Set*
begin

23.1 Cardinal stuff

context
includes *cardinal-syntax*
begin

lemma *countable-card-of-nat*: *countable A* $\longleftrightarrow$ $|A| \leq o$ $|UNIV::nat\ set|$
unfolding *countable-def card-of-ordLeq*[*symmetric*] **by** *auto*

lemma *countable-card-le-natLeq*: *countable A* $\longleftrightarrow$ $|A| \leq o$ *natLeq*
unfolding *countable-card-of-nat* **using** *card-of-nat ordLeq-ordIso-trans ordIso-symmetric*
by *blast*

lemma *countable-or-card-of*:
assumes *countable A*
shows (*finite A* $\wedge$ $|A| <o$ $|UNIV::nat\ set|$) $\vee$
(*infinite A* $\wedge$ $|A| =o$ $|UNIV::nat\ set|$)
by (*metis assms countable-card-of-nat infinite-iff-card-of-nat ordIso-iff-ordLeq*
ordLeq-iff-ordLess-or-ordIso)

lemma *countable-cases-card-of*[*elim*]:
assumes *countable A*
obtains (*Fin*) *finite A* $|A| <o$ $|UNIV::nat\ set|$
| (*Inf*) *infinite A* $|A| =o$ $|UNIV::nat\ set|$
using *assms countable-or-card-of* **by** *blast*

lemma *countable-or*:
countable A $\Longrightarrow$ ($\exists$ *f*::'*a*$\Rightarrow$*nat*. *finite A* $\wedge$ *inj-on f A*) $\vee$ ($\exists$ *f*::'*a*$\Rightarrow$*nat*. *infinite A*
$\wedge$ *bij-betw f A UNIV*)
by (*elim countable-enum-cases*) *fastforce*+

lemma *countable-cases*[*elim*]:
assumes *countable A*

obtains (*Fin*) $f :: 'a \Rightarrow nat$ **where** *finite A inj-on f A*
| (*Inf*) $f :: 'a \Rightarrow nat$ **where** *infinite A bij-betw f A UNIV*
using *assms countable-or* **by** *metis*

lemma *countable-ordLeq*:
assumes $|A| \leq_o |B|$ **and** *countable B*
shows *countable A*
using *assms* **unfolding** *countable-card-of-nat* **by**(*rule ordLeq-transitive*)

lemma *countable-ordLess*:
assumes *AB*: $|A| <_o |B|$ **and** *B*: *countable B*
shows *countable A*
using *countable-ordLeq*[*OF ordLess-imp-ordLeq*[*OF AB*] *B*] **.**

end

23.2 The type of countable sets

typedef $'a$ *cset* = $\{A :: 'a$ *set. countable A*$\}$ **morphisms** *rcset acset*
by (*rule exI*[*of* - {}]) *simp*

setup-lifting *type-definition-cset*

declare
rcset-inverse[*simp*]
acset-inverse[*Transfer.transferred*, *unfolded mem-Collect-eq*, *simp*]
acset-inject[*Transfer.transferred*, *unfolded mem-Collect-eq*, *simp*]
rcset[*Transfer.transferred*, *unfolded mem-Collect-eq*, *simp*]

instantiation *cset* :: (*type*) {*bounded-lattice-bot*, *distrib-lattice*, *minus*}
begin

lift-definition *bot-cset* :: $'a$ *cset* **is** {} **parametric** *empty-transfer* **by** *simp*

lift-definition *less-eq-cset* :: $'a$ *cset* $\Rightarrow$ $'a$ *cset* $\Rightarrow$ *bool*
is *subset-eq* **parametric** *subset-transfer* **.**

definition *less-cset* :: $'a$ *cset* $\Rightarrow$ $'a$ *cset* $\Rightarrow$ *bool*
where $xs < ys \equiv xs \leq ys \wedge xs \neq (ys::'a\ cset)$

lemma *less-cset-transfer*[*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *bi-unique A*
shows ((*pcr-cset A*) ===> (*pcr-cset A*) ===> (=)) ($\subset$) (<)
unfolding *less-cset-def*[*abs-def*] *psubset-eq*[*abs-def*] **by** *transfer-prover*

lift-definition *sup-cset* :: $'a$ *cset* $\Rightarrow$ $'a$ *cset* $\Rightarrow$ $'a$ *cset*
is *union* **parametric** *union-transfer* **by** *simp*

lift-definition *inf-cset* :: *'a cset ⇒ 'a cset ⇒ 'a cset*
is *inter* **parametric** *inter-transfer* **by** *simp*

lift-definition *minus-cset* :: *'a cset ⇒ 'a cset ⇒ 'a cset*
is *minus* **parametric** *Diff-transfer* **by** *simp*

instance by *standard* (*transfer*; *auto*)+

end

abbreviation *cempty* :: *'a cset* **where** *cempty* ≡ *bot*
abbreviation *csubset-eq* :: *'a cset ⇒ 'a cset ⇒ bool* **where** *csubset-eq xs ys* ≡ *xs* ≤ *ys*
abbreviation *csubset* :: *'a cset ⇒ 'a cset ⇒ bool* **where** *csubset xs ys* ≡ *xs* < *ys*
abbreviation *cUn* :: *'a cset ⇒ 'a cset ⇒ 'a cset* **where** *cUn xs ys* ≡ *sup xs ys*
abbreviation *cInt* :: *'a cset ⇒ 'a cset ⇒ 'a cset* **where** *cInt xs ys* ≡ *inf xs ys*
abbreviation *cDiff* :: *'a cset ⇒ 'a cset ⇒ 'a cset* **where** *cDiff xs ys* ≡ *minus xs ys*

lift-definition *cin* :: *'a ⇒ 'a cset ⇒ bool* **is** (∈) **parametric** *member-transfer*
.
lift-definition *cinsert* :: *'a ⇒ 'a cset ⇒ 'a cset* **is** *insert* **parametric** *Lifting-Set.insert-transfer*
by (*rule countable-insert*)
abbreviation *csingle* :: *'a ⇒ 'a cset* **where** *csingle x* ≡ *cinsert x cempty*
lift-definition *cimage* :: (*'a ⇒ 'b*) ⇒ *'a cset ⇒ 'b cset* **is** (‘) **parametric** *image-transfer*
by (*rule countable-image*)
lift-definition *cBall* :: *'a cset* ⇒ (*'a ⇒ bool*) ⇒ *bool* **is** *Ball* **parametric** *Ball-transfer*
.
lift-definition *cBex* :: *'a cset* ⇒ (*'a ⇒ bool*) ⇒ *bool* **is** *Bex* **parametric** *Bex-transfer*
.
lift-definition *cUnion* :: *'a cset cset ⇒ 'a cset* **is** *Union* **parametric** *Union-transfer*
using *countable-UN* [*of - id*] **by** *auto*
abbreviation (*input*) *cUNION* :: *'a cset ⇒* (*'a ⇒ 'b cset*) ⇒ *'b cset*
where *cUNION A f* ≡ *cUnion* (*cimage f A*)

lemma *Union-conv-UNION*: ⋃ *A* = ⋃ (*id ‘ A*)
by *simp*

lemmas *cset-eqI* = *set-eqI*[*Transfer.transferred*]
lemmas *cset-eq-iff*[*no-atp*] = *set-eq-iff*[*Transfer.transferred*]
lemmas *cBallI*[*intro*!] = *ballI*[*Transfer.transferred*]
lemmas *cbspec*[*dest?*] = *bspec*[*Transfer.transferred*]
lemmas *cBallE*[*elim*] = *ballE*[*Transfer.transferred*]
lemmas *cBexI*[*intro*] = *bexI*[*Transfer.transferred*]
lemmas *rev-cBexI*[*intro?*] = *rev-bexI*[*Transfer.transferred*]
lemmas *cBexCI* = *bexCI*[*Transfer.transferred*]
lemmas *cBexE*[*elim*!] = *bexE*[*Transfer.transferred*]
lemmas *cBall-triv*[*simp*] = *ball-triv*[*Transfer.transferred*]

lemmas *cBex-triv*[*simp*] = *bex-triv*[*Transfer.transferred*]
lemmas *cBex-triv-one-point1*[*simp*] = *bex-triv-one-point1*[*Transfer.transferred*]
lemmas *cBex-triv-one-point2*[*simp*] = *bex-triv-one-point2*[*Transfer.transferred*]
lemmas *cBex-one-point1*[*simp*] = *bex-one-point1*[*Transfer.transferred*]
lemmas *cBex-one-point2*[*simp*] = *bex-one-point2*[*Transfer.transferred*]
lemmas *cBall-one-point1*[*simp*] = *ball-one-point1*[*Transfer.transferred*]
lemmas *cBall-one-point2*[*simp*] = *ball-one-point2*[*Transfer.transferred*]
lemmas *cBall-conj-distrib* = *ball-conj-distrib*[*Transfer.transferred*]
lemmas *cBex-disj-distrib* = *bex-disj-distrib*[*Transfer.transferred*]
lemmas *cBall-cong* = *ball-cong*[*Transfer.transferred*]
lemmas *cBex-cong* = *bex-cong*[*Transfer.transferred*]
lemmas *csubsetI*[*intro*!] = *subsetI*[*Transfer.transferred*]
lemmas *csubsetD*[*elim*, *intro?*] = *subsetD*[*Transfer.transferred*]
lemmas *rev-csubsetD*[*no-atp*,*intro?*] = *rev-subsetD*[*Transfer.transferred*]
lemmas *csubsetCE*[*no-atp*,*elim*] = *subsetCE*[*Transfer.transferred*]
lemmas *csubset-eq*[*no-atp*] = *subset-eq*[*Transfer.transferred*]
lemmas *contra-csubsetD*[*no-atp*] = *contra-subsetD*[*Transfer.transferred*]
lemmas *csubset-refl* = *subset-refl*[*Transfer.transferred*]
lemmas *csubset-trans* = *subset-trans*[*Transfer.transferred*]
lemmas *cset-rev-mp* = *rev-subsetD*[*Transfer.transferred*]
lemmas *cset-mp* = *subsetD*[*Transfer.transferred*]
lemmas *csubset-not-fsubset-eq*[*code*] = *subset-not-subset-eq*[*Transfer.transferred*]
lemmas *eq-cmem-trans* = *eq-mem-trans*[*Transfer.transferred*]
lemmas *csubset-antisym*[*intro*!] = *subset-antisym*[*Transfer.transferred*]
lemmas *cequalityD1* = *equalityD1*[*Transfer.transferred*]
lemmas *cequalityD2* = *equalityD2*[*Transfer.transferred*]
lemmas *cequalityE* = *equalityE*[*Transfer.transferred*]
lemmas *cequalityCE*[*elim*] = *equalityCE*[*Transfer.transferred*]
lemmas *eqcset-imp-iff* = *eqset-imp-iff*[*Transfer.transferred*]
lemmas *eqcelem-imp-iff* = *eqelem-imp-iff*[*Transfer.transferred*]
lemmas *cempty-iff*[*simp*] = *empty-iff*[*Transfer.transferred*]
lemmas *cempty-fsubsetI*[*iff*] = *empty-subsetI*[*Transfer.transferred*]
lemmas *equals-cemptyI* = *equals0I*[*Transfer.transferred*]
lemmas *equals-cemptyD* = *equals0D*[*Transfer.transferred*]
lemmas *cBall-cempty*[*simp*] = *ball-empty*[*Transfer.transferred*]
lemmas *cBex-cempty*[*simp*] = *bex-empty*[*Transfer.transferred*]
lemmas *cInt-iff*[*simp*] = *Int-iff*[*Transfer.transferred*]
lemmas *cIntI*[*intro*!] = *IntI*[*Transfer.transferred*]
lemmas *cIntD1* = *IntD1*[*Transfer.transferred*]
lemmas *cIntD2* = *IntD2*[*Transfer.transferred*]
lemmas *cIntE*[*elim*!] = *IntE*[*Transfer.transferred*]
lemmas *cUn-iff*[*simp*] = *Un-iff*[*Transfer.transferred*]
lemmas *cUnI1*[*elim?*] = *UnI1*[*Transfer.transferred*]
lemmas *cUnI2*[*elim?*] = *UnI2*[*Transfer.transferred*]
lemmas *cUnCI*[*intro*!] = *UnCI*[*Transfer.transferred*]
lemmas *cuUnE*[*elim*!] = *UnE*[*Transfer.transferred*]
lemmas *cDiff-iff*[*simp*] = *Diff-iff*[*Transfer.transferred*]
lemmas *cDiffI*[*intro*!] = *DiffI*[*Transfer.transferred*]
lemmas *cDiffD1* = *DiffD1*[*Transfer.transferred*]

lemmas *cDiffD2* = *DiffD2*[*Transfer.transferred*]
lemmas *cDiffE*[*elim*!] = *DiffE*[*Transfer.transferred*]
lemmas *cinsert-iff*[*simp*] = *insert-iff*[*Transfer.transferred*]
lemmas *cinsertI1* = *insertI1*[*Transfer.transferred*]
lemmas *cinsertI2* = *insertI2*[*Transfer.transferred*]
lemmas *cinsertE*[*elim*!] = *insertE*[*Transfer.transferred*]
lemmas *cinsertCI*[*intro*!] = *insertCI*[*Transfer.transferred*]
lemmas *csubset-cinsert-iff* = *subset-insert-iff*[*Transfer.transferred*]
lemmas *cinsert-ident* = *insert-ident*[*Transfer.transferred*]
lemmas *csingletonI*[*intro*!,*no-atp*] = *singletonI*[*Transfer.transferred*]
lemmas *csingletonD*[*dest*!,*no-atp*] = *singletonD*[*Transfer.transferred*]
lemmas *fsingletonE* = *csingletonD* [*elim-format*]
lemmas *csingleton-iff* = *singleton-iff*[*Transfer.transferred*]
lemmas *csingleton-inject*[*dest*!] = *singleton-inject*[*Transfer.transferred*]
lemmas *csingleton-finsert-inj-eq*[*iff*,*no-atp*] = *singleton-insert-inj-eq*[*Transfer.transferred*]
lemmas *csingleton-finsert-inj-eq'*[*iff*,*no-atp*] = *singleton-insert-inj-eq'*[*Transfer.transferred*]
lemmas *csubset-csingletonD* = *subset-singletonD*[*Transfer.transferred*]
lemmas *cDiff-single-cinsert* = *Diff-single-insert*[*Transfer.transferred*]
lemmas *cdoubleton-eq-iff* = *doubleton-eq-iff*[*Transfer.transferred*]
lemmas *cUn-csingleton-iff* = *Un-singleton-iff*[*Transfer.transferred*]
lemmas *csingleton-cUn-iff* = *singleton-Un-iff*[*Transfer.transferred*]
lemmas *cimage-eqI*[*simp*, *intro*] = *image-eqI*[*Transfer.transferred*]
lemmas *cimageI* = *imageI*[*Transfer.transferred*]
lemmas *rev-cimage-eqI* = *rev-image-eqI*[*Transfer.transferred*]
lemmas *cimageE*[*elim*!] = *imageE*[*Transfer.transferred*]
lemmas *Compr-cimage-eq* = *Compr-image-eq*[*Transfer.transferred*]
lemmas *cimage-cUn* = *image-Un*[*Transfer.transferred*]
lemmas *cimage-iff* = *image-iff*[*Transfer.transferred*]
lemmas *cimage-csubset-iff*[*no-atp*] = *image-subset-iff*[*Transfer.transferred*]
lemmas *cimage-csubsetI* = *image-subsetI*[*Transfer.transferred*]
lemmas *cimage-ident*[*simp*] = *image-ident*[*Transfer.transferred*]
lemmas *if-split-cin1* = *if-split-mem1*[*Transfer.transferred*]
lemmas *if-split-cin2* = *if-split-mem2*[*Transfer.transferred*]
lemmas *cpsubsetI*[*intro*!,*no-atp*] = *psubsetI*[*Transfer.transferred*]
lemmas *cpsubsetE*[*elim*!,*no-atp*] = *psubsetE*[*Transfer.transferred*]
lemmas *cpsubset-finsert-iff* = *psubset-insert-iff*[*Transfer.transferred*]
lemmas *cpsubset-eq* = *psubset-eq*[*Transfer.transferred*]
lemmas *cpsubset-imp-fsubset* = *psubset-imp-subset*[*Transfer.transferred*]
lemmas *cpsubset-trans* = *psubset-trans*[*Transfer.transferred*]
lemmas *cpsubsetD* = *psubsetD*[*Transfer.transferred*]
lemmas *cpsubset-csubset-trans* = *psubset-subset-trans*[*Transfer.transferred*]
lemmas *csubset-cpsubset-trans* = *subset-psubset-trans*[*Transfer.transferred*]
lemmas *cpsubset-imp-ex-fmem* = *psubset-imp-ex-mem*[*Transfer.transferred*]
lemmas *csubset-cinsertI* = *subset-insertI*[*Transfer.transferred*]
lemmas *csubset-cinsertI2* = *subset-insertI2*[*Transfer.transferred*]
lemmas *csubset-cinsert* = *subset-insert*[*Transfer.transferred*]
lemmas *cUn-upper1* = *Un-upper1*[*Transfer.transferred*]
lemmas *cUn-upper2* = *Un-upper2*[*Transfer.transferred*]
lemmas *cUn-least* = *Un-least*[*Transfer.transferred*]

lemmas *cInt-lower1* = *Int-lower1*[*Transfer.transferred*]
lemmas *cInt-lower2* = *Int-lower2*[*Transfer.transferred*]
lemmas *cInt-greatest* = *Int-greatest*[*Transfer.transferred*]
lemmas *cDiff-csubset* = *Diff-subset*[*Transfer.transferred*]
lemmas *cDiff-csubset-conv* = *Diff-subset-conv*[*Transfer.transferred*]
lemmas *csubset-cempty*[*simp*] = *subset-empty*[*Transfer.transferred*]
lemmas *not-cpsubset-cempty*[*iff*] = *not-psubset-empty*[*Transfer.transferred*]
lemmas *cinsert-is-cUn* = *insert-is-Un*[*Transfer.transferred*]
lemmas *cinsert-not-cempty*[*simp*] = *insert-not-empty*[*Transfer.transferred*]
lemmas *cempty-not-cinsert* = *empty-not-insert*[*Transfer.transferred*]
lemmas *cinsert-absorb* = *insert-absorb*[*Transfer.transferred*]
lemmas *cinsert-absorb2*[*simp*] = *insert-absorb2*[*Transfer.transferred*]
lemmas *cinsert-commute* = *insert-commute*[*Transfer.transferred*]
lemmas *cinsert-csubset*[*simp*] = *insert-subset*[*Transfer.transferred*]
lemmas *cinsert-cinter-cinsert*[*simp*] = *insert-inter-insert*[*Transfer.transferred*]
lemmas *cinsert-disjoint*[*simp*,*no-atp*] = *insert-disjoint*[*Transfer.transferred*]
lemmas *disjoint-cinsert*[*simp*,*no-atp*] = *disjoint-insert*[*Transfer.transferred*]
lemmas *cimage-cempty*[*simp*] = *image-empty*[*Transfer.transferred*]
lemmas *cimage-cinsert*[*simp*] = *image-insert*[*Transfer.transferred*]
lemmas *cimage-constant* = *image-constant*[*Transfer.transferred*]
lemmas *cimage-constant-conv* = *image-constant-conv*[*Transfer.transferred*]
lemmas *cimage-cimage* = *image-image*[*Transfer.transferred*]
lemmas *cinsert-cimage*[*simp*] = *insert-image*[*Transfer.transferred*]
lemmas *cimage-is-cempty*[*iff*] = *image-is-empty*[*Transfer.transferred*]
lemmas *cempty-is-cimage*[*iff*] = *empty-is-image*[*Transfer.transferred*]
lemmas *cimage-cong* = *image-cong*[*Transfer.transferred*]
lemmas *cimage-cInt-csubset* = *image-Int-subset*[*Transfer.transferred*]
lemmas *cimage-cDiff-csubset* = *image-diff-subset*[*Transfer.transferred*]
lemmas *cInt-absorb* = *Int-absorb*[*Transfer.transferred*]
lemmas *cInt-left-absorb* = *Int-left-absorb*[*Transfer.transferred*]
lemmas *cInt-commute* = *Int-commute*[*Transfer.transferred*]
lemmas *cInt-left-commute* = *Int-left-commute*[*Transfer.transferred*]
lemmas *cInt-assoc* = *Int-assoc*[*Transfer.transferred*]
lemmas *cInt-ac* = *Int-ac*[*Transfer.transferred*]
lemmas *cInt-absorb1* = *Int-absorb1*[*Transfer.transferred*]
lemmas *cInt-absorb2* = *Int-absorb2*[*Transfer.transferred*]
lemmas *cInt-cempty-left* = *Int-empty-left*[*Transfer.transferred*]
lemmas *cInt-cempty-right* = *Int-empty-right*[*Transfer.transferred*]
lemmas *disjoint-iff-cnot-equal* = *disjoint-iff-not-equal*[*Transfer.transferred*]
lemmas *cInt-cUn-distrib* = *Int-Un-distrib*[*Transfer.transferred*]
lemmas *cInt-cUn-distrib2* = *Int-Un-distrib2*[*Transfer.transferred*]
lemmas *cInt-csubset-iff*[*no-atp*, *simp*] = *Int-subset-iff*[*Transfer.transferred*]
lemmas *cUn-absorb* = *Un-absorb*[*Transfer.transferred*]
lemmas *cUn-left-absorb* = *Un-left-absorb*[*Transfer.transferred*]
lemmas *cUn-commute* = *Un-commute*[*Transfer.transferred*]
lemmas *cUn-left-commute* = *Un-left-commute*[*Transfer.transferred*]
lemmas *cUn-assoc* = *Un-assoc*[*Transfer.transferred*]
lemmas *cUn-ac* = *Un-ac*[*Transfer.transferred*]
lemmas *cUn-absorb1* = *Un-absorb1*[*Transfer.transferred*]

lemmas *cUn-absorb2* = *Un-absorb2*[*Transfer.transferred*]
lemmas *cUn-cempty-left* = *Un-empty-left*[*Transfer.transferred*]
lemmas *cUn-cempty-right* = *Un-empty-right*[*Transfer.transferred*]
lemmas *cUn-cinsert-left*[*simp*] = *Un-insert-left*[*Transfer.transferred*]
lemmas *cUn-cinsert-right*[*simp*] = *Un-insert-right*[*Transfer.transferred*]
lemmas *cInt-cinsert-left* = *Int-insert-left*[*Transfer.transferred*]
lemmas *cInt-cinsert-left-if0*[*simp*] = *Int-insert-left-if0*[*Transfer.transferred*]
lemmas *cInt-cinsert-left-if1*[*simp*] = *Int-insert-left-if1*[*Transfer.transferred*]
lemmas *cInt-cinsert-right* = *Int-insert-right*[*Transfer.transferred*]
lemmas *cInt-cinsert-right-if0*[*simp*] = *Int-insert-right-if0*[*Transfer.transferred*]
lemmas *cInt-cinsert-right-if1*[*simp*] = *Int-insert-right-if1*[*Transfer.transferred*]
lemmas *cUn-cInt-distrib* = *Un-Int-distrib*[*Transfer.transferred*]
lemmas *cUn-cInt-distrib2* = *Un-Int-distrib2*[*Transfer.transferred*]
lemmas *cUn-cInt-crazy* = *Un-Int-crazy*[*Transfer.transferred*]
lemmas *csubset-cUn-eq* = *subset-Un-eq*[*Transfer.transferred*]
lemmas *cUn-cempty*[*iff*] = *Un-empty*[*Transfer.transferred*]
lemmas *cUn-csubset-iff*[*no-atp*, *simp*] = *Un-subset-iff*[*Transfer.transferred*]
lemmas *cUn-cDiff-cInt* = *Un-Diff-Int*[*Transfer.transferred*]
lemmas *cDiff-cInt2* = *Diff-Int2*[*Transfer.transferred*]
lemmas *cUn-cInt-assoc-eq* = *Un-Int-assoc-eq*[*Transfer.transferred*]
lemmas *cBall-cUn* = *ball-Un*[*Transfer.transferred*]
lemmas *cBex-cUn* = *bex-Un*[*Transfer.transferred*]
lemmas *cDiff-eq-cempty-iff*[*simp*,*no-atp*] = *Diff-eq-empty-iff*[*Transfer.transferred*]
lemmas *cDiff-cancel*[*simp*] = *Diff-cancel*[*Transfer.transferred*]
lemmas *cDiff-idemp*[*simp*] = *Diff-idemp*[*Transfer.transferred*]
lemmas *cDiff-triv* = *Diff-triv*[*Transfer.transferred*]
lemmas *cempty-cDiff*[*simp*] = *empty-Diff*[*Transfer.transferred*]
lemmas *cDiff-cempty*[*simp*] = *Diff-empty*[*Transfer.transferred*]
lemmas *cDiff-cinsert0*[*simp*,*no-atp*] = *Diff-insert0*[*Transfer.transferred*]
lemmas *cDiff-cinsert* = *Diff-insert*[*Transfer.transferred*]
lemmas *cDiff-cinsert2* = *Diff-insert2*[*Transfer.transferred*]
lemmas *cinsert-cDiff-if* = *insert-Diff-if*[*Transfer.transferred*]
lemmas *cinsert-cDiff1*[*simp*] = *insert-Diff1*[*Transfer.transferred*]
lemmas *cinsert-cDiff-single*[*simp*] = *insert-Diff-single*[*Transfer.transferred*]
lemmas *cinsert-cDiff* = *insert-Diff*[*Transfer.transferred*]
lemmas *cDiff-cinsert-absorb* = *Diff-insert-absorb*[*Transfer.transferred*]
lemmas *cDiff-disjoint*[*simp*] = *Diff-disjoint*[*Transfer.transferred*]
lemmas *cDiff-partition* = *Diff-partition*[*Transfer.transferred*]
lemmas *double-cDiff* = *double-diff*[*Transfer.transferred*]
lemmas *cUn-cDiff-cancel*[*simp*] = *Un-Diff-cancel*[*Transfer.transferred*]
lemmas *cUn-cDiff-cancel2*[*simp*] = *Un-Diff-cancel2*[*Transfer.transferred*]
lemmas *cDiff-cUn* = *Diff-Un*[*Transfer.transferred*]
lemmas *cDiff-cInt* = *Diff-Int*[*Transfer.transferred*]
lemmas *cUn-cDiff* = *Un-Diff*[*Transfer.transferred*]
lemmas *cInt-cDiff* = *Int-Diff*[*Transfer.transferred*]
lemmas *cDiff-cInt-distrib* = *Diff-Int-distrib*[*Transfer.transferred*]
lemmas *cDiff-cInt-distrib2* = *Diff-Int-distrib2*[*Transfer.transferred*]
lemmas *cset-eq-csubset* = *set-eq-subset*[*Transfer.transferred*]
lemmas *csubset-iff*[*no-atp*] = *subset-iff*[*Transfer.transferred*]

lemmas *csubset-iff-pfsubset-eq* = *subset-iff-psubset-eq*[*Transfer.transferred*]
lemmas *all-not-cin-conv*[*simp*] = *all-not-in-conv*[*Transfer.transferred*]
lemmas *ex-cin-conv* = *ex-in-conv*[*Transfer.transferred*]
lemmas *cimage-mono* = *image-mono*[*Transfer.transferred*]
lemmas *cinsert-mono* = *insert-mono*[*Transfer.transferred*]
lemmas *cunion-mono* = *Un-mono*[*Transfer.transferred*]
lemmas *cinter-mono* = *Int-mono*[*Transfer.transferred*]
lemmas *cminus-mono* = *Diff-mono*[*Transfer.transferred*]
lemmas *cin-mono* = *in-mono*[*Transfer.transferred*]
lemmas *cLeast-mono* = *Least-mono*[*Transfer.transferred*]
lemmas *cequalityI* = *equalityI*[*Transfer.transferred*]
lemmas *cUN-iff* [*simp*] = *UN-iff*[*Transfer.transferred*]
lemmas *cUN-I* [*intro*] = *UN-I*[*Transfer.transferred*]
lemmas *cUN-E* [*elim*!] = *UN-E*[*Transfer.transferred*]
lemmas *cUN-upper* = *UN-upper*[*Transfer.transferred*]
lemmas *cUN-least* = *UN-least*[*Transfer.transferred*]
lemmas *cUN-cinsert-distrib* = *UN-insert-distrib*[*Transfer.transferred*]
lemmas *cUN-empty* [*simp*] = *UN-empty*[*Transfer.transferred*]
lemmas *cUN-empty2* [*simp*] = *UN-empty2*[*Transfer.transferred*]
lemmas *cUN-absorb* = *UN-absorb*[*Transfer.transferred*]
lemmas *cUN-cinsert* [*simp*] = *UN-insert*[*Transfer.transferred*]
lemmas *cUN-cUn* [*simp*] = *UN-Un*[*Transfer.transferred*]
lemmas *cUN-cUN-flatten* = *UN-UN-flatten*[*Transfer.transferred*]
lemmas *cUN-csubset-iff* = *UN-subset-iff*[*Transfer.transferred*]
lemmas *cUN-constant* [*simp*] = *UN-constant*[*Transfer.transferred*]
lemmas *cimage-cUnion* = *image-Union*[*Transfer.transferred*]
lemmas *cUNION-cempty-conv* [*simp*] = *UNION-empty-conv*[*Transfer.transferred*]
lemmas *cBall-cUN* = *ball-UN*[*Transfer.transferred*]
lemmas *cBex-cUN* = *bex-UN*[*Transfer.transferred*]
lemmas *cUn-eq-cUN* = *Un-eq-UN*[*Transfer.transferred*]
lemmas *cUN-mono* = *UN-mono*[*Transfer.transferred*]
lemmas *cimage-cUN* = *image-UN*[*Transfer.transferred*]
lemmas *cUN-csingleton* [*simp*] = *UN-singleton*[*Transfer.transferred*]

23.3 Additional lemmas

23.3.1 *cempty*

lemma *cemptyE* [*elim*!]: *cin a cempty* $\Longrightarrow$ *P* **by** *simp*

23.3.2 *cinsert*

lemma *countable-insert-iff*: *countable* (*insert x A*) $\longleftrightarrow$ *countable A*
by (*metis Diff-eq-empty-iff countable-empty countable-insert subset-insertI uncountable-minus-countable*)

lemma *set-cinsert*:
assumes *cin x A*
obtains *B* **where** *A* = *cinsert x B* **and** ¬ *cin x B*
using *assms* **by** *transfer*(*erule Set.set-insert*, *simp add*: *countable-insert-iff*)

lemma *mk-disjoint-cinsert*: *cin a A* $\Longrightarrow \exists B.\ A = cinsert\ a\ B \wedge \neg\ cin\ a\ B$
by (*rule exI*[**where** $x = cDiff\ A\ (csingle\ a)$]) *blast*

23.3.3 *cimage*

lemma *subset-cimage-iff*: *csubset-eq B* (*cimage f A*) $\longleftrightarrow$ ($\exists AA.$ *csubset-eq AA A* $\wedge$ $B = cimage\ f\ AA$)
by *transfer* (*metis countable-subset image-mono mem-Collect-eq subset-imageE*)

23.3.4 bounded quantification

lemma *cBex-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ cBex\ A\ (\lambda x.\ P\ x \wedge Q) = (cBex\ A\ P \wedge Q)$
$\bigwedge A\ P\ Q.\ cBex\ A\ (\lambda x.\ P \wedge Q\ x) = (P \wedge cBex\ A\ Q)$
$\bigwedge P.\ cBex\ cempty\ P = False$
$\bigwedge a\ B\ P.\ cBex\ (cinsert\ a\ B)\ P = (P\ a \vee cBex\ B\ P)$
$\bigwedge A\ P\ f.\ cBex\ (cimage\ f\ A)\ P = cBex\ A\ (\lambda x.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg\ cBex\ A\ P) = cBall\ A\ (\lambda x.\ \neg\ P\ x)$
by *auto*

lemma *cBall-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ cBall\ A\ (\lambda x.\ P\ x \vee Q) = (cBall\ A\ P \vee Q)$
$\bigwedge A\ P\ Q.\ cBall\ A\ (\lambda x.\ P \vee Q\ x) = (P \vee cBall\ A\ Q)$
$\bigwedge A\ P\ Q.\ cBall\ A\ (\lambda x.\ P \longrightarrow Q\ x) = (P \longrightarrow cBall\ A\ Q)$
$\bigwedge A\ P\ Q.\ cBall\ A\ (\lambda x.\ P\ x \longrightarrow Q) = (cBex\ A\ P \longrightarrow Q)$
$\bigwedge P.\ cBall\ cempty\ P = True$
$\bigwedge a\ B\ P.\ cBall\ (cinsert\ a\ B)\ P = (P\ a \wedge cBall\ B\ P)$
$\bigwedge A\ P\ f.\ cBall\ (cimage\ f\ A)\ P = cBall\ A\ (\lambda x.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg\ cBall\ A\ P) = cBex\ A\ (\lambda x.\ \neg\ P\ x)$
by *auto*

lemma *atomize-cBall*:
$(\bigwedge x.\ cin\ x\ A \Longrightarrow P\ x) \equiv Trueprop\ (cBall\ A\ (\lambda x.\ P\ x))$
apply (*simp only*: *atomize-all atomize-imp*)
apply (*rule equal-intr-rule*)
by (*transfer*, *simp*)+

23.3.5 *cUnion*

lemma *cUNION-cimage*: *cUNION* (*cimage f A*) $g = cUNION\ A\ (g \circ f)$
by *transfer simp*

23.4 Setup for Lifting/Transfer

23.4.1 Relator and predicator properties

lift-definition *rel-cset* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ cset \Rightarrow 'b\ cset \Rightarrow bool$
is *rel-set* **parametric** *rel-set-transfer* **.**

lemma *rel-cset-alt-def*:
rel-cset R a b ⟷
(∀ *t* ∈ *rcset a*. ∃ *u* ∈ *rcset b*. *R t u*) ∧
(∀ *t* ∈ *rcset b*. ∃ *u* ∈ *rcset a*. *R u t*)
by(*simp add*: *rel-cset-def rel-set-def*)

lemma *rel-cset-iff*:
rel-cset R a b ⟷
(∀ *t*. *cin t a* ⟶ (∃ *u*. *cin u b* ∧ *R t u*)) ∧
(∀ *t*. *cin t b* ⟶ (∃ *u*. *cin u a* ∧ *R u t*))
by *transfer*(*auto simp add*: *rel-set-def*)

lemma *rel-cset-cUNION*:
⟦ *rel-cset Q A B*; *rel-fun Q* (*rel-cset R*) *f g* ⟧
⟹ *rel-cset R* (*cUnion* (*cimage f A*)) (*cUnion* (*cimage g B*))
unfolding *rel-fun-def* **by** *transfer*(*erule rel-set-UNION*, *simp add*: *rel-fun-def*)

lemma *rel-cset-csingle-iff* [*simp*]: *rel-cset R* (*csingle x*) (*csingle y*) ⟷ *R x y*
by *transfer*(*auto simp add*: *rel-set-def*)

23.4.2 Transfer rules for the Transfer package

Unconditional transfer rules

context includes *lifting-syntax*
begin

lemmas *cempty-parametric* [*transfer-rule*] = *empty-transfer*[*Transfer.transferred*]

lemma *cinsert-parametric* [*transfer-rule*]:
(*A* ===> *rel-cset A* ===> *rel-cset A*) *cinsert cinsert*
unfolding *rel-fun-def rel-cset-iff* **by** *blast*

lemma *cUn-parametric* [*transfer-rule*]:
(*rel-cset A* ===> *rel-cset A* ===> *rel-cset A*) *cUn cUn*
unfolding *rel-fun-def rel-cset-iff* **by** *blast*

lemma *cUnion-parametric* [*transfer-rule*]:
(*rel-cset* (*rel-cset A*) ===> *rel-cset A*) *cUnion cUnion*
unfolding *rel-fun-def*
by *transfer* (*auto simp*: *rel-set-def*, *metis*+)

lemma *cimage-parametric* [*transfer-rule*]:
((*A* ===> *B*) ===> *rel-cset A* ===> *rel-cset B*) *cimage cimage*
unfolding *rel-fun-def rel-cset-iff* **by** *blast*

lemma *cBall-parametric* [*transfer-rule*]:
(*rel-cset A* ===> (*A* ===> (=)) ===> (=)) *cBall cBall*
unfolding *rel-cset-iff rel-fun-def* **by** *blast*

lemma *cBex-parametric* [*transfer-rule*]:
(rel-cset A ===> (A ===> (=)) ===> (=)) cBex cBex
unfolding *rel-cset-iff rel-fun-def* **by** *blast*

lemma *rel-cset-parametric* [*transfer-rule*]:
((A ===> B ===> (=)) ===> rel-cset A ===> rel-cset B ===> (=)) rel-cset rel-cset
unfolding *rel-fun-def*
using *rel-set-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*, **where** *A = A* **and** *B = B*]
by *simp*

Rules requiring bi-unique, bi-total or right-total relations

lemma *cin-parametric* [*transfer-rule*]:
bi-unique A ⟹ (A ===> rel-cset A ===> (=)) cin cin
unfolding *rel-fun-def rel-cset-iff bi-unique-def* **by** *metis*

lemma *cInt-parametric* [*transfer-rule*]:
bi-unique A ⟹ (rel-cset A ===> rel-cset A ===> rel-cset A) cInt cInt
unfolding *rel-fun-def*
using *inter-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*]
by *blast*

lemma *cDiff-parametric* [*transfer-rule*]:
bi-unique A ⟹ (rel-cset A ===> rel-cset A ===> rel-cset A) cDiff cDiff
unfolding *rel-fun-def*
using *Diff-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by** *blast*

lemma *csubset-parametric* [*transfer-rule*]:
bi-unique A ⟹ (rel-cset A ===> rel-cset A ===> (=)) csubset-eq csubset-eq
unfolding *rel-fun-def*
using *subset-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by** *blast*

end

lifting-update *cset.lifting*
lifting-forget *cset.lifting*

23.5 Registration as BNF

context
includes *cardinal-syntax*
begin

lemma *card-of-countable-sets-range*:
fixes *A* :: *'a set*
shows *|{X. X ⊆ A ∧ countable X ∧ X ≠ {}}| ≤o |{f::nat ⇒ 'a. range f ⊆ A}|*
apply(*rule card-of-ordLeqI*[*of from-nat-into*]) **using** *inj-on-from-nat-into*
unfolding *inj-on-def* **by** *auto*

lemma *card-of-countable-sets-Func*:
$|\{X.\ X \subseteq A \wedge \mathit{countable}\ X \wedge X \neq \{\}\}| \leq o\ |A|\ \widehat{c}\ \mathit{natLeq}$
using *card-of-countable-sets-range card-of-Func-UNIV*[*THEN ordIso-symmetric*]
unfolding *cexp-def Field-natLeq Field-card-of*
by (*rule ordLeq-ordIso-trans*)

lemma *ordLeq-countable-subsets*:
$|A| \leq o\ |\{X.\ X \subseteq A \wedge \mathit{countable}\ X\}|$
apply (*rule card-of-ordLeqI*[*of* $\lambda\ a.\ \{a\}$]) **unfolding** *inj-on-def* **by** *auto*

end

lemma *finite-countable-subset*:
$\mathit{finite}\ \{X.\ X \subseteq A \wedge \mathit{countable}\ X\} \longleftrightarrow \mathit{finite}\ A$
apply (*rule iffI*)
apply (*erule contrapos-pp*)
apply (*rule card-of-ordLeq-infinite*)
apply (*rule ordLeq-countable-subsets*)
apply *assumption*
apply (*rule finite-Collect-conjI*)
apply (*rule disjI1*)
apply (*erule finite-Collect-subsets*)
done

lemma *rcset-to-rcset*: $\mathit{countable}\ A \Longrightarrow \mathit{rcset}\ (\mathit{the\text{-}inv}\ \mathit{rcset}\ A) = A$
including *cset.lifting*
apply (*rule f-the-inv-into-f*[*unfolded inj-on-def image-iff*])
apply *transfer'* **apply** *simp*
apply *transfer'* **apply** *simp*
done

lemma *Collect-Int-Times*: $\{(x,\ y).\ R\ x\ y\} \cap A \times B = \{(x,\ y).\ R\ x\ y \wedge x \in A \wedge y \in B\}$
by *auto*

lemma *rel-cset-aux*:
$(\forall\, t \in \mathit{rcset}\ a.\ \exists\, u \in \mathit{rcset}\ b.\ R\ t\ u) \wedge (\forall\, t \in \mathit{rcset}\ b.\ \exists\, u \in \mathit{rcset}\ a.\ R\ u\ t) \longleftrightarrow$
$((\mathit{BNF\text{-}Def.Grp}\ \{x.\ \mathit{rcset}\ x \subseteq \{(a,\ b).\ R\ a\ b\}\}\ (\mathit{cimage}\ \mathit{fst}))^{-1-1}\ \mathit{OO}$
$\mathit{BNF\text{-}Def.Grp}\ \{x.\ \mathit{rcset}\ x \subseteq \{(a,\ b).\ R\ a\ b\}\}\ (\mathit{cimage}\ \mathit{snd}))\ a\ b$ (**is** *?L* = *?R*)
proof
assume *?L*
define R' **where** $R' = \mathit{the\text{-}inv}\ \mathit{rcset}\ (\mathit{Collect}\ (\mathit{case\text{-}prod}\ R) \cap (\mathit{rcset}\ a \times \mathit{rcset}\ b))$
(**is** - = *the-inv rcset ?L'*)
have *L*: *countable ?L'* **by** *auto*
hence $*$: *rcset* R' = *?L'* **unfolding** R'*-def* **by** (*intro rcset-to-rcset*)
thus *?R* **unfolding** *Grp-def relcompp.simps conversep.simps* **including** *cset.lifting*
proof (*intro CollectI case-prodI exI*[*of* - *a*] *exI*[*of* - *b*] *exI*[*of* - R'] *conjI refl*)

from ∗ ‹*?L*› **show** *a* = *cimage fst R′* **by** *transfer* (*auto simp*: *image-def Collect-Int-Times*)
from ∗ ‹*?L*› **show** *b* = *cimage snd R′* **by** *transfer* (*auto simp*: *image-def Collect-Int-Times*)
qed *simp-all*
next
assume *?R* **thus** *?L* **unfolding** *Grp-def relcompp.simps conversep.simps*
by (*simp add*: *subset-eq Ball-def*)(*transfer*, *auto simp add*: *cimage.rep-eq*, *metis snd-conv*, *metis fst-conv*)
qed

context
includes *cardinal-syntax*
begin

bnf *′a cset*
map: *cimage*
sets: *rcset*
bd: *natLeq*
wits: *cempty*
rel: *rel-cset*
proof −
show *cimage id* = *id* **by** *auto*
next
fix *f g* **show** *cimage* (*g* ∘ *f*) = *cimage g* ∘ *cimage f* **by** *fastforce*
next
fix *C f g* **assume** *eq*: ⋀*a*. *a* ∈ *rcset C* ⟹ *f a* = *g a*
thus *cimage f C* = *cimage g C* **including** *cset.lifting* **by** *transfer force*
next
fix *f* **show** *rcset* ∘ *cimage f* = (‘) *f* ∘ *rcset* **including** *cset.lifting* **by** *transfer′ fastforce*
next
show *card-order natLeq* **by** (*rule natLeq-card-order*)
next
show *cinfinite natLeq* **by** (*rule natLeq-cinfinite*)
next
fix *C* **show** |*rcset C*| ≤*o natLeq*
including *cset.lifting* **by** *transfer* (*unfold countable-card-le-natLeq*)
next
fix *R S*
show *rel-cset R OO rel-cset S* ≤ *rel-cset* (*R OO S*)
unfolding *rel-cset-alt-def*[*abs-def*] **by** *fast*
next
fix *R*
show *rel-cset R* = (λ*x y*. ∃ *z*. *rcset z* ⊆ {(*x*, *y*). *R x y*} ∧
cimage fst z = *x* ∧ *cimage snd z* = *y*)
unfolding *rel-cset-alt-def*[*abs-def*] *rel-cset-aux*[*unfolded OO-Grp-alt*] **by** *simp*
qed(*simp add*: *bot-cset.rep-eq*)

end

end

24 Debugging facilities for code generated towards Isabelle/ML

theory *Debug*
imports *Main*
begin

context
begin

qualified definition *trace* :: *String.literal* ⇒ *unit* **where**
  [*simp*]: *trace s* = ()

qualified definition *tracing* :: *String.literal* ⇒ *'a* ⇒ *'a* **where**
  [*simp*]: *tracing s* = *id*

lemma [*code*]:
  tracing s = (*let u* = *trace s in id*)
  by *simp*

qualified definition *flush* :: *'a* ⇒ *unit* **where**
  [*simp*]: *flush x* = ()

qualified definition *flushing* :: *'a* ⇒ *'b* ⇒ *'b* **where**
  [*simp*]: *flushing x* = *id*

lemma [*code*, *code-unfold*]:
  flushing x = (*let u* = *flush x in id*)
  by *simp*

qualified definition *timing* :: *String.literal* ⇒ (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'b* **where**
  [*simp*]: *timing s f x* = *f x*

end

code-printing
  constant *Debug.trace* ⇀ (*Eval*) *Output.tracing*
| **constant** *Debug.flush* ⇀ (*Eval*) *Output.tracing/* (@{*make'-string*} -) — note indirection via antiquotation
| **constant** *Debug.timing* ⇀ (*Eval*) *Timing.timeap'-msg*

code-reserved *Eval Output Timing*

end

25 Sequence of Properties on Subsequences

theory *Diagonal-Subsequence*
imports *Complex-Main*
begin

locale *subseqs* =
  fixes *P*::*nat*⇒(*nat*⇒*nat*)⇒*bool*
  assumes *ex-subseq*: $\bigwedge n\ s.$ *strict-mono* (*s*::*nat*⇒*nat*) $\Longrightarrow \exists r'.$ *strict-mono* $r' \wedge$ *P n* (*s* ∘ r')
begin

definition *reduce* **where** *reduce s n* = (*SOME r'*::*nat*⇒*nat*. *strict-mono r'* ∧ *P n* (*s* ∘ r'))

lemma *subseq-reduce*[*intro*, *simp*]:
  strict-mono s ⟹ *strict-mono* (*reduce s n*)
  unfolding *reduce-def* **by** (*rule someI2-ex*[*OF ex-subseq*]) *auto*

lemma *reduce-holds*:
  strict-mono s ⟹ *P n* (*s* ∘ *reduce s n*)
  unfolding *reduce-def* **by** (*rule someI2-ex*[*OF ex-subseq*]) (*auto simp*: *o-def*)

primrec *seqseq* :: *nat* ⇒ *nat* ⇒ *nat* **where**
  seqseq 0 = *id*
| *seqseq* (*Suc n*) = *seqseq n* ∘ *reduce* (*seqseq n*) *n*

lemma *subseq-seqseq*[*intro*, *simp*]: *strict-mono* (*seqseq n*)
proof (*induct n*)
  case *0* **thus** *?case* **by** (*simp add*: *strict-mono-def*)
next
  case (*Suc n*) **thus** *?case* **by** (*subst seqseq.simps*) (*auto intro*!: *strict-mono-o*)
qed

lemma *seqseq-holds*:
  P n (*seqseq* (*Suc n*))
proof −
  have *P n* (*seqseq n* ∘ *reduce* (*seqseq n*) *n*)
    by (*intro reduce-holds subseq-seqseq*)
  thus *?thesis* **by** *simp*
qed

definition *diagseq* :: *nat* ⇒ *nat* **where** *diagseq i* = *seqseq i i*

lemma *diagseq-mono*: *diagseq n* < *diagseq* (*Suc n*)
proof −
  have *diagseq n* < *seqseq n* (*Suc n*)
    using *subseq-seqseq*[*of n*] **by** (*simp add*: *diagseq-def strict-mono-def*)
  also have ... ≤ *seqseq n* (*reduce* (*seqseq n*) *n* (*Suc n*))

using *strict-mono-less-eq seq-suble* **by** *blast*
also have ... = *diagseq* (*Suc n*) **by** (*simp add*: *diagseq-def*)
finally show *?thesis* **.**
qed

lemma *subseq-diagseq*: *strict-mono diagseq*
using *diagseq-mono* **by** (*simp add*: *strict-mono-Suc-iff diagseq-def*)

primrec *fold-reduce* **where**
fold-reduce n 0 = *id*
| *fold-reduce n* (*Suc k*) = *fold-reduce n k* ∘ *reduce* (*seqseq* (*n* + *k*)) (*n* + *k*)

lemma *subseq-fold-reduce*[*intro*, *simp*]: *strict-mono* (*fold-reduce n k*)
proof (*induct k*)
case (*Suc k*) **from** *strict-mono-o*[*OF this subseq-reduce*] **show** *?case* **by** (*simp add*: *o-def*)
qed (*simp add*: *strict-mono-def*)

lemma *ex-subseq-reduce-index*: *seqseq* (*n* + *k*) = *seqseq n* ∘ *fold-reduce n k*
by (*induct k*) *simp-all*

lemma *seqseq-fold-reduce*: *seqseq n* = *fold-reduce 0 n*
by (*induct n*) (*simp-all*)

lemma *diagseq-fold-reduce*: *diagseq n* = *fold-reduce 0 n n*
using *seqseq-fold-reduce* **by** (*simp add*: *diagseq-def*)

lemma *fold-reduce-add*: *fold-reduce 0* (*m* + *n*) = *fold-reduce 0 m* ∘ *fold-reduce m n*
by (*induct n*) *simp-all*

lemma *diagseq-add*: *diagseq* (*k* + *n*) = (*seqseq k* ∘ (*fold-reduce k n*)) (*k* + *n*)
proof −
have *diagseq* (*k* + *n*) = *fold-reduce 0* (*k* + *n*) (*k* + *n*)
by (*simp add*: *diagseq-fold-reduce*)
also have ... = (*seqseq k* ∘ *fold-reduce k n*) (*k* + *n*)
unfolding *fold-reduce-add seqseq-fold-reduce* **..**
finally show *?thesis* **.**
qed

lemma *diagseq-sub*:
assumes *m* ≤ *n* **shows** *diagseq n* = (*seqseq m* ∘ (*fold-reduce m* (*n* − *m*))) *n*
using *diagseq-add*[*of m n* − *m*] *assms* **by** *simp*

lemma *subseq-diagonal-rest*: *strict-mono* (λ*x*. *fold-reduce k x* (*k* + *x*))
unfolding *strict-mono-Suc-iff fold-reduce.simps o-def*
proof
fix *n*
have *fold-reduce k n* (*k* + *n*) < *fold-reduce k n* (*k* + *Suc n*) (**is** *?lhs* < -)

by (*auto intro*: *strict-monoD*)
also have ... ≤ *fold-reduce k n* (*reduce* (*seqseq* (*k* + *n*)) (*k* + *n*) (*k* + *Suc n*))
by (*auto intro*: *less-mono-imp-le-mono seq-suble strict-monoD*)
finally show *?lhs* < ... **.**
qed

lemma *diagseq-seqseq*: *diagseq* ∘ ((+) *k*) = (*seqseq k* ∘ (λ*x*. *fold-reduce k x* (*k* + *x*)))
by (*auto simp*: *o-def diagseq-add*)

lemma *diagseq-holds*:
assumes *subseq-stable*: ⋀*r s n*. *strict-mono r* ⟹ *P n s* ⟹ *P n* (*s* ∘ *r*)
shows *P k* (*diagseq* ∘ ((+) (*Suc k*)))
unfolding *diagseq-seqseq* **by** (*intro subseq-stable subseq-diagonal-rest seqseq-holds*)

end

end

26 Common discrete functions

theory *Discrete*
imports *Complex-Main*
begin

26.1 Discrete logarithm

context
begin

qualified fun *log* :: *nat* ⇒ *nat*
where [*simp del*]: *log n* = (*if n* < *2 then 0 else Suc* (*log* (*n div 2*)))

lemma *log-induct* [*consumes 1*, *case-names one double*]:
fixes *n* :: *nat*
assumes *n* > *0*
assumes *one*: *P 1*
assumes *double*: ⋀*n*. *n* ≥ *2* ⟹ *P* (*n div 2*) ⟹ *P n*
shows *P n*
using ‹*n* > *0*› **proof** (*induct n rule*: *log.induct*)
fix *n*
assume ¬ *n* < *2* ⟹
0 < *n div 2* ⟹ *P* (*n div 2*)
then have ∗: *n* ≥ *2* ⟹ *P* (*n div 2*) **by** *simp*
assume *n* > *0*
show *P n*
proof (*cases n* = *1*)
case *True*
with *one* **show** *?thesis* **by** *simp*

```
  next
    case False
    with ‹n > 0› have n ≥ 2 by auto
    with ∗ have P (n div 2) .
    with ‹n ≥ 2› show ?thesis by (rule double)
  qed
qed

lemma log-zero [simp]: log 0 = 0
  by (simp add: log.simps)

lemma log-one [simp]: log 1 = 0
  by (simp add: log.simps)

lemma log-Suc-zero [simp]: log (Suc 0) = 0
  using log-one by simp

lemma log-rec: n ≥ 2 ⟹ log n = Suc (log (n div 2))
  by (simp add: log.simps)

lemma log-twice [simp]: n ≠ 0 ⟹ log (2 ∗ n) = Suc (log n)
  by (simp add: log-rec)

lemma log-half [simp]: log (n div 2) = log n − 1
proof (cases n < 2)
  case True
  then have n = 0 ∨ n = 1 by arith
  then show ?thesis by (auto simp del: One-nat-def)
next
  case False
  then show ?thesis by (simp add: log-rec)
qed

lemma log-exp [simp]: log (2 ^ n) = n
  by (induct n) simp-all

lemma log-mono: mono log
proof
  fix m n :: nat
  assume m ≤ n
  then show log m ≤ log n
  proof (induct m arbitrary: n rule: log.induct)
    case (1 m)
    then have mn2: m div 2 ≤ n div 2 by arith
    show log m ≤ log n
    proof (cases m ≥ 2)
      case False
      then have m = 0 ∨ m = 1 by arith
      then show ?thesis by (auto simp del: One-nat-def)
```

```
    next
      case True then have ¬ m < 2 by simp
      with mn2 have n ≥ 2 by arith
      from True have m2-0: m div 2 ≠ 0 by arith
      with mn2 have n2-0: n div 2 ≠ 0 by arith
      from ‹¬ m < 2› 1.hyps mn2 have log (m div 2) ≤ log (n div 2) by blast
      with m2-0 n2-0 have log (2 * (m div 2)) ≤ log (2 * (n div 2)) by simp
     with m2-0 n2-0 ‹m ≥ 2› ‹n ≥ 2› show ?thesis by (simp only: log-rec [of m]
log-rec [of n]) simp
    qed
  qed
qed

lemma log-exp2-le:
  assumes n > 0
  shows 2 ^ log n ≤ n
  using assms
proof (induct n rule: log-induct)
  case one
  then show ?case by simp
next
  case (double n)
  with log-mono have log n ≥ Suc 0
    by (simp add: log.simps)
  assume 2 ^ log (n div 2) ≤ n div 2
  with ‹n ≥ 2› have 2 ^ (log n − Suc 0) ≤ n div 2 by simp
  then have 2 ^ (log n − Suc 0) * 2 ^ 1 ≤ n div 2 * 2 by simp
  with ‹log n ≥ Suc 0› have 2 ^ log n ≤ n div 2 * 2
    unfolding power-add [symmetric] by simp
  also have n div 2 * 2 ≤ n by (cases even n) simp-all
  finally show ?case .
qed

lemma log-exp2-gt: 2 * 2 ^ log n > n
proof (cases n > 0)
  case True
  thus ?thesis
  proof (induct n rule: log-induct)
    case (double n)
    thus ?case
      by (cases even n) (auto elim!: evenE oddE simp: field-simps log.simps)
  qed simp-all
qed simp-all

lemma log-exp2-ge: 2 * 2 ^ log n ≥ n
  using log-exp2-gt[of n] by simp

lemma log-le-iff: m ≤ n ⟹ log m ≤ log n
  by (rule monoD [OF log-mono])
```

lemma *log-eqI*:
assumes $n > 0$ $2\hat{}k \leq n$ $n < 2 * 2\hat{}k$
shows *log n = k*
proof (*rule antisym*)
from ‹$n > 0$› **have** $2 \hat{} \log n \leq n$ **by** (*rule log-exp2-le*)
also have ... $< 2 \hat{} Suc\ k$ **using** *assms* **by** *simp*
finally have *log n < Suc k* **by** (*subst* (*asm*) *power-strict-increasing-iff*) *simp-all*
thus *log n ≤ k* **by** *simp*
next
have $2\hat{}k \leq n$ **by** *fact*
also have ... $< 2\hat{}(Suc\ (\log n))$ **by** (*simp add*: *log-exp2-gt*)
finally have *k < Suc* (*log n*) **by** (*subst* (*asm*) *power-strict-increasing-iff*) *simp-all*
thus *k ≤ log n* **by** *simp*
qed

lemma *log-altdef*: *log n* = (*if n = 0 then 0 else nat* ⌊*Transcendental.log 2* (*real-of-nat n*)⌋)
proof (*cases n = 0*)
case *False*
have ⌊*Transcendental.log 2* (*real-of-nat n*)⌋ = *int* (*log n*)
proof (*rule floor-unique*)
from *False* **have** *2 powr* (*real* (*log n*)) ≤ *real n*
by (*simp add*: *powr-realpow log-exp2-le*)
hence *Transcendental.log 2* (*2 powr* (*real* (*log n*))) ≤ *Transcendental.log 2* (*real n*)
using *False* **by** (*subst Transcendental.log-le-cancel-iff*) *simp-all*
also have *Transcendental.log 2* (*2 powr* (*real* (*log n*))) = *real* (*log n*) **by** *simp*
finally show *real-of-int* (*int* (*log n*)) ≤ *Transcendental.log 2* (*real n*) **by** *simp*
next
have *real n < real* (*2 * 2 ^ log n*)
by (*subst of-nat-less-iff*) (*rule log-exp2-gt*)
also have ... = *2 powr* (*real* (*log n*) + *1*)
by (*simp add*: *powr-add powr-realpow*)
finally have *Transcendental.log 2* (*real n*) < *Transcendental.log 2* ...
using *False* **by** (*subst Transcendental.log-less-cancel-iff*) *simp-all*
also have ... = *real* (*log n*) + *1* **by** *simp*
finally show *Transcendental.log 2* (*real n*) < *real-of-int* (*int* (*log n*)) + *1* **by** *simp*
qed
thus *?thesis* **by** *simp*
qed *simp-all*

26.2 Discrete square root

qualified definition *sqrt* :: *nat* ⇒ *nat*
where *sqrt n* = *Max* {*m*. $m^2 \leq n$}

lemma *sqrt-aux*:

```
  fixes n :: nat
  shows finite {m. m^2 ≤ n} and {m. m^2 ≤ n} ≠ {}
proof −
  { fix m
    assume m^2 ≤ n
    then have m ≤ n
      by (cases m) (simp-all add: power2-eq-square)
  } note ** = this
  then have {m. m^2 ≤ n} ⊆ {m. m ≤ n} by auto
  then show finite {m. m^2 ≤ n} by (rule finite-subset) rule
  have 0^2 ≤ n by simp
  then show *: {m. m^2 ≤ n} ≠ {} by blast
qed

lemma sqrt-unique:
  assumes m^2 ≤ n n < (Suc m)^2
  shows   Discrete.sqrt n = m
proof −
  have m' ≤ m if m'^2 ≤ n for m'
  proof −
    note that
    also note assms(2)
    finally have m' < Suc m by (rule power-less-imp-less-base) simp-all
    thus m' ≤ m by simp
  qed
  with ‹m^2 ≤ n› sqrt-aux[of n] show ?thesis unfolding Discrete.sqrt-def
    by (intro antisym Max.boundedI Max.coboundedI) simp-all
qed


lemma sqrt-code[code]: sqrt n = Max (Set.filter (λm. m^2 ≤ n) {0..n})
proof −
  from power2-nat-le-imp-le [of - n] have {m. m ≤ n ∧ m^2 ≤ n} = {m. m^2 ≤ n}
by auto
  then show ?thesis by (simp add: sqrt-def Set.filter-def)
qed

lemma sqrt-inverse-power2 [simp]: sqrt (n^2) = n
proof −
  have {m. m ≤ n} ≠ {} by auto
  then have Max {m. m ≤ n} ≤ n by auto
  then show ?thesis
    by (auto simp add: sqrt-def power2-nat-le-eq-le intro: antisym)
qed

lemma sqrt-zero [simp]: sqrt 0 = 0
  using sqrt-inverse-power2 [of 0] by simp

lemma sqrt-one [simp]: sqrt 1 = 1
```

using *sqrt-inverse-power2* [*of 1*] **by** *simp*

lemma *mono-sqrt*: *mono sqrt*
proof
fix *m n* :: *nat*
have ∗: *0* ∗ *0* ≤ *m* **by** *simp*
assume *m* ≤ *n*
then show *sqrt m* ≤ *sqrt n*
by (*auto intro*!: *Max-mono* ‹*0* ∗ *0* ≤ *m*› *finite-less-ub simp add*: *power2-eq-square sqrt-def*)
qed

lemma *mono-sqrt'*: *m* ≤ *n* ⟹ *Discrete.sqrt m* ≤ *Discrete.sqrt n*
using *mono-sqrt* **unfolding** *mono-def* **by** *auto*

lemma *sqrt-greater-zero-iff* [*simp*]: *sqrt n* > *0* ⟷ *n* > *0*
proof −
have ∗: *0* < *Max* {*m*. *m*^{2} ≤ *n*} ⟷ (∃ *a*∈{*m*. *m*^{2} ≤ *n*}. *0* < *a*)
by (*rule Max-gr-iff*) (*fact sqrt-aux*)+
show *?thesis*
proof
assume *0* < *sqrt n*
then have *0* < *Max* {*m*. *m*^{2} ≤ *n*} **by** (*simp add*: *sqrt-def*)
with ∗ **show** *0* < *n* **by** (*auto dest*: *power2-nat-le-imp-le*)
next
assume *0* < *n*
then have *1*^{2} ≤ *n* ∧ *0* < (*1*::*nat*) **by** *simp*
then have ∃ *q*. *q*^{2} ≤ *n* ∧ *0* < *q* **..**
with ∗ **have** *0* < *Max* {*m*. *m*^{2} ≤ *n*} **by** *blast*
then show *0* < *sqrt n* **by** (*simp add*: *sqrt-def*)
qed
qed

lemma *sqrt-power2-le* [*simp*]: (*sqrt n*)^2 ≤ *n*
proof (*cases n* > *0*)
case *False* **then show** *?thesis* **by** *simp*
next
case *True* **then have** *sqrt n* > *0* **by** *simp*
then have *mono* (*times* (*Max* {*m*. *m*^{2} ≤ *n*})) **by** (*auto intro*: *mono-times-nat simp add*: *sqrt-def*)
then have ∗: *Max* {*m*. *m*^{2} ≤ *n*} ∗ *Max* {*m*. *m*^{2} ≤ *n*} = *Max* (*times* (*Max* {*m*. *m*^{2} ≤ *n*}) ‘ {*m*. *m*^{2} ≤ *n*})
using *sqrt-aux* [*of n*] **by** (*rule mono-Max-commute*)
have ⋀*a*. *a* ∗ *a* ≤ *n* ⟹ *Max* {*m*. *m* ∗ *m* ≤ *n*} ∗ *a* ≤ *n*
proof −
fix *q*
assume *q* ∗ *q* ≤ *n*
show *Max* {*m*. *m* ∗ *m* ≤ *n*} ∗ *q* ≤ *n*
proof (*cases q* > *0*)

```
    case False then show ?thesis by simp
   next
    case True then have mono (times q) by (rule mono-times-nat)
    then have q * Max {m. m * m ≤ n} = Max (times q ` {m. m * m ≤ n})
    using sqrt-aux [of n] by (auto simp add: power2-eq-square intro: mono-Max-commute)
     then have Max {m. m * m ≤ n} * q = Max (times q ` {m. m * m ≤ n})
by (simp add: ac-simps)
    moreover have finite ((*) q ` {m. m * m ≤ n})
     by (metis (mono-tags) finite-imageI finite-less-ub le-square)
    moreover have ∃x. x * x ≤ n
     by (metis ‹q * q ≤ n›)
    ultimately show ?thesis
    by simp (metis ‹q * q ≤ n› le-cases mult-le-mono1 mult-le-mono2 order-trans)
   qed
  qed
  then have Max ((*) (Max {m. m * m ≤ n}) ` {m. m * m ≤ n}) ≤ n
   apply (subst Max-le-iff)
    apply (metis (mono-tags) finite-imageI finite-less-ub le-square)
   apply auto
   apply (metis le0 mult-0-right)
   done
  with * show ?thesis by (simp add: sqrt-def power2-eq-square)
qed

lemma sqrt-le: sqrt n ≤ n
  using sqrt-aux [of n] by (auto simp add: sqrt-def intro: power2-nat-le-imp-le)
```

Additional facts about the discrete square root, thanks to Julian Biendarra, Manuel Eberl

```
lemma Suc-sqrt-power2-gt: n < (Suc (Discrete.sqrt n))^2
  using Max-ge[OF Discrete.sqrt-aux(1), of Discrete.sqrt n + 1 n]
  by (cases n < (Suc (Discrete.sqrt n))^2) (simp-all add: Discrete.sqrt-def)

lemma le-sqrt-iff: x ≤ Discrete.sqrt y ⟷ x^2 ≤ y
proof −
  have x ≤ Discrete.sqrt y ⟷ (∃z. z^2 ≤ y ∧ x ≤ z)
   using Max-ge-iff[OF Discrete.sqrt-aux, of x y] by (simp add: Discrete.sqrt-def)
  also have ... ⟷ x^2 ≤ y
  proof safe
   fix z assume x ≤ z z ^ 2 ≤ y
  thus x^2 ≤ y by (intro le-trans[of x^2 z^2 y]) (simp-all add: power2-nat-le-eq-le)
  qed auto
  finally show ?thesis .
qed

lemma le-sqrtI: x^2 ≤ y ⟹ x ≤ Discrete.sqrt y
  by (simp add: le-sqrt-iff)

lemma sqrt-le-iff: Discrete.sqrt y ≤ x ⟷ (∀z. z^2 ≤ y ⟶ z ≤ x)
```

using *Max.bounded-iff*[*OF Discrete.sqrt-aux*] **by** (*simp add*: *Discrete.sqrt-def*)

lemma *sqrt-leI*:
(⋀*z*. *z*^2 ≤ *y* ⟹ *z* ≤ *x*) ⟹ *Discrete.sqrt y* ≤ *x*
by (*simp add*: *sqrt-le-iff*)

lemma *sqrt-Suc*:
Discrete.sqrt (*Suc n*) = (*if* ∃ *m*. *Suc n* = *m*^2 *then Suc* (*Discrete.sqrt n*) *else Discrete.sqrt n*)
proof *cases*
assume ∃ *m*. *Suc n* = *m*^2
then obtain *m* **where** *m-def*: *Suc n* = *m*^2 **by** *blast*
then have *lhs*: *Discrete.sqrt* (*Suc n*) = *m* **by** *simp*
from *m-def sqrt-power2-le*[*of n*]
have (*Discrete.sqrt n*)^2 < *m*^2 **by** *linarith*
with *power2-less-imp-less* **have** *lt-m*: *Discrete.sqrt n* < *m* **by** *blast*
from *m-def Suc-sqrt-power2-gt*[*of n*]
have *m*^2 ≤ (*Suc*(*Discrete.sqrt n*))^2
by *linarith*
with *power2-nat-le-eq-le* **have** *m* ≤ *Suc* (*Discrete.sqrt n*) **by** *blast*
with *lt-m* **have** *m* = *Suc* (*Discrete.sqrt n*) **by** *simp*
with *lhs m-def* **show** *?thesis* **by** *fastforce*
next
assume *asm*: ¬ (∃ *m*. *Suc n* = *m*^2)
hence *Suc n* ≠ (*Discrete.sqrt* (*Suc n*))^2 **by** *simp*
with *sqrt-power2-le*[*of Suc n*]
have *Discrete.sqrt* (*Suc n*) ≤ *Discrete.sqrt n* **by** (*intro le-sqrtI*) *linarith*
moreover have *Discrete.sqrt* (*Suc n*) ≥ *Discrete.sqrt n*
by (*intro monoD*[*OF mono-sqrt*]) *simp-all*
ultimately show *?thesis* **using** *asm* **by** *simp*
qed

end

end

27 Pi and Function Sets

theory *FuncSet*
imports *Main*
abbrevs *PiE* = *Pi*$_E$
and *PIE* = Π$_E$
begin

definition *Pi* :: *'a set* ⇒ (*'a* ⇒ *'b set*) ⇒ (*'a* ⇒ *'b*) *set*
where *Pi A B* = {*f*. ∀ *x*. *x* ∈ *A* ⟶ *f x* ∈ *B x*}

definition *extensional* :: *'a set* ⇒ (*'a* ⇒ *'b*) *set*
where *extensional A* = {*f*. ∀ *x*. *x* ∉ *A* ⟶ *f x* = *undefined*}

definition *restrict* :: $('a \Rightarrow 'b) \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow 'b$
where $restrict\ f\ A = (\lambda x.\ if\ x \in A\ then\ f\ x\ else\ undefined)$

abbreviation *funcset* :: $'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$ (**infixr** $\rightarrow$ *60*)
where $A \rightarrow B \equiv Pi\ A\ (\lambda\text{-}.\ B)$

syntax
-Pi :: $pttrn \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$ (*3*Π -$\in$-./ -) *10*
-lam :: $pttrn \Rightarrow 'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'b)$ (*3*λ-$\in$-./ -) [*0*,*0*,*3*] *3*
translations
$\Pi\ x{\in}A.\ B \rightleftharpoons CONST\ Pi\ A\ (\lambda x.\ B)$
$\lambda x{\in}A.\ f \rightleftharpoons CONST\ restrict\ (\lambda x.\ f)\ A$

definition *compose* :: $'a\ set \Rightarrow ('b \Rightarrow 'c) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow 'c)$
where $compose\ A\ g\ f = (\lambda x{\in}A.\ g\ (f\ x))$

27.1 Basic Properties of *Pi*

lemma *Pi-I*[*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B\ x) \Longrightarrow f \in Pi\ A\ B$
by (*simp add*: *Pi-def*)

lemma *Pi-I'*[*simp*]: $(\bigwedge x.\ x \in A \longrightarrow f\ x \in B\ x) \Longrightarrow f \in Pi\ A\ B$
by (*simp add*:*Pi-def*)

lemma *funcsetI*: $(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B) \Longrightarrow f \in A \rightarrow B$
by (*simp add*: *Pi-def*)

lemma *Pi-mem*: $f \in Pi\ A\ B \Longrightarrow x \in A \Longrightarrow f\ x \in B\ x$
by (*simp add*: *Pi-def*)

lemma *Pi-iff*: $f \in Pi\ I\ X \longleftrightarrow (\forall i{\in}I.\ f\ i \in X\ i)$
unfolding *Pi-def* **by** *auto*

lemma *PiE* [*elim*]: $f \in Pi\ A\ B \Longrightarrow (f\ x \in B\ x \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
by (*auto simp*: *Pi-def*)

lemma *Pi-cong*: $(\bigwedge w.\ w \in A \Longrightarrow f\ w = g\ w) \Longrightarrow f \in Pi\ A\ B \longleftrightarrow g \in Pi\ A\ B$
by (*auto simp*: *Pi-def*)

lemma *funcset-id* [*simp*]: $(\lambda x.\ x) \in A \rightarrow A$
by *auto*

lemma *funcset-mem*: $f \in A \rightarrow B \Longrightarrow x \in A \Longrightarrow f\ x \in B$
by (*simp add*: *Pi-def*)

lemma *funcset-image*: $f \in A \rightarrow B \Longrightarrow f\ `\ A \subseteq B$
by *auto*

lemma *image-subset-iff-funcset*: $F \text{ ` } A \subseteq B \longleftrightarrow F \in A \to B$
by *auto*

lemma *funcset-to-empty-iff*: $A \to \{\} = (\textit{if } A{=}\{\} \textit{ then } \textit{UNIV} \textit{ else } \{\})$
by *auto*

lemma *Pi-eq-empty*[*simp*]: $(\Pi\ x \in A.\ B\ x) = \{\} \longleftrightarrow (\exists\, x{\in}A.\ B\ x = \{\})$
proof −
have $\exists\, x{\in}A.\ B\ x = \{\}$ **if** $\bigwedge f.\ \exists\, y.\ y \in A \wedge f\ y \notin B\ y$
using *that* [*of* $\lambda u.$ *SOME* $y.\ y \in B\ u$] *some-in-eq* **by** *blast*
then show *?thesis*
by *force*
qed

lemma *Pi-empty* [*simp*]: $\textit{Pi}\ \{\}\ B = \textit{UNIV}$
by (*simp add*: *Pi-def*)

lemma *Pi-Int*: $\textit{Pi}\ I\ E \cap \textit{Pi}\ I\ F = (\Pi\ i{\in}I.\ E\ i \cap F\ i)$
by *auto*

lemma *Pi-UN*:
fixes $A :: \textit{nat} \Rightarrow 'i \Rightarrow 'a\ \textit{set}$
assumes *finite I*
and *mono*: $\bigwedge i\ n\ m.\ i \in I \Longrightarrow n \leq m \Longrightarrow A\ n\ i \subseteq A\ m\ i$
shows $(\bigcup n.\ \textit{Pi}\ I\ (A\ n)) = (\Pi\ i{\in}I.\ \bigcup n.\ A\ n\ i)$
proof (*intro set-eqI iffI*)
fix *f*
assume $f \in (\Pi\ i{\in}I.\ \bigcup n.\ A\ n\ i)$
then have $\forall\, i{\in}I.\ \exists\, n.\ f\ i \in A\ n\ i$
by *auto*
from *bchoice*[*OF this*] **obtain** *n* **where** *n*: $f\ i \in A\ (n\ i)\ i$ **if** $i \in I$ **for** *i*
by *auto*
obtain *k* **where** *k*: $n\ i \leq k$ **if** $i \in I$ **for** *i*
using ‹*finite I*› *finite-nat-set-iff-bounded-le*[*of n'I*] **by** *auto*
have $f \in \textit{Pi}\ I\ (A\ k)$
proof (*intro Pi-I*)
fix *i*
assume $i \in I$
from *mono*[*OF this*, *of n i k*] *k*[*OF this*] *n*[*OF this*]
show $f\ i \in A\ k\ i$ **by** *auto*
qed
then show $f \in (\bigcup n.\ \textit{Pi}\ I\ (A\ n))$
by *auto*
qed *auto*

lemma *Pi-UNIV* [*simp*]: $A \to \textit{UNIV} = \textit{UNIV}$
by (*simp add*: *Pi-def*)

Covariance of Pi-sets in their second argument

lemma *Pi-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow Pi\ A\ B \subseteq Pi\ A\ C$
by *auto*

Contravariance of Pi-sets in their first argument

lemma *Pi-anti-mono*: $A' \subseteq A \Longrightarrow Pi\ A\ B \subseteq Pi\ A'\ B$
by *auto*

lemma *prod-final*:
assumes *1*: $fst \circ f \in Pi\ A\ B$
and *2*: $snd \circ f \in Pi\ A\ C$
shows $f \in (\Pi\ z \in A.\ B\ z \times C\ z)$
proof (*rule Pi-I*)
fix z
assume z: $z \in A$
have $f\ z = (fst\ (f\ z),\ snd\ (f\ z))$
by *simp*
also have $\ldots \in B\ z \times C\ z$
by (*metis SigmaI PiE o-apply 1 2 z*)
finally show $f\ z \in B\ z \times C\ z$ **.**
qed

lemma *Pi-split-domain*[*simp*]: $x \in Pi\ (I \cup J)\ X \longleftrightarrow x \in Pi\ I\ X \wedge x \in Pi\ J\ X$
by (*auto simp*: *Pi-def*)

lemma *Pi-split-insert-domain*[*simp*]: $x \in Pi\ (insert\ i\ I)\ X \longleftrightarrow x \in Pi\ I\ X \wedge x\ i \in X\ i$
by (*auto simp*: *Pi-def*)

lemma *Pi-cancel-fupd-range*[*simp*]: $i \notin I \Longrightarrow x \in Pi\ I\ (B(i := b)) \longleftrightarrow x \in Pi\ I\ B$
by (*auto simp*: *Pi-def*)

lemma *Pi-cancel-fupd*[*simp*]: $i \notin I \Longrightarrow x(i := a) \in Pi\ I\ B \longleftrightarrow x \in Pi\ I\ B$
by (*auto simp*: *Pi-def*)

lemma *Pi-fupd-iff*: $i \in I \Longrightarrow f \in Pi\ I\ (B(i := A)) \longleftrightarrow f \in Pi\ (I - \{i\})\ B \wedge f\ i \in A$
apply *auto*
apply (*metis PiE fun-upd-apply*)
by *force*

27.2 Composition With a Restricted Domain: *compose*

lemma *funcset-compose*: $f \in A \to B \Longrightarrow g \in B \to C \Longrightarrow compose\ A\ g\ f \in A \to C$
by (*simp add*: *Pi-def compose-def restrict-def*)

lemma *compose-assoc*:
assumes $f \in A \to B$
shows $compose\ A\ h\ (compose\ A\ g\ f) = compose\ A\ (compose\ B\ h\ g)\ f$

using *assms* **by** (*simp add*: *fun-eq-iff Pi-def compose-def restrict-def*)

lemma *compose-eq*: $x \in A \Longrightarrow$ *compose* A g f $x = g$ $(f$ $x)$
by (*simp add*: *compose-def restrict-def*)

lemma *surj-compose*: f ‘ $A = B \Longrightarrow g$ ‘ $B = C \Longrightarrow$ *compose* A g f ‘ $A = C$
by (*auto simp add*: *image-def compose-eq*)

27.3 Bounded Abstraction: *restrict*

lemma *restrict-cong*: $I = J \Longrightarrow (\bigwedge i.\ i \in J$ =*simp*=> f $i = g$ $i) \Longrightarrow$ *restrict* f I $=$ *restrict* g J
by (*auto simp*: *restrict-def fun-eq-iff simp-implies-def*)

lemma *restrictI*[*intro*!]: $(\bigwedge x.\ x \in A \Longrightarrow f$ $x \in B$ $x) \Longrightarrow (\lambda x{\in}A.\ f$ $x) \in Pi$ A B
by (*simp add*: *Pi-def restrict-def*)

lemma *restrict-apply*[*simp*]: $(\lambda y{\in}A.\ f$ $y)$ $x =$ (*if* $x \in A$ *then* f x *else undefined*)
by (*simp add*: *restrict-def*)

lemma *restrict-apply'*: $x \in A \Longrightarrow (\lambda y{\in}A.\ f$ $y)$ $x = f$ x
by *simp*

lemma *restrict-ext*: $(\bigwedge x.\ x \in A \Longrightarrow f$ $x = g$ $x) \Longrightarrow (\lambda x{\in}A.\ f$ $x) = (\lambda x{\in}A.\ g$ $x)$
by (*simp add*: *fun-eq-iff Pi-def restrict-def*)

lemma *restrict-UNIV*: *restrict* f *UNIV* $= f$
by (*simp add*: *restrict-def*)

lemma *inj-on-restrict-eq* [*simp*]: *inj-on* (*restrict* f A) $A =$ *inj-on* f A
by (*simp add*: *inj-on-def restrict-def*)

lemma *Id-compose*: $f \in A \to B \Longrightarrow f \in$ *extensional* $A \Longrightarrow$ *compose* A $(\lambda y{\in}B.\ y)$ $f = f$
by (*auto simp add*: *fun-eq-iff compose-def extensional-def Pi-def*)

lemma *compose-Id*: $g \in A \to B \Longrightarrow g \in$ *extensional* $A \Longrightarrow$ *compose* A g $(\lambda x{\in}A.$ $x) = g$
by (*auto simp add*: *fun-eq-iff compose-def extensional-def Pi-def*)

lemma *image-restrict-eq* [*simp*]: (*restrict* f A) ‘ $A = f$ ‘ A
by (*auto simp add*: *restrict-def*)

lemma *restrict-restrict*[*simp*]: *restrict* (*restrict* f A) $B =$ *restrict* f $(A \cap B)$
unfolding *restrict-def* **by** (*simp add*: *fun-eq-iff*)

lemma *restrict-fupd*[*simp*]: $i \notin I \Longrightarrow$ *restrict* $(f$ $(i := x))$ $I =$ *restrict* f I
by (*auto simp*: *restrict-def*)

lemma *restrict-upd*[*simp*]: $i \notin I \Longrightarrow (\mathit{restrict}\ f\ I)(i := y) = \mathit{restrict}\ (f(i := y))\ (\mathit{insert}\ i\ I)$
by (*auto simp*: *fun-eq-iff*)

lemma *restrict-Pi-cancel*: $\mathit{restrict}\ x\ I \in \mathit{Pi}\ I\ A \longleftrightarrow x \in \mathit{Pi}\ I\ A$
by (*auto simp*: *restrict-def Pi-def*)

lemma *sum-restrict'* [*simp*]: $\mathit{sum'}\ (\lambda i{\in}I.\ g\ i)\ I = \mathit{sum'}\ (\lambda i.\ g\ i)\ I$
by (*simp add*: *sum.G-def conj-commute cong*: *conj-cong*)

lemma *prod-restrict'* [*simp*]: $\mathit{prod'}\ (\lambda i{\in}I.\ g\ i)\ I = \mathit{prod'}\ (\lambda i.\ g\ i)\ I$
by (*simp add*: *prod.G-def conj-commute cong*: *conj-cong*)

27.4 Bijections Between Sets

The definition of *bij-betw* is in *Fun.thy*, but most of the theorems belong here, or need at least *Hilbert-Choice*.

lemma *bij-betwI*:
assumes $f \in A \to B$
and $g \in B \to A$
and *g-f*: $\bigwedge x.\ x{\in}A \Longrightarrow g\ (f\ x) = x$
and *f-g*: $\bigwedge y.\ y{\in}B \Longrightarrow f\ (g\ y) = y$
shows *bij-betw f A B*
unfolding *bij-betw-def*
proof
show *inj-on f A*
by (*metis g-f inj-on-def*)
have $f\ `\ A \subseteq B$
using ‹$f \in A \to B$› **by** *auto*
moreover
have $B \subseteq f\ `\ A$
by *auto* (*metis Pi-mem* ‹$g \in B \to A$› *f-g image-iff*)
ultimately show $f\ `\ A = B$
by *blast*
qed

lemma *bij-betw-imp-funcset*: *bij-betw f A B* $\Longrightarrow f \in A \to B$
by (*auto simp add*: *bij-betw-def*)

lemma *inj-on-compose*: *bij-betw f A B* $\Longrightarrow$ *inj-on g B* $\Longrightarrow$ *inj-on* (*compose A g f*) *A*
by (*auto simp add*: *bij-betw-def inj-on-def compose-eq*)

lemma *bij-betw-compose*: *bij-betw f A B* $\Longrightarrow$ *bij-betw g B C* $\Longrightarrow$ *bij-betw* (*compose A g f*) *A C*
apply (*simp add*: *bij-betw-def compose-eq inj-on-compose*)
apply (*auto simp add*: *compose-def image-def*)
done

lemma *bij-betw-restrict-eq* [*simp*]: *bij-betw* (*restrict f A*) *A B* = *bij-betw f A B*
by (*simp add*: *bij-betw-def*)

27.5 Extensionality

lemma *extensional-empty*[*simp*]: *extensional* {} = {λ*x*. *undefined*}
unfolding *extensional-def* **by** *auto*

lemma *extensional-arb*: *f* ∈ *extensional A* ⟹ *x* ∉ *A* ⟹ *f x* = *undefined*
by (*simp add*: *extensional-def*)

lemma *restrict-extensional* [*simp*]: *restrict f A* ∈ *extensional A*
by (*simp add*: *restrict-def extensional-def*)

lemma *compose-extensional* [*simp*]: *compose A f g* ∈ *extensional A*
by (*simp add*: *compose-def*)

lemma *extensionalityI*:
assumes *f* ∈ *extensional A*
and *g* ∈ *extensional A*
and ⋀*x*. *x* ∈ *A* ⟹ *f x* = *g x*
shows *f* = *g*
using *assms* **by** (*force simp add*: *fun-eq-iff extensional-def*)

lemma *extensional-restrict*: *f* ∈ *extensional A* ⟹ *restrict f A* = *f*
by (*rule extensionalityI*[*OF restrict-extensional*]) *auto*

lemma *extensional-subset*: *f* ∈ *extensional A* ⟹ *A* ⊆ *B* ⟹ *f* ∈ *extensional B*
unfolding *extensional-def* **by** *auto*

lemma *inv-into-funcset*: *f* ‘ *A* = *B* ⟹ (λ*x*∈*B*. *inv-into A f x*) ∈ *B* → *A*
by (*unfold inv-into-def*) (*fast intro*: *someI2*)

lemma *compose-inv-into-id*: *bij-betw f A B* ⟹ *compose A* (λ*y*∈*B*. *inv-into A f y*) *f* = (λ*x*∈*A*. *x*)
apply (*simp add*: *bij-betw-def compose-def*)
apply (*rule restrict-ext*, *auto*)
done

lemma *compose-id-inv-into*: *f* ‘ *A* = *B* ⟹ *compose B f* (λ*y*∈*B*. *inv-into A f y*) = (λ*x*∈*B*. *x*)
apply (*simp add*: *compose-def*)
apply (*rule restrict-ext*)
apply (*simp add*: *f-inv-into-f*)
done

lemma *extensional-insert*[*intro*, *simp*]:
assumes *a* ∈ *extensional* (*insert i I*)
shows *a*(*i* := *b*) ∈ *extensional* (*insert i I*)

using *assms* **unfolding** *extensional-def* **by** *auto*

lemma *extensional-Int*[*simp*]: *extensional* $I \cap$ *extensional* $I' =$ *extensional* $(I \cap I')$
unfolding *extensional-def* **by** *auto*

lemma *extensional-UNIV*[*simp*]: *extensional UNIV* $=$ *UNIV*
by (*auto simp*: *extensional-def*)

lemma *restrict-extensional-sub*[*intro*]: $A \subseteq B \Longrightarrow$ *restrict* $f\ A \in$ *extensional* B
unfolding *restrict-def extensional-def* **by** *auto*

lemma *extensional-insert-undefined*[*intro*, *simp*]:
$a \in$ *extensional* $(\mathit{insert}\ i\ I) \Longrightarrow a(i := \mathit{undefined}) \in$ *extensional* I
unfolding *extensional-def* **by** *auto*

lemma *extensional-insert-cancel*[*intro*, *simp*]:
$a \in$ *extensional* $I \Longrightarrow a \in$ *extensional* $(\mathit{insert}\ i\ I)$
unfolding *extensional-def* **by** *auto*

27.6 Cardinality

lemma *card-inj*: $f \in A \to B \Longrightarrow$ *inj-on* $f\ A \Longrightarrow$ *finite* $B \Longrightarrow$ *card* $A \leq$ *card* B
by (*rule card-inj-on-le*) *auto*

lemma *card-bij*:
assumes $f \in A \to B$ *inj-on* $f\ A$
and $g \in B \to A$ *inj-on* $g\ B$
and *finite* A *finite* B
shows *card* $A =$ *card* B
using *assms* **by** (*blast intro*: *card-inj order-antisym*)

27.7 Extensional Function Spaces

definition $PiE :: 'a\ set \Rightarrow ('a \Rightarrow 'b\ set) \Rightarrow ('a \Rightarrow 'b)\ set$
where $PiE\ S\ T = Pi\ S\ T \cap$ *extensional* S

abbreviation $Pi_E\ A\ B \equiv PiE\ A\ B$

syntax
$\text{-}PiE :: pttrn \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set\ \ ((3\Pi_E\ \text{-}\in\text{-}./\ \text{-})\ 10)$
translations
$\Pi_E\ x \in A.\ B \rightleftharpoons CONST\ Pi_E\ A\ (\lambda x.\ B)$

abbreviation *extensional-funcset* $:: 'a\ set \Rightarrow 'b\ set \Rightarrow ('a \Rightarrow 'b)\ set$ (**infixr** $\to_E$ *60*)
where $A \to_E B \equiv (\Pi_E\ i \in A.\ B)$

lemma *extensional-funcset-def*: *extensional-funcset* $S\ T = (S \to T) \cap$ *extensional* S

by (*simp add*: *PiE-def*)

lemma *PiE-empty-domain*[*simp*]: $Pi_E\ \{\}\ T = \{\lambda x.\ undefined\}$
unfolding *PiE-def* **by** *simp*

lemma *PiE-UNIV-domain*: $Pi_E\ UNIV\ T = Pi\ UNIV\ T$
unfolding *PiE-def* **by** *simp*

lemma *PiE-empty-range*[*simp*]: $i \in I \Longrightarrow F\ i = \{\} \Longrightarrow (\Pi_E\ i{\in}I.\ F\ i) = \{\}$
unfolding *PiE-def* **by** *auto*

lemma *PiE-eq-empty-iff*: $Pi_E\ I\ F = \{\} \longleftrightarrow (\exists\, i{\in}I.\ F\ i = \{\})$
proof
assume $Pi_E\ I\ F = \{\}$
show $\exists\, i{\in}I.\ F\ i = \{\}$
proof (*rule ccontr*)
assume $\neg$ *?thesis*
then have $\forall\, i.\ \exists\, y.\ (i \in I \longrightarrow y \in F\ i) \wedge (i \notin I \longrightarrow y = undefined)$
by *auto*
from *choice*[*OF this*]
obtain *f* **where** $\forall\, x.\ (x \in I \longrightarrow f\ x \in F\ x) \wedge (x \notin I \longrightarrow f\ x = undefined)$ **..**
then have $f \in Pi_E\ I\ F$
by (*auto simp*: *extensional-def PiE-def*)
with ‹$Pi_E\ I\ F = \{\}$› **show** *False*
by *auto*
qed
qed (*auto simp*: *PiE-def*)

lemma *PiE-arb*: $f \in Pi_E\ S\ T \Longrightarrow x \notin S \Longrightarrow f\ x = undefined$
unfolding *PiE-def* **by** *auto* (*auto dest*!: *extensional-arb*)

lemma *PiE-mem*: $f \in Pi_E\ S\ T \Longrightarrow x \in S \Longrightarrow f\ x \in T\ x$
unfolding *PiE-def* **by** *auto*

lemma *PiE-fun-upd*: $y \in T\ x \Longrightarrow f \in Pi_E\ S\ T \Longrightarrow f(x := y) \in Pi_E\ (insert\ x\ S)\ T$
unfolding *PiE-def extensional-def* **by** *auto*

lemma *fun-upd-in-PiE*: $x \notin S \Longrightarrow f \in Pi_E\ (insert\ x\ S)\ T \Longrightarrow f(x := undefined) \in Pi_E\ S\ T$
unfolding *PiE-def extensional-def* **by** *auto*

lemma *PiE-insert-eq*: $Pi_E\ (insert\ x\ S)\ T = (\lambda(y,\ g).\ g(x := y))\ `\ (T\ x \times Pi_E\ S\ T)$
proof −
{
fix *f* **assume** $f \in Pi_E\ (insert\ x\ S)\ T\ x \notin S$
then have $f \in (\lambda(y,\ g).\ g(x := y))\ `\ (T\ x \times Pi_E\ S\ T)$
by (*auto intro*!: *image-eqI*[**where** $x{=}(f\ x, f(x := undefined))$] *intro*: *fun-upd-in-PiE*

PiE-mem)
}
moreover
{
fix *f* **assume** $f \in Pi_E\ (insert\ x\ S)\ T\ x \in S$
then have $f \in (\lambda(y,\ g).\ g(x := y))\ `\ (T\ x \times Pi_E\ S\ T)$
by (*auto intro*!: *image-eqI*[**where** *x*=(*f x*, *f*)] *intro*: *fun-upd-in-PiE PiE-mem simp*: *insert-absorb*)
}
ultimately show *?thesis*
by (*auto intro*: *PiE-fun-upd*)
qed

lemma *PiE-Int*: $Pi_E\ I\ A \cap Pi_E\ I\ B = Pi_E\ I\ (\lambda x.\ A\ x \cap B\ x)$
by (*auto simp*: *PiE-def*)

lemma *PiE-cong*: $(\bigwedge i.\ i{\in}I \Longrightarrow A\ i = B\ i) \Longrightarrow Pi_E\ I\ A = Pi_E\ I\ B$
unfolding *PiE-def* **by** (*auto simp*: *Pi-cong*)

lemma *PiE-E* [*elim*]:
assumes $f \in Pi_E\ A\ B$
obtains $x \in A$ **and** $f\ x \in B\ x$
| $x \notin A$ **and** $f\ x = undefined$
using *assms* **by** (*auto simp*: *Pi-def PiE-def extensional-def*)

lemma *PiE-I*[*intro*!]:
$(\bigwedge x.\ x \in A \Longrightarrow f\ x \in B\ x) \Longrightarrow (\bigwedge x.\ x \notin A \Longrightarrow f\ x = undefined) \Longrightarrow f \in Pi_E\ A\ B$
by (*simp add*: *PiE-def extensional-def*)

lemma *PiE-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow Pi_E\ A\ B \subseteq Pi_E\ A\ C$
by *auto*

lemma *PiE-iff*: $f \in Pi_E\ I\ X \longleftrightarrow (\forall\, i{\in}I.\ f\ i \in X\ i) \wedge f \in extensional\ I$
by (*simp add*: *PiE-def Pi-iff*)

lemma *restrict-PiE-iff*: $restrict\ f\ I \in Pi_E\ I\ X \longleftrightarrow (\forall\, i \in I.\ f\ i \in X\ i)$
by (*simp add*: *PiE-iff*)

lemma *ext-funcset-to-sing-iff* [*simp*]: $A \rightarrow_E \{a\} = \{\lambda x{\in}A.\ a\}$
by (*auto simp*: *PiE-def Pi-iff extensionalityI*)

lemma *PiE-restrict*[*simp*]: $f \in Pi_E\ A\ B \Longrightarrow restrict\ f\ A = f$
by (*simp add*: *extensional-restrict PiE-def*)

lemma *restrict-PiE*[*simp*]: $restrict\ f\ I \in Pi_E\ I\ S \longleftrightarrow f \in Pi\ I\ S$
by (*auto simp*: *PiE-iff*)

lemma *PiE-eq-subset*:

assumes *ne*: $\bigwedge i.\ i \in I \Longrightarrow F\ i \neq \{\}$ $\bigwedge i.\ i \in I \Longrightarrow F'\ i \neq \{\}$
and *eq*: $Pi_E\ I\ F = Pi_E\ I\ F'$
and $i \in I$
shows $F\ i \subseteq F'\ i$
proof
fix x
assume $x \in F\ i$
with *ne* **have** $\forall j.\ \exists y.\ (j \in I \longrightarrow y \in F\ j \wedge (i = j \longrightarrow x = y)) \wedge (j \notin I \longrightarrow y = \mathit{undefined})$
by *auto*
from *choice*[*OF this*] **obtain** f
where f: $\forall j.\ (j \in I \longrightarrow f\ j \in F\ j \wedge (i = j \longrightarrow x = f\ j)) \wedge (j \notin I \longrightarrow f\ j = \mathit{undefined})$ **..**
then have $f \in Pi_E\ I\ F$
by (*auto simp*: *extensional-def PiE-def*)
then have $f \in Pi_E\ I\ F'$
using *assms* **by** *simp*
then show $x \in F'\ i$
using $f\ \langle i \in I \rangle$ **by** (*auto simp*: *PiE-def*)
qed

lemma *PiE-eq-iff-not-empty*:
assumes *ne*: $\bigwedge i.\ i \in I \Longrightarrow F\ i \neq \{\}$ $\bigwedge i.\ i \in I \Longrightarrow F'\ i \neq \{\}$
shows $Pi_E\ I\ F = Pi_E\ I\ F' \longleftrightarrow (\forall i{\in}I.\ F\ i = F'\ i)$
proof (*intro iffI ballI*)
fix i
assume *eq*: $Pi_E\ I\ F = Pi_E\ I\ F'$
assume *i*: $i \in I$
show $F\ i = F'\ i$
using *PiE-eq-subset*[*of I F F'*, *OF ne eq i*]
using *PiE-eq-subset*[*of I F' F*, *OF ne*(*2*,*1*) *eq*[*symmetric*] *i*]
by *auto*
qed (*auto simp*: *PiE-def*)

lemma *PiE-eq-iff*:
$Pi_E\ I\ F = Pi_E\ I\ F' \longleftrightarrow (\forall i{\in}I.\ F\ i = F'\ i) \vee ((\exists i{\in}I.\ F\ i = \{\}) \wedge (\exists i{\in}I.\ F'\ i = \{\}))$
proof (*intro iffI disjCI*)
assume *eq*[*simp*]: $Pi_E\ I\ F = Pi_E\ I\ F'$
assume $\neg\ ((\exists i{\in}I.\ F\ i = \{\}) \wedge (\exists i{\in}I.\ F'\ i = \{\}))$
then have $(\forall i{\in}I.\ F\ i \neq \{\}) \wedge (\forall i{\in}I.\ F'\ i \neq \{\})$
using *PiE-eq-empty-iff*[*of I F*] *PiE-eq-empty-iff*[*of I F'*] **by** *auto*
with *PiE-eq-iff-not-empty*[*of I F F'*] **show** $\forall i{\in}I.\ F\ i = F'\ i$
by *auto*
next
assume $(\forall i{\in}I.\ F\ i = F'\ i) \vee (\exists i{\in}I.\ F\ i = \{\}) \wedge (\exists i{\in}I.\ F'\ i = \{\})$
then show $Pi_E\ I\ F = Pi_E\ I\ F'$
using *PiE-eq-empty-iff*[*of I F*] *PiE-eq-empty-iff*[*of I F'*] **by** (*auto simp*: *PiE-def*)
qed

lemma *extensional-funcset-fun-upd-restricts-rangeI*:
$\forall\, y \in S.\ f\ x \neq f\ y \Longrightarrow f \in (\mathit{insert}\ x\ S) \rightarrow_E T \Longrightarrow f(x := \mathit{undefined}) \in S \rightarrow_E (T - \{f\ x\})$
unfolding *extensional-funcset-def extensional-def*
by (*auto split*: *if-split-asm*)

lemma *extensional-funcset-fun-upd-extends-rangeI*:
assumes $a \in T$ $f \in S \rightarrow_E (T - \{a\})$
shows $f(x := a) \in \mathit{insert}\ x\ S \rightarrow_E T$
using *assms* **unfolding** *extensional-funcset-def extensional-def* **by** *auto*

lemma *subset-PiE*:
$\mathit{PiE}\ I\ S \subseteq \mathit{PiE}\ I\ T \longleftrightarrow \mathit{PiE}\ I\ S = \{\} \vee (\forall\, i \in I.\ S\ i \subseteq T\ i)$ (**is** *?lhs* $\longleftrightarrow$ - $\vee$ *?rhs*)
proof (*cases* $\mathit{PiE}\ I\ S = \{\}$)
case *False*
moreover have *?lhs* = *?rhs*
proof
assume *L*: *?lhs*
have $\bigwedge i.\ i \in I \Longrightarrow S\ i \neq \{\}$
using *False PiE-eq-empty-iff* **by** *blast*
with *L* **show** *?rhs*
by (*simp add*: *PiE-Int PiE-eq-iff inf.absorb-iff2*)
qed *auto*
ultimately show *?thesis*
by *simp*
qed *simp*

lemma *PiE-eq*:
$\mathit{PiE}\ I\ S = \mathit{PiE}\ I\ T \longleftrightarrow \mathit{PiE}\ I\ S = \{\} \wedge \mathit{PiE}\ I\ T = \{\} \vee (\forall\, i \in I.\ S\ i = T\ i)$
by (*auto simp*: *PiE-eq-iff PiE-eq-empty-iff*)

lemma *PiE-UNIV* [*simp*]: $\mathit{PiE}\ \mathit{UNIV}\ (\lambda i.\ \mathit{UNIV}) = \mathit{UNIV}$
by *blast*

lemma *image-projection-PiE*:
$(\lambda f.\ f\ i)\ `\ (\mathit{PiE}\ I\ S) = (\mathit{if}\ \mathit{PiE}\ I\ S = \{\}\ \mathit{then}\ \{\}\ \mathit{else\ if}\ i \in I\ \mathit{then}\ S\ i\ \mathit{else}\ \{\mathit{undefined}\})$
proof −
have $(\lambda f.\ f\ i)\ `\ \mathit{Pi}_E\ I\ S = S\ i$ **if** $i \in I$ $f \in \mathit{PiE}\ I\ S$ **for** *f*
using *that* **apply** *auto*
by (*rule-tac* $x{=}(\lambda k.\ \mathit{if}\ k{=}i\ \mathit{then}\ x\ \mathit{else}\ f\ k)$ **in** *image-eqI*) *auto*
moreover have $(\lambda f.\ f\ i)\ `\ \mathit{Pi}_E\ I\ S = \{\mathit{undefined}\}$ **if** $f \in \mathit{PiE}\ I\ S$ $i \notin I$ **for** *f*
using *that* **by** (*blast intro*: *PiE-arb* [*OF that*, *symmetric*])
ultimately show *?thesis*
by *auto*
qed

lemma *PiE-singleton*:
assumes $f \in$ *extensional* A
shows *PiE* A $(\lambda x.\ \{f\ x\}) = \{f\}$
proof −
{
fix g **assume** $g \in$ *PiE* A $(\lambda x.\ \{f\ x\})$
hence $g\ x = f\ x$ **for** x
using *assms* **by** (*cases* $x \in A$) (*auto simp*: *extensional-def*)
hence $g = f$ **by** (*simp add*: *fun-eq-iff*)
}
thus *?thesis* **using** *assms* **by** (*auto simp*: *extensional-def*)
qed

lemma *PiE-eq-singleton*: $(\Pi_E\ i{\in}I.\ S\ i) = \{\lambda i{\in}I.\ f\ i\} \longleftrightarrow (\forall\, i{\in}I.\ S\ i = \{f\ i\})$
by (*metis* (*mono-tags*, *lifting*) *PiE-eq PiE-singleton insert-not-empty restrict-apply'* *restrict-extensional*)

lemma *PiE-over-singleton-iff*: $(\Pi_E\ x{\in}\{a\}.\ B\ x) = (\bigcup b \in B\ a.\ \{\lambda x \in \{a\}.\ b\})$
apply (*auto simp*: *PiE-iff split*: *if-split-asm*)
apply (*metis* (*no-types*, *lifting*) *extensionalityI restrict-apply' restrict-extensional* *singletonD*)
done

lemma *all-PiE-elements*:
$(\forall\, z \in PiE\ I\ S.\ \forall\, i \in I.\ P\ i\ (z\ i)) \longleftrightarrow PiE\ I\ S = \{\} \vee (\forall\, i \in I.\ \forall\, x \in S\ i.\ P\ i\ x)$ (**is** *?lhs* = *?rhs*)
proof (*cases PiE I S* = {})
case *False*
then obtain f **where** f: $\bigwedge i.\ i \in I \Longrightarrow f\ i \in S\ i$
by *fastforce*
show *?thesis*
proof
assume *L*: *?lhs*
have $P\ i\ x$
if $i \in I\ x \in S\ i$ **for** $i\ x$
proof −
have $(\lambda j \in I.\ \mathit{if}\ j{=}i\ \mathit{then}\ x\ \mathit{else}\ f\ j) \in PiE\ I\ S$
by (*simp add*: *f that*(*2*))
then have $P\ i\ ((\lambda j \in I.\ \mathit{if}\ j{=}i\ \mathit{then}\ x\ \mathit{else}\ f\ j)\ i)$
using *L that*(*1*) **by** *blast*
with *that* **show** *?thesis*
by *simp*
qed
then show *?rhs*
by (*simp add*: *False*)
qed *fastforce*
qed *simp*

lemma *PiE-ext*: $[\![x \in PiE\ k\ s;\ y \in PiE\ k\ s;\ \bigwedge i.\ i \in k \Longrightarrow x\ i = y\ i]\!] \Longrightarrow x = y$

by (*metis ext PiE-E*)

27.7.1 Injective Extensional Function Spaces

lemma *extensional-funcset-fun-upd-inj-onI*:
assumes $f \in S \rightarrow_E (T - \{a\})$
and *inj-on f S*
shows *inj-on* $(f(x := a))$ *S*
using *assms*
unfolding *extensional-funcset-def* **by** (*auto intro*!: *inj-on-fun-updI*)

lemma *extensional-funcset-extend-domain-inj-on-eq*:
assumes $x \notin S$
shows $\{f.\ f \in (insert\ x\ S) \rightarrow_E T \wedge inj\text{-}on\ f\ (insert\ x\ S)\} =$
$(\lambda(y,\ g).\ g(x{:=}y))$ ' $\{(y,\ g).\ y \in T \wedge g \in S \rightarrow_E (T - \{y\}) \wedge inj\text{-}on\ g\ S\}$
using *assms*
apply (*auto del*: *PiE-I PiE-E*)
apply (*auto intro*: *extensional-funcset-fun-upd-inj-onI*
extensional-funcset-fun-upd-extends-rangeI del: *PiE-I PiE-E*)
apply (*auto simp add*: *image-iff inj-on-def*)
apply (*rule-tac x=xa x* **in** *exI*)
apply (*auto intro*: *PiE-mem del*: *PiE-I PiE-E*)
apply (*rule-tac x=xa*$(x := undefined)$ **in** *exI*)
apply (*auto intro*!: *extensional-funcset-fun-upd-restricts-rangeI*)
apply (*auto dest*!: *PiE-mem split*: *if-split-asm*)
done

lemma *extensional-funcset-extend-domain-inj-onI*:
assumes $x \notin S$
shows *inj-on* $(\lambda(y,\ g).\ g(x := y))$ $\{(y,\ g).\ y \in T \wedge g \in S \rightarrow_E (T - \{y\}) \wedge$
inj-on g S$\}$
using *assms*
apply (*auto intro*!: *inj-onI*)
apply (*metis fun-upd-same*)
apply (*metis assms PiE-arb fun-upd-triv fun-upd-upd*)
done

27.7.2 Misc properties of functions, composition and restriction from HOL Light

lemma *function-factors-left-gen*:
$(\forall x\ y.\ P\ x \wedge P\ y \wedge g\ x = g\ y \longrightarrow f\ x = f\ y) \longleftrightarrow (\exists h.\ \forall x.\ P\ x \longrightarrow f\ x = h(g\ x))$
(**is** *?lhs* = *?rhs*)
proof
assume *L*: *?lhs*
then show *?rhs*
apply (*rule-tac x=f* $\circ$ *inv-into* (*Collect P*) *g* **in** *exI*)
unfolding *o-def*
by (*metis* (*mono-tags*, *opaque-lifting*) *f-inv-into-f imageI inv-into-into mem-Collect-eq*)
qed *auto*

lemma *function-factors-left*:
$(\forall x\ y.\ (g\ x = g\ y) \longrightarrow (f\ x = f\ y)) \longleftrightarrow (\exists h.\ f = h \circ g)$
using *function-factors-left-gen* [*of* $\lambda x.$ *True g f*] **unfolding** *o-def* **by** *blast*

lemma *function-factors-right-gen*:
$(\forall x.\ P\ x \longrightarrow (\exists y.\ g\ y = f\ x)) \longleftrightarrow (\exists h.\ \forall x.\ P\ x \longrightarrow f\ x = g(h\ x))$
by *metis*

lemma *function-factors-right*:
$(\forall x.\ \exists y.\ g\ y = f\ x) \longleftrightarrow (\exists h.\ f = g \circ h)$
unfolding *o-def* **by** *metis*

lemma *restrict-compose-right*:
restrict $(g \circ$ *restrict f S*$)\ S =$ *restrict* $(g \circ f)\ S$
by *auto*

lemma *restrict-compose-left*:
f ‘ $S \subseteq T \Longrightarrow$ *restrict* (*restrict g T* $\circ f)\ S =$ *restrict* $(g \circ f)\ S$
by *fastforce*

27.7.3 Cardinality

lemma *finite-PiE*: *finite* $S \Longrightarrow (\bigwedge i.\ i \in S \Longrightarrow$ *finite* $(T\ i)) \Longrightarrow$ *finite* $(\Pi_E\ i \in S.\ T\ i)$
by (*induct S arbitrary*: *T rule*: *finite-induct*) (*simp-all add*: *PiE-insert-eq*)

lemma *inj-combinator*: $x \notin S \Longrightarrow$ *inj-on* $(\lambda(y,\ g).\ g(x := y))\ (T\ x \times Pi_E\ S\ T)$
proof (*safe intro*!: *inj-onI ext*)
fix *f y g z*
assume $x \notin S$
assume *fg*: $f \in Pi_E\ S\ T\ g \in Pi_E\ S\ T$
assume $f(x := y) = g(x := z)$
then have $*$: $\bigwedge i.\ (f(x := y))\ i = (g(x := z))\ i$
unfolding *fun-eq-iff* **by** *auto*
from *this*[*of x*] **show** $y = z$ **by** *simp*
fix *i* **from** $*$[*of i*] ‹$x \notin S$› *fg* **show** $f\ i = g\ i$
by (*auto split*: *if-split-asm simp*: *PiE-def extensional-def*)
qed

lemma *card-PiE*: *finite* $S \Longrightarrow$ *card* $(\Pi_E\ i \in S.\ T\ i) = (\prod\ i \in S.$ *card* $(T\ i))$
proof (*induct rule*: *finite-induct*)
case *empty*
then show *?case* **by** *auto*
next
case (*insert x S*)
then show *?case*
by (*simp add*: *PiE-insert-eq inj-combinator card-image card-cartesian-product*)
qed

27.8 The pigeonhole principle

An alternative formulation of this is that for a function mapping a finite set A of cardinality m to a finite set B of cardinality n, there exists an element $y \in B$ that is hit at least $\lceil \frac{m}{n} \rceil$ times. However, since we do not have real numbers or rounding yet, we state it in the following equivalent form:

lemma *pigeonhole-card*:
  assumes $f \in A \to B$ *finite* A *finite* B $B \neq \{\}$
  shows $\exists\, y{\in}B.\ card\ (f -` \{y\} \cap A) * card\ B \geq card\ A$
proof −
  from *assms* **have** $card\ B > 0$
    by *auto*
  define M **where** $M = Max\ ((\lambda y.\ card\ (f -` \{y\} \cap A)) `` B)$
  have $A = (\bigcup y{\in}B.\ f -` \{y\} \cap A)$
    using *assms* **by** *auto*
  also have $card\ \ldots = (\sum i{\in}B.\ card\ (f -` \{i\} \cap A))$
    using *assms* **by** (*subst card-UN-disjoint*) *auto*
  also have $\ldots \leq (\sum i{\in}B.\ M)$
    unfolding *M-def* **using** *assms* **by** (*intro sum-mono Max.coboundedI*) *auto*
  also have $\ldots = card\ B * M$
    by *simp*
  finally have $M * card\ B \geq card\ A$
    by (*simp add*: *mult-ac*)
  moreover have $M \in (\lambda y.\ card\ (f -` \{y\} \cap A)) `` B$
    unfolding *M-def* **using** *assms* ‹$B \neq \{\}$› **by** (*intro Max-in*) *auto*
  ultimately show *?thesis*
    by *blast*
qed

end

28 Partitions and Disjoint Sets

theory *Disjoint-Sets*
  imports *FuncSet*
begin

lemma *mono-imp-UN-eq-last*: *mono* $A \Longrightarrow (\bigcup i{\leq}n.\ A\ i) = A\ n$
  unfolding *mono-def* **by** *auto*

28.1 Set of Disjoint Sets

abbreviation *disjoint* :: *'a set set* ⇒ *bool* **where** *disjoint* ≡ *pairwise disjnt*

lemma *disjoint-def*: *disjoint* $A \longleftrightarrow (\forall\, a{\in}A.\ \forall\, b{\in}A.\ a \neq b \longrightarrow a \cap b = \{\})$
  unfolding *pairwise-def disjnt-def* **by** *auto*

lemma *disjointI*:

$(\bigwedge a\ b.\ a \in A \Longrightarrow b \in A \Longrightarrow a \neq b \Longrightarrow a \cap b = \{\}) \Longrightarrow \mathit{disjoint}\ A$
unfolding *disjoint-def* **by** *auto*

lemma *disjointD*:
$\mathit{disjoint}\ A \Longrightarrow a \in A \Longrightarrow b \in A \Longrightarrow a \neq b \Longrightarrow a \cap b = \{\}$
unfolding *disjoint-def* **by** *auto*

lemma *disjoint-image*: *inj-on* $f\ (\bigcup A) \Longrightarrow \mathit{disjoint}\ A \Longrightarrow \mathit{disjoint}\ ((`)\ f\ `\ A)$
unfolding *inj-on-def disjoint-def* **by** *blast*

lemma assumes *disjoint* $(A \cup B)$
shows *disjoint-unionD1*: *disjoint* A **and** *disjoint-unionD2*: *disjoint* B
using *assms* **by** (*simp-all add*: *disjoint-def*)

lemma *disjoint-INT*:
assumes $*$: $\bigwedge i.\ i \in I \Longrightarrow \mathit{disjoint}\ (F\ i)$
shows *disjoint* $\{\bigcap i{\in}I.\ X\ i \mid X.\ \forall\, i{\in}I.\ X\ i \in F\ i\}$
proof (*safe intro*!: *disjointI del*: *equalityI*)
fix $A\ B :: 'a \Rightarrow 'b\ \mathit{set}$ **assume** $(\bigcap i{\in}I.\ A\ i) \neq (\bigcap i{\in}I.\ B\ i)$
then obtain i **where** $A\ i \neq B\ i\ i \in I$
by *auto*
moreover assume $\forall\, i{\in}I.\ A\ i \in F\ i\ \forall\, i{\in}I.\ B\ i \in F\ i$
ultimately show $(\bigcap i{\in}I.\ A\ i) \cap (\bigcap i{\in}I.\ B\ i) = \{\}$
using $*[\mathit{OF}\ \langle i{\in}I\rangle,\ \mathit{THEN}\ \mathit{disjointD},\ \mathit{of}\ A\ i\ B\ i]$
by (*auto simp flip*: *INT-Int-distrib*)
qed

lemma *diff-Union-pairwise-disjoint*:
assumes *pairwise disjnt* $\mathcal{A}\ \mathcal{B} \subseteq \mathcal{A}$
shows $\bigcup\mathcal{A} - \bigcup\mathcal{B} = \bigcup(\mathcal{A} - \mathcal{B})$
proof −
have *False*
if x: $x \in A\ x \in B$ **and** AB: $A \in \mathcal{A}\ A \notin \mathcal{B}\ B \in \mathcal{B}$ **for** $x\ A\ B$
proof −
have $A \cap B = \{\}$
using *assms disjointD AB* **by** *blast*
with x **show** *?thesis*
by *blast*
qed
then show *?thesis* **by** *auto*
qed

lemma *Int-Union-pairwise-disjoint*:
assumes *pairwise disjnt* $(\mathcal{A} \cup \mathcal{B})$
shows $\bigcup\mathcal{A} \cap \bigcup\mathcal{B} = \bigcup(\mathcal{A} \cap \mathcal{B})$
proof −
have *False*
if x: $x \in A\ x \in B$ **and** AB: $A \in \mathcal{A}\ A \notin \mathcal{B}\ B \in \mathcal{B}$ **for** $x\ A\ B$
proof −

```
    have A ∩ B = {}
      using assms disjointD AB by blast
  with x show ?thesis
    by blast
  qed
  then show ?thesis by auto
qed

lemma psubset-Union-pairwise-disjoint:
  assumes B: pairwise disjnt B and A ⊂ B − {{}}
  shows ⋃A ⊂ ⋃B
  unfolding psubset-eq
proof
  show ⋃A ⊆ ⋃B
    using assms by blast
  have A ⊆ B ⋃(B − A ∩ (B − {{}})) ≠ {}
    using assms by blast+
  then show ⋃A ≠ ⋃B
    using diff-Union-pairwise-disjoint [OF B] by blast
qed
```

28.1.1 Family of Disjoint Sets

```
definition disjoint-family-on :: ('i ⇒ 'a set) ⇒ 'i set ⇒ bool where
  disjoint-family-on A S ⟷ (∀ m∈S. ∀ n∈S. m ≠ n ⟶ A m ∩ A n = {})

abbreviation disjoint-family A ≡ disjoint-family-on A UNIV

lemma disjoint-family-elem-disjnt:
  assumes infinite A finite C
      and df: disjoint-family-on B A
  obtains x where x ∈ A disjnt C (B x)
proof −
  have False if ∗: ∀ x ∈ A. ∃ y. y ∈ C ∧ y ∈ B x
  proof −
    obtain g where g: ∀ x ∈ A. g x ∈ C ∧ g x ∈ B x
      using ∗ by metis
    with df have inj-on g A
      by (fastforce simp add: inj-on-def disjoint-family-on-def)
    then have infinite (g ‘ A)
      using ‹infinite A› finite-image-iff by blast
    then show False
      by (meson ‹finite C› finite-subset g image-subset-iff)
  qed
  then show ?thesis
    by (force simp: disjnt-iff intro: that)
qed

lemma disjoint-family-onD:
```

disjoint-family-on A I $\Longrightarrow$ $i \in I \Longrightarrow j \in I \Longrightarrow i \neq j \Longrightarrow A\ i \cap A\ j = \{\}$
by (*auto simp*: *disjoint-family-on-def*)

lemma *disjoint-family-subset*: *disjoint-family A* $\Longrightarrow$ $(\bigwedge x.\ B\ x \subseteq A\ x) \Longrightarrow$ *disjoint-family B*
by (*force simp add*: *disjoint-family-on-def*)

lemma *disjoint-family-on-insert*:
$i \notin I \Longrightarrow$ *disjoint-family-on A* (*insert i I*) $\longleftrightarrow A\ i \cap (\bigcup i \in I.\ A\ i) = \{\} \wedge$ *disjoint-family-on A I*
by (*fastforce simp*: *disjoint-family-on-def*)

lemma *disjoint-family-on-bisimulation*:
assumes *disjoint-family-on f S*
and $\bigwedge n\ m.\ n \in S \Longrightarrow m \in S \Longrightarrow n \neq m \Longrightarrow f\ n \cap f\ m = \{\} \Longrightarrow g\ n \cap g\ m = \{\}$
shows *disjoint-family-on g S*
using *assms* **unfolding** *disjoint-family-on-def* **by** *auto*

lemma *disjoint-family-on-mono*:
$A \subseteq B \Longrightarrow$ *disjoint-family-on f B* $\Longrightarrow$ *disjoint-family-on f A*
unfolding *disjoint-family-on-def* **by** *auto*

lemma *disjoint-family-Suc*:
$(\bigwedge n.\ A\ n \subseteq A\ (Suc\ n)) \Longrightarrow$ *disjoint-family* $(\lambda i.\ A\ (Suc\ i) - A\ i)$
using *lift-Suc-mono-le*[*of A*]
by (*auto simp add*: *disjoint-family-on-def*)
(*metis insert-absorb insert-subset le-SucE le-antisym not-le-imp-less less-imp-le*)

lemma *disjoint-family-on-disjoint-image*:
disjoint-family-on A I $\Longrightarrow$ *disjoint* $(A\ `\ I)$
unfolding *disjoint-family-on-def disjoint-def* **by** *force*

lemma *disjoint-family-on-vimageI*: *disjoint-family-on F I* $\Longrightarrow$ *disjoint-family-on* $(\lambda i.\ f -` F\ i)\ I$
by (*auto simp*: *disjoint-family-on-def*)

lemma *disjoint-image-disjoint-family-on*:
assumes *d*: *disjoint* $(A\ `\ I)$ **and** *i*: *inj-on A I*
shows *disjoint-family-on A I*
unfolding *disjoint-family-on-def*
proof (*intro ballI impI*)
fix *n m* **assume** *nm*: $m \in I\ n \in I$ **and** $n \neq m$
with *i*[*THEN inj-onD*, *of n m*] **show** $A\ n \cap A\ m = \{\}$
by (*intro disjointD*[*OF d*]) *auto*
qed

lemma *disjoint-family-on-iff-disjoint-image*:
assumes $\bigwedge i.\ i \in I \Longrightarrow A\ i \neq \{\}$

shows *disjoint-family-on A I* ⟷ *disjoint* (*A* ‘ *I*) ∧ *inj-on A I*
proof
assume *disjoint-family-on A I*
then show *disjoint* (*A* ‘ *I*) ∧ *inj-on A I*
by (*metis* (*mono-tags*, *lifting*) *assms disjoint-family-onD disjoint-family-on-disjoint-image inf.idem inj-onI*)
qed (*use disjoint-image-disjoint-family-on* **in** *metis*)

lemma *card-UN-disjoint'*:
assumes *disjoint-family-on A I* ⋀*i*. *i* ∈ *I* ⟹ *finite* (*A i*) *finite I*
shows *card* (⋃*i*∈*I*. *A i*) = (∑*i*∈*I*. *card* (*A i*))
using *assms* **by** (*simp add*: *card-UN-disjoint disjoint-family-on-def*)

lemma *disjoint-UN*:
assumes *F*: ⋀*i*. *i* ∈ *I* ⟹ *disjoint* (*F i*) **and** ∗: *disjoint-family-on* (λ*i*. ⋃(*F i*)) *I*
shows *disjoint* (⋃*i*∈*I*. *F i*)
proof (*safe intro*!: *disjointI del*: *equalityI*)
fix *A B i j* **assume** *A* ≠ *B A* ∈ *F i i* ∈ *I B* ∈ *F j j* ∈ *I*
show *A* ∩ *B* = {}
proof *cases*
assume *i* = *j* **with** *F*[*of i*] ‹*i* ∈ *I*› ‹*A* ∈ *F i*› ‹*B* ∈ *F j*› ‹*A* ≠ *B*› **show** *A* ∩ *B* = {}
by (*auto dest*: *disjointD*)
next
assume *i* ≠ *j*
with ∗ ‹*i*∈*I*› ‹*j*∈*I*› **have** (⋃(*F i*)) ∩ (⋃(*F j*)) = {}
by (*rule disjoint-family-onD*)
with ‹*A*∈*F i*› ‹*i*∈*I*› ‹*B*∈*F j*› ‹*j*∈*I*›
show *A* ∩ *B* = {}
by *auto*
qed
qed

lemma *distinct-list-bind*:
assumes *distinct xs* ⋀*x*. *x* ∈ *set xs* ⟹ *distinct* (*f x*)
disjoint-family-on (*set* ∘ *f*) (*set xs*)
shows *distinct* (*List.bind xs f*)
using *assms*
by (*induction xs*)
(*auto simp*: *disjoint-family-on-def distinct-map inj-on-def set-list-bind*)

lemma *bij-betw-UNION-disjoint*:
assumes *disj*: *disjoint-family-on A' I*
assumes *bij*: ⋀*i*. *i* ∈ *I* ⟹ *bij-betw f* (*A i*) (*A' i*)
shows *bij-betw f* (⋃*i*∈*I*. *A i*) (⋃*i*∈*I*. *A' i*)
unfolding *bij-betw-def*
proof
from *bij* **show** *eq*: *f* ‘ ⋃(*A* ‘ *I*) = ⋃(*A'* ‘ *I*)

by (*auto simp*: *bij-betw-def image-UN*)
show *inj-on f* (⋃(*A* ' *I*))
proof (*rule inj-onI*, *clarify*)
fix *i j x y* **assume** *A*: *i* ∈ *I j* ∈ *I x* ∈ *A i y* ∈ *A j* **and** *B*: *f x* = *f y*
from *A bij*[*of i*] *bij*[*of j*] **have** *f x* ∈ *A*′ *i f y* ∈ *A*′ *j*
by (*auto simp*: *bij-betw-def*)
with *B* **have** *A*′ *i* ∩ *A*′ *j* ≠ {} **by** *auto*
with *disj A* **have** *i* = *j* **unfolding** *disjoint-family-on-def* **by** *blast*
with *A B bij*[*of i*] **show** *x* = *y* **by** (*auto simp*: *bij-betw-def dest*: *inj-onD*)
qed
qed

lemma *disjoint-union*: *disjoint C* ⟹ *disjoint B* ⟹ ⋃ *C* ∩ ⋃ *B* = {} ⟹ *disjoint* (*C* ∪ *B*)
using *disjoint-UN*[*of* {*C*, *B*} λ*x*. *x*] **by** (*auto simp add*: *disjoint-family-on-def*)

Sum/product of the union of a finite disjoint family

context *comm-monoid-set*
begin

lemma *UNION-disjoint-family*:
assumes *finite I* **and** ∀ *i*∈*I*. *finite* (*A i*)
and *disjoint-family-on A I*
shows *F g* (⋃(*A* ' *I*)) = *F* (λ*x*. *F g* (*A x*)) *I*
using *assms* **unfolding** *disjoint-family-on-def* **by** (*rule UNION-disjoint*)

lemma *Union-disjoint-sets*:
assumes ∀ *A*∈*C*. *finite A* **and** *disjoint C*
shows *F g* (⋃ *C*) = (*F* ∘ *F*) *g C*
using *assms* **unfolding** *disjoint-def* **by** (*rule Union-disjoint*)

end

The union of an infinite disjoint family of non-empty sets is infinite.

lemma *infinite-disjoint-family-imp-infinite-UNION*:
assumes ¬*finite A* ⋀*x*. *x* ∈ *A* ⟹ *f x* ≠ {} *disjoint-family-on f A*
shows ¬*finite* (⋃(*f* ' *A*))
proof −
define *g* **where** *g x* = (*SOME y*. *y* ∈ *f x*) **for** *x*
have *g*: *g x* ∈ *f x* **if** *x* ∈ *A* **for** *x*
unfolding *g-def* **by** (*rule someI-ex*, *insert assms*(*2*) *that*) *blast*
have *inj-on-g*: *inj-on g A*
proof (*rule inj-onI*, *rule ccontr*)
fix *x y* **assume** *A*: *x* ∈ *A y* ∈ *A g x* = *g y x* ≠ *y*
with *g*[*of x*] *g*[*of y*] **have** *g x* ∈ *f x g x* ∈ *f y* **by** *auto*
with *A* ‹*x* ≠ *y*› *assms* **show** *False*
by (*auto simp*: *disjoint-family-on-def inj-on-def*)
qed
from *g* **have** *g* ' *A* ⊆ ⋃(*f* ' *A*) **by** *blast*

moreover from *inj-on-g* ‹¬*finite A*› **have** ¬*finite* (*g* ‘ *A*)
using *finite-imageD* **by** *blast*
ultimately show *?thesis* **using** *finite-subset* **by** *blast*
qed

28.2 Construct Disjoint Sequences

definition *disjointed* :: (*nat* ⇒ *'a set*) ⇒ *nat* ⇒ *'a set* **where**
disjointed A n = *A n* − (⋃ *i*∈{*0*..<*n*}. *A i*)

lemma *finite-UN-disjointed-eq*: (⋃ *i*∈{*0*..<*n*}. *disjointed A i*) = (⋃ *i*∈{*0*..<*n*}. *A i*)
proof (*induct n*)
case *0* **show** *?case* **by** *simp*
next
case (*Suc n*)
thus *?case* **by** (*simp add*: *atLeastLessThanSuc disjointed-def*)
qed

lemma *UN-disjointed-eq*: (⋃ *i*. *disjointed A i*) = (⋃ *i*. *A i*)
by (*rule UN-finite2-eq* [**where** *k*=*0*])
(*simp add*: *finite-UN-disjointed-eq*)

lemma *less-disjoint-disjointed*: *m* < *n* ⟹ *disjointed A m* ∩ *disjointed A n* = {}
by (*auto simp add*: *disjointed-def*)

lemma *disjoint-family-disjointed*: *disjoint-family* (*disjointed A*)
by (*simp add*: *disjoint-family-on-def*)
(*metis neq-iff Int-commute less-disjoint-disjointed*)

lemma *disjointed-subset*: *disjointed A n* ⊆ *A n*
by (*auto simp add*: *disjointed-def*)

lemma *disjointed-0*[*simp*]: *disjointed A 0* = *A 0*
by (*simp add*: *disjointed-def*)

lemma *disjointed-mono*: *mono A* ⟹ *disjointed A* (*Suc n*) = *A* (*Suc n*) − *A n*
using *mono-imp-UN-eq-last*[*of A*] **by** (*simp add*: *disjointed-def atLeastLessThanSuc-atLeastAtMost atLeast0AtMost*)

28.3 Partitions

Partitions P of a set A. We explicitly disallow empty sets.

definition *partition-on* :: *'a set* ⇒ *'a set set* ⇒ *bool*
where
partition-on A P ⟷ ⋃ *P* = *A* ∧ *disjoint P* ∧ {} ∉ *P*

lemma *partition-onI*:

$\bigcup P = A \Longrightarrow (\bigwedge p\ q.\ p \in P \Longrightarrow q \in P \Longrightarrow p \neq q \Longrightarrow \mathit{disjnt}\ p\ q) \Longrightarrow \{\} \notin P \Longrightarrow \mathit{partition\text{-}on}\ A\ P$
by (*auto simp*: *partition-on-def pairwise-def*)

lemma *partition-onD1*: $\mathit{partition\text{-}on}\ A\ P \Longrightarrow A = \bigcup P$
by (*auto simp*: *partition-on-def*)

lemma *partition-onD2*: $\mathit{partition\text{-}on}\ A\ P \Longrightarrow \mathit{disjoint}\ P$
by (*auto simp*: *partition-on-def*)

lemma *partition-onD3*: $\mathit{partition\text{-}on}\ A\ P \Longrightarrow \{\} \notin P$
by (*auto simp*: *partition-on-def*)

28.4 Constructions of partitions

lemma *partition-on-empty*: $\mathit{partition\text{-}on}\ \{\}\ P \longleftrightarrow P = \{\}$
unfolding *partition-on-def* **by** *fastforce*

lemma *partition-on-space*: $A \neq \{\} \Longrightarrow \mathit{partition\text{-}on}\ A\ \{A\}$
by (*auto simp*: *partition-on-def disjoint-def*)

lemma *partition-on-singletons*: $\mathit{partition\text{-}on}\ A\ ((\lambda x.\ \{x\})\ `\ A)$
by (*auto simp*: *partition-on-def disjoint-def*)

lemma *partition-on-transform*:
assumes *P*: $\mathit{partition\text{-}on}\ A\ P$
assumes *F-UN*: $\bigcup (F\ `\ P) = F\ (\bigcup P)$ **and** *F-disjnt*: $\bigwedge p\ q.\ p \in P \Longrightarrow q \in P \Longrightarrow \mathit{disjnt}\ p\ q \Longrightarrow \mathit{disjnt}\ (F\ p)\ (F\ q)$
shows $\mathit{partition\text{-}on}\ (F\ A)\ (F\ `\ P - \{\{\}\})$
proof −
have $\bigcup (F\ `\ P - \{\{\}\}) = F\ A$
unfolding *P*[*THEN partition-onD1*] *F-UN*[*symmetric*] **by** *auto*
with *P* **show** *?thesis*
by (*auto simp add*: *partition-on-def pairwise-def intro*!: *F-disjnt*)
qed

lemma *partition-on-restrict*: $\mathit{partition\text{-}on}\ A\ P \Longrightarrow \mathit{partition\text{-}on}\ (B \cap A)\ ((\cap)\ B\ `\ P - \{\{\}\})$
by (*intro partition-on-transform*) (*auto simp*: *disjnt-def*)

lemma *partition-on-vimage*: $\mathit{partition\text{-}on}\ A\ P \Longrightarrow \mathit{partition\text{-}on}\ (f\ -`\ A)\ ((-`)\ f\ `\ P - \{\{\}\})$
by (*intro partition-on-transform*) (*auto simp*: *disjnt-def*)

lemma *partition-on-inj-image*:
assumes *P*: $\mathit{partition\text{-}on}\ A\ P$ **and** *f*: $\mathit{inj\text{-}on}\ f\ A$
shows $\mathit{partition\text{-}on}\ (f\ `\ A)\ ((`)\ f\ `\ P - \{\{\}\})$
proof (*rule partition-on-transform*[*OF P*])
show $p \in P \Longrightarrow q \in P \Longrightarrow \mathit{disjnt}\ p\ q \Longrightarrow \mathit{disjnt}\ (f\ `\ p)\ (f\ `\ q)$ **for** $p\ q$

using *f*[*THEN inj-onD*] *P*[*THEN partition-onD1*] **by** (*auto simp*: *disjnt-def*)
qed *auto*

lemma *partition-on-insert*:
assumes *disjnt p* (⋃ *P*)
shows *partition-on A* (*insert p P*) ⟷ *partition-on* (*A*−*p*) *P* ∧ *p* ⊆ *A* ∧ *p* ≠ {}
using *assms*
by (*auto simp*: *partition-on-def disjnt-iff pairwise-insert*)

28.5 Finiteness of partitions

lemma *finitely-many-partition-on*:
assumes *finite A*
shows *finite* {*P*. *partition-on A P*}
proof (*rule finite-subset*)
show {*P*. *partition-on A P*} ⊆ *Pow* (*Pow A*)
unfolding *partition-on-def* **by** *auto*
show *finite* (*Pow* (*Pow A*))
using *assms* **by** *simp*
qed

lemma *finite-elements*: *finite A* ⟹ *partition-on A P* ⟹ *finite P*
using *partition-onD1*[*of A P*] **by** (*simp add*: *finite-UnionD*)

lemma *product-partition*:
assumes *partition-on A P* **and** ⋀*p*. *p* ∈ *P* ⟹ *finite p*
shows *card A* = (∑ *p*∈*P*. *card p*)
using *assms* **unfolding** *partition-on-def* **by** (*meson card-Union-disjoint*)

28.6 Equivalence of partitions and equivalence classes

lemma *partition-on-quotient*:
assumes *r*: *equiv A r*
shows *partition-on A* (*A* // *r*)
proof (*rule partition-onI*)
from *r* **have** *refl-on A r*
by (*auto elim*: *equivE*)
then show ⋃(*A* // *r*) = *A* {} ∉ *A* // *r*
by (*auto simp*: *refl-on-def quotient-def*)

fix *p q* **assume** *p* ∈ *A* // *r q* ∈ *A* // *r p* ≠ *q*
then obtain *x y* **where** *x* ∈ *A y* ∈ *A p* = *r* `` {*x*} *q* = *r* `` {*y*}
by (*auto simp*: *quotient-def*)
with *r equiv-class-eq-iff*[*OF r*, *of x y*] ‹*p* ≠ *q*› **show** *disjnt p q*
by (*auto simp*: *disjnt-equiv-class*)
qed

lemma *equiv-partition-on*:
assumes *P*: *partition-on A P*
shows *equiv A* {(*x*, *y*). ∃ *p* ∈ *P*. *x* ∈ *p* ∧ *y* ∈ *p*}

proof (*rule equivI*)
have $A = \bigcup P$ *disjoint* P $\{\} \notin P$
using P **by** (*auto simp*: *partition-on-def*)
then show *refl-on* A $\{(x, y).\ \exists p{\in}P.\ x \in p \wedge y \in p\}$
unfolding *refl-on-def* **by** *auto*
show *trans* $\{(x, y).\ \exists p{\in}P.\ x \in p \wedge y \in p\}$
using ‹*disjoint P*› **by** (*auto simp*: *trans-def disjoint-def*)
qed (*auto simp*: *sym-def*)

lemma *partition-on-eq-quotient*:
assumes P: *partition-on* A P
shows A // $\{(x, y).\ \exists p \in P.\ x \in p \wedge y \in p\} = P$
unfolding *quotient-def*
proof *safe*
fix x **assume** $x \in A$
then obtain p **where** $p \in P$ $x \in p$ $\bigwedge q.\ q \in P \Longrightarrow x \in q \Longrightarrow p = q$
using P **by** (*auto simp*: *partition-on-def disjoint-def*)
then have $\{y.\ \exists p{\in}P.\ x \in p \wedge y \in p\} = p$
by (*safe intro*!: *bexI*[*of* - *p*]) *simp*
then show $\{(x, y).\ \exists p{\in}P.\ x \in p \wedge y \in p\}$ `` $\{x\} \in P$
by (*simp add*: ‹$p \in P$›)
next
fix p **assume** $p \in P$
then have $p \neq \{\}$
using P **by** (*auto simp*: *partition-on-def*)
then obtain x **where** $x \in p$
by *auto*
then have $x \in A$ $\bigwedge q.\ q \in P \Longrightarrow x \in q \Longrightarrow p = q$
using P ‹$p \in P$› **by** (*auto simp*: *partition-on-def disjoint-def*)
with ‹$p{\in}P$› ‹$x \in p$› **have** $\{y.\ \exists p{\in}P.\ x \in p \wedge y \in p\} = p$
by (*safe intro*!: *bexI*[*of* - *p*]) *simp*
then show $p \in (\bigcup x{\in}A.\ \{\{(x, y).\ \exists p{\in}P.\ x \in p \wedge y \in p\}$ `` $\{x\}\})$
by (*auto intro*: ‹$x \in A$›)
qed

lemma *partition-on-alt*: *partition-on* A $P \longleftrightarrow (\exists r.\ \mathit{equiv}\ A\ r \wedge P = A\ //\ r)$
by (*auto simp*: *partition-on-eq-quotient intro*!: *partition-on-quotient intro*: *equiv-partition-on*)

28.7 Refinement of partitions

definition *refines* :: *'a set* ⇒ *'a set set* ⇒ *'a set set* ⇒ *bool*
where *refines* A P $Q \equiv$
partition-on A $P \wedge$ *partition-on* A $Q \wedge (\forall X{\in}P.\ \exists Y{\in}Q.\ X \subseteq Y)$

lemma *refines-refl*: *partition-on* A $P \Longrightarrow$ *refines* A P P
using *refines-def* **by** *blast*

lemma *refines-asym1*:
assumes *refines* A P Q *refines* A Q P

shows $P \subseteq Q$
proof
fix X
assume $X \in P$
then obtain $Y\ X'$ **where** $Y \in Q\ X \subseteq Y\ X' \in P\ Y \subseteq X'$
by (*meson assms refines-def*)
then have $X' = X$
using *assms*(*2*) **unfolding** *partition-on-def refines-def*
by (*metis* ‹$X \in P$› ‹$X \subseteq Y$› *disjnt-self-iff-empty disjnt-subset1 pairwiseD*)
then show $X \in Q$
using ‹$X \subseteq Y$› ‹$Y \in Q$› ‹$Y \subseteq X'$› **by** *force*
qed

lemma *refines-asym*: ⟦*refines A P Q*; *refines A Q P*⟧ ⟹ P=Q
by (*meson antisym-conv refines-asym1*)

lemma *refines-trans*: ⟦*refines A P Q*; *refines A Q R*⟧ ⟹ *refines A P R*
by (*meson order.trans refines-def*)

lemma *refines-obtains-subset*:
assumes *refines A P Q q* ∈ *Q*
shows *partition-on q* {$p \in P.\ p \subseteq q$}
proof −
have $p \subseteq q \vee$ *disjnt p q* **if** $p \in P$ **for** p
using *that assms* **unfolding** *refines-def partition-on-def disjoint-def*
by (*metis disjnt-def disjnt-subset1*)
with *assms* **have** *q* ⊆ *Union* {$p \in P.\ p \subseteq q$}
using *assms*
by (*clarsimp simp*: *refines-def disjnt-iff partition-on-def*) (*metis Union-iff*)
with *assms* **have** *q* = *Union* {$p \in P.\ p \subseteq q$}
by *auto*
then show *?thesis*
using *assms* **by** (*auto simp*: *refines-def disjoint-def partition-on-def*)
qed

28.8 The coarsest common refinement of a set of partitions

definition *common-refinement* :: *'a set set set* ⇒ *'a set set*
where *common-refinement* $\mathcal{P}$ ≡ (⋃*f* ∈ (Π$_E$ *P*∈$\mathcal{P}$. *P*). {⋂ (*f* ‘ $\mathcal{P}$)}) − {{}}

With non-extensional function space

lemma *common-refinement*: *common-refinement* $\mathcal{P}$ = (⋃*f* ∈ (Π *P*∈$\mathcal{P}$. *P*). {⋂ (*f* ‘ $\mathcal{P}$)}) − {{}}
(**is** *?lhs* = *?rhs*)
proof
show *?rhs* ⊆ *?lhs*
apply (*clarsimp simp add*: *common-refinement-def PiE-def Ball-def*)
by (*metis restrict-Pi-cancel image-restrict-eq restrict-extensional*)
qed (*auto simp add*: *common-refinement-def PiE-def*)

lemma *common-refinement-exists*: $[\![X \in \mathit{common\text{-}refinement}\ \mathcal{P};\ P \in \mathcal{P}]\!] \Longrightarrow \exists\, R{\in}P.\ X \subseteq R$
  by (*auto simp add*: *common-refinement*)

lemma *Union-common-refinement*: $\bigcup\ (\mathit{common\text{-}refinement}\ \mathcal{P}) = (\bigcap\ P{\in}\mathcal{P}.\ \bigcup P)$
proof
  show $(\bigcap\ P{\in}\mathcal{P}.\ \bigcup P) \subseteq \bigcup\ (\mathit{common\text{-}refinement}\ \mathcal{P})$
  proof (*clarsimp simp*: *common-refinement*)
    fix x
    assume $\forall\, P{\in}\mathcal{P}.\ \exists\, X{\in}P.\ x \in X$
    then obtain F **where** F: $\bigwedge P.\ P{\in}\mathcal{P} \Longrightarrow F\ P \in P \wedge x \in F\ P$
      by *metis*
    then have $x \in \bigcap\ (F\ `\ \mathcal{P})$
      by *force*
    with F **show** $\exists\, X{\in}(\bigcup x{\in}\Pi\ P{\in}\mathcal{P}.\ P.\ \{\bigcap\ (x\ `\ \mathcal{P})\}) - \{\{\}\}.\ x \in X$
      by (*auto simp add*: *Pi-iff Bex-def*)
  qed
qed (*auto simp*: *common-refinement-def*)

lemma *partition-on-common-refinement*:
  assumes A: $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow \mathit{partition\text{-}on}\ A\ P$ **and** $\mathcal{P} \neq \{\}$
  shows *partition-on* A (*common-refinement* $\mathcal{P}$)
proof (*rule partition-onI*)
  show $\bigcup\ (\mathit{common\text{-}refinement}\ \mathcal{P}) = A$
    using *assms* **by** (*simp add*: *partition-on-def Union-common-refinement*)
  fix $P\ Q$
  assume $P \in \mathit{common\text{-}refinement}\ \mathcal{P}$ **and** $Q \in \mathit{common\text{-}refinement}\ \mathcal{P}$ **and** $P \neq Q$
  then obtain $f\ g$ **where** f: $f \in (\Pi_E\ P{\in}\mathcal{P}.\ P)$ **and** P: $P = \bigcap\ (f\ `\ \mathcal{P})$ **and** $P \neq \{\}$
                and g: $g \in (\Pi_E\ P{\in}\mathcal{P}.\ P)$ **and** Q: $Q = \bigcap\ (g\ `\ \mathcal{P})$ **and** $Q \neq \{\}$
    by (*auto simp add*: *common-refinement-def*)
  have $f{=}g$ **if** $x \in P\ x \in Q$ **for** x
  proof (*rule extensionalityI* [*of* - $\mathcal{P}$])
    fix R
    assume $R \in \mathcal{P}$
    with *that* $P\ Q\ f\ g\ A$ [*unfolded partition-on-def*, *OF* ‹$R \in \mathcal{P}$›]
    show $f\ R = g\ R$
      by (*metis INT-E Int-iff PiE-iff disjointD emptyE*)
  qed (*use PiE-iff f g* **in** *auto*)
  then show *disjnt* $P\ Q$
    by (*metis* $P\ Q$ ‹$P \neq Q$› *disjnt-iff*)
qed (*simp add*: *common-refinement-def*)

lemma *refines-common-refinement*:
  assumes $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow \mathit{partition\text{-}on}\ A\ P\ P \in \mathcal{P}$
  shows *refines* A (*common-refinement* $\mathcal{P}$) P
  unfolding *refines-def*
proof (*intro conjI strip*)

fix X
assume $X \in$ *common-refinement* $\mathcal{P}$
with *assms* **show** $\exists\, Y{\in}P.\ X \subseteq Y$
by (*auto simp*: *common-refinement-def*)
qed (*use assms partition-on-common-refinement* **in** *auto*)

The common refinement is itself refined by any other

lemma *common-refinement-coarsest*:
assumes $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow$ *partition-on A P partition-on A R* $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow$ *refines A R P* $\mathcal{P} \neq \{\}$
shows *refines A R* (*common-refinement* $\mathcal{P}$)
unfolding *refines-def*
proof (*intro conjI ballI partition-on-common-refinement*)
fix X
assume $X \in R$
have $\exists\, p \in P.\ X \subseteq p$ **if** $P \in \mathcal{P}$ **for** P
by (*meson* ‹$X \in R$› *assms*(*3*) *refines-def that*)
then obtain F **where** f: $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow F\ P \in P \wedge X \subseteq F\ P$
by *metis*
with ‹*partition-on A R*› ‹$X \in R$› ‹$\mathcal{P} \neq \{\}$›
have $\bigcap\ (F\ `\ \mathcal{P}) \in$ *common-refinement* $\mathcal{P}$
apply (*simp add*: *partition-on-def common-refinement Pi-iff Bex-def*)
by (*metis* (*no-types*, *lifting*) *cINF-greatest subset-empty*)
with f **show** $\exists\, Y{\in}$*common-refinement* $\mathcal{P}.\ X \subseteq Y$
by (*metis* ‹$\mathcal{P} \neq \{\}$› *cINF-greatest*)
qed (*use assms* **in** *auto*)

lemma *finite-common-refinement*:
assumes *finite* $\mathcal{P}$ $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow$ *finite P*
shows *finite* (*common-refinement* $\mathcal{P}$)
proof −
have *finite* $(\Pi_E\ P{\in}\mathcal{P}.\ P)$
by (*simp add*: *assms finite-PiE*)
then show *?thesis*
by (*auto simp*: *common-refinement-def*)
qed

lemma *card-common-refinement*:
assumes *finite* $\mathcal{P}$ $\bigwedge P.\ P \in \mathcal{P} \Longrightarrow$ *finite P*
shows *card* (*common-refinement* $\mathcal{P}$) $\leq (\prod P \in \mathcal{P}.\ card\ P)$
proof −
have *card* (*common-refinement* $\mathcal{P}$) $\leq$ *card* $(\bigcup f \in (\Pi_E\ P{\in}\mathcal{P}.\ P).\ \{\bigcap\ (f\ `\ \mathcal{P})\})$
unfolding *common-refinement-def* **by** (*meson card-Diff1-le*)
also have $\ldots \leq (\sum f{\in}(\Pi_E\ P{\in}\mathcal{P}.\ P).\ card\{\bigcap\ (f\ `\ \mathcal{P})\})$
by (*metis assms finite-PiE card-UN-le*)
also have $\ldots =$ *card*$(\Pi_E\ P{\in}\mathcal{P}.\ P)$
by *simp*
also have $\ldots = (\prod P \in \mathcal{P}.\ card\ P)$
by (*simp add*: *assms*(*1*) *card-PiE dual-order.eq-iff*)

finally show *?thesis* **.**
qed

end

29 Type of finite sets defined as a subtype of sets

theory *FSet*
imports *Main Countable*
begin

29.1 Definition of the type

typedef *'a fset* = {*A* :: *'a set*. *finite A*} **morphisms** *fset Abs-fset*
by *auto*

setup-lifting *type-definition-fset*

29.2 Basic operations and type class instantiations

instantiation *fset* :: (*finite*) *finite*
begin
instance by (*standard*; *transfer*; *simp*)
end

instantiation *fset* :: (*type*) {*bounded-lattice-bot*, *distrib-lattice*, *minus*}
begin

lift-definition *bot-fset* :: *'a fset* **is** {} **parametric** *empty-transfer* **by** *simp*

lift-definition *less-eq-fset* :: *'a fset* ⇒ *'a fset* ⇒ *bool* **is** *subset-eq* **parametric** *subset-transfer*
.

definition *less-fset* :: *'a fset* ⇒ *'a fset* ⇒ *bool* **where** $xs < ys \equiv xs \leq ys \wedge xs \neq (ys::'a\ fset)$

lemma *less-fset-transfer*[*transfer-rule*]:
includes *lifting-syntax*
assumes [*transfer-rule*]: *bi-unique A*
shows ((*pcr-fset A*) ===> (*pcr-fset A*) ===> (=)) (⊂) (<)
unfolding *less-fset-def*[*abs-def*] *psubset-eq*[*abs-def*] **by** *transfer-prover*

lift-definition *sup-fset* :: *'a fset* ⇒ *'a fset* ⇒ *'a fset* **is** *union* **parametric** *union-transfer*
by *simp*

lift-definition *inf-fset* :: *'a fset* ⇒ *'a fset* ⇒ *'a fset* **is** *inter* **parametric** *inter-transfer*

by *simp*

lift-definition *minus-fset* :: *'a fset* ⇒ *'a fset* ⇒ *'a fset* **is** *minus* **parametric** *Diff-transfer*
by *simp*

instance
by (*standard*; *transfer*; *auto*)+

end

abbreviation *fempty* :: *'a fset* ({||}) **where** {||} ≡ *bot*
abbreviation *fsubset-eq* :: *'a fset* ⇒ *'a fset* ⇒ *bool* (**infix** |⊆| *50*) **where** *xs* |⊆| *ys* ≡ *xs* ≤ *ys*
abbreviation *fsubset* :: *'a fset* ⇒ *'a fset* ⇒ *bool* (**infix** |⊂| *50*) **where** *xs* |⊂| *ys* ≡ *xs* < *ys*
abbreviation *funion* :: *'a fset* ⇒ *'a fset* ⇒ *'a fset* (**infixl** |∪| *65*) **where** *xs* |∪| *ys* ≡ *sup xs ys*
abbreviation *finter* :: *'a fset* ⇒ *'a fset* ⇒ *'a fset* (**infixl** |∩| *65*) **where** *xs* |∩| *ys* ≡ *inf xs ys*
abbreviation *fminus* :: *'a fset* ⇒ *'a fset* ⇒ *'a fset* (**infixl** |−| *65*) **where** *xs* |−| *ys* ≡ *minus xs ys*

instantiation *fset* :: (*equal*) *equal*
begin
definition *HOL.equal A B* ⟷ *A* |⊆| *B* ∧ *B* |⊆| *A*
instance by *intro-classes* (*auto simp add*: *equal-fset-def*)
end

instantiation *fset* :: (*type*) *conditionally-complete-lattice*
begin

context includes *lifting-syntax*
begin

lemma *right-total-Inf-fset-transfer*:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *right-total A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*)
(λ*S*. *if finite* (⋂ *S* ∩ *Collect* (*Domainp A*)) *then* ⋂ *S* ∩ *Collect* (*Domainp A*) *else* {})
(λ*S*. *if finite* (*Inf S*) *then Inf S else* {})
by *transfer-prover*

lemma *Inf-fset-transfer*:
assumes [*transfer-rule*]: *bi-unique A* **and** [*transfer-rule*]: *bi-total A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*) (λ*A*. *if finite* (*Inf A*) *then Inf A else* {})
(λ*A*. *if finite* (*Inf A*) *then Inf A else* {})
by *transfer-prover*

lift-definition *Inf-fset* :: *'a fset set* ⇒ *'a fset* **is** λ*A*. *if finite* (*Inf A*) *then Inf A else* {}
parametric *right-total-Inf-fset-transfer Inf-fset-transfer* **by** *simp*

lemma *Sup-fset-transfer*:
assumes [*transfer-rule*]: *bi-unique A*
shows (*rel-set* (*rel-set A*) ===> *rel-set A*) (λ*A*. *if finite* (*Sup A*) *then Sup A else* {})
(λ*A*. *if finite* (*Sup A*) *then Sup A else* {}) **by** *transfer-prover*

lift-definition *Sup-fset* :: *'a fset set* ⇒ *'a fset* **is** λ*A*. *if finite* (*Sup A*) *then Sup A else* {}
parametric *Sup-fset-transfer* **by** *simp*

lemma *finite-Sup*: ∃ *z*. *finite z* ∧ (∀ *a*. *a* ∈ *X* ⟶ *a* ≤ *z*) ⟹ *finite* (*Sup X*)
by (*auto intro*: *finite-subset*)

lemma *transfer-bdd-below*[*transfer-rule*]: (*rel-set* (*pcr-fset* (=)) ===> (=)) *bdd-below bdd-below*
by *auto*

end

instance
proof
fix *x z* :: *'a fset*
fix *X* :: *'a fset set*
{
assume *x* ∈ *X bdd-below X*
then show *Inf X* |⊆| *x* **by** *transfer auto*
next
assume *X* ≠ {} (⋀*x*. *x* ∈ *X* ⟹ *z* |⊆| *x*)
then show *z* |⊆| *Inf X* **by** *transfer* (*clarsimp*, *blast*)
next
assume *x* ∈ *X bdd-above X*
then obtain *z* **where** *x* ∈ *X* (⋀*x*. *x* ∈ *X* ⟹ *x* |⊆| *z*)
by (*auto simp*: *bdd-above-def*)
then show *x* |⊆| *Sup X*
by *transfer* (*auto intro*!: *finite-Sup*)
next
assume *X* ≠ {} (⋀*x*. *x* ∈ *X* ⟹ *x* |⊆| *z*)
then show *Sup X* |⊆| *z* **by** *transfer* (*clarsimp*, *blast*)
}
qed
end

instantiation *fset* :: (*finite*) *complete-lattice*
begin

lift-definition *top-fset* :: *'a fset* **is** *UNIV* **parametric** *right-total-UNIV-transfer UNIV-transfer*
by *simp*

instance
by (*standard*; *transfer*; *auto*)

end

instantiation *fset* :: (*finite*) *complete-boolean-algebra*
begin

lift-definition *uminus-fset* :: *'a fset* ⇒ *'a fset* **is** *uminus*
parametric *right-total-Compl-transfer Compl-transfer* **by** *simp*

instance
by (*standard*; *transfer*) (*simp-all add*: *Inf-Sup Diff-eq*)
end

abbreviation *fUNIV* :: *'a*::*finite fset* **where** *fUNIV* ≡ *top*
abbreviation *fuminus* :: *'a*::*finite fset* ⇒ *'a fset* (|−| - [*81*] *80*) **where** |−| *x* ≡ *uminus x*

declare *top-fset.rep-eq*[*simp*]

29.3 Other operations

lift-definition *finsert* :: *'a* ⇒ *'a fset* ⇒ *'a fset* **is** *insert* **parametric** *Lifting-Set.insert-transfer*
by *simp*

syntax
-insert-fset :: *args* => *'a fset* ({|(-)|})

translations
{|*x*, *xs*|} == *CONST finsert x* {|*xs*|}
{|*x*|} == *CONST finsert x* {||}

lift-definition *fmember* :: *'a* ⇒ *'a fset* ⇒ *bool* (**infix** |∈| *50*) **is** *Set.member*
parametric *member-transfer* **.**

abbreviation *notin-fset* :: *'a* ⇒ *'a fset* ⇒ *bool* (**infix** |∉| *50*) **where** *x* |∉| *S* ≡ ¬ (*x* |∈| *S*)

context includes *lifting-syntax*
begin

lift-definition *ffilter* :: (*'a* ⇒ *bool*) ⇒ *'a fset* ⇒ *'a fset* **is** *Set.filter*
parametric *Lifting-Set.filter-transfer* **unfolding** *Set.filter-def* **by** *simp*

lift-definition *fPow* :: *'a fset ⇒ 'a fset fset* **is** *Pow* **parametric** *Pow-transfer*
by (*simp add*: *finite-subset*)

lift-definition *fcard* :: *'a fset ⇒ nat* **is** *card* **parametric** *card-transfer* **.**

lift-definition *fimage* :: *('a ⇒ 'b) ⇒ 'a fset ⇒ 'b fset* (**infixr** *|`|* *90*) **is** *image*
parametric *image-transfer* **by** *simp*

lift-definition *fthe-elem* :: *'a fset ⇒ 'a* **is** *the-elem* **.**

lift-definition *fbind* :: *'a fset ⇒ ('a ⇒ 'b fset) ⇒ 'b fset* **is** *Set.bind* **parametric**
bind-transfer
by (*simp add*: *Set.bind-def*)

lift-definition *ffUnion* :: *'a fset fset ⇒ 'a fset* **is** *Union* **parametric** *Union-transfer*
by *simp*

lift-definition *fBall* :: *'a fset ⇒ ('a ⇒ bool) ⇒ bool* **is** *Ball* **parametric** *Ball-transfer*
.
lift-definition *fBex* :: *'a fset ⇒ ('a ⇒ bool) ⇒ bool* **is** *Bex* **parametric** *Bex-transfer*
.

lift-definition *ffold* :: *('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a fset ⇒ 'b* **is** *Finite-Set.fold* **.**

lift-definition *fset-of-list* :: *'a list ⇒ 'a fset* **is** *set* **by** (*rule finite-set*)

lift-definition *sorted-list-of-fset* :: *'a*::*linorder fset ⇒ 'a list* **is** *sorted-list-of-set* **.**

29.4 Transferred lemmas from Set.thy

lemmas *fset-eqI* = *set-eqI*[*Transfer.transferred*]
lemmas *fset-eq-iff*[*no-atp*] = *set-eq-iff*[*Transfer.transferred*]
lemmas *fBallI*[*intro*!] = *ballI*[*Transfer.transferred*]
lemmas *fbspec*[*dest?*] = *bspec*[*Transfer.transferred*]
lemmas *fBallE*[*elim*] = *ballE*[*Transfer.transferred*]
lemmas *fBexI*[*intro*] = *bexI*[*Transfer.transferred*]
lemmas *rev-fBexI*[*intro?*] = *rev-bexI*[*Transfer.transferred*]
lemmas *fBexCI* = *bexCI*[*Transfer.transferred*]
lemmas *fBexE*[*elim*!] = *bexE*[*Transfer.transferred*]
lemmas *fBall-triv*[*simp*] = *ball-triv*[*Transfer.transferred*]
lemmas *fBex-triv*[*simp*] = *bex-triv*[*Transfer.transferred*]
lemmas *fBex-triv-one-point1*[*simp*] = *bex-triv-one-point1*[*Transfer.transferred*]
lemmas *fBex-triv-one-point2*[*simp*] = *bex-triv-one-point2*[*Transfer.transferred*]
lemmas *fBex-one-point1*[*simp*] = *bex-one-point1*[*Transfer.transferred*]
lemmas *fBex-one-point2*[*simp*] = *bex-one-point2*[*Transfer.transferred*]
lemmas *fBall-one-point1*[*simp*] = *ball-one-point1*[*Transfer.transferred*]
lemmas *fBall-one-point2*[*simp*] = *ball-one-point2*[*Transfer.transferred*]
lemmas *fBall-conj-distrib* = *ball-conj-distrib*[*Transfer.transferred*]

lemmas *fBex-disj-distrib* = *bex-disj-distrib*[*Transfer.transferred*]
lemmas *fBall-cong*[*fundef-cong*] = *ball-cong*[*Transfer.transferred*]
lemmas *fBex-cong*[*fundef-cong*] = *bex-cong*[*Transfer.transferred*]
lemmas *fsubsetI*[*intro*!] = *subsetI*[*Transfer.transferred*]
lemmas *fsubsetD*[*elim*, *intro?*] = *subsetD*[*Transfer.transferred*]
lemmas *rev-fsubsetD*[*no-atp*,*intro?*] = *rev-subsetD*[*Transfer.transferred*]
lemmas *fsubsetCE*[*no-atp*,*elim*] = *subsetCE*[*Transfer.transferred*]
lemmas *fsubset-eq*[*no-atp*] = *subset-eq*[*Transfer.transferred*]
lemmas *contra-fsubsetD*[*no-atp*] = *contra-subsetD*[*Transfer.transferred*]
lemmas *fsubset-refl* = *subset-refl*[*Transfer.transferred*]
lemmas *fsubset-trans* = *subset-trans*[*Transfer.transferred*]
lemmas *fset-rev-mp* = *rev-subsetD*[*Transfer.transferred*]
lemmas *fset-mp* = *subsetD*[*Transfer.transferred*]
lemmas *fsubset-not-fsubset-eq*[*code*] = *subset-not-subset-eq*[*Transfer.transferred*]
lemmas *eq-fmem-trans* = *eq-mem-trans*[*Transfer.transferred*]
lemmas *fsubset-antisym*[*intro*!] = *subset-antisym*[*Transfer.transferred*]
lemmas *fequalityD1* = *equalityD1*[*Transfer.transferred*]
lemmas *fequalityD2* = *equalityD2*[*Transfer.transferred*]
lemmas *fequalityE* = *equalityE*[*Transfer.transferred*]
lemmas *fequalityCE*[*elim*] = *equalityCE*[*Transfer.transferred*]
lemmas *eqfset-imp-iff* = *eqset-imp-iff*[*Transfer.transferred*]
lemmas *eqfelem-imp-iff* = *eqelem-imp-iff*[*Transfer.transferred*]
lemmas *fempty-iff*[*simp*] = *empty-iff*[*Transfer.transferred*]
lemmas *fempty-fsubsetI*[*iff*] = *empty-subsetI*[*Transfer.transferred*]
lemmas *equalsffemptyI* = *equals0I*[*Transfer.transferred*]
lemmas *equalsffemptyD* = *equals0D*[*Transfer.transferred*]
lemmas *fBall-fempty*[*simp*] = *ball-empty*[*Transfer.transferred*]
lemmas *fBex-fempty*[*simp*] = *bex-empty*[*Transfer.transferred*]
lemmas *fPow-iff*[*iff*] = *Pow-iff*[*Transfer.transferred*]
lemmas *fPowI* = *PowI*[*Transfer.transferred*]
lemmas *fPowD* = *PowD*[*Transfer.transferred*]
lemmas *fPow-bottom* = *Pow-bottom*[*Transfer.transferred*]
lemmas *fPow-top* = *Pow-top*[*Transfer.transferred*]
lemmas *fPow-not-fempty* = *Pow-not-empty*[*Transfer.transferred*]
lemmas *finter-iff*[*simp*] = *Int-iff*[*Transfer.transferred*]
lemmas *finterI*[*intro*!] = *IntI*[*Transfer.transferred*]
lemmas *finterD1* = *IntD1*[*Transfer.transferred*]
lemmas *finterD2* = *IntD2*[*Transfer.transferred*]
lemmas *finterE*[*elim*!] = *IntE*[*Transfer.transferred*]
lemmas *funion-iff*[*simp*] = *Un-iff*[*Transfer.transferred*]
lemmas *funionI1*[*elim?*] = *UnI1*[*Transfer.transferred*]
lemmas *funionI2*[*elim?*] = *UnI2*[*Transfer.transferred*]
lemmas *funionCI*[*intro*!] = *UnCI*[*Transfer.transferred*]
lemmas *funionE*[*elim*!] = *UnE*[*Transfer.transferred*]
lemmas *fminus-iff*[*simp*] = *Diff-iff*[*Transfer.transferred*]
lemmas *fminusI*[*intro*!] = *DiffI*[*Transfer.transferred*]
lemmas *fminusD1* = *DiffD1*[*Transfer.transferred*]
lemmas *fminusD2* = *DiffD2*[*Transfer.transferred*]
lemmas *fminusE*[*elim*!] = *DiffE*[*Transfer.transferred*]

lemmas *finsert-iff*[*simp*] = *insert-iff*[*Transfer.transferred*]
lemmas *finsertI1* = *insertI1*[*Transfer.transferred*]
lemmas *finsertI2* = *insertI2*[*Transfer.transferred*]
lemmas *finsertE*[*elim*!] = *insertE*[*Transfer.transferred*]
lemmas *finsertCI*[*intro*!] = *insertCI*[*Transfer.transferred*]
lemmas *fsubset-finsert-iff* = *subset-insert-iff*[*Transfer.transferred*]
lemmas *finsert-ident* = *insert-ident*[*Transfer.transferred*]
lemmas *fsingletonI*[*intro*!,*no-atp*] = *singletonI*[*Transfer.transferred*]
lemmas *fsingletonD*[*dest*!,*no-atp*] = *singletonD*[*Transfer.transferred*]
lemmas *fsingleton-iff* = *singleton-iff*[*Transfer.transferred*]
lemmas *fsingleton-inject*[*dest*!] = *singleton-inject*[*Transfer.transferred*]
lemmas *fsingleton-finsert-inj-eq*[*iff*,*no-atp*] = *singleton-insert-inj-eq*[*Transfer.transferred*]
lemmas *fsingleton-finsert-inj-eq'*[*iff*,*no-atp*] = *singleton-insert-inj-eq'*[*Transfer.transferred*]
lemmas *fsubset-fsingletonD* = *subset-singletonD*[*Transfer.transferred*]
lemmas *fminus-single-finsert* = *Diff-single-insert*[*Transfer.transferred*]
lemmas *fdoubleton-eq-iff* = *doubleton-eq-iff*[*Transfer.transferred*]
lemmas *funion-fsingleton-iff* = *Un-singleton-iff*[*Transfer.transferred*]
lemmas *fsingleton-funion-iff* = *singleton-Un-iff*[*Transfer.transferred*]
lemmas *fimage-eqI*[*simp*, *intro*] = *image-eqI*[*Transfer.transferred*]
lemmas *fimageI* = *imageI*[*Transfer.transferred*]
lemmas *rev-fimage-eqI* = *rev-image-eqI*[*Transfer.transferred*]
lemmas *fimageE*[*elim*!] = *imageE*[*Transfer.transferred*]
lemmas *Compr-fimage-eq* = *Compr-image-eq*[*Transfer.transferred*]
lemmas *fimage-funion* = *image-Un*[*Transfer.transferred*]
lemmas *fimage-iff* = *image-iff*[*Transfer.transferred*]
lemmas *fimage-fsubset-iff*[*no-atp*] = *image-subset-iff*[*Transfer.transferred*]
lemmas *fimage-fsubsetI* = *image-subsetI*[*Transfer.transferred*]
lemmas *fimage-ident*[*simp*] = *image-ident*[*Transfer.transferred*]
lemmas *if-split-fmem1* = *if-split-mem1*[*Transfer.transferred*]
lemmas *if-split-fmem2* = *if-split-mem2*[*Transfer.transferred*]
lemmas *pfsubsetI*[*intro*!,*no-atp*] = *psubsetI*[*Transfer.transferred*]
lemmas *pfsubsetE*[*elim*!,*no-atp*] = *psubsetE*[*Transfer.transferred*]
lemmas *pfsubset-finsert-iff* = *psubset-insert-iff*[*Transfer.transferred*]
lemmas *pfsubset-eq* = *psubset-eq*[*Transfer.transferred*]
lemmas *pfsubset-imp-fsubset* = *psubset-imp-subset*[*Transfer.transferred*]
lemmas *pfsubset-trans* = *psubset-trans*[*Transfer.transferred*]
lemmas *pfsubsetD* = *psubsetD*[*Transfer.transferred*]
lemmas *pfsubset-fsubset-trans* = *psubset-subset-trans*[*Transfer.transferred*]
lemmas *fsubset-pfsubset-trans* = *subset-psubset-trans*[*Transfer.transferred*]
lemmas *pfsubset-imp-ex-fmem* = *psubset-imp-ex-mem*[*Transfer.transferred*]
lemmas *fimage-fPow-mono* = *image-Pow-mono*[*Transfer.transferred*]
lemmas *fimage-fPow-surj* = *image-Pow-surj*[*Transfer.transferred*]
lemmas *fsubset-finsertI* = *subset-insertI*[*Transfer.transferred*]
lemmas *fsubset-finsertI2* = *subset-insertI2*[*Transfer.transferred*]
lemmas *fsubset-finsert* = *subset-insert*[*Transfer.transferred*]
lemmas *funion-upper1* = *Un-upper1*[*Transfer.transferred*]
lemmas *funion-upper2* = *Un-upper2*[*Transfer.transferred*]
lemmas *funion-least* = *Un-least*[*Transfer.transferred*]
lemmas *finter-lower1* = *Int-lower1*[*Transfer.transferred*]

lemmas *finter-lower2* = *Int-lower2*[*Transfer.transferred*]
lemmas *finter-greatest* = *Int-greatest*[*Transfer.transferred*]
lemmas *fminus-fsubset* = *Diff-subset*[*Transfer.transferred*]
lemmas *fminus-fsubset-conv* = *Diff-subset-conv*[*Transfer.transferred*]
lemmas *fsubset-fempty*[*simp*] = *subset-empty*[*Transfer.transferred*]
lemmas *not-pfsubset-fempty*[*iff*] = *not-psubset-empty*[*Transfer.transferred*]
lemmas *finsert-is-funion* = *insert-is-Un*[*Transfer.transferred*]
lemmas *finsert-not-fempty*[*simp*] = *insert-not-empty*[*Transfer.transferred*]
lemmas *fempty-not-finsert* = *empty-not-insert*[*Transfer.transferred*]
lemmas *finsert-absorb* = *insert-absorb*[*Transfer.transferred*]
lemmas *finsert-absorb2*[*simp*] = *insert-absorb2*[*Transfer.transferred*]
lemmas *finsert-commute* = *insert-commute*[*Transfer.transferred*]
lemmas *finsert-fsubset*[*simp*] = *insert-subset*[*Transfer.transferred*]
lemmas *finsert-inter-finsert*[*simp*] = *insert-inter-insert*[*Transfer.transferred*]
lemmas *finsert-disjoint*[*simp*,*no-atp*] = *insert-disjoint*[*Transfer.transferred*]
lemmas *disjoint-finsert*[*simp*,*no-atp*] = *disjoint-insert*[*Transfer.transferred*]
lemmas *fimage-fempty*[*simp*] = *image-empty*[*Transfer.transferred*]
lemmas *fimage-finsert*[*simp*] = *image-insert*[*Transfer.transferred*]
lemmas *fimage-constant* = *image-constant*[*Transfer.transferred*]
lemmas *fimage-constant-conv* = *image-constant-conv*[*Transfer.transferred*]
lemmas *fimage-fimage* = *image-image*[*Transfer.transferred*]
lemmas *finsert-fimage*[*simp*] = *insert-image*[*Transfer.transferred*]
lemmas *fimage-is-fempty*[*iff*] = *image-is-empty*[*Transfer.transferred*]
lemmas *fempty-is-fimage*[*iff*] = *empty-is-image*[*Transfer.transferred*]
lemmas *fimage-cong* = *image-cong*[*Transfer.transferred*]
lemmas *fimage-finter-fsubset* = *image-Int-subset*[*Transfer.transferred*]
lemmas *fimage-fminus-fsubset* = *image-diff-subset*[*Transfer.transferred*]
lemmas *finter-absorb* = *Int-absorb*[*Transfer.transferred*]
lemmas *finter-left-absorb* = *Int-left-absorb*[*Transfer.transferred*]
lemmas *finter-commute* = *Int-commute*[*Transfer.transferred*]
lemmas *finter-left-commute* = *Int-left-commute*[*Transfer.transferred*]
lemmas *finter-assoc* = *Int-assoc*[*Transfer.transferred*]
lemmas *finter-ac* = *Int-ac*[*Transfer.transferred*]
lemmas *finter-absorb1* = *Int-absorb1*[*Transfer.transferred*]
lemmas *finter-absorb2* = *Int-absorb2*[*Transfer.transferred*]
lemmas *finter-fempty-left* = *Int-empty-left*[*Transfer.transferred*]
lemmas *finter-fempty-right* = *Int-empty-right*[*Transfer.transferred*]
lemmas *disjoint-iff-fnot-equal* = *disjoint-iff-not-equal*[*Transfer.transferred*]
lemmas *finter-funion-distrib* = *Int-Un-distrib*[*Transfer.transferred*]
lemmas *finter-funion-distrib2* = *Int-Un-distrib2*[*Transfer.transferred*]
lemmas *finter-fsubset-iff*[*no-atp*, *simp*] = *Int-subset-iff*[*Transfer.transferred*]
lemmas *funion-absorb* = *Un-absorb*[*Transfer.transferred*]
lemmas *funion-left-absorb* = *Un-left-absorb*[*Transfer.transferred*]
lemmas *funion-commute* = *Un-commute*[*Transfer.transferred*]
lemmas *funion-left-commute* = *Un-left-commute*[*Transfer.transferred*]
lemmas *funion-assoc* = *Un-assoc*[*Transfer.transferred*]
lemmas *funion-ac* = *Un-ac*[*Transfer.transferred*]
lemmas *funion-absorb1* = *Un-absorb1*[*Transfer.transferred*]
lemmas *funion-absorb2* = *Un-absorb2*[*Transfer.transferred*]

lemmas *funion-fempty-left* = *Un-empty-left*[*Transfer.transferred*]
lemmas *funion-fempty-right* = *Un-empty-right*[*Transfer.transferred*]
lemmas *funion-finsert-left*[*simp*] = *Un-insert-left*[*Transfer.transferred*]
lemmas *funion-finsert-right*[*simp*] = *Un-insert-right*[*Transfer.transferred*]
lemmas *finter-finsert-left* = *Int-insert-left*[*Transfer.transferred*]
lemmas *finter-finsert-left-ifffempty*[*simp*] = *Int-insert-left-if0*[*Transfer.transferred*]
lemmas *finter-finsert-left-if1*[*simp*] = *Int-insert-left-if1*[*Transfer.transferred*]
lemmas *finter-finsert-right* = *Int-insert-right*[*Transfer.transferred*]
lemmas *finter-finsert-right-ifffempty*[*simp*] = *Int-insert-right-if0*[*Transfer.transferred*]
lemmas *finter-finsert-right-if1*[*simp*] = *Int-insert-right-if1*[*Transfer.transferred*]
lemmas *funion-finter-distrib* = *Un-Int-distrib*[*Transfer.transferred*]
lemmas *funion-finter-distrib2* = *Un-Int-distrib2*[*Transfer.transferred*]
lemmas *funion-finter-crazy* = *Un-Int-crazy*[*Transfer.transferred*]
lemmas *fsubset-funion-eq* = *subset-Un-eq*[*Transfer.transferred*]
lemmas *funion-fempty*[*iff*] = *Un-empty*[*Transfer.transferred*]
lemmas *funion-fsubset-iff*[*no-atp, simp*] = *Un-subset-iff*[*Transfer.transferred*]
lemmas *funion-fminus-finter* = *Un-Diff-Int*[*Transfer.transferred*]
lemmas *ffunion-empty*[*simp*] = *Union-empty*[*Transfer.transferred*]
lemmas *ffunion-mono* = *Union-mono*[*Transfer.transferred*]
lemmas *ffunion-insert*[*simp*] = *Union-insert*[*Transfer.transferred*]
lemmas *fminus-finter2* = *Diff-Int2*[*Transfer.transferred*]
lemmas *funion-finter-assoc-eq* = *Un-Int-assoc-eq*[*Transfer.transferred*]
lemmas *fBall-funion* = *ball-Un*[*Transfer.transferred*]
lemmas *fBex-funion* = *bex-Un*[*Transfer.transferred*]
lemmas *fminus-eq-fempty-iff*[*simp,no-atp*] = *Diff-eq-empty-iff*[*Transfer.transferred*]
lemmas *fminus-cancel*[*simp*] = *Diff-cancel*[*Transfer.transferred*]
lemmas *fminus-idemp*[*simp*] = *Diff-idemp*[*Transfer.transferred*]
lemmas *fminus-triv* = *Diff-triv*[*Transfer.transferred*]
lemmas *fempty-fminus*[*simp*] = *empty-Diff*[*Transfer.transferred*]
lemmas *fminus-fempty*[*simp*] = *Diff-empty*[*Transfer.transferred*]
lemmas *fminus-finsertffempty*[*simp,no-atp*] = *Diff-insert0*[*Transfer.transferred*]
lemmas *fminus-finsert* = *Diff-insert*[*Transfer.transferred*]
lemmas *fminus-finsert2* = *Diff-insert2*[*Transfer.transferred*]
lemmas *finsert-fminus-if* = *insert-Diff-if*[*Transfer.transferred*]
lemmas *finsert-fminus1*[*simp*] = *insert-Diff1*[*Transfer.transferred*]
lemmas *finsert-fminus-single*[*simp*] = *insert-Diff-single*[*Transfer.transferred*]
lemmas *finsert-fminus* = *insert-Diff*[*Transfer.transferred*]
lemmas *fminus-finsert-absorb* = *Diff-insert-absorb*[*Transfer.transferred*]
lemmas *fminus-disjoint*[*simp*] = *Diff-disjoint*[*Transfer.transferred*]
lemmas *fminus-partition* = *Diff-partition*[*Transfer.transferred*]
lemmas *double-fminus* = *double-diff*[*Transfer.transferred*]
lemmas *funion-fminus-cancel*[*simp*] = *Un-Diff-cancel*[*Transfer.transferred*]
lemmas *funion-fminus-cancel2*[*simp*] = *Un-Diff-cancel2*[*Transfer.transferred*]
lemmas *fminus-funion* = *Diff-Un*[*Transfer.transferred*]
lemmas *fminus-finter* = *Diff-Int*[*Transfer.transferred*]
lemmas *funion-fminus* = *Un-Diff*[*Transfer.transferred*]
lemmas *finter-fminus* = *Int-Diff*[*Transfer.transferred*]
lemmas *fminus-finter-distrib* = *Diff-Int-distrib*[*Transfer.transferred*]
lemmas *fminus-finter-distrib2* = *Diff-Int-distrib2*[*Transfer.transferred*]

lemmas *fUNIV-bool*[*no-atp*] = *UNIV-bool*[*Transfer.transferred*]
lemmas *fPow-fempty*[*simp*] = *Pow-empty*[*Transfer.transferred*]
lemmas *fPow-finsert* = *Pow-insert*[*Transfer.transferred*]
lemmas *funion-fPow-fsubset* = *Un-Pow-subset*[*Transfer.transferred*]
lemmas *fPow-finter-eq*[*simp*] = *Pow-Int-eq*[*Transfer.transferred*]
lemmas *fset-eq-fsubset* = *set-eq-subset*[*Transfer.transferred*]
lemmas *fsubset-iff*[*no-atp*] = *subset-iff*[*Transfer.transferred*]
lemmas *fsubset-iff-pfsubset-eq* = *subset-iff-psubset-eq*[*Transfer.transferred*]
lemmas *all-not-fin-conv*[*simp*] = *all-not-in-conv*[*Transfer.transferred*]
lemmas *ex-fin-conv* = *ex-in-conv*[*Transfer.transferred*]
lemmas *fimage-mono* = *image-mono*[*Transfer.transferred*]
lemmas *fPow-mono* = *Pow-mono*[*Transfer.transferred*]
lemmas *finsert-mono* = *insert-mono*[*Transfer.transferred*]
lemmas *funion-mono* = *Un-mono*[*Transfer.transferred*]
lemmas *finter-mono* = *Int-mono*[*Transfer.transferred*]
lemmas *fminus-mono* = *Diff-mono*[*Transfer.transferred*]
lemmas *fin-mono* = *in-mono*[*Transfer.transferred*]
lemmas *fthe-felem-eq*[*simp*] = *the-elem-eq*[*Transfer.transferred*]
lemmas *fLeast-mono* = *Least-mono*[*Transfer.transferred*]
lemmas *fbind-fbind* = *bind-bind*[*Transfer.transferred*]
lemmas *fempty-fbind*[*simp*] = *empty-bind*[*Transfer.transferred*]
lemmas *nonfempty-fbind-const* = *nonempty-bind-const*[*Transfer.transferred*]
lemmas *fbind-const* = *bind-const*[*Transfer.transferred*]
lemmas *ffmember-filter*[*simp*] = *member-filter*[*Transfer.transferred*]
lemmas *fequalityI* = *equalityI*[*Transfer.transferred*]
lemmas *fset-of-list-simps*[*simp*] = *set-simps*[*Transfer.transferred*]
lemmas *fset-of-list-append*[*simp*] = *set-append*[*Transfer.transferred*]
lemmas *fset-of-list-rev*[*simp*] = *set-rev*[*Transfer.transferred*]
lemmas *fset-of-list-map*[*simp*] = *set-map*[*Transfer.transferred*]

29.5 Additional lemmas

29.5.1 *ffUnion*

lemmas *ffUnion-funion-distrib*[*simp*] = *Union-Un-distrib*[*Transfer.transferred*]

29.5.2 *fbind*

lemma *fbind-cong*[*fundef-cong*]: $A = B \Longrightarrow (\bigwedge x.\ x\ |\in|\ B \Longrightarrow f\ x = g\ x) \Longrightarrow$ *fbind* $A\ f =$ *fbind* $B\ g$
by *transfer force*

29.5.3 *fsingleton*

lemmas *fsingletonE* = *fsingletonD* [*elim-format*]

29.5.4 *fempty*

lemma *fempty-ffilter*[*simp*]: *ffilter* (λ-. *False*) $A = \{||\}$
by *transfer auto*

lemma *femptyE* [*elim*!]: $a \mid\in\mid \{\mid\mid\} \Longrightarrow P$
by *simp*

29.5.5 *fset*

lemmas *fset-simps*[*simp*] = *bot-fset.rep-eq finsert.rep-eq*

lemma *finite-fset* [*simp*]:
shows *finite* (*fset S*)
by *transfer simp*

lemmas *fset-cong* = *fset-inject*

lemma *filter-fset* [*simp*]:
shows *fset* (*ffilter P xs*) = *Collect P* ∩ *fset xs*
by *transfer auto*

lemma *notin-fset*: $x \mid\notin\mid S \longleftrightarrow x \notin$ *fset S* **by** (*simp add*: *fmember.rep-eq*)

lemmas *inter-fset*[*simp*] = *inf-fset.rep-eq*

lemmas *union-fset*[*simp*] = *sup-fset.rep-eq*

lemmas *minus-fset*[*simp*] = *minus-fset.rep-eq*

29.5.6 *ffilter*

lemma *subset-ffilter*:
ffilter P A $\mid\subseteq\mid$ *ffilter Q A* = (∀ *x*. *x* $\mid\in\mid$ *A* ⟶ *P x* ⟶ *Q x*)
by *transfer auto*

lemma *eq-ffilter*:
(*ffilter P A* = *ffilter Q A*) = (∀ *x*. *x* $\mid\in\mid$ *A* ⟶ *P x* = *Q x*)
by *transfer auto*

lemma *pfsubset-ffilter*:
(⋀*x*. *x* $\mid\in\mid$ *A* ⟹ *P x* ⟹ *Q x*) ⟹ (*x* $\mid\in\mid$ *A* ∧ ¬ *P x* ∧ *Q x*) ⟹
ffilter P A $\mid\subset\mid$ *ffilter Q A*
unfolding *less-fset-def* **by** (*auto simp add*: *subset-ffilter eq-ffilter*)

29.5.7 *fset-of-list*

lemma *fset-of-list-filter*[*simp*]:
fset-of-list (*filter P xs*) = *ffilter P* (*fset-of-list xs*)
by *transfer* (*auto simp*: *Set.filter-def*)

lemma *fset-of-list-subset*[*intro*]:
set xs ⊆ *set ys* ⟹ *fset-of-list xs* $\mid\subseteq\mid$ *fset-of-list ys*

by *transfer simp*

lemma *fset-of-list-elem*: $(x \mid\in\mid \textit{fset-of-list}\ xs) \longleftrightarrow (x \in \textit{set}\ xs)$
by *transfer simp*

29.5.8 *finsert*

lemma *set-finsert*:
assumes $x \mid\in\mid A$
obtains B **where** $A = \textit{finsert}\ x\ B$ **and** $x \mid\notin\mid B$
using *assms* **by** *transfer* (*metis Set.set-insert finite-insert*)

lemma *mk-disjoint-finsert*: $a \mid\in\mid A \Longrightarrow \exists B.\ A = \textit{finsert}\ a\ B \wedge a \mid\notin\mid B$
by (*rule exI* [**where** $x = A \mid-\mid \{\mid a \mid\}$]) *blast*

lemma *finsert-eq-iff*:
assumes $a \mid\notin\mid A$ **and** $b \mid\notin\mid B$
shows (*finsert a A* = *finsert b B*) =
(*if a* = *b then A* = *B else* $\exists C.\ A = \textit{finsert}\ b\ C \wedge b \mid\notin\mid C \wedge B = \textit{finsert}\ a\ C \wedge a \mid\notin\mid C$)
using *assms* **by** *transfer* (*force simp*: *insert-eq-iff*)

29.5.9 *fimage*

lemma *subset-fimage-iff*: $(B \mid\subseteq\mid f\mid`\mid A) = (\exists\ AA.\ AA \mid\subseteq\mid A \wedge B = f\mid`\mid AA)$
by *transfer* (*metis mem-Collect-eq rev-finite-subset subset-image-iff*)

29.5.10 bounded quantification

lemma *bex-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ \textit{fBex}\ A\ (\lambda x.\ P\ x \wedge Q) = (\textit{fBex}\ A\ P \wedge Q)$
$\bigwedge A\ P\ Q.\ \textit{fBex}\ A\ (\lambda x.\ P \wedge Q\ x) = (P \wedge \textit{fBex}\ A\ Q)$
$\bigwedge P.\ \textit{fBex}\ \{||\}\ P = \textit{False}$
$\bigwedge a\ B\ P.\ \textit{fBex}\ (\textit{finsert}\ a\ B)\ P = (P\ a \vee \textit{fBex}\ B\ P)$
$\bigwedge A\ P\ f.\ \textit{fBex}\ (f\mid`\mid A)\ P = \textit{fBex}\ A\ (\lambda x.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg\ \textit{fBex}\ A\ P) = \textit{fBall}\ A\ (\lambda x.\ \neg\ P\ x)$
by *auto*

lemma *ball-simps* [*simp*, *no-atp*]:
$\bigwedge A\ P\ Q.\ \textit{fBall}\ A\ (\lambda x.\ P\ x \vee Q) = (\textit{fBall}\ A\ P \vee Q)$
$\bigwedge A\ P\ Q.\ \textit{fBall}\ A\ (\lambda x.\ P \vee Q\ x) = (P \vee \textit{fBall}\ A\ Q)$
$\bigwedge A\ P\ Q.\ \textit{fBall}\ A\ (\lambda x.\ P \longrightarrow Q\ x) = (P \longrightarrow \textit{fBall}\ A\ Q)$
$\bigwedge A\ P\ Q.\ \textit{fBall}\ A\ (\lambda x.\ P\ x \longrightarrow Q) = (\textit{fBex}\ A\ P \longrightarrow Q)$
$\bigwedge P.\ \textit{fBall}\ \{||\}\ P = \textit{True}$
$\bigwedge a\ B\ P.\ \textit{fBall}\ (\textit{finsert}\ a\ B)\ P = (P\ a \wedge \textit{fBall}\ B\ P)$
$\bigwedge A\ P\ f.\ \textit{fBall}\ (f\mid`\mid A)\ P = \textit{fBall}\ A\ (\lambda x.\ P\ (f\ x))$
$\bigwedge A\ P.\ (\neg\ \textit{fBall}\ A\ P) = \textit{fBex}\ A\ (\lambda x.\ \neg\ P\ x)$
by *auto*

lemma *atomize-fBall*:

$(\bigwedge x.\ x\ |\in|\ A \Longrightarrow P\ x) \equiv \mathit{Trueprop}\ (\mathit{fBall}\ A\ (\lambda x.\ P\ x))$
apply (*simp only*: *atomize-all atomize-imp*)
apply (*rule equal-intr-rule*)
by (*transfer*, *simp*)+

lemma *fBall-mono*[*mono*]: $P \leq Q \Longrightarrow \mathit{fBall}\ S\ P \leq \mathit{fBall}\ S\ Q$
by *auto*

lemma *fBex-mono*[*mono*]: $P \leq Q \Longrightarrow \mathit{fBex}\ S\ P \leq \mathit{fBex}\ S\ Q$
by *auto*

end

29.5.11 *fcard*

lemma *fcard-fempty*:
$\mathit{fcard}\ \{||\} = 0$
by *transfer* (*rule card.empty*)

lemma *fcard-finsert-disjoint*:
$x\ |\notin|\ A \Longrightarrow \mathit{fcard}\ (\mathit{finsert}\ x\ A) = \mathit{Suc}\ (\mathit{fcard}\ A)$
by *transfer* (*rule card-insert-disjoint*)

lemma *fcard-finsert-if*:
$\mathit{fcard}\ (\mathit{finsert}\ x\ A) = (\mathit{if}\ x\ |\in|\ A\ \mathit{then}\ \mathit{fcard}\ A\ \mathit{else}\ \mathit{Suc}\ (\mathit{fcard}\ A))$
by *transfer* (*rule card-insert-if*)

lemma *fcard-0-eq* [*simp*, *no-atp*]:
$\mathit{fcard}\ A = 0 \longleftrightarrow A = \{||\}$
by *transfer* (*rule card-0-eq*)

lemma *fcard-Suc-fminus1*:
$x\ |\in|\ A \Longrightarrow \mathit{Suc}\ (\mathit{fcard}\ (A\ |-|\ \{|x|\})) = \mathit{fcard}\ A$
by *transfer* (*rule card-Suc-Diff1*)

lemma *fcard-fminus-fsingleton*:
$x\ |\in|\ A \Longrightarrow \mathit{fcard}\ (A\ |-|\ \{|x|\}) = \mathit{fcard}\ A - 1$
by *transfer* (*rule card-Diff-singleton*)

lemma *fcard-fminus-fsingleton-if*:
$\mathit{fcard}\ (A\ |-|\ \{|x|\}) = (\mathit{if}\ x\ |\in|\ A\ \mathit{then}\ \mathit{fcard}\ A - 1\ \mathit{else}\ \mathit{fcard}\ A)$
by *transfer* (*rule card-Diff-singleton-if*)

lemma *fcard-fminus-finsert*[*simp*]:
assumes $a\ |\in|\ A$ **and** $a\ |\notin|\ B$
shows $\mathit{fcard}\ (A\ |-|\ \mathit{finsert}\ a\ B) = \mathit{fcard}\ (A\ |-|\ B) - 1$
using *assms* **by** *transfer* (*rule card-Diff-insert*)

lemma *fcard-finsert*: $\mathit{fcard}\ (\mathit{finsert}\ x\ A) = \mathit{Suc}\ (\mathit{fcard}\ (A\ |-|\ \{|x|\}))$

by *transfer* (*rule card.insert-remove*)

lemma *fcard-finsert-le*: *fcard A* ≤ *fcard* (*finsert x A*)
by *transfer* (*rule card-insert-le*)

lemma *fcard-mono*:
A |⊆| *B* ⟹ *fcard A* ≤ *fcard B*
by *transfer* (*rule card-mono*)

lemma *fcard-seteq*: *A* |⊆| *B* ⟹ *fcard B* ≤ *fcard A* ⟹ *A* = *B*
by *transfer* (*rule card-seteq*)

lemma *pfsubset-fcard-mono*: *A* |⊂| *B* ⟹ *fcard A* < *fcard B*
by *transfer* (*rule psubset-card-mono*)

lemma *fcard-funion-finter*:
fcard A + *fcard B* = *fcard* (*A* |∪| *B*) + *fcard* (*A* |∩| *B*)
by *transfer* (*rule card-Un-Int*)

lemma *fcard-funion-disjoint*:
A |∩| *B* = {||} ⟹ *fcard* (*A* |∪| *B*) = *fcard A* + *fcard B*
by *transfer* (*rule card-Un-disjoint*)

lemma *fcard-funion-fsubset*:
B |⊆| *A* ⟹ *fcard* (*A* |−| *B*) = *fcard A* − *fcard B*
by *transfer* (*rule card-Diff-subset*)

lemma *diff-fcard-le-fcard-fminus*:
fcard A − *fcard B* ≤ *fcard*(*A* |−| *B*)
by *transfer* (*rule diff-card-le-card-Diff*)

lemma *fcard-fminus1-less*: *x* |∈| *A* ⟹ *fcard* (*A* |−| {|*x*|}) < *fcard A*
by *transfer* (*rule card-Diff1-less*)

lemma *fcard-fminus2-less*:
x |∈| *A* ⟹ *y* |∈| *A* ⟹ *fcard* (*A* |−| {|*x*|} |−| {|*y*|}) < *fcard A*
by *transfer* (*rule card-Diff2-less*)

lemma *fcard-fminus1-le*: *fcard* (*A* |−| {|*x*|}) ≤ *fcard A*
by *transfer* (*rule card-Diff1-le*)

lemma *fcard-pfsubset*: *A* |⊆| *B* ⟹ *fcard A* < *fcard B* ⟹ *A* < *B*
by *transfer* (*rule card-psubset*)

29.5.12 *sorted-list-of-fset*

lemma *sorted-list-of-fset-simps*[*simp*]:
set (*sorted-list-of-fset S*) = *fset S*
fset-of-list (*sorted-list-of-fset S*) = *S*

by (*transfer*, *simp*)+

29.5.13 *ffold*

context *comp-fun-commute*
begin
lemmas *ffold-empty*[*simp*] = *fold-empty*[*Transfer.transferred*]

lemma *ffold-finsert* [*simp*]:
assumes x $|\notin|$ A
shows *ffold f z* (*finsert x A*) = *f x* (*ffold f z A*)
using *assms* **by** (*transfer fixing*: *f*) (*rule fold-insert*)

lemma *ffold-fun-left-comm*:
f x (*ffold f z A*) = *ffold f* (*f x z*) *A*
by (*transfer fixing*: *f*) (*rule fold-fun-left-comm*)

lemma *ffold-finsert2*:
x $|\notin|$ A $\Longrightarrow$ *ffold f z* (*finsert x A*) = *ffold f* (*f x z*) *A*
by (*transfer fixing*: *f*) (*rule fold-insert2*)

lemma *ffold-rec*:
assumes x $|\in|$ A
shows *ffold f z A* = *f x* (*ffold f z* (*A* $|-|$ {$|x|$}))
using *assms* **by** (*transfer fixing*: *f*) (*rule fold-rec*)

lemma *ffold-finsert-fremove*:
ffold f z (*finsert x A*) = *f x* (*ffold f z* (*A* $|-|$ {$|x|$}))
by (*transfer fixing*: *f*) (*rule fold-insert-remove*)
end

lemma *ffold-fimage*:
assumes *inj-on g* (*fset A*)
shows *ffold f z* (*g* $|`|$ *A*) = *ffold* (*f* $\circ$ *g*) *z A*
using *assms* **by** *transfer'* (*rule fold-image*)

lemma *ffold-cong*:
assumes *comp-fun-commute f comp-fun-commute g*
$\bigwedge x.$ x $|\in|$ A $\Longrightarrow$ *f x* = *g x*
and *s* = *t* **and** *A* = *B*
shows *ffold f s A* = *ffold g t B*
using *assms*[*unfolded comp-fun-commute-def'*]
by *transfer* (*meson Finite-Set.fold-cong subset-UNIV*)

context *comp-fun-idem*
begin

lemma *ffold-finsert-idem*:
ffold f z (*finsert x A*) = *f x* (*ffold f z A*)

by (*transfer fixing*: *f*) (*rule fold-insert-idem*)

declare *ffold-finsert* [*simp del*] *ffold-finsert-idem* [*simp*]

lemma *ffold-finsert-idem2*:
ffold f z (*finsert x A*) = *ffold f* (*f x z*) *A*
by (*transfer fixing*: *f*) (*rule fold-insert-idem2*)

end

29.5.14 Group operations

locale *comm-monoid-fset* = *comm-monoid*
begin

sublocale *set*: *comm-monoid-set* **..**

lift-definition *F* :: ('*b* ⇒ '*a*) ⇒ '*b fset* ⇒ '*a* **is** *set.F* **.**

lemmas *cong*[*fundef-cong*] = *set.cong*[*Transfer.transferred*]

lemma *cong-simp*[*cong*]:
⟦ *A* = *B*; ⋀*x*. *x* |∈| *B* =*simp*=> *g x* = *h x* ⟧ ⟹ *F g A* = *F h B*
unfolding *simp-implies-def* **by** (*auto cong*: *cong*)

end

context *comm-monoid-add* **begin**

sublocale *fsum*: *comm-monoid-fset plus 0*
rewrites *comm-monoid-set.F plus 0* = *sum*
defines *fsum* = *fsum.F*
proof −
show *comm-monoid-fset* (+) *0* **by** *standard*

show *comm-monoid-set.F* (+) *0* = *sum* **unfolding** *sum-def* **..**
qed

end

29.5.15 Semilattice operations

locale *semilattice-fset* = *semilattice*
begin

sublocale *set*: *semilattice-set* **..**

lift-definition *F* :: '*a fset* ⇒ '*a* **is** *set.F* **.**

lemma *eq-fold*: *F* (*finsert x A*) = *ffold f x A*

by *transfer* (*rule set.eq-fold*)

lemma *singleton* [*simp*]: $F\ \{|x|\} = x$
by *transfer* (*rule set.singleton*)

lemma *insert-not-elem*: $x\ |\notin|\ A \Longrightarrow A \neq \{||\} \Longrightarrow F\ (\mathit{finsert}\ x\ A) = x * F\ A$
by *transfer* (*rule set.insert-not-elem*)

lemma *in-idem*: $x\ |\in|\ A \Longrightarrow x * F\ A = F\ A$
by *transfer* (*rule set.in-idem*)

lemma *insert* [*simp*]: $A \neq \{||\} \Longrightarrow F\ (\mathit{finsert}\ x\ A) = x * F\ A$
by *transfer* (*rule set.insert*)

end

locale *semilattice-order-fset* = *binary?*: *semilattice-order* + *semilattice-fset*
begin

end

context *linorder* **begin**

sublocale *fMin*: *semilattice-order-fset min less-eq less*
rewrites *semilattice-set.F min* = *Min*
defines *fMin* = *fMin.F*
proof −
show *semilattice-order-fset min* (≤) (<) **by** *standard*

show *semilattice-set.F min* = *Min* **unfolding** *Min-def* **..**
qed

sublocale *fMax*: *semilattice-order-fset max greater-eq greater*
rewrites *semilattice-set.F max* = *Max*
defines *fMax* = *fMax.F*
proof −
show *semilattice-order-fset max* (≥) (>)
by *standard*

show *semilattice-set.F max* = *Max*
unfolding *Max-def* **..**
qed

end

lemma *mono-fMax-commute*: *mono f* $\Longrightarrow A \neq \{||\} \Longrightarrow f\ (\mathit{fMax}\ A) = \mathit{fMax}\ (f\ |`|\ A)$
by *transfer* (*rule mono-Max-commute*)

lemma *mono-fMin-commute*: *mono f* $\Longrightarrow$ *A* $\neq$ {||} $\Longrightarrow$ *f* (*fMin A*) = *fMin* (*f* |‘| *A*)
by *transfer* (*rule mono-Min-commute*)

lemma *fMax-in*[*simp*]: *A* $\neq$ {||} $\Longrightarrow$ *fMax A* |∈| *A*
by *transfer* (*rule Max-in*)

lemma *fMin-in*[*simp*]: *A* $\neq$ {||} $\Longrightarrow$ *fMin A* |∈| *A*
by *transfer* (*rule Min-in*)

lemma *fMax-ge*[*simp*]: *x* |∈| *A* $\Longrightarrow$ *x* $\leq$ *fMax A*
by *transfer* (*rule Max-ge*)

lemma *fMin-le*[*simp*]: *x* |∈| *A* $\Longrightarrow$ *fMin A* $\leq$ *x*
by *transfer* (*rule Min-le*)

lemma *fMax-eqI*: ($\bigwedge$*y*. *y* |∈| *A* $\Longrightarrow$ *y* $\leq$ *x*) $\Longrightarrow$ *x* |∈| *A* $\Longrightarrow$ *fMax A* = *x*
by *transfer* (*rule Max-eqI*)

lemma *fMin-eqI*: ($\bigwedge$*y*. *y* |∈| *A* $\Longrightarrow$ *x* $\leq$ *y*) $\Longrightarrow$ *x* |∈| *A* $\Longrightarrow$ *fMin A* = *x*
by *transfer* (*rule Min-eqI*)

lemma *fMax-finsert*[*simp*]: *fMax* (*finsert x A*) = (*if A* = {||} *then x else max x* (*fMax A*))
by *transfer simp*

lemma *fMin-finsert*[*simp*]: *fMin* (*finsert x A*) = (*if A* = {||} *then x else min x* (*fMin A*))
by *transfer simp*

context *linorder* **begin**

lemma *fset-linorder-max-induct*[*case-names fempty finsert*]:
assumes *P* {||}
and $\bigwedge$*x S*. ⟦∀ *y*. *y* |∈| *S* ⟶ *y* < *x*; *P S*⟧ $\Longrightarrow$ *P* (*finsert x S*)
shows *P S*
proof −

note *Domainp-forall-transfer*[*transfer-rule*]
show *?thesis*
using *assms* **by** (*transfer fixing*: *less*) (*auto intro*: *finite-linorder-max-induct*)
qed

lemma *fset-linorder-min-induct*[*case-names fempty finsert*]:
assumes *P* {||}
and $\bigwedge$*x S*. ⟦∀ *y*. *y* |∈| *S* ⟶ *y* > *x*; *P S*⟧ $\Longrightarrow$ *P* (*finsert x S*)
shows *P S*
proof −

note *Domainp-forall-transfer*[*transfer-rule*]
show *?thesis*
using *assms* **by** (*transfer fixing*: *less*) (*auto intro*: *finite-linorder-min-induct*)
qed

end

29.6 Choice in fsets

lemma *fset-choice*:
assumes $\forall x.\ x\ |\in|\ A \longrightarrow (\exists y.\ P\ x\ y)$
shows $\exists f.\ \forall x.\ x\ |\in|\ A \longrightarrow P\ x\ (f\ x)$
using *assms* **by** *transfer metis*

29.7 Induction and Cases rules for fsets

lemma *fset-exhaust* [*case-names empty insert*, *cases type*: *fset*]:
assumes *fempty-case*: $S = \{||\} \Longrightarrow P$
and *finsert-case*: $\bigwedge x\ S'.\ S = \mathit{finsert}\ x\ S' \Longrightarrow P$
shows P
using *assms* **by** *transfer blast*

lemma *fset-induct* [*case-names empty insert*]:
assumes *fempty-case*: $P\ \{||\}$
and *finsert-case*: $\bigwedge x\ S.\ P\ S \Longrightarrow P\ (\mathit{finsert}\ x\ S)$
shows $P\ S$
proof −

note *Domainp-forall-transfer*[*transfer-rule*]
show *?thesis*
using *assms* **by** *transfer* (*auto intro*: *finite-induct*)
qed

lemma *fset-induct-stronger* [*case-names empty insert*, *induct type*: *fset*]:
assumes *empty-fset-case*: $P\ \{||\}$
and *insert-fset-case*: $\bigwedge x\ S.\ [\![x\ |\notin|\ S;\ P\ S]\!] \Longrightarrow P\ (\mathit{finsert}\ x\ S)$
shows $P\ S$
proof −

note *Domainp-forall-transfer*[*transfer-rule*]
show *?thesis*
using *assms* **by** *transfer* (*auto intro*: *finite-induct*)
qed

lemma *fset-card-induct*:
assumes *empty-fset-case*: $P\ \{||\}$
and *card-fset-Suc-case*: $\bigwedge S\ T.\ \mathit{Suc}\ (\mathit{fcard}\ S) = (\mathit{fcard}\ T) \Longrightarrow P\ S \Longrightarrow P\ T$
shows $P\ S$
proof (*induct S*)

case *empty*
show *P* {||} **by** (*rule empty-fset-case*)
next
case (*insert x S*)
have *h*: *P S* **by** *fact*
have *x* |∉| *S* **by** *fact*
then have *Suc* (*fcard S*) = *fcard* (*finsert x S*)
by *transfer auto*
then show *P* (*finsert x S*)
using *h card-fset-Suc-case* **by** *simp*
qed

lemma *fset-strong-cases*:
obtains *xs* = {||}
| *ys x* **where** *x* |∉| *ys* **and** *xs* = *finsert x ys*
by *transfer blast*

lemma *fset-induct2*:
P {||} {||} ⟹
(⋀*x xs*. *x* |∉| *xs* ⟹ *P* (*finsert x xs*) {||}) ⟹
(⋀*y ys*. *y* |∉| *ys* ⟹ *P* {||} (*finsert y ys*)) ⟹
(⋀*x xs y ys*. ⟦*P xs ys*; *x* |∉| *xs*; *y* |∉| *ys*⟧ ⟹ *P* (*finsert x xs*) (*finsert y ys*)) ⟹
P xsa ysa
apply (*induct xsa arbitrary*: *ysa*)
apply (*induct-tac x rule*: *fset-induct-stronger*)
apply *simp-all*
apply (*induct-tac xa rule*: *fset-induct-stronger*)
apply *simp-all*
done

29.8 Setup for Lifting/Transfer

29.8.1 Relator and predicator properties

lift-definition *rel-fset* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ '*a fset* ⇒ '*b fset* ⇒ *bool* **is** *rel-set*
parametric *rel-set-transfer* **.**

lemma *rel-fset-alt-def*: *rel-fset R* = (λ*A B*. (∀ *x*.∃ *y*. *x*|∈|*A* ⟶ *y*|∈|*B* ∧ *R x y*)
∧ (∀ *y*. ∃ *x*. *y*|∈|*B* ⟶ *x*|∈|*A* ∧ *R x y*))
apply (*rule ext*)+
apply *transfer'*
apply (*subst rel-set-def*[*unfolded fun-eq-iff*])
by *blast*

lemma *finite-rel-set*:
assumes *fin*: *finite X finite Z*
assumes *R-S*: *rel-set* (*R OO S*) *X Z*
shows ∃ *Y*. *finite Y* ∧ *rel-set R X Y* ∧ *rel-set S Y Z*
proof −
obtain *f* **where** *f*: ∀ *x*∈*X*. *R x* (*f x*) ∧ (∃ *z*∈*Z*. *S* (*f x*) *z*)

apply *atomize-elim*
apply (*subst bchoice-iff*[*symmetric*])
using *R-S*[*unfolded rel-set-def OO-def*] **by** *blast*

obtain *g* **where** *g*: ∀ *z*∈*Z*. *S* (*g z*) *z* ∧ (∃ *x*∈*X*. *R x* (*g z*))
apply *atomize-elim*
apply (*subst bchoice-iff*[*symmetric*])
using *R-S*[*unfolded rel-set-def OO-def*] **by** *blast*

let *?Y* = *f* ‘ *X* ∪ *g* ‘ *Z*
have *finite ?Y* **by** (*simp add*: *fin*)
moreover have *rel-set R X ?Y*
unfolding *rel-set-def*
using *f g* **by** *clarsimp blast*
moreover have *rel-set S ?Y Z*
unfolding *rel-set-def*
using *f g* **by** *clarsimp blast*
ultimately show *?thesis* **by** *metis*
qed

29.8.2 Transfer rules for the Transfer package

Unconditional transfer rules

context includes *lifting-syntax*
begin

lemmas *fempty-transfer* [*transfer-rule*] = *empty-transfer*[*Transfer.transferred*]

lemma *finsert-transfer* [*transfer-rule*]:
(*A* ===> *rel-fset A* ===> *rel-fset A*) *finsert finsert*
unfolding *rel-fun-def rel-fset-alt-def* **by** *blast*

lemma *funion-transfer* [*transfer-rule*]:
(*rel-fset A* ===> *rel-fset A* ===> *rel-fset A*) *funion funion*
unfolding *rel-fun-def rel-fset-alt-def* **by** *blast*

lemma *ffUnion-transfer* [*transfer-rule*]:
(*rel-fset* (*rel-fset A*) ===> *rel-fset A*) *ffUnion ffUnion*
unfolding *rel-fun-def rel-fset-alt-def* **by** *transfer* (*simp*, *fast*)

lemma *fimage-transfer* [*transfer-rule*]:
((*A* ===> *B*) ===> *rel-fset A* ===> *rel-fset B*) *fimage fimage*
unfolding *rel-fun-def rel-fset-alt-def* **by** *simp blast*

lemma *fBall-transfer* [*transfer-rule*]:
(*rel-fset A* ===> (*A* ===> (=)) ===> (=)) *fBall fBall*
unfolding *rel-fset-alt-def rel-fun-def* **by** *blast*

lemma *fBex-transfer* [*transfer-rule*]:

(*rel-fset A* ===> (*A* ===> (=)) ===> (=)) *fBex fBex*
unfolding *rel-fset-alt-def rel-fun-def* **by** *blast*

lemma *fPow-transfer* [*transfer-rule*]:
(*rel-fset A* ===> *rel-fset* (*rel-fset A*)) *fPow fPow*
unfolding *rel-fun-def*
using *Pow-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*]
by *blast*

lemma *rel-fset-transfer* [*transfer-rule*]:
((*A* ===> *B* ===> (=)) ===> *rel-fset A* ===> *rel-fset B* ===> (=))
rel-fset rel-fset
unfolding *rel-fun-def*
using *rel-set-transfer*[*unfolded rel-fun-def*,*rule-format*, *Transfer.transferred*, **where** *A* = *A* **and** *B* = *B*]
by *simp*

lemma *bind-transfer* [*transfer-rule*]:
(*rel-fset A* ===> (*A* ===> *rel-fset B*) ===> *rel-fset B*) *fbind fbind*
unfolding *rel-fun-def*
using *bind-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by** *blast*

Rules requiring bi-unique, bi-total or right-total relations

lemma *fmember-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*A* ===> *rel-fset A* ===> (=)) (|∈|) (|∈|)
using *assms* **unfolding** *rel-fun-def rel-fset-alt-def bi-unique-def* **by** *metis*

lemma *finter-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-fset A* ===> *rel-fset A* ===> *rel-fset A*) *finter finter*
using *assms* **unfolding** *rel-fun-def*
using *inter-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by** *blast*

lemma *fminus-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-fset A* ===> *rel-fset A* ===> *rel-fset A*) (|−|) (|−|)
using *assms* **unfolding** *rel-fun-def*
using *Diff-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by** *blast*

lemma *fsubset-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows (*rel-fset A* ===> *rel-fset A* ===> (=)) (|⊆|) (|⊆|)
using *assms* **unfolding** *rel-fun-def*
using *subset-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by**

blast

lemma *fSup-transfer* [*transfer-rule*]:
bi-unique A ⟹ (*rel-set* (*rel-fset A*) ===> *rel-fset A*) *Sup Sup*
unfolding *rel-fun-def*
apply *clarify*
apply *transfer'*
using *Sup-fset-transfer*[*unfolded rel-fun-def*] **by** *blast*

lemma *fInf-transfer* [*transfer-rule*]:
assumes *bi-unique A* **and** *bi-total A*
shows (*rel-set* (*rel-fset A*) ===> *rel-fset A*) *Inf Inf*
using *assms* **unfolding** *rel-fun-def*
apply *clarify*
apply *transfer'*
using *Inf-fset-transfer*[*unfolded rel-fun-def*] **by** *blast*

lemma *ffilter-transfer* [*transfer-rule*]:
assumes *bi-unique A*
shows ((*A* ===> (=)) ===> *rel-fset A* ===> *rel-fset A*) *ffilter ffilter*
using *assms* **unfolding** *rel-fun-def*
using *Lifting-Set.filter-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*]
by *blast*

lemma *card-transfer* [*transfer-rule*]:
bi-unique A ⟹ (*rel-fset A* ===> (=)) *fcard fcard*
unfolding *rel-fun-def*
using *card-transfer*[*unfolded rel-fun-def*, *rule-format*, *Transfer.transferred*] **by** *blast*

end

lifting-update *fset.lifting*
lifting-forget *fset.lifting*

29.9 BNF setup

context
includes *fset.lifting*
begin

lemma *rel-fset-alt*:
rel-fset R a b ⟷ (∀ *t* ∈ *fset a*. ∃ *u* ∈ *fset b*. *R t u*) ∧ (∀ *t* ∈ *fset b*. ∃ *u* ∈ *fset a*. *R u t*)
by *transfer* (*simp add*: *rel-set-def*)

lemma *fset-to-fset*: *finite A* ⟹ *fset* (*the-inv fset A*) = *A*

apply (*rule f-the-inv-into-f*[*unfolded inj-on-def*])
apply (*simp add*: *fset-inject*)
apply (*rule range-eqI Abs-fset-inverse*[*symmetric*] *CollectI*)+
.

lemma *rel-fset-aux*:
$(\forall\, t \in \mathit{fset}\ a.\ \exists\, u \in \mathit{fset}\ b.\ R\ t\ u) \wedge (\forall\, u \in \mathit{fset}\ b.\ \exists\, t \in \mathit{fset}\ a.\ R\ t\ u) \longleftrightarrow$
$((\mathit{BNF\text{-}Def.Grp}\ \{a.\ \mathit{fset}\ a \subseteq \{(a,\ b).\ R\ a\ b\}\}\ (\mathit{fimage}\ \mathit{fst}))^{-1-1}\ \mathit{OO}$
$\mathit{BNF\text{-}Def.Grp}\ \{a.\ \mathit{fset}\ a \subseteq \{(a,\ b).\ R\ a\ b\}\}\ (\mathit{fimage}\ \mathit{snd}))\ a\ b$ (**is** *?L* = *?R*)
proof
assume *?L*
define R' **where** $R' =$
the-inv fset (*Collect* (*case-prod R*) ∩ (*fset a* × *fset b*)) (**is** - = *the-inv fset ?L'*)
have *finite ?L'* **by** (*intro finite-Int*[*OF disjI2*] *finite-cartesian-product*) (*transfer*, *simp*)+
hence ∗: *fset* R' = *?L'* **unfolding** R'*-def* **by** (*intro fset-to-fset*)
show *?R* **unfolding** *Grp-def relcompp.simps conversep.simps*
proof (*intro CollectI case-prodI exI*[*of - a*] *exI*[*of - b*] *exI*[*of - R'*] *conjI refl*)
from ∗ **show** *a* = *fimage fst* R' **using** *conjunct1*[*OF* ‹*?L*›]
by (*transfer*, *auto simp add*: *image-def Int-def split*: *prod.splits*)
from ∗ **show** *b* = *fimage snd* R' **using** *conjunct2*[*OF* ‹*?L*›]
by (*transfer*, *auto simp add*: *image-def Int-def split*: *prod.splits*)
qed (*auto simp add*: ∗)
next
assume *?R* **thus** *?L* **unfolding** *Grp-def relcompp.simps conversep.simps*
apply (*simp add*: *subset-eq Ball-def*)
apply (*rule conjI*)
apply (*transfer*, *clarsimp*, *metis snd-conv*)
by (*transfer*, *clarsimp*, *metis fst-conv*)
qed

bnf *'a fset*
map: *fimage*
sets: *fset*
bd: *natLeq*
wits: {||}
rel: *rel-fset*
apply −
apply *transfer'* **apply** *simp*
apply *transfer'* **apply** *force*
apply *transfer* **apply** *force*
apply *transfer'* **apply** *force*
apply (*rule natLeq-card-order*)
apply (*rule natLeq-cinfinite*)
apply *transfer* **apply** (*metis ordLess-imp-ordLeq finite-iff-ordLess-natLeq*)
apply (*fastforce simp*: *rel-fset-alt*)
apply (*simp add*: *Grp-def relcompp.simps conversep.simps fun-eq-iff rel-fset-alt*
rel-fset-aux[*unfolded OO-Grp-alt*])
apply *transfer* **apply** *simp*

done

lemma *rel-fset-fset*: *rel-set* χ (*fset A1*) (*fset A2*) = *rel-fset* χ *A1 A2*
by *transfer* (*rule refl*)

end

lemmas [*simp*] = *fset.map-comp fset.map-id fset.set-map*

29.10 Size setup

context includes *fset.lifting* **begin**
lift-definition *size-fset* :: ('*a* ⇒ *nat*) ⇒ '*a fset* ⇒ *nat* **is** λ*f*. *sum* (*Suc* ∘ *f*) **.**
end

instantiation *fset* :: (*type*) *size* **begin**
definition *size-fset* **where**
size-fset-overloaded-def: *size-fset* = *FSet.size-fset* (λ-. *0*)
instance ..
end

lemmas *size-fset-simps*[*simp*] =
size-fset-def[*THEN meta-eq-to-obj-eq*, *THEN fun-cong*, *THEN fun-cong*,
unfolded map-fun-def comp-def id-apply]

lemmas *size-fset-overloaded-simps*[*simp*] =
size-fset-simps[*of* λ-. *0*, *unfolded add-0-left add-0-right*,
folded size-fset-overloaded-def]

lemma *fset-size-o-map*: *inj f* ⟹ *size-fset g* ∘ *fimage f* = *size-fset* (*g* ∘ *f*)
apply (*subst fun-eq-iff*)
including *fset.lifting* **by** *transfer* (*auto intro*: *sum.reindex-cong subset-inj-on*)

setup ‹
BNF-LFP-Size.register-size-global ***type-name*** ‹*fset*› ***const-name*** ‹*size-fset*›
@{*thm size-fset-overloaded-def*} @{*thms size-fset-simps size-fset-overloaded-simps*}
@{*thms fset-size-o-map*}
›

lifting-update *fset.lifting*
lifting-forget *fset.lifting*

29.11 Advanced relator customization

Set vs. sum relators:

lemma *rel-set-rel-sum*[*simp*]:
rel-set (*rel-sum* χ φ) *A1 A2* ⟷
rel-set χ (*Inl* −‘ *A1*) (*Inl* −‘ *A2*) ∧ *rel-set* φ (*Inr* −‘ *A1*) (*Inr* −‘ *A2*)
(**is** *?L* ⟷ *?Rl* ∧ *?Rr*)

proof *safe*
assume *L*: *?L*
show *?Rl* **unfolding** *rel-set-def Bex-def vimage-eq* **proof** *safe*
fix *l1* **assume** *Inl l1 ∈ A1*
then obtain *a2* **where** *a2*: *a2 ∈ A2* **and** *rel-sum χ φ* (*Inl l1*) *a2*
using *L* **unfolding** *rel-set-def* **by** *auto*
then obtain *l2* **where** *a2 = Inl l2 ∧ χ l1 l2* **by** (*cases a2*, *auto*)
thus ∃ *l2*. *Inl l2 ∈ A2 ∧ χ l1 l2* **using** *a2* **by** *auto*
next
fix *l2* **assume** *Inl l2 ∈ A2*
then obtain *a1* **where** *a1*: *a1 ∈ A1* **and** *rel-sum χ φ a1* (*Inl l2*)
using *L* **unfolding** *rel-set-def* **by** *auto*
then obtain *l1* **where** *a1 = Inl l1 ∧ χ l1 l2* **by** (*cases a1*, *auto*)
thus ∃ *l1*. *Inl l1 ∈ A1 ∧ χ l1 l2* **using** *a1* **by** *auto*
qed
show *?Rr* **unfolding** *rel-set-def Bex-def vimage-eq* **proof** *safe*
fix *r1* **assume** *Inr r1 ∈ A1*
then obtain *a2* **where** *a2*: *a2 ∈ A2* **and** *rel-sum χ φ* (*Inr r1*) *a2*
using *L* **unfolding** *rel-set-def* **by** *auto*
then obtain *r2* **where** *a2 = Inr r2 ∧ φ r1 r2* **by** (*cases a2*, *auto*)
thus ∃ *r2*. *Inr r2 ∈ A2 ∧ φ r1 r2* **using** *a2* **by** *auto*
next
fix *r2* **assume** *Inr r2 ∈ A2*
then obtain *a1* **where** *a1*: *a1 ∈ A1* **and** *rel-sum χ φ a1* (*Inr r2*)
using *L* **unfolding** *rel-set-def* **by** *auto*
then obtain *r1* **where** *a1 = Inr r1 ∧ φ r1 r2* **by** (*cases a1*, *auto*)
thus ∃ *r1*. *Inr r1 ∈ A1 ∧ φ r1 r2* **using** *a1* **by** *auto*
qed
next
assume *Rl*: *?Rl* **and** *Rr*: *?Rr*
show *?L* **unfolding** *rel-set-def Bex-def vimage-eq* **proof** *safe*
fix *a1* **assume** *a1*: *a1 ∈ A1*
show ∃ *a2*. *a2 ∈ A2 ∧ rel-sum χ φ a1 a2*
proof(*cases a1*)
case (*Inl l1*) **then obtain** *l2* **where** *Inl l2 ∈ A2 ∧ χ l1 l2*
using *Rl a1* **unfolding** *rel-set-def* **by** *blast*
thus *?thesis* **unfolding** *Inl* **by** *auto*
next
case (*Inr r1*) **then obtain** *r2* **where** *Inr r2 ∈ A2 ∧ φ r1 r2*
using *Rr a1* **unfolding** *rel-set-def* **by** *blast*
thus *?thesis* **unfolding** *Inr* **by** *auto*
qed
next
fix *a2* **assume** *a2*: *a2 ∈ A2*
show ∃ *a1*. *a1 ∈ A1 ∧ rel-sum χ φ a1 a2*
proof(*cases a2*)
case (*Inl l2*) **then obtain** *l1* **where** *Inl l1 ∈ A1 ∧ χ l1 l2*
using *Rl a2* **unfolding** *rel-set-def* **by** *blast*
thus *?thesis* **unfolding** *Inl* **by** *auto*

```
    next
      case (Inr r2) then obtain r1 where Inr r1 ∈ A1 ∧ φ r1 r2
      using Rr a2 unfolding rel-set-def by blast
      thus ?thesis unfolding Inr by auto
    qed
  qed
qed
```

29.11.1 Countability

```
lemma exists-fset-of-list: ∃ xs. fset-of-list xs = S
including fset.lifting
by transfer (rule finite-list)

lemma fset-of-list-surj[simp, intro]: surj fset-of-list
proof −
  have x ∈ range fset-of-list for x :: 'a fset
    unfolding image-iff
    using exists-fset-of-list by fastforce
  thus ?thesis by auto
qed

instance fset :: (countable) countable
proof
  obtain to-nat :: 'a list ⇒ nat where inj to-nat
    by (metis ex-inj)
  moreover have inj (inv fset-of-list)
    using fset-of-list-surj by (rule surj-imp-inj-inv)
  ultimately have inj (to-nat ∘ inv fset-of-list)
    by (rule inj-compose)
  thus ∃ to-nat::'a fset ⇒ nat. inj to-nat
    by auto
qed
```

29.12 Quickcheck setup

Setup adapted from sets.

```
notation Quickcheck-Exhaustive.orelse (infixr orelse 55)

context
  includes term-syntax
begin

definition [code-unfold]:
valterm-femptyset = Code-Evaluation.valtermify ({||} :: ('a :: typerep) fset)

definition [code-unfold]:
valtermify-finsert x s = Code-Evaluation.valtermify finsert {·} (x :: ('a :: typerep ∗
-)) {·} s
```

end

instantiation *fset* :: (*exhaustive*) *exhaustive*
begin

fun *exhaustive-fset* **where**
exhaustive-fset f i = (*if i* = *0 then None else* (*f* {||} *orelse exhaustive-fset* (λ*A*. *f A orelse Quickcheck-Exhaustive.exhaustive* (λ*x*. *if x* |∈| *A then None else f* (*finsert x A*)) (*i* − *1*)) (*i* − *1*)))

instance ..

end

instantiation *fset* :: (*full-exhaustive*) *full-exhaustive*
begin

fun *full-exhaustive-fset* **where**
full-exhaustive-fset f i = (*if i* = *0 then None else* (*f valterm-femptyset orelse full-exhaustive-fset* (λ*A*. *f A orelse Quickcheck-Exhaustive.full-exhaustive* (λ*x*. *if fst x* |∈| *fst A then None else f* (*valtermify-finsert x A*)) (*i* − *1*)) (*i* − *1*)))

instance ..

end

no-notation *Quickcheck-Exhaustive.orelse* (**infixr** *orelse 55*)

instantiation *fset* :: (*random*) *random*
begin

context
includes *state-combinator-syntax*
begin

fun *random-aux-fset* :: *natural* ⇒ *natural* ⇒ *natural* × *natural* ⇒ ('*a fset* × (*unit* ⇒ *term*)) × *natural* × *natural* **where**
random-aux-fset 0 j = *Quickcheck-Random.collapse* (*Random.select-weight* [(*1*, *Pair valterm-femptyset*)]) |
random-aux-fset (*Code-Numeral.Suc i*) *j* =
Quickcheck-Random.collapse (*Random.select-weight*
[(*1*, *Pair valterm-femptyset*),
(*Code-Numeral.Suc i*,
Quickcheck-Random.random j ∘→ (λ*x*. *random-aux-fset i j* ∘→ (λ*s*. *Pair* (*valtermify-finsert x s*))))])

lemma [*code*]:
random-aux-fset i j =

Quickcheck-Random.collapse (*Random.select-weight* [(*1*, *Pair valterm-femptyset*), (*i*, *Quickcheck-Random.random j* ∘→ (λ*x*. *random-aux-fset* (*i* − *1*) *j* ∘→ (λ*s*. *Pair* (*valtermify-finsert x s*))))])
proof (*induct i rule*: *natural.induct*)
case *zero*
show *?case* **by** (*subst select-weight-drop-zero*[*symmetric*]) (*simp add*: *less-natural-def*)
next
case (*Suc i*)
show *?case* **by** (*simp only*: *random-aux-fset.simps Suc-natural-minus-one*)
qed

definition *random-fset i* = *random-aux-fset i i*

instance ..

end

end

end

30 Type of finite maps defined as a subtype of maps

theory *Finite-Map*
imports *FSet AList Conditional-Parametricity*
abbrevs (= = ⊆$_f$
begin

30.1 Auxiliary constants and lemmas over *map*

parametric-constant *map-add-transfer*[*transfer-rule*]: *map-add-def*
parametric-constant *map-of-transfer*[*transfer-rule*]: *map-of-def*

context includes *lifting-syntax* **begin**

abbreviation *rel-map* :: ('*b* ⇒ '*c* ⇒ *bool*) ⇒ ('*a* ⇀ '*b*) ⇒ ('*a* ⇀ '*c*) ⇒ *bool* **where**
rel-map f ≡ (=) ===> *rel-option f*

lemma *ran-transfer*[*transfer-rule*]: (*rel-map A* ===> *rel-set A*) *ran ran*
proof
fix *m n*
assume *rel-map A m n*
show *rel-set A* (*ran m*) (*ran n*)
proof (*rule rel-setI*)
fix *x*
assume *x* ∈ *ran m*
then obtain *a* **where** *m a* = *Some x*

unfolding *ran-def* **by** *auto*

have *rel-option A* (*m a*) (*n a*)
using ‹*rel-map A m n*›
by (*auto dest*: *rel-funD*)
then obtain *y* **where** *n a* = *Some y A x y*
unfolding ‹*m a* = -›
by *cases auto*
then show ∃ *y* ∈ *ran n*. *A x y*
unfolding *ran-def* **by** *blast*
next
fix *y*
assume *y* ∈ *ran n*
then obtain *a* **where** *n a* = *Some y*
unfolding *ran-def* **by** *auto*

have *rel-option A* (*m a*) (*n a*)
using ‹*rel-map A m n*›
by (*auto dest*: *rel-funD*)
then obtain *x* **where** *m a* = *Some x A x y*
unfolding ‹*n a* = -›
by *cases auto*
then show ∃ *x* ∈ *ran m*. *A x y*
unfolding *ran-def* **by** *blast*
qed
qed

lemma *ran-alt-def*: *ran m* = (*the* ∘ *m*) ‘ *dom m*
unfolding *ran-def dom-def* **by** *force*

parametric-constant *dom-transfer*[*transfer-rule*]: *dom-def*

definition *map-upd* :: *'a* ⇒ *'b* ⇒ (*'a* ⇀ *'b*) ⇒ (*'a* ⇀ *'b*) **where**
map-upd k v m = *m*(*k* ↦ *v*)

parametric-constant *map-upd-transfer*[*transfer-rule*]: *map-upd-def*

definition *map-filter* :: (*'a* ⇒ *bool*) ⇒ (*'a* ⇀ *'b*) ⇒ (*'a* ⇀ *'b*) **where**
map-filter P m = (λ*x*. *if P x then m x else None*)

parametric-constant *map-filter-transfer*[*transfer-rule*]: *map-filter-def*

lemma *map-filter-map-of*[*simp*]: *map-filter P* (*map-of m*) = *map-of* [(*k*, -) ← *m*. *P k*]
proof
fix *x*
show *map-filter P* (*map-of m*) *x* = *map-of* [(*k*, -) ← *m*. *P k*] *x*
by (*induct m*) (*auto simp*: *map-filter-def*)
qed

lemma *map-filter-finite*[*intro*]:
assumes *finite* (*dom m*)
shows *finite* (*dom* (*map-filter P m*))
proof −
have *dom* (*map-filter P m*) = *Set.filter P* (*dom m*)
unfolding *map-filter-def Set.filter-def dom-def*
by *auto*
then show *?thesis*
using *assms*
by (*simp add*: *Set.filter-def*)
qed

definition *map-drop* :: *'a* ⇒ (*'a* ⇀ *'b*) ⇒ (*'a* ⇀ *'b*) **where**
map-drop a = *map-filter* (λ*a'*. *a'* ≠ *a*)

parametric-constant *map-drop-transfer*[*transfer-rule*]: *map-drop-def*

definition *map-drop-set* :: *'a set* ⇒ (*'a* ⇀ *'b*) ⇒ (*'a* ⇀ *'b*) **where**
map-drop-set A = *map-filter* (λ*a*. *a* ∉ *A*)

parametric-constant *map-drop-set-transfer*[*transfer-rule*]: *map-drop-set-def*

definition *map-restrict-set* :: *'a set* ⇒ (*'a* ⇀ *'b*) ⇒ (*'a* ⇀ *'b*) **where**
map-restrict-set A = *map-filter* (λ*a*. *a* ∈ *A*)

parametric-constant *map-restrict-set-transfer*[*transfer-rule*]: *map-restrict-set-def*

definition *map-pred* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ (*'a* ⇀ *'b*) ⇒ *bool* **where**
map-pred P m ⟷ (∀ *x*. *case m x of None* ⇒ *True* | *Some y* ⇒ *P x y*)

parametric-constant *map-pred-transfer*[*transfer-rule*]: *map-pred-def*

definition *rel-map-on-set* :: *'a set* ⇒ (*'b* ⇒ *'c* ⇒ *bool*) ⇒ (*'a* ⇀ *'b*) ⇒ (*'a* ⇀ *'c*)
⇒ *bool* **where**
rel-map-on-set S P = *eq-onp* (λ*x*. *x* ∈ *S*) ===> *rel-option P*

definition *set-of-map* :: (*'a* ⇀ *'b*) ⇒ (*'a* × *'b*) *set* **where**
set-of-map m = {(*k*, *v*)|*k v*. *m k* = *Some v*}

lemma *set-of-map-alt-def*: *set-of-map m* = (λ*k*. (*k*, *the* (*m k*))) ‘ *dom m*
unfolding *set-of-map-def dom-def*
by *auto*

lemma *set-of-map-finite*: *finite* (*dom m*) ⟹ *finite* (*set-of-map m*)
unfolding *set-of-map-alt-def*
by *auto*

lemma *set-of-map-inj*: *inj set-of-map*

proof
  fix *x y*
  assume *set-of-map x = set-of-map y*
  hence *(x a = Some b) = (y a = Some b)* **for** *a b*
    unfolding *set-of-map-def* **by** *auto*
  hence *x k = y k* **for** *k*
    by (*metis not-None-eq*)
  thus *x = y* **..**
qed

lemma *dom-comp*: *dom (m ∘$_m$ n) ⊆ dom n*
unfolding *map-comp-def dom-def*
by (*auto split*: *option.splits*)

lemma *dom-comp-finite*: *finite (dom n) ⟹ finite (dom (map-comp m n))*
by (*metis finite-subset dom-comp*)

parametric-constant *map-comp-transfer*[*transfer-rule*]: *map-comp-def*

end

30.2 Abstract characterisation

typedef *('a, 'b) fmap = {m. finite (dom m)} :: ('a ⇀ 'b) set*
  morphisms *fmlookup Abs-fmap*
proof
  show *Map.empty ∈ {m. finite (dom m)}*
    by *auto*
qed

setup-lifting *type-definition-fmap*

lemma *dom-fmlookup-finite*[*intro*, *simp*]: *finite (dom (fmlookup m))*
using *fmap.fmlookup* **by** *auto*

lemma *fmap-ext*:
  assumes *⋀x. fmlookup m x = fmlookup n x*
  shows *m = n*
using *assms*
by *transfer' auto*

30.3 Operations

context
  includes *fset.lifting*
begin

lift-definition *fmran* :: *('a, 'b) fmap ⇒ 'b fset*
  is *ran*
  parametric *ran-transfer*

by (*rule finite-ran*)

lemma *fmlookup-ran-iff*: $y \mid\in\mid fmran\ m \longleftrightarrow (\exists x.\ fmlookup\ m\ x = Some\ y)$
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmranI*: $fmlookup\ m\ x = Some\ y \Longrightarrow y \mid\in\mid fmran\ m$ **by** (*auto simp*: *fmlookup-ran-iff*)

lemma *fmranE*[*elim*]:
  assumes $y \mid\in\mid fmran\ m$
  obtains x **where** $fmlookup\ m\ x = Some\ y$
using *assms* **by** (*auto simp*: *fmlookup-ran-iff*)

lift-definition *fmdom* :: ('*a*, '*b*) *fmap* ⇒ '*a fset*
  is *dom*
  parametric *dom-transfer*
.

lemma *fmlookup-dom-iff*: $x \mid\in\mid fmdom\ m \longleftrightarrow (\exists a.\ fmlookup\ m\ x = Some\ a)$
by *transfer' auto*

lemma *fmdom-notI*: $fmlookup\ m\ x = None \Longrightarrow x \mid\notin\mid fmdom\ m$ **by** (*simp add*: *fmlookup-dom-iff*)
lemma *fmdomI*: $fmlookup\ m\ x = Some\ y \Longrightarrow x \mid\in\mid fmdom\ m$ **by** (*simp add*: *fmlookup-dom-iff*)
lemma *fmdom-notD*[*dest*]: $x \mid\notin\mid fmdom\ m \Longrightarrow fmlookup\ m\ x = None$ **by** (*simp add*: *fmlookup-dom-iff*)

lemma *fmdomE*[*elim*]:
  assumes $x \mid\in\mid fmdom\ m$
  obtains y **where** $fmlookup\ m\ x = Some\ y$
using *assms* **by** (*auto simp*: *fmlookup-dom-iff*)

lift-definition *fmdom'* :: ('*a*, '*b*) *fmap* ⇒ '*a set*
  is *dom*
  parametric *dom-transfer*
.

lemma *fmlookup-dom'-iff*: $x \in fmdom'\ m \longleftrightarrow (\exists a.\ fmlookup\ m\ x = Some\ a)$
by *transfer' auto*

lemma *fmdom'-notI*: $fmlookup\ m\ x = None \Longrightarrow x \notin fmdom'\ m$ **by** (*simp add*: *fmlookup-dom'-iff*)
lemma *fmdom'I*: $fmlookup\ m\ x = Some\ y \Longrightarrow x \in fmdom'\ m$ **by** (*simp add*: *fmlookup-dom'-iff*)
lemma *fmdom'-notD*[*dest*]: $x \notin fmdom'\ m \Longrightarrow fmlookup\ m\ x = None$ **by** (*simp add*: *fmlookup-dom'-iff*)

lemma *fmdom'E*[*elim*]:

assumes $x \in fmdom'\ m$
obtains $x\ y$ **where** $fmlookup\ m\ x = Some\ y$
using *assms* **by** (*auto simp*: *fmlookup-dom'-iff*)

lemma *fmdom'-alt-def*: $fmdom'\ m = fset\ (fmdom\ m)$
by *transfer' force*

lemma *finite-fmdom'*[*simp*]: *finite* $(fmdom'\ m)$
unfolding *fmdom'-alt-def* **by** *simp*

lemma *dom-fmlookup*[*simp*]: $dom\ (fmlookup\ m) = fmdom'\ m$
by *transfer' simp*

lift-definition *fmempty* :: $('a, 'b)\ fmap$
is *Map.empty*
by *simp*

lemma *fmempty-lookup*[*simp*]: $fmlookup\ fmempty\ x = None$
by *transfer' simp*

lemma *fmdom-empty*[*simp*]: $fmdom\ fmempty = \{||\}$ **by** *transfer' simp*
lemma *fmdom'-empty*[*simp*]: $fmdom'\ fmempty = \{\}$ **by** *transfer' simp*
lemma *fmran-empty*[*simp*]: $fmran\ fmempty = fempty$ **by** *transfer'* (*auto simp*: *ran-def map-filter-def*)

lift-definition *fmupd* :: $'a \Rightarrow 'b \Rightarrow ('a, 'b)\ fmap \Rightarrow ('a, 'b)\ fmap$
is *map-upd*
parametric *map-upd-transfer*
unfolding *map-upd-def*[*abs-def*]
by *simp*

lemma *fmupd-lookup*[*simp*]: $fmlookup\ (fmupd\ a\ b\ m)\ a' = (if\ a = a'\ then\ Some\ b\ else\ fmlookup\ m\ a')$
by *transfer'* (*auto simp*: *map-upd-def*)

lemma *fmdom-fmupd*[*simp*]: $fmdom\ (fmupd\ a\ b\ m) = finsert\ a\ (fmdom\ m)$ **by** *transfer* (*simp add*: *map-upd-def*)
lemma *fmdom'-fmupd*[*simp*]: $fmdom'\ (fmupd\ a\ b\ m) = insert\ a\ (fmdom'\ m)$ **by** *transfer* (*simp add*: *map-upd-def*)

lemma *fmupd-reorder-neq*:
assumes $a \neq b$
shows $fmupd\ a\ x\ (fmupd\ b\ y\ m) = fmupd\ b\ y\ (fmupd\ a\ x\ m)$
using *assms*
by *transfer'* (*auto simp*: *map-upd-def*)

lemma *fmupd-idem*[*simp*]: $fmupd\ a\ x\ (fmupd\ a\ y\ m) = fmupd\ a\ x\ m$
by *transfer'* (*auto simp*: *map-upd-def*)

lift-definition *fmfilter* :: $('a \Rightarrow bool) \Rightarrow ('a, 'b)\ fmap \Rightarrow ('a, 'b)\ fmap$
  is *map-filter*
  parametric *map-filter-transfer*
by *auto*

lemma *fmdom-filter*[*simp*]: *fmdom* (*fmfilter P m*) = *ffilter P* (*fmdom m*)
by *transfer'* (*auto simp*: *map-filter-def Set.filter-def split*: *if-splits*)

lemma *fmdom'-filter*[*simp*]: *fmdom'* (*fmfilter P m*) = *Set.filter P* (*fmdom' m*)
by *transfer'* (*auto simp*: *map-filter-def Set.filter-def split*: *if-splits*)

lemma *fmlookup-filter*[*simp*]: *fmlookup* (*fmfilter P m*) *x* = (*if P x then fmlookup m x else None*)
by *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmfilter-empty*[*simp*]: *fmfilter P fmempty* = *fmempty*
by *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmfilter-true*[*simp*]:
  assumes $\bigwedge x\ y.$ *fmlookup m x* = *Some y* $\Longrightarrow$ *P x*
  shows *fmfilter P m* = *m*
proof (*rule fmap-ext*)
  fix *x*
  have *fmlookup m x* = *None* **if** $\neg$ *P x*
    using *that assms* **by** *fastforce*
  then show *fmlookup* (*fmfilter P m*) *x* = *fmlookup m x*
    by *simp*
qed

lemma *fmfilter-false*[*simp*]:
  assumes $\bigwedge x\ y.$ *fmlookup m x* = *Some y* $\Longrightarrow$ $\neg$ *P x*
  shows *fmfilter P m* = *fmempty*
using *assms* **by** *transfer'* (*fastforce simp*: *map-filter-def*)

lemma *fmfilter-comp*[*simp*]: *fmfilter P* (*fmfilter Q m*) = *fmfilter* ($\lambda x.$ *P x* $\wedge$ *Q x*) *m*
by *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmfilter-comm*: *fmfilter P* (*fmfilter Q m*) = *fmfilter Q* (*fmfilter P m*)
unfolding *fmfilter-comp* **by** *meson*

lemma *fmfilter-cong*[*cong*]:
  assumes $\bigwedge x\ y.$ *fmlookup m x* = *Some y* $\Longrightarrow$ *P x* = *Q x*
  shows *fmfilter P m* = *fmfilter Q m*
proof (*rule fmap-ext*)
  fix *x*
  have *fmlookup m x* = *None* **if** *P x* $\neq$ *Q x*
    using *that assms* **by** *fastforce*
  then show *fmlookup* (*fmfilter P m*) *x* = *fmlookup* (*fmfilter Q m*) *x*

by *auto*
qed

lemma *fmfilter-cong'*[*fundef-cong*]:
assumes $m = n$ $\bigwedge x.\ x \in fmdom'\ m \Longrightarrow P\ x = Q\ x$
shows *fmfilter P m* = *fmfilter Q n*
using *assms*(*2*) **unfolding** *assms*(*1*)
by (*rule fmfilter-cong*) (*metis fmdom'I*)

lemma *fmfilter-upd*[*simp*]:
fmfilter P (*fmupd x y m*) = (*if P x then fmupd x y* (*fmfilter P m*) *else fmfilter P m*)
by *transfer'* (*auto simp*: *map-upd-def map-filter-def*)

lift-definition *fmdrop* :: *'a* ⇒ (*'a*, *'b*) *fmap* ⇒ (*'a*, *'b*) *fmap*
is *map-drop*
parametric *map-drop-transfer*
unfolding *map-drop-def* **by** *auto*

lemma *fmdrop-lookup*[*simp*]: *fmlookup* (*fmdrop a m*) *a* = *None*
by *transfer'* (*auto simp*: *map-drop-def map-filter-def*)

lift-definition *fmdrop-set* :: *'a set* ⇒ (*'a*, *'b*) *fmap* ⇒ (*'a*, *'b*) *fmap*
is *map-drop-set*
parametric *map-drop-set-transfer*
unfolding *map-drop-set-def* **by** *auto*

lift-definition *fmdrop-fset* :: *'a fset* ⇒ (*'a*, *'b*) *fmap* ⇒ (*'a*, *'b*) *fmap*
is *map-drop-set*
parametric *map-drop-set-transfer*
unfolding *map-drop-set-def* **by** *auto*

lift-definition *fmrestrict-set* :: *'a set* ⇒ (*'a*, *'b*) *fmap* ⇒ (*'a*, *'b*) *fmap*
is *map-restrict-set*
parametric *map-restrict-set-transfer*
unfolding *map-restrict-set-def* **by** *auto*

lift-definition *fmrestrict-fset* :: *'a fset* ⇒ (*'a*, *'b*) *fmap* ⇒ (*'a*, *'b*) *fmap*
is *map-restrict-set*
parametric *map-restrict-set-transfer*
unfolding *map-restrict-set-def* **by** *auto*

lemma *fmfilter-alt-defs*:
fmdrop a = *fmfilter* (λ*a'*. *a'* ≠ *a*)
fmdrop-set A = *fmfilter* (λ*a*. *a* ∉ *A*)
fmdrop-fset B = *fmfilter* (λ*a*. *a* |∉| *B*)
fmrestrict-set A = *fmfilter* (λ*a*. *a* ∈ *A*)
fmrestrict-fset B = *fmfilter* (λ*a*. *a* |∈| *B*)
by (*transfer'*; *simp add*: *map-drop-def map-drop-set-def map-restrict-set-def*)+

lemma *fmdom-drop*[*simp*]: *fmdom* (*fmdrop a m*) = *fmdom m* − {|*a*|} **unfolding** *fmfilter-alt-defs* **by** *auto*
lemma *fmdom'-drop*[*simp*]: *fmdom'* (*fmdrop a m*) = *fmdom' m* − {*a*} **unfolding** *fmfilter-alt-defs* **by** *auto*
lemma *fmdom'-drop-set*[*simp*]: *fmdom'* (*fmdrop-set A m*) = *fmdom' m* − *A* **unfolding** *fmfilter-alt-defs* **by** *auto*
lemma *fmdom-drop-fset*[*simp*]: *fmdom* (*fmdrop-fset A m*) = *fmdom m* − *A* **unfolding** *fmfilter-alt-defs* **by** *auto*
lemma *fmdom'-restrict-set*: *fmdom'* (*fmrestrict-set A m*) ⊆ *A* **unfolding** *fmfilter-alt-defs* **by** *auto*
lemma *fmdom-restrict-fset*: *fmdom* (*fmrestrict-fset A m*) |⊆| *A* **unfolding** *fmfilter-alt-defs* **by** *auto*

lemma *fmdrop-fmupd*: *fmdrop x* (*fmupd y z m*) = (*if x* = *y then fmdrop x m else fmupd y z* (*fmdrop x m*))
by *transfer'* (*auto simp*: *map-drop-def map-filter-def map-upd-def*)

lemma *fmdrop-idle*: *x* |∉| *fmdom B* ⟹ *fmdrop x B* = *B*
by *transfer'* (*auto simp*: *map-drop-def map-filter-def*)

lemma *fmdrop-idle'*: *x* ∉ *fmdom' B* ⟹ *fmdrop x B* = *B*
by *transfer'* (*auto simp*: *map-drop-def map-filter-def*)

lemma *fmdrop-fmupd-same*: *fmdrop x* (*fmupd x y m*) = *fmdrop x m*
by *transfer'* (*auto simp*: *map-drop-def map-filter-def map-upd-def*)

lemma *fmdom'-restrict-set-precise*: *fmdom'* (*fmrestrict-set A m*) = *fmdom' m* ∩ *A*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmdom'-restrict-fset-precise*: *fmdom* (*fmrestrict-fset A m*) = *fmdom m* |∩| *A*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmdom'-drop-fset*[*simp*]: *fmdom'* (*fmdrop-fset A m*) = *fmdom' m* − *fset A*
unfolding *fmfilter-alt-defs* **by** *transfer'* (*auto simp*: *map-filter-def split*: *if-splits*)

lemma *fmdom'-restrict-fset*: *fmdom'* (*fmrestrict-fset A m*) ⊆ *fset A*
unfolding *fmfilter-alt-defs* **by** *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmlookup-drop*[*simp*]:
fmlookup (*fmdrop a m*) *x* = (*if x* ≠ *a then fmlookup m x else None*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmlookup-drop-set*[*simp*]:
fmlookup (*fmdrop-set A m*) *x* = (*if x* ∉ *A then fmlookup m x else None*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmlookup-drop-fset*[*simp*]:
fmlookup (*fmdrop-fset A m*) *x* = (*if x* |∉| *A then fmlookup m x else None*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmlookup-restrict-set*[*simp*]:
fmlookup (*fmrestrict-set A m*) *x* = (*if x* ∈ *A then fmlookup m x else None*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmlookup-restrict-fset*[*simp*]:
fmlookup (*fmrestrict-fset A m*) *x* = (*if x* |∈| *A then fmlookup m x else None*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-set-dom*[*simp*]: *fmrestrict-set* (*fmdom' m*) *m* = *m*
by (*rule fmap-ext*) *auto*

lemma *fmrestrict-fset-dom*[*simp*]: *fmrestrict-fset* (*fmdom m*) *m* = *m*
by (*rule fmap-ext*) *auto*

lemma *fmdrop-empty*[*simp*]: *fmdrop a fmempty* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-set-empty*[*simp*]: *fmdrop-set A fmempty* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-fset-empty*[*simp*]: *fmdrop-fset A fmempty* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-fset-fmdom*[*simp*]: *fmdrop-fset* (*fmdom A*) *A* = *fmempty*
by *transfer'* (*auto simp*: *map-drop-set-def map-filter-def*)

lemma *fmdrop-set-fmdom*[*simp*]: *fmdrop-set* (*fmdom' A*) *A* = *fmempty*
by *transfer'* (*auto simp*: *map-drop-set-def map-filter-def*)

lemma *fmrestrict-set-empty*[*simp*]: *fmrestrict-set A fmempty* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-fset-empty*[*simp*]: *fmrestrict-fset A fmempty* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-set-null*[*simp*]: *fmdrop-set* {} *m* = *m*
by (*rule fmap-ext*) *auto*

lemma *fmdrop-fset-null*[*simp*]: *fmdrop-fset* {||} *m* = *m*
by (*rule fmap-ext*) *auto*

lemma *fmdrop-set-single*[*simp*]: *fmdrop-set* {*a*} *m* = *fmdrop a m*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-fset-single*[*simp*]: *fmdrop-fset* {|*a*|} *m* = *fmdrop a m*

unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-set-null*[*simp*]: *fmrestrict-set* {} *m* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-fset-null*[*simp*]: *fmrestrict-fset* {||} *m* = *fmempty*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-comm*: *fmdrop a* (*fmdrop b m*) = *fmdrop b* (*fmdrop a m*)
unfolding *fmfilter-alt-defs* **by** (*rule fmfilter-comm*)

lemma *fmdrop-set-insert*[*simp*]: *fmdrop-set* (*insert x S*) *m* = *fmdrop x* (*fmdrop-set S m*)
by (*rule fmap-ext*) *auto*

lemma *fmdrop-fset-insert*[*simp*]: *fmdrop-fset* (*finsert x S*) *m* = *fmdrop x* (*fmdrop-fset S m*)
by (*rule fmap-ext*) *auto*

lemma *fmrestrict-set-twice*[*simp*]: *fmrestrict-set S* (*fmrestrict-set T m*) = *fmrestrict-set* (*S* ∩ *T*) *m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmrestrict-fset-twice*[*simp*]: *fmrestrict-fset S* (*fmrestrict-fset T m*) = *fmrestrict-fset* (*S* |∩| *T*) *m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmrestrict-set-drop*[*simp*]: *fmrestrict-set S* (*fmdrop b m*) = *fmrestrict-set* (*S* − {*b*}) *m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmrestrict-fset-drop*[*simp*]: *fmrestrict-fset S* (*fmdrop b m*) = *fmrestrict-fset* (*S* − {| *b* |}) *m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmdrop-fmrestrict-set*[*simp*]: *fmdrop b* (*fmrestrict-set S m*) = *fmrestrict-set* (*S* − {*b*}) *m*
by (*rule fmap-ext*) *auto*

lemma *fmdrop-fmrestrict-fset*[*simp*]: *fmdrop b* (*fmrestrict-fset S m*) = *fmrestrict-fset* (*S* − {| *b* |}) *m*
by (*rule fmap-ext*) *auto*

lemma *fmdrop-idem*[*simp*]: *fmdrop a* (*fmdrop a m*) = *fmdrop a m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmdrop-set-twice*[*simp*]: *fmdrop-set S* (*fmdrop-set T m*) = *fmdrop-set* (*S* ∪ *T*) *m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmdrop-fset-twice*[*simp*]: *fmdrop-fset S* (*fmdrop-fset T m*) = *fmdrop-fset* (*S* |∪| *T*) *m*
unfolding *fmfilter-alt-defs* **by** *auto*

lemma *fmdrop-set-fmdrop*[*simp*]: *fmdrop-set S* (*fmdrop b m*) = *fmdrop-set* (*insert b S*) *m*
by (*rule fmap-ext*) *auto*

lemma *fmdrop-fset-fmdrop*[*simp*]: *fmdrop-fset S* (*fmdrop b m*) = *fmdrop-fset* (*finsert b S*) *m*
by (*rule fmap-ext*) *auto*

lift-definition *fmadd* :: ('*a*, '*b*) *fmap* ⇒ ('*a*, '*b*) *fmap* ⇒ ('*a*, '*b*) *fmap* (**infixl** $++_f$ *100*)
  is *map-add*
  parametric *map-add-transfer*
by *simp*

lemma *fmlookup-add*[*simp*]:
  fmlookup (*m* $++_f$ *n*) *x* = (*if x* |∈| *fmdom n then fmlookup n x else fmlookup m x*)
  by *transfer'* (*auto simp*: *map-add-def split*: *option.splits*)

lemma *fmdom-add*[*simp*]: *fmdom* (*m* $++_f$ *n*) = *fmdom m* |∪| *fmdom n* **by** *transfer' auto*
lemma *fmdom'-add*[*simp*]: *fmdom'* (*m* $++_f$ *n*) = *fmdom' m* ∪ *fmdom' n* **by** *transfer' auto*

lemma *fmadd-drop-left-dom*: *fmdrop-fset* (*fmdom n*) *m* $++_f$ *n* = *m* $++_f$ *n*
by (*rule fmap-ext*) *auto*

lemma *fmadd-restrict-right-dom*: *fmrestrict-fset* (*fmdom n*) (*m* $++_f$ *n*) = *n*
by (*rule fmap-ext*) *auto*

lemma *fmfilter-add-distrib*[*simp*]: *fmfilter P* (*m* $++_f$ *n*) = *fmfilter P m* $++_f$ *fmfilter P n*
by *transfer'* (*auto simp*: *map-filter-def map-add-def*)

lemma *fmdrop-add-distrib*[*simp*]: *fmdrop a* (*m* $++_f$ *n*) = *fmdrop a m* $++_f$ *fmdrop a n*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-set-add-distrib*[*simp*]: *fmdrop-set A* (*m* $++_f$ *n*) = *fmdrop-set A m* $++_f$ *fmdrop-set A n*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-fset-add-distrib*[*simp*]: *fmdrop-fset A* (*m* $++_f$ *n*) = *fmdrop-fset A m* $++_f$ *fmdrop-fset A n*

unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-set-add-distrib*[*simp*]:
fmrestrict-set A (*m* ++$_f$ *n*) = *fmrestrict-set A m* ++$_f$ *fmrestrict-set A n*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-fset-add-distrib*[*simp*]:
fmrestrict-fset A (*m* ++$_f$ *n*) = *fmrestrict-fset A m* ++$_f$ *fmrestrict-fset A n*
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmadd-empty*[*simp*]: *fmempty* ++$_f$ *m* = *m m* ++$_f$ *fmempty* = *m*
by (*transfer'*; *auto*)+

lemma *fmadd-idempotent*[*simp*]: *m* ++$_f$ *m* = *m*
by *transfer'* (*auto simp*: *map-add-def split*: *option.splits*)

lemma *fmadd-assoc*[*simp*]: *m* ++$_f$ (*n* ++$_f$ *p*) = *m* ++$_f$ *n* ++$_f$ *p*
by *transfer' simp*

lemma *fmadd-fmupd*[*simp*]: *m* ++$_f$ *fmupd a b n* = *fmupd a b* (*m* ++$_f$ *n*)
by (*rule fmap-ext*) *simp*

lift-definition *fmpred* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ ('*a*, '*b*) *fmap* ⇒ *bool*
is *map-pred*
parametric *map-pred-transfer*
.

lemma *fmpredI*[*intro*]:
assumes ⋀*x y*. *fmlookup m x* = *Some y* ⟹ *P x y*
shows *fmpred P m*
using *assms*
by *transfer'* (*auto simp*: *map-pred-def split*: *option.splits*)

lemma *fmpredD*[*dest*]: *fmpred P m* ⟹ *fmlookup m x* = *Some y* ⟹ *P x y*
by *transfer'* (*auto simp*: *map-pred-def split*: *option.split-asm*)

lemma *fmpred-iff*: *fmpred P m* ⟷ (∀ *x y*. *fmlookup m x* = *Some y* ⟶ *P x y*)
by *auto*

lemma *fmpred-alt-def*: *fmpred P m* ⟷ *fBall* (*fmdom m*) (λ*x*. *P x* (*the* (*fmlookup m x*)))
unfolding *fmpred-iff*
apply *auto*
apply (*rename-tac x y*)
apply (*erule-tac x* = *x* **in** *fBallE*)
apply *simp*
by (*simp add*: *fmlookup-dom-iff*)

lemma *fmpred-mono-strong*:

assumes $\bigwedge x\ y.\ fmlookup\ m\ x = Some\ y \Longrightarrow P\ x\ y \Longrightarrow Q\ x\ y$
shows $fmpred\ P\ m \Longrightarrow fmpred\ Q\ m$
using *assms* **unfolding** *fmpred-iff* **by** *auto*

lemma *fmpred-mono*[*mono*]: $P \leq Q \Longrightarrow fmpred\ P \leq fmpred\ Q$
apply *rule*
apply (*rule fmpred-mono-strong*[**where** $P = P$ **and** $Q = Q$])
apply *auto*
done

lemma *fmpred-empty*[*intro*!, *simp*]: *fmpred P fmempty*
by *auto*

lemma *fmpred-upd*[*intro*]: $fmpred\ P\ m \Longrightarrow P\ x\ y \Longrightarrow fmpred\ P\ (fmupd\ x\ y\ m)$
by *transfer'* (*auto simp*: *map-pred-def map-upd-def*)

lemma *fmpred-updD*[*dest*]: $fmpred\ P\ (fmupd\ x\ y\ m) \Longrightarrow P\ x\ y$
by *auto*

lemma *fmpred-add*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ n \Longrightarrow fmpred\ P\ (m\ ++_f\ n)$
by *transfer'* (*auto simp*: *map-pred-def map-add-def split*: *option.splits*)

lemma *fmpred-filter*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ (fmfilter\ Q\ m)$
by *transfer'* (*auto simp*: *map-pred-def map-filter-def*)

lemma *fmpred-drop*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ (fmdrop\ a\ m)$
by (*auto simp*: *fmfilter-alt-defs*)

lemma *fmpred-drop-set*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ (fmdrop\text{-}set\ A\ m)$
by (*auto simp*: *fmfilter-alt-defs*)

lemma *fmpred-drop-fset*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ (fmdrop\text{-}fset\ A\ m)$
by (*auto simp*: *fmfilter-alt-defs*)

lemma *fmpred-restrict-set*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ (fmrestrict\text{-}set\ A\ m)$
by (*auto simp*: *fmfilter-alt-defs*)

lemma *fmpred-restrict-fset*[*intro*]: $fmpred\ P\ m \Longrightarrow fmpred\ P\ (fmrestrict\text{-}fset\ A\ m)$
by (*auto simp*: *fmfilter-alt-defs*)

lemma *fmpred-cases*[*consumes 1*]:
assumes *fmpred P m*
obtains (*none*) $fmlookup\ m\ x = None$ | (*some*) *y* **where** $fmlookup\ m\ x = Some\ y\ P\ x\ y$
using *assms* **by** *auto*

lift-definition $fmsubset :: ('a, 'b)\ fmap \Rightarrow ('a, 'b)\ fmap \Rightarrow bool$ (**infix** $\subseteq_f$ *50*)
is *map-le*
.

lemma *fmsubset-alt-def*: $m \subseteq_f n \longleftrightarrow$ *fmpred* $(\lambda k\ v.$ *fmlookup* $n\ k =$ *Some* $v)\ m$
by *transfer'* (*auto simp*: *map-pred-def map-le-def dom-def split*: *option.splits*)

lemma *fmsubset-pred*: *fmpred* $P\ m \Longrightarrow n \subseteq_f m \Longrightarrow$ *fmpred* $P\ n$
unfolding *fmsubset-alt-def fmpred-iff*
by *auto*

lemma *fmsubset-filter-mono*: $m \subseteq_f n \Longrightarrow$ *fmfilter* $P\ m \subseteq_f$ *fmfilter* $P\ n$
unfolding *fmsubset-alt-def fmpred-iff*
by *auto*

lemma *fmsubset-drop-mono*: $m \subseteq_f n \Longrightarrow$ *fmdrop* $a\ m \subseteq_f$ *fmdrop* $a\ n$
unfolding *fmfilter-alt-defs* **by** (*rule fmsubset-filter-mono*)

lemma *fmsubset-drop-set-mono*: $m \subseteq_f n \Longrightarrow$ *fmdrop-set* $A\ m \subseteq_f$ *fmdrop-set* $A\ n$
unfolding *fmfilter-alt-defs* **by** (*rule fmsubset-filter-mono*)

lemma *fmsubset-drop-fset-mono*: $m \subseteq_f n \Longrightarrow$ *fmdrop-fset* $A\ m \subseteq_f$ *fmdrop-fset* A n
unfolding *fmfilter-alt-defs* **by** (*rule fmsubset-filter-mono*)

lemma *fmsubset-restrict-set-mono*: $m \subseteq_f n \Longrightarrow$ *fmrestrict-set* $A\ m \subseteq_f$ *fmrestrict-set* $A\ n$
unfolding *fmfilter-alt-defs* **by** (*rule fmsubset-filter-mono*)

lemma *fmsubset-restrict-fset-mono*: $m \subseteq_f n \Longrightarrow$ *fmrestrict-fset* $A\ m \subseteq_f$ *fmrestrict-fset* $A\ n$
unfolding *fmfilter-alt-defs* **by** (*rule fmsubset-filter-mono*)

lemma *fmfilter-subset*[*simp*]: *fmfilter* $P\ m \subseteq_f m$
unfolding *fmsubset-alt-def fmpred-iff* **by** *auto*

lemma *fmsubset-drop*[*simp*]: *fmdrop* $a\ m \subseteq_f m$
unfolding *fmfilter-alt-defs* **by** (*rule fmfilter-subset*)

lemma *fmsubset-drop-set*[*simp*]: *fmdrop-set* $S\ m \subseteq_f m$
unfolding *fmfilter-alt-defs* **by** (*rule fmfilter-subset*)

lemma *fmsubset-drop-fset*[*simp*]: *fmdrop-fset* $S\ m \subseteq_f m$
unfolding *fmfilter-alt-defs* **by** (*rule fmfilter-subset*)

lemma *fmsubset-restrict-set*[*simp*]: *fmrestrict-set* $S\ m \subseteq_f m$
unfolding *fmfilter-alt-defs* **by** (*rule fmfilter-subset*)

lemma *fmsubset-restrict-fset*[*simp*]: *fmrestrict-fset* $S\ m \subseteq_f m$
unfolding *fmfilter-alt-defs* **by** (*rule fmfilter-subset*)

lift-definition *fset-of-fmap* :: $('a, 'b)$ *fmap* $\Rightarrow$ $('a \times 'b)$ *fset* **is** *set-of-map*

by (*rule set-of-map-finite*)

lemma *fset-of-fmap-inj*[*intro*, *simp*]: *inj fset-of-fmap*
apply *rule*
apply *transfer'*
using *set-of-map-inj* **unfolding** *inj-def* **by** *auto*

lemma *fset-of-fmap-iff*[*simp*]: (*a*, *b*) |∈| *fset-of-fmap m* ⟷ *fmlookup m a* = *Some b*
by *transfer'* (*auto simp*: *set-of-map-def*)

lemma *fset-of-fmap-iff'*[*simp*]: (*a*, *b*) ∈ *fset* (*fset-of-fmap m*) ⟷ *fmlookup m a* = *Some b*
by *transfer'* (*auto simp*: *set-of-map-def*)

lift-definition *fmap-of-list* :: ('*a* × '*b*) *list* ⇒ ('*a*, '*b*) *fmap*
  is *map-of*
  parametric *map-of-transfer*
by (*rule finite-dom-map-of*)

lemma *fmap-of-list-simps*[*simp*]:
  fmap-of-list [] = *fmempty*
  fmap-of-list ((*k*, *v*) # *kvs*) = *fmupd k v* (*fmap-of-list kvs*)
by (*transfer*, *simp add*: *map-upd-def*)+

lemma *fmap-of-list-app*[*simp*]: *fmap-of-list* (*xs* @ *ys*) = *fmap-of-list ys* ++$_f$ *fmap-of-list xs*
by *transfer' simp*

lemma *fmupd-alt-def*: *fmupd k v m* = *m* ++$_f$ *fmap-of-list* [(*k*, *v*)]
by *transfer'* (*auto simp*: *map-upd-def*)

lemma *fmpred-of-list*[*intro*]:
  assumes ⋀*k v*. (*k*, *v*) ∈ *set xs* ⟹ *P k v*
  shows *fmpred P* (*fmap-of-list xs*)
using *assms*
by (*induction xs*) (*transfer'*; *auto simp*: *map-pred-def*)+

lemma *fmap-of-list-SomeD*: *fmlookup* (*fmap-of-list xs*) *k* = *Some v* ⟹ (*k*, *v*) ∈ *set xs*
by *transfer'* (*auto dest*: *map-of-SomeD*)

lemma *fmdom-fmap-of-list*[*simp*]: *fmdom* (*fmap-of-list xs*) = *fset-of-list* (*map fst xs*)
apply *transfer'*
apply (*subst dom-map-of-conv-image-fst*)
apply *auto*
done

lift-definition *fmrel-on-fset* :: *'a fset* ⇒ (*'b* ⇒ *'c* ⇒ *bool*) ⇒ (*'a*, *'b*) *fmap* ⇒ (*'a*, *'c*) *fmap* ⇒ *bool*
  is *rel-map-on-set*
.

lemma *fmrel-on-fset-alt-def*: *fmrel-on-fset S P m n* ⟷ *fBall S* (λ*x*. *rel-option P* (*fmlookup m x*) (*fmlookup n x*))
by *transfer'* (*auto simp*: *rel-map-on-set-def eq-onp-def rel-fun-def*)

lemma *fmrel-on-fsetI*[*intro*]:
  assumes ⋀*x*. *x* |∈| *S* ⟹ *rel-option P* (*fmlookup m x*) (*fmlookup n x*)
  shows *fmrel-on-fset S P m n*
using *assms*
unfolding *fmrel-on-fset-alt-def* **by** *auto*

lemma *fmrel-on-fset-mono*[*mono*]: *R* ≤ *Q* ⟹ *fmrel-on-fset S R* ≤ *fmrel-on-fset S Q*
unfolding *fmrel-on-fset-alt-def*[*abs-def*]
apply (*intro le-funI fBall-mono*)
using *option.rel-mono* **by** *auto*

lemma *fmrel-on-fsetD*: *x* |∈| *S* ⟹ *fmrel-on-fset S P m n* ⟹ *rel-option P* (*fmlookup m x*) (*fmlookup n x*)
unfolding *fmrel-on-fset-alt-def*
by *auto*

lemma *fmrel-on-fsubset*: *fmrel-on-fset S R m n* ⟹ *T* |⊆| *S* ⟹ *fmrel-on-fset T R m n*
unfolding *fmrel-on-fset-alt-def*
by *auto*

lemma *fmrel-on-fset-unionI*:
  fmrel-on-fset A R m n ⟹ *fmrel-on-fset B R m n* ⟹ *fmrel-on-fset* (*A* |∪| *B*) *R m n*
unfolding *fmrel-on-fset-alt-def*
by *auto*

lemma *fmrel-on-fset-updateI*:
  assumes *fmrel-on-fset S P m n P v*$_1$ *v*$_2$
  shows *fmrel-on-fset* (*finsert k S*) *P* (*fmupd k v*$_1$ *m*) (*fmupd k v*$_2$ *n*)
using *assms*
unfolding *fmrel-on-fset-alt-def*
by *auto*

lift-definition *fmimage* :: (*'a*, *'b*) *fmap* ⇒ *'a fset* ⇒ *'b fset* **is** λ*m S*. {*b*|*a b*. *m a* = *Some b* ∧ *a* ∈ *S*}
subgoal for *m S*
  apply (*rule finite-subset*[**where** *B* = *ran m*])
  apply (*auto simp*: *ran-def*)[]

by (*rule finite-ran*)
done

lemma *fmimage-alt-def*: *fmimage m S* = *fmran* (*fmrestrict-fset S m*)
by *transfer'* (*auto simp*: *ran-def map-restrict-set-def map-filter-def*)

lemma *fmimage-empty*[*simp*]: *fmimage m fempty* = *fempty*
by *transfer'* *auto*

lemma *fmimage-subset-ran*[*simp*]: *fmimage m S* |⊆| *fmran m*
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmimage-dom*[*simp*]: *fmimage m* (*fmdom m*) = *fmran m*
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmimage-inter*: *fmimage m* (*A* |∩| *B*) |⊆| *fmimage m A* |∩| *fmimage m B*
by *transfer'* *auto*

lemma *fimage-inter-dom*[*simp*]:
fmimage m (*fmdom m* |∩| *A*) = *fmimage m A*
fmimage m (*A* |∩| *fmdom m*) = *fmimage m A*
by (*transfer'*; *auto*)+

lemma *fmimage-union*[*simp*]: *fmimage m* (*A* |∪| *B*) = *fmimage m A* |∪| *fmimage m B*
by *transfer'* *auto*

lemma *fmimage-Union*[*simp*]: *fmimage m* (*ffUnion A*) = *ffUnion* (*fmimage m* |‘| *A*)
by *transfer'* *auto*

lemma *fmimage-filter*[*simp*]: *fmimage* (*fmfilter P m*) *A* = *fmimage m* (*ffilter P A*)
by *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmimage-drop*[*simp*]: *fmimage* (*fmdrop a m*) *A* = *fmimage m* (*A* − {|*a*|})
by *transfer'* (*auto simp*: *map-filter-def map-drop-def*)

lemma *fmimage-drop-fset*[*simp*]: *fmimage* (*fmdrop-fset B m*) *A* = *fmimage m* (*A* − *B*)
by *transfer'* (*auto simp*: *map-filter-def map-drop-set-def*)

lemma *fmimage-restrict-fset*[*simp*]: *fmimage* (*fmrestrict-fset B m*) *A* = *fmimage m* (*A* |∩| *B*)
by *transfer'* (*auto simp*: *map-filter-def map-restrict-set-def*)

lemma *fmfilter-ran*[*simp*]: *fmran* (*fmfilter P m*) = *fmimage m* (*ffilter P* (*fmdom m*))
by *transfer'* (*auto simp*: *ran-def map-filter-def*)

lemma *fmran-drop*[*simp*]: *fmran* (*fmdrop a m*) = *fmimage m* (*fmdom m* − {|*a*|})
by *transfer'* (*auto simp*: *ran-def map-drop-def map-filter-def*)

lemma *fmran-drop-fset*[*simp*]: *fmran* (*fmdrop-fset A m*) = *fmimage m* (*fmdom m* − *A*)
by *transfer'* (*auto simp*: *ran-def map-drop-set-def map-filter-def*)

lemma *fmran-restrict-fset*: *fmran* (*fmrestrict-fset A m*) = *fmimage m* (*fmdom m* |∩| *A*)
by *transfer'* (*auto simp*: *ran-def map-restrict-set-def map-filter-def*)

lemma *fmlookup-image-iff*: *y* |∈| *fmimage m A* ⟷ (∃ *x*. *fmlookup m x* = *Some y* ∧ *x* |∈| *A*)
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmimageI*: *fmlookup m x* = *Some y* ⟹ *x* |∈| *A* ⟹ *y* |∈| *fmimage m A*
by (*auto simp*: *fmlookup-image-iff*)

lemma *fmimageE*[*elim*]:
  assumes *y* |∈| *fmimage m A*
  obtains *x* **where** *fmlookup m x* = *Some y x* |∈| *A*
using *assms* **by** (*auto simp*: *fmlookup-image-iff*)

lift-definition *fmcomp* :: ('*b*, '*c*) *fmap* ⇒ ('*a*, '*b*) *fmap* ⇒ ('*a*, '*c*) *fmap* (**infixl** ∘$_f$ *55*)
  is *map-comp*
  parametric *map-comp-transfer*
by (*rule dom-comp-finite*)

lemma *fmlookup-comp*[*simp*]: *fmlookup* (*m* ∘$_f$ *n*) *x* = *Option.bind* (*fmlookup n x*) (*fmlookup m*)
by *transfer'* (*auto simp*: *map-comp-def split*: *option.splits*)

end

30.4 BNF setup

lift-bnf ('*a*, *fmran'*: '*b*) *fmap* [*wits*: *Map.empty*]
  for *map*: *fmmap*
    rel: *fmrel*
  by *auto*

declare *fmap.pred-mono*[*mono*]

lemma *fmran'-alt-def*: *fmran' m* = *fset* (*fmran m*)
including *fset.lifting*
by *transfer'* (*auto simp*: *ran-def fun-eq-iff*)

lemma *fmlookup-ran'-iff*: $y \in fmran'\ m \longleftrightarrow (\exists x.\ fmlookup\ m\ x = Some\ y)$
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmran'I*: $fmlookup\ m\ x = Some\ y \Longrightarrow y \in fmran'\ m$ **by** (*auto simp*: *fmlookup-ran'-iff*)

lemma *fmran'E*[*elim*]:
assumes $y \in fmran'\ m$
obtains x **where** $fmlookup\ m\ x = Some\ y$
using *assms* **by** (*auto simp*: *fmlookup-ran'-iff*)

lemma *fmrel-iff*: $fmrel\ R\ m\ n \longleftrightarrow (\forall x.\ rel\text{-}option\ R\ (fmlookup\ m\ x)\ (fmlookup\ n\ x))$
by *transfer'* (*auto simp*: *rel-fun-def*)

lemma *fmrelI*[*intro*]:
assumes $\bigwedge x.\ rel\text{-}option\ R\ (fmlookup\ m\ x)\ (fmlookup\ n\ x)$
shows $fmrel\ R\ m\ n$
using *assms*
by *transfer' auto*

lemma *fmrel-upd*[*intro*]: $fmrel\ P\ m\ n \Longrightarrow P\ x\ y \Longrightarrow fmrel\ P\ (fmupd\ k\ x\ m)\ (fmupd\ k\ y\ n)$
by *transfer'* (*auto simp*: *map-upd-def rel-fun-def*)

lemma *fmrelD*[*dest*]: $fmrel\ P\ m\ n \Longrightarrow rel\text{-}option\ P\ (fmlookup\ m\ x)\ (fmlookup\ n\ x)$
by *transfer'* (*auto simp*: *rel-fun-def*)

lemma *fmrel-addI*[*intro*]:
assumes *fmrel P m n fmrel P a b*
shows $fmrel\ P\ (m +\!\!+_f a)\ (n +\!\!+_f b)$
using *assms*
apply *transfer'*
apply (*auto simp*: *rel-fun-def map-add-def*)
by (*metis option.case-eq-if option.collapse option.rel-sel*)

lemma *fmrel-cases*[*consumes 1*]:
assumes *fmrel P m n*
obtains (*none*) *fmlookup m x = None fmlookup n x = None*
| (*some*) *a b* **where** *fmlookup m x = Some a fmlookup n x = Some b P a b*
proof −
from *assms* **have** *rel-option P* (*fmlookup m x*) (*fmlookup n x*)
by *auto*
then show *thesis*
using *none some*
by (*cases rule*: *option.rel-cases*) *auto*
qed

lemma *fmrel-filter*[*intro*]: $fmrel\ P\ m\ n \Longrightarrow fmrel\ P\ (fmfilter\ Q\ m)\ (fmfilter\ Q\ n)$

unfolding *fmrel-iff* **by** *auto*

lemma *fmrel-drop*[*intro*]: *fmrel P m n* ⟹ *fmrel P* (*fmdrop a m*) (*fmdrop a n*)
unfolding *fmfilter-alt-defs* **by** *blast*

lemma *fmrel-drop-set*[*intro*]: *fmrel P m n* ⟹ *fmrel P* (*fmdrop-set A m*) (*fmdrop-set A n*)
unfolding *fmfilter-alt-defs* **by** *blast*

lemma *fmrel-drop-fset*[*intro*]: *fmrel P m n* ⟹ *fmrel P* (*fmdrop-fset A m*) (*fmdrop-fset A n*)
unfolding *fmfilter-alt-defs* **by** *blast*

lemma *fmrel-restrict-set*[*intro*]: *fmrel P m n* ⟹ *fmrel P* (*fmrestrict-set A m*) (*fmrestrict-set A n*)
unfolding *fmfilter-alt-defs* **by** *blast*

lemma *fmrel-restrict-fset*[*intro*]: *fmrel P m n* ⟹ *fmrel P* (*fmrestrict-fset A m*) (*fmrestrict-fset A n*)
unfolding *fmfilter-alt-defs* **by** *blast*

lemma *fmrel-on-fset-fmrel-restrict*:
fmrel-on-fset S P m n ⟷ *fmrel P* (*fmrestrict-fset S m*) (*fmrestrict-fset S n*)
unfolding *fmrel-on-fset-alt-def fmrel-iff*
by *auto*

lemma *fmrel-on-fset-refl-strong*:
assumes ⋀*x y*. *x* |∈| *S* ⟹ *fmlookup m x* = *Some y* ⟹ *P y y*
shows *fmrel-on-fset S P m m*
unfolding *fmrel-on-fset-fmrel-restrict fmrel-iff*
using *assms*
by (*simp add*: *option.rel-sel*)

lemma *fmrel-on-fset-addI*:
assumes *fmrel-on-fset S P m n fmrel-on-fset S P a b*
shows *fmrel-on-fset S P* (*m* ++$_f$ *a*) (*n* ++$_f$ *b*)
using *assms*
unfolding *fmrel-on-fset-fmrel-restrict*
by *auto*

lemma *fmrel-fmdom-eq*:
assumes *fmrel P x y*
shows *fmdom x* = *fmdom y*
proof −
have *a* |∈| *fmdom x* ⟷ *a* |∈| *fmdom y* **for** *a*
proof −
have *rel-option P* (*fmlookup x a*) (*fmlookup y a*)
using *assms* **by** (*simp add*: *fmrel-iff*)
thus *?thesis*

```
      by cases (auto intro: fmdomI)
    qed
  thus ?thesis
    by auto
qed

lemma fmrel-fmdom'-eq: fmrel P x y ⟹ fmdom' x = fmdom' y
unfolding fmdom'-alt-def
by (metis fmrel-fmdom-eq)

lemma fmrel-rel-fmran:
  assumes fmrel P x y
  shows rel-fset P (fmran x) (fmran y)
proof −
  {
    fix b
    assume b |∈| fmran x
    then obtain a where fmlookup x a = Some b
      by auto
    moreover have rel-option P (fmlookup x a) (fmlookup y a)
      using assms by auto
    ultimately have ∃ b'. b' |∈| fmran y ∧ P b b'
      by (metis option-rel-Some1 fmranI)
  }
  moreover
  {
    fix b
    assume b |∈| fmran y
    then obtain a where fmlookup y a = Some b
      by auto
    moreover have rel-option P (fmlookup x a) (fmlookup y a)
      using assms by auto
    ultimately have ∃ b'. b' |∈| fmran x ∧ P b' b
      by (metis option-rel-Some2 fmranI)
  }
  ultimately show ?thesis
    unfolding rel-fset-alt-def
    by auto
qed

lemma fmrel-rel-fmran': fmrel P x y ⟹ rel-set P (fmran' x) (fmran' y)
unfolding fmran'-alt-def
by (metis fmrel-rel-fmran rel-fset-fset)

lemma pred-fmap-fmpred[simp]: pred-fmap P = fmpred (λ-. P)
unfolding fmap.pred-set fmran'-alt-def
including fset.lifting
apply transfer'
apply (rule ext)
```

apply (*auto simp*: *map-pred-def ran-def split*: *option.splits dest*:)
done

lemma *pred-fmap-id*[*simp*]: *pred-fmap id* (*fmmap f m*) ⟷ *pred-fmap f m*
unfolding *fmap.pred-set fmap.set-map*
by *simp*

lemma *pred-fmapD*: *pred-fmap P m* ⟹ *x* |∈| *fmran m* ⟹ *P x*
by *auto*

lemma *fmlookup-map*[*simp*]: *fmlookup* (*fmmap f m*) *x* = *map-option f* (*fmlookup m x*)
by *transfer' auto*

lemma *fmpred-map*[*simp*]: *fmpred P* (*fmmap f m*) ⟷ *fmpred* (λ*k v*. *P k* (*f v*)) *m*
unfolding *fmpred-iff pred-fmap-def fmap.set-map*
by *auto*

lemma *fmpred-id*[*simp*]: *fmpred* (λ-. *id*) (*fmmap f m*) ⟷ *fmpred* (λ-. *f*) *m*
by *simp*

lemma *fmmap-add*[*simp*]: *fmmap f* (*m* ++$_f$ *n*) = *fmmap f m* ++$_f$ *fmmap f n*
by *transfer'* (*auto simp*: *map-add-def fun-eq-iff split*: *option.splits*)

lemma *fmmap-empty*[*simp*]: *fmmap f fmempty* = *fmempty*
by *transfer auto*

lemma *fmdom-map*[*simp*]: *fmdom* (*fmmap f m*) = *fmdom m*
including *fset.lifting*
by *transfer' simp*

lemma *fmdom'-map*[*simp*]: *fmdom'* (*fmmap f m*) = *fmdom' m*
by *transfer' simp*

lemma *fmran-fmmap*[*simp*]: *fmran* (*fmmap f m*) = *f* |‘| *fmran m*
including *fset.lifting*
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmran'-fmmap*[*simp*]: *fmran'* (*fmmap f m*) = *f* ‘ *fmran' m*
by *transfer'* (*auto simp*: *ran-def*)

lemma *fmfilter-fmmap*[*simp*]: *fmfilter P* (*fmmap f m*) = *fmmap f* (*fmfilter P m*)
by *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmdrop-fmmap*[*simp*]: *fmdrop a* (*fmmap f m*) = *fmmap f* (*fmdrop a m*)
unfolding *fmfilter-alt-defs* **by** *simp*
lemma *fmdrop-set-fmmap*[*simp*]: *fmdrop-set A* (*fmmap f m*) = *fmmap f* (*fmdrop-set A m*) **unfolding** *fmfilter-alt-defs* **by** *simp*
lemma *fmdrop-fset-fmmap*[*simp*]: *fmdrop-fset A* (*fmmap f m*) = *fmmap f* (*fmdrop-fset*

A m) **unfolding** *fmfilter-alt-defs* **by** *simp*
lemma *fmrestrict-set-fmmap*[*simp*]: *fmrestrict-set A* (*fmmap f m*) = *fmmap f* (*fmrestrict-set A m*) **unfolding** *fmfilter-alt-defs* **by** *simp*
lemma *fmrestrict-fset-fmmap*[*simp*]: *fmrestrict-fset A* (*fmmap f m*) = *fmmap f* (*fmrestrict-fset A m*) **unfolding** *fmfilter-alt-defs* **by** *simp*

lemma *fmmap-subset*[*intro*]: *m* $\subseteq_f$ *n* $\Longrightarrow$ *fmmap f m* $\subseteq_f$ *fmmap f n*
by *transfer'* (*auto simp*: *map-le-def*)

lemma *fmmap-fset-of-fmap*: *fset-of-fmap* (*fmmap f m*) = (λ(*k*, *v*). (*k*, *f v*)) |‘| *fset-of-fmap m*
including *fset.lifting*
by *transfer'* (*auto simp*: *set-of-map-def*)

lemma *fmmap-fmupd*: *fmmap f* (*fmupd x y m*) = *fmupd x* (*f y*) (*fmmap f m*)
by *transfer'* (*auto simp*: *fun-eq-iff map-upd-def*)

30.5 *size* setup

definition *size-fmap* :: ('*a* ⇒ *nat*) ⇒ ('*b* ⇒ *nat*) ⇒ ('*a*, '*b*) *fmap* ⇒ *nat* **where**
[*simp*]: *size-fmap f g m* = *size-fset* (λ(*a*, *b*). *f a* + *g b*) (*fset-of-fmap m*)

instantiation *fmap* :: (*type*, *type*) *size* **begin**

definition *size-fmap* **where**
size-fmap-overloaded-def: *size-fmap* = *Finite-Map.size-fmap* (λ-. *0*) (λ-. *0*)

instance ..

end

lemma *size-fmap-overloaded-simps*[*simp*]: *size x* = *size* (*fset-of-fmap x*)
unfolding *size-fmap-overloaded-def*
by *simp*

lemma *fmap-size-o-map*: *inj h* $\Longrightarrow$ *size-fmap f g* ∘ *fmmap h* = *size-fmap f* (*g* ∘ *h*)
unfolding *size-fmap-def*
apply (*auto simp*: *fun-eq-iff fmmap-fset-of-fmap*)
apply (*subst sum.reindex*)
subgoal for *m*
using *prod.inj-map*[*unfolded map-prod-def*, *of* λ*x*. *x h*]
unfolding *inj-on-def*
by *auto*
subgoal
by (*rule sum.cong*) (*auto split*: *prod.splits*)
done

setup ‹
BNF-LFP-Size.register-size-global ***type-name*** ‹*fmap*› ***const-name*** ‹*size-fmap*›

@{*thm size-fmap-overloaded-def*} @{*thms size-fmap-def size-fmap-overloaded-simps*}
@{*thms fmap-size-o-map*}
›

30.6 Additional operations

lift-definition *fmmap-keys* :: ('*a* ⇒ '*b* ⇒ '*c*) ⇒ ('*a*, '*b*) *fmap* ⇒ ('*a*, '*c*) *fmap* **is**
λ*f m a*. *map-option* (*f a*) (*m a*)
unfolding *dom-def*
by *simp*

lemma *fmpred-fmmap-keys*[*simp*]: *fmpred P* (*fmmap-keys f m*) = *fmpred* (λ*a b*. *P a* (*f a b*)) *m*
by *transfer'* (*auto simp*: *map-pred-def split*: *option.splits*)

lemma *fmdom-fmmap-keys*[*simp*]: *fmdom* (*fmmap-keys f m*) = *fmdom m*
including *fset.lifting*
by *transfer' auto*

lemma *fmlookup-fmmap-keys*[*simp*]: *fmlookup* (*fmmap-keys f m*) *x* = *map-option* (*f x*) (*fmlookup m x*)
by *transfer' simp*

lemma *fmfilter-fmmap-keys*[*simp*]: *fmfilter P* (*fmmap-keys f m*) = *fmmap-keys f* (*fmfilter P m*)
by *transfer'* (*auto simp*: *map-filter-def*)

lemma *fmdrop-fmmap-keys*[*simp*]: *fmdrop a* (*fmmap-keys f m*) = *fmmap-keys f* (*fmdrop a m*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-set-fmmap-keys*[*simp*]: *fmdrop-set A* (*fmmap-keys f m*) = *fmmap-keys f* (*fmdrop-set A m*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmdrop-fset-fmmap-keys*[*simp*]: *fmdrop-fset A* (*fmmap-keys f m*) = *fmmap-keys f* (*fmdrop-fset A m*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-set-fmmap-keys*[*simp*]: *fmrestrict-set A* (*fmmap-keys f m*) = *fmmap-keys f* (*fmrestrict-set A m*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmrestrict-fset-fmmap-keys*[*simp*]: *fmrestrict-fset A* (*fmmap-keys f m*) = *fmmap-keys f* (*fmrestrict-fset A m*)
unfolding *fmfilter-alt-defs* **by** *simp*

lemma *fmmap-keys-subset*[*intro*]: *m* ⊆$_f$ *n* ⟹ *fmmap-keys f m* ⊆$_f$ *fmmap-keys f n*

by *transfer'* (*auto simp*: *map-le-def dom-def*)

definition *sorted-list-of-fmap* :: ('*a*::*linorder*, '*b*) *fmap* ⇒ ('*a* × '*b*) *list* **where**
sorted-list-of-fmap m = *map* (λ*k*. (*k*, *the* (*fmlookup m k*))) (*sorted-list-of-fset* (*fmdom m*))

lemma *list-all-sorted-list*[*simp*]: *list-all P* (*sorted-list-of-fmap m*) = *fmpred* (*curry P*) *m*
unfolding *sorted-list-of-fmap-def curry-def list.pred-map*
apply (*auto simp*: *list-all-iff*)
including *fset.lifting*
by (*transfer*; *auto simp*: *dom-def map-pred-def split*: *option.splits*)+

lemma *map-of-sorted-list*[*simp*]: *map-of* (*sorted-list-of-fmap m*) = *fmlookup m*
unfolding *sorted-list-of-fmap-def*
including *fset.lifting*
by *transfer* (*simp add*: *map-of-map-keys*)

30.7 Additional properties

lemma *fmchoice'*:
assumes *finite S* ∀ *x* ∈ *S*. ∃ *y*. *Q x y*
shows ∃ *m*. *fmdom' m* = *S* ∧ *fmpred Q m*
proof −
obtain *f* **where** *f*: *Q x* (*f x*) **if** *x* ∈ *S* **for** *x*
using *assms* **by** (*metis bchoice*)
define *f'* **where** *f' x* = (*if x* ∈ *S then Some* (*f x*) *else None*) **for** *x*

have *eq-onp* (λ*m*. *finite* (*dom m*)) *f' f'*
unfolding *eq-onp-def f'-def dom-def* **using** *assms* **by** *auto*

show *?thesis*
apply (*rule exI*[**where** *x* = *Abs-fmap f'*])
apply (*subst fmpred.abs-eq*, *fact*)
apply (*subst fmdom'.abs-eq*, *fact*)
unfolding *f'-def dom-def map-pred-def* **using** *f*
by *auto*
qed

30.8 Lifting/transfer setup

context includes *lifting-syntax* **begin**

lemma *fmempty-transfer*[*simp*, *intro*, *transfer-rule*]: *fmrel P fmempty fmempty*
by *transfer auto*

lemma *fmadd-transfer*[*transfer-rule*]:
(*fmrel P* ===> *fmrel P* ===> *fmrel P*) *fmadd fmadd*
by (*intro fmrel-addI rel-funI*)

lemma *fmupd-transfer*[*transfer-rule*]:
  ((=) ===> *P* ===> *fmrel P* ===> *fmrel P*) *fmupd fmupd*
by *auto*

end

lemma *Quotient-fmap-bnf*[*quot-map*]:
  assumes *Quotient R Abs Rep T*
  shows *Quotient* (*fmrel R*) (*fmmap Abs*) (*fmmap Rep*) (*fmrel T*)
unfolding *Quotient-alt-def4* **proof** *safe*
  fix *m n*
  assume *fmrel T m n*
  then have *fmlookup* (*fmmap Abs m*) *x* = *fmlookup n x* **for** *x*
    apply (*cases rule*: *fmrel-cases*[**where** *x* = *x*])
    using *assms* **unfolding** *Quotient-alt-def* **by** *auto*
  then show *fmmap Abs m* = *n*
    by (*rule fmap-ext*)
next
  fix *m*
  show *fmrel T* (*fmmap Rep m*) *m*
    unfolding *fmap.rel-map*
    apply (*rule fmap.rel-refl*)
    using *assms* **unfolding** *Quotient-alt-def*
    by *auto*
next
  from *assms* **have** *R* = *T OO* T^{-1-1}
    unfolding *Quotient-alt-def4* **by** *simp*

  then show *fmrel R* = *fmrel T OO* $(\mathit{fmrel}\ T)^{-1-1}$
    by (*simp add*: *fmap.rel-compp fmap.rel-conversep*)
qed

30.9 View as datatype

lemma *fmap-distinct*[*simp*]:
  fmempty ≠ *fmupd k v m*
  fmupd k v m ≠ *fmempty*
by (*transfer'*; *auto simp*: *map-upd-def fun-eq-iff*)+

lifting-update *fmap.lifting*

lemma *fmap-exhaust*[*cases type*: *fmap*]:
  obtains (*fmempty*) *m* = *fmempty*
      | (*fmupd*) *x y m'* **where** *m* = *fmupd x y m' x* |∉| *fmdom m'*
using *that* **including** *fmap.lifting fset.lifting*
proof *transfer*
  fix *m P*
  assume *finite* (*dom m*)
  assume *empty*: *P* **if** *m* = *Map.empty*

assume *map-upd*: *P* **if** *finite* (*dom m'*) *m* = *map-upd x y m' x* ∉ *dom m'* **for** *x y m'*

show *P*
proof (*cases m* = *Map.empty*)
case *True* **thus** *?thesis* **using** *empty* **by** *simp*
next
case *False*
hence *dom m* ≠ {} **by** *simp*
then obtain *x* **where** *x* ∈ *dom m* **by** *blast*

let *?m'* = *map-drop x m*

show *?thesis*
proof (*rule map-upd*)
show *finite* (*dom ?m'*)
using ‹*finite* (*dom m*)›
unfolding *map-drop-def*
by *auto*
next
show *m* = *map-upd x* (*the* (*m x*)) *?m'*
using ‹*x* ∈ *dom m*› **unfolding** *map-drop-def map-filter-def map-upd-def*
by *auto*
next
show *x* ∉ *dom ?m'*
unfolding *map-drop-def map-filter-def*
by *auto*
qed
qed
qed

lemma *fmap-induct*[*case-names fmempty fmupd*, *induct type*: *fmap*]:
assumes *P fmempty*
assumes (⋀*x y m*. *P m* ⟹ *fmlookup m x* = *None* ⟹ *P* (*fmupd x y m*))
shows *P m*
proof (*induction fmdom m arbitrary*: *m rule*: *fset-induct-stronger*)
case *empty*
hence *m* = *fmempty*
by (*metis fmrestrict-fset-dom fmrestrict-fset-null*)
with *assms* **show** *?case*
by *simp*
next
case (*insert x S*)
hence *S* = *fmdom* (*fmdrop x m*)
by *auto*
with *insert* **have** *P* (*fmdrop x m*)
by *auto*

have *x* |∈| *fmdom m*

using *insert* **by** *auto*
then obtain *y* **where** *fmlookup m x = Some y*
by *auto*
hence *m = fmupd x y (fmdrop x m)*
by (*auto intro*: *fmap-ext*)

show *?case*
apply (*subst ‹m = -›*)
apply (*rule assms*)
apply *fact*
apply *simp*
done
qed

30.10 Code setup

instantiation *fmap* :: (*type*, *equal*) *equal* **begin**

definition *equal-fmap* ≡ *fmrel HOL.equal*

instance proof
fix *m n* :: (*'a*, *'b*) *fmap*
have *fmrel* (=) *m n* ⟷ (*m = n*)
by *transfer'* (*simp add*: *option.rel-eq rel-fun-eq*)
then show *equal-class.equal m n* ⟷ (*m = n*)
unfolding *equal-fmap-def*
by (*simp add*: *equal-eq*[*abs-def*])
qed

end

lemma *fBall-alt-def*: *fBall S P* ⟷ (∀ *x*. *x* |∈| *S* ⟶ *P x*)
by *force*

lemma *fmrel-code*:
fmrel R m n ⟷
fBall (*fmdom m*) (λ*x*. *rel-option R* (*fmlookup m x*) (*fmlookup n x*)) ∧
fBall (*fmdom n*) (λ*x*. *rel-option R* (*fmlookup m x*) (*fmlookup n x*))
unfolding *fmrel-iff fmlookup-dom-iff fBall-alt-def*
by (*metis option.collapse option.rel-sel*)

lemmas [*code*] =
fmrel-code
fmran'-alt-def
fmdom'-alt-def
fmfilter-alt-defs
pred-fmap-fmpred
fmsubset-alt-def
fmupd-alt-def

fmrel-on-fset-alt-def
fmpred-alt-def

code-datatype *fmap-of-list*
quickcheck-generator *fmap constructors*: *fmap-of-list*

context includes *fset.lifting* **begin**

lemma *fmlookup-of-list*[*code*]: *fmlookup* (*fmap-of-list m*) = *map-of m*
by *transfer simp*

lemma *fmempty-of-list*[*code*]: *fmempty* = *fmap-of-list* []
by *transfer simp*

lemma *fmran-of-list*[*code*]: *fmran* (*fmap-of-list m*) = *snd* |‘| *fset-of-list* (*AList.clearjunk m*)
by *transfer* (*auto simp*: *ran-map-of*)

lemma *fmdom-of-list*[*code*]: *fmdom* (*fmap-of-list m*) = *fst* |‘| *fset-of-list m*
by *transfer* (*auto simp*: *dom-map-of-conv-image-fst*)

lemma *fmfilter-of-list*[*code*]: *fmfilter P* (*fmap-of-list m*) = *fmap-of-list* (*filter* (λ(*k*, -). *P k*) *m*)
by *transfer′ auto*

lemma *fmadd-of-list*[*code*]: *fmap-of-list m* ++$_f$ *fmap-of-list n* = *fmap-of-list* (*AList.merge m n*)
by *transfer* (*simp add*: *merge-conv′*)

lemma *fmmap-of-list*[*code*]: *fmmap f* (*fmap-of-list m*) = *fmap-of-list* (*map* (*apsnd f*) *m*)
apply *transfer*
apply (*subst map-of-map*[*symmetric*])
apply (*auto simp*: *apsnd-def map-prod-def*)
done

lemma *fmmap-keys-of-list*[*code*]:
fmmap-keys f (*fmap-of-list m*) = *fmap-of-list* (*map* (λ(*a*, *b*). (*a*, *f a b*)) *m*)
apply *transfer*
subgoal for *f m* **by** (*induction m*) (*auto simp*: *apsnd-def map-prod-def fun-eq-iff*)
done

lemma *fmimage-of-list*[*code*]:
fmimage (*fmap-of-list m*) *A* = *fset-of-list* (*map snd* (*filter* (λ(*k*, -). *k* |∈| *A*) (*AList.clearjunk m*)))
apply (*subst fmimage-alt-def*)
apply (*subst fmfilter-alt-defs*)
apply (*subst fmfilter-of-list*)

apply (*subst fmran-of-list*)
apply *transfer'*
apply (*subst AList.restrict-eq*[*symmetric*])
apply (*subst clearjunk-restrict*)
apply (*subst AList.restrict-eq*)
by *auto*

lemma *fmcomp-list*[*code*]:
fmap-of-list m $\circ_f$ *fmap-of-list n* = *fmap-of-list* (*AList.compose n m*)
by (*rule fmap-ext*) (*simp add*: *fmlookup-of-list compose-conv map-comp-def split*: *option.splits*)

end

30.11 Instances

lemma *exists-map-of*:
assumes *finite* (*dom m*) **shows** ∃ *xs*. *map-of xs* = *m*
using *assms*
proof (*induction dom m arbitrary*: *m*)
case *empty*
hence *m* = *Map.empty*
by *auto*
moreover have *map-of* [] = *Map.empty*
by *simp*
ultimately show *?case*
by *blast*
next
case (*insert x F*)
hence *F* = *dom* (*map-drop x m*)
unfolding *map-drop-def map-filter-def dom-def* **by** *auto*
with *insert* **have** ∃ *xs'*. *map-of xs'* = *map-drop x m*
by *auto*
then obtain *xs'* **where** *map-of xs'* = *map-drop x m*
..
moreover obtain *y* **where** *m x* = *Some y*
using *insert* **unfolding** *dom-def* **by** *blast*
ultimately have *map-of* ((*x*, *y*) # *xs'*) = *m*
using ‹*insert x F* = *dom m*›
unfolding *map-drop-def map-filter-def*
by *auto*
thus *?case*
..
qed

lemma *exists-fmap-of-list*: ∃ *xs*. *fmap-of-list xs* = *m*
by *transfer* (*rule exists-map-of*)

lemma *fmap-of-list-surj*[*simp*, *intro*]: *surj fmap-of-list*

proof −
  have $x \in$ *range fmap-of-list* **for** x :: $('a, 'b)$ *fmap*
    unfolding *image-iff*
    using *exists-fmap-of-list* **by** (*metis UNIV-I*)
  thus *?thesis* **by** *auto*
qed

instance *fmap* :: (*countable*, *countable*) *countable*
proof
  obtain *to-nat* :: $('a \times 'b)$ *list* ⇒ *nat* **where** *inj to-nat*
    by (*metis ex-inj*)
  moreover have *inj* (*inv fmap-of-list*)
    using *fmap-of-list-surj* **by** (*rule surj-imp-inj-inv*)
  ultimately have *inj* (*to-nat* ∘ *inv fmap-of-list*)
    by (*rule inj-compose*)
  thus ∃ *to-nat*::$('a, 'b)$ *fmap* ⇒ *nat*. *inj to-nat*
    by *auto*
qed

instance *fmap* :: (*finite*, *finite*) *finite*
proof
  show *finite* (*UNIV* :: $('a, 'b)$ *fmap set*)
    by (*rule finite-imageD*) *auto*
qed

lifting-update *fmap.lifting*
lifting-forget *fmap.lifting*

30.12 Tests

export-code
  fBall fmrel fmran fmran′ fmdom fmdom′ fmpred pred-fmap fmsubset fmupd fmrel-on-fset
  fmdrop fmdrop-set fmdrop-fset fmrestrict-set fmrestrict-fset fmimage fmlookup fmempty
  fmfilter fmadd fmmap fmmap-keys fmcomp
  checking *SML Scala Haskell? OCaml?*

— *lifting* through *fmap*

experiment begin

context includes *fset.lifting* **begin**

lift-definition *test1* :: $('a, 'b\ fset)$ *fmap* **is** *fmempty* :: $('a, 'b\ set)$ *fmap*
  by *auto*

lift-definition *test2* :: $'a \Rightarrow 'b \Rightarrow ('a, 'b\ fset)$ *fmap* **is** $\lambda a\ b.$ *fmupd a* $\{b\}$ *fmempty*
  by *auto*

end

end

end

31 Disjoint FSets

theory *Disjoint-FSets*
imports
HOL−Library.Finite-Map
Disjoint-Sets
begin

context
includes *fset.lifting*
begin

lift-definition *fdisjnt* :: *'a fset* ⇒ *'a fset* ⇒ *bool* **is** *disjnt* .

lemma *fdisjnt-alt-def*: *fdisjnt M N* ⟷ (*M* |∩| *N* = {||})
by *transfer* (*simp add*: *disjnt-def*)

lemma *fdisjnt-insert*: *x* |∉| *N* ⟹ *fdisjnt M N* ⟹ *fdisjnt* (*finsert x M*) *N*
by *transfer'* (*rule disjnt-insert*)

lemma *fdisjnt-subset-right*: *N'* |⊆| *N* ⟹ *fdisjnt M N* ⟹ *fdisjnt M N'*
unfolding *fdisjnt-alt-def* **by** *auto*

lemma *fdisjnt-subset-left*: *N'* |⊆| *N* ⟹ *fdisjnt N M* ⟹ *fdisjnt N' M*
unfolding *fdisjnt-alt-def* **by** *auto*

lemma *fdisjnt-union-right*: *fdisjnt M A* ⟹ *fdisjnt M B* ⟹ *fdisjnt M* (*A* |∪| *B*)
unfolding *fdisjnt-alt-def* **by** *auto*

lemma *fdisjnt-union-left*: *fdisjnt A M* ⟹ *fdisjnt B M* ⟹ *fdisjnt* (*A* |∪| *B*) *M*
unfolding *fdisjnt-alt-def* **by** *auto*

lemma *fdisjnt-swap*: *fdisjnt M N* ⟹ *fdisjnt N M*
including *fset.lifting* **by** *transfer'* (*auto simp*: *disjnt-def*)

lemma *distinct-append-fset*:
assumes *distinct xs distinct ys fdisjnt* (*fset-of-list xs*) (*fset-of-list ys*)
shows *distinct* (*xs* @ *ys*)
using *assms*
by *transfer'* (*simp add*: *disjnt-def*)

lemma *fdisjnt-contrI*:

assumes $\bigwedge x.\ x \mid\in\mid M \Longrightarrow x \mid\in\mid N \Longrightarrow \mathit{False}$
shows *fdisjnt M N*
using *assms*
by *transfer'* (*auto simp*: *disjnt-def*)

lemma *fdisjnt-Union-left*: *fdisjnt* (*ffUnion S*) *T* $\longleftrightarrow$ *fBall S* ($\lambda S.$ *fdisjnt S T*)
by *transfer'* (*auto simp*: *disjnt-def*)

lemma *fdisjnt-Union-right*: *fdisjnt T* (*ffUnion S*) $\longleftrightarrow$ *fBall S* ($\lambda S.$ *fdisjnt T S*)
by *transfer'* (*auto simp*: *disjnt-def*)

lemma *fdisjnt-ge-max*: *fBall X* ($\lambda x.\ x >$ *fMax Y*) $\Longrightarrow$ *fdisjnt X Y*
by *transfer* (*auto intro*: *disjnt-ge-max*)

end

lemma *fmadd-disjnt*: *fdisjnt* (*fmdom m*) (*fmdom n*) $\Longrightarrow$ $m +\!\!+_f n = n +\!\!+_f m$
unfolding *fdisjnt-alt-def*
including *fset.lifting fmap.lifting*
apply *transfer*
apply (*rule ext*)
apply (*auto simp*: *map-add-def split*: *option.splits*)
done

end

32 Lists with elements distinct as canonical example for datatype invariants

theory *Dlist*
imports *Confluent-Quotient*
begin

32.1 The type of distinct lists

typedef *'a dlist* = {*xs*::*'a list*. *distinct xs*}
morphisms *list-of-dlist Abs-dlist*
proof
show [] $\in$ {*xs*. *distinct xs*} **by** *simp*
qed

context begin

qualified definition *dlist-eq* **where** *dlist-eq* = *BNF-Def.vimage2p remdups remdups* (=)

qualified lemma *equivp-dlist-eq*: *equivp dlist-eq*

unfolding *dlist-eq-def* **by**(*rule equivp-vimage2p*)(*rule identity-equivp*)

qualified definition *abs-dlist* :: *'a list* ⇒ *'a dlist* **where** *abs-dlist* = *Abs-dlist o remdups*

definition *qcr-dlist* :: *'a list* ⇒ *'a dlist* ⇒ *bool* **where** *qcr-dlist x y* ⟷ *y* = *abs-dlist x*

qualified lemma *Quotient-dlist-remdups*: *Quotient dlist-eq abs-dlist list-of-dlist qcr-dlist*
unfolding *Quotient-def dlist-eq-def qcr-dlist-def vimage2p-def abs-dlist-def*
by (*auto simp add*: *fun-eq-iff Abs-dlist-inject*
list-of-dlist[*simplified*] *list-of-dlist-inverse distinct-remdups-id*)

end

locale *Quotient-dlist* **begin**
setup-lifting *Dlist.Quotient-dlist-remdups Dlist.equivp-dlist-eq*[*THEN equivp-reflp2*]
end

setup-lifting *type-definition-dlist*

lemma *dlist-eq-iff*:
dxs = *dys* ⟷ *list-of-dlist dxs* = *list-of-dlist dys*
by (*simp add*: *list-of-dlist-inject*)

lemma *dlist-eqI*:
list-of-dlist dxs = *list-of-dlist dys* ⟹ *dxs* = *dys*
by (*simp add*: *dlist-eq-iff*)

Formal, totalized constructor for *'a dlist*:

definition *Dlist* :: *'a list* ⇒ *'a dlist* **where**
Dlist xs = *Abs-dlist* (*remdups xs*)

lemma *distinct-list-of-dlist* [*simp*, *intro*]:
distinct (*list-of-dlist dxs*)
using *list-of-dlist* [*of dxs*] **by** *simp*

lemma *list-of-dlist-Dlist* [*simp*]:
list-of-dlist (*Dlist xs*) = *remdups xs*
by (*simp add*: *Dlist-def Abs-dlist-inverse*)

lemma *remdups-list-of-dlist* [*simp*]:
remdups (*list-of-dlist dxs*) = *list-of-dlist dxs*
by *simp*

lemma *Dlist-list-of-dlist* [*simp*, *code abstype*]:
Dlist (*list-of-dlist dxs*) = *dxs*
by (*simp add*: *Dlist-def list-of-dlist-inverse distinct-remdups-id*)

Fundamental operations:

context
begin

qualified definition *empty* :: *'a dlist* **where**
empty = *Dlist* []

qualified definition *insert* :: *'a* ⇒ *'a dlist* ⇒ *'a dlist* **where**
insert x dxs = *Dlist* (*List.insert x* (*list-of-dlist dxs*))

qualified definition *remove* :: *'a* ⇒ *'a dlist* ⇒ *'a dlist* **where**
remove x dxs = *Dlist* (*remove1 x* (*list-of-dlist dxs*))

qualified definition *map* :: (*'a* ⇒ *'b*) ⇒ *'a dlist* ⇒ *'b dlist* **where**
map f dxs = *Dlist* (*remdups* (*List.map f* (*list-of-dlist dxs*)))

qualified definition *filter* :: (*'a* ⇒ *bool*) ⇒ *'a dlist* ⇒ *'a dlist* **where**
filter P dxs = *Dlist* (*List.filter P* (*list-of-dlist dxs*))

qualified definition *rotate* :: *nat* ⇒ *'a dlist* ⇒ *'a dlist* **where**
rotate n dxs = *Dlist* (*List.rotate n* (*list-of-dlist dxs*))

end

Derived operations:

context
begin

qualified definition *null* :: *'a dlist* ⇒ *bool* **where**
null dxs = *List.null* (*list-of-dlist dxs*)

qualified definition *member* :: *'a dlist* ⇒ *'a* ⇒ *bool* **where**
member dxs = *List.member* (*list-of-dlist dxs*)

qualified definition *length* :: *'a dlist* ⇒ *nat* **where**
length dxs = *List.length* (*list-of-dlist dxs*)

qualified definition *fold* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a dlist* ⇒ *'b* ⇒ *'b* **where**
fold f dxs = *List.fold f* (*list-of-dlist dxs*)

qualified definition *foldr* :: (*'a* ⇒ *'b* ⇒ *'b*) ⇒ *'a dlist* ⇒ *'b* ⇒ *'b* **where**
foldr f dxs = *List.foldr f* (*list-of-dlist dxs*)

end

32.2 Executable version obeying invariant

lemma *list-of-dlist-empty* [*simp*, *code abstract*]:
list-of-dlist Dlist.empty = []

by (*simp add*: *Dlist.empty-def*)

lemma *list-of-dlist-insert* [*simp*, *code abstract*]:
list-of-dlist (*Dlist.insert x dxs*) = *List.insert x* (*list-of-dlist dxs*)
by (*simp add*: *Dlist.insert-def*)

lemma *list-of-dlist-remove* [*simp*, *code abstract*]:
list-of-dlist (*Dlist.remove x dxs*) = *remove1 x* (*list-of-dlist dxs*)
by (*simp add*: *Dlist.remove-def*)

lemma *list-of-dlist-map* [*simp*, *code abstract*]:
list-of-dlist (*Dlist.map f dxs*) = *remdups* (*List.map f* (*list-of-dlist dxs*))
by (*simp add*: *Dlist.map-def*)

lemma *list-of-dlist-filter* [*simp*, *code abstract*]:
list-of-dlist (*Dlist.filter P dxs*) = *List.filter P* (*list-of-dlist dxs*)
by (*simp add*: *Dlist.filter-def*)

lemma *list-of-dlist-rotate* [*simp*, *code abstract*]:
list-of-dlist (*Dlist.rotate n dxs*) = *List.rotate n* (*list-of-dlist dxs*)
by (*simp add*: *Dlist.rotate-def*)

Explicit executable conversion

definition *dlist-of-list* [*simp*]:
dlist-of-list = *Dlist*

lemma [*code abstract*]:
list-of-dlist (*dlist-of-list xs*) = *remdups xs*
by *simp*

Equality

instantiation *dlist* :: (*equal*) *equal*
begin

definition *HOL.equal dxs dys* ⟷ *HOL.equal* (*list-of-dlist dxs*) (*list-of-dlist dys*)

instance
by *standard* (*simp add*: *equal-dlist-def equal list-of-dlist-inject*)

end

declare *equal-dlist-def* [*code*]

lemma [*code nbe*]: *HOL.equal* (*dxs* :: *'a*::*equal dlist*) *dxs* ⟷ *True*
by (*fact equal-refl*)

32.3 Induction principle and case distinction

lemma *dlist-induct* [*case-names empty insert*, *induct type*: *dlist*]:
assumes *empty*: *P Dlist.empty*

assumes *insrt*: $\bigwedge x\ dxs.\ \neg$ *Dlist.member dxs x* $\Longrightarrow$ *P dxs* $\Longrightarrow$ *P* (*Dlist.insert x dxs*)
shows *P dxs*
proof (*cases dxs*)
case (*Abs-dlist xs*)
then have *distinct xs* **and** *dxs*: *dxs* = *Dlist xs*
by (*simp-all add*: *Dlist-def distinct-remdups-id*)
from ‹*distinct xs*› **have** *P* (*Dlist xs*)
proof (*induct xs*)
case *Nil* **from** *empty* **show** *?case* **by** (*simp add*: *Dlist.empty-def*)
next
case (*Cons x xs*)
then have ¬ *Dlist.member* (*Dlist xs*) *x* **and** *P* (*Dlist xs*)
by (*simp-all add*: *Dlist.member-def List.member-def*)
with *insrt* **have** *P* (*Dlist.insert x* (*Dlist xs*)) **.**
with *Cons* **show** *?case* **by** (*simp add*: *Dlist.insert-def distinct-remdups-id*)
qed
with *dxs* **show** *P dxs* **by** *simp*
qed

lemma *dlist-case* [*cases type*: *dlist*]:
obtains (*empty*) *dxs* = *Dlist.empty*
| (*insert*) *x dys* **where** ¬ *Dlist.member dys x* **and** *dxs* = *Dlist.insert x dys*
proof (*cases dxs*)
case (*Abs-dlist xs*)
then have *dxs*: *dxs* = *Dlist xs* **and** *distinct*: *distinct xs*
by (*simp-all add*: *Dlist-def distinct-remdups-id*)
show *thesis*
proof (*cases xs*)
case *Nil* **with** *dxs*
have *dxs* = *Dlist.empty* **by** (*simp add*: *Dlist.empty-def*)
with *empty* **show** *?thesis* **.**
next
case (*Cons x xs*)
with *dxs distinct* **have** ¬ *Dlist.member* (*Dlist xs*) *x*
and *dxs* = *Dlist.insert x* (*Dlist xs*)
by (*simp-all add*: *Dlist.member-def List.member-def Dlist.insert-def distinct-remdups-id*)
with *insert* **show** *?thesis* **.**
qed
qed

32.4 Functorial structure

functor *map*: *map*
by (*simp-all add*: *remdups-map-remdups fun-eq-iff dlist-eq-iff*)

32.5 Quickcheck generators

quickcheck-generator *dlist predicate*: *distinct constructors*: *Dlist.empty*, *Dlist.insert*

32.6 BNF instance

context begin

qualified inductive *double* :: *'a list* ⇒ *'a list* ⇒ *bool* **where**
double (*xs* @ *ys*) (*xs* @ *x* # *ys*) **if** *x* ∈ *set ys*

qualified lemma *strong-confluentp-double*: *strong-confluentp double*
proof
fix *xs ys zs* :: *'a list*
assume *ys*: *double xs ys* **and** *zs*: *double xs zs*
consider (*left*) *as y bs z cs* **where** *xs* = *as* @ *bs* @ *cs ys* = *as* @ *y* # *bs* @ *cs zs* = *as* @ *bs* @ *z* # *cs y* ∈ *set* (*bs* @ *cs*) *z* ∈ *set cs*
| (*right*) *as y bs z cs* **where** *xs* = *as* @ *bs* @ *cs ys* = *as* @ *bs* @ *y* # *cs zs* = *as* @ *z* # *bs* @ *cs y* ∈ *set cs z* ∈ *set* (*bs* @ *cs*)
proof −
show *thesis* **using** *ys zs*
by(*clarsimp simp add*: *double.simps append-eq-append-conv2*)(*auto intro*: *that*)
qed
then show ∃ *us*. *double*** *ys us* ∧ *double*== *zs us*
proof *cases*
case *left*
let *?us* = *as* @ *y* # *bs* @ *z* # *cs*
have *double ys ?us double zs ?us* **using** *left*
by(*auto 4 4 simp add*: *double.simps*)(*metis append-Cons append-assoc*)+
then show *?thesis* **by** *blast*
next
case *right*
let *?us* = *as* @ *z* # *bs* @ *y* # *cs*
have *double ys ?us double zs ?us* **using** *right*
by(*auto 4 4 simp add*: *double.simps*)(*metis append-Cons append-assoc*)+
then show *?thesis* **by** *blast*
qed
qed

qualified lemma *double-Cons1* [*simp*]: *double xs* (*x* # *xs*) **if** *x* ∈ *set xs*
using *double.intros*[*of x xs* []] *that* **by** *simp*

qualified lemma *double-Cons-same* [*simp*]: *double xs ys* ⟹ *double* (*x* # *xs*) (*x* # *ys*)
by(*auto simp add*: *double.simps Cons-eq-append-conv*)

qualified lemma *doubles-Cons-same*: *double*** *xs ys* ⟹ *double*** (*x* # *xs*) (*x* # *ys*)
by(*induction rule*: *rtranclp-induct*)(*auto intro*: *rtranclp.rtrancl-into-rtrancl*)

qualified lemma *remdups-into-doubles*: *double*** (*remdups xs*) *xs*
by(*induction xs*)(*auto intro*: *doubles-Cons-same rtranclp.rtrancl-into-rtrancl*)

qualified lemma *dlist-eq-into-doubles*: *Dlist.dlist-eq* ≤ *equivclp double*

by(*auto 4 4 simp add*: *Dlist.dlist-eq-def vimage2p-def*
intro: *equivclp-trans converse-rtranclp-into-equivclp rtranclp-into-equivclp remdups-into-doubles*)

qualified lemma *factor-double-map*: *double* (*map f xs*) *ys* $\Longrightarrow$ $\exists$ *zs*. *Dlist.dlist-eq xs zs* $\wedge$ *ys* = *map f zs* $\wedge$ *set zs* $\subseteq$ *set xs*
by(*auto simp add*: *double.simps Dlist.dlist-eq-def vimage2p-def map-eq-append-conv*)
(*metis* (*no-types*, *opaque-lifting*) *list.simps*(*9*) *map-append remdups.simps*(*2*) *remdups-append2 set-append set-eq-subset set-remdups*)

qualified lemma *dlist-eq-set-eq*: *Dlist.dlist-eq xs ys* $\Longrightarrow$ *set xs* = *set ys*
by(*simp add*: *Dlist.dlist-eq-def vimage2p-def*)(*metis set-remdups*)

qualified lemma *dlist-eq-map-respect*: *Dlist.dlist-eq xs ys* $\Longrightarrow$ *Dlist.dlist-eq* (*map f xs*) (*map f ys*)
by(*clarsimp simp add*: *Dlist.dlist-eq-def vimage2p-def*)(*metis remdups-map-remdups*)

qualified lemma *confluent-quotient-dlist*:
confluent-quotient double Dlist.dlist-eq Dlist.dlist-eq Dlist.dlist-eq Dlist.dlist-eq Dlist.dlist-eq
(*map fst*) (*map snd*) (*map fst*) (*map snd*) *list-all2 list-all2 list-all2 set set*
by(*unfold-locales*)(*auto intro*: *strong-confluentp-imp-confluentp strong-confluentp-double*
dest: *factor-double-map dlist-eq-into-doubles*[*THEN predicate2D*] *dlist-eq-set-eq*
simp add: *list.in-rel list.rel-compp dlist-eq-map-respect Dlist.equivp-dlist-eq equivp-imp-transp*)

lifting-update *dlist.lifting*
lifting-forget *dlist.lifting*

end

context begin
interpretation *Quotient-dlist*: *Quotient-dlist* **.**

lift-bnf (*plugins del*: *code*) *'a dlist*
subgoal for *A B* **by**(*rule confluent-quotient.subdistributivity*[*OF Dlist.confluent-quotient-dlist*])
subgoal by(*force dest*: *Dlist.dlist-eq-set-eq intro*: *equivp-reflp*[*OF Dlist.equivp-dlist-eq*])
done

qualified lemma *list-of-dlist-transfer*[*transfer-rule*]:
bi-unique R $\Longrightarrow$ (*rel-fun* (*Quotient-dlist.pcr-dlist R*) (*list-all2 R*)) *remdups list-of-dlist*
unfolding *rel-fun-def Quotient-dlist.pcr-dlist-def qcr-dlist-def Dlist.abs-dlist-def*
by (*auto simp*: *Abs-dlist-inverse intro*!: *remdups-transfer*[*THEN rel-funD*])

lemma *list-of-dlist-map-dlist*[*simp*]:
list-of-dlist (*map-dlist f xs*) = *remdups* (*map f* (*list-of-dlist xs*))
by *transfer* (*auto simp*: *remdups-map-remdups*)

end

end

33 Type of dual ordered lattices

theory *Dual-Ordered-Lattice*
imports *Main*
begin

The *dual* of an ordered structure is an isomorphic copy of the underlying type, with the $\leq$ relation defined as the inverse of the original one.

The class of lattices is closed under formation of dual structures. This means that for any theorem of lattice theory, the dualized statement holds as well; this important fact simplifies many proofs of lattice theory.

typedef *'a dual = UNIV :: 'a set*
  morphisms *undual dual* **..**

setup-lifting *type-definition-dual*

code-datatype *dual*

lemma *dual-eqI*:
  $x = y$ **if** *undual x = undual y*
  using *that* **by** *transfer assumption*

lemma *dual-eq-iff*:
  $x = y \longleftrightarrow$ *undual x = undual y*
  by *transfer simp*

lemma *eq-dual-iff* [*iff*]:
  dual x = dual y $\longleftrightarrow$ $x = y$
  by *transfer simp*

lemma *undual-dual* [*simp*, *code*]:
  undual (*dual x*) $= x$
  by *transfer rule*

lemma *dual-undual* [*simp*]:
  dual (*undual x*) $= x$
  by *transfer rule*

lemma *undual-comp-dual* [*simp*]:
  undual $\circ$ *dual* $= id$
  by (*simp add*: *fun-eq-iff*)

lemma *dual-comp-undual* [*simp*]:
  dual $\circ$ *undual* $= id$
  by (*simp add*: *fun-eq-iff*)

lemma *inj-dual*:
inj dual
by (*rule injI*) *simp*

lemma *inj-undual*:
inj undual
by (*rule injI*) (*rule dual-eqI*)

lemma *surj-dual*:
surj dual
by (*rule surjI* [*of* - *undual*]) *simp*

lemma *surj-undual*:
surj undual
by (*rule surjI* [*of* - *dual*]) *simp*

lemma *bij-dual*:
bij dual
using *inj-dual surj-dual* **by** (*rule bijI*)

lemma *bij-undual*:
bij undual
using *inj-undual surj-undual* **by** (*rule bijI*)

instance *dual* :: (*finite*) *finite*
proof
from *finite* **have** *finite* (*range dual* :: *'a dual set*)
by (*rule finite-imageI*)
then show *finite* (*UNIV* :: *'a dual set*)
by (*simp add*: *surj-dual*)
qed

instantiation *dual* :: (*equal*) *equal*
begin

lift-definition *equal-dual* :: *'a dual* ⇒ *'a dual* ⇒ *bool*
is *HOL.equal* **.**

instance
by (*standard*; *transfer*) (*simp add*: *equal*)

end

33.1 Pointwise ordering

instantiation *dual* :: (*ord*) *ord*
begin

lift-definition *less-eq-dual* :: *'a dual* ⇒ *'a dual* ⇒ *bool*

is $(\geq)$ **.**

lift-definition *less-dual* :: *'a dual* ⇒ *'a dual* ⇒ *bool*
is $(>)$ **.**

instance ..

end

lemma *dual-less-eqI*:
$x \leq y$ **if** *undual* $y \leq$ *undual* x
using *that* **by** *transfer assumption*

lemma *dual-less-eq-iff*:
$x \leq y \longleftrightarrow$ *undual* $y \leq$ *undual* x
by *transfer simp*

lemma *less-eq-dual-iff* [*iff*]:
dual $x \leq$ *dual* $y \longleftrightarrow y \leq x$
by *transfer simp*

lemma *dual-lessI*:
$x < y$ **if** *undual* $y <$ *undual* x
using *that* **by** *transfer assumption*

lemma *dual-less-iff*:
$x < y \longleftrightarrow$ *undual* $y <$ *undual* x
by *transfer simp*

lemma *less-dual-iff* [*iff*]:
dual $x <$ *dual* $y \longleftrightarrow y < x$
by *transfer simp*

instance *dual* :: (*preorder*) *preorder*
by (*standard*; *transfer*) (*auto simp add*: *less-le-not-le intro*: *order-trans*)

instance *dual* :: (*order*) *order*
by (*standard*; *transfer*) *simp*

33.2 Binary infimum and supremum

instantiation *dual* :: (*sup*) *inf*
begin

lift-definition *inf-dual* :: *'a dual* ⇒ *'a dual* ⇒ *'a dual*
is *sup* **.**

instance ..

end

lemma *undual-inf-eq* [*simp*]:
undual (*inf x y*) = *sup* (*undual x*) (*undual y*)
by (*fact inf-dual.rep-eq*)

lemma *dual-sup-eq* [*simp*]:
dual (*sup x y*) = *inf* (*dual x*) (*dual y*)
by *transfer rule*

instantiation *dual* :: (*inf*) *sup*
begin

lift-definition *sup-dual* :: *'a dual* ⇒ *'a dual* ⇒ *'a dual*
is *inf* **.**

instance ..

end

lemma *undual-sup-eq* [*simp*]:
undual (*sup x y*) = *inf* (*undual x*) (*undual y*)
by (*fact sup-dual.rep-eq*)

lemma *dual-inf-eq* [*simp*]:
dual (*inf x y*) = *sup* (*dual x*) (*dual y*)
by *transfer simp*

instance *dual* :: (*semilattice-sup*) *semilattice-inf*
by (*standard*; *transfer*) *simp-all*

instance *dual* :: (*semilattice-inf*) *semilattice-sup*
by (*standard*; *transfer*) *simp-all*

instance *dual* :: (*lattice*) *lattice* **..**

instance *dual* :: (*distrib-lattice*) *distrib-lattice*
by (*standard*; *transfer*) (*fact inf-sup-distrib1*)

33.3 Top and bottom elements

instantiation *dual* :: (*top*) *bot*
begin

lift-definition *bot-dual* :: *'a dual*
is *top* **.**

instance ..

end

lemma *undual-bot-eq* [*simp*]:
undual bot = *top*
by (*fact bot-dual.rep-eq*)

lemma *dual-top-eq* [*simp*]:
dual top = *bot*
by *transfer rule*

instantiation *dual* :: (*bot*) *top*
begin

lift-definition *top-dual* :: *'a dual*
is *bot* **.**

instance ..

end

lemma *undual-top-eq* [*simp*]:
undual top = *bot*
by (*fact top-dual.rep-eq*)

lemma *dual-bot-eq* [*simp*]:
dual bot = *top*
by *transfer rule*

instance *dual* :: (*order-top*) *order-bot*
by (*standard*; *transfer*) *simp*

instance *dual* :: (*order-bot*) *order-top*
by (*standard*; *transfer*) *simp*

instance *dual* :: (*bounded-lattice-top*) *bounded-lattice-bot* **..**

instance *dual* :: (*bounded-lattice-bot*) *bounded-lattice-top* **..**

instance *dual* :: (*bounded-lattice*) *bounded-lattice* **..**

33.4 Complement

instantiation *dual* :: (*uminus*) *uminus*
begin

lift-definition *uminus-dual* :: *'a dual* ⇒ *'a dual*
is *uminus* **.**

instance ..

end

lemma *undual-uminus-eq* [*simp*]:
undual (− *x*) = − *undual x*
by (*fact uminus-dual.rep-eq*)

lemma *dual-uminus-eq* [*simp*]:
dual (− *x*) = − *dual x*
by *transfer rule*

instantiation *dual* :: (*boolean-algebra*) *boolean-algebra*
begin

lift-definition *minus-dual* :: *'a dual* ⇒ *'a dual* ⇒ *'a dual*
is λ*x y*. − (*y* − *x*) **.**

instance
by (*standard*; *transfer*) (*simp-all add*: *diff-eq ac-simps*)

end

lemma *undual-minus-eq* [*simp*]:
undual (*x* − *y*) = − (*undual y* − *undual x*)
by (*fact minus-dual.rep-eq*)

lemma *dual-minus-eq* [*simp*]:
dual (*x* − *y*) = − (*dual y* − *dual x*)
by *transfer simp*

33.5 Complete lattice operations

The class of complete lattices is closed under formation of dual structures.

instantiation *dual* :: (*Sup*) *Inf*
begin

lift-definition *Inf-dual* :: *'a dual set* ⇒ *'a dual*
is *Sup* **.**

instance ..

end

lemma *undual-Inf-eq* [*simp*]:
undual (*Inf A*) = *Sup* (*undual* ‘ *A*)
by (*fact Inf-dual.rep-eq*)

lemma *dual-Sup-eq* [*simp*]:
dual (*Sup A*) = *Inf* (*dual* ‘ *A*)

by *transfer simp*

instantiation *dual* :: (*Inf*) *Sup*
begin

lift-definition *Sup-dual* :: *'a dual set* ⇒ *'a dual*
is *Inf* .

instance ..

end

lemma *undual-Sup-eq* [*simp*]:
undual (*Sup A*) = *Inf* (*undual* ‘ *A*)
by (*fact Sup-dual.rep-eq*)

lemma *dual-Inf-eq* [*simp*]:
dual (*Inf A*) = *Sup* (*dual* ‘ *A*)
by *transfer simp*

instance *dual* :: (*complete-lattice*) *complete-lattice*
by (*standard*; *transfer*) (*auto intro*: *Inf-lower Sup-upper Inf-greatest Sup-least*)

context
fixes *f* :: *'a*::*complete-lattice* ⇒ *'a*
and *g* :: *'a dual* ⇒ *'a dual*
assumes *mono f*
defines *g* ≡ *dual* ∘ *f* ∘ *undual*
begin

private lemma *mono-dual*:
mono g
proof
fix *x y* :: *'a dual*
assume *x* ≤ *y*
then have *undual y* ≤ *undual x*
by (*simp add*: *dual-less-eq-iff*)
with ‹*mono f*› **have** *f* (*undual y*) ≤ *f* (*undual x*)
by (*rule monoD*)
then have (*dual* ∘ *f* ∘ *undual*) *x* ≤ (*dual* ∘ *f* ∘ *undual*) *y*
by *simp*
then show *g x* ≤ *g y*
by (*simp add*: *g-def*)
qed

lemma *lfp-dual-gfp*:
lfp f = *undual* (*gfp g*) (**is** *?lhs* = *?rhs*)
proof (*rule antisym*)
have *dual* (*undual* (*g* (*gfp g*))) ≤ *dual* (*f* (*undual* (*gfp g*)))

by (*simp add*: *g-def*)
with *mono-dual* **have** *f* (*undual* (*gfp g*)) ≤ *undual* (*gfp g*)
by (*simp add*: *gfp-unfold* [**where** *f* = *g*, *symmetric*] *dual-less-eq-iff*)
then show *?lhs* ≤ *?rhs*
by (*rule lfp-lowerbound*)
from ‹*mono f*› **have** *dual* (*lfp f*) ≤ *dual* (*undual* (*gfp g*))
by (*simp add*: *lfp-fixpoint gfp-upperbound g-def*)
then show *?rhs* ≤ *?lhs*
by (*simp only*: *less-eq-dual-iff*)
qed

lemma *gfp-dual-lfp*:
gfp f = *undual* (*lfp g*)
proof −
have *mono* (λ*x*. *undual* (*undual x*))
by (*rule monoI*) (*simp add*: *dual-less-eq-iff*)
moreover have *mono* (λ*a*. *dual* (*dual* (*f a*)))
using ‹*mono f*› **by** (*auto intro*: *monoI dest*: *monoD*)
moreover have *gfp f* = *gfp* (λ*x*. *undual* (*undual* (*dual* (*dual* (*f x*)))))
by *simp*
ultimately have *undual* (*undual* (*gfp* (λ*x*. *dual*
(*dual* (*f* (*undual* (*undual x*)))))) =
gfp (λ*x*. *undual* (*undual* (*dual* (*dual* (*f x*)))))
by (*subst gfp-rolling* [**where** *g* = λ*x*. *undual* (*undual x*)]) *simp-all*
then have *gfp f* =
undual
(*undual*
(*gfp* (λ*x*. *dual* (*dual* (*f* (*undual* (*undual x*))))))
by *simp*
also have ... = *undual* (*undual* (*gfp* (*dual* ∘ *g* ∘ *undual*)))
by (*simp add*: *comp-def g-def*)
also have ... = *undual* (*lfp g*)
using *mono-dual* **by** (*simp only*: *Dual-Ordered-Lattice.lfp-dual-gfp*)
finally show *?thesis* **.**
qed

end

Finally

lifting-update *dual.lifting*
lifting-forget *dual.lifting*

end

34 Equipollence and Other Relations Connected with Cardinality

theory *Equipollence*

imports *FuncSet*
begin

34.1 Eqpoll

definition *eqpoll* :: *'a set* ⇒ *'b set* ⇒ *bool* (**infixl** ≈ *50*)
where *eqpoll A B* ≡ ∃*f*. *bij-betw f A B*

definition *lepoll* :: *'a set* ⇒ *'b set* ⇒ *bool* (**infixl** ≲ *50*)
where *lepoll A B* ≡ ∃*f*. *inj-on f A* ∧ *f* ‘ *A* ⊆ *B*

definition *lesspoll* :: *'a set* ⇒ *'b set* ⇒ *bool* (**infixl** ‹≺› *50*)
where *A* ≺ *B* == *A* ≲ *B* ∧ ~(*A* ≈ *B*)

lemma *lepoll-empty-iff-empty* [*simp*]: *A* ≲ {} ⟷ *A* = {}
by (*auto simp*: *lepoll-def*)

lemma *eqpoll-iff-card-of-ordIso*: *A* ≈ *B* ⟷ *ordIso2* (*card-of A*) (*card-of B*)
by (*simp add*: *card-of-ordIso eqpoll-def*)

lemma *eqpoll-refl* [*iff*]: *A* ≈ *A*
by (*simp add*: *card-of-refl eqpoll-iff-card-of-ordIso*)

lemma *eqpoll-finite-iff*: *A* ≈ *B* ⟹ *finite A* ⟷ *finite B*
by (*meson bij-betw-finite eqpoll-def*)

lemma *eqpoll-iff-card*:
assumes *finite A finite B*
shows *A* ≈ *B* ⟷ *card A* = *card B*
using *assms* **by** (*auto simp*: *bij-betw-iff-card eqpoll-def*)

lemma *lepoll-antisym*:
assumes *A* ≲ *B B* ≲ *A* **shows** *A* ≈ *B*
using *assms* **unfolding** *eqpoll-def lepoll-def* **by** (*metis Schroeder-Bernstein*)

lemma *lepoll-trans* [*trans*]: ⟦*A* ≲ *B*; *B* ≲ *C*⟧ ⟹ *A* ≲ *C*
apply (*clarsimp simp*: *lepoll-def*)
apply (*rename-tac f g*)
apply (*rule-tac x*=*g* ∘ *f* **in** *exI*)
apply (*auto simp*: *image-subset-iff inj-on-def*)
done

lemma *lepoll-trans1* [*trans*]: ⟦*A* ≈ *B*; *B* ≲ *C*⟧ ⟹ *A* ≲ *C*
by (*meson card-of-ordLeq eqpoll-iff-card-of-ordIso lepoll-def lepoll-trans ordIso-iff-ordLeq*)

lemma *lepoll-trans2* [*trans*]: ⟦*A* ≲ *B*; *B* ≈ *C*⟧ ⟹ *A* ≲ *C*
apply (*clarsimp simp*: *eqpoll-def lepoll-def bij-betw-def*)
apply (*rename-tac f g*)
apply (*rule-tac x*=*g* ∘ *f* **in** *exI*)

apply (*auto simp*: *image-subset-iff inj-on-def*)
done

lemma *eqpoll-sym*: $A \approx B \Longrightarrow B \approx A$
unfolding *eqpoll-def*
using *bij-betw-the-inv-into* **by** *auto*

lemma *eqpoll-trans* [*trans*]: $[\![A \approx B; B \approx C]\!] \Longrightarrow A \approx C$
unfolding *eqpoll-def* **using** *bij-betw-trans* **by** *blast*

lemma *eqpoll-imp-lepoll*: $A \approx B \Longrightarrow A \lesssim B$
unfolding *eqpoll-def lepoll-def* **by** (*metis bij-betw-def order-refl*)

lemma *subset-imp-lepoll*: $A \subseteq B \Longrightarrow A \lesssim B$
by (*force simp*: *lepoll-def*)

lemma *lepoll-refl* [*iff*]: $A \lesssim A$
by (*simp add*: *subset-imp-lepoll*)

lemma *lepoll-iff*: $A \lesssim B \longleftrightarrow (\exists\, g.\ A \subseteq g\ `\ B)$
unfolding *lepoll-def*
proof *safe*
fix *g* **assume** $A \subseteq g\ `\ B$
then show $\exists f.\ \mathit{inj\text{-}on}\ f\ A \wedge f\ `\ A \subseteq B$
by (*rule-tac x*=*inv-into B g* **in** *exI*) (*auto simp*: *inv-into-into inj-on-inv-into*)
qed (*metis image-mono the-inv-into-onto*)

lemma *empty-lepoll* [*iff*]: $\{\} \lesssim A$
by (*simp add*: *lepoll-iff*)

lemma *subset-image-lepoll*: $B \subseteq f\ `\ A \Longrightarrow B \lesssim A$
by (*auto simp*: *lepoll-iff*)

lemma *image-lepoll*: $f\ `\ A \lesssim A$
by (*auto simp*: *lepoll-iff*)

lemma *infinite-le-lepoll*: *infinite* $A \longleftrightarrow (\mathit{UNIV}{::}\mathit{nat\ set}) \lesssim A$
apply (*auto simp*: *lepoll-def*)
apply (*simp add*: *infinite-countable-subset*)
using *infinite-iff-countable-subset* **by** *blast*

lemma *lepoll-Pow-self*: $A \lesssim \mathit{Pow}\ A$
unfolding *lepoll-def inj-def*
proof (*intro exI conjI*)
show *inj-on* $(\lambda x.\ \{x\})\ A$
by (*auto simp*: *inj-on-def*)
qed *auto*

lemma *eqpoll-iff-bijections*:

$A \approx B \longleftrightarrow (\exists f\ g.\ (\forall x \in A.\ f\ x \in B \wedge g(f\ x) = x) \wedge (\forall y \in B.\ g\ y \in A \wedge f(g\ y) = y))$
by (*auto simp*: *eqpoll-def bij-betw-iff-bijections*)

lemma *lepoll-restricted-funspace*:
$\{f.\ f\ `\ A \subseteq B \wedge \{x.\ f\ x \neq k\ x\} \subseteq A \wedge \mathit{finite}\ \{x.\ f\ x \neq k\ x\}\} \lesssim \mathit{Fpow}\ (A \times B)$
proof −
have ∗: $\exists\ U \in \mathit{Fpow}\ (A \times B).\ f = (\lambda x.\ \mathit{if}\ \exists\ y.\ (x,\ y) \in U\ \mathit{then}\ \mathit{SOME}\ y.\ (x,y) \in U\ \mathit{else}\ k\ x)$
if $f\ `\ A \subseteq B\ \{x.\ f\ x \neq k\ x\} \subseteq A\ \mathit{finite}\ \{x.\ f\ x \neq k\ x\}$ **for** f
apply (*rule-tac* $x=(\lambda x.\ (x,\ f\ x))\ `\ \{x.\ f\ x \neq k\ x\}$ **in** *bexI*)
using *that* **by** (*auto simp*: *image-def Fpow-def*)
show *?thesis*
apply (*rule subset-image-lepoll* [**where** $f = \lambda U\ x.\ \mathit{if}\ \exists\ y.\ (x,y) \in U\ \mathit{then}\ @y.\ (x,y) \in U\ \mathit{else}\ k\ x$])
using ∗ **by** (*auto simp*: *image-def*)
qed

lemma *singleton-lepoll*: $\{x\} \lesssim \mathit{insert}\ y\ A$
by (*force simp*: *lepoll-def*)

lemma *singleton-eqpoll*: $\{x\} \approx \{y\}$
by (*blast intro*: *lepoll-antisym singleton-lepoll*)

lemma *subset-singleton-iff-lepoll*: $(\exists\ x.\ S \subseteq \{x\}) \longleftrightarrow S \lesssim \{()\}$
proof *safe*
show $S \lesssim \{()\}$ **if** $S \subseteq \{x\}$ **for** x
using *subset-imp-lepoll* [*OF that*] **by** (*simp add*: *singleton-eqpoll lepoll-trans2*)
show $\exists\ x.\ S \subseteq \{x\}$ **if** $S \lesssim \{()\}$
by (*metis* (*no-types*, *opaque-lifting*) *image-empty image-insert lepoll-iff that*)
qed

lemma *infinite-insert-lepoll*:
assumes *infinite A* **shows** *insert a A* $\lesssim$ *A*
proof −
obtain *f* :: *nat* ⇒ *'a* **where** *inj f* **and** *f*: *range f* ⊆ *A*
using *assms infinite-countable-subset* **by** *blast*
let *?g* = (λ*z*. *if z=a then f 0 else if z* ∈ *range f then f* (*Suc* (*inv f z*)) *else z*)
show *?thesis*
unfolding *lepoll-def*
proof (*intro exI conjI*)
show *inj-on ?g* (*insert a A*)
using *inj-on-eq-iff* [*OF* ‹*inj f*›]
by (*auto simp*: *inj-on-def*)
show *?g ` insert a A* ⊆ *A*
using *f* **by** *auto*
qed
qed

lemma *infinite-insert-eqpoll*: *infinite A* $\Longrightarrow$ *insert a A* $\approx$ *A*
by (*simp add*: *lepoll-antisym infinite-insert-lepoll subset-imp-lepoll subset-insertI*)

lemma *finite-lepoll-infinite*:
assumes *infinite A finite B* **shows** $B \lesssim A$
proof −
have $B \lesssim$ (*UNIV*::*nat set*)
unfolding *lepoll-def*
using *finite-imp-inj-to-nat-seg* [*OF* ‹*finite B*›] **by** *blast*
then show *?thesis*
using ‹*infinite A*› *infinite-le-lepoll lepoll-trans* **by** *auto*
qed

34.2 The strict relation

lemma *lesspoll-not-refl* [*iff*]: ~ ($i \prec i$)
by (*simp add*: *lepoll-antisym lesspoll-def*)

lemma *lesspoll-imp-lepoll*: $A \prec B$ ==> $A \lesssim B$
by (*unfold lesspoll-def*, *blast*)

lemma *lepoll-iff-leqpoll*: $A \lesssim B \longleftrightarrow A \prec B \mid A \approx B$
using *eqpoll-imp-lepoll lesspoll-def* **by** *blast*

lemma *lesspoll-trans* [*trans*]: $[\![X \prec Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
by (*meson eqpoll-sym lepoll-antisym lepoll-trans lepoll-trans1 lesspoll-def*)

lemma *lesspoll-trans1* [*trans*]: $[\![X \lesssim Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
by (*meson eqpoll-sym lepoll-antisym lepoll-trans lepoll-trans1 lesspoll-def*)

lemma *lesspoll-trans2* [*trans*]: $[\![X \prec Y;\ Y \lesssim Z]\!] \Longrightarrow X \prec Z$
by (*meson eqpoll-imp-lepoll eqpoll-sym lepoll-antisym lepoll-trans lesspoll-def*)

lemma *eq-lesspoll-trans* [*trans*]: $[\![X \approx Y;\ Y \prec Z]\!] \Longrightarrow X \prec Z$
using *eqpoll-imp-lepoll lesspoll-trans1* **by** *blast*

lemma *lesspoll-eq-trans* [*trans*]: $[\![X \prec Y;\ Y \approx Z]\!] \Longrightarrow X \prec Z$
using *eqpoll-imp-lepoll lesspoll-trans2* **by** *blast*

lemma *lesspoll-Pow-self*: $A \prec Pow\ A$
unfolding *lesspoll-def bij-betw-def eqpoll-def*
by (*meson lepoll-Pow-self Cantors-paradox*)

lemma *finite-lesspoll-infinite*:
assumes *infinite A finite B* **shows** $B \prec A$
by (*meson assms eqpoll-finite-iff finite-lepoll-infinite lesspoll-def*)

34.3 Mapping by an injection

lemma *inj-on-image-eqpoll-self*: *inj-on f A* $\Longrightarrow$ *f* ‘ *A* $\approx$ *A*

by (*meson bij-betw-def eqpoll-def eqpoll-sym*)

lemma *inj-on-image-lepoll-1* [*simp*]:
assumes *inj-on f A* **shows** $f \mathbin{`} A \lesssim B \longleftrightarrow A \lesssim B$
by (*meson assms image-lepoll lepoll-def lepoll-trans order-refl*)

lemma *inj-on-image-lepoll-2* [*simp*]:
assumes *inj-on f B* **shows** $A \lesssim f \mathbin{`} B \longleftrightarrow A \lesssim B$
by (*meson assms eq-iff image-lepoll lepoll-def lepoll-trans*)

lemma *inj-on-image-lesspoll-1* [*simp*]:
assumes *inj-on f A* **shows** $f \mathbin{`} A \prec B \longleftrightarrow A \prec B$
by (*meson assms image-lepoll le-less lepoll-def lesspoll-trans1*)

lemma *inj-on-image-lesspoll-2* [*simp*]:
assumes *inj-on f B* **shows** $A \prec f \mathbin{`} B \longleftrightarrow A \prec B$
by (*meson assms eqpoll-sym inj-on-image-eqpoll-self lesspoll-eq-trans*)

lemma *inj-on-image-eqpoll-1* [*simp*]:
assumes *inj-on f A* **shows** $f \mathbin{`} A \approx B \longleftrightarrow A \approx B$
by (*metis assms eqpoll-trans inj-on-image-eqpoll-self eqpoll-sym*)

lemma *inj-on-image-eqpoll-2* [*simp*]:
assumes *inj-on f B* **shows** $A \approx f \mathbin{`} B \longleftrightarrow A \approx B$
by (*metis assms inj-on-image-eqpoll-1 eqpoll-sym*)

34.4 Inserting elements into sets

lemma *insert-lepoll-insertD*:
assumes *insert u A* $\lesssim$ *insert v B u* $\notin$ *A v* $\notin$ *B* **shows** $A \lesssim B$
proof −
obtain *f* **where** *inj*: *inj-on f* (*insert u A*) **and** *fim*: $f \mathbin{`} (\mathit{insert}\ u\ A) \subseteq \mathit{insert}\ v\ B$
by (*meson assms lepoll-def*)
show *?thesis*
unfolding *lepoll-def*
proof (*intro exI conjI*)
let $?g = \lambda x{\in}A.\ \mathit{if}\ f\ x = v\ \mathit{then}\ f\ u\ \mathit{else}\ f\ x$
show *inj-on ?g A*
using *inj* ‹$u \notin A$› **by** (*auto simp*: *inj-on-def*)
show $?g \mathbin{`} A \subseteq B$
using *fim* ‹$u \notin A$› *image-subset-iff inj inj-on-image-mem-iff* **by** *fastforce*
qed
qed

lemma *insert-eqpoll-insertD*: ⟦*insert u A* $\approx$ *insert v B*; $u \notin A$; $v \notin B$⟧ $\Longrightarrow$ $A \approx B$
by (*meson insert-lepoll-insertD eqpoll-imp-lepoll eqpoll-sym lepoll-antisym*)

lemma *insert-lepoll-cong*:
assumes $A \lesssim B$ $b \notin B$ **shows** *insert a A* $\lesssim$ *insert b B*

proof −
obtain *f* **where** *f*: *inj-on f A f ‘ A ⊆ B*
by (*meson assms lepoll-def*)
let *?f* = *λu ∈ insert a A. if u=a then b else f u*
show *?thesis*
unfolding *lepoll-def*
proof (*intro exI conjI*)
show *inj-on ?f* (*insert a A*)
using *f ‹b ∉ B›* **by** (*auto simp*: *inj-on-def*)
show *?f ‘ insert a A ⊆ insert b B*
using *f ‹b ∉ B›* **by** *auto*
qed
qed

lemma *insert-eqpoll-cong*:
⟦A ≈ B; a ∉ A; b ∉ B⟧ ⟹ insert a A ≈ insert b B
apply (*rule lepoll-antisym*)
apply (*simp add*: *eqpoll-imp-lepoll insert-lepoll-cong*)+
by (*meson eqpoll-imp-lepoll eqpoll-sym insert-lepoll-cong*)

lemma *insert-eqpoll-insert-iff*:
⟦a ∉ A; b ∉ B⟧ ⟹ insert a A ≈ insert b B ⟷ A ≈ B
by (*meson insert-eqpoll-insertD insert-eqpoll-cong*)

lemma *insert-lepoll-insert-iff*:
⟦a ∉ A; b ∉ B⟧ ⟹ (insert a A ≲ insert b B) ⟷ (A ≲ B)
by (*meson insert-lepoll-insertD insert-lepoll-cong*)

lemma *less-imp-insert-lepoll*:
assumes *A ≺ B* **shows** *insert a A ≲ B*
proof −
obtain *f* **where** *inj-on f A f ‘ A ⊂ B*
using *assms* **by** (*metis bij-betw-def eqpoll-def lepoll-def lesspoll-def psubset-eq*)
then obtain *b* **where** *b*: *b ∈ B b ∉ f ‘ A*
by *auto*
show *?thesis*
unfolding *lepoll-def*
proof (*intro exI conjI*)
show *inj-on* (*f*(*a*:=*b*)) (*insert a A*)
using *b ‹inj-on f A›* **by** (*auto simp*: *inj-on-def*)
show (*f*(*a*:=*b*)) ‘ *insert a A ⊆ B*
using *‹f ‘ A ⊂ B›* **by** (*auto simp*: *b*)
qed
qed

lemma *finite-insert-lepoll*: *finite A ⟹* (*insert a A ≲ A*) *⟷* (*a ∈ A*)
proof (*induction A rule*: *finite-induct*)
case (*insert x A*)
then show *?case*

apply (*auto simp*: *insert-absorb*)
by (*metis insert-commute insert-iff insert-lepoll-insertD*)
qed *auto*

34.5 Binary sums and unions

lemma *Un-lepoll-mono*:
assumes $A \lesssim C$ $B \lesssim D$ *disjnt C D* **shows** $A \cup B \lesssim C \cup D$
proof −
obtain *f g* **where** *inj*: *inj-on f A inj-on g B* **and** *fg*: *f ' A* $\subseteq$ *C g ' B* $\subseteq$ *D*
by (*meson assms lepoll-def*)
have *inj-on* (λx. *if x* $\in$ *A then f x else g x*) (*A* $\cup$ *B*)
using *inj* ‹*disjnt C D*› *fg* **unfolding** *disjnt-iff*
by (*fastforce intro*: *inj-onI dest*: *inj-on-contraD split*: *if-split-asm*)
with *fg* **show** *?thesis*
unfolding *lepoll-def*
by (*rule-tac x*=λx. *if x* $\in$ *A then f x else g x* **in** *exI*) *auto*
qed

lemma *Un-eqpoll-cong*: $[\![A \approx C; B \approx D;$ *disjnt A B*; *disjnt C D*$]\!] \Longrightarrow A \cup B \approx C \cup D$
by (*meson Un-lepoll-mono eqpoll-imp-lepoll eqpoll-sym lepoll-antisym*)

lemma *sum-lepoll-mono*:
assumes $A \lesssim C$ $B \lesssim D$ **shows** *A <+> B* $\lesssim$ *C <+> D*
proof −
obtain *f g* **where** *inj-on f A f ' A* $\subseteq$ *C inj-on g B g ' B* $\subseteq$ *D*
by (*meson assms lepoll-def*)
then show *?thesis*
unfolding *lepoll-def*
by (*rule-tac x*=*case-sum* (*Inl* $\circ$ *f*) (*Inr* $\circ$ *g*) **in** *exI*) (*force simp*: *inj-on-def*)
qed

lemma *sum-eqpoll-cong*: $[\![A \approx C; B \approx D]\!] \Longrightarrow$ *A <+> B* $\approx$ *C <+> D*
by (*meson eqpoll-imp-lepoll eqpoll-sym lepoll-antisym sum-lepoll-mono*)

34.6 Binary Cartesian products

lemma *times-square-lepoll*: $A \lesssim A \times A$
unfolding *lepoll-def inj-def*
proof (*intro exI conjI*)
show *inj-on* (λx. (*x*,*x*)) *A*
by (*auto simp*: *inj-on-def*)
qed *auto*

lemma *times-commute-eqpoll*: $A \times B \approx B \times A$
unfolding *eqpoll-def*
by (*force intro*: *bij-betw-byWitness* [**where** $f = \lambda(x,y).$ (*y*,*x*) **and** $f' = \lambda(x,y).$ (*y*,*x*)])

lemma *times-assoc-eqpoll*: $(A \times B) \times C \approx A \times (B \times C)$
unfolding *eqpoll-def*
by (*force intro*: *bij-betw-byWitness* [**where** $f = \lambda((x,y),z).\ (x,(y,z))$ **and** $f' = \lambda(x,(y,z)).\ ((x,y),z)$])

lemma *times-singleton-eqpoll*: $\{a\} \times A \approx A$
proof −
have $\{a\} \times A = (\lambda x.\ (a,x))\ `\ A$
by *auto*
also have $\ldots \approx A$
proof (*rule inj-on-image-eqpoll-self*)
show *inj-on* (*Pair a*) *A*
by (*auto simp*: *inj-on-def*)
qed
finally show *?thesis* **.**
qed

lemma *times-lepoll-mono*:
assumes $A \lesssim C$ $B \lesssim D$ **shows** $A \times B \lesssim C \times D$
proof −
obtain *f g* **where** *inj-on f A* $f\ `\ A \subseteq C$ *inj-on g B* $g\ `\ B \subseteq D$
by (*meson assms lepoll-def*)
then show *?thesis*
unfolding *lepoll-def*
by (*rule-tac x*=$\lambda(x,y).\ (f\ x,\ g\ y)$ **in** *exI*) (*auto simp*: *inj-on-def*)
qed

lemma *times-eqpoll-cong*: $[\![A \approx C;\ B \approx D]\!] \Longrightarrow A \times B \approx C \times D$
by (*metis eqpoll-imp-lepoll eqpoll-sym lepoll-antisym times-lepoll-mono*)

lemma
assumes $B \neq \{\}$ **shows** *lepoll-times1*: $A \lesssim A \times B$ **and** *lepoll-times2*: $A \lesssim B \times A$
using *assms lepoll-iff* **by** *fastforce+*

lemma *times-0-eqpoll*: $\{\} \times A \approx \{\}$
by (*simp add*: *eqpoll-iff-bijections*)

lemma *Sigma-lepoll-mono*:
assumes $A \subseteq C$ $\bigwedge x.\ x \in A \Longrightarrow B\ x \lesssim D\ x$ **shows** *Sigma A B* $\lesssim$ *Sigma C D*
proof −
have $\bigwedge x.\ x \in A \Longrightarrow \exists f.$ *inj-on f* $(B\ x) \wedge f\ `\ (B\ x) \subseteq D\ x$
by (*meson assms lepoll-def*)
then obtain *f* **where** $\bigwedge x.\ x \in A \Longrightarrow$ *inj-on* $(f\ x)\ (B\ x) \wedge f\ x\ `\ B\ x \subseteq D\ x$
by *metis*
with ‹$A \subseteq C$› **show** *?thesis*
unfolding *lepoll-def inj-on-def*
by (*rule-tac x*=$\lambda(x,y).\ (x,\ f\ x\ y)$ **in** *exI*) *force*
qed

lemma *sum-times-distrib-eqpoll*: $(A <+> B) \times C \approx (A \times C) <+> (B \times C)$
unfolding *eqpoll-def*
proof
show *bij-betw* $(\lambda(x,z).\ case\text{-}sum(\lambda y.\ Inl(y,z))\ (\lambda y.\ Inr(y,z))\ x)\ ((A <+> B) \times C)\ (A \times C <+> B \times C)$
by (*rule bij-betw-byWitness* [**where** $f' = case\text{-}sum\ (\lambda(x,z).\ (Inl\ x,\ z))\ (\lambda(y,z).\ (Inr\ y,\ z))$]) *auto*
qed

lemma *prod-insert-eqpoll*:
assumes $a \notin A$ **shows** $insert\ a\ A \times B \approx B <+> A \times B$
unfolding *eqpoll-def*
proof
show *bij-betw* $(\lambda(x,y).\ if\ x{=}a\ then\ Inl\ y\ else\ Inr\ (x,y))\ (insert\ a\ A \times B)\ (B <+> A \times B)$
by (*rule bij-betw-byWitness* [**where** $f' = case\text{-}sum\ (\lambda y.\ (a,y))\ id$]) (*auto simp*: *assms*)
qed

34.7 General Unions

lemma *Union-eqpoll-Times*:
assumes *B*: $\bigwedge x.\ x \in A \Longrightarrow F\ x \approx B$ **and** *disj*: *pairwise* $(\lambda x\ y.\ disjnt\ (F\ x)\ (F\ y))\ A$
shows $(\bigcup x{\in}A.\ F\ x) \approx A \times B$
proof (*rule lepoll-antisym*)
obtain *b* **where** *b*: $\bigwedge x.\ x \in A \Longrightarrow bij\text{-}betw\ (b\ x)\ (F\ x)\ B$
using *B* **unfolding** *eqpoll-def* **by** *metis*
show $\bigcup(F\ `\ A) \lesssim A \times B$
unfolding *lepoll-def*
proof (*intro exI conjI*)
define χ **where** $\chi \equiv \lambda z.\ THE\ x.\ x \in A \wedge z \in F\ x$
have χ: $\chi\ z = x$ **if** $x \in A\ z \in F\ x$ **for** $x\ z$
unfolding χ*-def*
apply (*rule the-equality*)
apply (*simp add*: *that*)
by (*metis disj disjnt-iff pairwiseD that*)
let *?f* $= \lambda z.\ (\chi\ z,\ b\ (\chi\ z)\ z)$
show *inj-on ?f* $(\bigcup(F\ `\ A))$
unfolding *inj-on-def*
by *clarify* (*metis* χ *b bij-betw-inv-into-left*)
show *?f* $`\bigcup(F\ `\ A) \subseteq A \times B$
using χ *b bij-betwE* **by** *blast*
qed
show $A \times B \lesssim \bigcup(F\ `\ A)$
unfolding *lepoll-def*
proof (*intro exI conjI*)
let *?f* $= \lambda(x,y).\ inv\text{-}into\ (F\ x)\ (b\ x)\ y$

have $*$: *inv-into* (*F x*) (*b x*) *y* ∈ *F x* **if** *x* ∈ *A y* ∈ *B* **for** *x y*
by (*metis b bij-betw-imp-surj-on inv-into-into that*)
then show *inj-on ?f* (*A* × *B*)
unfolding *inj-on-def*
by *clarsimp* (*metis* (*mono-tags*, *lifting*) *b bij-betw-inv-into-right disj disjnt-iff pairwiseD*)
show *?f* ‘ (*A* × *B*) ⊆ ⋃ (*F* ‘ *A*)
by *clarsimp* (*metis b bij-betw-imp-surj-on inv-into-into*)
qed
qed

lemma *UN-lepoll-UN*:
assumes *A*: ⋀*x*. *x* ∈ *A* ⟹ *B x* ≲ *C x*
and *disj*: *pairwise* (λ*x y*. *disjnt* (*C x*) (*C y*)) *A*
shows ⋃ (*B*‘*A*) ≲ ⋃ (*C*‘*A*)
proof −
obtain *f* **where** *f*: ⋀*x*. *x* ∈ *A* ⟹ *inj-on* (*f x*) (*B x*) ∧ *f x* ‘ (*B x*) ⊆ (*C x*)
using *A* **unfolding** *lepoll-def* **by** *metis*
show *?thesis*
unfolding *lepoll-def*
proof (*intro exI conjI*)
define χ **where** χ ≡ λ*z*. @*x*. *x* ∈ *A* ∧ *z* ∈ *B x*
have χ: χ *z* ∈ *A* ∧ *z* ∈ *B* (χ *z*) **if** *x* ∈ *A z* ∈ *B x* **for** *x z*
unfolding χ*-def* **by** (*metis* (*mono-tags*, *lifting*) *someI-ex that*)
let *?f* = λ*z*. (*f* (χ *z*) *z*)
show *inj-on ?f* (⋃(*B* ‘ *A*))
using *disj f* **unfolding** *inj-on-def disjnt-iff pairwise-def image-subset-iff*
by (*metis UN-iff* χ)
show *?f* ‘ ⋃ (*B* ‘ *A*) ⊆ ⋃ (*C* ‘ *A*)
using χ *f* **unfolding** *image-subset-iff* **by** *blast*
qed
qed

lemma *UN-eqpoll-UN*:
assumes *A*: ⋀*x*. *x* ∈ *A* ⟹ *B x* ≈ *C x*
and *B*: *pairwise* (λ*x y*. *disjnt* (*B x*) (*B y*)) *A*
and *C*: *pairwise* (λ*x y*. *disjnt* (*C x*) (*C y*)) *A*
shows (⋃*x*∈*A*. *B x*) ≈ (⋃*x*∈*A*. *C x*)
proof (*rule lepoll-antisym*)
show ⋃ (*B* ‘ *A*) ≲ ⋃ (*C* ‘ *A*)
by (*meson A C UN-lepoll-UN eqpoll-imp-lepoll*)
show ⋃ (*C* ‘ *A*) ≲ ⋃ (*B* ‘ *A*)
by (*simp add*: *A B UN-lepoll-UN eqpoll-imp-lepoll eqpoll-sym*)
qed

34.8 General Cartesian products (Pi)

lemma *PiE-sing-eqpoll-self*: ({*a*} →$_E$ *B*) ≈ *B*
proof −

```
  have 1: x = y
    if x ∈ {a} →E B y ∈ {a} →E B x a = y a for x y
    by (metis IntD2 PiE-def extensionalityI singletonD that)
  have 2: x ∈ (λh. h a) ` ({a} →E B) if x ∈ B for x
    using that by (rule-tac x=λz∈{a}. x in image-eqI) auto
  show ?thesis
  unfolding eqpoll-def bij-betw-def inj-on-def
  by (force intro: 1 2)
qed

lemma lepoll-funcset-right:
   B ≲ B' ⟹ A →E B ≲ A →E B'
  apply (auto simp: lepoll-def inj-on-def)
  apply (rule-tac x = λg. λz ∈ A. f(g z) in exI)
  apply (auto simp: fun-eq-iff)
  apply (metis PiE-E)
  by blast

lemma lepoll-funcset-left:
  assumes B ≠ {} A ≲ A'
  shows A →E B ≲ A' →E B
proof −
  obtain b where b ∈ B
    using assms by blast
  obtain f where inj-on f A and fim: f ` A ⊆ A'
    using assms by (auto simp: lepoll-def)
  then obtain h where h: ⋀x. x ∈ A ⟹ h (f x) = x
    using the-inv-into-f-f by fastforce
  let ?F = λg. λu ∈ A'. if h u ∈ A then g(h u) else b
  show ?thesis
    unfolding lepoll-def inj-on-def
  proof (intro exI conjI ballI impI ext)
    fix k l x
    assume k: k ∈ A →E B and l: l ∈ A →E B and ?F k = ?F l
    then have ?F k (f x) = ?F l (f x)
      by simp
    then show k x = l x
      apply (auto simp: h split: if-split-asm)
      apply (metis PiE-arb h k l)
      apply (metis (full-types) PiE-E h k l)
      using fim k l by fastforce
  next
    show ?F ` (A →E B) ⊆ A' →E B
      using ‹b ∈ B› by force
  qed
qed

lemma lepoll-funcset:
   ⟦B ≠ {}; A ≲ A'; B ≲ B'⟧ ⟹ A →E B ≲ A' →E B'
```

by (*rule lepoll-trans* [*OF lepoll-funcset-right lepoll-funcset-left*]) *auto*

lemma *lepoll-PiE*:
assumes $\bigwedge i.\ i \in A \Longrightarrow B\ i \lesssim C\ i$
shows $PiE\ A\ B \lesssim PiE\ A\ C$
proof −
obtain f **where** f: $\bigwedge i.\ i \in A \Longrightarrow inj\text{-}on\ (f\ i)\ (B\ i) \wedge (f\ i)\ `\ B\ i \subseteq C\ i$
using *assms* **unfolding** *lepoll-def* **by** *metis*
then show *?thesis*
unfolding *lepoll-def*
apply (*rule-tac* $x = \lambda g.\ \lambda i \in A.\ f\ i\ (g\ i)$ **in** *exI*)
apply (*auto simp*: *inj-on-def*)
apply (*rule PiE-ext*, *auto*)
apply (*metis* (*full-types*) *PiE-mem restrict-apply'*)
by *blast*
qed

lemma *card-le-PiE-subindex*:
assumes $A \subseteq A'\ Pi_E\ A'\ B \neq \{\}$
shows $PiE\ A\ B \lesssim PiE\ A'\ B$
proof −
have $\bigwedge x.\ x \in A' \Longrightarrow \exists\, y.\ y \in B\ x$
using *assms* **by** *blast*
then obtain g **where** g: $\bigwedge x.\ x \in A' \Longrightarrow g\ x \in B\ x$
by *metis*
let $?F = \lambda f\ x.\ if\ x \in A\ then\ f\ x\ else\ if\ x \in A'\ then\ g\ x\ else\ undefined$
have $Pi_E\ A\ B \subseteq (\lambda f.\ restrict\ f\ A)\ `\ Pi_E\ A'\ B$
proof
show $f \in Pi_E\ A\ B \Longrightarrow f \in (\lambda f.\ restrict\ f\ A)\ `\ Pi_E\ A'\ B$ **for** f
using ‹$A \subseteq A'$›
by (*rule-tac x=?F f* **in** *image-eqI*) (*auto simp*: *g fun-eq-iff*)
qed
then have $Pi_E\ A\ B \lesssim (\lambda f.\ \lambda i \in A.\ f\ i)\ `\ Pi_E\ A'\ B$
by (*simp add*: *subset-imp-lepoll*)
also have $\ldots \lesssim PiE\ A'\ B$
by (*rule image-lepoll*)
finally show *?thesis* **.**
qed

lemma *finite-restricted-funspace*:
assumes *finite A finite B*
shows *finite* $\{f.\ f\ `\ A \subseteq B \wedge \{x.\ f\ x \neq k\ x\} \subseteq A\}$ (**is** *finite ?F*)
proof (*rule finite-subset*)
show *finite* $((\lambda U\ x.\ if\ \exists\, y.\ (x,y) \in U\ then\ @y.\ (x,y) \in U\ else\ k\ x)\ `\ Pow(A \times B))$ (**is** *finite ?G*)
using *assms* **by** *auto*
show $?F \subseteq ?G$

```
  proof
    fix f
    assume f ∈ ?F
    then show f ∈ ?G
     by (rule-tac x=(λx. (x,f x)) ` {x. f x ≠ k x} in image-eqI) (auto simp: fun-eq-iff
image-def)
  qed
qed


proposition finite-PiE-iff:
   finite(PiE I S) ⟷ PiE I S = {} ∨ finite {i ∈ I. ~(∃ a. S i ⊆ {a})} ∧ (∀ i ∈
I. finite(S i))
 (is ?lhs = ?rhs)
proof (cases PiE I S = {})
  case False
  define J where J ≡ {i ∈ I. ∄ a. S i ⊆ {a}}
  show ?thesis
  proof
    assume L: ?lhs
    have infinite (Pi_E I S) if infinite J
    proof −
      have (UNIV::nat set) ≲ (UNIV::(nat⇒bool) set)
      proof −
        have ∀ N::nat set. inj-on (=) N
          by (simp add: inj-on-def)
        then show ?thesis
          by (meson infinite-iff-countable-subset infinite-le-lepoll top.extremum)
      qed
      also have ... = (UNIV::nat set) →_E (UNIV::bool set)
        by auto
      also have ... ≲ J →_E (UNIV::bool set)
        apply (rule lepoll-funcset-left)
        using infinite-le-lepoll that by auto
      also have ... ≲ Pi_E J S
      proof −
        have ∗: (UNIV::bool set) ≲ S i if i ∈ I and ∀ a. ¬ S i ⊆ {a} for i
        proof −
          obtain a b where {a,b} ⊆ S i a ≠ b
              by (metis ‹∀ a. ¬ S i ⊆ {a}› all-not-in-conv empty-subsetI insertCI
insert-subset set-eq-subset subsetI)
          then show ?thesis
            apply (clarsimp simp: lepoll-def inj-on-def)
            apply (rule-tac x=λx. if x then a else b in exI, auto)
            done
        qed
        show ?thesis
          by (auto simp: ∗ J-def intro: lepoll-PiE)
      qed
```

```
    also have ... ≲ Pi_E I S
      using False by (auto simp: J-def intro: card-le-PiE-subindex)
    finally have (UNIV::nat set) ≲ Pi_E I S .
    then show ?thesis
      by (simp add: infinite-le-lepoll)
  qed
  moreover have finite (S i) if i ∈ I for i
  proof (rule finite-subset)
    obtain f where f: f ∈ PiE I S
      using False by blast
    show S i ⊆ (λf. f i) ` Pi_E I S
    proof
      show s ∈ (λf. f i) ` Pi_E I S if s ∈ S i for s
        using that f ‹i ∈ I›
        by (rule-tac x=λj. if j = i then s else f j in image-eqI) auto
    qed
  next
    show finite ((λx. x i) ` Pi_E I S)
      using L by blast
  qed
  ultimately show ?rhs
    using L
    by (auto simp: J-def False)
next
  assume R: ?rhs
  have ∀ i ∈ I - J. ∃ a. S i = {a}
    using False J-def by blast
  then obtain a where a: ∀ i ∈ I - J. S i = {a i}
    by metis
  let ?F = {f. f ` J ⊆ (⋃ i ∈ J. S i) ∧ {i. f i ≠ (if i ∈ I then a i else undefined)}
⊆ J}
  have ∗: finite (Pi_E I S)
    if finite J and ∀ i∈I. finite (S i)
  proof (rule finite-subset)
    show Pi_E I S ⊆ ?F
      apply safe
      using J-def apply blast
      by (metis DiffI PiE-E a singletonD)
    show finite ?F
    proof (rule finite-restricted-funspace [OF ‹finite J›])
      show finite (⋃ (S ` J))
        using that J-def by blast
    qed
  qed
  show ?lhs
    using R by (auto simp: ∗ J-def)
qed
qed auto
```

corollary *finite-funcset-iff*:
finite$(I \rightarrow_E S) \longleftrightarrow (\exists\, a.\ S \subseteq \{a\}) \vee I = \{\} \vee$ *finite* $I \wedge$ *finite* S
apply (*auto simp*: *finite-PiE-iff PiE-eq-empty-iff dest*: *not-finite-existsD*)
using *finite.simps* **by** *auto*

lemma *lists-lepoll-mono*:
assumes $A \lesssim B$ **shows** *lists* $A \lesssim$ *lists* B
proof −
obtain f **where** f: *inj-on* f A f ‘ $A \subseteq B$
by (*meson assms lepoll-def*)
moreover have *inj-on* (*map* f) (*lists* A)
using f **unfolding** *inj-on-def*
by *clarsimp* (*metis list.inj-map-strong*)
ultimately show *?thesis*
unfolding *lepoll-def* **by** *force*
qed

lemma *lepoll-lists*: $A \lesssim$ *lists* A
unfolding *lepoll-def inj-on-def* **by**(*rule-tac* x=$\lambda x.\ [x]$ **in** *exI*) *auto*

end

theory *Simps-Case-Conv*
imports *Case-Converter*
keywords *simps-of-case case-of-simps* :: *thy-decl*
abbrevs *simps-of-case case-of-simps* =
begin

ML-file ‹*simps-case-conv.ML*›

end

theory *Extended*
imports *Simps-Case-Conv*
begin

datatype *'a extended* = *Fin 'a* | *Pinf* (∞) | *Minf* $(-\infty)$

instantiation *extended* :: (*order*)*order*
begin

fun *less-eq-extended* :: *'a extended* ⇒ *'a extended* ⇒ *bool* **where**
Fin $x \leq$ *Fin* $y = (x \leq y)$ |
- $\leq$ *Pinf* = *True* |
Minf $\leq$ - = *True* |

$(\text{-}::'a\ extended) \leq \text{-} \quad = False$

case-of-simps *less-eq-extended-case*: *less-eq-extended.simps*

definition *less-extended* :: $'a\ extended \Rightarrow 'a\ extended \Rightarrow bool$ **where**
$((x::'a\ extended) < y) = (x \leq y \wedge \neg\ y \leq x)$

instance
by *intro-classes* (*auto simp*: *less-extended-def less-eq-extended-case split*: *extended.splits*)

end

instance *extended* :: (*linorder*)*linorder*
by *intro-classes* (*auto simp*: *less-eq-extended-case split*:*extended.splits*)

lemma *Minf-le*[*simp*]: $Minf \leq y$
by(*cases y*) *auto*
lemma *le-Pinf*[*simp*]: $x \leq Pinf$
by(*cases x*) *auto*
lemma *le-Minf*[*simp*]: $x \leq Minf \longleftrightarrow x = Minf$
by(*cases x*) *auto*
lemma *Pinf-le*[*simp*]: $Pinf \leq x \longleftrightarrow x = Pinf$
by(*cases x*) *auto*

lemma *less-extended-simps*[*simp*]:
$Fin\ x < Fin\ y = (x < y)$
$Fin\ x < Pinf\ = True$
$Fin\ x < Minf\ = False$
$Pinf < h\ \quad = False$
$Minf < Fin\ x\ = True$
$Minf < Pinf\ = True$
$l\ \quad < Minf\ = False$
by (*auto simp add*: *less-extended-def*)

lemma *min-extended-simps*[*simp*]:
$min\ (Fin\ x)\ (Fin\ y) = Fin(min\ x\ y)$
$min\ xx\ \quad Pinf\ \quad = xx$
$min\ xx\ \quad Minf\ \quad = Minf$
$min\ Pinf\ \quad yy\ \quad = yy$
$min\ Minf\ \quad yy\ \quad = Minf$
by (*auto simp add*: *min-def*)

lemma *max-extended-simps*[*simp*]:
$max\ (Fin\ x)\ (Fin\ y) = Fin(max\ x\ y)$
$max\ xx\ \quad Pinf\ \quad = Pinf$
$max\ xx\ \quad Minf\ \quad = xx$
$max\ Pinf\ \quad yy\ \quad = Pinf$
$max\ Minf\ \quad yy\ \quad = yy$
by (*auto simp add*: *max-def*)

instantiation *extended* :: (*zero*)*zero*
begin
definition *0* = *Fin*(*0*::'*a*)
instance ..
end

declare *zero-extended-def*[*symmetric*, *code-post*]

instantiation *extended* :: (*one*)*one*
begin
definition *1* = *Fin*(*1*::'*a*)
instance ..
end

declare *one-extended-def*[*symmetric*, *code-post*]

instantiation *extended* :: (*plus*)*plus*
begin

The following definition of of addition is totalized to make it asociative and commutative. Normally the sum of plus and minus infinity is undefined.

fun *plus-extended* **where**
Fin x + *Fin y* = *Fin*(*x*+*y*) |
Fin x + *Pinf* = *Pinf* |
Pinf + *Fin x* = *Pinf* |
Pinf + *Pinf* = *Pinf* |
Minf + *Fin y* = *Minf* |
Fin x + *Minf* = *Minf* |
Minf + *Minf* = *Minf* |
Minf + *Pinf* = *Pinf* |
Pinf + *Minf* = *Pinf*

case-of-simps *plus-case*: *plus-extended.simps*

instance ..

end

instance *extended* :: (*ab-semigroup-add*)*ab-semigroup-add*
by *intro-classes* (*simp-all add*: *ac-simps plus-case split*: *extended.splits*)

instance *extended* :: (*ordered-ab-semigroup-add*)*ordered-ab-semigroup-add*
by *intro-classes* (*auto simp*: *add-left-mono plus-case split*: *extended.splits*)

instance *extended* :: (*comm-monoid-add*)*comm-monoid-add*

proof
fix *x* :: *'a extended* **show** *0 + x = x* **unfolding** *zero-extended-def* **by**(*cases x*)*auto*
qed

instantiation *extended* :: (*uminus*)*uminus*
begin

fun *uminus-extended* **where**
− (Fin x) = Fin (− x) |
− Pinf = Minf |
− Minf = Pinf

instance ..

end

instantiation *extended* :: (*ab-group-add*)*minus*
begin
definition *x − y = x + −*(*y*::*'a extended*)
instance ..
end

lemma *minus-extended-simps*[*simp*]:
Fin x − Fin y = Fin(*x − y*)
Fin x − Pinf = Minf
Fin x − Minf = Pinf
Pinf − Fin y = Pinf
Pinf − Minf = Pinf
Minf − Fin y = Minf
Minf − Pinf = Minf
Minf − Minf = Pinf
Pinf − Pinf = Pinf
by (*simp-all add*: *minus-extended-def*)

Numerals:

instance *extended* :: ({*ab-semigroup-add*,*one*})*numeral* **..**

lemma *Fin-numeral*[*code-post*]: *Fin*(*numeral w*) = *numeral w*
apply (*induct w rule*: *num-induct*)
apply (*simp only*: *numeral-One one-extended-def*)
apply (*simp only*: *numeral-inc one-extended-def plus-extended.simps*(*1*)[*symmetric*])
done

lemma *Fin-neg-numeral*[*code-post*]: *Fin* (*− numeral w*) = *− numeral w*
by (*simp only*: *Fin-numeral uminus-extended.simps*[*symmetric*])

instantiation *extended* :: (*lattice*)*bounded-lattice*
begin

definition *bot* = *Minf*
definition *top* = *Pinf*

fun *inf-extended* :: *'a extended* ⇒ *'a extended* ⇒ *'a extended* **where**
inf-extended (*Fin i*) (*Fin j*) = *Fin* (*inf i j*) |
inf-extended a Minf = *Minf* |
inf-extended Minf a = *Minf* |
inf-extended Pinf a = *a* |
inf-extended a Pinf = *a*

fun *sup-extended* :: *'a extended* ⇒ *'a extended* ⇒ *'a extended* **where**
sup-extended (*Fin i*) (*Fin j*) = *Fin* (*sup i j*) |
sup-extended a Pinf = *Pinf* |
sup-extended Pinf a = *Pinf* |
sup-extended Minf a = *a* |
sup-extended a Minf = *a*

case-of-simps *inf-extended-case*: *inf-extended.simps*
case-of-simps *sup-extended-case*: *sup-extended.simps*

instance
by (*intro-classes*) (*auto simp*: *inf-extended-case sup-extended-case less-eq-extended-case*
bot-extended-def top-extended-def split: *extended.splits*)
end

end

35 Continuity and iterations

theory *Order-Continuity*
imports *Complex-Main Countable-Complete-Lattices*
begin

lemma *SUP-nat-binary*:
(*sup A* (*SUP x*∈*Collect* ((<) (*0*::*nat*)). *B*)) = (*sup A B*::*'a*::*countable-complete-lattice*)
apply (*subst image-constant*)
apply *auto*
done

lemma *INF-nat-binary*:
inf A (*INF x*∈*Collect* ((<) (*0*::*nat*)). *B*) = (*inf A B*::*'a*::*countable-complete-lattice*)
apply (*subst image-constant*)
apply *auto*
done

The name *continuous* is already taken in *Complex-Main*, so we use *sup-continuous* and *inf-continuous*. These names appear sometimes in literature and have the advantage that these names are duals.

named-theorems *order-continuous-intros*

35.1 Continuity for complete lattices

definition
sup-continuous :: ('*a*::*countable-complete-lattice* ⇒ '*b*::*countable-complete-lattice*) ⇒ *bool*
where
sup-continuous F ⟷ (∀ *M*::*nat* ⇒ '*a*. *mono M* ⟶ *F* (*SUP i*. *M i*) = (*SUP i*. *F* (*M i*)))

lemma *sup-continuousD*: *sup-continuous F* ⟹ *mono M* ⟹ *F* (*SUP i*::*nat*. *M i*) = (*SUP i*. *F* (*M i*))
by (*auto simp*: *sup-continuous-def*)

lemma *sup-continuous-mono*:
mono F **if** *sup-continuous F*
proof
fix *A B* :: '*a*
assume *A* ≤ *B*
let *?f* = λ*n*::*nat*. *if n* = *0 then A else B*
from ‹*A* ≤ *B*› **have** *incseq ?f*
by (*auto intro*: *monoI*)
with ‹*sup-continuous F*› **have** ∗: *F* (*SUP i*. *?f i*) = (*SUP i*. *F* (*?f i*))
by (*auto dest*: *sup-continuousD*)
from ‹*A* ≤ *B*› **have** *B* = *sup A B*
by (*simp add*: *le-iff-sup*)
then have *F B* = *F* (*sup A B*)
by *simp*
also have ... = *sup* (*F A*) (*F B*)
using ∗ **by** (*simp add*: *if-distrib SUP-nat-binary cong del*: *SUP-cong*)
finally show *F A* ≤ *F B*
by (*simp add*: *le-iff-sup*)
qed

lemma [*order-continuous-intros*]:
shows *sup-continuous-const*: *sup-continuous* (λ*x*. *c*)
and *sup-continuous-id*: *sup-continuous* (λ*x*. *x*)
and *sup-continuous-apply*: *sup-continuous* (λ*f*. *f x*)
and *sup-continuous-fun*: (⋀*s*. *sup-continuous* (λ*x*. *P x s*)) ⟹ *sup-continuous P*
and *sup-continuous-If*: *sup-continuous F* ⟹ *sup-continuous G* ⟹ *sup-continuous* (λ*f*. *if C then F f else G f*)
by (*auto simp*: *sup-continuous-def image-comp*)

lemma *sup-continuous-compose*:

assumes *f*: *sup-continuous f* **and** *g*: *sup-continuous g*
shows *sup-continuous* (λ*x*. *f* (*g x*))
unfolding *sup-continuous-def*
proof *safe*
fix *M* :: *nat* ⇒ *'c*
assume *M*: *mono M*
then have *mono* (λ*i*. *g* (*M i*))
using *sup-continuous-mono*[*OF g*] **by** (*auto simp*: *mono-def*)
with *M* **show** *f* (*g* (*Sup* (*M* ' *UNIV*))) = (*SUP i*. *f* (*g* (*M i*)))
by (*auto simp*: *sup-continuous-def g*[*THEN sup-continuousD*] *f*[*THEN sup-continuousD*])
qed

lemma *sup-continuous-sup*[*order-continuous-intros*]:
sup-continuous f ⟹ *sup-continuous g* ⟹ *sup-continuous* (λ*x*. *sup* (*f x*) (*g x*))
by (*simp add*: *sup-continuous-def ccSUP-sup-distrib*)

lemma *sup-continuous-inf*[*order-continuous-intros*]:
fixes *P Q* :: *'a* :: *countable-complete-lattice* ⇒ *'b* :: *countable-complete-distrib-lattice*
assumes *P*: *sup-continuous P* **and** *Q*: *sup-continuous Q*
shows *sup-continuous* (λ*x*. *inf* (*P x*) (*Q x*))
unfolding *sup-continuous-def*
proof (*safe intro*!: *antisym*)
fix *M* :: *nat* ⇒ *'a* **assume** *M*: *incseq M*
have *inf* (*P* (*SUP i*. *M i*)) (*Q* (*SUP i*. *M i*)) ≤ (*SUP j i*. *inf* (*P* (*M i*)) (*Q* (*M j*)))
by (*simp add*: *sup-continuousD*[*OF P M*] *sup-continuousD*[*OF Q M*] *inf-ccSUP ccSUP-inf*)
also have ... ≤ (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*)))
proof (*intro ccSUP-least*)
fix *i j* **from** *M assms*[*THEN sup-continuous-mono*] **show** *inf* (*P* (*M i*)) (*Q* (*M j*)) ≤ (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*)))
by (*intro ccSUP-upper2*[*of - sup i j*] *inf-mono*) (*auto simp*: *mono-def*)
qed *auto*
finally show *inf* (*P* (*SUP i*. *M i*)) (*Q* (*SUP i*. *M i*)) ≤ (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*))) **.**

show (*SUP i*. *inf* (*P* (*M i*)) (*Q* (*M i*))) ≤ *inf* (*P* (*SUP i*. *M i*)) (*Q* (*SUP i*. *M i*))
unfolding *sup-continuousD*[*OF P M*] *sup-continuousD*[*OF Q M*] **by** (*intro ccSUP-least inf-mono ccSUP-upper*) *auto*
qed

lemma *sup-continuous-and*[*order-continuous-intros*]:
sup-continuous P ⟹ *sup-continuous Q* ⟹ *sup-continuous* (λ*x*. *P x* ∧ *Q x*)
using *sup-continuous-inf*[*of P Q*] **by** *simp*

lemma *sup-continuous-or*[*order-continuous-intros*]:
sup-continuous P ⟹ *sup-continuous Q* ⟹ *sup-continuous* (λ*x*. *P x* ∨ *Q x*)
by (*auto simp*: *sup-continuous-def*)

lemma *sup-continuous-lfp*:
assumes *sup-continuous F* **shows** *lfp F* = (*SUP i*. (*F* ^^ *i*) *bot*) (**is** *lfp F* = *?U*)
proof (*rule antisym*)
note *mono* = *sup-continuous-mono*[*OF* ‹*sup-continuous F*›]
show *?U* ≤ *lfp F*
proof (*rule SUP-least*)
fix *i* **show** (*F* ^^ *i*) *bot* ≤ *lfp F*
proof (*induct i*)
case (*Suc i*)
have (*F* ^^ *Suc i*) *bot* = *F* ((*F* ^^ *i*) *bot*) **by** *simp*
also have ... ≤ *F* (*lfp F*) **by** (*rule monoD*[*OF mono Suc*])
also have ... = *lfp F* **by** (*simp add*: *lfp-fixpoint*[*OF mono*])
finally show *?case* **.**
qed *simp*
qed
show *lfp F* ≤ *?U*
proof (*rule lfp-lowerbound*)
have *mono* (λ*i*::*nat*. (*F* ^^ *i*) *bot*)
proof −
{ fix *i*::*nat* **have** (*F* ^^ *i*) *bot* ≤ (*F* ^^ (*Suc i*)) *bot*
proof (*induct i*)
case *0* **show** *?case* **by** *simp*
next
case *Suc* **thus** *?case* **using** *monoD*[*OF mono Suc*] **by** *auto*
qed }
thus *?thesis* **by** (*auto simp add*: *mono-iff-le-Suc*)
qed
hence *F ?U* = (*SUP i*. (*F* ^^ *Suc i*) *bot*)
using ‹*sup-continuous F*› **by** (*simp add*: *sup-continuous-def*)
also have ... ≤ *?U*
by (*fast intro*: *SUP-least SUP-upper*)
finally show *F ?U* ≤ *?U* **.**
qed
qed

lemma *lfp-transfer-bounded*:
assumes *P*: *P bot* ⋀*x*. *P x* ⟹ *P* (*f x*) ⋀*M*. (⋀*i*. *P* (*M i*)) ⟹ *P* (*SUP i*::*nat*. *M i*)
assumes α: ⋀*M*. *mono M* ⟹ (⋀*i*::*nat*. *P* (*M i*)) ⟹ α (*SUP i*. *M i*) = (*SUP i*. α (*M i*))
assumes *f*: *sup-continuous f* **and** *g*: *sup-continuous g*
assumes [*simp*]: ⋀*x*. *P x* ⟹ *x* ≤ *lfp f* ⟹ α (*f x*) = *g* (α *x*)
assumes *g-bound*: ⋀*x*. α *bot* ≤ *g x*
shows α (*lfp f*) = *lfp g*
proof (*rule antisym*)
note *mono-g* = *sup-continuous-mono*[*OF g*]
note *mono-f* = *sup-continuous-mono*[*OF f*]
have *lfp-bound*: α *bot* ≤ *lfp g*

```
    by (subst lfp-unfold[OF mono-g]) (rule g-bound)

  have P-pow: P ((f ^^ i) bot) for i
    by (induction i) (auto intro!: P)
  have incseq-pow: mono (λi. (f ^^ i) bot)
    unfolding mono-iff-le-Suc
  proof
    fix i show (f ^^ i) bot ≤ (f ^^ (Suc i)) bot
    proof (induct i)
      case Suc thus ?case using monoD[OF sup-continuous-mono[OF f] Suc] by
auto
    qed (simp add: le-fun-def)
  qed
  have P-lfp: P (lfp f)
    using P-pow unfolding sup-continuous-lfp[OF f] by (auto intro!: P)

  have iter-le-lfp: (f ^^ n) bot ≤ lfp f for n
    apply (induction n)
    apply simp
    apply (subst lfp-unfold[OF mono-f])
    apply (auto intro!: monoD[OF mono-f])
    done

  have α (lfp f) = (SUP i. α ((f^^i) bot))
    unfolding sup-continuous-lfp[OF f] using incseq-pow P-pow by (rule α)
  also have ... ≤ lfp g
  proof (rule SUP-least)
    fix i show α ((f^^i) bot) ≤ lfp g
    proof (induction i)
      case (Suc n) then show ?case
        by (subst lfp-unfold[OF mono-g]) (simp add: monoD[OF mono-g] P-pow
iter-le-lfp)
    qed (simp add: lfp-bound)
  qed
  finally show α (lfp f) ≤ lfp g .

  show lfp g ≤ α (lfp f)
  proof (induction rule: lfp-ordinal-induct[OF mono-g])
    case (1 S) then show ?case
      by (subst lfp-unfold[OF sup-continuous-mono[OF f]])
        (simp add: monoD[OF mono-g] P-lfp)
  qed (auto intro: Sup-least)
qed

lemma lfp-transfer:
  sup-continuous α ⟹ sup-continuous f ⟹ sup-continuous g ⟹
    (⋀x. α bot ≤ g x) ⟹ (⋀x. x ≤ lfp f ⟹ α (f x) = g (α x)) ⟹ α (lfp f) =
lfp g
  by (rule lfp-transfer-bounded[where P=top]) (auto dest: sup-continuousD)
```

definition
inf-continuous :: (*'a*::*countable-complete-lattice* ⇒ *'b*::*countable-complete-lattice*) ⇒ *bool*
where
inf-continuous F ⟷ (∀ *M*::*nat* ⇒ *'a. antimono M* ⟶ *F* (*INF i. M i*) = (*INF i. F* (*M i*)))

lemma *inf-continuousD*: *inf-continuous F* ⟹ *antimono M* ⟹ *F* (*INF i*::*nat. M i*) = (*INF i. F* (*M i*))
by (*auto simp*: *inf-continuous-def*)

lemma *inf-continuous-mono*:
mono F **if** *inf-continuous F*
proof
fix *A B* :: *'a*
assume *A* ≤ *B*
let *?f* = λ*n*::*nat. if n* = *0 then B else A*
from ‹*A* ≤ *B*› **have** *decseq ?f*
by (*auto intro*: *antimonoI*)
with ‹*inf-continuous F*› **have** ∗: *F* (*INF i. ?f i*) = (*INF i. F* (*?f i*))
by (*auto dest*: *inf-continuousD*)
from ‹*A* ≤ *B*› **have** *A* = *inf B A*
by (*simp add*: *inf.absorb-iff2*)
then have *F A* = *F* (*inf B A*)
by *simp*
also have ... = *inf* (*F B*) (*F A*)
using ∗ **by** (*simp add*: *if-distrib INF-nat-binary cong del*: *INF-cong*)
finally show *F A* ≤ *F B*
by (*simp add*: *inf.absorb-iff2*)
qed

lemma [*order-continuous-intros*]:
shows *inf-continuous-const*: *inf-continuous* (λ*x. c*)
and *inf-continuous-id*: *inf-continuous* (λ*x. x*)
and *inf-continuous-apply*: *inf-continuous* (λ*f. f x*)
and *inf-continuous-fun*: (⋀*s. inf-continuous* (λ*x. P x s*)) ⟹ *inf-continuous P*
and *inf-continuous-If*: *inf-continuous F* ⟹ *inf-continuous G* ⟹ *inf-continuous* (λ*f. if C then F f else G f*)
by (*auto simp*: *inf-continuous-def image-comp*)

lemma *inf-continuous-inf*[*order-continuous-intros*]:
inf-continuous f ⟹ *inf-continuous g* ⟹ *inf-continuous* (λ*x. inf* (*f x*) (*g x*))
by (*simp add*: *inf-continuous-def ccINF-inf-distrib*)

lemma *inf-continuous-sup*[*order-continuous-intros*]:
fixes *P Q* :: *'a* :: *countable-complete-lattice* ⇒ *'b* :: *countable-complete-distrib-lattice*
assumes *P*: *inf-continuous P* **and** *Q*: *inf-continuous Q*
shows *inf-continuous* (λ*x. sup* (*P x*) (*Q x*))

unfolding *inf-continuous-def*
proof (*safe intro*!: *antisym*)
fix $M :: nat \Rightarrow 'a$ **assume** M: *decseq M*
show *sup* (*P* (*INF i. M i*)) (*Q* (*INF i. M i*)) ≤ (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*)))
unfolding *inf-continuousD*[*OF P M*] *inf-continuousD*[*OF Q M*] **by** (*intro ccINF-greatest sup-mono ccINF-lower*) *auto*

have (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*))) ≤ (*INF j i. sup* (*P* (*M i*)) (*Q* (*M j*)))
proof (*intro ccINF-greatest*)
fix *i j* **from** *M assms*[*THEN inf-continuous-mono*] **show** *sup* (*P* (*M i*)) (*Q* (*M j*)) ≥ (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*)))
by (*intro ccINF-lower2*[*of - sup i j*] *sup-mono*) (*auto simp*: *mono-def anti-mono-def*)
qed *auto*
also have ... ≤ *sup* (*P* (*INF i. M i*)) (*Q* (*INF i. M i*))
by (*simp add*: *inf-continuousD*[*OF P M*] *inf-continuousD*[*OF Q M*] *ccINF-sup sup-ccINF*)
finally show *sup* (*P* (*INF i. M i*)) (*Q* (*INF i. M i*)) ≥ (*INF i. sup* (*P* (*M i*)) (*Q* (*M i*))) **.**
qed

lemma *inf-continuous-and*[*order-continuous-intros*]:
inf-continuous P ⟹ *inf-continuous Q* ⟹ *inf-continuous* (λ*x*. *P x* ∧ *Q x*)
using *inf-continuous-inf*[*of P Q*] **by** *simp*

lemma *inf-continuous-or*[*order-continuous-intros*]:
inf-continuous P ⟹ *inf-continuous Q* ⟹ *inf-continuous* (λ*x*. *P x* ∨ *Q x*)
using *inf-continuous-sup*[*of P Q*] **by** *simp*

lemma *inf-continuous-compose*:
assumes *f*: *inf-continuous f* **and** *g*: *inf-continuous g*
shows *inf-continuous* (λ*x*. *f* (*g x*))
unfolding *inf-continuous-def*
proof *safe*
fix $M :: nat \Rightarrow 'c$
assume *M*: *antimono M*
then have *antimono* (λ*i*. *g* (*M i*))
using *inf-continuous-mono*[*OF g*] **by** (*auto simp*: *mono-def antimono-def*)
with *M* **show** *f* (*g* (*Inf* (*M* ‘ *UNIV*))) = (*INF i. f* (*g* (*M i*)))
by (*auto simp*: *inf-continuous-def g*[*THEN inf-continuousD*] *f*[*THEN inf-continuousD*])
qed

lemma *inf-continuous-gfp*:
assumes *inf-continuous F* **shows** *gfp F* = (*INF i.* (*F* ^^ *i*) *top*) (**is** *gfp F* = *?U*)
proof (*rule antisym*)
note *mono* = *inf-continuous-mono*[*OF* ‹*inf-continuous F*›]
show *gfp F* ≤ *?U*
proof (*rule INF-greatest*)

```
    fix i show gfp F ≤ (F ^^ i) top
    proof (induct i)
      case (Suc i)
      have gfp F = F (gfp F) by (simp add: gfp-fixpoint[OF mono])
      also have ... ≤ F ((F ^^ i) top) by (rule monoD[OF mono Suc])
      also have ... = (F ^^ Suc i) top by simp
      finally show ?case .
    qed simp
  qed
  show ?U ≤ gfp F
  proof (rule gfp-upperbound)
    have ∗: antimono (λi::nat. (F ^^ i) top)
    proof −
      { fix i::nat have (F ^^ Suc i) top ≤ (F ^^ i) top
        proof (induct i)
          case 0 show ?case by simp
        next
          case Suc thus ?case using monoD[OF mono Suc] by auto
        qed }
      thus ?thesis by (auto simp add: antimono-iff-le-Suc)
    qed
    have ?U ≤ (INF i. (F ^^ Suc i) top)
      by (fast intro: INF-greatest INF-lower)
    also have ... ≤ F ?U
      by (simp add: inf-continuousD ‹inf-continuous F› ∗)
    finally show ?U ≤ F ?U .
  qed
qed

lemma gfp-transfer:
  assumes α: inf-continuous α and f: inf-continuous f and g: inf-continuous g
  assumes [simp]: α top = top ⋀x. α (f x) = g (α x)
  shows α (gfp f) = gfp g
proof −
  have α (gfp f) = (INF i. α ((f^^i) top))
   unfolding inf-continuous-gfp[OF f] by (intro f α inf-continuousD antimono-funpow
inf-continuous-mono)
  moreover have α ((f^^i) top) = (g^^i) top for i
    by (induction i; simp)
  ultimately show ?thesis
    unfolding inf-continuous-gfp[OF g] by simp
qed

lemma gfp-transfer-bounded:
  assumes P: P (f top) ⋀x. P x ⟹ P (f x) ⋀M. antimono M ⟹ (⋀i. P (M
i)) ⟹ P (INF i::nat. M i)
  assumes α: ⋀M. antimono M ⟹ (⋀i::nat. P (M i)) ⟹ α (INF i. M i) =
(INF i. α (M i))
  assumes f: inf-continuous f and g: inf-continuous g
```

```
  assumes [simp]: ⋀x. P x ⟹ α (f x) = g (α x)
  assumes g-bound: ⋀x. g x ≤ α (f top)
  shows α (gfp f) = gfp g
proof (rule antisym)
  note mono-g = inf-continuous-mono[OF g]

  have P-pow: P ((f ^^ i) (f top)) for i
    by (induction i) (auto intro!: P)

  have antimono-pow: antimono (λi. (f ^^ i) top)
    unfolding antimono-iff-le-Suc
  proof
    fix i show (f ^^ Suc i) top ≤ (f ^^ i) top
    proof (induct i)
      case Suc thus ?case using monoD[OF inf-continuous-mono[OF f] Suc] by
auto
    qed (simp add: le-fun-def)
  qed
  have antimono-pow2: antimono (λi. (f ^^ i) (f top))
  proof
    show x ≤ y ⟹ (f ^^ y) (f top) ≤ (f ^^ x) (f top) for x y
      using antimono-pow[THEN antimonoD, of Suc x Suc y]
      unfolding funpow-Suc-right by simp
  qed

  have gfp-f: gfp f = (INF i. (f ^^ i) (f top))
    unfolding inf-continuous-gfp[OF f]
  proof (rule INF-eq)
    show ∃j∈UNIV. (f ^^ j) (f top) ≤ (f ^^ i) top for i
      by (intro bexI[of - i − 1]) (auto simp: diff-Suc funpow-Suc-right simp del:
funpow.simps(2) split: nat.split)
    show ∃j∈UNIV. (f ^^ j) top ≤ (f ^^ i) (f top) for i
      by (intro bexI[of - Suc i]) (auto simp: funpow-Suc-right simp del: fun-
pow.simps(2))
  qed

  have P-lfp: P (gfp f)
    unfolding gfp-f by (auto intro!: P P-pow antimono-pow2)

  have α (gfp f) = (INF i. α ((f^^i) (f top)))
    unfolding gfp-f by (rule α) (auto intro!: P-pow antimono-pow2)
  also have ... ≥ gfp g
  proof (rule INF-greatest)
    fix i show gfp g ≤ α ((f^^i) (f top))
    proof (induction i)
      case (Suc n) then show ?case
        by (subst gfp-unfold[OF mono-g]) (simp add: monoD[OF mono-g] P-pow)
    next
      case 0
```

```
    have gfp g ≤ α (f top)
      by (subst gfp-unfold[OF mono-g]) (rule g-bound)
    then show ?case
      by simp
  qed
 qed
 finally show gfp g ≤ α (gfp f) .

 show α (gfp f) ≤ gfp g
 proof (induction rule: gfp-ordinal-induct[OF mono-g])
  case (1 S) then show ?case
    by (subst gfp-unfold[OF inf-continuous-mono[OF f]])
      (simp add: monoD[OF mono-g] P-lfp)
 qed (auto intro: Inf-greatest)
qed
```

35.1.1 Least fixed points in countable complete lattices

```
definition (in countable-complete-lattice) cclfp :: ('a ⇒ 'a) ⇒ 'a
 where cclfp f = (SUP i. (f ^^ i) bot)

lemma cclfp-unfold:
 assumes sup-continuous F shows cclfp F = F (cclfp F)
proof −
 have cclfp F = (SUP i. F ((F ^^ i) bot))
  unfolding cclfp-def
  by (subst UNIV-nat-eq) (simp add: image-comp)
 also have ... = F (cclfp F)
  unfolding cclfp-def
 by (intro sup-continuousD[symmetric] assms mono-funpow sup-continuous-mono)
 finally show ?thesis .
qed

lemma cclfp-lowerbound: assumes f: mono f and A: f A ≤ A shows cclfp f ≤ A
 unfolding cclfp-def
proof (intro ccSUP-least)
 fix i show (f ^^ i) bot ≤ A
 proof (induction i)
  case (Suc i) from monoD[OF f this] A show ?case
    by auto
 qed simp
qed simp

lemma cclfp-transfer:
 assumes sup-continuous α mono f
 assumes α bot = bot ⋀x. α (f x) = g (α x)
 shows α (cclfp f) = cclfp g
proof −
 have α (cclfp f) = (SUP i. α ((f ^^ i) bot))
```

unfolding *cclfp-def* **by** (*intro sup-continuousD assms mono-funpow sup-continuous-mono*)
moreover have α $((f \frown\frown i)\ bot) = (g \frown\frown i)\ bot$ **for** i
by (*induction i*) (*simp-all add*: *assms*)
ultimately show *?thesis*
by (*simp add*: *cclfp-def*)
qed

end

36 Extended natural numbers (i.e. with infinity)

theory *Extended-Nat*
imports *Main Countable Order-Continuity*
begin

class *infinity* =
fixes *infinity* :: *'a* (∞)

context
fixes *f* :: *nat* $\Rightarrow$ *'a*::{*canonically-ordered-monoid-add*, *linorder-topology*, *complete-linorder*}
begin

lemma *sums-SUP*[*simp*, *intro*]: *f sums* (*SUP n.* $\sum$ *i*<*n. f i*)
unfolding *sums-def* **by** (*intro LIMSEQ-SUP monoI sum-mono2 zero-le*) *auto*

lemma *suminf-eq-SUP*: *suminf f* = (*SUP n.* $\sum$ *i*<*n. f i*)
using *sums-SUP* **by** (*rule sums-unique*[*symmetric*])

end

36.1 Type definition

We extend the standard natural numbers by a special value indicating infinity.

typedef *enat* = *UNIV* :: *nat option set* **..**

TODO: introduce enat as coinductive datatype, enat is just *of-nat*

definition *enat* :: *nat* $\Rightarrow$ *enat* **where**
enat n = *Abs-enat* (*Some n*)

instantiation *enat* :: *infinity*
begin

definition ∞ = *Abs-enat None*
instance ..

end

instance *enat* :: *countable*
proof
  show ∃ *to-nat*::*enat* ⇒ *nat*. *inj to-nat*
    by (*rule exI*[*of* - *to-nat* ∘ *Rep-enat*]) (*simp add*: *inj-on-def Rep-enat-inject*)
qed

old-rep-datatype *enat* ∞ :: *enat*
proof −
  fix *P i* **assume** ⋀*j*. *P* (*enat j*) *P* ∞
  then show *P i*
  proof *induct*
    case (*Abs-enat y*) **then show** *?case*
      by (*cases y rule*: *option.exhaust*)
        (*auto simp*: *enat-def infinity-enat-def*)
  qed
qed (*auto simp add*: *enat-def infinity-enat-def Abs-enat-inject*)

declare [[*coercion enat*::*nat*⇒*enat*]]

lemmas *enat2-cases* = *enat.exhaust*[*case-product enat.exhaust*]
lemmas *enat3-cases* = *enat.exhaust*[*case-product enat.exhaust enat.exhaust*]

lemma *not-infinity-eq* [*iff*]: (*x* ≠ ∞) = (∃ *i*. *x* = *enat i*)
  by (*cases x*) *auto*

lemma *not-enat-eq* [*iff*]: (∀ *y*. *x* ≠ *enat y*) = (*x* = ∞)
  by (*cases x*) *auto*

lemma *enat-ex-split*: (∃ *c*::*enat*. *P c*) ⟷ *P* ∞ ∨ (∃ *c*::*nat*. *P c*)
  by (*metis enat.exhaust*)

primrec *the-enat* :: *enat* ⇒ *nat*
  where *the-enat* (*enat n*) = *n*

36.2 Constructors and numbers

instantiation *enat* :: *zero-neq-one*
begin

definition
  0 = *enat 0*

definition
  1 = *enat 1*

instance
  proof qed (*simp add*: *zero-enat-def one-enat-def*)

end

definition *eSuc* :: *enat* ⇒ *enat* **where**
eSuc i = (*case i of enat n* ⇒ *enat* (*Suc n*) | ∞ ⇒ ∞)

lemma *enat-0* [*code-post*]: *enat 0* = *0*
by (*simp add*: *zero-enat-def*)

lemma *enat-1* [*code-post*]: *enat 1* = *1*
by (*simp add*: *one-enat-def*)

lemma *enat-0-iff*: *enat x* = *0* ⟷ *x* = *0 0* = *enat x* ⟷ *x* = *0*
by (*auto simp add*: *zero-enat-def*)

lemma *enat-1-iff*: *enat x* = *1* ⟷ *x* = *1 1* = *enat x* ⟷ *x* = *1*
by (*auto simp add*: *one-enat-def*)

lemma *one-eSuc*: *1* = *eSuc 0*
by (*simp add*: *zero-enat-def one-enat-def eSuc-def*)

lemma *infinity-ne-i0* [*simp*]: (∞::*enat*) ≠ *0*
by (*simp add*: *zero-enat-def*)

lemma *i0-ne-infinity* [*simp*]: *0* ≠ (∞::*enat*)
by (*simp add*: *zero-enat-def*)

lemma *zero-one-enat-neq*:
¬ *0* = (*1*::*enat*)
¬ *1* = (*0*::*enat*)
unfolding *zero-enat-def one-enat-def* **by** *simp-all*

lemma *infinity-ne-i1* [*simp*]: (∞::*enat*) ≠ *1*
by (*simp add*: *one-enat-def*)

lemma *i1-ne-infinity* [*simp*]: *1* ≠ (∞::*enat*)
by (*simp add*: *one-enat-def*)

lemma *eSuc-enat*: *eSuc* (*enat n*) = *enat* (*Suc n*)
by (*simp add*: *eSuc-def*)

lemma *eSuc-infinity* [*simp*]: *eSuc* ∞ = ∞
by (*simp add*: *eSuc-def*)

lemma *eSuc-ne-0* [*simp*]: *eSuc n* ≠ *0*
by (*simp add*: *eSuc-def zero-enat-def split*: *enat.splits*)

lemma *zero-ne-eSuc* [*simp*]: *0* ≠ *eSuc n*
by (*rule eSuc-ne-0* [*symmetric*])

lemma *eSuc-inject* [*simp*]: *eSuc m = eSuc n ⟷ m = n*
by (*simp add*: *eSuc-def split*: *enat.splits*)

lemma *eSuc-enat-iff*: *eSuc x = enat y ⟷ (∃ n. y = Suc n ∧ x = enat n)*
by (*cases y*) (*auto simp*: *enat-0 eSuc-enat*[*symmetric*])

lemma *enat-eSuc-iff*: *enat y = eSuc x ⟷ (∃ n. y = Suc n ∧ enat n = x)*
by (*cases y*) (*auto simp*: *enat-0 eSuc-enat*[*symmetric*])

36.3 Addition

instantiation *enat* :: *comm-monoid-add*
begin

definition [*nitpick-simp*]:
m + n = (case m of ∞ ⇒ ∞ | enat m ⇒ (case n of ∞ ⇒ ∞ | enat n ⇒ enat (m + n)))

lemma *plus-enat-simps* [*simp*, *code*]:
fixes *q* :: *enat*
shows *enat m + enat n = enat (m + n)*
and *∞ + q = ∞*
and *q + ∞ = ∞*
by (*simp-all add*: *plus-enat-def split*: *enat.splits*)

instance
proof
fix *n m q* :: *enat*
show *n + m + q = n + (m + q)*
by (*cases n m q rule*: *enat3-cases*) *auto*
show *n + m = m + n*
by (*cases n m rule*: *enat2-cases*) *auto*
show *0 + n = n*
by (*cases n*) (*simp-all add*: *zero-enat-def*)
qed

end

lemma *eSuc-plus-1*:
eSuc n = n + 1
by (*cases n*) (*simp-all add*: *eSuc-enat one-enat-def*)

lemma *plus-1-eSuc*:
1 + q = eSuc q
q + 1 = eSuc q
by (*simp-all add*: *eSuc-plus-1 ac-simps*)

lemma *iadd-Suc*: *eSuc m + n = eSuc (m + n)*
by (*simp-all add*: *eSuc-plus-1 ac-simps*)

lemma *iadd-Suc-right*: $m + eSuc\ n = eSuc\ (m + n)$
by (*simp only*: *add.commute*[*of m*] *iadd-Suc*)

36.4 Multiplication

instantiation *enat* :: {*comm-semiring-1*, *semiring-no-zero-divisors*}
begin

definition *times-enat-def* [*nitpick-simp*]:
$m * n = (case\ m\ of\ \infty \Rightarrow if\ n = 0\ then\ 0\ else\ \infty \mid enat\ m \Rightarrow$
$(case\ n\ of\ \infty \Rightarrow if\ m = 0\ then\ 0\ else\ \infty \mid enat\ n \Rightarrow enat\ (m * n)))$

lemma *times-enat-simps* [*simp*, *code*]:
$enat\ m * enat\ n = enat\ (m * n)$
$\infty * \infty = (\infty :: enat)$
$\infty * enat\ n = (if\ n = 0\ then\ 0\ else\ \infty)$
$enat\ m * \infty = (if\ m = 0\ then\ 0\ else\ \infty)$
unfolding *times-enat-def zero-enat-def*
by (*simp-all split*: *enat.split*)

instance
proof
fix $a\ b\ c :: enat$
show $(a * b) * c = a * (b * c)$
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show *comm*: $a * b = b * a$
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show $1 * a = a$
unfolding *times-enat-def zero-enat-def one-enat-def*
by (*simp split*: *enat.split*)
show *distr*: $(a + b) * c = a * c + b * c$
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split add*: *distrib-right*)
show $0 * a = 0$
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show $a * 0 = 0$
unfolding *times-enat-def zero-enat-def*
by (*simp split*: *enat.split*)
show $a * (b + c) = a * b + a * c$
by (*cases a b c rule*: *enat3-cases*) (*auto simp*: *times-enat-def zero-enat-def distrib-left*)
show $a \neq 0 \Longrightarrow b \neq 0 \Longrightarrow a * b \neq 0$
by (*cases a b rule*: *enat2-cases*) (*auto simp*: *times-enat-def zero-enat-def*)
qed

end

lemma *mult-eSuc*: *eSuc m * n = n + m * n*
unfolding *eSuc-plus-1* **by** (*simp add*: *algebra-simps*)

lemma *mult-eSuc-right*: *m * eSuc n = m + m * n*
unfolding *eSuc-plus-1* **by** (*simp add*: *algebra-simps*)

lemma *of-nat-eq-enat*: *of-nat n = enat n*
apply (*induct n*)
apply (*simp add*: *enat-0*)
apply (*simp add*: *plus-1-eSuc eSuc-enat*)
done

instance *enat* :: *semiring-char-0*
proof
have *inj enat* **by** (*rule injI*) *simp*
then show *inj* (λ*n*. *of-nat n* :: *enat*) **by** (*simp add*: *of-nat-eq-enat*)
qed

lemma *imult-is-infinity*: ((*a*::*enat*) * *b* = ∞) = (*a* = ∞ ∧ *b* ≠ *0* ∨ *b* = ∞ ∧ *a* ≠ *0*)
by (*auto simp add*: *times-enat-def zero-enat-def split*: *enat.split*)

36.5 Numerals

lemma *numeral-eq-enat*:
numeral k = enat (*numeral k*)
using *of-nat-eq-enat* [*of numeral k*] **by** *simp*

lemma *enat-numeral* [*code-abbrev*]:
enat (*numeral k*) = *numeral k*
using *numeral-eq-enat* **..**

lemma *infinity-ne-numeral* [*simp*]: (∞::*enat*) ≠ *numeral k*
by (*simp add*: *numeral-eq-enat*)

lemma *numeral-ne-infinity* [*simp*]: *numeral k* ≠ (∞::*enat*)
by (*simp add*: *numeral-eq-enat*)

lemma *eSuc-numeral* [*simp*]: *eSuc* (*numeral k*) = *numeral* (*k* + *Num.One*)
by (*simp only*: *eSuc-plus-1 numeral-plus-one*)

36.6 Subtraction

instantiation *enat* :: *minus*
begin

definition *diff-enat-def*:
a − *b* = (*case a of* (*enat x*) ⇒ (*case b of* (*enat y*) ⇒ *enat* (*x* − *y*) | ∞ ⇒ *0*)

$\mid \infty \Rightarrow \infty)$

instance ..

end

lemma *idiff-enat-enat* [*simp*, *code*]: *enat a − enat b = enat* (*a − b*)
by (*simp add*: *diff-enat-def*)

lemma *idiff-infinity* [*simp*, *code*]: $\infty - n = (\infty::enat)$
by (*simp add*: *diff-enat-def*)

lemma *idiff-infinity-right* [*simp*, *code*]: *enat a − ∞ = 0*
by (*simp add*: *diff-enat-def*)

lemma *idiff-0* [*simp*]: (*0*::*enat*) *− n = 0*
by (*cases n*, *simp-all add*: *zero-enat-def*)

lemmas *idiff-enat-0* [*simp*] = *idiff-0* [*unfolded zero-enat-def*]

lemma *idiff-0-right* [*simp*]: (*n*::*enat*) *− 0 = n*
by (*cases n*) (*simp-all add*: *zero-enat-def*)

lemmas *idiff-enat-0-right* [*simp*] = *idiff-0-right* [*unfolded zero-enat-def*]

lemma *idiff-self* [*simp*]: $n \neq \infty \Longrightarrow (n::enat) - n = 0$
by (*auto simp*: *zero-enat-def*)

lemma *eSuc-minus-eSuc* [*simp*]: *eSuc n − eSuc m = n − m*
by (*simp add*: *eSuc-def split*: *enat.split*)

lemma *eSuc-minus-1* [*simp*]: *eSuc n − 1 = n*
by (*simp add*: *one-enat-def flip*: *eSuc-enat zero-enat-def*)

36.7 Ordering

instantiation *enat* :: *linordered-ab-semigroup-add*
begin

definition [*nitpick-simp*]:
$m \leq n = (case\ n\ of\ enat\ n1 \Rightarrow (case\ m\ of\ enat\ m1 \Rightarrow m1 \leq n1 \mid \infty \Rightarrow False) \mid \infty \Rightarrow True)$

definition [*nitpick-simp*]:
$m < n = (case\ m\ of\ enat\ m1 \Rightarrow (case\ n\ of\ enat\ n1 \Rightarrow m1 < n1 \mid \infty \Rightarrow True) \mid \infty \Rightarrow False)$

lemma *enat-ord-simps* [*simp*]:
enat m ≤ enat n $\longleftrightarrow$ *m ≤ n*

enat m < enat n ⟷ m < n
q ≤ (∞::enat)
q < (∞::enat) ⟷ q ≠ ∞
(∞::enat) ≤ q ⟷ q = ∞
(∞::enat) < q ⟷ False
by (*simp-all add*: *less-eq-enat-def less-enat-def split*: *enat.splits*)

lemma *numeral-le-enat-iff*[*simp*]:
shows *numeral m ≤ enat n ⟷ numeral m ≤ n*
by (*auto simp*: *numeral-eq-enat*)

lemma *numeral-less-enat-iff*[*simp*]:
shows *numeral m < enat n ⟷ numeral m < n*
by (*auto simp*: *numeral-eq-enat*)

lemma *enat-ord-code* [*code*]:
enat m ≤ enat n ⟷ m ≤ n
enat m < enat n ⟷ m < n
q ≤ (∞::enat) ⟷ True
enat m < ∞ ⟷ True
∞ ≤ enat n ⟷ False
(∞::enat) < q ⟷ False
by *simp-all*

instance
by *standard* (*auto simp add*: *less-eq-enat-def less-enat-def plus-enat-def split*: *enat.splits*)

end

instance *enat* :: *dioid*
proof
fix *a b* :: *enat* **show** *(a ≤ b) = (∃ c. b = a + c)*
by (*cases a b rule*: *enat2-cases*) (*auto simp*: *le-iff-add enat-ex-split*)
qed

instance *enat* :: {*linordered-nonzero-semiring*, *strict-ordered-comm-monoid-add*}
proof
fix *a b c* :: *enat*
show *a ≤ b ⟹ 0 ≤ c ⟹c * a ≤ c * b*
unfolding *times-enat-def less-eq-enat-def zero-enat-def*
by (*simp split*: *enat.splits*)
show *a < b ⟹ c < d ⟹ a + c < b + d* **for** *a b c d* :: *enat*
by (*cases a b c d rule*: *enat2-cases*[*case-product enat2-cases*]) *auto*
show *a < b ⟹ a + 1 < b + 1*
by (*metis add-right-mono eSuc-minus-1 eSuc-plus-1 less-le*)
qed (*simp add*: *zero-enat-def one-enat-def*)

lemma *add-diff-assoc-enat*: $z \leq y \Longrightarrow x + (y - z) = x + y - (z::enat)$
by(*cases x*)(*auto simp add*: *diff-enat-def split*: *enat.split*)

lemma *enat-ord-number* [*simp*]:
$(numeral\ m :: enat) \leq numeral\ n \longleftrightarrow (numeral\ m :: nat) \leq numeral\ n$
$(numeral\ m :: enat) < numeral\ n \longleftrightarrow (numeral\ m :: nat) < numeral\ n$
by (*simp-all add*: *numeral-eq-enat*)

lemma *infinity-ileE* [*elim*!]: $\infty \leq enat\ m \Longrightarrow R$
by (*simp add*: *zero-enat-def less-eq-enat-def split*: *enat.splits*)

lemma *infinity-ilessE* [*elim*!]: $\infty < enat\ m \Longrightarrow R$
by *simp*

lemma *eSuc-ile-mono* [*simp*]: $eSuc\ n \leq eSuc\ m \longleftrightarrow n \leq m$
by (*simp add*: *eSuc-def less-eq-enat-def split*: *enat.splits*)

lemma *eSuc-mono* [*simp*]: $eSuc\ n < eSuc\ m \longleftrightarrow n < m$
by (*simp add*: *eSuc-def less-enat-def split*: *enat.splits*)

lemma *ile-eSuc* [*simp*]: $n \leq eSuc\ n$
by (*simp add*: *eSuc-def less-eq-enat-def split*: *enat.splits*)

lemma *not-eSuc-ilei0* [*simp*]: $\neg\ eSuc\ n \leq 0$
by (*simp add*: *zero-enat-def eSuc-def less-eq-enat-def split*: *enat.splits*)

lemma *i0-iless-eSuc* [*simp*]: $0 < eSuc\ n$
by (*simp add*: *zero-enat-def eSuc-def less-enat-def split*: *enat.splits*)

lemma *iless-eSuc0*[*simp*]: $(n < eSuc\ 0) = (n = 0)$
by (*simp add*: *zero-enat-def eSuc-def less-enat-def split*: *enat.split*)

lemma *ileI1*: $m < n \Longrightarrow eSuc\ m \leq n$
by (*simp add*: *eSuc-def less-eq-enat-def less-enat-def split*: *enat.splits*)

lemma *Suc-ile-eq*: $enat\ (Suc\ m) \leq n \longleftrightarrow enat\ m < n$
by (*cases n*) *auto*

lemma *iless-Suc-eq* [*simp*]: $enat\ m < eSuc\ n \longleftrightarrow enat\ m \leq n$
by (*auto simp add*: *eSuc-def less-enat-def split*: *enat.splits*)

lemma *imult-infinity*: $(0::enat) < n \Longrightarrow \infty * n = \infty$
by (*simp add*: *zero-enat-def less-enat-def split*: *enat.splits*)

lemma *imult-infinity-right*: $(0::enat) < n \Longrightarrow n * \infty = \infty$
by (*simp add*: *zero-enat-def less-enat-def split*: *enat.splits*)

lemma *enat-0-less-mult-iff*: $(0 < (m::enat) * n) = (0 < m \wedge 0 < n)$

by (*simp only*: *zero-less-iff-neq-zero mult-eq-0-iff*, *simp*)

lemma *mono-eSuc*: *mono eSuc*
by (*simp add*: *mono-def*)

lemma *min-enat-simps* [*simp*]:
min (*enat m*) (*enat n*) = *enat* (*min m n*)
min q 0 = *0*
min 0 q = *0*
min q (∞::*enat*) = *q*
min (∞::*enat*) *q* = *q*
by (*auto simp add*: *min-def*)

lemma *max-enat-simps* [*simp*]:
max (*enat m*) (*enat n*) = *enat* (*max m n*)
max q 0 = *q*
max 0 q = *q*
max q ∞ = (∞::*enat*)
max ∞ *q* = (∞::*enat*)
by (*simp-all add*: *max-def*)

lemma *enat-ile*: $n \leq$ *enat m* $\Longrightarrow \exists k.$ *n* = *enat k*
by (*cases n*) *simp-all*

lemma *enat-iless*: $n <$ *enat m* $\Longrightarrow \exists k.$ *n* = *enat k*
by (*cases n*) *simp-all*

lemma *iadd-le-enat-iff*:
$x + y \leq$ *enat n* $\longleftrightarrow (\exists y' x'.$ *x* = *enat x'* $\wedge$ *y* = *enat y'* $\wedge$ $x' + y' \leq$ *n*)
by(*cases x y rule*: *enat.exhaust*[*case-product enat.exhaust*]) *simp-all*

lemma *chain-incr*: $\forall i. \exists j.$ *Y i* $<$ *Y j* $\Longrightarrow \exists j.$ *enat k* $<$ *Y j*
apply (*induct-tac k*)
apply (*simp* (*no-asm*) *only*: *enat-0*)
apply (*fast intro*: *le-less-trans* [*OF zero-le*])
apply (*erule exE*)
apply (*drule spec*)
apply (*erule exE*)
apply (*drule ileI1*)
apply (*rule eSuc-enat* [*THEN subst*])
apply (*rule exI*)
apply (*erule* (*1*) *le-less-trans*)
done

lemma *eSuc-max*: *eSuc* (*max x y*) = *max* (*eSuc x*) (*eSuc y*)
by (*simp add*: *eSuc-def split*: *enat.split*)

lemma *eSuc-Max*:
assumes *finite A A* $\neq \{\}$

shows *eSuc* (*Max A*) = *Max* (*eSuc* ' *A*)
using *assms* **proof** *induction*
case (*insert x A*)
thus *?case* **by**(*cases A* = {})(*simp-all add*: *eSuc-max*)
qed *simp*

instantiation *enat* :: {*order-bot*, *order-top*}
begin

definition *bot-enat* :: *enat* **where** *bot-enat* = *0*
definition *top-enat* :: *enat* **where** *top-enat* = ∞

instance
by *standard* (*simp-all add*: *bot-enat-def top-enat-def*)

end

lemma *finite-enat-bounded*:
assumes *le-fin*: ⋀*y*. *y* ∈ *A* ⟹ *y* ≤ *enat n*
shows *finite A*
proof (*rule finite-subset*)
show *finite* (*enat* ' {..*n*}) **by** *blast*
have *A* ⊆ {..*enat n*} **using** *le-fin* **by** *fastforce*
also have ... ⊆ *enat* ' {..*n*}
apply (*rule subsetI*)
subgoal for *x* **by** (*cases x*) *auto*
done
finally show *A* ⊆ *enat* ' {..*n*} **.**
qed

36.8 Cancellation simprocs

lemma *add-diff-cancel-enat*[*simp*]: *x* ≠ ∞ ⟹ *x* + *y* − *x* = (*y*::*enat*)
by (*metis add.commute add.right-neutral add-diff-assoc-enat idiff-self order-refl*)

lemma *enat-add-left-cancel*: *a* + *b* = *a* + *c* ⟷ *a* = (∞::*enat*) ∨ *b* = *c*
unfolding *plus-enat-def* **by** (*simp split*: *enat.split*)

lemma *enat-add-left-cancel-le*: *a* + *b* ≤ *a* + *c* ⟷ *a* = (∞::*enat*) ∨ *b* ≤ *c*
unfolding *plus-enat-def* **by** (*simp split*: *enat.split*)

lemma *enat-add-left-cancel-less*: *a* + *b* < *a* + *c* ⟷ *a* ≠ (∞::*enat*) ∧ *b* < *c*
unfolding *plus-enat-def* **by** (*simp split*: *enat.split*)

lemma *plus-eq-infty-iff-enat*: (*m*::*enat*) + *n* = ∞ ⟷ *m*=∞ ∨ *n*=∞
using *enat-add-left-cancel* **by** *fastforce*

ML ‹
structure Cancel-Enat-Common =

```
struct
  (* copied from src/HOL/Tools/nat-numeral-simprocs.ML *)
  fun find-first-t -    - []          = raise TERM(find-first-t, [])
    | find-first-t past u (t::terms) =
          if u aconv t then (rev past @ terms)
          else find-first-t (t::past) u terms

  fun dest-summing (Const (const-name‹Groups.plus›, -) $ t $ u, ts) =
        dest-summing (t, dest-summing (u, ts))
    | dest-summing (t, ts) = t :: ts

  val mk-sum = Arith-Data.long-mk-sum
  fun dest-sum t = dest-summing (t, [])
  val find-first = find-first-t []
  val trans-tac = Numeral-Simprocs.trans-tac
  val norm-ss =
    simpset-of (put-simpset HOL-basic-ss context
      addsimps @{thms ac-simps add-0-left add-0-right})
  fun norm-tac ctxt = ALLGOALS (simp-tac (put-simpset norm-ss ctxt))
  fun simplify-meta-eq ctxt cancel-th th =
    Arith-Data.simplify-meta-eq [] ctxt
      ([th, cancel-th] MRS trans)
  fun mk-eq (a, b) = HOLogic.mk-Trueprop (HOLogic.mk-eq (a, b))
end

structure Eq-Enat-Cancel = ExtractCommonTermFun
(open Cancel-Enat-Common
  val mk-bal = HOLogic.mk-eq
  val dest-bal = HOLogic.dest-bin const-name‹HOL.eq› typ‹enat›
  fun simp-conv - - = SOME @{thm enat-add-left-cancel}
)

structure Le-Enat-Cancel = ExtractCommonTermFun
(open Cancel-Enat-Common
  val mk-bal = HOLogic.mk-binrel const-name‹Orderings.less-eq›
  val dest-bal = HOLogic.dest-bin const-name‹Orderings.less-eq› typ‹enat›
  fun simp-conv - - = SOME @{thm enat-add-left-cancel-le}
)

structure Less-Enat-Cancel = ExtractCommonTermFun
(open Cancel-Enat-Common
  val mk-bal = HOLogic.mk-binrel const-name‹Orderings.less›
  val dest-bal = HOLogic.dest-bin const-name‹Orderings.less› typ‹enat›
  fun simp-conv - - = SOME @{thm enat-add-left-cancel-less}
)
›

simproc-setup enat-eq-cancel
  ((l::enat) + m = n | (l::enat) = m + n) =
```

‹*fn phi => fn ctxt => fn ct => Eq-Enat-Cancel.proc ctxt (Thm.term-of ct)*›

simproc-setup *enat-le-cancel*
((*l*::*enat*) + *m* ≤ *n* | (*l*::*enat*) ≤ *m* + *n*) =
‹*fn phi => fn ctxt => fn ct => Le-Enat-Cancel.proc ctxt (Thm.term-of ct)*›

simproc-setup *enat-less-cancel*
((*l*::*enat*) + *m* < *n* | (*l*::*enat*) < *m* + *n*) =
‹*fn phi => fn ctxt => fn ct => Less-Enat-Cancel.proc ctxt (Thm.term-of ct)*›

TODO: add regression tests for these simprocs

TODO: add simprocs for combining and cancelling numerals

36.9 Well-ordering

lemma *less-enatE*:
[| *n* < *enat m*; !!*k*. *n* = *enat k* ==> *k* < *m* ==> *P* |] ==> *P*
by (*induct n*) *auto*

lemma *less-infinityE*:
[| *n* < ∞; !!*k*. *n* = *enat k* ==> *P* |] ==> *P*
by (*induct n*) *auto*

lemma *enat-less-induct*:
assumes *prem*: ⋀*n*. ∀ *m*::*enat*. *m* < *n* ⟶ *P m* ⟹ *P n* **shows** *P n*
proof −
have *P-enat*: ⋀*k*. *P* (*enat k*)
apply (*rule nat-less-induct*)
apply (*rule prem*, *clarify*)
apply (*erule less-enatE*, *simp*)
done
show *?thesis*
proof (*induct n*)
fix *nat*
show *P* (*enat nat*) **by** (*rule P-enat*)
next
show *P* ∞
apply (*rule prem*, *clarify*)
apply (*erule less-infinityE*)
apply (*simp add*: *P-enat*)
done
qed
qed

instance *enat* :: *wellorder*
proof
fix *P* **and** *n*
assume *hyp*: (⋀*n*::*enat*. (⋀*m*::*enat*. *m* < *n* ⟹ *P m*) ⟹ *P n*)
show *P n* **by** (*blast intro*: *enat-less-induct hyp*)

qed

36.10 Complete Lattice

instantiation *enat* :: *complete-lattice*
begin

definition *inf-enat* :: *enat* ⇒ *enat* ⇒ *enat* **where**
inf-enat = *min*

definition *sup-enat* :: *enat* ⇒ *enat* ⇒ *enat* **where**
sup-enat = *max*

definition *Inf-enat* :: *enat set* ⇒ *enat* **where**
Inf-enat A = (*if A* = {} *then* ∞ *else* (*LEAST x*. *x* ∈ *A*))

definition *Sup-enat* :: *enat set* ⇒ *enat* **where**
Sup-enat A = (*if A* = {} *then 0 else if finite A then Max A else* ∞)
instance
proof
fix *x* :: *enat* **and** *A* :: *enat set*
{ assume *x* ∈ *A* **then show** *Inf A* ≤ *x*
unfolding *Inf-enat-def* **by** (*auto intro*: *Least-le*) **}**
{ assume ⋀*y*. *y* ∈ *A* ⟹ *x* ≤ *y* **then show** *x* ≤ *Inf A*
unfolding *Inf-enat-def*
by (*cases A* = {}) (*auto intro*: *LeastI2-ex*) **}**
{ assume *x* ∈ *A* **then show** *x* ≤ *Sup A*
unfolding *Sup-enat-def* **by** (*cases finite A*) *auto* **}**
{ assume ⋀*y*. *y* ∈ *A* ⟹ *y* ≤ *x* **then show** *Sup A* ≤ *x*
unfolding *Sup-enat-def* **using** *finite-enat-bounded* **by** *auto* **}**
qed (*simp-all add*:
inf-enat-def sup-enat-def bot-enat-def top-enat-def Inf-enat-def Sup-enat-def)
end

instance *enat* :: *complete-linorder* **..**

lemma *eSuc-Sup*: *A* ≠ {} ⟹ *eSuc* (*Sup A*) = *Sup* (*eSuc* ‘ *A*)
by(*auto simp add*: *Sup-enat-def eSuc-Max inj-on-def dest*: *finite-imageD*)

lemma *sup-continuous-eSuc*: *sup-continuous f* ⟹ *sup-continuous* (λ*x*. *eSuc* (*f x*))
using *eSuc-Sup* [*of* - ‘ *UNIV*] **by** (*auto simp*: *sup-continuous-def image-comp*)

36.11 Traditional theorem names

lemmas *enat-defs* = *zero-enat-def one-enat-def eSuc-def*
plus-enat-def less-eq-enat-def less-enat-def

lemma *iadd-is-0*: (*m* + *n* = (*0*::*enat*)) = (*m* = *0* ∧ *n* = *0*)
by (*rule add-eq-0-iff-both-eq-0*)

lemma *i0-lb* : $(0::enat) \leq n$
by (*rule zero-le*)

lemma *ile0-eq*: $n \leq (0::enat) \longleftrightarrow n = 0$
by (*rule le-zero-eq*)

lemma *not-iless0*: $\neg\ n < (0::enat)$
by (*rule not-less-zero*)

lemma *i0-less*[*simp*]: $(0::enat) < n \longleftrightarrow n \neq 0$
by (*rule zero-less-iff-neq-zero*)

lemma *imult-is-0*: $((m::enat) * n = 0) = (m = 0 \vee n = 0)$
by (*rule mult-eq-0-iff*)

end

37 Liminf and Limsup on conditionally complete lattices

theory *Liminf-Limsup*
imports *Complex-Main*
begin

lemma (**in** *conditionally-complete-linorder*) *le-cSup-iff*:
assumes $A \neq \{\}$ *bdd-above A*
shows $x \leq Sup\ A \longleftrightarrow (\forall\, y{<}x.\ \exists\, a{\in}A.\ y < a)$
proof *safe*
fix *y* **assume** $x \leq Sup\ A\ y < x$
then have $y < Sup\ A$ **by** *auto*
then show $\exists\, a{\in}A.\ y < a$
unfolding *less-cSup-iff*[*OF assms*] **.**
qed (*auto elim*!: *allE*[*of - Sup A*] *simp add*: *not-le*[*symmetric*] *cSup-upper assms*)

lemma (**in** *conditionally-complete-linorder*) *le-cSUP-iff*:
$A \neq \{\} \Longrightarrow$ *bdd-above* $(f`A) \Longrightarrow x \leq Sup\ (f\ `\ A) \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y < f\ i)$
using *le-cSup-iff* [*of f ` A*] **by** *simp*

lemma *le-cSup-iff-less*:
fixes $x :: 'a :: \{$*conditionally-complete-linorder*, *dense-linorder*$\}$
shows $A \neq \{\} \Longrightarrow$ *bdd-above* $(f`A) \Longrightarrow x \leq (SUP\ i{\in}A.\ f\ i) \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y \leq f\ i)$
by (*simp add*: *le-cSUP-iff*)
(*blast intro*: *less-imp-le less-trans less-le-trans dest*: *dense*)

lemma *le-Sup-iff-less*:
fixes $x :: 'a :: \{$*complete-linorder*, *dense-linorder*$\}$
shows $x \leq (SUP\ i{\in}A.\ f\ i) \longleftrightarrow (\forall\, y{<}x.\ \exists\, i{\in}A.\ y \leq f\ i)$ (**is** *?lhs = ?rhs*)

unfolding *le-SUP-iff*
by (*blast intro*: *less-imp-le less-trans less-le-trans dest*: *dense*)

lemma (**in** *conditionally-complete-linorder*) *cInf-le-iff*:
assumes $A \neq \{\}$ *bdd-below A*
shows *Inf A* $\leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, a{\in}A.\ y > a)$
proof *safe*
fix *y* **assume** $x \geq$ *Inf A* $y > x$
then have $y >$ *Inf A* **by** *auto*
then show $\exists\, a{\in}A.\ y > a$
unfolding *cInf-less-iff*[*OF assms*] **.**
qed (*auto elim*!: *allE*[*of - Inf A*] *simp add*: *not-le*[*symmetric*] *cInf-lower assms*)

lemma (**in** *conditionally-complete-linorder*) *cINF-le-iff*:
$A \neq \{\} \Longrightarrow$ *bdd-below* (*f*`*A*) $\Longrightarrow$ *Inf* (*f* ` *A*) $\leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ y > f\ i)$
using *cInf-le-iff* [*of f* ` *A*] **by** *simp*

lemma *cInf-le-iff-less*:
fixes *x* :: *'a* :: {*conditionally-complete-linorder*, *dense-linorder*}
shows $A \neq \{\} \Longrightarrow$ *bdd-below* (*f*`*A*) $\Longrightarrow$ (*INF* $i{\in}A.\ f\ i) \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ f\ i \leq y)$
by (*simp add*: *cINF-le-iff*)
(*blast intro*: *less-imp-le less-trans le-less-trans dest*: *dense*)

lemma *Inf-le-iff-less*:
fixes *x* :: *'a* :: {*complete-linorder*, *dense-linorder*}
shows (*INF* $i{\in}A.\ f\ i) \leq x \longleftrightarrow (\forall\, y{>}x.\ \exists\, i{\in}A.\ f\ i \leq y)$
unfolding *INF-le-iff*
by (*blast intro*: *less-imp-le less-trans le-less-trans dest*: *dense*)

lemma *SUP-pair*:
fixes *f* :: - ⇒ - ⇒ - :: *complete-lattice*
shows (*SUP* $i \in A.$ *SUP* $j \in B.\ f\ i\ j) =$ (*SUP* $p \in A \times B.\ f$ (*fst p*) (*snd p*))
by (*rule antisym*) (*auto intro*!: *SUP-least SUP-upper2*)

lemma *INF-pair*:
fixes *f* :: - ⇒ - ⇒ - :: *complete-lattice*
shows (*INF* $i \in A.$ *INF* $j \in B.\ f\ i\ j) =$ (*INF* $p \in A \times B.\ f$ (*fst p*) (*snd p*))
by (*rule antisym*) (*auto intro*!: *INF-greatest INF-lower2*)

lemma *INF-Sigma*:
fixes *f* :: - ⇒ - ⇒ - :: *complete-lattice*
shows (*INF* $i \in A.$ *INF* $j \in B\ i.\ f\ i\ j) =$ (*INF* $p \in$ *Sigma A B*. *f* (*fst p*) (*snd p*))
by (*rule antisym*) (*auto intro*!: *INF-greatest INF-lower2*)

37.0.1 *Liminf* and *Limsup*

definition *Liminf* :: *'a filter* ⇒ (*'a* ⇒ *'b*) ⇒ *'b* :: *complete-lattice* **where**
Liminf F f = (*SUP* $P{\in}\{P.$ *eventually P F*$\}$. *INF* $x{\in}\{x.\ P\ x\}.\ f\ x)$

definition *Limsup* :: *'a filter* ⇒ (*'a* ⇒ *'b*) ⇒ *'b* :: *complete-lattice* **where**
Limsup F f = (*INF P*∈{*P*. *eventually P F*}. *SUP x*∈{*x*. *P x*}. *f x*)

abbreviation *liminf* ≡ *Liminf sequentially*

abbreviation *limsup* ≡ *Limsup sequentially*

lemma *Liminf-eqI*:
(⋀*P*. *eventually P F* ⟹ *Inf* (*f* ‘ (*Collect P*)) ≤ *x*) ⟹
(⋀*y*. (⋀*P*. *eventually P F* ⟹ *Inf* (*f* ‘ (*Collect P*)) ≤ *y*) ⟹ *x* ≤ *y*) ⟹ *Liminf F f* = *x*
unfolding *Liminf-def* **by** (*auto intro*!: *SUP-eqI*)

lemma *Limsup-eqI*:
(⋀*P*. *eventually P F* ⟹ *x* ≤ *Sup* (*f* ‘ (*Collect P*))) ⟹
(⋀*y*. (⋀*P*. *eventually P F* ⟹ *y* ≤ *Sup* (*f* ‘ (*Collect P*))) ⟹ *y* ≤ *x*) ⟹ *Limsup F f* = *x*
unfolding *Limsup-def* **by** (*auto intro*!: *INF-eqI*)

lemma *liminf-SUP-INF*: *liminf f* = (*SUP n*. *INF m*∈{*n*..}. *f m*)
unfolding *Liminf-def eventually-sequentially*
by (*rule SUP-eq*) (*auto simp*: *atLeast-def intro*!: *INF-mono*)

lemma *limsup-INF-SUP*: *limsup f* = (*INF n*. *SUP m*∈{*n*..}. *f m*)
unfolding *Limsup-def eventually-sequentially*
by (*rule INF-eq*) (*auto simp*: *atLeast-def intro*!: *SUP-mono*)

lemma *mem-limsup-iff*: *x* ∈ *limsup A* ⟷ (∃$_F$ *n in sequentially*. *x* ∈ *A n*)
by (*simp add*: *Limsup-def*) (*metis* (*mono-tags*) *eventually-mono not-frequently*)

lemma *mem-liminf-iff*: *x* ∈ *liminf A* ⟷ (∀$_F$ *n in sequentially*. *x* ∈ *A n*)
by (*simp add*: *Liminf-def*) (*metis* (*mono-tags*) *eventually-mono*)

lemma *Limsup-const*:
assumes *ntriv*: ¬ *trivial-limit F*
shows *Limsup F* (λ*x*. *c*) = *c*
proof −
have ∗: ⋀*P*. *Ex P* ⟷ *P* ≠ (λ*x*. *False*) **by** *auto*
have ⋀*P*. *eventually P F* ⟹ (*SUP x* ∈ {*x*. *P x*}. *c*) = *c*
using *ntriv* **by** (*intro SUP-const*) (*auto simp*: *eventually-False* ∗)
then show *?thesis*
apply (*auto simp add*: *Limsup-def*)
apply (*rule INF-const*)
apply *auto*
using *eventually-True* **apply** *blast*
done
qed

lemma *Liminf-const*:
assumes *ntriv*: ¬ *trivial-limit F*
shows *Liminf F* (λ*x*. *c*) = *c*
proof −
have ∗: ⋀*P*. *Ex P* ⟷ *P* ≠ (λ*x*. *False*) **by** *auto*
have ⋀*P*. *eventually P F* ⟹ (*INF x* ∈ {*x*. *P x*}. *c*) = *c*
using *ntriv* **by** (*intro INF-const*) (*auto simp*: *eventually-False* ∗)
then show *?thesis*
apply (*auto simp add*: *Liminf-def*)
apply (*rule SUP-const*)
apply *auto*
using *eventually-True* **apply** *blast*
done
qed

lemma *Liminf-mono*:
assumes *ev*: *eventually* (λ*x*. *f x* ≤ *g x*) *F*
shows *Liminf F f* ≤ *Liminf F g*
unfolding *Liminf-def*
proof (*safe intro*!: *SUP-mono*)
fix *P* **assume** *eventually P F*
with *ev* **have** *eventually* (λ*x*. *f x* ≤ *g x* ∧ *P x*) *F* (**is** *eventually ?Q F*) **by** (*rule eventually-conj*)
then show ∃ *Q*∈{*P*. *eventually P F*}. *Inf* (*f* ‘ (*Collect P*)) ≤ *Inf* (*g* ‘ (*Collect Q*))
by (*intro bexI*[*of* - *?Q*]) (*auto intro*!: *INF-mono*)
qed

lemma *Liminf-eq*:
assumes *eventually* (λ*x*. *f x* = *g x*) *F*
shows *Liminf F f* = *Liminf F g*
by (*intro antisym Liminf-mono eventually-mono*[*OF assms*]) *auto*

lemma *Limsup-mono*:
assumes *ev*: *eventually* (λ*x*. *f x* ≤ *g x*) *F*
shows *Limsup F f* ≤ *Limsup F g*
unfolding *Limsup-def*
proof (*safe intro*!: *INF-mono*)
fix *P* **assume** *eventually P F*
with *ev* **have** *eventually* (λ*x*. *f x* ≤ *g x* ∧ *P x*) *F* (**is** *eventually ?Q F*) **by** (*rule eventually-conj*)
then show ∃ *Q*∈{*P*. *eventually P F*}. *Sup* (*f* ‘ (*Collect Q*)) ≤ *Sup* (*g* ‘ (*Collect P*))
by (*intro bexI*[*of* - *?Q*]) (*auto intro*!: *SUP-mono*)
qed

lemma *Limsup-eq*:
assumes *eventually* (λ*x*. *f x* = *g x*) *net*
shows *Limsup net f* = *Limsup net g*

by (*intro antisym Limsup-mono eventually-mono*[*OF assms*]) *auto*

lemma *Liminf-bot*[*simp*]: *Liminf bot f* = *top*
unfolding *Liminf-def top-unique*[*symmetric*]
by (*rule SUP-upper2*[**where** *i*=λ*x*. *False*]) *simp-all*

lemma *Limsup-bot*[*simp*]: *Limsup bot f* = *bot*
unfolding *Limsup-def bot-unique*[*symmetric*]
by (*rule INF-lower2*[**where** *i*=λ*x*. *False*]) *simp-all*

lemma *Liminf-le-Limsup*:
assumes *ntriv*: ¬ *trivial-limit F*
shows *Liminf F f* ≤ *Limsup F f*
unfolding *Limsup-def Liminf-def*
apply (*rule SUP-least*)
apply (*rule INF-greatest*)
proof *safe*
fix *P Q* **assume** *eventually P F eventually Q F*
then have *eventually* (λ*x*. *P x* ∧ *Q x*) *F* (**is** *eventually ?C F*) **by** (*rule eventually-conj*)
then have *not-False*: (λ*x*. *P x* ∧ *Q x*) ≠ (λ*x*. *False*)
using *ntriv* **by** (*auto simp add*: *eventually-False*)
have *Inf* (*f* ‘ (*Collect P*)) ≤ *Inf* (*f* ‘ (*Collect ?C*))
by (*rule INF-mono*) *auto*
also have ... ≤ *Sup* (*f* ‘ (*Collect ?C*))
using *not-False* **by** (*intro INF-le-SUP*) *auto*
also have ... ≤ *Sup* (*f* ‘ (*Collect Q*))
by (*rule SUP-mono*) *auto*
finally show *Inf* (*f* ‘ (*Collect P*)) ≤ *Sup* (*f* ‘ (*Collect Q*)) .
qed

lemma *Liminf-bounded*:
assumes *le*: *eventually* (λ*n*. *C* ≤ *X n*) *F*
shows *C* ≤ *Liminf F X*
using *Liminf-mono*[*OF le*] *Liminf-const*[*of F C*]
by (*cases F* = *bot*) *simp-all*

lemma *Limsup-bounded*:
assumes *le*: *eventually* (λ*n*. *X n* ≤ *C*) *F*
shows *Limsup F X* ≤ *C*
using *Limsup-mono*[*OF le*] *Limsup-const*[*of F C*]
by (*cases F* = *bot*) *simp-all*

lemma *le-Limsup*:
assumes *F*: *F* ≠ *bot* **and** *x*: ∀ *F x in F*. *l* ≤ *f x*
shows *l* ≤ *Limsup F f*
using *F Liminf-bounded*[*of l f F*] *Liminf-le-Limsup*[*of F f*] *order.trans x* **by** *blast*

lemma *Liminf-le*:

```
  assumes F: F ≠ bot and x: ∀ F x in F. f x ≤ l
  shows Liminf F f ≤ l
  using F Liminf-le-Limsup Limsup-bounded order.trans x by blast

lemma le-Liminf-iff:
  fixes X :: - ⇒ - :: complete-linorder
  shows C ≤ Liminf F X ⟷ (∀ y<C. eventually (λx. y < X x) F)
proof −
  have eventually (λx. y < X x) F
    if eventually P F y < Inf (X ‘ (Collect P)) for y P
    using that by (auto elim!: eventually-mono dest: less-INF-D)
  moreover
  have ∃ P. eventually P F ∧ y < Inf (X ‘ (Collect P))
    if y < C and y: ∀ y<C. eventually (λx. y < X x) F for y P
  proof (cases ∃ z. y < z ∧ z < C)
    case True
    then obtain z where z: y < z ∧ z < C ..
    moreover from z have z ≤ Inf (X ‘ {x. z < X x})
      by (auto intro!: INF-greatest)
    ultimately show ?thesis
      using y by (intro exI[of - λx. z < X x]) auto
  next
    case False
    then have C ≤ Inf (X ‘ {x. y < X x})
      by (intro INF-greatest) auto
    with ‹y < C› show ?thesis
      using y by (intro exI[of - λx. y < X x]) auto
  qed
  ultimately show ?thesis
    unfolding Liminf-def le-SUP-iff by auto
qed

lemma Limsup-le-iff:
  fixes X :: - ⇒ - :: complete-linorder
  shows C ≥ Limsup F X ⟷ (∀ y>C. eventually (λx. y > X x) F)
proof −
  { fix y P assume eventually P F y > Sup (X ‘ (Collect P))
    then have eventually (λx. y > X x) F
      by (auto elim!: eventually-mono dest: SUP-lessD) }
  moreover
  { fix y P assume y > C and y: ∀ y>C. eventually (λx. y > X x) F
    have ∃ P. eventually P F ∧ y > Sup (X ‘ (Collect P))
    proof (cases ∃ z. C < z ∧ z < y)
      case True
      then obtain z where z: C < z ∧ z < y ..
      moreover from z have z ≥ Sup (X ‘ {x. X x < z})
        by (auto intro!: SUP-least)
      ultimately show ?thesis
        using y by (intro exI[of - λx. z > X x]) auto
```

```
    next
      case False
      then have C ≥ Sup (X ‘ {x. X x < y})
        by (intro SUP-least) (auto simp: not-less)
      with ‹y > C› show ?thesis
        using y by (intro exI[of - λx. y > X x]) auto
    qed }
  ultimately show ?thesis
    unfolding Limsup-def INF-le-iff by auto
qed

lemma less-LiminfD:
  y < Liminf F (f :: - ⇒ 'a :: complete-linorder) ⟹ eventually (λx. f x > y) F
  using le-Liminf-iff[of Liminf F f F f] by simp

lemma Limsup-lessD:
  y > Limsup F (f :: - ⇒ 'a :: complete-linorder) ⟹ eventually (λx. f x < y) F
  using Limsup-le-iff[of F f Limsup F f] by simp

lemma lim-imp-Liminf:
  fixes f :: 'a ⇒ - :: {complete-linorder,linorder-topology}
  assumes ntriv: ¬ trivial-limit F
  assumes lim: (f ⟶ f0) F
  shows Liminf F f = f0
proof (intro Liminf-eqI)
  fix P assume P: eventually P F
  then have eventually (λx. Inf (f ‘ (Collect P)) ≤ f x) F
    by eventually-elim (auto intro!: INF-lower)
  then show Inf (f ‘ (Collect P)) ≤ f0
    by (rule tendsto-le[OF ntriv lim tendsto-const])
next
  fix y assume upper: ⋀P. eventually P F ⟹ Inf (f ‘ (Collect P)) ≤ y
  show f0 ≤ y
  proof cases
    assume ∃z. y < z ∧ z < f0
    then obtain z where y < z ∧ z < f0 ..
    moreover have z ≤ Inf (f ‘ {x. z < f x})
      by (rule INF-greatest) simp
    ultimately show ?thesis
      using lim[THEN topological-tendstoD, THEN upper, of {z <..}] by auto
  next
    assume discrete: ¬ (∃z. y < z ∧ z < f0)
    show ?thesis
    proof (rule classical)
      assume ¬ f0 ≤ y
      then have eventually (λx. y < f x) F
        using lim[THEN topological-tendstoD, of {y <..}] by auto
      then have eventually (λx. f0 ≤ f x) F
        using discrete by (auto elim!: eventually-mono)
```

```
    then have Inf (f ‘ {x. f0 ≤ f x}) ≤ y
      by (rule upper)
    moreover have f0 ≤ Inf (f ‘ {x. f0 ≤ f x})
      by (intro INF-greatest) simp
    ultimately show f0 ≤ y by simp
   qed
  qed
qed

lemma lim-imp-Limsup:
  fixes f :: 'a ⇒ - :: {complete-linorder,linorder-topology}
  assumes ntriv: ¬ trivial-limit F
  assumes lim: (f ⟶ f0) F
  shows Limsup F f = f0
proof (intro Limsup-eqI)
  fix P assume P: eventually P F
  then have eventually (λx. f x ≤ Sup (f ‘ (Collect P))) F
    by eventually-elim (auto intro!: SUP-upper)
  then show f0 ≤ Sup (f ‘ (Collect P))
    by (rule tendsto-le[OF ntriv tendsto-const lim])
next
  fix y assume lower: ⋀P. eventually P F ⟹ y ≤ Sup (f ‘ (Collect P))
  show y ≤ f0
  proof (cases ∃ z. f0 < z ∧ z < y)
    case True
    then obtain z where f0 < z ∧ z < y ..
    moreover have Sup (f ‘ {x. f x < z}) ≤ z
      by (rule SUP-least) simp
    ultimately show ?thesis
      using lim[THEN topological-tendstoD, THEN lower, OF {..< z}] by auto
  next
    case False
    show ?thesis
    proof (rule classical)
      assume ¬ y ≤ f0
      then have eventually (λx. f x < y) F
        using lim[THEN topological-tendstoD, OF {..< y}] by auto
      then have eventually (λx. f x ≤ f0) F
        using False by (auto elim!: eventually-mono simp: not-less)
      then have y ≤ Sup (f ‘ {x. f x ≤ f0})
        by (rule lower)
      moreover have Sup (f ‘ {x. f x ≤ f0}) ≤ f0
        by (intro SUP-least) simp
      ultimately show y ≤ f0 by simp
    qed
  qed
qed

lemma Liminf-eq-Limsup:
```

fixes *f0* :: *'a* :: {*complete-linorder*,*linorder-topology*}
assumes *ntriv*: ¬ *trivial-limit F*
and *lim*: *Liminf F f* = *f0 Limsup F f* = *f0*
shows (*f* ⟶ *f0*) *F*
proof (*rule order-tendstoI*)
fix *a* **assume** *f0* < *a*
with *assms* **have** *Limsup F f* < *a* **by** *simp*
then obtain *P* **where** *eventually P F Sup* (*f* ‘ (*Collect P*)) < *a*
unfolding *Limsup-def INF-less-iff* **by** *auto*
then show *eventually* (λ*x*. *f x* < *a*) *F*
by (*auto elim*!: *eventually-mono dest*: *SUP-lessD*)
next
fix *a* **assume** *a* < *f0*
with *assms* **have** *a* < *Liminf F f* **by** *simp*
then obtain *P* **where** *eventually P F a* < *Inf* (*f* ‘ (*Collect P*))
unfolding *Liminf-def less-SUP-iff* **by** *auto*
then show *eventually* (λ*x*. *a* < *f x*) *F*
by (*auto elim*!: *eventually-mono dest*: *less-INF-D*)
qed

lemma *tendsto-iff-Liminf-eq-Limsup*:
fixes *f0* :: *'a* :: {*complete-linorder*,*linorder-topology*}
shows ¬ *trivial-limit F* ⟹ (*f* ⟶ *f0*) *F* ⟷ (*Liminf F f* = *f0* ∧ *Limsup F f* = *f0*)
by (*metis Liminf-eq-Limsup lim-imp-Limsup lim-imp-Liminf*)

lemma *liminf-subseq-mono*:
fixes *X* :: *nat* ⇒ *'a* :: *complete-linorder*
assumes *strict-mono r*
shows *liminf X* ≤ *liminf* (*X* ∘ *r*)
proof−
have ⋀*n*. (*INF m*∈{*n*..}. *X m*) ≤ (*INF m*∈{*n*..}. (*X* ∘ *r*) *m*)
proof (*safe intro*!: *INF-mono*)
fix *n m* :: *nat* **assume** *n* ≤ *m* **then show** ∃ *ma*∈{*n*..}. *X ma* ≤ (*X* ∘ *r*) *m*
using *seq-suble*[*OF* ‹*strict-mono r*›, *of m*] **by** (*intro bexI*[*of* - *r m*]) *auto*
qed
then show *?thesis* **by** (*auto intro*!: *SUP-mono simp*: *liminf-SUP-INF comp-def*)
qed

lemma *limsup-subseq-mono*:
fixes *X* :: *nat* ⇒ *'a* :: *complete-linorder*
assumes *strict-mono r*
shows *limsup* (*X* ∘ *r*) ≤ *limsup X*
proof−
have (*SUP m*∈{*n*..}. (*X* ∘ *r*) *m*) ≤ (*SUP m*∈{*n*..}. *X m*) **for** *n*
proof (*safe intro*!: *SUP-mono*)
fix *m* :: *nat*
assume *n* ≤ *m*
then show ∃ *ma*∈{*n*..}. (*X* ∘ *r*) *m* ≤ *X ma*

```
    using seq-suble[OF ‹strict-mono r›, of m] by (intro bexI[of - r m]) auto
  qed
  then show ?thesis
    by (auto intro!: INF-mono simp: limsup-INF-SUP comp-def)
qed

lemma continuous-on-imp-continuous-within:
  continuous-on s f ⟹ t ⊆ s ⟹ x ∈ s ⟹ continuous (at x within t) f
  unfolding continuous-on-eq-continuous-within
  by (auto simp: continuous-within intro: tendsto-within-subset)

lemma Liminf-compose-continuous-mono:
  fixes f :: 'a::{complete-linorder, linorder-topology} ⇒ 'b::{complete-linorder, linorder-topology}
  assumes c: continuous-on UNIV f and am: mono f and F: F ≠ bot
  shows Liminf F (λn. f (g n)) = f (Liminf F g)
proof −
  { fix P assume eventually P F
    have ∃ x. P x
    proof (rule ccontr)
      assume ¬ (∃ x. P x) then have P = (λx. False)
        by auto
      with ‹eventually P F› F show False
        by auto
    qed }
  note ∗ = this

  have f (SUP P∈{P. eventually P F}. Inf (g ‘ Collect P)) =
    Sup (f ‘ (λP. Inf (g ‘ Collect P)) ‘ {P. eventually P F})
    using am continuous-on-imp-continuous-within [OF c]
    by (rule continuous-at-Sup-mono) (auto intro: eventually-True)
  then have f (Liminf F g) = (SUP P ∈ {P. eventually P F}. f (Inf (g ‘ Collect
P)))
    by (simp add: Liminf-def image-comp)
  also have ... = (SUP P ∈ {P. eventually P F}. Inf (f ‘ (g ‘ Collect P)))
    using ∗ continuous-at-Inf-mono [OF am continuous-on-imp-continuous-within
[OF c]]
    by auto
  finally show ?thesis by (auto simp: Liminf-def image-comp)
qed

lemma Limsup-compose-continuous-mono:
  fixes f :: 'a::{complete-linorder, linorder-topology} ⇒ 'b::{complete-linorder, linorder-topology}
  assumes c: continuous-on UNIV f and am: mono f and F: F ≠ bot
  shows Limsup F (λn. f (g n)) = f (Limsup F g)
proof −
  { fix P assume eventually P F
    have ∃ x. P x
    proof (rule ccontr)
      assume ¬ (∃ x. P x) then have P = (λx. False)
```

```
      by auto
    with ‹eventually P F› F show False
      by auto
  qed }
 note ∗ = this

 have f (INF P∈{P. eventually P F}. Sup (g ‘ Collect P)) =
   Inf (f ‘ (λP. Sup (g ‘ Collect P)) ‘ {P. eventually P F})
   using am continuous-on-imp-continuous-within [OF c]
   by (rule continuous-at-Inf-mono) (auto intro: eventually-True)
 then have f (Limsup F g) = (INF P ∈ {P. eventually P F}. f (Sup (g ‘ Collect
P)))
   by (simp add: Limsup-def image-comp)
 also have ... = (INF P ∈ {P. eventually P F}. Sup (f ‘ (g ‘ Collect P)))
   using ∗ continuous-at-Sup-mono [OF am continuous-on-imp-continuous-within
[OF c]]
   by auto
 finally show ?thesis by (auto simp: Limsup-def image-comp)
qed

lemma Liminf-compose-continuous-antimono:
 fixes f :: 'a::{complete-linorder,linorder-topology} ⇒ 'b::{complete-linorder,linorder-topology}
 assumes c: continuous-on UNIV f
   and am: antimono f
   and F: F ≠ bot
 shows Liminf F (λn. f (g n)) = f (Limsup F g)
proof −
 have ∗: ∃ x. P x if eventually P F for P
 proof (rule ccontr)
   assume ¬ (∃ x. P x) then have P = (λx. False)
     by auto
   with ‹eventually P F› F show False
     by auto
 qed

 have f (INF P∈{P. eventually P F}. Sup (g ‘ Collect P)) =
   Sup (f ‘ (λP. Sup (g ‘ Collect P)) ‘ {P. eventually P F})
   using am continuous-on-imp-continuous-within [OF c]
   by (rule continuous-at-Inf-antimono) (auto intro: eventually-True)
 then have f (Limsup F g) = (SUP P ∈ {P. eventually P F}. f (Sup (g ‘ Collect
P)))
   by (simp add: Limsup-def image-comp)
 also have ... = (SUP P ∈ {P. eventually P F}. Inf (f ‘ (g ‘ Collect P)))
  using ∗ continuous-at-Sup-antimono [OF am continuous-on-imp-continuous-within
[OF c]]
   by auto
 finally show ?thesis
   by (auto simp: Liminf-def image-comp)
qed
```

lemma *Limsup-compose-continuous-antimono*:
fixes $f :: 'a::\{complete\text{-}linorder, linorder\text{-}topology\} \Rightarrow 'b::\{complete\text{-}linorder, linorder\text{-}topology\}$
assumes *c*: *continuous-on UNIV f* **and** *am*: *antimono f* **and** *F*: $F \neq bot$
shows *Limsup F* $(\lambda n.\ f\ (g\ n)) = f\ (Liminf\ F\ g)$
proof −
{ fix *P* **assume** *eventually P F*
have $\exists x.\ P\ x$
proof (*rule ccontr*)
assume $\neg\ (\exists x.\ P\ x)$ **then have** $P = (\lambda x.\ False)$
by *auto*
with ‹*eventually P F*› *F* **show** *False*
by *auto*
qed }
note $* = this$

have *f* (*SUP P*∈{*P*. *eventually P F*}. *Inf* (*g* ‘ *Collect P*)) =
Inf (*f* ‘ (λ*P*. *Inf* (*g* ‘ *Collect P*)) ‘ {*P*. *eventually P F*})
using *am continuous-on-imp-continuous-within* [*OF c*]
by (*rule continuous-at-Sup-antimono*) (*auto intro*: *eventually-True*)
then have *f* (*Liminf F g*) = (*INF P* ∈ {*P*. *eventually P F*}. *f* (*Inf* (*g* ‘ *Collect P*)))
by (*simp add*: *Liminf-def image-comp*)
also have ... = (*INF P* ∈ {*P*. *eventually P F*}. *Sup* (*f* ‘ (*g* ‘ *Collect P*)))
using ∗ *continuous-at-Inf-antimono* [*OF am continuous-on-imp-continuous-within* [*OF c*]]
by *auto*
finally show *?thesis*
by (*auto simp*: *Limsup-def image-comp*)
qed

lemma *Liminf-filtermap-le*: *Liminf* (*filtermap f F*) *g* ≤ *Liminf F* (λ*x*. *g* (*f x*))
apply (*cases F* = *bot*, *simp*)
by (*subst Liminf-def*)
(*auto simp add*: *INF-lower Liminf-bounded eventually-filtermap eventually-mono intro*!: *SUP-least*)

lemma *Limsup-filtermap-ge*: *Limsup* (*filtermap f F*) *g* ≥ *Limsup F* (λ*x*. *g* (*f x*))
apply (*cases F* = *bot*, *simp*)
by (*subst Limsup-def*)
(*auto simp add*: *SUP-upper Limsup-bounded eventually-filtermap eventually-mono intro*!: *INF-greatest*)

lemma *Liminf-least*: (⋀*P*. *eventually P F* ⟹ (*INF x*∈*Collect P*. *f x*) ≤ *x*) ⟹ *Liminf F f* ≤ *x*
by (*auto intro*!: *SUP-least simp*: *Liminf-def*)

lemma *Limsup-greatest*: (⋀*P*. *eventually P F* ⟹ *x* ≤ (*SUP x*∈*Collect P*. *f x*)) ⟹ *Limsup F f* ≥ *x*

by (*auto intro*!: *INF-greatest simp*: *Limsup-def*)

lemma *Liminf-filtermap-ge*: *inj f* $\Longrightarrow$ *Liminf* (*filtermap f F*) *g* $\geq$ *Liminf F* (λx. *g* (*f x*))
apply (*cases F* = *bot*, *simp*)
apply (*rule Liminf-least*)
subgoal for *P*
by (*auto simp*: *eventually-filtermap the-inv-f-f*
intro!: *Liminf-bounded INF-lower2 eventually-mono*[*of P*])
done

lemma *Limsup-filtermap-le*: *inj f* $\Longrightarrow$ *Limsup* (*filtermap f F*) *g* $\leq$ *Limsup F* (λx. *g* (*f x*))
apply (*cases F* = *bot*, *simp*)
apply (*rule Limsup-greatest*)
subgoal for *P*
by (*auto simp*: *eventually-filtermap the-inv-f-f*
intro!: *Limsup-bounded SUP-upper2 eventually-mono*[*of P*])
done

lemma *Liminf-filtermap-eq*: *inj f* $\Longrightarrow$ *Liminf* (*filtermap f F*) *g* = *Liminf F* (λx. *g* (*f x*))
using *Liminf-filtermap-le*[*of f F g*] *Liminf-filtermap-ge*[*of f F g*]
by *simp*

lemma *Limsup-filtermap-eq*: *inj f* $\Longrightarrow$ *Limsup* (*filtermap f F*) *g* = *Limsup F* (λx. *g* (*f x*))
using *Limsup-filtermap-le*[*of f F g*] *Limsup-filtermap-ge*[*of F g f*]
by *simp*

37.1 More Limits

lemma *convergent-limsup-cl*:
fixes *X* :: *nat* $\Rightarrow$ *'a*::{*complete-linorder*,*linorder-topology*}
shows *convergent X* $\Longrightarrow$ *limsup X* = *lim X*
by (*auto simp*: *convergent-def limI lim-imp-Limsup*)

lemma *convergent-liminf-cl*:
fixes *X* :: *nat* $\Rightarrow$ *'a*::{*complete-linorder*,*linorder-topology*}
shows *convergent X* $\Longrightarrow$ *liminf X* = *lim X*
by (*auto simp*: *convergent-def limI lim-imp-Liminf*)

lemma *lim-increasing-cl*:
assumes $\bigwedge n\ m.\ n \geq m \Longrightarrow f\ n \geq f\ m$
obtains *l* **where** *f* $\longrightarrow$ (*l*::*'a*::{*complete-linorder*,*linorder-topology*})
proof
show *f* $\longrightarrow$ (*SUP n. f n*)
using *assms*
by (*intro increasing-tendsto*)

(*auto simp*: *SUP-upper eventually-sequentially less-SUP-iff intro*: *less-le-trans*)
qed

lemma *lim-decreasing-cl*:
assumes $\bigwedge n\ m.\ n \geq m \Longrightarrow f\ n \leq f\ m$
obtains l **where** $f \longrightarrow (l::'a::\{complete\text{-}linorder,linorder\text{-}topology\})$
proof
show $f \longrightarrow (INF\ n.\ f\ n)$
using *assms*
by (*intro decreasing-tendsto*)
(*auto simp*: *INF-lower eventually-sequentially INF-less-iff intro*: *le-less-trans*)
qed

lemma *compact-complete-linorder*:
fixes $X :: nat \Rightarrow 'a::\{complete\text{-}linorder,linorder\text{-}topology\}$
shows $\exists l\ r.\ strict\text{-}mono\ r \wedge (X \circ r) \longrightarrow l$
proof −
obtain r **where** *strict-mono r* **and** *mono*: *monoseq* $(X \circ r)$
using *seq-monosub*[*of X*]
unfolding *comp-def*
by *auto*
then have $(\forall n\ m.\ m \leq n \longrightarrow (X \circ r)\ m \leq (X \circ r)\ n) \vee (\forall n\ m.\ m \leq n \longrightarrow (X \circ r)\ n \leq (X \circ r)\ m)$
by (*auto simp add*: *monoseq-def*)
then obtain l **where** $(X \circ r) \longrightarrow l$
using *lim-increasing-cl*[*of X* ∘ *r*] *lim-decreasing-cl*[*of X* ∘ *r*]
by *auto*
then show *?thesis*
using ‹*strict-mono r*› **by** *auto*
qed

lemma *tendsto-Limsup*:
fixes $f :: - \Rightarrow 'a :: \{complete\text{-}linorder,linorder\text{-}topology\}$
shows $F \neq bot \Longrightarrow Limsup\ F\ f = Liminf\ F\ f \Longrightarrow (f \longrightarrow Limsup\ F\ f)\ F$
by (*subst tendsto-iff-Liminf-eq-Limsup*) *auto*

lemma *tendsto-Liminf*:
fixes $f :: - \Rightarrow 'a :: \{complete\text{-}linorder,linorder\text{-}topology\}$
shows $F \neq bot \Longrightarrow Limsup\ F\ f = Liminf\ F\ f \Longrightarrow (f \longrightarrow Liminf\ F\ f)\ F$
by (*subst tendsto-iff-Liminf-eq-Limsup*) *auto*

end

38 Extended real number line

theory *Extended-Real*
imports *Complex-Main Extended-Nat Liminf-Limsup*
begin

This should be part of *HOL−Library.Extended-Nat* or *HOL−Library.Order-Continuity*, but then the AFP-entry *Jinja-Thread* fails, as it does overload certain named from *Complex-Main*.

lemma *incseq-sumI2*:
fixes $f :: 'i \Rightarrow nat \Rightarrow 'a$::*ordered-comm-monoid-add*
shows $(\bigwedge n.\ n \in A \Longrightarrow mono\ (f\ n)) \Longrightarrow mono\ (\lambda i.\ \sum n{\in}A.\ f\ n\ i)$
unfolding *incseq-def* **by** (*auto intro*: *sum-mono*)

lemma *incseq-sumI*:
fixes $f :: nat \Rightarrow 'a$::*ordered-comm-monoid-add*
assumes $\bigwedge i.\ 0 \le f\ i$
shows *incseq* $(\lambda i.\ sum\ f\ \{..< i\})$
proof (*intro incseq-SucI*)
fix *n*
have $sum\ f\ \{..< n\} + 0 \le sum\ f\ \{..<n\} + f\ n$
using *assms* **by** (*rule add-left-mono*)
then show $sum\ f\ \{..< n\} \le sum\ f\ \{..< Suc\ n\}$
by *auto*
qed

lemma *continuous-at-left-imp-sup-continuous*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*} $\Rightarrow 'b$::{*complete-linorder*, *linorder-topology*}
assumes *mono f* $\bigwedge x.$ *continuous* (*at-left x*) *f*
shows *sup-continuous f*
unfolding *sup-continuous-def*
proof *safe*
fix $M :: nat \Rightarrow 'a$ **assume** *incseq M* **then show** $f\ (SUP\ i.\ M\ i) = (SUP\ i.\ f\ (M\ i))$
using *continuous-at-Sup-mono* [*OF assms*, *of range M*] **by** (*simp add*: *image-comp*)
qed

lemma *sup-continuous-at-left*:
fixes $f :: 'a$::{*complete-linorder*, *linorder-topology*, *first-countable-topology*} $\Rightarrow$
$'b$::{*complete-linorder*, *linorder-topology*}
assumes *f*: *sup-continuous f*
shows *continuous* (*at-left x*) *f*
proof *cases*
assume $x = bot$ **then show** *?thesis*
by (*simp add*: *trivial-limit-at-left-bot*)
next
assume *x*: $x \neq bot$
show *?thesis*
unfolding *continuous-within*
proof (*intro tendsto-at-left-sequentially*[*of bot*])
fix $S :: nat \Rightarrow 'a$ **assume** *S*: *incseq S* **and** *S-x*: $S \longrightarrow x$
from *S-x* **have** *x-eq*: $x = (SUP\ i.\ S\ i)$
by (*rule LIMSEQ-unique*) (*intro LIMSEQ-SUP S*)
show $(\lambda n.\ f\ (S\ n)) \longrightarrow f\ x$

```
    unfolding x-eq sup-continuousD[OF f S]
     using S sup-continuous-mono[OF f] by (intro LIMSEQ-SUP) (auto simp:
mono-def)
  qed (insert x, auto simp: bot-less)
qed

lemma sup-continuous-iff-at-left:
  fixes f :: 'a::{complete-linorder, linorder-topology, first-countable-topology} ⇒
    'b::{complete-linorder, linorder-topology}
  shows sup-continuous f ⟷ (∀ x. continuous (at-left x) f) ∧ mono f
  using sup-continuous-at-left[of f] continuous-at-left-imp-sup-continuous[of f]
    sup-continuous-mono[of f] by auto

lemma continuous-at-right-imp-inf-continuous:
 fixes f :: 'a::{complete-linorder, linorder-topology} ⇒ 'b::{complete-linorder, linorder-topology}
  assumes mono f ⋀x. continuous (at-right x) f
  shows inf-continuous f
  unfolding inf-continuous-def
proof safe
  fix M :: nat ⇒ 'a assume decseq M then show f (INF i. M i) = (INF i. f (M
i))
    using continuous-at-Inf-mono [OF assms, of range M] by (simp add: im-
age-comp)
qed

lemma inf-continuous-at-right:
  fixes f :: 'a::{complete-linorder, linorder-topology, first-countable-topology} ⇒
    'b::{complete-linorder, linorder-topology}
  assumes f: inf-continuous f
  shows continuous (at-right x) f
proof cases
  assume x = top then show ?thesis
    by (simp add: trivial-limit-at-right-top)
next
  assume x: x ≠ top
  show ?thesis
    unfolding continuous-within
  proof (intro tendsto-at-right-sequentially[of - top])
    fix S :: nat ⇒ 'a assume S: decseq S and S-x: S ⟶ x
    from S-x have x-eq: x = (INF i. S i)
      by (rule LIMSEQ-unique) (intro LIMSEQ-INF S)
    show (λn. f (S n)) ⟶ f x
      unfolding x-eq inf-continuousD[OF f S]
       using S inf-continuous-mono[OF f] by (intro LIMSEQ-INF) (auto simp:
mono-def antimono-def)
  qed (insert x, auto simp: less-top)
qed

lemma inf-continuous-iff-at-right:
```

fixes f :: $'a$::{*complete-linorder*, *linorder-topology*, *first-countable-topology*} ⇒ $'b$::{*complete-linorder*, *linorder-topology*}
shows *inf-continuous* $f \longleftrightarrow (\forall x.\ continuous\ (at\text{-}right\ x)\ f) \wedge mono\ f$
using *inf-continuous-at-right*[*of f*] *continuous-at-right-imp-inf-continuous*[*of f*] *inf-continuous-mono*[*of f*] **by** *auto*

instantiation *enat* :: *linorder-topology*
begin

definition *open-enat* :: *enat set* ⇒ *bool* **where**
open-enat = *generate-topology* (*range lessThan* ∪ *range greaterThan*)

instance
proof qed (*rule open-enat-def*)

end

lemma *open-enat*: *open* {*enat n*}
proof (*cases n*)
case *0*
then have {*enat n*} = {..< *eSuc 0*}
by (*auto simp*: *enat-0*)
then show *?thesis*
by *simp*
next
case (*Suc n'*)
then have {*enat n*} = {*enat n'* <..< *enat* (*Suc n*)}
using *enat-iless* **by** (*fastforce simp*: *set-eq-iff*)
then show *?thesis*
by *simp*
qed

lemma *open-enat-iff*:
fixes *A* :: *enat set*
shows *open A* ⟷ (∞ ∈ *A* ⟶ (∃ *n*::*nat*. {*n* <..} ⊆ *A*))
proof *safe*
assume ∞ ∉ *A*
then have *A* = (⋃ *n*∈{*n*. *enat n* ∈ *A*}. {*enat n*})
by (*simp add*: *set-eq-iff*) (*metis not-enat-eq*)
moreover have *open* ...
by (*auto intro*: *open-enat*)
ultimately show *open A*
by *simp*
next
fix *n* **assume** {*enat n* <..} ⊆ *A*
then have *A* = (⋃ *n*∈{*n*. *enat n* ∈ *A*}. {*enat n*}) ∪ {*enat n* <..}
using *enat-ile leI* **by** (*simp add*: *set-eq-iff*) *blast*
moreover have *open* ...
by (*intro open-Un open-UN ballI open-enat open-greaterThan*)

```
  ultimately show open A
    by simp
next
  assume open A ∞ ∈ A
  then have generate-topology (range lessThan ∪ range greaterThan) A ∞ ∈ A
    unfolding open-enat-def by auto
  then show ∃ n::nat. {n <..} ⊆ A
  proof induction
    case (Int A B)
    then obtain n m where {enat n<..} ⊆ A {enat m<..} ⊆ B
      by auto
    then have {enat (max n m) <..} ⊆ A ∩ B
      by (auto simp add: subset-eq Ball-def max-def simp flip: enat-ord-code(1))
    then show ?case
      by auto
  next
    case (UN K)
    then obtain k where k ∈ K ∞ ∈ k
      by auto
    with UN.IH[OF this] show ?case
      by auto
  qed auto
qed

lemma nhds-enat: nhds x = (if x = ∞ then INF i. principal {enat i..} else principal
{x})
proof auto
  show nhds ∞ = (INF i. principal {enat i..})
  proof (rule antisym)
    show nhds ∞ ≤ (INF i. principal {enat i..})
      unfolding nhds-def
    using Ioi-le-Ico by (intro INF-greatest INF-lower) (auto simp add: open-enat-iff)
    show (INF i. principal {enat i..}) ≤ nhds ∞
      unfolding nhds-def
    by (intro INF-greatest) (force intro: INF-lower2[of Suc -] simp add: open-enat-iff
Suc-ile-eq)
  qed
  show nhds (enat i) = principal {enat i} for i
    by (simp add: nhds-discrete-open open-enat)
qed

instance enat :: topological-comm-monoid-add
proof
  have [simp]: enat i ≤ aa ⟹ enat i ≤ aa + ba for aa ba i
    by (rule order-trans[OF - add-mono[of aa aa 0 ba]]) auto
  then have [simp]: enat i ≤ ba ⟹ enat i ≤ aa + ba for aa ba i
    by (metis add.commute)
  fix a b :: enat show ((λx. fst x + snd x) ⟶ a + b) (nhds a ×_F nhds b)
    apply (auto simp: nhds-enat filterlim-INF prod-filter-INF1 prod-filter-INF2
```

filterlim-principal principal-prod-principal eventually-principal)
subgoal for *i*
by (*auto intro*!: *eventually-INF1*[*of i*] *simp*: *eventually-principal*)
subgoal for *j i*
by (*auto intro*!: *eventually-INF1*[*of i*] *simp*: *eventually-principal*)
subgoal for *j i*
by (*auto intro*!: *eventually-INF1*[*of i*] *simp*: *eventually-principal*)
done
qed

For more lemmas about the extended real numbers see `~~/src/HOL/Analysis/Extended_Real_Limits.thy`.

38.1 Definition and basic properties

datatype *ereal* = *ereal real* | *PInfty* | *MInfty*

lemma *ereal-cong*: $x = y \Longrightarrow$ *ereal* $x =$ *ereal* y **by** *simp*

instantiation *ereal* :: *uminus*
begin

fun *uminus-ereal* **where**
$-$ (*ereal* r) = *ereal* ($-$ r)
| $-$ *PInfty* = *MInfty*
| $-$ *MInfty* = *PInfty*

instance ..

end

instantiation *ereal* :: *infinity*
begin

definition (∞::*ereal*) = *PInfty*
instance ..

end

declare [[*coercion ereal* :: *real* $\Rightarrow$ *ereal*]]

lemma *ereal-uminus-uminus*[*simp*]:
fixes a :: *ereal*
shows $-$ ($-$ a) = a
by (*cases a*) *simp-all*

lemma
shows *PInfty-eq-infinity*[*simp*]: *PInfty* = ∞
and *MInfty-eq-minfinity*[*simp*]: *MInfty* = $-$ ∞

and *MInfty-neq-PInfty*[*simp*]: $\infty \neq -(\infty::ereal) - \infty \neq (\infty::ereal)$
and *MInfty-neq-ereal*[*simp*]: $ereal\ r \neq -\infty - \infty \neq ereal\ r$
and *PInfty-neq-ereal*[*simp*]: $ereal\ r \neq \infty\ \infty \neq ereal\ r$
and *PInfty-cases*[*simp*]: $(case\ \infty\ of\ ereal\ r \Rightarrow f\ r \mid PInfty \Rightarrow y \mid MInfty \Rightarrow z) = y$
and *MInfty-cases*[*simp*]: $(case\ -\infty\ of\ ereal\ r \Rightarrow f\ r \mid PInfty \Rightarrow y \mid MInfty \Rightarrow z) = z$
by (*simp-all add*: *infinity-ereal-def*)

declare
PInfty-eq-infinity[*code-post*]
MInfty-eq-minfinity[*code-post*]

lemma [*code-unfold*]:
$\infty = PInfty$
$-\ PInfty = MInfty$
by *simp-all*

lemma *inj-ereal*[*simp*]: *inj-on ereal A*
unfolding *inj-on-def* **by** *auto*

lemma *ereal-cases*[*cases type*: *ereal*]:
obtains (*real*) r **where** $x = ereal\ r$
| (*PInf*) $x = \infty$
| (*MInf*) $x = -\infty$
by (*cases x*) *auto*

lemmas *ereal2-cases* = *ereal-cases*[*case-product ereal-cases*]
lemmas *ereal3-cases* = *ereal2-cases*[*case-product ereal-cases*]

lemma *ereal-all-split*: $\bigwedge P.\ (\forall x::ereal.\ P\ x) \longleftrightarrow P\ \infty \wedge (\forall x.\ P\ (ereal\ x)) \wedge P\ (-\infty)$
by (*metis ereal-cases*)

lemma *ereal-ex-split*: $\bigwedge P.\ (\exists x::ereal.\ P\ x) \longleftrightarrow P\ \infty \vee (\exists x.\ P\ (ereal\ x)) \vee P\ (-\infty)$
by (*metis ereal-cases*)

lemma *ereal-uminus-eq-iff*[*simp*]:
fixes $a\ b :: ereal$
shows $-a = -b \longleftrightarrow a = b$
by (*cases rule*: *ereal2-cases*[*of a b*]) *simp-all*

function *real-of-ereal* :: $ereal \Rightarrow real$ **where**
real-of-ereal (*ereal r*) = *r*
| *real-of-ereal* ∞ = *0*
| *real-of-ereal* $(-\infty)$ = *0*
by (*auto intro*: *ereal-cases*)
termination by *standard* (*rule wf-empty*)

lemma *real-of-ereal*[*simp*]:
  real-of-ereal (− *x* :: *ereal*) = − (*real-of-ereal x*)
  by (*cases x*) *simp-all*

lemma *range-ereal*[*simp*]: *range ereal* = *UNIV* − {∞, −∞}
proof *safe*
  fix *x*
  assume *x* ∉ *range ereal x* ≠ ∞
  then show *x* = −∞
    by (*cases x*) *auto*
qed *auto*

lemma *ereal-range-uminus*[*simp*]: *range uminus* = (*UNIV*::*ereal set*)
proof *safe*
  fix *x* :: *ereal*
  show *x* ∈ *range uminus*
    by (*intro image-eqI*[*of* - - −*x*]) *auto*
qed *auto*

instantiation *ereal* :: *abs*
begin

function *abs-ereal* **where**
  |*ereal r*| = *ereal* |*r*|
| |−∞| = (∞::*ereal*)
| |∞| = (∞::*ereal*)
by (*auto intro*: *ereal-cases*)
termination proof qed (*rule wf-empty*)

instance ..

end

lemma *abs-eq-infinity-cases*[*elim*!]:
  fixes *x* :: *ereal*
  assumes |*x*| = ∞
  obtains *x* = ∞ | *x* = −∞
  using *assms* **by** (*cases x*) *auto*

lemma *abs-neq-infinity-cases*[*elim*!]:
  fixes *x* :: *ereal*
  assumes |*x*| ≠ ∞
  obtains *r* **where** *x* = *ereal r*
  using *assms* **by** (*cases x*) *auto*

lemma *abs-ereal-uminus*[*simp*]:
  fixes *x* :: *ereal*
  shows |− *x*| = |*x*|

by (*cases x*) *auto*

lemma *ereal-infinity-cases*:
fixes *a* :: *ereal*
shows $a \neq \infty \Longrightarrow a \neq -\infty \Longrightarrow |a| \neq \infty$
by *auto*

38.1.1 Addition

instantiation *ereal* :: {*one*,*comm-monoid-add*,*zero-neq-one*}
begin

definition *0* = *ereal 0*
definition *1* = *ereal 1*

function *plus-ereal* **where**
ereal r + *ereal p* = *ereal* (*r* + *p*)
| ∞ + *a* = (∞::*ereal*)
| *a* + ∞ = (∞::*ereal*)
| *ereal r* + −∞ = − ∞
| −∞ + *ereal p* = −(∞::*ereal*)
| −∞ + −∞ = −(∞::*ereal*)
proof *goal-cases*
case *prems*: (*1 P x*)
then obtain *a b* **where** *x* = (*a*, *b*)
by (*cases x*) *auto*
with *prems* **show** *P*
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*
qed *auto*
termination by *standard* (*rule wf-empty*)

lemma *Infty-neq-0*[*simp*]:
(∞::*ereal*) ≠ *0 0* ≠ (∞::*ereal*)
−(∞::*ereal*) ≠ *0 0* ≠ −(∞::*ereal*)
by (*simp-all add*: *zero-ereal-def*)

lemma *ereal-eq-0*[*simp*]:
ereal r = *0* ⟷ *r* = *0*
0 = *ereal r* ⟷ *r* = *0*
unfolding *zero-ereal-def* **by** *simp-all*

lemma *ereal-eq-1*[*simp*]:
ereal r = *1* ⟷ *r* = *1*
1 = *ereal r* ⟷ *r* = *1*
unfolding *one-ereal-def* **by** *simp-all*

instance
proof
fix *a b c* :: *ereal*

show $0 + a = a$
by (*cases a*) (*simp-all add*: *zero-ereal-def*)
show $a + b = b + a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *simp-all*
show $a + b + c = a + (b + c)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) *simp-all*
show $0 \neq (1::ereal)$
by (*simp add*: *one-ereal-def zero-ereal-def*)
qed

end

lemma *ereal-0-plus* [*simp*]: *ereal 0* + *x* = *x*
and *plus-ereal-0* [*simp*]: *x* + *ereal 0* = *x*
by(*simp-all flip*: *zero-ereal-def*)

instance *ereal* :: *numeral* **..**

lemma *real-of-ereal-0*[*simp*]: *real-of-ereal* (*0*::*ereal*) = *0*
unfolding *zero-ereal-def* **by** *simp*

lemma *abs-ereal-zero*[*simp*]: |*0*| = (*0*::*ereal*)
unfolding *zero-ereal-def abs-ereal.simps* **by** *simp*

lemma *ereal-uminus-zero*[*simp*]: − *0* = (*0*::*ereal*)
by (*simp add*: *zero-ereal-def*)

lemma *ereal-uminus-zero-iff*[*simp*]:
fixes *a* :: *ereal*
shows $-a = 0 \longleftrightarrow a = 0$
by (*cases a*) *simp-all*

lemma *ereal-plus-eq-PInfty*[*simp*]:
fixes *a b* :: *ereal*
shows $a + b = \infty \longleftrightarrow a = \infty \vee b = \infty$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-plus-eq-MInfty*[*simp*]:
fixes *a b* :: *ereal*
shows $a + b = -\infty \longleftrightarrow (a = -\infty \vee b = -\infty) \wedge a \neq \infty \wedge b \neq \infty$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-add-cancel-left*:
fixes *a b* :: *ereal*
assumes $a \neq -\infty$
shows $a + b = a + c \longleftrightarrow a = \infty \vee b = c$
using *assms* **by** (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*

lemma *ereal-add-cancel-right*:

fixes $a\ b :: ereal$
assumes $a \neq -\infty$
shows $b + a = c + a \longleftrightarrow a = \infty \vee b = c$
using *assms* **by** (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*

lemma *ereal-real*: $ereal\ (real\text{-}of\text{-}ereal\ x) = (if\ |x| = \infty\ then\ 0\ else\ x)$
by (*cases x*) *simp-all*

lemma *real-of-ereal-add*:
fixes $a\ b :: ereal$
shows $real\text{-}of\text{-}ereal\ (a + b) =$
$(if\ (|a| = \infty) \wedge (|b| = \infty) \vee (|a| \neq \infty) \wedge (|b| \neq \infty)\ then\ real\text{-}of\text{-}ereal\ a + real\text{-}of\text{-}ereal\ b\ else\ 0)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

38.1.2 Linear order on *ereal*

instantiation *ereal* :: *linorder*
begin

function *less-ereal*
where
$ereal\ x < ereal\ y \longleftrightarrow x < y$
| $(\infty::ereal) < a \longleftrightarrow False$
| $a < -(\infty::ereal) \longleftrightarrow False$
| $ereal\ x < \infty \longleftrightarrow True$
| $-\infty < ereal\ r \longleftrightarrow True$
| $-\infty < (\infty::ereal) \longleftrightarrow True$
proof *goal-cases*
case *prems*: (*1 P x*)
then obtain *a b* **where** $x = (a,b)$ **by** (*cases x*) *auto*
with *prems* **show** *P* **by** (*cases rule*: *ereal2-cases*[*of a b*]) *auto*
qed *simp-all*
termination by (*relation* {}) *simp*

definition $x \leq (y::ereal) \longleftrightarrow x < y \vee x = y$

lemma *ereal-infty-less*[*simp*]:
fixes $x :: ereal$
shows $x < \infty \longleftrightarrow (x \neq \infty)$
$-\infty < x \longleftrightarrow (x \neq -\infty)$
by (*cases x*, *simp-all*) (*cases x*, *simp-all*)

lemma *ereal-infty-less-eq*[*simp*]:
fixes $x :: ereal$
shows $\infty \leq x \longleftrightarrow x = \infty$
and $x \leq -\infty \longleftrightarrow x = -\infty$
by (*auto simp add*: *less-eq-ereal-def*)

lemma *ereal-less*[*simp*]:
ereal r < 0 ⟷ (r < 0)
0 < ereal r ⟷ (0 < r)
ereal r < 1 ⟷ (r < 1)
1 < ereal r ⟷ (1 < r)
0 < (∞::ereal)
−(∞::ereal) < 0
by (*simp-all add*: *zero-ereal-def one-ereal-def*)

lemma *ereal-less-eq*[*simp*]:
x ≤ (∞::ereal)
−(∞::ereal) ≤ x
ereal r ≤ ereal p ⟷ r ≤ p
ereal r ≤ 0 ⟷ r ≤ 0
0 ≤ ereal r ⟷ 0 ≤ r
ereal r ≤ 1 ⟷ r ≤ 1
1 ≤ ereal r ⟷ 1 ≤ r
by (*auto simp add*: *less-eq-ereal-def zero-ereal-def one-ereal-def*)

lemma *ereal-infty-less-eq2*:
a ≤ b ⟹ a = ∞ ⟹ b = (∞::ereal)
a ≤ b ⟹ b = −∞ ⟹ a = −(∞::ereal)
by *simp-all*

instance
proof
fix *x y z* :: *ereal*
show *x ≤ x*
by (*cases x*) *simp-all*
show *x < y ⟷ x ≤ y ∧ ¬ y ≤ x*
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*
show *x ≤ y ∨ y ≤ x*
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*
{
assume *x ≤ y y ≤ x*
then show *x = y*
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*
}
{
assume *x ≤ y y ≤ z*
then show *x ≤ z*
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*
}
qed

end

lemma *ereal-dense2*: *x < y ⟹ ∃ z. x < ereal z ∧ ereal z < y*
using *lt-ex gt-ex dense* **by** (*cases x y rule*: *ereal2-cases*) *auto*

instance *ereal* :: *dense-linorder*
by *standard* (*blast dest*: *ereal-dense2*)

instance *ereal* :: *ordered-comm-monoid-add*
proof
fix *a b c* :: *ereal*
assume $a \leq b$
then show $c + a \leq c + b$
by (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*
qed

lemma *ereal-one-not-less-zero-ereal*[*simp*]: $\neg\ 1 < (0::ereal)$
by (*simp add*: *zero-ereal-def*)

lemma *real-of-ereal-positive-mono*:
fixes *x y* :: *ereal*
shows $0 \leq x \Longrightarrow x \leq y \Longrightarrow y \neq \infty \Longrightarrow$ *real-of-ereal x* $\leq$ *real-of-ereal y*
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*

lemma *ereal-MInfty-lessI*[*intro*, *simp*]:
fixes *a* :: *ereal*
shows $a \neq -\infty \Longrightarrow -\infty < a$
by (*cases a*) *auto*

lemma *ereal-less-PInfty*[*intro*, *simp*]:
fixes *a* :: *ereal*
shows $a \neq \infty \Longrightarrow a < \infty$
by (*cases a*) *auto*

lemma *ereal-less-ereal-Ex*:
fixes *a b* :: *ereal*
shows $x <$ *ereal r* $\longleftrightarrow x = -\infty \vee (\exists\, p.\ p < r \wedge x =$ *ereal p*$)$
by (*cases x*) *auto*

lemma *less-PInf-Ex-of-nat*: $x \neq \infty \longleftrightarrow (\exists\, n::nat.\ x <$ *ereal* (*real n*))
proof (*cases x*)
case (*real r*)
then show *?thesis*
using *reals-Archimedean2*[*of r*] **by** *simp*
qed *simp-all*

lemma *ereal-add-strict-mono2*:
fixes *a b c d* :: *ereal*
assumes $a < b$ **and** $c < d$
shows $a + c < b + d$
using *assms*
by (*cases a*; *force simp add*: *elim*: *less-ereal.elims*)

lemma *ereal-minus-le-minus*[*simp*]:
fixes *a b* :: *ereal*
shows $-a \leq -b \longleftrightarrow b \leq a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-minus-less-minus*[*simp*]:
fixes *a b* :: *ereal*
shows $-a < -b \longleftrightarrow b < a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-le-real-iff*:
$x \leq$ *real-of-ereal* $y \longleftrightarrow (|y| \neq \infty \longrightarrow$ *ereal* $x \leq y) \wedge (|y| = \infty \longrightarrow x \leq 0)$
by (*cases y*) *auto*

lemma *real-le-ereal-iff*:
real-of-ereal $y \leq x \longleftrightarrow (|y| \neq \infty \longrightarrow y \leq$ *ereal* $x) \wedge (|y| = \infty \longrightarrow 0 \leq x)$
by (*cases y*) *auto*

lemma *ereal-less-real-iff*:
$x <$ *real-of-ereal* $y \longleftrightarrow (|y| \neq \infty \longrightarrow$ *ereal* $x < y) \wedge (|y| = \infty \longrightarrow x < 0)$
by (*cases y*) *auto*

lemma *real-less-ereal-iff*:
real-of-ereal $y < x \longleftrightarrow (|y| \neq \infty \longrightarrow y <$ *ereal* $x) \wedge (|y| = \infty \longrightarrow 0 < x)$
by (*cases y*) *auto*

To help with inferences like $[\![a <$ *ereal* $x; x < y]\!] \Longrightarrow a <$ *ereal* y, where x and y are real.

lemma *le-ereal-le*: $a \leq$ *ereal* $x \Longrightarrow x \leq y \Longrightarrow a \leq$ *ereal* y
using *ereal-less-eq*(*3*) *order.trans* **by** *blast*

lemma *le-ereal-less*: $a \leq$ *ereal* $x \Longrightarrow x < y \Longrightarrow a <$ *ereal* y
by (*simp add*: *le-less-trans*)

lemma *less-ereal-le*: $a <$ *ereal* $x \Longrightarrow x \leq y \Longrightarrow a <$ *ereal* y
using *ereal-less-ereal-Ex* **by** *auto*

lemma *ereal-le-le*: *ereal* $y \leq a \Longrightarrow x \leq y \Longrightarrow$ *ereal* $x \leq a$
by (*simp add*: *order-subst2*)

lemma *ereal-le-less*: *ereal* $y \leq a \Longrightarrow x < y \Longrightarrow$ *ereal* $x < a$
by (*simp add*: *dual-order.strict-trans1*)

lemma *ereal-less-le*: *ereal* $y < a \Longrightarrow x \leq y \Longrightarrow$ *ereal* $x < a$
using *ereal-less-eq*(*3*) *le-less-trans* **by** *blast*

lemma *real-of-ereal-pos*:
fixes *x* :: *ereal*
shows $0 \leq x \Longrightarrow 0 \leq$ *real-of-ereal* x **by** (*cases x*) *auto*

lemmas *real-of-ereal-ord-simps* =
ereal-le-real-iff real-le-ereal-iff ereal-less-real-iff real-less-ereal-iff

lemma *abs-ereal-ge0*[*simp*]: $0 \leq x \Longrightarrow |x :: ereal| = x$
by (*cases x*) *auto*

lemma *abs-ereal-less0*[*simp*]: $x < 0 \Longrightarrow |x :: ereal| = -x$
by (*cases x*) *auto*

lemma *abs-ereal-pos*[*simp*]: $0 \leq |x :: ereal|$
by (*cases x*) *auto*

lemma *ereal-abs-leI*:
fixes *x y* :: *ereal*
shows $[\![x \leq y; -x \leq y]\!] \Longrightarrow |x| \leq y$
by(*cases x y rule*: *ereal2-cases*)(*simp-all*)

lemma *ereal-abs-add*:
fixes *a b*::*ereal*
shows *abs*(*a*+*b*) ≤ *abs a* + *abs b*
by (*cases rule*: *ereal2-cases*[*of a b*]) (*auto*)

lemma *real-of-ereal-le-0*[*simp*]: *real-of-ereal* (*x* :: *ereal*) ≤ *0* ⟷ *x* ≤ *0* ∨ *x* = ∞
by (*cases x*) *auto*

lemma *abs-real-of-ereal*[*simp*]: |*real-of-ereal* (*x* :: *ereal*)| = *real-of-ereal* |*x*|
by (*cases x*) *auto*

lemma *zero-less-real-of-ereal*:
fixes *x* :: *ereal*
shows *0* < *real-of-ereal x* ⟷ *0* < *x* ∧ *x* ≠ ∞
by (*cases x*) *auto*

lemma *ereal-0-le-uminus-iff*[*simp*]:
fixes *a* :: *ereal*
shows *0* ≤ − *a* ⟷ *a* ≤ *0*
by (*cases rule*: *ereal2-cases*[*of a*]) *auto*

lemma *ereal-uminus-le-0-iff*[*simp*]:
fixes *a* :: *ereal*
shows − *a* ≤ *0* ⟷ *0* ≤ *a*
by (*cases rule*: *ereal2-cases*[*of a*]) *auto*

lemma *ereal-add-strict-mono*:
fixes *a b c d* :: *ereal*
assumes *a* ≤ *b*
and *0* ≤ *a*
and *a* ≠ ∞

and $c < d$
shows $a + c < b + d$
using *assms*
by (*cases rule*: *ereal3-cases*[*case-product ereal-cases*, *of a b c d*]) *auto*

lemma *ereal-less-add*:
fixes $a\ b\ c$ :: *ereal*
shows $|a| \neq \infty \Longrightarrow c < b \Longrightarrow a + c < a + b$
by (*cases rule*: *ereal2-cases*[*of b c*]) *auto*

lemma *ereal-add-nonneg-eq-0-iff*:
fixes $a\ b$ :: *ereal*
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a + b = 0 \longleftrightarrow a = 0 \wedge b = 0$
by (*cases a b rule*: *ereal2-cases*) *auto*

lemma *ereal-uminus-eq-reorder*: $-\ a = b \longleftrightarrow a = (-b::ereal)$
by *auto*

lemma *ereal-uminus-less-reorder*: $-\ a < b \longleftrightarrow -b < (a::ereal)$
by (*subst* (*3*) *ereal-uminus-uminus*[*symmetric*]) (*simp only*: *ereal-minus-less-minus*)

lemma *ereal-less-uminus-reorder*: $a < -\ b \longleftrightarrow b < -\ (a::ereal)$
by (*subst* (*3*) *ereal-uminus-uminus*[*symmetric*]) (*simp only*: *ereal-minus-less-minus*)

lemma *ereal-uminus-le-reorder*: $-\ a \leq b \longleftrightarrow -b \leq (a::ereal)$
by (*subst* (*3*) *ereal-uminus-uminus*[*symmetric*]) (*simp only*: *ereal-minus-le-minus*)

lemmas *ereal-uminus-reorder* =
ereal-uminus-eq-reorder ereal-uminus-less-reorder ereal-uminus-le-reorder

lemma *ereal-bot*:
fixes x :: *ereal*
assumes $\bigwedge B.\ x \leq ereal\ B$
shows $x = -\ \infty$
proof (*cases x*)
case (*real r*)
with *assms*[*of r − 1*] **show** *?thesis*
by *auto*
next
case *PInf*
with *assms*[*of 0*] **show** *?thesis*
by *auto*
next
case *MInf*
then show *?thesis*
by *simp*
qed

lemma *ereal-top*:

fixes x :: *ereal*
assumes $\bigwedge B.\ x \geq ereal\ B$
shows $x = \infty$
proof (*cases x*)
case (*real r*)
with *assms*[*of r + 1*] **show** *?thesis*
by *auto*
next
case *MInf*
with *assms*[*of 0*] **show** *?thesis*
by *auto*
next
case *PInf*
then show *?thesis*
by *simp*
qed

lemma
shows *ereal-max*[*simp*]: *ereal* (*max x y*) = *max* (*ereal x*) (*ereal y*)
and *ereal-min*[*simp*]: *ereal* (*min x y*) = *min* (*ereal x*) (*ereal y*)
by (*simp-all add*: *min-def max-def*)

lemma *ereal-max-0*: *max 0* (*ereal r*) = *ereal* (*max 0 r*)
by (*auto simp*: *zero-ereal-def*)

lemma
fixes f :: *nat* $\Rightarrow$ *ereal*
shows *ereal-incseq-uminus*[*simp*]: *incseq* ($\lambda x.\ -f\ x$) $\longleftrightarrow$ *decseq f*
and *ereal-decseq-uminus*[*simp*]: *decseq* ($\lambda x.\ -f\ x$) $\longleftrightarrow$ *incseq f*
unfolding *decseq-def incseq-def* **by** *auto*

lemma *incseq-ereal*: *incseq f* $\Longrightarrow$ *incseq* ($\lambda x.$ *ereal* (*f x*))
unfolding *incseq-def* **by** *auto*

lemma *sum-ereal*[*simp*]: ($\sum x{\in}A.$ *ereal* (*f x*)) = *ereal* ($\sum x{\in}A.$ *f x*)
proof (*cases finite A*)
case *True*
then show *?thesis* **by** *induct auto*
next
case *False*
then show *?thesis* **by** *simp*
qed

lemma *sum-list-ereal* [*simp*]: *sum-list* (*map* ($\lambda x.$ *ereal* (*f x*)) *xs*) = *ereal* (*sum-list* (*map f xs*))
by (*induction xs*) *simp-all*

lemma *sum-Pinfty*:
fixes f :: $'a \Rightarrow$ *ereal*

```
  shows (∑x∈P. f x) = ∞ ⟷ finite P ∧ (∃ i∈P. f i = ∞)
proof safe
  assume ∗: sum f P = ∞
  show finite P
  proof (rule ccontr)
    assume ¬ finite P
    with ∗ show False
      by auto
  qed
  show ∃ i∈P. f i = ∞
  proof (rule ccontr)
    assume ¬ ?thesis
    then have ⋀i. i ∈ P ⟹ f i ≠ ∞
      by auto
    with ‹finite P› have sum f P ≠ ∞
      by induct auto
    with ∗ show False
      by auto
  qed
next
  fix i
  assume finite P and i ∈ P and f i = ∞
  then show sum f P = ∞
  proof induct
    case (insert x A)
    show ?case using insert by (cases x = i) auto
  qed simp
qed

lemma sum-Inf:
  fixes f :: 'a ⇒ ereal
  shows |sum f A| = ∞ ⟷ finite A ∧ (∃ i∈A. |f i| = ∞)
proof
  assume ∗: |sum f A| = ∞
  have finite A
    by (rule ccontr) (insert ∗, auto)
  moreover have ∃ i∈A. |f i| = ∞
  proof (rule ccontr)
    assume ¬ ?thesis
    then have ∀ i∈A. ∃ r. f i = ereal r
      by auto
    from bchoice[OF this] obtain r where ∀ x∈A. f x = ereal (r x) ..
    with ∗ show False
      by auto
  qed
  ultimately show finite A ∧ (∃ i∈A. |f i| = ∞)
    by auto
next
  assume finite A ∧ (∃ i∈A. |f i| = ∞)
```

```
  then obtain i where finite A i ∈ A and |f i| = ∞
    by auto
  then show |sum f A| = ∞
  proof induct
    case (insert j A)
    then show ?case
      by (cases rule: ereal3-cases[of f i f j sum f A]) auto
  qed simp
qed

lemma sum-real-of-ereal:
  fixes f :: 'i ⇒ ereal
  assumes ⋀x. x ∈ S ⟹ |f x| ≠ ∞
  shows (∑ x∈S. real-of-ereal (f x)) = real-of-ereal (sum f S)
proof −
  have ∀ x∈S. ∃ r. f x = ereal r
  proof
    fix x
    assume x ∈ S
    from assms[OF this] show ∃ r. f x = ereal r
      by (cases f x) auto
  qed
  from bchoice[OF this] obtain r where ∀ x∈S. f x = ereal (r x) ..
  then show ?thesis
    by simp
qed

lemma sum-ereal-0:
  fixes f :: 'a ⇒ ereal
  assumes finite A
    and ⋀i. i ∈ A ⟹ 0 ≤ f i
  shows (∑ x∈A. f x) = 0 ⟷ (∀ i∈A. f i = 0)
proof
  assume sum f A = 0 with assms show ∀ i∈A. f i = 0
  proof (induction A)
    case (insert a A)
    then have f a = 0 ∧ (∑ a∈A. f a) = 0
      by (subst ereal-add-nonneg-eq-0-iff[symmetric]) (simp-all add: sum-nonneg)
    with insert show ?case
      by simp
  qed simp
qed auto
```

38.1.3 Multiplication

```
instantiation ereal :: {comm-monoid-mult,sgn}
begin

function sgn-ereal :: ereal ⇒ ereal where
```

```
  sgn (ereal r) = ereal (sgn r)
| sgn (∞::ereal) = 1
| sgn (−∞::ereal) = −1
by (auto intro: ereal-cases)
termination by standard (rule wf-empty)

function times-ereal where
  ereal r * ereal p = ereal (r * p)
| ereal r * ∞ = (if r = 0 then 0 else if r > 0 then ∞ else −∞)
| ∞ * ereal r = (if r = 0 then 0 else if r > 0 then ∞ else −∞)
| ereal r * −∞ = (if r = 0 then 0 else if r > 0 then −∞ else ∞)
| −∞ * ereal r = (if r = 0 then 0 else if r > 0 then −∞ else ∞)
| (∞::ereal) * ∞ = ∞
| −(∞::ereal) * ∞ = −∞
| (∞::ereal) * −∞ = −∞
| −(∞::ereal) * −∞ = ∞
proof goal-cases
  case prems: (1 P x)
  then obtain a b where x = (a, b)
    by (cases x) auto
  with prems show P
    by (cases rule: ereal2-cases[of a b]) auto
qed simp-all
termination by (relation {}) simp

instance
proof
  fix a b c :: ereal
  show 1 * a = a
    by (cases a) (simp-all add: one-ereal-def)
  show a * b = b * a
    by (cases rule: ereal2-cases[of a b]) simp-all
  show a * b * c = a * (b * c)
    by (cases rule: ereal3-cases[of a b c])
       (simp-all add: zero-ereal-def zero-less-mult-iff)
qed

end

lemma [simp]:
  shows ereal-1-times: ereal 1 * x = x
  and times-ereal-1: x * ereal 1 = x
by(simp-all flip: one-ereal-def)

lemma one-not-le-zero-ereal[simp]: ¬ (1 ≤ (0::ereal))
  by (simp add: one-ereal-def zero-ereal-def)

lemma real-ereal-1[simp]: real-of-ereal (1::ereal) = 1
  unfolding one-ereal-def by simp
```

lemma *real-of-ereal-le-1*:
fixes a :: *ereal*
shows $a \leq 1 \Longrightarrow$ *real-of-ereal* $a \leq 1$
by (*cases a*) (*auto simp*: *one-ereal-def*)

lemma *abs-ereal-one*[*simp*]: $|1| = (1::ereal)$
unfolding *one-ereal-def* **by** *simp*

lemma *ereal-mult-zero*[*simp*]:
fixes a :: *ereal*
shows $a * 0 = 0$
by (*cases a*) (*simp-all add*: *zero-ereal-def*)

lemma *ereal-zero-mult*[*simp*]:
fixes a :: *ereal*
shows $0 * a = 0$
by (*cases a*) (*simp-all add*: *zero-ereal-def*)

lemma *ereal-m1-less-0*[*simp*]: $-(1::ereal) < 0$
by (*simp add*: *zero-ereal-def one-ereal-def*)

lemma *ereal-times*[*simp*]:
$1 \neq (\infty::ereal)$ $(\infty::ereal) \neq 1$
$1 \neq -(\infty::ereal)$ $-(\infty::ereal) \neq 1$
by (*auto simp*: *one-ereal-def*)

lemma *ereal-plus-1*[*simp*]:
$1 + ereal\ r = ereal\ (r + 1)$
$ereal\ r + 1 = ereal\ (r + 1)$
$1 + -(\infty::ereal) = -\infty$
$-(\infty::ereal) + 1 = -\infty$
unfolding *one-ereal-def* **by** *auto*

lemma *ereal-zero-times*[*simp*]:
fixes $a\ b$ :: *ereal*
shows $a * b = 0 \longleftrightarrow a = 0 \vee b = 0$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-eq-PInfty*[*simp*]:
$a * b = (\infty::ereal) \longleftrightarrow$
$(a = \infty \wedge b > 0) \vee (a > 0 \wedge b = \infty) \vee (a = -\infty \wedge b < 0) \vee (a < 0 \wedge b = -\infty)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-eq-MInfty*[*simp*]:
$a * b = -(\infty::ereal) \longleftrightarrow$
$(a = \infty \wedge b < 0) \vee (a < 0 \wedge b = \infty) \vee (a = -\infty \wedge b > 0) \vee (a > 0 \wedge b = -\infty)$

by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-abs-mult*: $|x * y :: ereal| = |x| * |y|$
by (*cases x y rule*: *ereal2-cases*) (*auto simp*: *abs-mult*)

lemma *ereal-0-less-1*[*simp*]: $0 < (1::ereal)$
by (*simp-all add*: *zero-ereal-def one-ereal-def*)

lemma *ereal-mult-minus-left*[*simp*]:
fixes $a\ b :: ereal$
shows $-a * b = -(a * b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-minus-right*[*simp*]:
fixes $a\ b :: ereal$
shows $a * -b = -(a * b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-mult-infty*[*simp*]:
$a * (\infty::ereal) = (if\ a = 0\ then\ 0\ else\ if\ 0 < a\ then\ \infty\ else\ -\infty)$
by (*cases a*) *auto*

lemma *ereal-infty-mult*[*simp*]:
$(\infty::ereal) * a = (if\ a = 0\ then\ 0\ else\ if\ 0 < a\ then\ \infty\ else\ -\infty)$
by (*cases a*) *auto*

lemma *ereal-mult-strict-right-mono*:
assumes $a < b$
and $0 < c$
and $c < (\infty::ereal)$
shows $a * c < b * c$
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*auto simp*: *zero-le-mult-iff*)

lemma *ereal-mult-strict-left-mono*:
$a < b \Longrightarrow 0 < c \Longrightarrow c < (\infty::ereal) \Longrightarrow c * a < c * b$
using *ereal-mult-strict-right-mono*
by (*simp add*: *mult.commute*[*of c*])

lemma *ereal-mult-right-mono*:
fixes $a\ b\ c :: ereal$
assumes $a \leq b\ 0 \leq c$
shows $a * c \leq b * c$
proof (*cases* $c = 0$)
case *False*
with *assms* **show** *?thesis*
by (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*
qed *auto*

lemma *ereal-mult-left-mono*:
fixes *a b c* :: *ereal*
shows $a \leq b \Longrightarrow 0 \leq c \Longrightarrow c * a \leq c * b$
using *ereal-mult-right-mono*
by (*simp add*: *mult.commute*[*of c*])

lemma *ereal-mult-mono*:
fixes *a b c d*::*ereal*
assumes $b \geq 0$ $c \geq 0$ $a \leq b$ $c \leq d$
shows $a * c \leq b * d$
by (*metis ereal-mult-right-mono mult.commute order-trans assms*)

lemma *ereal-mult-mono'*:
fixes *a b c d*::*ereal*
assumes $a \geq 0$ $c \geq 0$ $a \leq b$ $c \leq d$
shows $a * c \leq b * d$
by (*metis ereal-mult-right-mono mult.commute order-trans assms*)

lemma *ereal-mult-mono-strict*:
fixes *a b c d*::*ereal*
assumes $b > 0$ $c > 0$ $a < b$ $c < d$
shows $a * c < b * d$
proof −
have $c < \infty$ **using** ‹$c < d$› **by** *auto*
then have $a * c < b * c$ **by** (*metis ereal-mult-strict-left-mono*[*OF assms*(*3*) *assms*(*2*)] *mult.commute*)
moreover have $b * c \leq b * d$ **using** *assms*(*2*) *assms*(*4*) **by** (*simp add*: *assms*(*1*) *ereal-mult-left-mono less-imp-le*)
ultimately show *?thesis* **by** *simp*
qed

lemma *ereal-mult-mono-strict'*:
fixes *a b c d*::*ereal*
assumes $a > 0$ $c > 0$ $a < b$ $c < d$
shows $a * c < b * d$
using *assms ereal-mult-mono-strict* **by** *auto*

lemma *zero-less-one-ereal*[*simp*]: $0 \leq (1{::}ereal)$
by (*simp add*: *one-ereal-def zero-ereal-def*)

lemma *ereal-0-le-mult*[*simp*]: $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow 0 \leq a * (b :: ereal)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-right-distrib*:
fixes *r a b* :: *ereal*
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow r * (a + b) = r * a + r * b$
by (*cases rule*: *ereal3-cases*[*of r a b*]) (*simp-all add*: *field-simps*)

lemma *ereal-left-distrib*:

fixes r a b :: *ereal*
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow (a + b) * r = a * r + b * r$
by (*cases rule*: *ereal3-cases*[*of r a b*]) (*simp-all add*: *field-simps*)

lemma *ereal-mult-le-0-iff*:
fixes a b :: *ereal*
shows $a * b \leq 0 \longleftrightarrow (0 \leq a \wedge b \leq 0) \vee (a \leq 0 \wedge 0 \leq b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *mult-le-0-iff*)

lemma *ereal-zero-le-0-iff*:
fixes a b :: *ereal*
shows $0 \leq a * b \longleftrightarrow (0 \leq a \wedge 0 \leq b) \vee (a \leq 0 \wedge b \leq 0)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *zero-le-mult-iff*)

lemma *ereal-mult-less-0-iff*:
fixes a b :: *ereal*
shows $a * b < 0 \longleftrightarrow (0 < a \wedge b < 0) \vee (a < 0 \wedge 0 < b)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *mult-less-0-iff*)

lemma *ereal-zero-less-0-iff*:
fixes a b :: *ereal*
shows $0 < a * b \longleftrightarrow (0 < a \wedge 0 < b) \vee (a < 0 \wedge b < 0)$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*simp-all add*: *zero-less-mult-iff*)

lemma *ereal-left-mult-cong*:
fixes a b c :: *ereal*
shows $c = d \Longrightarrow (d \neq 0 \Longrightarrow a = b) \Longrightarrow a * c = b * d$
by (*cases* $c = 0$) *simp-all*

lemma *ereal-right-mult-cong*:
fixes a b c :: *ereal*
shows $c = d \Longrightarrow (d \neq 0 \Longrightarrow a = b) \Longrightarrow c * a = d * b$
by (*cases* $c = 0$) *simp-all*

lemma *ereal-distrib*:
fixes a b c :: *ereal*
assumes $a \neq \infty \vee b \neq -\infty$
and $a \neq -\infty \vee b \neq \infty$
and $|c| \neq \infty$
shows $(a + b) * c = a * c + b * c$
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

lemma *numeral-eq-ereal* [*simp*]: *numeral w* = *ereal* (*numeral w*)
proof (*induct w rule*: *num-induct*)
case *One*
then show *?case*
by *simp*
next

case (*inc x*)
then show *?case*
by (*simp add*: *inc numeral-inc*)
qed

lemma *distrib-left-ereal-nn*:
$c \geq 0 \Longrightarrow (x + y) * ereal\ c = x * ereal\ c + y * ereal\ c$
by(*cases x y rule*: *ereal2-cases*)(*simp-all add*: *ring-distribs*)

lemma *sum-ereal-right-distrib*:
fixes $f :: 'a \Rightarrow ereal$
shows $(\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i) \Longrightarrow r * sum\ f\ A = (\sum n{\in}A.\ r * f\ n)$
by (*induct A rule*: *infinite-finite-induct*) (*auto simp*: *ereal-right-distrib sum-nonneg*)

lemma *sum-ereal-left-distrib*:
$(\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i) \Longrightarrow sum\ f\ A * r = (\sum n{\in}A.\ f\ n * r :: ereal)$
using *sum-ereal-right-distrib*[*of A f r*] **by** (*simp add*: *mult-ac*)

lemma *sum-distrib-right-ereal*:
$c \geq 0 \Longrightarrow sum\ f\ A * ereal\ c = (\sum x{\in}A.\ f\ x * c :: ereal)$
by(*subst sum-comp-morphism*[**where** $h{=}\lambda x.\ x * ereal\ c$, *symmetric*])(*simp-all add*: *distrib-left-ereal-nn*)

lemma *ereal-le-epsilon*:
fixes $x\ y :: ereal$
assumes $\bigwedge e.\ 0 < e \Longrightarrow x \leq y + e$
shows $x \leq y$
proof (*cases* $x = -\infty \vee x = \infty \vee y = -\infty \vee y = \infty$)
case *True*
then show *?thesis*
using *assms*[*of 1*] **by** *auto*
next
case *False*
then obtain *p q* **where** $x = ereal\ p\ y = ereal\ q$
by (*metis MInfty-eq-minfinity ereal.distinct*(*3*) *uminus-ereal.elims*)
then show *?thesis*
by (*metis assms field-le-epsilon ereal-less*(*2*) *ereal-less-eq*(*3*) *plus-ereal.simps*(*1*))
qed

lemma *ereal-le-epsilon2*:
fixes $x\ y :: ereal$
assumes $\bigwedge e{::}real.\ 0 < e \Longrightarrow x \leq y + ereal\ e$
shows $x \leq y$
proof (*rule ereal-le-epsilon*)
show $\bigwedge \varepsilon{::}ereal.\ 0 < \varepsilon \Longrightarrow x \leq y + \varepsilon$
using *assms less-ereal.elims*(*2*) *zero-less-real-of-ereal* **by** *fastforce*
qed

lemma *ereal-le-real*:

fixes *x y* :: *ereal*
assumes $\bigwedge z.\ x \leq ereal\ z \Longrightarrow y \leq ereal\ z$
shows $y \leq x$
by (*metis assms ereal-bot ereal-cases ereal-infty-less-eq*(*2*) *ereal-less-eq*(*1*) *linorder-le-cases*)

lemma *prod-ereal-0*:
fixes *f* :: *'a* ⇒ *ereal*
shows $(\prod i{\in}A.\ f\ i) = 0 \longleftrightarrow finite\ A \wedge (\exists\, i{\in}A.\ f\ i = 0)$
proof (*cases finite A*)
case *True*
then show *?thesis* **by** (*induct A*) *auto*
qed *auto*

lemma *prod-ereal-pos*:
fixes *f* :: *'a* ⇒ *ereal*
assumes *pos*: $\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i$
shows $0 \leq (\prod i{\in}I.\ f\ i)$
proof (*cases finite I*)
case *True*
from *this pos* **show** *?thesis*
by *induct auto*
qed *auto*

lemma *prod-PInf*:
fixes *f* :: *'a* ⇒ *ereal*
assumes $\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i$
shows $(\prod i{\in}I.\ f\ i) = \infty \longleftrightarrow finite\ I \wedge (\exists\, i{\in}I.\ f\ i = \infty) \wedge (\forall\, i{\in}I.\ f\ i \neq 0)$
proof (*cases finite I*)
case *True*
from *this assms* **show** *?thesis*
proof (*induct I*)
case (*insert i I*)
then have *pos*: $0 \leq f\ i\ 0 \leq prod\ f\ I$
by (*auto intro*!: *prod-ereal-pos*)
from *insert* **have** $(\prod j{\in}insert\ i\ I.\ f\ j) = \infty \longleftrightarrow prod\ f\ I * f\ i = \infty$
by *auto*
also have ... $\longleftrightarrow (prod\ f\ I = \infty \vee f\ i = \infty) \wedge f\ i \neq 0 \wedge prod\ f\ I \neq 0$
using *prod-ereal-pos*[*of I f*] *pos*
by (*cases rule*: *ereal2-cases*[*of f i prod f I*]) *auto*
also have ... $\longleftrightarrow$ *finite* (*insert i I*) ∧ (∃ *j*∈*insert i I*. *f j* = ∞) ∧ (∀ *j*∈*insert i I*. *f j* ≠ *0*)
using *insert* **by** (*auto simp*: *prod-ereal-0*)
finally show *?case* **.**
qed *simp*
qed *auto*

lemma *prod-ereal*: $(\prod i{\in}A.\ ereal\ (f\ i)) = ereal\ (prod\ f\ A)$
proof (*cases finite A*)
case *True*

then show *?thesis*
by *induct* (*auto simp*: *one-ereal-def*)
next
case *False*
then show *?thesis*
by (*simp add*: *one-ereal-def*)
qed

38.1.4 Power

lemma *ereal-power*[*simp*]: (*ereal x*) ^ *n* = *ereal* (*x*^*n*)
by (*induct n*) (*auto simp*: *one-ereal-def*)

lemma *ereal-power-PInf*[*simp*]: (∞::*ereal*) ^ *n* = (*if n* = *0 then 1 else* ∞)
by (*induct n*) (*auto simp*: *one-ereal-def*)

lemma *ereal-power-uminus*[*simp*]:
fixes *x* :: *ereal*
shows (− *x*) ^ *n* = (*if even n then x* ^ *n else* − (*x*^*n*))
by (*induct n*) (*auto simp*: *one-ereal-def*)

lemma *ereal-power-numeral*[*simp*]:
(*numeral num* :: *ereal*) ^ *n* = *ereal* (*numeral num* ^ *n*)
by (*induct n*) (*auto simp*: *one-ereal-def*)

lemma *zero-le-power-ereal*[*simp*]:
fixes *a* :: *ereal*
assumes *0* ≤ *a*
shows *0* ≤ *a* ^ *n*
using *assms* **by** (*induct n*) (*auto simp*: *ereal-zero-le-0-iff*)

38.1.5 Subtraction

lemma *ereal-minus-minus-image*[*simp*]:
fixes *S* :: *ereal set*
shows *uminus* ‘ *uminus* ‘ *S* = *S*
by (*auto simp*: *image-iff*)

lemma *ereal-uminus-lessThan*[*simp*]:
fixes *a* :: *ereal*
shows *uminus* ‘ {..<*a*} = {−*a*<..}
proof −
{
fix *x*
assume −*a* < *x*
then have − *x* < − (− *a*)
by (*simp del*: *ereal-uminus-uminus*)
then have − *x* < *a*
by *simp*
}

then show *?thesis*
by *force*
qed

lemma *ereal-uminus-greaterThan*[*simp*]: *uminus* ' $\{(a::ereal)<..\} = \{..<-a\}$
by (*metis ereal-uminus-lessThan ereal-uminus-uminus ereal-minus-minus-image*)

instantiation *ereal* :: *minus*
begin

definition $x - y = x + -(y::ereal)$
instance ..

end

lemma *ereal-minus*[*simp*]:
$ereal\ r - ereal\ p = ereal\ (r - p)$
$-\infty - ereal\ r = -\infty$
$ereal\ r - \infty = -\infty$
$(\infty::ereal) - x = \infty$
$-(\infty::ereal) - \infty = -\infty$
$x - -y = x + y$
$x - 0 = x$
$0 - x = -x$
by (*simp-all add*: *minus-ereal-def*)

lemma *ereal-x-minus-x*[*simp*]: $x - x = (if\ |x| = \infty\ then\ \infty\ else\ 0::ereal)$
by (*cases x*) *simp-all*

lemma *ereal-eq-minus-iff*:
fixes $x\ y\ z :: ereal$
shows $x = z - y \longleftrightarrow$
$(|y| \neq \infty \longrightarrow x + y = z) \wedge$
$(y = -\infty \longrightarrow x = \infty) \wedge$
$(y = \infty \longrightarrow z = \infty \longrightarrow x = \infty) \wedge$
$(y = \infty \longrightarrow z \neq \infty \longrightarrow x = -\infty)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-eq-minus*:
fixes $x\ y\ z :: ereal$
shows $|y| \neq \infty \Longrightarrow x = z - y \longleftrightarrow x + y = z$
by (*auto simp*: *ereal-eq-minus-iff*)

lemma *ereal-less-minus-iff*:
fixes $x\ y\ z :: ereal$
shows $x < z - y \longleftrightarrow$
$(y = \infty \longrightarrow z = \infty \wedge x \neq \infty) \wedge$
$(y = -\infty \longrightarrow x \neq \infty) \wedge$
$(|y| \neq \infty \longrightarrow x + y < z)$

by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-less-minus*:
fixes *x y z* :: *ereal*
shows $|y| \neq \infty \Longrightarrow x < z - y \longleftrightarrow x + y < z$
by (*auto simp*: *ereal-less-minus-iff*)

lemma *ereal-le-minus-iff*:
fixes *x y z* :: *ereal*
shows $x \leq z - y \longleftrightarrow (y = \infty \longrightarrow z \neq \infty \longrightarrow x = -\infty) \wedge (|y| \neq \infty \longrightarrow x + y \leq z)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-le-minus*:
fixes *x y z* :: *ereal*
shows $|y| \neq \infty \Longrightarrow x \leq z - y \longleftrightarrow x + y \leq z$
by (*auto simp*: *ereal-le-minus-iff*)

lemma *ereal-minus-less-iff*:
fixes *x y z* :: *ereal*
shows $x - y < z \longleftrightarrow y \neq -\infty \wedge (y = \infty \longrightarrow x \neq \infty \wedge z \neq -\infty) \wedge (y \neq \infty \longrightarrow x < z + y)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-minus-less*:
fixes *x y z* :: *ereal*
shows $|y| \neq \infty \Longrightarrow x - y < z \longleftrightarrow x < z + y$
by (*auto simp*: *ereal-minus-less-iff*)

lemma *ereal-minus-le-iff*:
fixes *x y z* :: *ereal*
shows $x - y \leq z \longleftrightarrow$
$(y = -\infty \longrightarrow z = \infty) \wedge$
$(y = \infty \longrightarrow x = \infty \longrightarrow z = \infty) \wedge$
$(|y| \neq \infty \longrightarrow x \leq z + y)$
by (*cases rule*: *ereal3-cases*[*of x y z*]) *auto*

lemma *ereal-minus-le*:
fixes *x y z* :: *ereal*
shows $|y| \neq \infty \Longrightarrow x - y \leq z \longleftrightarrow x \leq z + y$
by (*auto simp*: *ereal-minus-le-iff*)

lemma *ereal-minus-eq-minus-iff*:
fixes *a b c* :: *ereal*
shows $a - b = a - c \longleftrightarrow$
$b = c \vee a = \infty \vee (a = -\infty \wedge b \neq -\infty \wedge c \neq -\infty)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) *auto*

lemma *ereal-add-le-add-iff*:

fixes *a b c* :: *ereal*
shows $c + a \leq c + b \longleftrightarrow$
$a \leq b \vee c = \infty \vee (c = -\infty \wedge a \neq \infty \wedge b \neq \infty)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

lemma *ereal-add-le-add-iff2*:
fixes *a b c* :: *ereal*
shows $a + c \leq b + c \longleftrightarrow a \leq b \vee c = \infty \vee (c = -\infty \wedge a \neq \infty \wedge b \neq \infty)$
by(*cases rule*: *ereal3-cases*[*of a b c*])(*simp-all add*: *field-simps*)

lemma *ereal-mult-le-mult-iff*:
fixes *a b c* :: *ereal*
shows $|c| \neq \infty \Longrightarrow c * a \leq c * b \longleftrightarrow (0 < c \longrightarrow a \leq b) \wedge (c < 0 \longrightarrow b \leq a)$
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *mult-le-cancel-left*)

lemma *ereal-minus-mono*:
fixes *A B C D* :: *ereal* **assumes** $A \leq B$ $D \leq C$
shows $A - C \leq B - D$
using *assms*
by (*cases rule*: *ereal3-cases*[*case-product ereal-cases*, *of A B C D*]) *simp-all*

lemma *ereal-mono-minus-cancel*:
fixes *a b c* :: *ereal*
shows $c - a \leq c - b \Longrightarrow 0 \leq c \Longrightarrow c < \infty \Longrightarrow b \leq a$
by (*cases a b c rule*: *ereal3-cases*) *auto*

lemma *real-of-ereal-minus*:
fixes *a b* :: *ereal*
shows *real-of-ereal* $(a - b) =$ (*if* $|a| = \infty \vee |b| = \infty$ *then 0 else real-of-ereal a* $-$ *real-of-ereal b*)
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *real-of-ereal-minus'*: $|x| = \infty \longleftrightarrow |y| = \infty \Longrightarrow$ *real-of-ereal x* $-$ *real-of-ereal* $y =$ *real-of-ereal* $(x - y ::$ *ereal*$)$
by(*subst real-of-ereal-minus*) *auto*

lemma *ereal-diff-positive*:
fixes *a b* :: *ereal* **shows** $a \leq b \Longrightarrow 0 \leq b - a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-between*:
fixes *x e* :: *ereal*
assumes $|x| \neq \infty$
and $0 < e$
shows $x - e < x$
and $x < x + e$
using *assms* **by** (*cases x*, *cases e*, *auto*)+

lemma *ereal-minus-eq-PInfty-iff*:

fixes $x\ y$ :: *ereal*
shows $x - y = \infty \longleftrightarrow y = -\infty \vee x = \infty$
by (*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-diff-add-eq-diff-diff-swap*:
fixes $x\ y\ z$ :: *ereal*
shows $|y| \neq \infty \Longrightarrow x - (y + z) = x - y - z$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *ereal-diff-add-assoc2*:
fixes $x\ y\ z$ :: *ereal*
shows $x + y - z = x - z + y$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *ereal-add-uminus-conv-diff*: **fixes** $x\ y\ z$ :: *ereal* **shows** $-x + y = y - x$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-minus-diff-eq*:
fixes $x\ y$ :: *ereal*
shows $[\![x = \infty \longrightarrow y \neq \infty; x = -\infty \longrightarrow y \neq -\infty]\!] \Longrightarrow -(x - y) = y - x$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ediff-le-self* [*simp*]: $x - y \leq (x :: enat)$
by(*cases x y rule*: *enat.exhaust*[*case-product enat.exhaust*]) *simp-all*

lemma *ereal-abs-diff*:
fixes $a\ b$::*ereal*
shows $abs(a-b) \leq abs\ a + abs\ b$
by (*cases rule*: *ereal2-cases*[*of a b*]) (*auto*)

38.1.6 Division

instantiation *ereal* :: *inverse*
begin

function *inverse-ereal* **where**
inverse (*ereal r*) = (*if r = 0 then* ∞ *else ereal* (*inverse r*))
| *inverse* (∞::*ereal*) = *0*
| *inverse* ($-\infty$::*ereal*) = *0*
by (*auto intro*: *ereal-cases*)
termination by (*relation* {}) *simp*

definition $x\ div\ y = x * inverse\ (y :: ereal)$

instance ..

end

lemma *real-of-ereal-inverse*[*simp*]:

fixes a :: *ereal*
shows *real-of-ereal* (*inverse* a) = *1* / *real-of-ereal* a
by (*cases* a) (*auto simp*: *inverse-eq-divide*)

lemma *ereal-inverse*[*simp*]:
inverse (*0*::*ereal*) = ∞
inverse (*1*::*ereal*) = *1*
by (*simp-all add*: *one-ereal-def zero-ereal-def*)

lemma *ereal-divide*[*simp*]:
ereal r / *ereal p* = (*if p* = *0 then ereal r* * ∞ *else ereal* (*r* / *p*))
unfolding *divide-ereal-def* **by** (*auto simp*: *divide-real-def*)

lemma *ereal-divide-same*[*simp*]:
fixes x :: *ereal*
shows x / x = (*if* |x| = ∞ $\vee$ x = *0 then 0 else 1*)
by (*cases* x) (*simp-all add*: *divide-real-def divide-ereal-def one-ereal-def*)

lemma *ereal-inv-inv*[*simp*]:
fixes x :: *ereal*
shows *inverse* (*inverse* x) = (*if* $x \neq -\infty$ *then* x *else* ∞)
by (*cases* x) *auto*

lemma *ereal-inverse-minus*[*simp*]:
fixes x :: *ereal*
shows *inverse* (− x) = (*if* x = *0 then* ∞ *else* −*inverse* x)
by (*cases* x) *simp-all*

lemma *ereal-uminus-divide*[*simp*]:
fixes x y :: *ereal*
shows − x / y = − (x / y)
unfolding *divide-ereal-def* **by** *simp*

lemma *ereal-divide-Infty*[*simp*]:
fixes x :: *ereal*
shows x / ∞ = *0* x / $-\infty$ = *0*
unfolding *divide-ereal-def* **by** *simp-all*

lemma *ereal-divide-one*[*simp*]: x / *1* = (x::*ereal*)
unfolding *divide-ereal-def* **by** *simp*

lemma *ereal-divide-ereal*[*simp*]: ∞ / *ereal r* = (*if 0* ≤ *r then* ∞ *else* $-\infty$)
unfolding *divide-ereal-def* **by** *simp*

lemma *ereal-inverse-nonneg-iff*: *0* ≤ *inverse* (x :: *ereal*) ⟷ *0* ≤ x ∨ x = $-\infty$
by (*cases* x) *auto*

lemma *inverse-ereal-ge0I*: *0* ≤ (x :: *ereal*) ⟹ *0* ≤ *inverse* x
by(*cases* x) *simp-all*

lemma *zero-le-divide-ereal*[*simp*]:
fixes a :: *ereal*
assumes $0 \leq a$
and $0 \leq b$
shows $0 \leq a \,/\, b$
using *assms* **by** (*cases rule*: *ereal2-cases*[*of a b*]) (*auto simp*: *zero-le-divide-iff*)

lemma *ereal-le-divide-pos*:
fixes $x\ y\ z$ :: *ereal*
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow y \leq z \,/\, x \longleftrightarrow x * y \leq z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-le-pos*:
fixes $x\ y\ z$ :: *ereal*
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow z \,/\, x \leq y \longleftrightarrow z \leq x * y$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-le-divide-neg*:
fixes $x\ y\ z$ :: *ereal*
shows $x < 0 \Longrightarrow x \neq -\infty \Longrightarrow y \leq z \,/\, x \longleftrightarrow z \leq x * y$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-le-neg*:
fixes $x\ y\ z$ :: *ereal*
shows $x < 0 \Longrightarrow x \neq -\infty \Longrightarrow z \,/\, x \leq y \longleftrightarrow x * y \leq z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-inverse-antimono-strict*:
fixes $x\ y$ :: *ereal*
shows $0 \leq x \Longrightarrow x < y \Longrightarrow \mathit{inverse}\ y < \mathit{inverse}\ x$
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*

lemma *ereal-inverse-antimono*:
fixes $x\ y$ :: *ereal*
shows $0 \leq x \Longrightarrow x \leq y \Longrightarrow \mathit{inverse}\ y \leq \mathit{inverse}\ x$
by (*cases rule*: *ereal2-cases*[*of x y*]) *auto*

lemma *inverse-inverse-Pinfty-iff*[*simp*]:
fixes x :: *ereal*
shows $\mathit{inverse}\ x = \infty \longleftrightarrow x = 0$
by (*cases x*) *auto*

lemma *ereal-inverse-eq-0*:
fixes x :: *ereal*
shows $\mathit{inverse}\ x = 0 \longleftrightarrow x = \infty \vee x = -\infty$
by (*cases x*) *auto*

lemma *ereal-0-gt-inverse*:

fixes x :: *ereal*
shows $0 < inverse\ x \longleftrightarrow x \neq \infty \wedge 0 \leq x$
by (*cases x*) *auto*

lemma *ereal-inverse-le-0-iff*:
fixes x :: *ereal*
shows $inverse\ x \leq 0 \longleftrightarrow x < 0 \vee x = \infty$
by(*cases x*) *auto*

lemma *ereal-divide-eq-0-iff*: $x\ /\ y = 0 \longleftrightarrow x = 0 \vee |y :: ereal| = \infty$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-mult-less-right*:
fixes $a\ b\ c$ :: *ereal*
assumes $b * a < c * a$
and $0 < a$
and $a < \infty$
shows $b < c$
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*])
(*auto split*: *if-split-asm simp*: *zero-less-mult-iff zero-le-mult-iff*)

lemma *ereal-mult-divide*: **fixes** $a\ b$ :: *ereal* **shows** $0 < b \Longrightarrow b < \infty \Longrightarrow b * (a\ /\ b) = a$
by (*cases a b rule*: *ereal2-cases*) *auto*

lemma *ereal-power-divide*:
fixes $x\ y$ :: *ereal*
shows $y \neq 0 \Longrightarrow (x\ /\ y)\ \hat{}\ n = x\hat{}n\ /\ y\hat{}n$
by (*cases rule*: *ereal2-cases* [*of x y*])
(*auto simp*: *one-ereal-def zero-ereal-def power-divide zero-le-power-eq*)

lemma *ereal-le-mult-one-interval*:
fixes $x\ y$ :: *ereal*
assumes *y*: $y \neq -\infty$
assumes *z*: $\bigwedge z.\ 0 < z \Longrightarrow z < 1 \Longrightarrow z * x \leq y$
shows $x \leq y$
proof (*cases x*)
case *PInf*
with *z*[*of 1 / 2*] **show** $x \leq y$
by (*simp add*: *one-ereal-def*)
next
case (*real r*)
note $r = this$
show $x \leq y$
proof (*cases y*)
case (*real p*)
note $p = this$
have $r \leq p$

```
    proof (rule field-le-mult-one-interval)
      fix z :: real
      assume 0 < z and z < 1
      with z[of ereal z] show z * r ≤ p
        using p r by (auto simp: zero-le-mult-iff one-ereal-def)
    qed
    then show x ≤ y
      using p r by simp
  qed (insert y, simp-all)
qed simp

lemma ereal-divide-right-mono[simp]:
  fixes x y z :: ereal
  assumes x ≤ y
    and 0 < z
  shows x / z ≤ y / z
  using assms by (cases x y z rule: ereal3-cases) (auto intro: divide-right-mono)

lemma ereal-divide-left-mono[simp]:
  fixes x y z :: ereal
  assumes y ≤ x
    and 0 < z
    and 0 < x * y
  shows z / x ≤ z / y
  using assms
  by (cases x y z rule: ereal3-cases)
     (auto intro: divide-left-mono simp: field-simps zero-less-mult-iff mult-less-0-iff
split: if-split-asm)

lemma ereal-divide-zero-left[simp]:
  fixes a :: ereal
  shows 0 / a = 0
  by (cases a) (auto simp: zero-ereal-def)

lemma ereal-times-divide-eq-left[simp]:
  fixes a b c :: ereal
  shows b / c * a = b * a / c
 by (cases a b c rule: ereal3-cases) (auto simp: field-simps zero-less-mult-iff mult-less-0-iff)

lemma ereal-times-divide-eq: a * (b / c :: ereal) = a * b / c
  by (cases a b c rule: ereal3-cases)
     (auto simp: field-simps zero-less-mult-iff)

lemma ereal-inverse-real [simp]: |z| ≠ ∞ ⟹ z ≠ 0 ⟹ ereal (inverse (real-of-ereal
z)) = inverse z
  by auto

lemma ereal-inverse-mult:
  a ≠ 0 ⟹ b ≠ 0 ⟹ inverse (a * (b::ereal)) = inverse a * inverse b
```

by (*cases a*; *cases b*) *auto*

lemma *inverse-eq-infinity-iff-eq-zero* [*simp*]:
1/(*x*::*ereal*) = ∞ ⟷ *x* = *0*
by (*simp add*: *divide-ereal-def*)

lemma *ereal-distrib-left*:
fixes *a b c* :: *ereal*
assumes *a* ≠ ∞ ∨ *b* ≠ −∞
and *a* ≠ −∞ ∨ *b* ≠ ∞
and |*c*| ≠ ∞
shows *c* ∗ (*a* + *b*) = *c* ∗ *a* + *c* ∗ *b*
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

lemma *ereal-distrib-minus-left*:
fixes *a b c* :: *ereal*
assumes *a* ≠ ∞ ∨ *b* ≠ ∞
and *a* ≠ −∞ ∨ *b* ≠ −∞
and |*c*| ≠ ∞
shows *c* ∗ (*a* − *b*) = *c* ∗ *a* − *c* ∗ *b*
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

lemma *ereal-distrib-minus-right*:
fixes *a b c* :: *ereal*
assumes *a* ≠ ∞ ∨ *b* ≠ ∞
and *a* ≠ −∞ ∨ *b* ≠ −∞
and |*c*| ≠ ∞
shows (*a* − *b*) ∗ *c* = *a* ∗ *c* − *b* ∗ *c*
using *assms*
by (*cases rule*: *ereal3-cases*[*of a b c*]) (*simp-all add*: *field-simps*)

38.2 Complete lattice

instantiation *ereal* :: *lattice*
begin

definition [*simp*]: *sup x y* = (*max x y* :: *ereal*)
definition [*simp*]: *inf x y* = (*min x y* :: *ereal*)
instance by *standard simp-all*

end

instantiation *ereal* :: *complete-lattice*
begin

definition *bot* = (−∞::*ereal*)
definition *top* = (∞::*ereal*)

definition *Sup* $S = (SOME\ x :: ereal.\ (\forall\, y{\in}S.\ y \leq x) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ y \leq z) \longrightarrow x \leq z))$
definition *Inf* $S = (SOME\ x :: ereal.\ (\forall\, y{\in}S.\ x \leq y) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ z \leq y) \longrightarrow z \leq x))$

lemma *ereal-complete-Sup*:
fixes S :: *ereal set*
shows $\exists\, x.\ (\forall\, y{\in}S.\ y \leq x) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ y \leq z) \longrightarrow x \leq z)$
proof (*cases* $\exists\, x.\ \forall\, a{\in}S.\ a \leq ereal\ x$)
case *True*
then obtain y **where** y: $a \leq ereal\ y$ **if** $a{\in}S$ **for** a
by *auto*
then have $\infty \notin S$
by *force*
show *?thesis*
proof (*cases* $S \neq \{-\infty\} \wedge S \neq \{\}$)
case *True*
with ‹$\infty \notin S$› **obtain** x **where** x: $x \in S$ $|x| \neq \infty$
by *auto*
obtain s **where** s: $\forall\, x{\in}ereal\ -`\ S.\ x \leq s$ $(\forall\, x{\in}ereal\ -`\ S.\ x \leq z) \Longrightarrow s \leq z$ **for** z
proof (*atomize-elim*, *rule complete-real*)
show $\exists\, x.\ x \in ereal\ -`\ S$
using x **by** *auto*
show $\exists\, z.\ \forall\, x{\in}ereal\ -`\ S.\ x \leq z$
by (*auto dest*: *y intro*!: *exI*[*of* - *y*])
qed
show *?thesis*
proof (*safe intro*!: *exI*[*of* - *ereal s*])
fix y
assume $y \in S$
with s ‹$\infty \notin S$› **show** $y \leq ereal\ s$
by (*cases y*) *auto*
next
fix z
assume $\forall\, y{\in}S.\ y \leq z$
with ‹$S \neq \{-\infty\} \wedge S \neq \{\}$› **show** $ereal\ s \leq z$
by (*cases z*) (*auto intro*!: *s*)
qed
next
case *False*
then show *?thesis*
by (*auto intro*!: *exI*[*of* - $-\infty$])
qed
next
case *False*
then show *?thesis*
by (*fastforce intro*!: *exI*[*of* - ∞] *ereal-top intro*: *order-trans dest*: *less-imp-le*

simp: *not-le*)
qed

lemma *ereal-complete-uminus-eq*:
fixes S :: *ereal set*
shows $(\forall\, y{\in}uminus`S.\ y \le x) \wedge (\forall\, z.\ (\forall\, y{\in}uminus`S.\ y \le z) \longrightarrow x \le z)$
$\longleftrightarrow (\forall\, y{\in}S.\ -x \le y) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ z \le y) \longrightarrow z \le -x)$
by *simp* (*metis ereal-minus-le-minus ereal-uminus-uminus*)

lemma *ereal-complete-Inf*:
$\exists\, x.\ (\forall\, y{\in}S{::}ereal\ set.\ x \le y) \wedge (\forall\, z.\ (\forall\, y{\in}S.\ z \le y) \longrightarrow z \le x)$
using *ereal-complete-Sup*[*of uminus ` S*]
unfolding *ereal-complete-uminus-eq*
by *auto*

instance
proof
show *Sup* $\{\}$ = (*bot*::*ereal*)
using *ereal-bot* **by** (*auto simp*: *bot-ereal-def Sup-ereal-def*)
show *Inf* $\{\}$ = (*top*::*ereal*)
unfolding *top-ereal-def Inf-ereal-def*
using *ereal-infty-less-eq*(*1*) *ereal-less-eq*(*1*) **by** *blast*
qed (*auto intro*: *someI2-ex ereal-complete-Sup ereal-complete-Inf*
simp: *Sup-ereal-def Inf-ereal-def bot-ereal-def top-ereal-def*)

end

instance *ereal* :: *complete-linorder* **..**

instance *ereal* :: *linear-continuum*
proof
show $\exists\, a\ b{::}ereal.\ a \neq b$
using *zero-neq-one* **by** *blast*
qed

lemma *min-PInf* [*simp*]: *min* (∞::*ereal*) $x = x$
by (*metis min-top top-ereal-def*)

lemma *min-PInf2* [*simp*]: *min* x (∞::*ereal*) $= x$
by (*metis min-top2 top-ereal-def*)

lemma *max-PInf* [*simp*]: *max* (∞::*ereal*) $x = \infty$
by (*metis max-top top-ereal-def*)

lemma *max-PInf2* [*simp*]: *max* x (∞::*ereal*) $= \infty$
by (*metis max-top2 top-ereal-def*)

lemma *min-MInf* [*simp*]: *min* ($-\infty$::*ereal*) $x = -\infty$
by (*metis min-bot bot-ereal-def*)

lemma *min-MInf2* [*simp*]: *min x* (−∞::*ereal*) = −∞
by (*metis min-bot2 bot-ereal-def*)

lemma *max-MInf* [*simp*]: *max* (−∞::*ereal*) *x* = *x*
by (*metis max-bot bot-ereal-def*)

lemma *max-MInf2* [*simp*]: *max x* (−∞::*ereal*) = *x*
by (*metis max-bot2 bot-ereal-def*)

38.3 Extended real intervals

lemma *real-greaterThanLessThan-infinity-eq*:
real-of-ereal ' {*N*::*ereal*<..<∞} =
(*if N* = ∞ *then* {} *else if N* = −∞ *then UNIV else* {*real-of-ereal N*<..})
by (*force simp*: *real-less-ereal-iff intro*!: *image-eqI*[**where** *x*=*ereal* -] *elim*!: *less-ereal.elims*)

lemma *real-greaterThanLessThan-minus-infinity-eq*:
real-of-ereal ' {−∞<..<*N*::*ereal*} =
(*if N* = ∞ *then UNIV else if N* = −∞ *then* {} *else* {..<*real-of-ereal N*})
proof −
have *real-of-ereal* ' {−∞<..<*N*::*ereal*} = *uminus* ' *real-of-ereal* ' {−*N*<..<∞}
by (*auto simp*: *ereal-uminus-less-reorder intro*!: *image-eqI*[**where** *x*=−*x* **for** *x*])
also note *real-greaterThanLessThan-infinity-eq*
finally show *?thesis* **by** (*auto intro*!: *image-eqI*[**where** *x*=−*x* **for** *x*])
qed

lemma *real-greaterThanLessThan-inter*:
real-of-ereal ' {*N*<..<*M*::*ereal*} = *real-of-ereal* ' {−∞<..<*M*} ∩ *real-of-ereal* '
{*N*<..<∞}
by (*force elim*!: *less-ereal.elims*)

lemma *real-atLeastGreaterThan-eq*: *real-of-ereal* ' {*N*<..<*M*::*ereal*} =
(*if N* = ∞ *then* {} *else*
if N = −∞ *then*
(*if M* = ∞ *then UNIV*
else if M = −∞ *then* {}
else {..< *real-of-ereal M*})
else if M = − ∞ *then* {}
else if M = ∞ *then* {*real-of-ereal N*<..}
else {*real-of-ereal N* <..< *real-of-ereal M*})
proof (*cases M* = −∞ ∨ *M* = ∞ ∨ *N* = −∞ ∨ *N* = ∞)
case *True*
then show *?thesis*
by (*auto simp*: *real-greaterThanLessThan-minus-infinity-eq real-greaterThanLessThan-infinity-eq*
)
next
case *False*
then obtain *p q* **where** *M* = *ereal p N* = *ereal q*

by (*metis MInfty-eq-minfinity ereal.distinct*(*3*) *uminus-ereal.elims*)
moreover have $\bigwedge x.\ [\![q < x;\ x < p]\!] \Longrightarrow x \in$ *real-of-ereal* ' {*ereal q*<..<*ereal p*}
by (*metis greaterThanLessThan-iff imageI less-ereal.simps*(*1*) *real-of-ereal.simps*(*1*))
ultimately show *?thesis*
by (*auto elim*!: *less-ereal.elims*)
qed

lemma *real-image-ereal-ivl*:
fixes *a b*::*ereal*
shows
real-of-ereal ' {*a*<..<*b*} =
(*if a* < *b then* (*if a* = − ∞ *then if b* = ∞ *then UNIV else* {..<*real-of-ereal b*}
else if b = ∞ *then* {*real-of-ereal a*<..} *else* {*real-of-ereal a* <..< *real-of-ereal b*})
else {})
by (*cases a*; *cases b*; *simp add*: *real-atLeastGreaterThan-eq not-less*)

lemma fixes *a b c*::*ereal*
shows *not-inftyI*: *a* < *b* ⟹ *b* < *c* ⟹ *abs b* ≠ ∞
by *force*

lemma
interval-neqs:
fixes *r s t*::*real*
shows {*r*<..<*s*} ≠ {*t*<..}
and {*r*<..<*s*} ≠ {..<*t*}
and {*r*<..<*ra*} ≠ *UNIV*
and {*r*<..} ≠ {..<*s*}
and {*r*<..} ≠ *UNIV*
and {..<*r*} ≠ *UNIV*
and {} ≠ {*r*<..}
and {} ≠ {..<*r*}
subgoal
by (*metis dual-order.strict-trans greaterThanLessThan-iff greaterThan-iff gt-ex*
not-le order-refl)
subgoal
by (*metis* (*no-types*, *opaque-lifting*) *greaterThanLessThan-empty-iff greaterThanLessThan-iff*
gt-ex
lessThan-iff minus-minus neg-less-iff-less not-less order-less-irrefl)
subgoal by *force*
subgoal
by (*metis greaterThanLessThan-empty-iff greaterThanLessThan-eq greaterThan-iff*
inf.idem
lessThan-iff lessThan-non-empty less-irrefl not-le)
subgoal by *force*
subgoal by *force*
subgoal using *greaterThan-non-empty* **by** *blast*
subgoal using *lessThan-non-empty* **by** *blast*
done

lemma *greaterThanLessThan-eq-iff*:
fixes *r s t u*::*real*
shows $(\{r<..<s\} = \{t<..<u\}) = (r \geq s \wedge u \leq t \vee r = t \wedge s = u)$
by (*metis cInf-greaterThanLessThan cSup-greaterThanLessThan greaterThanLessThan-empty-iff not-le*)

lemma *real-of-ereal-image-greaterThanLessThan-iff*:
$real\text{-}of\text{-}ereal\ `\ \{a <..< b\} = real\text{-}of\text{-}ereal\ `\ \{c <..< d\} \longleftrightarrow (a \geq b \wedge c \geq d \vee a = c \wedge b = d)$
unfolding *real-atLeastGreaterThan-eq*
by (*cases a*; *cases b*; *cases c*; *cases d*;
simp add: *greaterThanLessThan-eq-iff interval-neqs interval-neqs*[*symmetric*])

lemma *uminus-image-real-of-ereal-image-greaterThanLessThan*:
$uminus\ `\ real\text{-}of\text{-}ereal\ `\ \{l <..< u\} = real\text{-}of\text{-}ereal\ `\ \{-u <..< -l\}$
by (*force simp*: *algebra-simps ereal-less-uminus-reorder ereal-uminus-less-reorder intro*: *image-eqI*[**where** $x{=}-x$ **for** x])

lemma *add-image-real-of-ereal-image-greaterThanLessThan*:
$(+)\ c\ `\ real\text{-}of\text{-}ereal\ `\ \{l <..< u\} = real\text{-}of\text{-}ereal\ `\ \{c + l <..< c + u\}$
apply *safe*
subgoal for x
using *ereal-less-add*[*of c*]
by (*force simp*: *real-of-ereal-add add.commute*)
subgoal for - x
by (*force simp*: *add.commute real-of-ereal-minus ereal-minus-less ereal-less-minus intro*: *image-eqI*[**where** $x{=}x - c$])
done

lemma *add2-image-real-of-ereal-image-greaterThanLessThan*:
$(\lambda x.\ x + c)\ `\ real\text{-}of\text{-}ereal\ `\ \{l <..< u\} = real\text{-}of\text{-}ereal\ `\ \{l + c <..< u + c\}$
using *add-image-real-of-ereal-image-greaterThanLessThan*[*of c l u*]
by (*metis add.commute image-cong*)

lemma *minus-image-real-of-ereal-image-greaterThanLessThan*:
$(-)\ c\ `\ real\text{-}of\text{-}ereal\ `\ \{l <..< u\} = real\text{-}of\text{-}ereal\ `\ \{c - u <..< c - l\}$
(**is** *?l* = *?r*)
proof −
have $?l = (+)\ c\ `\ uminus\ `\ real\text{-}of\text{-}ereal\ `\ \{l <..< u\}$ **by** *auto*
also note *uminus-image-real-of-ereal-image-greaterThanLessThan*
also note *add-image-real-of-ereal-image-greaterThanLessThan*
finally show *?thesis* **by** (*simp add*: *minus-ereal-def*)
qed

lemma *real-ereal-bound-lemma-up*:
assumes $s \in real\text{-}of\text{-}ereal\ `\ \{a<..<b\}$
assumes $t \notin real\text{-}of\text{-}ereal\ `\ \{a<..<b\}$
assumes $s \leq t$
shows $b \neq \infty$

proof (*cases b*)
case *PInf*
then show *?thesis*
using *assms*
apply *clarsimp*
by (*metis UNIV-I assms*(*1*) *ereal-less-PInfty greaterThan-iff less-eq-ereal-def less-le-trans real-image-ereal-ivl*)
qed *auto*

lemma *real-ereal-bound-lemma-down*:
assumes *s*: $s \in$ *real-of-ereal* ' $\{a<..<b\}$
and *t*: $t \notin$ *real-of-ereal* ' $\{a<..<b\}$
and $t \leq s$
shows $a \neq -\infty$
proof (*cases b*)
case (*real r*)
then show *?thesis*
using *assms real-greaterThanLessThan-minus-infinity-eq* **by** *force*
next
case *PInf*
then show *?thesis*
using *t real-greaterThanLessThan-infinity-eq* **by** *auto*
next
case *MInf*
then show *?thesis*
using *s* **by** *auto*
qed

38.4 Topological space

instantiation *ereal* :: *linear-continuum-topology*
begin

definition *open-ereal* :: *ereal set* ⇒ *bool* **where**
open-ereal-generated: *open-ereal* = *generate-topology* (*range lessThan* ∪ *range greaterThan*)

instance
by *standard* (*simp add*: *open-ereal-generated*)

end

lemma *continuous-on-ereal*[*continuous-intros*]:
assumes *f*: *continuous-on s f* **shows** *continuous-on s* (λ*x*. *ereal* (*f x*))
by (*rule continuous-on-compose2* [*OF continuous-onI-mono*[*of ereal UNIV*] *f*]) *auto*

lemma *tendsto-ereal*[*tendsto-intros*, *simp*, *intro*]: (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *ereal* (*f x*)) ⟶ *ereal x*) *F*

using *isCont-tendsto-compose*[*of x ereal f F*] *continuous-on-ereal*[*of UNIV* $\lambda x.\ x$]
by (*simp add*: *continuous-on-eq-continuous-at*)

lemma *tendsto-uminus-ereal*[*tendsto-intros*, *simp*, *intro*]:
assumes $(f \longrightarrow x)\ F$
shows $((\lambda x.\ -\ f\ x{::}ereal) \longrightarrow -\ x)\ F$
proof (*rule tendsto-compose*[*OF order-tendstoI assms*])
show $\bigwedge a.\ a < -\ x \Longrightarrow \forall_F\ x\ in\ at\ x.\ a < -\ x$
by (*metis ereal-less-uminus-reorder eventually-at-topological lessThan-iff open-lessThan*)
show $\bigwedge a.\ -\ x < a \Longrightarrow \forall_F\ x\ in\ at\ x.\ -\ x < a$
by (*metis ereal-uminus-reorder*(*2*) *eventually-at-topological greaterThan-iff open-greaterThan*)
qed

lemma *at-infty-ereal-eq-at-top*: *at* ∞ = *filtermap ereal at-top*
unfolding *filter-eq-iff eventually-at-filter eventually-at-top-linorder eventually-filtermap top-ereal-def*[*symmetric*]
apply (*subst eventually-nhds-top*[*of 0*])
apply (*auto simp*: *top-ereal-def less-le ereal-all-split ereal-ex-split*)
apply (*metis PInfty-neq-ereal*(*2*) *ereal-less-eq*(*3*) *ereal-top le-cases order-trans*)
done

lemma *ereal-Lim-uminus*: $(f \longrightarrow f0)\ net \longleftrightarrow ((\lambda x.\ -\ f\ x{::}ereal) \longrightarrow -\ f0)\ net$
using *tendsto-uminus-ereal*[*of f f0 net*] *tendsto-uminus-ereal*[*of* $\lambda x.\ -\ f\ x\ -\ f0\ net$]
by *auto*

lemma *ereal-divide-less-iff*: $0 < (c{::}ereal) \Longrightarrow c < \infty \Longrightarrow a\ /\ c < b \longleftrightarrow a < b * c$
by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps*)

lemma *ereal-less-divide-iff*: $0 < (c{::}ereal) \Longrightarrow c < \infty \Longrightarrow a < b\ /\ c \longleftrightarrow a * c < b$
by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps*)

lemma *tendsto-cmult-ereal*[*tendsto-intros*, *simp*, *intro*]:
assumes *c*: $|c| \neq \infty$ **and** *f*: $(f \longrightarrow x)\ F$ **shows** $((\lambda x.\ c * f\ x{::}ereal) \longrightarrow c * x)\ F$
proof −
{ fix *c* :: *ereal* **assume** $0 < c\ c < \infty$
then have $((\lambda x.\ c * f\ x{::}ereal) \longrightarrow c * x)\ F$
apply (*intro tendsto-compose*[*OF - f*])
apply (*auto intro*!: *order-tendstoI simp*: *eventually-at-topological*)
apply (*rule-tac x*={*a*/*c* <..} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-divide-less-iff mult.commute*) []
apply (*rule-tac x*={..< *a*/*c*} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-less-divide-iff mult.commute*) []
done }
note $* = this$

```
  have ((0 < c ∧ c < ∞) ∨ (−∞ < c ∧ c < 0) ∨ c = 0)
    using c by (cases c) auto
  then show ?thesis
  proof (elim disjE conjE)
    assume − ∞ < c c < 0
    then have 0 < − c − c < ∞
      by (auto simp: ereal-uminus-reorder ereal-less-uminus-reorder[of 0])
    then have ((λx. (− c) ∗ f x) ⟶ (− c) ∗ x) F
      by (rule ∗)
    from tendsto-uminus-ereal[OF this] show ?thesis
      by simp
  qed (auto intro!: ∗)
qed

lemma tendsto-cmult-ereal-not-0[tendsto-intros, simp, intro]:
  assumes x ≠ 0 and f: (f ⟶ x) F shows ((λx. c ∗ f x::ereal) ⟶ c ∗ x) F
proof cases
  assume |c| = ∞
  show ?thesis
  proof (rule filterlim-cong[THEN iffD1, OF refl refl - tendsto-const])
    have 0 < x ∨ x < 0
      using ‹x ≠ 0› by (auto simp add: neq-iff)
    then show eventually (λx'. c ∗ x = c ∗ f x') F
    proof
      assume 0 < x from order-tendstoD(1)[OF f this] show ?thesis
        by eventually-elim (insert ‹0<x› ‹|c| = ∞›, auto)
    next
      assume x < 0 from order-tendstoD(2)[OF f this] show ?thesis
        by eventually-elim (insert ‹x<0› ‹|c| = ∞›, auto)
    qed
  qed
qed (rule tendsto-cmult-ereal[OF - f])

lemma tendsto-cadd-ereal[tendsto-intros, simp, intro]:
  assumes c: y ≠ − ∞ x ≠ − ∞ and f: (f ⟶ x) F shows ((λx. f x + y::ereal)
⟶ x + y) F
  apply (intro tendsto-compose[OF - f])
  apply (auto intro!: order-tendstoI simp: eventually-at-topological)
  apply (rule-tac x={a − y <..} in exI)
  apply (auto split: ereal.split simp: ereal-minus-less-iff c) []
  apply (rule-tac x={..< a − y} in exI)
  apply (auto split: ereal.split simp: ereal-less-minus-iff c) []
  done

lemma tendsto-add-left-ereal[tendsto-intros, simp, intro]:
  assumes c: |y| ≠ ∞ and f: (f ⟶ x) F shows ((λx. f x + y::ereal) ⟶ x
+ y) F
  apply (intro tendsto-compose[OF - f])
```

apply (*auto intro*!: *order-tendstoI simp*: *eventually-at-topological*)
apply (*rule-tac x*={*a* − *y* <..} **in** *exI*)
apply (*insert c*, *auto split*: *ereal.split simp*: *ereal-minus-less-iff*) []
apply (*rule-tac x*={..< *a* − *y*} **in** *exI*)
apply (*auto split*: *ereal.split simp*: *ereal-less-minus-iff c*) []
done

lemma *continuous-at-ereal*[*continuous-intros*]: *continuous F f* ⟹ *continuous F* (λ*x*. *ereal* (*f x*))
unfolding *continuous-def* **by** *auto*

lemma *ereal-Sup*:
assumes ∗: |*SUP a*∈*A*. *ereal a*| ≠ ∞
shows *ereal* (*Sup A*) = (*SUP a*∈*A*. *ereal a*)
proof (*rule continuous-at-Sup-mono*)
obtain *r* **where** *r*: *ereal r* = (*SUP a*∈*A*. *ereal a*) *A* ≠ {}
using ∗ **by** (*force simp*: *bot-ereal-def*)
then show *bdd-above A A* ≠ {}
by (*auto intro*!: *SUP-upper bdd-aboveI*[*of* - *r*] *simp flip*: *ereal-less-eq*)
qed (*auto simp*: *mono-def continuous-at-imp-continuous-at-within continuous-at-ereal*)

lemma *ereal-SUP*: |*SUP a*∈*A*. *ereal* (*f a*)| ≠ ∞ ⟹ *ereal* (*SUP a*∈*A*. *f a*) = (*SUP a*∈*A*. *ereal* (*f a*))
by (*simp add*: *ereal-Sup image-comp*)

lemma *ereal-Inf*:
assumes ∗: |*INF a*∈*A*. *ereal a*| ≠ ∞
shows *ereal* (*Inf A*) = (*INF a*∈*A*. *ereal a*)
proof (*rule continuous-at-Inf-mono*)
obtain *r* **where** *r*: *ereal r* = (*INF a*∈*A*. *ereal a*) *A* ≠ {}
using ∗ **by** (*force simp*: *top-ereal-def*)
then show *bdd-below A A* ≠ {}
by (*auto intro*!: *INF-lower bdd-belowI*[*of* - *r*] *simp flip*: *ereal-less-eq*)
qed (*auto simp*: *mono-def continuous-at-imp-continuous-at-within continuous-at-ereal*)

lemma *ereal-Inf′*:
assumes ∗: *bdd-below A A* ≠ {}
shows *ereal* (*Inf A*) = (*INF a*∈*A*. *ereal a*)
proof (*rule ereal-Inf*)
from ∗ **obtain** *l u* **where** *x* ∈ *A* ⟹ *l* ≤ *x u* ∈ *A* **for** *x*
by (*auto simp*: *bdd-below-def*)
then have *l* ≤ (*INF x*∈*A*. *ereal x*) (*INF x*∈*A*. *ereal x*) ≤ *u*
by (*auto intro*!: *INF-greatest INF-lower*)
then show |*INF a*∈*A*. *ereal a*| ≠ ∞
by *auto*
qed

lemma *ereal-INF*: |*INF a*∈*A*. *ereal* (*f a*)| ≠ ∞ ⟹ *ereal* (*INF a*∈*A*. *f a*) = (*INF a*∈*A*. *ereal* (*f a*))

by (*simp add*: *ereal-Inf image-comp*)

lemma *ereal-Sup-uminus-image-eq*: *Sup* (*uminus* ‘ *S*::*ereal set*) = − *Inf S*
by (*auto intro*!: *SUP-eqI*
simp: *Ball-def*[*symmetric*] *ereal-uminus-le-reorder le-Inf-iff*
intro!: *complete-lattice-class.Inf-lower2*)

lemma *ereal-SUP-uminus-eq*:
fixes *f* :: *'a* ⇒ *ereal*
shows (*SUP x*∈*S*. *uminus* (*f x*)) = − (*INF x*∈*S*. *f x*)
using *ereal-Sup-uminus-image-eq* [*of f* ‘ *S*] **by** (*simp add*: *image-comp*)

lemma *ereal-inj-on-uminus*[*intro*, *simp*]: *inj-on uminus* (*A* :: *ereal set*)
by (*auto intro*!: *inj-onI*)

lemma *ereal-Inf-uminus-image-eq*: *Inf* (*uminus* ‘ *S*::*ereal set*) = − *Sup S*
using *ereal-Sup-uminus-image-eq*[*of uminus* ‘ *S*] **by** *simp*

lemma *ereal-INF-uminus-eq*:
fixes *f* :: *'a* ⇒ *ereal*
shows (*INF x*∈*S*. − *f x*) = − (*SUP x*∈*S*. *f x*)
using *ereal-Inf-uminus-image-eq* [*of f* ‘ *S*] **by** (*simp add*: *image-comp*)

lemma *ereal-SUP-uminus*:
fixes *f* :: *'a* ⇒ *ereal*
shows (*SUP i* ∈ *R*. − *f i*) = − (*INF i* ∈ *R*. *f i*)
using *ereal-Sup-uminus-image-eq*[*of f*‘*R*]
by (*simp add*: *image-image*)

lemma *ereal-SUP-not-infty*:
fixes *f* :: - ⇒ *ereal*
shows *A* ≠ {} ⟹ *l* ≠ −∞ ⟹ *u* ≠ ∞ ⟹ ∀ *a*∈*A*. *l* ≤ *f a* ∧ *f a* ≤ *u* ⟹ |*Sup* (*f* ‘ *A*)| ≠ ∞
using *SUP-upper2*[*of* - *A l f*] *SUP-least*[*of A f u*]
by (*cases Sup* (*f* ‘ *A*)) *auto*

lemma *ereal-INF-not-infty*:
fixes *f* :: - ⇒ *ereal*
shows *A* ≠ {} ⟹ *l* ≠ −∞ ⟹ *u* ≠ ∞ ⟹ ∀ *a*∈*A*. *l* ≤ *f a* ∧ *f a* ≤ *u* ⟹ |*Inf* (*f* ‘ *A*)| ≠ ∞
using *INF-lower2*[*of* - *A f u*] *INF-greatest*[*of A l f*]
by (*cases Inf* (*f* ‘ *A*)) *auto*

lemma *ereal-image-uminus-shift*:
fixes *X Y* :: *ereal set*
shows *uminus* ‘ *X* = *Y* ⟷ *X* = *uminus* ‘ *Y*
proof
assume *uminus* ‘ *X* = *Y*
then have *uminus* ‘ *uminus* ‘ *X* = *uminus* ‘ *Y*

by (*simp add*: *inj-image-eq-iff*)
then show $X = uminus\ `\ Y$
by (*simp add*: *image-image*)
qed (*simp add*: *image-image*)

lemma *Sup-eq-MInfty*:
fixes S :: *ereal set*
shows $Sup\ S = -\infty \longleftrightarrow S = \{\} \vee S = \{-\infty\}$
unfolding *bot-ereal-def*[*symmetric*] **by** *auto*

lemma *Inf-eq-PInfty*:
fixes S :: *ereal set*
shows $Inf\ S = \infty \longleftrightarrow S = \{\} \vee S = \{\infty\}$
using *Sup-eq-MInfty*[*of uminus`S*]
unfolding *ereal-Sup-uminus-image-eq ereal-image-uminus-shift* **by** *simp*

lemma *Inf-eq-MInfty*:
fixes S :: *ereal set*
shows $-\infty \in S \Longrightarrow Inf\ S = -\infty$
unfolding *bot-ereal-def*[*symmetric*] **by** *auto*

lemma *Sup-eq-PInfty*:
fixes S :: *ereal set*
shows $\infty \in S \Longrightarrow Sup\ S = \infty$
unfolding *top-ereal-def*[*symmetric*] **by** *auto*

lemma *not-MInfty-nonneg*[*simp*]: $0 \leq (x::ereal) \Longrightarrow x \neq -\infty$
by *auto*

lemma *Sup-ereal-close*:
fixes e :: *ereal*
assumes $0 < e$
and S: $|Sup\ S| \neq \infty\ S \neq \{\}$
shows $\exists x{\in}S.\ Sup\ S - e < x$
using *assms* **by** (*cases e*) (*auto intro*!: *less-Sup-iff*[*THEN iffD1*])

lemma *Inf-ereal-close*:
fixes e :: *ereal*
assumes $|Inf\ X| \neq \infty$
and $0 < e$
shows $\exists x{\in}X.\ x < Inf\ X + e$
proof (*rule Inf-less-iff*[*THEN iffD1*])
show $Inf\ X < Inf\ X + e$
using *assms* **by** (*cases e*) *auto*
qed

lemma *SUP-PInfty*:
$(\bigwedge n::nat.\ \exists i{\in}A.\ ereal\ (real\ n) \leq f\ i) \Longrightarrow (SUP\ i{\in}A.\ f\ i :: ereal) = \infty$
unfolding *top-ereal-def*[*symmetric*] *SUP-eq-top-iff*

by (*metis MInfty-neq-PInfty*(2) *PInfty-neq-ereal*(2) *less-PInf-Ex-of-nat less-ereal.elims*(2) *less-le-trans*)

lemma *SUP-nat-Infty*: (*SUP i. ereal* (*real i*)) = ∞
by (*rule SUP-PInfty*) *auto*

lemma *SUP-ereal-add-left*:
assumes $I \neq \{\}$ $c \neq -\infty$
shows (*SUP i∈I. f i* + *c* :: *ereal*) = (*SUP i∈I. f i*) + *c*
proof (*cases* (*SUP i∈I. f i*) = − ∞)
case *True*
then have $\bigwedge i.\ i \in I \Longrightarrow f\ i = -\infty$
unfolding *Sup-eq-MInfty* **by** *auto*
with *True* **show** *?thesis*
by (*cases c*) (*auto simp*: ‹$I \neq \{\}$›)
next
case *False*
then show *?thesis*
by (*subst continuous-at-Sup-mono*[**where** *f*=λ*x. x* + *c*])
(*auto simp*: *continuous-at-imp-continuous-at-within continuous-at mono-def add-mono* ‹$I \neq \{\}$› ‹$c \neq -\infty$› *image-comp*)
qed

lemma *SUP-ereal-add-right*:
fixes *c* :: *ereal*
shows $I \neq \{\} \Longrightarrow c \neq -\infty \Longrightarrow$ (*SUP i∈I. c* + *f i*) = *c* + (*SUP i∈I. f i*)
using *SUP-ereal-add-left*[*of I c f*] **by** (*simp add*: *add.commute*)

lemma *SUP-ereal-minus-right*:
assumes $I \neq \{\}$ $c \neq -\infty$
shows (*SUP i∈I. c* − *f i* :: *ereal*) = *c* − (*INF i∈I. f i*)
using *SUP-ereal-add-right*[*OF assms*, *of* λ*i.* − *f i*]
by (*simp add*: *ereal-SUP-uminus minus-ereal-def*)

lemma *SUP-ereal-minus-left*:
assumes $I \neq \{\}$ $c \neq \infty$
shows (*SUP i∈I. f i* − *c*:: *ereal*) = (*SUP i∈I. f i*) − *c*
using *SUP-ereal-add-left*[*OF* ‹$I \neq \{\}$›, *of* −*c f*] **by** (*simp add*: ‹$c \neq \infty$› *minus-ereal-def*)

lemma *INF-ereal-minus-right*:
assumes $I \neq \{\}$ **and** $|c| \neq \infty$
shows (*INF i∈I. c* − *f i*) = *c* − (*SUP i∈I. f i*::*ereal*)
proof −
{ fix *b* **have** (−*c*) + *b* = − (*c* − *b*)
using ‹$|c| \neq \infty$› **by** (*cases c b rule*: *ereal2-cases*) *auto* **}**
note ∗ = *this*
show *?thesis*

```
    using SUP-ereal-add-right[OF ‹I ≠ {}›, of −c f] ‹|c| ≠ ∞›
    by (auto simp add: ∗ ereal-SUP-uminus-eq)
qed

lemma SUP-ereal-le-addI:
  fixes f :: 'i ⇒ ereal
  assumes ⋀i. f i + y ≤ z and y ≠ −∞
  shows Sup (f ‘ UNIV) + y ≤ z
  unfolding SUP-ereal-add-left[OF UNIV-not-empty ‹y ≠ −∞›, symmetric]
  by (rule SUP-least assms)+

lemma SUP-combine:
  fixes f :: 'a::semilattice-sup ⇒ 'a::semilattice-sup ⇒ 'b::complete-lattice
  assumes mono: ⋀a b c d. a ≤ b ⟹ c ≤ d ⟹ f a c ≤ f b d
  shows (SUP i∈UNIV. SUP j∈UNIV. f i j) = (SUP i. f i i)
proof (rule antisym)
  show (SUP i j. f i j) ≤ (SUP i. f i i)
    by (rule SUP-least SUP-upper2[where i=sup i j for i j] UNIV-I mono sup-ge1
sup-ge2)+
  show (SUP i. f i i) ≤ (SUP i j. f i j)
    by (rule SUP-least SUP-upper2 UNIV-I mono order-refl)+
qed

lemma SUP-ereal-add:
  fixes f g :: nat ⇒ ereal
  assumes inc: incseq f incseq g
    and pos: ⋀i. f i ≠ −∞ ⋀i. g i ≠ −∞
  shows (SUP i. f i + g i) = Sup (f ‘ UNIV) + Sup (g ‘ UNIV)
  apply (subst SUP-ereal-add-left[symmetric, OF UNIV-not-empty])
   apply (metis SUP-upper UNIV-I assms(4) ereal-infty-less-eq(2))
  apply (subst (2) add.commute)
  apply (subst SUP-ereal-add-left[symmetric, OF UNIV-not-empty assms(3)])
  apply (subst (2) add.commute)
   apply (rule SUP-combine[symmetric] add-mono inc[THEN monoD] | assump-
tion)+
  done

lemma INF-eq-minf: (INF i∈I. f i::ereal) ≠ −∞ ⟷ (∃ b>−∞. ∀ i∈I. b ≤ f i)
  unfolding bot-ereal-def[symmetric] INF-eq-bot-iff by (auto simp: not-less)

lemma INF-ereal-add-left:
  assumes I ≠ {} c ≠ −∞ ⋀x. x ∈ I ⟹ 0 ≤ f x
  shows (INF i∈I. f i + c :: ereal) = (INF i∈I. f i) + c
proof −
  have (INF i∈I. f i) ≠ −∞
    unfolding INF-eq-minf using assms by (intro exI[of - 0]) auto
  then show ?thesis
    by (subst continuous-at-Inf-mono[where f=λx. x + c])
     (auto simp: mono-def add-mono ‹I ≠ {}› ‹c ≠ −∞› continuous-at-imp-continuous-at-within
```

```
      continuous-at image-comp)
qed

lemma INF-ereal-add-right:
  assumes I ≠ {} c ≠ −∞ ⋀x. x ∈ I ⟹ 0 ≤ f x
  shows (INF i∈I. c + f i :: ereal) = c + (INF i∈I. f i)
  using INF-ereal-add-left[OF assms] by (simp add: ac-simps)

lemma INF-ereal-add-directed:
  fixes f g :: 'a ⇒ ereal
  assumes nonneg: ⋀i. i ∈ I ⟹ 0 ≤ f i ⋀i. i ∈ I ⟹ 0 ≤ g i
  assumes directed: ⋀i j. i ∈ I ⟹ j ∈ I ⟹ ∃k∈I. f i + g j ≥ f k + g k
  shows (INF i∈I. f i + g i) = (INF i∈I. f i) + (INF i∈I. g i)
proof cases
  assume I = {} then show ?thesis
    by (simp add: top-ereal-def)
next
  assume I ≠ {}
  show ?thesis
  proof (rule antisym)
    show (INF i∈I. f i) + (INF i∈I. g i) ≤ (INF i∈I. f i + g i)
      by (rule INF-greatest; intro add-mono INF-lower)
  next
    have (INF i∈I. f i + g i) ≤ (INF i∈I. (INF j∈I. f i + g j))
      using directed by (intro INF-greatest) (blast intro: INF-lower2)
    also have ... = (INF i∈I. f i + (INF i∈I. g i))
      using nonneg ‹I ≠ {}› by (auto simp: INF-ereal-add-right)
    also have ... = (INF i∈I. f i) + (INF i∈I. g i)
     using nonneg by (intro INF-ereal-add-left ‹I ≠ {}›) (auto simp: INF-eq-minf
intro!: exI[of - 0])
    finally show (INF i∈I. f i + g i) ≤ (INF i∈I. f i) + (INF i∈I. g i) .
  qed
qed

lemma INF-ereal-add:
  fixes f :: nat ⇒ ereal
  assumes decseq f decseq g
    and fin: ⋀i. f i ≠ ∞ ⋀i. g i ≠ ∞
  shows (INF i. f i + g i) = Inf (f ‘ UNIV) + Inf (g ‘ UNIV)
proof −
  have INF-less: (INF i. f i) < ∞ (INF i. g i) < ∞
    using assms unfolding INF-less-iff by auto
  { fix a b :: ereal assume a ≠ ∞ b ≠ ∞
    then have − ((− a) + (− b)) = a + b
      by (cases a b rule: ereal2-cases) auto }
  note ∗ = this
  have (INF i. f i + g i) = (INF i. − ((− f i) + (− g i)))
    by (simp add: fin ∗)
  also have ... = Inf (f ‘ UNIV) + Inf (g ‘ UNIV)
```

```
    unfolding ereal-INF-uminus-eq
    using assms INF-less
    by (subst SUP-ereal-add) (auto simp: ereal-SUP-uminus fin ∗)
  finally show ?thesis .
qed

lemma SUP-ereal-add-pos:
  fixes f g :: nat ⇒ ereal
  assumes inc: incseq f incseq g
    and pos: ⋀i. 0 ≤ f i ⋀i. 0 ≤ g i
  shows (SUP i. f i + g i) = Sup (f ‘ UNIV) + Sup (g ‘ UNIV)
proof (intro SUP-ereal-add inc)
  fix i
  show f i ≠ −∞ g i ≠ −∞
    using pos[of i] by auto
qed

lemma SUP-ereal-sum:
  fixes f g :: 'a ⇒ nat ⇒ ereal
  assumes ⋀n. n ∈ A ⟹ incseq (f n)
    and pos: ⋀n i. n ∈ A ⟹ 0 ≤ f n i
  shows (SUP i. ∑ n∈A. f n i) = (∑ n∈A. Sup ((f n) ‘ UNIV))
proof (cases finite A)
  case True
  then show ?thesis using assms
    by induct (auto simp: incseq-sumI2 sum-nonneg SUP-ereal-add-pos)
next
  case False
  then show ?thesis by simp
qed

lemma SUP-ereal-mult-left:
  fixes f :: 'a ⇒ ereal
  assumes I ≠ {}
  assumes f: ⋀i. i ∈ I ⟹ 0 ≤ f i and c: 0 ≤ c
  shows (SUP i∈I. c ∗ f i) = c ∗ (SUP i∈I. f i)
proof (cases (SUP i ∈ I. f i) = 0)
  case True
  then have ⋀i. i ∈ I ⟹ f i = 0
    by (metis SUP-upper f antisym)
  with True show ?thesis
    by simp
next
  case False
  then show ?thesis
    by (subst continuous-at-Sup-mono[where f=λx. c ∗ x])
       (auto simp: mono-def continuous-at continuous-at-imp-continuous-at-within
‹I ≠ {}› image-comp
             intro!: ereal-mult-left-mono c)
```

qed

lemma *countable-approach*:
fixes x :: *ereal*
assumes $x \neq -\infty$
shows $\exists f.$ *incseq* $f \wedge (\forall i{::}nat.\ f\ i < x) \wedge (f \longrightarrow x)$
proof (*cases* x)
case (*real* r)
moreover have $(\lambda n.\ r - inverse\ (real\ (Suc\ n))) \longrightarrow r - 0$
by (*intro tendsto-intros LIMSEQ-inverse-real-of-nat*)
ultimately show *?thesis*
by (*intro exI*[*of* - $\lambda n.\ x - inverse\ (Suc\ n)$]) (*auto simp*: *incseq-def*)
next
case *PInf* **with** *LIMSEQ-SUP*[*of* $\lambda n{::}nat.\ ereal\ (real\ n)$] **show** *?thesis*
by (*intro exI*[*of* - $\lambda n.\ ereal\ (real\ n)$]) (*auto simp*: *incseq-def SUP-nat-Infty*)
qed (*simp add*: *assms*)

lemma *Sup-countable-SUP*:
assumes $A \neq \{\}$
shows $\exists f{::}nat \Rightarrow ereal.\ incseq\ f \wedge range\ f \subseteq A \wedge Sup\ A = (SUP\ i.\ f\ i)$
proof *cases*
assume $Sup\ A = -\infty$
with ‹$A \neq \{\}$› **have** $A = \{-\infty\}$
by (*auto simp*: *Sup-eq-MInfty*)
then show *?thesis*
by (*auto intro*!: *exI*[*of* - λ-. $-\infty$] *simp*: *bot-ereal-def*)
next
assume $Sup\ A \neq -\infty$
then obtain l **where** *incseq* l **and** l: $l\ i < Sup\ A$ **and** *l-Sup*: $l \longrightarrow Sup\ A$
for i :: *nat*
by (*auto dest*: *countable-approach*)

have $\exists f.\ \forall n.\ (f\ n \in A \wedge l\ n \leq f\ n) \wedge (f\ n \leq f\ (Suc\ n))$ (**is** $\exists f.\ ?P\ f$)
proof (*rule dependent-nat-choice*)
show $\exists x.\ x \in A \wedge l\ 0 \leq x$
using *l*[*of 0*] **by** (*auto simp*: *less-Sup-iff*)
next
fix $x\ n$ **assume** $x \in A \wedge l\ n \leq x$
moreover from *l*[*of Suc n*] **obtain** y **where** $y \in A\ l\ (Suc\ n) < y$
by (*auto simp*: *less-Sup-iff*)
ultimately show $\exists y.\ (y \in A \wedge l\ (Suc\ n) \leq y) \wedge x \leq y$
by (*auto intro*!: *exI*[*of* - *max x y*] *split*: *split-max*)
qed
then obtain f **where** f: *?P f* **..**
then have *range* $f \subseteq A$ *incseq* f
by (*auto simp*: *incseq-Suc-iff*)
moreover
have $(SUP\ i.\ f\ i) = Sup\ A$
proof (*rule tendsto-unique*)

```
    show f ⟶ (SUP i. f i)
      by (rule LIMSEQ-SUP ‹incseq f›)+
    show f ⟶ Sup A
      using l f
      by (intro tendsto-sandwich[OF - - l-Sup tendsto-const])
        (auto simp: Sup-upper)
  qed simp
  ultimately show ?thesis
    by auto
qed

lemma Inf-countable-INF:
  assumes A ≠ {} shows ∃f::nat ⇒ ereal. decseq f ∧ range f ⊆ A ∧ Inf A =
(INF i. f i)
proof −
  obtain f where incseq f range f ⊆ uminus`A Sup (uminus`A) = (SUP i. f i)
    using Sup-countable-SUP[of uminus ` A] ‹A ≠ {}› by auto
  then show ?thesis
    by (intro exI[of - λx. − f x])
       (auto simp: ereal-Sup-uminus-image-eq ereal-INF-uminus-eq eq-commute[of
− -])
qed

lemma SUP-countable-SUP:
  A ≠ {} ⟹ ∃f::nat ⇒ ereal. range f ⊆ g`A ∧ Sup (g ` A) = Sup (f ` UNIV)
  using Sup-countable-SUP [of g`A] by auto
```

38.5 Relation to *enat*

```
definition ereal-of-enat n = (case n of enat n ⇒ ereal (real n) | ∞ ⇒ ∞)

declare [[coercion ereal-of-enat :: enat ⇒ ereal]]
declare [[coercion (λn. ereal (real n)) :: nat ⇒ ereal]]

lemma ereal-of-enat-simps[simp]:
  ereal-of-enat (enat n) = ereal n
  ereal-of-enat ∞ = ∞
  by (simp-all add: ereal-of-enat-def)

lemma ereal-of-enat-le-iff[simp]: ereal-of-enat m ≤ ereal-of-enat n ⟷ m ≤ n
  by (cases m n rule: enat2-cases) auto

lemma ereal-of-enat-less-iff[simp]: ereal-of-enat m < ereal-of-enat n ⟷ m < n
  by (cases m n rule: enat2-cases) auto

lemma numeral-le-ereal-of-enat-iff[simp]: numeral m ≤ ereal-of-enat n ⟷ nu-
meral m ≤ n
by (cases n) (auto)
```

lemma *numeral-less-ereal-of-enat-iff*[*simp*]: *numeral m* < *ereal-of-enat n* ⟷ *numeral m* < *n*
by (*cases n*) *auto*

lemma *ereal-of-enat-ge-zero-cancel-iff*[*simp*]: *0* ≤ *ereal-of-enat n* ⟷ *0* ≤ *n*
by (*cases n*) (*auto simp flip*: *enat-0*)

lemma *ereal-of-enat-gt-zero-cancel-iff*[*simp*]: *0* < *ereal-of-enat n* ⟷ *0* < *n*
by (*cases n*) (*auto simp flip*: *enat-0*)

lemma *ereal-of-enat-zero*[*simp*]: *ereal-of-enat 0* = *0*
by (*auto simp flip*: *enat-0*)

lemma *ereal-of-enat-inf*[*simp*]: *ereal-of-enat n* = ∞ ⟷ *n* = ∞
by (*cases n*) *auto*

lemma *ereal-of-enat-add*: *ereal-of-enat* (*m* + *n*) = *ereal-of-enat m* + *ereal-of-enat n*
by (*cases m n rule*: *enat2-cases*) *auto*

lemma *ereal-of-enat-sub*:
assumes *n* ≤ *m*
shows *ereal-of-enat* (*m* − *n*) = *ereal-of-enat m* − *ereal-of-enat n*
using *assms* **by** (*cases m n rule*: *enat2-cases*) *auto*

lemma *ereal-of-enat-mult*:
ereal-of-enat (*m* ∗ *n*) = *ereal-of-enat m* ∗ *ereal-of-enat n*
by (*cases m n rule*: *enat2-cases*) *auto*

lemmas *ereal-of-enat-pushin* = *ereal-of-enat-add ereal-of-enat-sub ereal-of-enat-mult*
lemmas *ereal-of-enat-pushout* = *ereal-of-enat-pushin*[*symmetric*]

lemma *ereal-of-enat-nonneg*: *ereal-of-enat n* ≥ *0*
by(*cases n*) *simp-all*

lemma *ereal-of-enat-Sup*:
assumes *A* ≠ {} **shows** *ereal-of-enat* (*Sup A*) = (*SUP a* ∈ *A*. *ereal-of-enat a*)
proof (*intro antisym mono-Sup*)
show *ereal-of-enat* (*Sup A*) ≤ (*SUP a* ∈ *A*. *ereal-of-enat a*)
proof *cases*
assume *finite A*
with ‹*A* ≠ {}› **obtain** *a* **where** *a* ∈ *A ereal-of-enat* (*Sup A*) = *ereal-of-enat a*
using *Max-in*[*of A*] **by** (*auto simp*: *Sup-enat-def simp del*: *Max-in*)
then show *?thesis*
by (*auto intro*: *SUP-upper*)
next
assume ¬ *finite A*
have [*simp*]: (*SUP a* ∈ *A*. *ereal-of-enat a*) = *top*
unfolding *SUP-eq-top-iff*

```
    proof safe
      fix x :: ereal assume x < top
      then obtain n :: nat where x < n
        using less-PInf-Ex-of-nat top-ereal-def by auto
      obtain a where a ∈ A − enat ' {.. n}
       by (metis ‹¬ finite A› all-not-in-conv finite-Diff2 finite-atMost finite-imageI
finite.emptyI)
      then have a ∈ A ereal n ≤ ereal-of-enat a
        by (auto simp: image-iff Ball-def)
        (metis enat-iless enat-ord-simps(1) ereal-of-enat-less-iff ereal-of-enat-simps(1)
less-le not-less)
      with ‹x < n› show ∃ i∈A. x < ereal-of-enat i
        by (auto intro!: bexI[of - a])
    qed
    show ?thesis
      by simp
  qed
qed (simp add: mono-def)

lemma ereal-of-enat-SUP:
  A ≠ {} ⟹ ereal-of-enat (SUP a∈A. f a) = (SUP a ∈ A. ereal-of-enat (f a))
  by (simp add: ereal-of-enat-Sup image-comp)
```

38.6 Limits on *ereal*

```
lemma open-PInfty: open A ⟹ ∞ ∈ A ⟹ (∃ x. {ereal x<..} ⊆ A)
  unfolding open-ereal-generated
proof (induct rule: generate-topology.induct)
  case (Int A B)
  then obtain x z where ∞ ∈ A ⟹ {ereal x <..} ⊆ A ∞ ∈ B ⟹ {ereal z <..}
⊆ B
    by auto
  with Int show ?case
    by (intro exI[of - max x z]) fastforce
next
  case (Basis S)
  {
    fix x
    have x ≠ ∞ ⟹ ∃ t. x ≤ ereal t
      by (cases x) auto
  }
  moreover note Basis
  ultimately show ?case
    by (auto split: ereal.split)
qed (fastforce simp add: vimage-Union)+

lemma open-MInfty: open A ⟹ −∞ ∈ A ⟹ (∃ x. {..<ereal x} ⊆ A)
  unfolding open-ereal-generated
proof (induct rule: generate-topology.induct)
```

case (*Int A B*)
then obtain *x z* **where** $-\infty \in A \Longrightarrow \{..< ereal\ x\} \subseteq A\ -\infty \in B \Longrightarrow \{..< ereal\ z\} \subseteq B$
by *auto*
with *Int* **show** *?case*
by (*intro exI*[*of - min x z*]) *fastforce*
next
case (*Basis S*)
{
fix *x*
have $x \neq -\infty \Longrightarrow \exists\, t.\ ereal\ t \leq x$
by (*cases x*) *auto*
}
moreover note *Basis*
ultimately show *?case*
by (*auto split*: *ereal.split*)
qed (*fastforce simp add*: *vimage-Union*)+

lemma *open-ereal-vimage*: *open S* $\Longrightarrow$ *open* (*ereal* $-$‘ *S*)
by (*intro open-vimage continuous-intros*)

lemma *open-ereal*: *open S* $\Longrightarrow$ *open* (*ereal* ‘ *S*)
unfolding *open-generated-order*[**where** *'a=real*]
proof (*induct rule*: *generate-topology.induct*)
case (*Basis S*)
moreover have $\bigwedge x.\ ereal$ ‘ $\{..< x\} = \{ -\infty <..< ereal\ x \}$
using *ereal-less-ereal-Ex* **by** *auto*
moreover have $\bigwedge x.\ ereal$ ‘ $\{x <..\} = \{ ereal\ x <..< \infty \}$
using *less-ereal.elims*(*2*) **by** *fastforce*
ultimately show *?case*
by *auto*
qed (*auto simp add*: *image-Union image-Int*)

lemma *open-image-real-of-ereal*:
fixes *X*::*ereal set*
assumes *open X*
assumes *infty*: $\infty \notin X\ -\infty \notin X$
shows *open* (*real-of-ereal* ‘ *X*)
proof −
have *real-of-ereal* ‘ *X* = *ereal* $-$‘ *X*
using *infty ereal-real* **by** (*force simp*: *set-eq-iff*)
thus *?thesis*
by (*auto intro*!: *open-ereal-vimage assms*)
qed

lemma *eventually-finite*:
fixes *x* :: *ereal*
assumes $|x| \neq \infty$ $(f \longrightarrow x)\ F$
shows *eventually* $(\lambda x.\ |f\ x| \neq \infty)\ F$

proof −
have $(f \longrightarrow ereal\ (real\text{-}of\text{-}ereal\ x))\ F$
using *assms* **by** (*cases x*) *auto*
then have *eventually* $(\lambda x.\ f\ x \in ereal\ `\ UNIV)\ F$
by (*rule topological-tendstoD*) (*auto intro*: *open-ereal*)
also have $(\lambda x.\ f\ x \in ereal\ `\ UNIV) = (\lambda x.\ |f\ x| \neq \infty)$
by *auto*
finally show *?thesis* **.**
qed

lemma *open-ereal-def*:
$open\ A \longleftrightarrow open\ (ereal\ -`\ A) \wedge (\infty \in A \longrightarrow (\exists x.\ \{ereal\ x <..\} \subseteq A)) \wedge (-\infty \in A \longrightarrow (\exists x.\ \{..< ereal\ x\} \subseteq A))$
(**is** $open\ A \longleftrightarrow$ *?rhs*)
proof
assume *open A*
then show *?rhs*
using *open-PInfty open-MInfty open-ereal-vimage* **by** *auto*
next
assume *?rhs*
then obtain *x y* **where** *A*: $open\ (ereal\ -`\ A)\ \infty \in A \Longrightarrow \{ereal\ x<..\} \subseteq A\ -\infty \in A \Longrightarrow \{..<\ ereal\ y\} \subseteq A$
by *auto*
have $*$: $A = ereal\ `\ (ereal\ -`\ A) \cup (if\ \infty \in A\ then\ \{ereal\ x<..\}\ else\ \{\}) \cup (if\ -\infty \in A\ then\ \{..<\ ereal\ y\}\ else\ \{\})$
using *A*(*2*,*3*) **by** *auto*
from *open-ereal*[*OF A*(*1*)] **show** *open A*
by (*subst* $*$) (*auto simp*: *open-Un*)
qed

lemma *open-PInfty2*:
assumes *open A*
and $\infty \in A$
obtains *x* **where** $\{ereal\ x<..\} \subseteq A$
using *open-PInfty*[*OF assms*] **by** *auto*

lemma *open-MInfty2*:
assumes *open A*
and $-\infty \in A$
obtains *x* **where** $\{..<ereal\ x\} \subseteq A$
using *open-MInfty*[*OF assms*] **by** *auto*

lemma *ereal-openE*:
assumes *open A*
obtains *x y* **where** *open* $(ereal\ -`\ A)$
and $\infty \in A \Longrightarrow \{ereal\ x<..\} \subseteq A$
and $-\infty \in A \Longrightarrow \{..<ereal\ y\} \subseteq A$
using *assms open-ereal-def* **by** *auto*

lemmas *open-ereal-lessThan* = *open-lessThan*[**where** *'a=ereal*]
lemmas *open-ereal-greaterThan* = *open-greaterThan*[**where** *'a=ereal*]
lemmas *ereal-open-greaterThanLessThan* = *open-greaterThanLessThan*[**where** *'a=ereal*]
lemmas *closed-ereal-atLeast* = *closed-atLeast*[**where** *'a=ereal*]
lemmas *closed-ereal-atMost* = *closed-atMost*[**where** *'a=ereal*]
lemmas *closed-ereal-atLeastAtMost* = *closed-atLeastAtMost*[**where** *'a=ereal*]
lemmas *closed-ereal-singleton* = *closed-singleton*[**where** *'a=ereal*]

lemma *ereal-open-cont-interval*:
  fixes *S* :: *ereal set*
  assumes *open S*
    and $x \in S$
    and $|x| \neq \infty$
  obtains *e* **where** $e > 0$ **and** $\{x-e <..< x+e\} \subseteq S$
proof −
  from ‹*open S*›
  have *open* (*ereal* −‘ *S*)
    by (*rule ereal-openE*)
  then obtain *e* **where** $e > 0$ **and** *e*: *dist y* (*real-of-ereal x*) $< e \Longrightarrow$ *ereal y* $\in S$
for *y*
    using *assms* **unfolding** *open-dist* **by** *force*
  show *thesis*
  proof (*intro that subsetI*)
    show $0 <$ *ereal e*
      using ‹$0 < e$› **by** *auto*
    fix *y*
    assume $y \in \{x -$ *ereal e*$<..<x +$ *ereal e*$\}$
    with *assms* **obtain** *t* **where** $y =$ *ereal t dist t* (*real-of-ereal x*) $< e$
      by (*cases y*) (*auto simp*: *dist-real-def*)
    then show $y \in S$
      using *e*[*of t*] **by** *auto*
  qed
qed

lemma *ereal-open-cont-interval2*:
  fixes *S* :: *ereal set*
  assumes *open S*
    and $x \in S$
    and *x*: $|x| \neq \infty$
  obtains *a b* **where** $a < x$ **and** $x < b$ **and** $\{a <..< b\} \subseteq S$
proof −
  obtain *e* **where** $0 < e$ $\{x -$ *e*$<..<x +$ *e*$\} \subseteq S$
    using *assms* **by** (*rule ereal-open-cont-interval*)
  with *that*[*of x* − *e x* + *e*] *ereal-between*[*OF x*, *of e*]
  show *thesis*
    by *auto*
qed

38.6.1 Convergent sequences

lemma *lim-real-of-ereal*[*simp*]:
assumes *lim*: $(f \longrightarrow ereal\ x)\ net$
shows $((\lambda x.\ real\text{-}of\text{-}ereal\ (f\ x)) \longrightarrow x)\ net$
proof (*intro topological-tendstoI*)
fix S
assume *open* S **and** $x \in S$
then have S: *open* S $ereal\ x \in ereal\ `\ S$
by (*simp-all add*: *inj-image-mem-iff*)
show *eventually* $(\lambda x.\ real\text{-}of\text{-}ereal\ (f\ x)) \in S)\ net$
by (*auto intro*: *eventually-mono* [*OF lim*[*THEN topological-tendstoD*, *OF open-ereal*, *OF S*]])
qed

lemma *lim-ereal*[*simp*]: $((\lambda n.\ ereal\ (f\ n)) \longrightarrow ereal\ x)\ net \longleftrightarrow (f \longrightarrow x)\ net$
by (*auto dest*!: *lim-real-of-ereal*)

lemma *convergent-real-imp-convergent-ereal*:
assumes *convergent a*
shows *convergent* $(\lambda n.\ ereal\ (a\ n))$ **and** $lim\ (\lambda n.\ ereal\ (a\ n)) = ereal\ (lim\ a)$
proof −
from *assms* **obtain** L **where** L: $a \longrightarrow L$ **unfolding** *convergent-def* **..**
hence *lim*: $(\lambda n.\ ereal\ (a\ n)) \longrightarrow ereal\ L$ **using** *lim-ereal* **by** *auto*
thus *convergent* $(\lambda n.\ ereal\ (a\ n))$ **unfolding** *convergent-def* **..**
thus $lim\ (\lambda n.\ ereal\ (a\ n)) = ereal\ (lim\ a)$ **using** *lim L limI* **by** *metis*
qed

lemma *tendsto-PInfty*: $(f \longrightarrow \infty)\ F \longleftrightarrow (\forall r.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F)$
proof −
{
fix $l :: ereal$
assume $\forall r.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
from *this*[*THEN spec*, *of real-of-ereal l*] **have** $l \neq \infty \Longrightarrow eventually\ (\lambda x.\ l < f\ x)\ F$
by (*cases l*) (*auto elim*: *eventually-mono*)
}
then show *?thesis*
by (*auto simp*: *order-tendsto-iff*)
qed

lemma *tendsto-PInfty'*: $(f \longrightarrow \infty)\ F = (\forall r{>}c.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F)$
proof (*subst tendsto-PInfty*, *intro iffI allI impI*)
assume A: $\forall r{>}c.\ eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
fix $r :: real$
from A **have** A: $eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$ **if** $r > c$ **for** r **using** *that* **by** *blast*
show $eventually\ (\lambda x.\ ereal\ r < f\ x)\ F$
proof (*cases* $r > c$)

```
    case False
    hence B: ereal r ≤ ereal (c + 1) by simp
    have c < c + 1 by simp
    from A[OF this] show eventually (λx. ereal r < f x) F
      by eventually-elim (rule le-less-trans[OF B])
  qed (simp add: A)
qed simp

lemma tendsto-PInfty-eq-at-top:
  ((λz. ereal (f z)) ⟶ ∞) F ⟷ (LIM z F. f z :> at-top)
  unfolding tendsto-PInfty filterlim-at-top-dense by simp

lemma tendsto-MInfty: (f ⟶ −∞) F ⟷ (∀ r. eventually (λx. f x < ereal r)
F)
  unfolding tendsto-def
proof safe
  fix S :: ereal set
  assume open S −∞ ∈ S
  from open-MInfty[OF this] obtain B where {..<ereal B} ⊆ S ..
  moreover
  assume ∀ r::real. eventually (λz. f z < r) F
  then have eventually (λz. f z ∈ {..< B}) F
    by auto
  ultimately show eventually (λz. f z ∈ S) F
    by (auto elim!: eventually-mono)
next
  fix x
  assume ∀ S. open S ⟶ −∞ ∈ S ⟶ eventually (λx. f x ∈ S) F
  from this[rule-format, of {..< ereal x}] show eventually (λy. f y < ereal x) F
    by auto
qed

lemma tendsto-MInfty': (f ⟶ −∞) F = (∀ r<c. eventually (λx. ereal r > f x)
F)
proof (subst tendsto-MInfty, intro iffI allI impI)
  assume A: ∀ r<c. eventually (λx. ereal r > f x) F
  fix r :: real
  from A have A: eventually (λx. ereal r > f x) F if r < c for r using that by
blast
  show eventually (λx. ereal r > f x) F
  proof (cases r < c)
    case False
    hence B: ereal r ≥ ereal (c − 1) by simp
    have c > c − 1 by simp
    from A[OF this] show eventually (λx. ereal r > f x) F
      by eventually-elim (erule less-le-trans[OF - B])
  qed (simp add: A)
qed simp
```

lemma *Lim-PInfty*: $f \longrightarrow \infty \longleftrightarrow (\forall B.\ \exists N.\ \forall n{\geq}N.\ f\ n \geq ereal\ B)$
unfolding *tendsto-PInfty eventually-sequentially*
proof *safe*
fix *r*
assume $\forall r.\ \exists N.\ \forall n{\geq}N.\ ereal\ r \leq f\ n$
then obtain *N* **where** $\forall n{\geq}N.\ ereal\ (r + 1) \leq f\ n$
by *blast*
moreover have $ereal\ r < ereal\ (r + 1)$
by *auto*
ultimately show $\exists N.\ \forall n{\geq}N.\ ereal\ r < f\ n$
by (*blast intro*: *less-le-trans*)
qed (*blast intro*: *less-imp-le*)

lemma *Lim-MInfty*: $f \longrightarrow -\infty \longleftrightarrow (\forall B.\ \exists N.\ \forall n{\geq}N.\ ereal\ B \geq f\ n)$
unfolding *tendsto-MInfty eventually-sequentially*
proof *safe*
fix *r*
assume $\forall r.\ \exists N.\ \forall n{\geq}N.\ f\ n \leq ereal\ r$
then obtain *N* **where** $\forall n{\geq}N.\ f\ n \leq ereal\ (r - 1)$
by *blast*
moreover have $ereal\ (r - 1) < ereal\ r$
by *auto*
ultimately show $\exists N.\ \forall n{\geq}N.\ f\ n < ereal\ r$
by (*blast intro*: *le-less-trans*)
qed (*blast intro*: *less-imp-le*)

lemma *Lim-bounded-PInfty*: $f \longrightarrow l \Longrightarrow (\bigwedge n.\ f\ n \leq ereal\ B) \Longrightarrow l \neq \infty$
using *LIMSEQ-le-const2*[*of f l ereal B*] **by** *auto*

lemma *Lim-bounded-MInfty*: $f \longrightarrow l \Longrightarrow (\bigwedge n.\ ereal\ B \leq f\ n) \Longrightarrow l \neq -\infty$
using *LIMSEQ-le-const*[*of f l ereal B*] **by** *auto*

lemma *tendsto-zero-erealI*:
assumes $\bigwedge e.\ e > 0 \Longrightarrow eventually\ (\lambda x.\ |f\ x| < ereal\ e)\ F$
shows $(f \longrightarrow 0)\ F$
proof (*subst filterlim-cong*[*OF refl refl*])
from *assms*[*OF zero-less-one*] **show** *eventually* ($\lambda x.\ f\ x = ereal\ (real\text{-}of\text{-}ereal\ (f\ x)))\ F$
by *eventually-elim* (*auto simp*: *ereal-real*)
hence *eventually* ($\lambda x.\ abs\ (real\text{-}of\text{-}ereal\ (f\ x)) < e$) *F* **if** $e > 0$ **for** *e* **using** *assms*[*OF that*]
by *eventually-elim* (*simp add*: *real-less-ereal-iff that*)
hence $((\lambda x.\ real\text{-}of\text{-}ereal\ (f\ x)) \longrightarrow 0)\ F$ **unfolding** *tendsto-iff*
by (*auto simp*: *tendsto-iff dist-real-def*)
thus $((\lambda x.\ ereal\ (real\text{-}of\text{-}ereal\ (f\ x))) \longrightarrow 0)\ F$ **by** (*simp add*: *zero-ereal-def*)
qed

lemma *Lim-bounded-PInfty2*: $f \longrightarrow l \Longrightarrow \forall n{\geq}N.\ f\ n \leq ereal\ B \Longrightarrow l \neq \infty$
using *LIMSEQ-le-const2*[*of f l ereal B*] **by** *fastforce*

lemma *real-of-ereal-mult*[*simp*]:
fixes $a\ b$ :: *ereal*
shows *real-of-ereal* $(a * b) =$ *real-of-ereal* $a *$ *real-of-ereal* b
by (*cases rule*: *ereal2-cases*[*of* $a\ b$]) *auto*

lemma *real-of-ereal-eq-0*:
fixes x :: *ereal*
shows *real-of-ereal* $x = 0 \longleftrightarrow x = \infty \vee x = -\infty \vee x = 0$
by (*cases* x) *auto*

lemma *tendsto-ereal-realD*:
fixes $f :: 'a \Rightarrow$ *ereal*
assumes $x \neq 0$
and *tendsto*: $((\lambda x.$ *ereal* (*real-of-ereal* $(f\ x))) \longrightarrow x)$ *net*
shows $(f \longrightarrow x)$ *net*
proof (*intro topological-tendstoI*)
fix S
assume S: *open* $S\ x \in S$
with ‹$x \neq 0$› **have** *open* $(S - \{0\})\ x \in S - \{0\}$
by *auto*
from *tendsto*[*THEN topological-tendstoD*, *OF this*]
show *eventually* $(\lambda x.\ f\ x \in S)$ *net*
by (*rule eventually-rev-mp*) (*auto simp*: *ereal-real*)
qed

lemma *tendsto-ereal-realI*:
fixes $f :: 'a \Rightarrow$ *ereal*
assumes x: $|x| \neq \infty$ **and** *tendsto*: $(f \longrightarrow x)$ *net*
shows $((\lambda x.$ *ereal* (*real-of-ereal* $(f\ x))) \longrightarrow x)$ *net*
proof (*intro topological-tendstoI*)
fix S
assume *open* S **and** $x \in S$
with x **have** *open* $(S - \{\infty, -\infty\})\ x \in S - \{\infty, -\infty\}$
by *auto*
from *tendsto*[*THEN topological-tendstoD*, *OF this*]
show *eventually* $(\lambda x.$ *ereal* (*real-of-ereal* $(f\ x)) \in S)$ *net*
by (*elim eventually-mono*) (*auto simp*: *ereal-real*)
qed

lemma *ereal-mult-cancel-left*:
fixes $a\ b\ c$ :: *ereal*
shows $a * b = a * c \longleftrightarrow (|a| = \infty \wedge 0 < b * c) \vee a = 0 \vee b = c$
by (*cases rule*: *ereal3-cases*[*of* $a\ b\ c$]) (*simp-all add*: *zero-less-mult-iff*)

lemma *tendsto-add-ereal*:
fixes $x\ y$ :: *ereal*
assumes x: $|x| \neq \infty$ **and** y: $|y| \neq \infty$
assumes f: $(f \longrightarrow x)\ F$ **and** g: $(g \longrightarrow y)\ F$

```
  shows ((λx. f x + g x) ⟶ x + y) F
proof −
  from x obtain r where x': x = ereal r by (cases x) auto
  with f have ((λi. real-of-ereal (f i)) ⟶ r) F by simp
  moreover
  from y obtain p where y': y = ereal p by (cases y) auto
  with g have ((λi. real-of-ereal (g i)) ⟶ p) F by simp
  ultimately have ((λi. real-of-ereal (f i) + real-of-ereal (g i)) ⟶ r + p) F
    by (rule tendsto-add)
  moreover
  from eventually-finite[OF x f] eventually-finite[OF y g]
  have eventually (λx. f x + g x = ereal (real-of-ereal (f x) + real-of-ereal (g x)))
F
    by eventually-elim auto
  ultimately show ?thesis
    by (simp add: x' y' cong: filterlim-cong)
qed

lemma tendsto-add-ereal-nonneg:
  fixes x y :: ereal
  assumes x ≠ −∞ y ≠ −∞ (f ⟶ x) F (g ⟶ y) F
  shows ((λx. f x + g x) ⟶ x + y) F
proof cases
  assume x = ∞ ∨ y = ∞
  moreover
  { fix y :: ereal and f g :: 'a ⇒ ereal assume y ≠ −∞ (f ⟶ ∞) F (g ⟶
y) F
    then obtain y' where −∞ < y' y' < y
      using dense[of −∞ y] by auto
    have ((λx. f x + g x) ⟶ ∞) F
    proof (rule tendsto-sandwich)
      have ∀F x in F. y' < g x
        using order-tendstoD(1)[OF ‹(g ⟶ y) F› ‹y' < y›] by auto
      then show ∀F x in F. f x + y' ≤ f x + g x
        by eventually-elim (auto intro!: add-mono)
      show ∀F n in F. f n + g n ≤ ∞ ((λn. ∞) ⟶ ∞) F
        by auto
      show ((λx. f x + y') ⟶ ∞) F
        using tendsto-cadd-ereal[of y' ∞ f F] ‹(f ⟶ ∞) F› ‹−∞ < y'› by auto
    qed }
  note this[of y f g] this[of x g f]
  ultimately show ?thesis
    using assms by (auto simp: add-ac)
next
  assume ¬ (x = ∞ ∨ y = ∞)
  with assms tendsto-add-ereal[of x y f F g]
  show ?thesis
    by auto
qed
```

lemma *ereal-inj-affinity*:
fixes $m\ t :: ereal$
assumes $|m| \neq \infty$
and $m \neq 0$
and $|t| \neq \infty$
shows *inj-on* $(\lambda x.\ m * x + t)\ A$
using *assms*
by (*cases rule*: *ereal2-cases*[*of m t*])
(*auto intro*!: *inj-onI simp*: *ereal-add-cancel-right ereal-mult-cancel-left*)

lemma *ereal-PInfty-eq-plus*[*simp*]:
fixes $a\ b :: ereal$
shows $\infty = a + b \longleftrightarrow a = \infty \vee b = \infty$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-MInfty-eq-plus*[*simp*]:
fixes $a\ b :: ereal$
shows $-\infty = a + b \longleftrightarrow (a = -\infty \wedge b \neq \infty) \vee (b = -\infty \wedge a \neq \infty)$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-less-divide-pos*:
fixes $x\ y :: ereal$
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow y < z\ /\ x \longleftrightarrow x * y < z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-less-pos*:
fixes $x\ y\ z :: ereal$
shows $x > 0 \Longrightarrow x \neq \infty \Longrightarrow y\ /\ x < z \longleftrightarrow y < x * z$
by (*cases rule*: *ereal3-cases*[*of x y z*]) (*auto simp*: *field-simps*)

lemma *ereal-divide-eq*:
fixes $a\ b\ c :: ereal$
shows $b \neq 0 \Longrightarrow |b| \neq \infty \Longrightarrow a\ /\ b = c \longleftrightarrow a = b * c$
by (*cases rule*: *ereal3-cases*[*of a b c*])
(*simp-all add*: *field-simps*)

lemma *ereal-inverse-not-MInfty*[*simp*]: *inverse* $(a::ereal) \neq -\infty$
by (*cases a*) *auto*

lemma *ereal-mult-m1*[*simp*]: $x * ereal\ (-1) = -x$
by (*cases x*) *auto*

lemma *ereal-real'*:
assumes $|x| \neq \infty$
shows *ereal* (*real-of-ereal x*) $= x$
using *assms* **by** *auto*

lemma *real-ereal-id*: *real-of-ereal* $\circ$ *ereal* $= id$

proof −
  {
    fix *x*
    have (*real-of-ereal* ∘ *ereal*) *x* = *id x*
      by *auto*
  }
  then show *?thesis*
    using *ext* **by** *blast*
qed

lemma *open-image-ereal*: *open*(*UNIV*−{ ∞ , (−∞ :: *ereal*)})
  by (*metis range-ereal open-ereal open-UNIV*)

lemma *ereal-le-distrib*:
  fixes *a b c* :: *ereal*
  shows *c* ∗ (*a* + *b*) ≤ *c* ∗ *a* + *c* ∗ *b*
  by (*cases rule*: *ereal3-cases*[*of a b c*])
     (*auto simp add*: *field-simps not-le mult-le-0-iff mult-less-0-iff*)

lemma *ereal-pos-distrib*:
  fixes *a b c* :: *ereal*
  assumes *0* ≤ *c*
    and *c* ≠ ∞
  shows *c* ∗ (*a* + *b*) = *c* ∗ *a* + *c* ∗ *b*
  using *assms*
  by (*cases rule*: *ereal3-cases*[*of a b c*])
    (*auto simp add*: *field-simps not-le mult-le-0-iff mult-less-0-iff*)

lemma *ereal-LimI-finite*:
  fixes *x* :: *ereal*
  assumes |*x*| ≠ ∞
    and ⋀*r*. *0* < *r* ⟹ ∃ *N*. ∀ *n*≥*N*. *u n* < *x* + *r* ∧ *x* < *u n* + *r*
  shows *u* ⟶ *x*
proof (*rule topological-tendstoI*, *unfold eventually-sequentially*)
  obtain *rx* **where** *rx*: *x* = *ereal rx*
    using *assms* **by** (*cases x*) *auto*
  fix *S*
  assume *open S* **and** *x* ∈ *S*
  then have *open* (*ereal* −‘ *S*)
    unfolding *open-ereal-def* **by** *auto*
  with ‹*x* ∈ *S*› **obtain** *r* **where** *0* < *r* **and** *dist*: *dist y rx* < *r* ⟹ *ereal y* ∈ *S*
for *y*
    unfolding *open-dist rx* **by** *auto*
  then obtain *n*
    where *upper*: *u N* < *x* + *ereal r*
      and *lower*: *x* < *u N* + *ereal r*
      if *n* ≤ *N* **for** *N*
    using *assms*(*2*)[*of ereal r*] **by** *auto*
  show ∃ *N*. ∀ *n*≥*N*. *u n* ∈ *S*

```
  proof (safe intro!: exI[of - n])
    fix N
    assume n ≤ N
    from upper[OF this] lower[OF this] assms ‹0 < r›
    have u N ∉ {∞,(−∞)}
      by auto
    then obtain ra where ra-def: (u N) = ereal ra
      by (cases u N) auto
    then have rx < ra + r and ra < rx + r
      using rx assms ‹0 < r› lower[OF ‹n ≤ N›] upper[OF ‹n ≤ N›]
      by auto
    then have dist (real-of-ereal (u N)) rx < r
      using rx ra-def
      by (auto simp: dist-real-def abs-diff-less-iff field-simps)
    from dist[OF this] show u N ∈ S
      using ‹u N ∉ {∞, −∞}›
      by (auto simp: ereal-real split: if-split-asm)
  qed
qed

lemma tendsto-obtains-N:
  assumes f ⟶ f0
  assumes open S
    and f0 ∈ S
  obtains N where ∀ n≥N. f n ∈ S
  using assms using tendsto-def
  using lim-explicit[of f f0] assms by auto

lemma ereal-LimI-finite-iff:
  fixes x :: ereal
  assumes |x| ≠ ∞
  shows u ⟶ x ⟷ (∀ r. 0 < r ⟶ (∃ N. ∀ n≥N. u n < x + r ∧ x < u n
+ r))
  (is ?lhs ⟷ ?rhs)
proof
  assume lim: u ⟶ x
  {
    fix r :: ereal
    assume r > 0
    then obtain N where ∀ n≥N. u n ∈ {x − r <..< x + r}
      apply (subst tendsto-obtains-N[of u x {x − r <..< x + r}])
      using lim ereal-between[of x r] assms ‹r > 0›
      apply auto
      done
    then have ∃ N. ∀ n≥N. u n < x + r ∧ x < u n + r
      using ereal-minus-less[of r x]
      by (cases r) auto
  }
  then show ?rhs
```

```
    by auto
next
  assume ?rhs
  then show u ⟶ x
    using ereal-LimI-finite[of x] assms by auto
qed

lemma ereal-Limsup-uminus:
  fixes f :: 'a ⇒ ereal
  shows Limsup net (λx. − (f x)) = − Liminf net f
  unfolding Limsup-def Liminf-def ereal-SUP-uminus ereal-INF-uminus-eq ..

lemma liminf-bounded-iff:
  fixes x :: nat ⇒ ereal
  shows C ≤ liminf x ⟷ (∀ B<C. ∃ N. ∀ n≥N. B < x n)
  (is ?lhs ⟷ ?rhs)
  unfolding le-Liminf-iff eventually-sequentially ..

lemma Liminf-add-le:
  fixes f g :: - ⇒ ereal
  assumes F: F ≠ bot
  assumes ev: eventually (λx. 0 ≤ f x) F eventually (λx. 0 ≤ g x) F
  shows Liminf F f + Liminf F g ≤ Liminf F (λx. f x + g x)
  unfolding Liminf-def
proof (subst SUP-ereal-add-left[symmetric])
  let ?F = {P. eventually P F}
  let ?INF = λP g. Inf (g ' (Collect P))
  show ?F ≠ {}
    by (auto intro: eventually-True)
  show (SUP P∈?F. ?INF P g) ≠ − ∞
    unfolding bot-ereal-def[symmetric] SUP-bot-conv INF-eq-bot-iff
    by (auto intro!: exI[of - 0] ev simp: bot-ereal-def)
  have (SUP P∈?F. ?INF P f + (SUP P∈?F. ?INF P g)) ≤ (SUP P∈?F. (SUP
P'∈?F. ?INF P f + ?INF P' g))
  proof (safe intro!: SUP-mono bexI[of - λx. P x ∧ 0 ≤ f x for P])
    fix P let ?P' = λx. P x ∧ 0 ≤ f x
    assume eventually P F
    with ev show eventually ?P' F
      by eventually-elim auto
    have ?INF P f + (SUP P∈?F. ?INF P g) ≤ ?INF ?P' f + (SUP P∈?F. ?INF
P g)
      by (intro add-mono INF-mono) auto
    also have ... = (SUP P'∈?F. ?INF ?P' f + ?INF P' g)
    proof (rule SUP-ereal-add-right[symmetric])
      show Inf (f ' {x. P x ∧ 0 ≤ f x}) ≠ − ∞
        unfolding bot-ereal-def[symmetric] INF-eq-bot-iff
        by (auto intro!: exI[of - 0] ev simp: bot-ereal-def)
    qed fact
    finally show ?INF P f + (SUP P∈?F. ?INF P g) ≤ (SUP P'∈?F. ?INF ?P'
```

f + ?INF P' g) **.**
qed
also have ... ≤ (*SUP P∈?F. INF x∈Collect P. f x + g x*)
proof (*safe intro*!: *SUP-least*)
fix *P Q* **assume** ∗: *eventually P F eventually Q F*
show *?INF P f + ?INF Q g* ≤ (*SUP P∈?F. INF x∈Collect P. f x + g x*)
proof (*rule SUP-upper2*)
show (λ*x. P x* ∧ *Q x*) ∈ *?F*
using ∗ **by** (*auto simp*: *eventually-conj*)
show *?INF P f + ?INF Q g* ≤ (*INF x∈*{*x. P x* ∧ *Q x*}. *f x + g x*)
by (*intro INF-greatest add-mono*) (*auto intro*: *INF-lower*)
qed
qed
finally show (*SUP P∈?F. ?INF P f* + (*SUP P∈?F. ?INF P g*)) ≤ (*SUP P∈?F. INF x∈Collect P. f x + g x*) **.**
qed

lemma *Sup-ereal-mult-right'*:
assumes *nonempty*: *Y* ≠ {}
and *x*: *x* ≥ *0*
shows (*SUP i∈Y. f i*) ∗ *ereal x* = (*SUP i∈Y. f i* ∗ *ereal x*) (**is** *?lhs* = *?rhs*)
proof(*cases x* = *0*)
case *True* **thus** *?thesis* **by**(*auto simp add*: *nonempty zero-ereal-def*[*symmetric*])
next
case *False*
show *?thesis*
proof(*rule antisym*)
show *?rhs* ≤ *?lhs*
by(*rule SUP-least*)(*simp add*: *ereal-mult-right-mono SUP-upper x*)
next
have *?lhs / ereal x* = (*SUP i∈Y. f i*) ∗ (*ereal x / ereal x*) **by**(*simp only*: *ereal-times-divide-eq*)
also have ... = (*SUP i∈Y. f i*) **using** *False* **by** *simp*
also have ... ≤ *?rhs / x*
proof(*rule SUP-least*)
fix *i*
assume *i* ∈ *Y*
have *f i* = *f i* ∗ (*ereal x / ereal x*) **using** *False* **by** *simp*
also have ... = *f i* ∗ *x / x* **by**(*simp only*: *ereal-times-divide-eq*)
also from ‹*i* ∈ *Y*› **have** *f i* ∗ *x* ≤ *?rhs* **by**(*rule SUP-upper*)
hence *f i* ∗ *x / x* ≤ *?rhs / x* **using** *x False* **by** *simp*
finally show *f i* ≤ *?rhs / x* **.**
qed
finally have (*?lhs / x*) ∗ *x* ≤ (*?rhs / x*) ∗ *x*
by(*rule ereal-mult-right-mono*)(*simp add*: *x*)
also have ... = *?rhs* **using** *False ereal-divide-eq mult.commute* **by** *force*
also have (*?lhs / x*) ∗ *x* = *?lhs* **using** *False ereal-divide-eq mult.commute* **by** *force*
finally show *?lhs* ≤ *?rhs* **.**

qed
qed

lemma *Sup-ereal-mult-left'*:
⟦ $Y \neq \{\}$; $x \geq 0$ ⟧ ⟹ *ereal* $x * (SUP\ i{\in}Y.\ f\ i) = (SUP\ i{\in}Y.\ ereal\ x * f\ i)$
by(*subst* (*1 2*) *mult.commute*)(*rule Sup-ereal-mult-right'*)

lemma *sup-continuous-add*[*order-continuous-intros*]:
fixes *f g* :: *'a*::*complete-lattice* ⇒ *ereal*
assumes *nn*: ⋀*x*. *0* ≤ *f x* ⋀*x*. *0* ≤ *g x* **and** *cont*: *sup-continuous f sup-continuous g*
shows *sup-continuous* (λ*x*. *f x* + *g x*)
unfolding *sup-continuous-def*
proof *safe*
fix *M* :: *nat* ⇒ *'a* **assume** *incseq M*
then show *f* (*SUP i*. *M i*) + *g* (*SUP i*. *M i*) = (*SUP i*. *f* (*M i*) + *g* (*M i*))
using *SUP-ereal-add-pos*[*of* λ*i*. *f* (*M i*) λ*i*. *g* (*M i*)] *nn*
cont[*THEN sup-continuous-mono*] *cont*[*THEN sup-continuousD*]
by (*auto simp*: *mono-def*)
qed

lemma *sup-continuous-mult-right*[*order-continuous-intros*]:
0 ≤ *c* ⟹ *c* < ∞ ⟹ *sup-continuous f* ⟹ *sup-continuous* (λ*x*. *f x* * *c* :: *ereal*)
by (*cases c*) (*auto simp*: *sup-continuous-def fun-eq-iff Sup-ereal-mult-right'*)

lemma *sup-continuous-mult-left*[*order-continuous-intros*]:
0 ≤ *c* ⟹ *c* < ∞ ⟹ *sup-continuous f* ⟹ *sup-continuous* (λ*x*. *c* * *f x* :: *ereal*)
using *sup-continuous-mult-right*[*of c f*] **by** (*simp add*: *mult-ac*)

lemma *sup-continuous-ereal-of-enat*[*order-continuous-intros*]:
assumes *f*: *sup-continuous f* **shows** *sup-continuous* (λ*x*. *ereal-of-enat* (*f x*))
by (*rule sup-continuous-compose*[*OF - f*])
(*auto simp*: *sup-continuous-def ereal-of-enat-SUP*)

38.6.2 Sums

lemma *sums-ereal-positive*:
fixes *f* :: *nat* ⇒ *ereal*
assumes ⋀*i*. *0* ≤ *f i*
shows *f sums* (*SUP n*. ∑ *i*<*n*. *f i*)
proof −
have *incseq* (λ*i*. ∑ *j*=*0*..<*i*. *f j*)
using *add-mono*[*OF - assms*]
by (*auto intro*!: *incseq-SucI*)
from *LIMSEQ-SUP*[*OF this*]
show *?thesis* **unfolding** *sums-def*
by (*simp add*: *atLeast0LessThan*)
qed

lemma *summable-ereal-pos*:
fixes $f :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i$
shows *summable f*
using *sums-ereal-positive*[*of f*, *OF assms*]
unfolding *summable-def*
by *auto*

lemma *sums-ereal*: $(\lambda x.\ ereal\ (f\ x))\ sums\ ereal\ x \longleftrightarrow f\ sums\ x$
unfolding *sums-def* **by** *simp*

lemma *suminf-ereal-eq-SUP*:
fixes $f :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i$
shows $(\sum x.\ f\ x) = (SUP\ n.\ \sum i{<}n.\ f\ i)$
using *sums-ereal-positive*[*of f*, *OF assms*, *THEN sums-unique*]
by *simp*

lemma *suminf-bound*:
fixes $f :: nat \Rightarrow ereal$
assumes $\forall N.\ (\sum n{<}N.\ f\ n) \le x$
and *pos*: $\bigwedge n.\ 0 \le f\ n$
shows $suminf\ f \le x$
proof (*rule Lim-bounded*)
have *summable f* **using** *pos*[*THEN summable-ereal-pos*] **.**
then show $(\lambda N.\ \sum n{<}N.\ f\ n) \longrightarrow suminf\ f$
by (*auto dest*!: *summable-sums simp*: *sums-def atLeast0LessThan*)
show $\forall n{\ge}0.\ sum\ f\ \{..{<}n\} \le x$
using *assms* **by** *auto*
qed

lemma *suminf-bound-add*:
fixes $f :: nat \Rightarrow ereal$
assumes $\forall N.\ (\sum n{<}N.\ f\ n) + y \le x$
and *pos*: $\bigwedge n.\ 0 \le f\ n$
and $y \neq -\infty$
shows $suminf\ f + y \le x$
proof (*cases y*)
case (*real r*)
then have $\forall N.\ (\sum n{<}N.\ f\ n) \le x - y$
using *assms* **by** (*simp add*: *ereal-le-minus*)
then have $(\sum\ n.\ f\ n) \le x - y$
using *pos* **by** (*rule suminf-bound*)
then show $(\sum\ n.\ f\ n) + y \le x$
using *assms real* **by** (*simp add*: *ereal-le-minus*)
qed (*insert assms*, *auto*)

lemma *suminf-upper*:
fixes $f :: nat \Rightarrow ereal$

assumes $\bigwedge n.\ 0 \le f\ n$
shows $(\sum n{<}N.\ f\ n) \le (\sum n.\ f\ n)$
unfolding *suminf-ereal-eq-SUP* [*OF assms*]
by (*auto intro*: *complete-lattice-class.SUP-upper*)

lemma *suminf-0-le*:
fixes $f :: nat \Rightarrow ereal$
assumes $\bigwedge n.\ 0 \le f\ n$
shows $0 \le (\sum n.\ f\ n)$
using *suminf-upper*[*of f 0*, *OF assms*]
by *simp*

lemma *suminf-le-pos*:
fixes $f\ g :: nat \Rightarrow ereal$
assumes $\bigwedge N.\ f\ N \le g\ N$
and $\bigwedge N.\ 0 \le f\ N$
shows *suminf f* $\le$ *suminf g*
proof (*safe intro*!: *suminf-bound*)
fix n
{
fix N
have $0 \le g\ N$
using *assms*(*2*,*1*)[*of N*] **by** *auto*
}
have *sum f* $\{..{<}n\} \le$ *sum g* $\{..{<}n\}$
using *assms* **by** (*auto intro*: *sum-mono*)
also have $\ldots \le$ *suminf g*
using ‹$\bigwedge N.\ 0 \le g\ N$›
by (*rule suminf-upper*)
finally show *sum f* $\{..{<}n\} \le$ *suminf g* **.**
qed (*rule assms*(*2*))

lemma *suminf-half-series-ereal*: $(\sum n.\ (1/2 :: ereal) \mathbin{\hat{}} Suc\ n) = 1$
using *sums-ereal*[*THEN iffD2*, *OF power-half-series*, *THEN sums-unique*, *symmetric*]
by (*simp add*: *one-ereal-def*)

lemma *suminf-add-ereal*:
fixes $f\ g :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i\ \bigwedge i.\ 0 \le g\ i$
shows $(\sum i.\ f\ i + g\ i) =$ *suminf f* + *suminf g*
proof −
have $(SUP\ n.\ \sum i{<}n.\ f\ i + g\ i) = (SUP\ n.\ sum\ f\ \{..{<}n\}) + (SUP\ n.\ sum\ g\ \{..{<}n\})$
unfolding *sum.distrib*
by (*intro assms add-nonneg-nonneg SUP-ereal-add-pos incseq-sumI sum-nonneg ballI*)
with *assms* **show** *?thesis*
by (*subst* (*1 2 3*) *suminf-ereal-eq-SUP*) *auto*

qed

lemma *suminf-cmult-ereal*:
fixes $f\ g :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i$
and $0 \le a$
shows $(\sum i.\ a * f\ i) = a * suminf\ f$
by (*auto simp*: *sum-ereal-right-distrib*[*symmetric*] *assms*
ereal-zero-le-0-iff sum-nonneg suminf-ereal-eq-SUP
intro!: *SUP-ereal-mult-left*)

lemma *suminf-PInfty*:
fixes $f :: nat \Rightarrow ereal$
assumes $\bigwedge i.\ 0 \le f\ i$
and $suminf\ f \neq \infty$
shows $f\ i \neq \infty$
proof −
from *suminf-upper*[*of f Suc i*, *OF assms*(*1*)] *assms*(*2*)
have $(\sum i{<}Suc\ i.\ f\ i) \neq \infty$
by *auto*
then show *?thesis*
unfolding *sum-Pinfty* **by** *simp*
qed

lemma *suminf-PInfty-fun*:
assumes $\bigwedge i.\ 0 \le f\ i$
and $suminf\ f \neq \infty$
shows $\exists f'.\ f = (\lambda x.\ ereal\ (f'\ x))$
proof −
have $\forall i.\ \exists r.\ f\ i = ereal\ r$
proof
fix *i*
show $\exists r.\ f\ i = ereal\ r$
using *suminf-PInfty*[*OF assms*] *assms*(*1*)[*of i*]
by (*cases f i*) *auto*
qed
from *choice*[*OF this*] **show** *?thesis*
by *auto*
qed

lemma *summable-ereal*:
assumes $\bigwedge i.\ 0 \le f\ i$
and $(\sum i.\ ereal\ (f\ i)) \neq \infty$
shows *summable f*
proof −
have $0 \le (\sum i.\ ereal\ (f\ i))$
using *assms* **by** (*intro suminf-0-le*) *auto*
with *assms* **obtain** *r* **where** *r*: $(\sum i.\ ereal\ (f\ i)) = ereal\ r$
by (*cases* $\sum i.\ ereal\ (f\ i)$) *auto*

```
  from summable-ereal-pos[of λx. ereal (f x)]
  have summable (λx. ereal (f x))
    using assms by auto
  from summable-sums[OF this]
  have (λx. ereal (f x)) sums (∑ x. ereal (f x))
    by auto
  then show summable f
    unfolding r sums-ereal summable-def ..
qed

lemma suminf-ereal:
  assumes ⋀i. 0 ≤ f i
    and (∑ i. ereal (f i)) ≠ ∞
  shows (∑ i. ereal (f i)) = ereal (suminf f)
proof (rule sums-unique[symmetric])
  from summable-ereal[OF assms]
  show (λx. ereal (f x)) sums (ereal (suminf f))
    unfolding sums-ereal
    using assms
    by (intro summable-sums summable-ereal)
qed

lemma suminf-ereal-minus:
  fixes f g :: nat ⇒ ereal
  assumes ord: ⋀i. g i ≤ f i ⋀i. 0 ≤ g i
    and fin: suminf f ≠ ∞ suminf g ≠ ∞
  shows (∑ i. f i − g i) = suminf f − suminf g
proof −
  {
    fix i
    have 0 ≤ f i
      using ord[of i] by auto
  }
  moreover
  from suminf-PInfty-fun[OF ‹⋀i. 0 ≤ f i› fin(1)] obtain f' where [simp]: f =
(λx. ereal (f' x)) ..
  from suminf-PInfty-fun[OF ‹⋀i. 0 ≤ g i› fin(2)] obtain g' where [simp]: g =
(λx. ereal (g' x)) ..
  {
    fix i
    have 0 ≤ f i − g i
      using ord[of i] by (auto simp: ereal-le-minus-iff)
  }
  moreover
  have suminf (λi. f i − g i) ≤ suminf f
    using assms by (auto intro!: suminf-le-pos simp: field-simps)
  then have suminf (λi. f i − g i) ≠ ∞
    using fin by auto
  ultimately show ?thesis
```

```
    using assms ‹⋀i. 0 ≤ f i›
    apply simp
    apply (subst (1 2 3) suminf-ereal)
    apply (auto intro!: suminf-diff[symmetric] summable-ereal)
    done
qed

lemma suminf-ereal-PInf [simp]: (∑ x. ∞::ereal) = ∞
proof −
  have (∑ i<Suc 0. ∞) ≤ (∑ x. ∞::ereal)
    by (rule suminf-upper) auto
  then show ?thesis
    by simp
qed

lemma summable-real-of-ereal:
  fixes f :: nat ⇒ ereal
  assumes f: ⋀i. 0 ≤ f i
    and fin: (∑ i. f i) ≠ ∞
  shows summable (λi. real-of-ereal (f i))
proof (rule summable-def[THEN iffD2])
  have 0 ≤ (∑ i. f i)
    using assms by (auto intro: suminf-0-le)
  with fin obtain r where r: ereal r = (∑ i. f i)
    by (cases (∑ i. f i)) auto
  {
    fix i
    have f i ≠ ∞
      using f by (intro suminf-PInfty[OF - fin]) auto
    then have |f i| ≠ ∞
      using f[of i] by auto
  }
  note fin = this
  have (λi. ereal (real-of-ereal (f i))) sums (∑ i. ereal (real-of-ereal (f i)))
    using f
    by (auto intro!: summable-ereal-pos simp: ereal-le-real-iff zero-ereal-def)
  also have ... = ereal r
    using fin r by (auto simp: ereal-real)
  finally show ∃ r. (λi. real-of-ereal (f i)) sums r
    by (auto simp: sums-ereal)
qed

lemma suminf-SUP-eq:
  fixes f :: nat ⇒ nat ⇒ ereal
  assumes ⋀i. incseq (λn. f n i)
    and ⋀n i. 0 ≤ f n i
  shows (∑ i. SUP n. f n i) = (SUP n. ∑ i. f n i)
proof −
  have ∗: ⋀n. (∑ i<n. SUP k. f k i) = (SUP k. ∑ i<n. f k i)
```

```
    using assms
    by (auto intro!: SUP-ereal-sum [symmetric])
  show ?thesis
    using assms
    apply (subst (1 2) suminf-ereal-eq-SUP)
    apply (auto intro!: SUP-upper2 SUP-commute simp: *)
    done
qed

lemma suminf-sum-ereal:
  fixes f :: - ⇒ - ⇒ ereal
  assumes nonneg: ⋀i a. a ∈ A ⟹ 0 ≤ f i a
  shows (∑ i. ∑ a∈A. f i a) = (∑ a∈A. ∑ i. f i a)
proof (cases finite A)
  case True
  then show ?thesis
    using nonneg
    by induct (simp-all add: suminf-add-ereal sum-nonneg)
next
  case False
  then show ?thesis by simp
qed

lemma suminf-ereal-eq-0:
  fixes f :: nat ⇒ ereal
  assumes nneg: ⋀i. 0 ≤ f i
  shows (∑ i. f i) = 0 ⟷ (∀ i. f i = 0)
proof
  assume (∑ i. f i) = 0
  {
    fix i
    assume f i ≠ 0
    with nneg have 0 < f i
      by (auto simp: less-le)
    also have f i = (∑ j. if j = i then f i else 0)
      by (subst suminf-finite[where N={i}]) auto
    also have ... ≤ (∑ i. f i)
      using nneg
      by (auto intro!: suminf-le-pos)
    finally have False
      using ‹(∑ i. f i) = 0› by auto
  }
  then show ∀ i. f i = 0
    by auto
qed simp

lemma suminf-ereal-offset-le:
  fixes f :: nat ⇒ ereal
  assumes f: ⋀i. 0 ≤ f i
```

```
  shows (∑ i. f (i + k)) ≤ suminf f
proof −
  have (λn. ∑ i<n. f (i + k)) ⟶ (∑ i. f (i + k))
    using summable-sums[OF summable-ereal-pos]
    by (simp add: sums-def atLeast0LessThan f)
  moreover have (λn. ∑ i<n. f i) ⟶ (∑ i. f i)
    using summable-sums[OF summable-ereal-pos]
    by (simp add: sums-def atLeast0LessThan f)
  then have (λn. ∑ i<n + k. f i) ⟶ (∑ i. f i)
    by (rule LIMSEQ-ignore-initial-segment)
  ultimately show ?thesis
  proof (rule LIMSEQ-le, safe intro!: exI[of - k])
    fix n assume k ≤ n
    have (∑ i<n. f (i + k)) = (∑ i<n. (f ∘ plus k) i)
      by (simp add: ac-simps)
    also have ... = (∑ i∈(plus k) ` {..<n}. f i)
      by (rule sum.reindex [symmetric]) simp
    also have ... ≤ sum f {..<n + k}
      by (intro sum-mono2) (auto simp: f)
    finally show (∑ i<n. f (i + k)) ≤ sum f {..<n + k} .
  qed
qed

lemma sums-suminf-ereal: f sums x ⟹ (∑ i. ereal (f i)) = ereal x
  by (metis sums-ereal sums-unique)

lemma suminf-ereal': summable f ⟹ (∑ i. ereal (f i)) = ereal (∑ i. f i)
  by (metis sums-ereal sums-unique summable-def)

lemma suminf-ereal-finite: summable f ⟹ (∑ i. ereal (f i)) ≠ ∞
  by (auto simp: summable-def simp flip: sums-ereal sums-unique)

lemma suminf-ereal-finite-neg:
  assumes summable f
  shows (∑ x. ereal (f x)) ≠ −∞
proof−
  from assms obtain x where f sums x by blast
  hence (λx. ereal (f x)) sums ereal x by (simp add: sums-ereal)
  from sums-unique[OF this] have (∑ x. ereal (f x)) = ereal x ..
  thus (∑ x. ereal (f x)) ≠ −∞ by simp-all
qed

lemma SUP-ereal-add-directed:
  fixes f g :: 'a ⇒ ereal
  assumes nonneg: ⋀i. i ∈ I ⟹ 0 ≤ f i ⋀i. i ∈ I ⟹ 0 ≤ g i
  assumes directed: ⋀i j. i ∈ I ⟹ j ∈ I ⟹ ∃ k∈I. f i + g j ≤ f k + g k
  shows (SUP i∈I. f i + g i) = (SUP i∈I. f i) + (SUP i∈I. g i)
proof cases
  assume I = {} then show ?thesis
```

```
    by (simp add: bot-ereal-def)
next
  assume I ≠ {}
  show ?thesis
  proof (rule antisym)
    show (SUP i∈I. f i + g i) ≤ (SUP i∈I. f i) + (SUP i∈I. g i)
      by (rule SUP-least; intro add-mono SUP-upper)
  next
    have bot < (SUP i∈I. g i)
      using ‹I ≠ {}› nonneg(2) by (auto simp: bot-ereal-def less-SUP-iff)
    then have (SUP i∈I. f i) + (SUP i∈I. g i) = (SUP i∈I. f i + (SUP i∈I. g
i))
      by (intro SUP-ereal-add-left[symmetric] ‹I ≠ {}›) auto
    also have ... = (SUP i∈I. (SUP j∈I. f i + g j))
      using nonneg(1) ‹I ≠ {}› by (simp add: SUP-ereal-add-right)
    also have ... ≤ (SUP i∈I. f i + g i)
      using directed by (intro SUP-least) (blast intro: SUP-upper2)
    finally show (SUP i∈I. f i) + (SUP i∈I. g i) ≤ (SUP i∈I. f i + g i) .
  qed
qed

lemma SUP-ereal-sum-directed:
  fixes f g :: 'a ⇒ 'b ⇒ ereal
  assumes I ≠ {}
  assumes directed: ⋀N i j. N ⊆ A ⟹ i ∈ I ⟹ j ∈ I ⟹ ∃k∈I. ∀n∈N. f n i
≤ f n k ∧ f n j ≤ f n k
  assumes nonneg: ⋀n i. i ∈ I ⟹ n ∈ A ⟹ 0 ≤ f n i
  shows (SUP i∈I. ∑ n∈A. f n i) = (∑ n∈A. SUP i∈I. f n i)
proof −
  have N ⊆ A ⟹ (SUP i∈I. ∑ n∈N. f n i) = (∑ n∈N. SUP i∈I. f n i) for N
  proof (induction N rule: infinite-finite-induct)
    case (insert n N)
     have (SUP i∈I. f n i + (∑ l∈N. f l i)) = (SUP i∈I. f n i) + (SUP i∈I.
∑ l∈N. f l i)
    proof (rule SUP-ereal-add-directed)
      fix i assume i ∈ I then show 0 ≤ f n i 0 ≤ (∑ l∈N. f l i)
        using insert by (auto intro!: sum-nonneg nonneg)
    next
      fix i j assume i ∈ I j ∈ I
      from directed[OF insert(4) this]
      show ∃k∈I. f n i + (∑ l∈N. f l j) ≤ f n k + (∑ l∈N. f l k)
        by (auto intro!: add-mono sum-mono)
    qed
    with insert show ?case
      by simp
  qed (simp-all add: SUP-constant ‹I ≠ {}›)
  from this[of A] show ?thesis by simp
qed
```

lemma *suminf-SUP-eq-directed*:
fixes $f :: - \Rightarrow nat \Rightarrow ereal$
assumes $I \neq \{\}$
assumes *directed*: $\bigwedge N\ i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow finite\ N \Longrightarrow \exists k{\in}I.\ \forall n{\in}N.\ f\ i\ n \leq f\ k\ n \wedge f\ j\ n \leq f\ k\ n$
assumes *nonneg*: $\bigwedge n\ i.\ 0 \leq f\ n\ i$
shows $(\sum i.\ SUP\ n{\in}I.\ f\ n\ i) = (SUP\ n{\in}I.\ \sum i.\ f\ n\ i)$
proof (*subst* (*1 2*) *suminf-ereal-eq-SUP*)
show $\bigwedge n\ i.\ 0 \leq f\ n\ i\ \bigwedge i.\ 0 \leq (SUP\ n{\in}I.\ f\ n\ i)$
using ‹$I \neq \{\}$› *nonneg* **by** (*auto intro*: *SUP-upper2*)
show $(SUP\ n.\ \sum i{<}n.\ SUP\ n{\in}I.\ f\ n\ i) = (SUP\ n{\in}I.\ SUP\ j.\ \sum i{<}j.\ f\ n\ i)$
by (*auto simp*: *finite-subset SUP-commute SUP-ereal-sum-directed assms*)
qed

lemma *ereal-dense3*:
fixes $x\ y :: ereal$
shows $x < y \Longrightarrow \exists r{::}rat.\ x < real\text{-}of\text{-}rat\ r \wedge real\text{-}of\text{-}rat\ r < y$
proof (*cases x y rule*: *ereal2-cases*, *simp-all*)
fix $r\ q :: real$
assume $r < q$
from *Rats-dense-in-real*[*OF this*] **show** $\exists x.\ r < real\text{-}of\text{-}rat\ x \wedge real\text{-}of\text{-}rat\ x < q$
by (*fastforce simp*: *Rats-def*)
next
fix $r :: real$
show $\exists x.\ r < real\text{-}of\text{-}rat\ x\ \exists x.\ real\text{-}of\text{-}rat\ x < r$
using *gt-ex*[*of r*] *lt-ex*[*of r*] *Rats-dense-in-real*
by (*auto simp*: *Rats-def*)
qed

lemma *continuous-within-ereal*[*intro*, *simp*]: $x \in A \Longrightarrow continuous\ (at\ x\ within\ A)\ ereal$
using *continuous-on-eq-continuous-within*[*of A ereal*]
by (*auto intro*: *continuous-on-ereal continuous-on-id*)

lemma *ereal-open-uminus*:
fixes $S :: ereal\ set$
assumes *open S*
shows *open* (*uminus ‘ S*)
using ‹*open S*›[*unfolded open-generated-order*]
proof *induct*
have *range uminus* = (*UNIV* :: *ereal set*)
by (*auto simp*: *image-iff ereal-uminus-eq-reorder*)
then show *open* (*range uminus* :: *ereal set*)
by *simp*
qed (*auto simp add*: *image-Union image-Int*)

lemma *ereal-uminus-complement*:
fixes $S :: ereal\ set$
shows $uminus\ ‘\ (-\ S) = -\ uminus\ ‘\ S$

by (*auto intro*!: *bij-image-Compl-eq surjI*[*of - uminus*] *simp*: *bij-betw-def*)

lemma *ereal-closed-uminus*:
fixes S :: *ereal set*
assumes *closed S*
shows *closed* (*uminus* ‘ *S*)
using *assms*
unfolding *closed-def ereal-uminus-complement*[*symmetric*]
by (*rule ereal-open-uminus*)

lemma *ereal-open-affinity-pos*:
fixes S :: *ereal set*
assumes *open S*
and *m*: $m \neq \infty$ $0 < m$
and *t*: $|t| \neq \infty$
shows *open* ((λ*x*. *m* ∗ *x* + *t*) ‘ *S*)
proof −
have *continuous-on UNIV* (λ*x*. *inverse m* ∗ (*x* + − *t*))
using *m t*
by (*intro continuous-at-imp-continuous-on ballI continuous-at*[*THEN iffD2*];
force)
then have *open* ((λ*x*. *inverse m* ∗ (*x* + −*t*)) −‘ *S*)
using ‹*open S*› *open-vimage* **by** *blast*
also have (λ*x*. *inverse m* ∗ (*x* + −*t*)) −‘ *S* = (λ*x*. (*x* − *t*) / *m*) −‘ *S*
using *m t* **by** (*auto simp*: *divide-ereal-def mult.commute minus-ereal-def*
simp flip: *uminus-ereal.simps*)
also have (λ*x*. (*x* − *t*) / *m*) −‘ *S* = (λ*x*. *m* ∗ *x* + *t*) ‘ *S*
using *m t*
by (*simp add*: *set-eq-iff image-iff*)
(*metis abs-ereal-less0 abs-ereal-uminus ereal-divide-eq ereal-eq-minus ereal-minus*(*7*,*8*)
ereal-minus-less-minus ereal-mult-eq-PInfty ereal-uminus-uminus
ereal-zero-mult)
finally show *?thesis* **.**
qed

lemma *ereal-open-affinity*:
fixes S :: *ereal set*
assumes *open S*
and *m*: $|m| \neq \infty$ $m \neq 0$
and *t*: $|t| \neq \infty$
shows *open* ((λ*x*. *m* ∗ *x* + *t*) ‘ *S*)
proof *cases*
assume $0 < m$
then show *?thesis*
using *ereal-open-affinity-pos*[*OF* ‹*open S*› - - *t*, *of m*] *m*
by *auto*
next
assume ¬ $0 < m$ **then**
have $0 < -m$

```
    using ‹m ≠ 0›
    by (cases m) auto
  then have m: −m ≠ ∞ 0 < −m
    using ‹|m| ≠ ∞›
    by (auto simp: ereal-uminus-eq-reorder)
 from ereal-open-affinity-pos[OF ereal-open-uminus[OF ‹open S›] m t] show ?thesis
    unfolding image-image by simp
qed

lemma open-uminus-iff:
  fixes S :: ereal set
  shows open (uminus ‘ S) ⟷ open S
  using ereal-open-uminus[of S] ereal-open-uminus[of uminus ‘ S]
  by auto

lemma ereal-Liminf-uminus:
  fixes f :: 'a ⇒ ereal
  shows Liminf net (λx. − (f x)) = − Limsup net f
  using ereal-Limsup-uminus[of - (λx. − (f x))] by auto

lemma Liminf-PInfty:
  fixes f :: 'a ⇒ ereal
  assumes ¬ trivial-limit net
  shows (f ⟶ ∞) net ⟷ Liminf net f = ∞
  unfolding tendsto-iff-Liminf-eq-Limsup[OF assms]
  using Liminf-le-Limsup[OF assms, of f]
  by auto

lemma Limsup-MInfty:
  fixes f :: 'a ⇒ ereal
  assumes ¬ trivial-limit net
  shows (f ⟶ −∞) net ⟷ Limsup net f = −∞
  unfolding tendsto-iff-Liminf-eq-Limsup[OF assms]
  using Liminf-le-Limsup[OF assms, of f]
  by auto

lemma convergent-ereal: — RENAME
  fixes X :: nat ⇒ 'a :: {complete-linorder,linorder-topology}
  shows convergent X ⟷ limsup X = liminf X
  using tendsto-iff-Liminf-eq-Limsup[of sequentially]
  by (auto simp: convergent-def)

lemma limsup-le-liminf-real:
  fixes X :: nat ⇒ real and L :: real
  assumes 1: limsup X ≤ L and 2: L ≤ liminf X
  shows X ⟶ L
proof −
  from 1 2 have limsup X ≤ liminf X by auto
  hence 3: limsup X = liminf X
```

```
    by (simp add: Liminf-le-Limsup order-class.order.antisym)
  hence 4: convergent (λn. ereal (X n))
    by (subst convergent-ereal)
  hence limsup X = lim (λn. ereal(X n))
    by (rule convergent-limsup-cl)
  also from 1 2 3 have limsup X = L by auto
  finally have lim (λn. ereal(X n)) = L ..
  hence (λn. ereal (X n)) ⟶ L
    using 4 convergent-LIMSEQ-iff by force
  thus ?thesis by simp
qed

lemma liminf-PInfty:
  fixes X :: nat ⇒ ereal
  shows X ⟶ ∞ ⟷ liminf X = ∞
  by (metis Liminf-PInfty trivial-limit-sequentially)

lemma limsup-MInfty:
  fixes X :: nat ⇒ ereal
  shows X ⟶ −∞ ⟷ limsup X = −∞
  by (metis Limsup-MInfty trivial-limit-sequentially)

lemma SUP-eq-LIMSEQ:
  assumes mono f
  shows (SUP n. ereal (f n)) = ereal x ⟷ f ⟶ x
proof
  have inc: incseq (λi. ereal (f i))
    using ‹mono f› unfolding mono-def incseq-def by auto
  {
    assume f ⟶ x
    then have (λi. ereal (f i)) ⟶ ereal x
      by auto
    from SUP-Lim[OF inc this] show (SUP n. ereal (f n)) = ereal x .
  next
    assume (SUP n. ereal (f n)) = ereal x
    with LIMSEQ-SUP[OF inc] show f ⟶ x by auto
  }
qed

lemma liminf-ereal-cminus:
  fixes f :: nat ⇒ ereal
  assumes c ≠ −∞
  shows liminf (λx. c − f x) = c − limsup f
proof (cases c)
  case PInf
  then show ?thesis
    by (simp add: Liminf-const)
next
  case (real r)
```

then show *?thesis*
by (*simp add*: *liminf-SUP-INF limsup-INF-SUP INF-ereal-minus-right SUP-ereal-minus-right*)
qed (*use* ‹$c \neq -\infty$› **in** *simp*)

38.6.3 Continuity

lemma *continuous-at-of-ereal*:
$|x0 :: ereal| \neq \infty \Longrightarrow$ *continuous* (*at x0*) *real-of-ereal*
unfolding *continuous-at*
by (*rule lim-real-of-ereal*) (*simp add*: *ereal-real*)

lemma *nhds-ereal*: *nhds* (*ereal r*) = *filtermap ereal* (*nhds r*)
by (*simp add*: *filtermap-nhds-open-map open-ereal continuous-at-of-ereal*)

lemma *at-ereal*: *at* (*ereal r*) = *filtermap ereal* (*at r*)
by (*simp add*: *filter-eq-iff eventually-at-filter nhds-ereal eventually-filtermap*)

lemma *at-left-ereal*: *at-left* (*ereal r*) = *filtermap ereal* (*at-left r*)
by (*simp add*: *filter-eq-iff eventually-at-filter nhds-ereal eventually-filtermap*)

lemma *at-right-ereal*: *at-right* (*ereal r*) = *filtermap ereal* (*at-right r*)
by (*simp add*: *filter-eq-iff eventually-at-filter nhds-ereal eventually-filtermap*)

lemma
shows *at-left-PInf*: *at-left* ∞ = *filtermap ereal at-top*
and *at-right-MInf*: *at-right* $(-\infty)$ = *filtermap ereal at-bot*
unfolding *filter-eq-iff eventually-filtermap eventually-at-top-dense eventually-at-bot-dense*
eventually-at-left[*OF ereal-less*(*5*)] *eventually-at-right*[*OF ereal-less*(*6*)]
by (*auto simp add*: *ereal-all-split ereal-ex-split*)

lemma *ereal-tendsto-simps1*:
$((f \circ real\text{-}of\text{-}ereal) \longrightarrow y)\ (at\text{-}left\ (ereal\ x)) \longleftrightarrow (f \longrightarrow y)\ (at\text{-}left\ x)$
$((f \circ real\text{-}of\text{-}ereal) \longrightarrow y)\ (at\text{-}right\ (ereal\ x)) \longleftrightarrow (f \longrightarrow y)\ (at\text{-}right\ x)$
$((f \circ real\text{-}of\text{-}ereal) \longrightarrow y)\ (at\text{-}left\ (\infty::ereal)) \longleftrightarrow (f \longrightarrow y)\ at\text{-}top$
$((f \circ real\text{-}of\text{-}ereal) \longrightarrow y)\ (at\text{-}right\ (-\infty::ereal)) \longleftrightarrow (f \longrightarrow y)\ at\text{-}bot$
unfolding *tendsto-compose-filtermap at-left-ereal at-right-ereal at-left-PInf at-right-MInf*
by (*auto simp*: *filtermap-filtermap filtermap-ident*)

lemma *ereal-tendsto-simps2*:
$((ereal \circ f) \longrightarrow ereal\ a)\ F \longleftrightarrow (f \longrightarrow a)\ F$
$((ereal \circ f) \longrightarrow \infty)\ F \longleftrightarrow (LIM\ x\ F.\ f\ x :> at\text{-}top)$
$((ereal \circ f) \longrightarrow -\infty)\ F \longleftrightarrow (LIM\ x\ F.\ f\ x :> at\text{-}bot)$
unfolding *tendsto-PInfty filterlim-at-top-dense tendsto-MInfty filterlim-at-bot-dense*
using *lim-ereal* **by** (*simp-all add*: *comp-def*)

lemma *inverse-infty-ereal-tendsto-0*: *inverse* $-\infty\rightarrow$ (*0*::*ereal*)
proof −
have $**$: $((\lambda x.\ ereal\ (inverse\ x)) \longrightarrow ereal\ 0)\ at\text{-}infinity$
by (*intro tendsto-intros tendsto-inverse-0*)

```
  then have ((λx. if x = 0 then ∞ else ereal (inverse x)) ⟶ 0) at-top
  proof (rule filterlim-mono-eventually)
    show nhds (ereal 0) ≤ nhds 0
      by (simp add: zero-ereal-def)
    show (at-top::real filter) ≤ at-infinity
      by (simp add: at-top-le-at-infinity)
  qed auto
  then show ?thesis
   unfolding at-infty-ereal-eq-at-top tendsto-compose-filtermap[symmetric] comp-def
by auto
qed

lemma inverse-ereal-tendsto-pos:
  fixes x :: ereal assumes 0 < x
  shows inverse -x→ inverse x
proof (cases x)
  case (real r)
  with ‹0 < x› have **: (λx. ereal (inverse x)) -r→ ereal (inverse r)
    by (auto intro!: tendsto-inverse)
  from real ‹0 < x› show ?thesis
   by (auto simp: at-ereal tendsto-compose-filtermap[symmetric] eventually-at-filter
            intro!: Lim-transform-eventually[OF **] t1-space-nhds)
qed (insert ‹0 < x›, auto intro!: inverse-infty-ereal-tendsto-0)

lemma inverse-ereal-tendsto-at-right-0: (inverse ⟶ ∞) (at-right (0::ereal))
  unfolding tendsto-compose-filtermap[symmetric] at-right-ereal zero-ereal-def
  by (subst filterlim-cong[OF refl refl, where g=λx. ereal (inverse x)])
    (auto simp: eventually-at-filter tendsto-PInfty-eq-at-top filterlim-inverse-at-top-right)

lemmas ereal-tendsto-simps = ereal-tendsto-simps1 ereal-tendsto-simps2

lemma continuous-at-iff-ereal:
  fixes f :: 'a::t2-space ⇒ real
  shows continuous (at x0 within s) f ⟷ continuous (at x0 within s) (ereal ∘ f)
  unfolding continuous-within comp-def lim-ereal ..

lemma continuous-on-iff-ereal:
  fixes f :: 'a::t2-space => real
  assumes open A
  shows continuous-on A f ⟷ continuous-on A (ereal ∘ f)
  unfolding continuous-on-def comp-def lim-ereal ..

lemma continuous-on-real: continuous-on (UNIV - {∞, -∞::ereal}) real-of-ereal
  using continuous-at-of-ereal continuous-on-eq-continuous-at open-image-ereal
  by auto

lemma continuous-on-iff-real:
  fixes f :: 'a::t2-space ⇒ ereal
  assumes ⋀x. x ∈ A ⟹ |f x| ≠ ∞
```

shows *continuous-on A f* $\longleftrightarrow$ *continuous-on A* (*real-of-ereal* $\circ$ *f*)
proof
assume *L*: *continuous-on A f*
have *f* ' *A* $\subseteq$ *UNIV* $-$ $\{\infty, -\infty\}$
using *assms* **by** *force*
then show *continuous-on A* (*real-of-ereal* $\circ$ *f*)
by (*meson L continuous-on-compose continuous-on-real continuous-on-subset*)
next
assume *R*: *continuous-on A* (*real-of-ereal* $\circ$ *f*)
then have *continuous-on A* (*ereal* $\circ$ (*real-of-ereal* $\circ$ *f*))
by (*meson continuous-at-iff-ereal continuous-on-eq-continuous-within*)
then show *continuous-on A f*
using *assms ereal-real'* **by** *auto*
qed

lemma *continuous-uminus-ereal* [*continuous-intros*]: *continuous-on* (*A* :: *ereal set*) *uminus*
unfolding *continuous-on-def*
by (*intro ballI tendsto-uminus-ereal*[*of* $\lambda x.$ *x*::*ereal*]) *simp*

lemma *ereal-uminus-atMost* [*simp*]: *uminus* ' $\{..(a::ereal)\} = \{-a..\}$
proof (*intro equalityI subsetI*)
fix *x* :: *ereal* **assume** $x \in \{-a..\}$
hence $-(-x) \in$ *uminus* ' $\{..a\}$ **by** (*intro imageI*) (*simp add*: *ereal-uminus-le-reorder*)
thus $x \in$ *uminus* ' $\{..a\}$ **by** *simp*
qed *auto*

lemma *continuous-on-inverse-ereal* [*continuous-intros*]:
continuous-on $\{0::ereal ..\}$ *inverse*
unfolding *continuous-on-def*
proof *clarsimp*
fix *x* :: *ereal* **assume** $0 \leq x$
moreover have *at 0 within* $\{0 ..\}$ = *at-right* (0::*ereal*)
by (*auto simp*: *filter-eq-iff eventually-at-filter le-less*)
moreover have *at x within* $\{0 ..\}$ = *at x* **if** $0 < x$
using *that* **by** (*intro at-within-nhd*[*of* - $\{0<..\}$]) *auto*
ultimately show (*inverse* $\longrightarrow$ *inverse x*) (*at x within* $\{0..\}$)
by (*auto simp*: *le-less inverse-ereal-tendsto-at-right-0 inverse-ereal-tendsto-pos*)
qed

lemma *continuous-inverse-ereal-nonpos*: *continuous-on* ($\{..<0\}$ :: *ereal set*) *inverse*
proof (*subst continuous-on-cong*[*OF refl*])
have *continuous-on* $\{(0::ereal)<..\}$ *inverse*
by (*rule continuous-on-subset*[*OF continuous-on-inverse-ereal*]) *auto*
thus *continuous-on* $\{..<(0::ereal)\}$ (*uminus* $\circ$ *inverse* $\circ$ *uminus*)
by (*intro continuous-intros*) *simp-all*
qed *simp*

lemma *tendsto-inverse-ereal*:
assumes $(f \longrightarrow (c :: ereal))\ F$
assumes *eventually* $(\lambda x.\ f\ x \geq 0)\ F$
shows $((\lambda x.\ inverse\ (f\ x)) \longrightarrow inverse\ c)\ F$
by (*cases* $F = bot$)
(*auto intro*!: *tendsto-lowerbound assms*
continuous-on-tendsto-compose[*OF continuous-on-inverse-ereal*])

38.6.4 liminf and limsup

lemma *Limsup-ereal-mult-right*:
assumes $F \neq bot\ (c::real) \geq 0$
shows $Limsup\ F\ (\lambda n.\ f\ n * ereal\ c) = Limsup\ F\ f * ereal\ c$
proof (*rule Limsup-compose-continuous-mono*)
from *assms* **show** *continuous-on UNIV* $(\lambda a.\ a * ereal\ c)$
using *tendsto-cmult-ereal*[*of ereal c* $\lambda x.\ x$]
by (*force simp*: *continuous-on-def mult-ac*)
qed (*insert assms*, *auto simp*: *mono-def ereal-mult-right-mono*)

lemma *Liminf-ereal-mult-right*:
assumes $F \neq bot\ (c::real) \geq 0$
shows $Liminf\ F\ (\lambda n.\ f\ n * ereal\ c) = Liminf\ F\ f * ereal\ c$
proof (*rule Liminf-compose-continuous-mono*)
from *assms* **show** *continuous-on UNIV* $(\lambda a.\ a * ereal\ c)$
using *tendsto-cmult-ereal*[*of ereal c* $\lambda x.\ x$]
by (*force simp*: *continuous-on-def mult-ac*)
qed (*use assms* **in** ‹*auto simp*: *mono-def ereal-mult-right-mono*›)

lemma *Liminf-ereal-mult-left*:
assumes $F \neq bot\ (c::real) \geq 0$
shows $Liminf\ F\ (\lambda n.\ ereal\ c * f\ n) = ereal\ c * Liminf\ F\ f$
using *Liminf-ereal-mult-right*[*OF assms*] **by** (*subst* (*1 2*) *mult.commute*)

lemma *Limsup-ereal-mult-left*:
assumes $F \neq bot\ (c::real) \geq 0$
shows $Limsup\ F\ (\lambda n.\ ereal\ c * f\ n) = ereal\ c * Limsup\ F\ f$
using *Limsup-ereal-mult-right*[*OF assms*] **by** (*subst* (*1 2*) *mult.commute*)

lemma *limsup-ereal-mult-right*:
$(c::real) \geq 0 \Longrightarrow limsup\ (\lambda n.\ f\ n * ereal\ c) = limsup\ f * ereal\ c$
by (*rule Limsup-ereal-mult-right*) *simp-all*

lemma *limsup-ereal-mult-left*:
$(c::real) \geq 0 \Longrightarrow limsup\ (\lambda n.\ ereal\ c * f\ n) = ereal\ c * limsup\ f$
by (*subst* (*1 2*) *mult.commute*, *rule limsup-ereal-mult-right*) *simp-all*

lemma *Limsup-add-ereal-right*:
$F \neq bot \Longrightarrow abs\ c \neq \infty \Longrightarrow Limsup\ F\ (\lambda n.\ g\ n + (c :: ereal)) = Limsup\ F\ g + c$

by (*rule Limsup-compose-continuous-mono*) (*auto simp*: *mono-def add-mono continuous-on-def*)

lemma *Limsup-add-ereal-left*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Limsup F* (λ*n*. (*c* :: *ereal*) + *g n*) = *c* + *Limsup F g*
by (*subst* (*1 2*) *add.commute*) (*rule Limsup-add-ereal-right*)

lemma *Liminf-add-ereal-right*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Liminf F* (λ*n*. *g n* + (*c* :: *ereal*)) = *Liminf F g* + *c*
by (*rule Liminf-compose-continuous-mono*) (*auto simp*: *mono-def add-mono continuous-on-def*)

lemma *Liminf-add-ereal-left*:
F ≠ *bot* ⟹ *abs c* ≠ ∞ ⟹ *Liminf F* (λ*n*. (*c* :: *ereal*) + *g n*) = *c* + *Liminf F g*
by (*subst* (*1 2*) *add.commute*) (*rule Liminf-add-ereal-right*)

lemma
assumes *F* ≠ *bot*
assumes *nonneg*: *eventually* (λ*x*. *f x* ≥ (*0*::*ereal*)) *F*
shows *Liminf-inverse-ereal*: *Liminf F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Limsup F f*)
and *Limsup-inverse-ereal*: *Limsup F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Liminf F f*)
proof −
define *inv* **where** [*abs-def*]: *inv x* = (*if x* ≤ *0 then* ∞ *else inverse x*) **for** *x* :: *ereal*
have *continuous-on* ({..*0*} ∪ {*0*..}) *inv* **unfolding** *inv-def*
by (*intro continuous-on-If*) (*auto intro*!: *continuous-intros*)
also have {..*0*} ∪ {*0*..} = (*UNIV* :: *ereal set*) **by** *auto*
finally have *cont*: *continuous-on UNIV inv* **.**
have *antimono*: *antimono inv* **unfolding** *inv-def antimono-def*
by (*auto intro*!: *ereal-inverse-antimono*)

have *Liminf F* (λ*x*. *inverse* (*f x*)) = *Liminf F* (λ*x*. *inv* (*f x*)) **using** *nonneg*
by (*auto intro*!: *Liminf-eq elim*!: *eventually-mono simp*: *inv-def*)
also have ... = *inv* (*Limsup F f*)
by (*simp add*: *assms*(*1*) *Liminf-compose-continuous-antimono*[*OF cont antimono*])
also from *assms* **have** *Limsup F f* ≥ *0* **by** (*intro le-Limsup*) *simp-all*
hence *inv* (*Limsup F f*) = *inverse* (*Limsup F f*) **by** (*simp add*: *inv-def*)
finally show *Liminf F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Limsup F f*) **.**

have *Limsup F* (λ*x*. *inverse* (*f x*)) = *Limsup F* (λ*x*. *inv* (*f x*)) **using** *nonneg*
by (*auto intro*!: *Limsup-eq elim*!: *eventually-mono simp*: *inv-def*)
also have ... = *inv* (*Liminf F f*)
by (*simp add*: *assms*(*1*) *Limsup-compose-continuous-antimono*[*OF cont antimono*])
also from *assms* **have** *Liminf F f* ≥ *0* **by** (*intro Liminf-bounded*) *simp-all*

hence *inv* (*Liminf F f*) = *inverse* (*Liminf F f*) **by** (*simp add*: *inv-def*)
finally show *Limsup F* (λ*x*. *inverse* (*f x*)) = *inverse* (*Liminf F f*) .
qed

lemma *ereal-diff-le-mono-left*: ⟦ *x* ≤ *z*; *0* ≤ *y* ⟧ ⟹ *x* − *y* ≤ (*z* :: *ereal*)
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *neg-0-less-iff-less-erea* [*simp*]: *0* < − *a* ⟷ (*a* :: *ereal*) < *0*
by(*cases a*) *simp-all*

lemma *not-infty-ereal*: |*x*| ≠ ∞ ⟷ (∃ *x'*. *x* = *ereal x'*)
by(*cases x*) *simp-all*

lemma *neq-PInf-trans*: **fixes** *x y* :: *ereal* **shows** ⟦ *y* ≠ ∞; *x* ≤ *y* ⟧ ⟹ *x* ≠ ∞
by *auto*

lemma *mult-2-ereal*: *ereal 2* ∗ *x* = *x* + *x*
by(*cases x*) *simp-all*

lemma *ereal-diff-le-self*: *0* ≤ *y* ⟹ *x* − *y* ≤ (*x* :: *ereal*)
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-le-add-self*: *0* ≤ *y* ⟹ *x* ≤ *x* + (*y* :: *ereal*)
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-le-add-self2*: *0* ≤ *y* ⟹ *x* ≤ *y* + (*x* :: *ereal*)
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-le-add-mono1*: ⟦ *x* ≤ *y*; *0* ≤ (*z* :: *ereal*) ⟧ ⟹ *x* ≤ *y* + *z*
using *add-mono* **by** *fastforce*

lemma *ereal-le-add-mono2*: ⟦ *x* ≤ *z*; *0* ≤ (*y* :: *ereal*) ⟧ ⟹ *x* ≤ *y* + *z*
using *add-mono* **by** *fastforce*

lemma *ereal-diff-nonpos*:
fixes *a b* :: *ereal* **shows** ⟦ *a* ≤ *b*; *a* = ∞ ⟹ *b* ≠ ∞; *a* = −∞ ⟹ *b* ≠ −∞ ⟧
⟹ *a* − *b* ≤ *0*
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *minus-ereal-0* [*simp*]: *x* − *ereal 0* = *x*
by(*simp flip*: *zero-ereal-def*)

lemma *ereal-diff-eq-0-iff*: **fixes** *a b* :: *ereal*
shows (|*a*| = ∞ ⟹ |*b*| ≠ ∞) ⟹ *a* − *b* = *0* ⟷ *a* = *b*
by(*cases a b rule*: *ereal2-cases*) *simp-all*

lemma *SUP-ereal-eq-0-iff-nonneg*:
fixes *f* :: - ⇒ *ereal* **and** *A*
assumes *nonneg*: ∀ *x*∈*A*. *f x* ≥ *0*

and A:$A \neq \{\}$
shows $(SUP\ x{\in}A.\ f\ x) = 0 \longleftrightarrow (\forall x{\in}A.\ f\ x = 0)$ (**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof(*intro iffI ballI*)
fix x
assume *?lhs* $x \in A$
from ‹$x \in A$› **have** $f\ x \leq (SUP\ x{\in}A.\ f\ x)$ **by**(*rule SUP-upper*)
with ‹*?lhs*› **show** $f\ x = 0$ **using** *nonneg* ‹$x \in A$› **by** *auto*
qed (*simp add*: A)

lemma *ereal-divide-le-posI*:
fixes $x\ y\ z$ :: *ereal*
shows $x > 0 \Longrightarrow z \neq -\infty \Longrightarrow z \leq x * y \Longrightarrow z\ /\ x \leq y$
by (*cases rule*: *ereal3-cases*[*of x y z*])(*auto simp*: *field-simps split*: *if-split-asm*)

lemma *add-diff-eq-ereal*: **fixes** $x\ y\ z$ :: *ereal*
shows $x + (y - z) = x + y - z$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *ereal-diff-gr0*:
fixes $a\ b$ :: *ereal* **shows** $a < b \Longrightarrow 0 < b - a$
by (*cases rule*: *ereal2-cases*[*of a b*]) *auto*

lemma *ereal-minus-minus*: **fixes** $x\ y\ z$ :: *ereal* **shows**
$(|y| = \infty \Longrightarrow |z| \neq \infty) \Longrightarrow x - (y - z) = x + z - y$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *diff-add-eq-ereal*: **fixes** $a\ b\ c$ :: *ereal* **shows** $a - b + c = a + c - b$
by(*cases a b c rule*: *ereal3-cases*) *simp-all*

lemma *diff-diff-commute-ereal*: **fixes** $x\ y\ z$ :: *ereal* **shows** $x - y - z = x - z - y$
by(*cases x y z rule*: *ereal3-cases*) *simp-all*

lemma *ereal-diff-eq-MInfty-iff*: **fixes** $x\ y$ :: *ereal* **shows** $x - y = -\infty \longleftrightarrow x = -\infty \wedge y \neq -\infty \vee y = \infty \wedge |x| \neq \infty$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *ereal-diff-add-inverse*: **fixes** $x\ y$ :: *ereal* **shows** $|x| \neq \infty \Longrightarrow x + y - x = y$
by(*cases x y rule*: *ereal2-cases*) *simp-all*

lemma *tendsto-diff-ereal*:
fixes $x\ y$ :: *ereal*
assumes x: $|x| \neq \infty$ **and** y: $|y| \neq \infty$
assumes f: $(f \longrightarrow x)\ F$ **and** g: $(g \longrightarrow y)\ F$
shows $((\lambda x.\ f\ x - g\ x) \longrightarrow x - y)\ F$
proof −
from x **obtain** r **where** x': $x = ereal\ r$ **by** (*cases x*) *auto*
with f **have** $((\lambda i.\ real\text{-}of\text{-}ereal\ (f\ i)) \longrightarrow r)\ F$ **by** *simp*
moreover

from y **obtain** p **where** y': $y = ereal\ p$ **by** (*cases* y) *auto*
with g **have** $((\lambda i.\ real\text{-}of\text{-}ereal\ (g\ i)) \longrightarrow p)\ F$ **by** *simp*
ultimately have $((\lambda i.\ real\text{-}of\text{-}ereal\ (f\ i) - real\text{-}of\text{-}ereal\ (g\ i)) \longrightarrow r - p)\ F$
by (*rule tendsto-diff*)
moreover
from *eventually-finite*[*OF x f*] *eventually-finite*[*OF y g*]
have *eventually* $(\lambda x.\ f\ x - g\ x = ereal\ (real\text{-}of\text{-}ereal\ (f\ x) - real\text{-}of\text{-}ereal\ (g\ x)))$ *F*
by *eventually-elim auto*
ultimately show *?thesis*
by (*simp add*: x' y' *cong*: *filterlim-cong*)
qed

lemma *continuous-on-diff-ereal*:
continuous-on A f $\Longrightarrow$ *continuous-on A g* $\Longrightarrow$ $(\bigwedge x.\ x \in A \Longrightarrow |f\ x| \neq \infty) \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow |g\ x| \neq \infty) \Longrightarrow$ *continuous-on A* $(\lambda z.\ f\ z - g\ z$::*ereal*)
by (*auto simp*: *tendsto-diff-ereal continuous-on-def*)

38.6.5 Tests for code generator

A small list of simple arithmetic expressions.

value $-\infty$ :: *ereal*
value $|-\infty|$ :: *ereal*
value *4 + 5 / 4 − ereal 2* :: *ereal*
value *ereal 3* $< \infty$
value *real-of-ereal* $(\infty$::*ereal*$) = 0$

end

39 Indicator Function

theory *Indicator-Function*
imports *Complex-Main Disjoint-Sets*
begin

definition *indicator S x* = *of-bool* $(x \in S)$

Type constrained version

abbreviation *indicat-real* :: *'a set* $\Rightarrow$ *'a* $\Rightarrow$ *real* **where** *indicat-real S* $\equiv$ *indicator S*

lemma *indicator-simps*[*simp*]:
$x \in S \Longrightarrow$ *indicator S x* = *1*
$x \notin S \Longrightarrow$ *indicator S x* = *0*
unfolding *indicator-def* **by** *auto*

lemma *indicator-pos-le*[*intro*, *simp*]: $(0$::*'a*::*linordered-semidom*$) \leq$ *indicator S x*
and *indicator-le-1*[*intro*, *simp*]: *indicator S x* $\leq (1$::*'a*::*linordered-semidom*$)$

unfolding *indicator-def* **by** *auto*

lemma *indicator-abs-le-1*: $|indicator\ S\ x| \leq (1::'a::linordered\text{-}idom)$
unfolding *indicator-def* **by** *auto*

lemma *indicator-eq-0-iff*: $indicator\ A\ x = (0::'a::zero\text{-}neq\text{-}one) \longleftrightarrow x \notin A$
by (*auto simp*: *indicator-def*)

lemma *indicator-eq-1-iff*: $indicator\ A\ x = (1::'a::zero\text{-}neq\text{-}one) \longleftrightarrow x \in A$
by (*auto simp*: *indicator-def*)

lemma *indicator-UNIV* [*simp*]: $indicator\ UNIV = (\lambda x.\ 1)$
by *auto*

lemma *indicator-leI*:
$(x \in A \Longrightarrow y \in B) \Longrightarrow (indicator\ A\ x :: 'a::linordered\text{-}nonzero\text{-}semiring) \leq indicator\ B\ y$
by (*auto simp*: *indicator-def*)

lemma *split-indicator*: $P\ (indicator\ S\ x) \longleftrightarrow ((x \in S \longrightarrow P\ 1) \wedge (x \notin S \longrightarrow P\ 0))$
unfolding *indicator-def* **by** *auto*

lemma *split-indicator-asm*: $P\ (indicator\ S\ x) \longleftrightarrow (\neg\ (x \in S \wedge \neg\ P\ 1 \vee x \notin S \wedge \neg\ P\ 0))$
unfolding *indicator-def* **by** *auto*

lemma *indicator-inter-arith*: $indicator\ (A \cap B)\ x = indicator\ A\ x * (indicator\ B\ x::'a::semiring\text{-}1)$
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-union-arith*:
$indicator\ (A \cup B)\ x = indicator\ A\ x + indicator\ B\ x - indicator\ A\ x * (indicator\ B\ x :: 'a::ring\text{-}1)$
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-inter-min*: $indicator\ (A \cap B)\ x = min\ (indicator\ A\ x)\ (indicator\ B\ x::'a::linordered\text{-}semidom)$
and *indicator-union-max*: $indicator\ (A \cup B)\ x = max\ (indicator\ A\ x)\ (indicator\ B\ x::'a::linordered\text{-}semidom)$
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-disj-union*:
$A \cap B = \{\} \Longrightarrow indicator\ (A \cup B)\ x = (indicator\ A\ x + indicator\ B\ x :: 'a::linordered\text{-}semidom)$
by (*auto split*: *split-indicator*)

lemma *indicator-compl*: $indicator\ (-\ A)\ x = 1 - (indicator\ A\ x :: 'a::ring\text{-}1)$
and *indicator-diff*: $indicator\ (A - B)\ x = indicator\ A\ x * (1 - indicator\ B\ x$

::′a::*ring-1*)
unfolding *indicator-def* **by** (*auto simp*: *min-def max-def*)

lemma *indicator-times*:
indicator (*A* × *B*) *x* = *indicator A* (*fst x*) ∗ (*indicator B* (*snd x*) :: ′a::*semiring-1*)
unfolding *indicator-def* **by** (*cases x*) *auto*

lemma *indicator-sum*:
indicator (*A* <+> *B*) *x* = (*case x of Inl x* ⇒ *indicator A x* | *Inr x* ⇒ *indicator B x*)
unfolding *indicator-def* **by** (*cases x*) *auto*

lemma *indicator-image*: *inj f* ⟹ *indicator* (*f* ‘ *X*) (*f x*) = (*indicator X x*::-::*zero-neq-one*)
by (*auto simp*: *indicator-def inj-def*)

lemma *indicator-vimage*: *indicator* (*f* −‘ *A*) *x* = *indicator A* (*f x*)
by (*auto split*: *split-indicator*)

lemma *mult-indicator-cong*:
fixes *f g* :: - ⇒ ′a :: *semiring-1*
shows (⋀*x*. *x* ∈ *A* ⟹ *f x* = *g x*) ⟹ *indicator A x* ∗ *f x* = *indicator A x* ∗ *g x*
by (*auto simp*: *indicator-def*)

lemma
fixes *f* :: ′a ⇒ ′b::*semiring-1*
assumes *finite A*
shows *sum-mult-indicator*[*simp*]: (∑ *x* ∈ *A*. *f x* ∗ *indicator B x*) = (∑ *x* ∈ *A* ∩ *B*. *f x*)
and *sum-indicator-mult*[*simp*]: (∑ *x* ∈ *A*. *indicator B x* ∗ *f x*) = (∑ *x* ∈ *A* ∩ *B*. *f x*)
unfolding *indicator-def*
using *assms* **by** (*auto intro*!: *sum.mono-neutral-cong-right split*: *if-split-asm*)

lemma *sum-indicator-eq-card*:
assumes *finite A*
shows (∑ *x* ∈ *A*. *indicator B x*) = *card* (*A Int B*)
using *sum-mult-indicator* [*OF assms*, *of* λ*x*. *1*::*nat*]
unfolding *card-eq-sum* **by** *simp*

lemma *sum-indicator-scaleR*[*simp*]:
finite A ⟹
(∑ *x* ∈ *A*. *indicator* (*B x*) (*g x*) ∗*R* *f x*) = (∑ *x* ∈ {*x*∈*A*. *g x* ∈ *B x*}. *f x* :: ′a::*real-vector*)
by (*auto intro*!: *sum.mono-neutral-cong-right split*: *if-split-asm simp*: *indicator-def*)

lemma *LIMSEQ-indicator-incseq*:
assumes *incseq A*
shows (λ*i*. *indicator* (*A i*) *x* :: ′a::{*topological-space*,*zero-neq-one*}) ⟶ *indi-*

cator $(\bigcup i.\ A\ i)\ x$
proof (*cases* $\exists i.\ x \in A\ i$)
case *True*
then obtain i **where** $x \in A\ i$
by *auto*
then have $*$:
$\bigwedge n.\ (indicator\ (A\ (n + i))\ x :: 'a) = 1$
$(indicator\ (\bigcup i.\ A\ i)\ x :: 'a) = 1$
using *incseqD*[*OF* ‹*incseq A*›, *of i n* + *i* **for** *n*] ‹$x \in A\ i$› **by** (*auto simp*: *indicator-def*)
show *?thesis*
by (*rule LIMSEQ-offset*[*of* - *i*]) (*use* $*$ **in** *simp*)
next
case *False*
then show *?thesis* **by** (*simp add*: *indicator-def*)
qed

lemma *LIMSEQ-indicator-UN*:
$(\lambda k.\ indicator\ (\bigcup i{<}k.\ A\ i)\ x :: 'a{::}\{topological\text{-}space, zero\text{-}neq\text{-}one\}) \longrightarrow indicator\ (\bigcup i.\ A\ i)\ x$
proof −
have $(\lambda k.\ indicator\ (\bigcup i{<}k.\ A\ i)\ x{::}'a) \longrightarrow indicator\ (\bigcup k.\ \bigcup i{<}k.\ A\ i)\ x$
by (*intro LIMSEQ-indicator-incseq*) (*auto simp*: *incseq-def intro*: *less-le-trans*)
also have $(\bigcup k.\ \bigcup i{<}k.\ A\ i) = (\bigcup i.\ A\ i)$
by *auto*
finally show *?thesis* **.**
qed

lemma *LIMSEQ-indicator-decseq*:
assumes *decseq A*
shows $(\lambda i.\ indicator\ (A\ i)\ x :: 'a{::}\{topological\text{-}space, zero\text{-}neq\text{-}one\}) \longrightarrow indicator\ (\bigcap i.\ A\ i)\ x$
proof (*cases* $\exists i.\ x \notin A\ i$)
case *True*
then obtain i **where** $x \notin A\ i$
by *auto*
then have $*$:
$\bigwedge n.\ (indicator\ (A\ (n + i))\ x :: 'a) = 0$
$(indicator\ (\bigcap i.\ A\ i)\ x :: 'a) = 0$
using *decseqD*[*OF* ‹*decseq A*›, *of i n* + *i* **for** *n*] ‹$x \notin A\ i$› **by** (*auto simp*: *indicator-def*)
show *?thesis*
by (*rule LIMSEQ-offset*[*of* - *i*]) (*use* $*$ **in** *simp*)
next
case *False*
then show *?thesis* **by** (*simp add*: *indicator-def*)
qed

lemma *LIMSEQ-indicator-INT*:

(λk. *indicator* (⋂ i<k. *A i*) *x* :: 'a::{*topological-space*,*zero-neq-one*}) ⟶ *indicator* (⋂ *i*. *A i*) *x*
proof −
have (λk. *indicator* (⋂ i<k. *A i*) *x*::'a) ⟶ *indicator* (⋂ *k*. ⋂ i<k. *A i*) *x*
by (*intro LIMSEQ-indicator-decseq*) (*auto simp*: *decseq-def intro*: *less-le-trans*)
also have (⋂ *k*. ⋂ i<k. *A i*) = (⋂ *i*. *A i*)
by *auto*
finally show *?thesis* **.**
qed

lemma *indicator-add*:
A ∩ *B* = {} ⟹ (*indicator A x*::-::*monoid-add*) + *indicator B x* = *indicator* (*A* ∪ *B*) *x*
unfolding *indicator-def* **by** *auto*

lemma *of-real-indicator*: *of-real* (*indicator A x*) = *indicator A x*
by (*simp split*: *split-indicator*)

lemma *real-of-nat-indicator*: *real* (*indicator A x* :: *nat*) = *indicator A x*
by (*simp split*: *split-indicator*)

lemma *abs-indicator*: |*indicator A x* :: 'a::*linordered-idom*| = *indicator A x*
by (*simp split*: *split-indicator*)

lemma *mult-indicator-subset*:
A ⊆ *B* ⟹ *indicator A x* ∗ *indicator B x* = (*indicator A x* :: 'a::*comm-semiring-1*)
by (*auto split*: *split-indicator simp*: *fun-eq-iff*)

lemma *indicator-times-eq-if*:
fixes *f* :: 'a ⇒ 'b::*comm-ring-1*
shows *indicator S x* ∗ *f x* = (*if x* ∈ *S then f x else 0*) *f x* ∗ *indicator S x* = (*if x* ∈ *S then f x else 0*)
by *auto*

lemma *indicator-scaleR-eq-if*:
fixes *f* :: 'a ⇒ 'b::*real-vector*
shows *indicator S x* ∗*R* *f x* = (*if x* ∈ *S then f x else 0*)
by *simp*

lemma *indicator-sums*:
assumes ⋀*i j*. *i* ≠ *j* ⟹ *A i* ∩ *A j* = {}
shows (λi. *indicator* (*A i*) *x*::*real*) *sums indicator* (⋃ *i*. *A i*) *x*
proof (*cases* ∃ *i*. *x* ∈ *A i*)
case *True*
then obtain *i* **where** *i*: *x* ∈ *A i* **..**
with *assms* **have** (λi. *indicator* (*A i*) *x*::*real*) *sums* (∑ i∈{*i*}. *indicator* (*A i*) *x*)
by (*intro sums-finite*) (*auto split*: *split-indicator*)
also have (∑ i∈{*i*}. *indicator* (*A i*) *x*) = *indicator* (⋃ *i*. *A i*) *x*
using *i* **by** (*auto split*: *split-indicator*)

finally show *?thesis* **.**
next
case *False*
then show *?thesis* **by** *simp*
qed

The indicator function of the union of a disjoint family of sets is the sum over all the individual indicators.

lemma *indicator-UN-disjoint*:
finite A $\Longrightarrow$ *disjoint-family-on f A* $\Longrightarrow$ *indicator* $(\bigcup (f \text{`} A))\ x = (\sum y{\in}A.\ \textit{indicator}\ (f\ y)\ x)$
by (*induct A rule*: *finite-induct*)
(*auto simp*: *disjoint-family-on-def indicator-def split*: *if-splits*)

end

39.1 The type of non-negative extended real numbers

theory *Extended-Nonnegative-Real*
imports *Extended-Real Indicator-Function*
begin

lemma *ereal-ineq-diff-add*:
assumes $b \neq (-\infty{::}ereal)$ $a \geq b$
shows $a = b + (a-b)$
by (*metis add.commute assms ereal-eq-minus-iff ereal-minus-le-iff ereal-plus-eq-PInfty*)

lemma *Limsup-const-add*:
fixes *c* :: *'a*::{*complete-linorder*, *linorder-topology*, *topological-monoid-add*, *ordered-ab-semigroup-add*}
shows $F \neq bot \Longrightarrow Limsup\ F\ (\lambda x.\ c + f\ x) = c + Limsup\ F\ f$
by (*rule Limsup-compose-continuous-mono*)
(*auto intro*!: *monoI add-mono continuous-on-add continuous-on-id continuous-on-const*)

lemma *Liminf-const-add*:
fixes *c* :: *'a*::{*complete-linorder*, *linorder-topology*, *topological-monoid-add*, *ordered-ab-semigroup-add*}
shows $F \neq bot \Longrightarrow Liminf\ F\ (\lambda x.\ c + f\ x) = c + Liminf\ F\ f$
by (*rule Liminf-compose-continuous-mono*)
(*auto intro*!: *monoI add-mono continuous-on-add continuous-on-id continuous-on-const*)

lemma *Liminf-add-const*:
fixes *c* :: *'a*::{*complete-linorder*, *linorder-topology*, *topological-monoid-add*, *ordered-ab-semigroup-add*}
shows $F \neq bot \Longrightarrow Liminf\ F\ (\lambda x.\ f\ x + c) = Liminf\ F\ f + c$
by (*rule Liminf-compose-continuous-mono*)
(*auto intro*!: *monoI add-mono continuous-on-add continuous-on-id continuous-on-const*)

lemma *sums-offset*:
fixes *f g* :: *nat* ⇒ *'a* :: {*t2-space*, *topological-comm-monoid-add*}
assumes (λ*n*. *f* (*n* + *i*)) *sums l* **shows** *f sums* (*l* + (∑*j*<*i*. *f j*))
proof −
have (λ*k*. (∑*n*<*k*. *f* (*n* + *i*)) + (∑*j*<*i*. *f j*)) ⟶ *l* + (∑*j*<*i*. *f j*)
using *assms* **by** (*auto intro*!: *tendsto-add simp*: *sums-def*)
moreover
{ fix *k* :: *nat*
have (∑*j*<*k* + *i*. *f j*) = (∑*j*=*i*..<*k* + *i*. *f j*) + (∑*j*=*0*..<*i*. *f j*)
by (*subst sum.union-disjoint*[*symmetric*]) (*auto intro*!: *sum.cong*)
also have (∑*j*=*i*..<*k* + *i*. *f j*) = (∑*j*∈(λ*n*. *n* + *i*) ‘{*0*..<*k*}. *f j*)
unfolding *image-add-atLeastLessThan* **by** *simp*
finally have (∑*j*<*k* + *i*. *f j*) = (∑*n*<*k*. *f* (*n* + *i*)) + (∑*j*<*i*. *f j*)
by (*auto simp*: *inj-on-def atLeast0LessThan sum.reindex*) **}**
ultimately have (λ*k*. (∑*n*<*k* + *i*. *f n*)) ⟶ *l* + (∑*j*<*i*. *f j*)
by *simp*
then show *?thesis*
unfolding *sums-def* **by** (*rule LIMSEQ-offset*)
qed

lemma *suminf-offset*:
fixes *f g* :: *nat* ⇒ *'a* :: {*t2-space*, *topological-comm-monoid-add*}
shows *summable* (λ*j*. *f* (*j* + *i*)) ⟹ *suminf f* = (∑*j*. *f* (*j* + *i*)) + (∑*j*<*i*. *f j*)
by (*intro sums-unique*[*symmetric*] *sums-offset summable-sums*)

lemma *eventually-at-left-1*: (⋀*z*::*real*. *0* < *z* ⟹ *z* < *1* ⟹ *P z*) ⟹ *eventually P* (*at-left 1*)
by (*subst eventually-at-left*[*of 0*]) (*auto intro*: *exI*[*of - 0*])

lemma *mult-eq-1*:
fixes *a b* :: *'a* :: {*ordered-semiring*, *comm-monoid-mult*}
shows *0* ≤ *a* ⟹ *a* ≤ *1* ⟹ *b* ≤ *1* ⟹ *a* ∗ *b* = *1* ⟷ (*a* = *1* ∧ *b* = *1*)
by (*metis mult.left-neutral eq-iff mult.commute mult-right-mono*)

lemma *ereal-add-diff-cancel*:
fixes *a b* :: *ereal*
shows |*b*| ≠ ∞ ⟹ (*a* + *b*) − *b* = *a*
by (*cases a b rule*: *ereal2-cases*) *auto*

lemma *add-top*:
fixes *x* :: *'a*::{*order-top*, *ordered-comm-monoid-add*}
shows *0* ≤ *x* ⟹ *x* + *top* = *top*
by (*intro top-le add-increasing order-refl*)

lemma *top-add*:
fixes *x* :: *'a*::{*order-top*, *ordered-comm-monoid-add*}
shows *0* ≤ *x* ⟹ *top* + *x* = *top*
by (*intro top-le add-increasing2 order-refl*)

lemma *le-lfp*: *mono f* $\Longrightarrow$ $x \leq$ *lfp f* $\Longrightarrow$ *f x* $\leq$ *lfp f*
by (*subst lfp-unfold*) (*auto dest*: *monoD*)

lemma *lfp-transfer*:
assumes α: *sup-continuous* α **and** *f*: *sup-continuous f* **and** *mg*: *mono g*
assumes *bot*: α *bot* $\leq$ *lfp g* **and** *eq*: $\bigwedge x.\ x \leq$ *lfp f* $\Longrightarrow$ α (*f x*) = *g* (α *x*)
shows α (*lfp f*) = *lfp g*
proof (*rule antisym*)
note *mf* = *sup-continuous-mono*[*OF f*]
have *f-le-lfp*: (*f* ^^ *i*) *bot* $\leq$ *lfp f* **for** *i*
by (*induction i*) (*auto intro*: *le-lfp mf*)

have α ((*f* ^^ *i*) *bot*) $\leq$ *lfp g* **for** *i*
by (*induction i*) (*auto simp*: *bot eq f-le-lfp intro*!: *le-lfp mg*)
then show α (*lfp f*) $\leq$ *lfp g*
unfolding *sup-continuous-lfp*[*OF f*]
by (*subst* α[*THEN sup-continuousD*])
(*auto intro*!: *mono-funpow sup-continuous-mono*[*OF f*] *SUP-least*)

show *lfp g* $\leq$ α (*lfp f*)
by (*rule lfp-lowerbound*) (*simp add*: *eq*[*symmetric*] *lfp-fixpoint*[*OF mf*])
qed

lemma *sup-continuous-applyD*: *sup-continuous f* $\Longrightarrow$ *sup-continuous* ($\lambda x.$ *f x h*)
using *sup-continuous-apply*[*THEN sup-continuous-compose*] **.**

lemma *sup-continuous-SUP*[*order-continuous-intros*]:
fixes *M* :: - $\Rightarrow$ - $\Rightarrow$ *'a*::*complete-lattice*
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow$ *sup-continuous* (*M i*)
shows *sup-continuous* (*SUP i*$\in$*I*. *M i*)
unfolding *sup-continuous-def* **by** (*auto simp add*: *sup-continuousD* [*OF M*] *image-comp intro*: *SUP-commute*)

lemma *sup-continuous-apply-SUP*[*order-continuous-intros*]:
fixes *M* :: - $\Rightarrow$ - $\Rightarrow$ *'a*::*complete-lattice*
shows ($\bigwedge i.\ i \in I \Longrightarrow$ *sup-continuous* (*M i*)) $\Longrightarrow$ *sup-continuous* ($\lambda x.$ *SUP i*$\in$*I*. *M i x*)
unfolding *SUP-apply*[*symmetric*] **by** (*rule sup-continuous-SUP*)

lemma *sup-continuous-lfp'*[*order-continuous-intros*]:
assumes *1*: *sup-continuous f*
assumes *2*: $\bigwedge g.$ *sup-continuous g* $\Longrightarrow$ *sup-continuous* (*f g*)
shows *sup-continuous* (*lfp f*)
proof −
have *sup-continuous* ((*f* ^^ *i*) *bot*) **for** *i*
proof (*induction i*)
case (*Suc i*) **then show** *?case*
by (*auto intro*!: *2*)

qed (*simp add*: *bot-fun-def sup-continuous-const*)
then show *?thesis*
unfolding *sup-continuous-lfp*[*OF 1*] **by** (*intro order-continuous-intros*)
qed

lemma *sup-continuous-lfp''*[*order-continuous-intros*]:
assumes *1*: $\bigwedge s$. *sup-continuous* (*f s*)
assumes *2*: $\bigwedge g$. *sup-continuous g* $\Longrightarrow$ *sup-continuous* (λs. *f s* (*g s*))
shows *sup-continuous* (λx. *lfp* (*f x*))
proof −
have *sup-continuous* (λx. (*f x* ^^ *i*) *bot*) **for** *i*
proof (*induction i*)
case (*Suc i*) **then show** *?case*
by (*auto intro*!: *2*)
qed (*simp add*: *bot-fun-def sup-continuous-const*)
then show *?thesis*
unfolding *sup-continuous-lfp*[*OF 1*] **by** (*intro order-continuous-intros*)
qed

lemma *mono-INF-fun*:
($\bigwedge x\ y$. *mono* (*F x y*)) $\Longrightarrow$ *mono* ($\lambda z\ x$. *INF y* $\in$ *X x*. *F x y z* :: *'a* :: *complete-lattice*)
by (*auto intro*!: *INF-mono*[*OF bexI*] *simp*: *le-fun-def mono-def*)

lemma *continuous-on-cmult-ereal*:
|*c*::*ereal*| $\neq \infty \Longrightarrow$ *continuous-on A f* $\Longrightarrow$ *continuous-on A* (λx. *c* $*$ *f x*)
using *tendsto-cmult-ereal*[*of c f f x at x within A* **for** *x*]
by (*auto simp*: *continuous-on-def simp del*: *tendsto-cmult-ereal*)

lemma *real-of-nat-Sup*:
assumes *A* $\neq \{\}$ *bdd-above A*
shows *of-nat* (*Sup A*) = (*SUP a*$\in$*A*. *of-nat a* :: *real*)
proof (*intro antisym*)
show (*SUP a*$\in$*A*. *of-nat a*::*real*) $\leq$ *of-nat* (*Sup A*)
using *assms* **by** (*intro cSUP-least of-nat-mono*) (*auto intro*: *cSup-upper*)
have *Sup A* $\in$ *A*
using *assms* **by** (*auto simp*: *Sup-nat-def bdd-above-nat*)
then show *of-nat* (*Sup A*) $\leq$ (*SUP a*$\in$*A*. *of-nat a*::*real*)
by (*intro cSUP-upper bdd-above-image-mono assms*) (*auto simp*: *mono-def*)
qed

lemma (**in** *complete-lattice*) *SUP-sup-const1*:
I $\neq \{\} \Longrightarrow$ (*SUP i*$\in$*I*. *sup c* (*f i*)) = *sup c* (*SUP i*$\in$*I*. *f i*)
using *SUP-sup-distrib*[*of* λ-. *c I f*] **by** *simp*

lemma (**in** *complete-lattice*) *SUP-sup-const2*:
I $\neq \{\} \Longrightarrow$ (*SUP i*$\in$*I*. *sup* (*f i*) *c*) = *sup* (*SUP i*$\in$*I*. *f i*) *c*
using *SUP-sup-distrib*[*of f I* λ-. *c*] **by** *simp*

lemma *one-less-of-natD*:
assumes $(1::'a::linordered\text{-}semidom) < of\text{-}nat\ n$ **shows** $1 < n$
by (*cases n*) (*use assms* **in** *auto*)

39.2 Defining the extended non-negative reals

Basic definitions and type class setup

typedef $ennreal = \{x :: ereal.\ 0 \leq x\}$
morphisms *enn2ereal e2ennreal'*
by *auto*

definition $e2ennreal\ x = e2ennreal'\ (max\ 0\ x)$

lemma *enn2ereal-range*: $e2ennreal\ `\ \{0..\} = UNIV$
proof −
have $\exists y{\geq}0.\ x = e2ennreal\ y$ **for** x
by (*cases x*) (*auto simp*: *e2ennreal-def max-absorb2*)
then show *?thesis*
by (*auto simp*: *image-iff Bex-def*)
qed

lemma *type-definition-ennreal'*: *type-definition enn2ereal e2ennreal* $\{x.\ 0 \leq x\}$
using *type-definition-ennreal*
by (*auto simp*: *type-definition-def e2ennreal-def max-absorb2*)

setup-lifting *type-definition-ennreal'*

declare [[*coercion e2ennreal*]]

instantiation *ennreal* :: *complete-linorder*
begin

lift-definition *top-ennreal* :: *ennreal* **is** *top* **by** (*rule top-greatest*)
lift-definition *bot-ennreal* :: *ennreal* **is** *0* **by** (*rule order-refl*)
lift-definition *sup-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** *sup* **by** (*rule le-supI1*)
lift-definition *inf-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal* **is** *inf* **by** (*rule le-infI*)

lift-definition *Inf-ennreal* :: *ennreal set* ⇒ *ennreal* **is** *Inf*
by (*rule Inf-greatest*)

lift-definition *Sup-ennreal* :: *ennreal set* ⇒ *ennreal* **is** *sup 0* ∘ *Sup*
by *auto*

lift-definition *less-eq-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *bool* **is** (≤) **.**
lift-definition *less-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *bool* **is** (<) **.**

instance
by *standard*
(*transfer* ; *auto simp*: *Inf-lower Inf-greatest Sup-upper Sup-least le-max-iff-disj*

max.absorb1)+

end

lemma *pcr-ennreal-enn2ereal*[*simp*]: *pcr-ennreal* (*enn2ereal x*) *x*
by (*simp add*: *ennreal.pcr-cr-eq cr-ennreal-def*)

lemma *rel-fun-eq-pcr-ennreal*: *rel-fun* (=) *pcr-ennreal f g* $\longleftrightarrow$ $f = enn2ereal \circ g$
by (*auto simp*: *rel-fun-def ennreal.pcr-cr-eq cr-ennreal-def*)

instantiation *ennreal* :: *infinity*
begin

definition *infinity-ennreal* :: *ennreal*
where
[*simp*]: $\infty = (top::ennreal)$

instance ..

end

instantiation *ennreal* :: {*semiring-1-no-zero-divisors*, *comm-semiring-1*}
begin

lift-definition *one-ennreal* :: *ennreal* **is** *1* **by** *simp*
lift-definition *zero-ennreal* :: *ennreal* **is** *0* **by** *simp*
lift-definition *plus-ennreal* :: *ennreal* $\Rightarrow$ *ennreal* $\Rightarrow$ *ennreal* **is** (+) **by** *simp*
lift-definition *times-ennreal* :: *ennreal* $\Rightarrow$ *ennreal* $\Rightarrow$ *ennreal* **is** (∗) **by** *simp*

instance
by *standard* (*transfer*; *auto simp*: *field-simps ereal-right-distrib*)+

end

instantiation *ennreal* :: *minus*
begin

lift-definition *minus-ennreal* :: *ennreal* $\Rightarrow$ *ennreal* $\Rightarrow$ *ennreal* **is** $\lambda a\ b.\ max\ 0\ (a - b)$
by *simp*

instance ..

end

instance *ennreal* :: *numeral* **..**

instantiation *ennreal* :: *inverse*
begin

lift-definition *inverse-ennreal* :: *ennreal* ⇒ *ennreal* **is** *inverse*
by (*rule inverse-ereal-ge0I*)

definition *divide-ennreal* :: *ennreal* ⇒ *ennreal* ⇒ *ennreal*
where *x div y* = *x* ∗ *inverse* (*y* :: *ennreal*)

instance ..

end

lemma *ennreal-zero-less-one*: *0* < (*1*::*ennreal*) — TODO: remove
by *transfer auto*

instance *ennreal* :: *dioid*
proof (*standard*; *transfer*)
fix *a b* :: *ereal* **assume** *0* ≤ *a 0* ≤ *b* **then show** (*a* ≤ *b*) = (∃ *c*∈*Collect* ((≤) *0*). *b* = *a* + *c*)
unfolding *ereal-ex-split Bex-def*
by (*cases a b rule*: *ereal2-cases*) (*auto intro*!: *exI*[*of* - *real-of-ereal* (*b* − *a*)])
qed

instance *ennreal* :: *ordered-comm-semiring*
by *standard*
(*transfer* ; *auto intro*: *add-mono mult-mono mult-ac ereal-left-distrib ereal-mult-left-mono*)+

instance *ennreal* :: *linordered-nonzero-semiring*
proof
fix *a b*::*ennreal*
show *a* < *b* ⟹ *a* + *1* < *b* + *1*
by *transfer* (*simp add*: *add-right-mono ereal-add-cancel-right less-le*)
qed (*transfer*; *simp*)

instance *ennreal* :: *strict-ordered-ab-semigroup-add*
proof
fix *a b c d* :: *ennreal* **show** *a* < *b* ⟹ *c* < *d* ⟹ *a* + *c* < *b* + *d*
by *transfer* (*auto intro*!: *ereal-add-strict-mono*)
qed

declare [[*coercion of-nat* :: *nat* ⇒ *ennreal*]]

lemma *e2ennreal-neg*: *x* ≤ *0* ⟹ *e2ennreal x* = *0*
unfolding *zero-ennreal-def e2ennreal-def* **by** (*simp add*: *max-absorb1*)

lemma *e2ennreal-mono*: *x* ≤ *y* ⟹ *e2ennreal x* ≤ *e2ennreal y*
by (*cases 0* ≤ *x 0* ≤ *y rule*: *bool.exhaust*[*case-product bool.exhaust*])
(*auto simp*: *e2ennreal-neg less-eq-ennreal.abs-eq eq-onp-def*)

lemma *enn2ereal-nonneg*[*simp*]: *0* ≤ *enn2ereal x*

using *ennreal.enn2ereal*[*of x*] **by** *simp*

lemma *ereal-ennreal-cases*:
obtains *b* **where** $0 \leq a$ $a = enn2ereal\ b \mid a < 0$
using *e2ennreal'-inverse*[*of a*, *symmetric*] **by** (*cases* $0 \leq a$) (*auto intro*: *enn2ereal-nonneg*)

lemma *rel-fun-liminf*[*transfer-rule*]: *rel-fun* (*rel-fun* (=) *pcr-ennreal*) *pcr-ennreal liminf liminf*
proof −
have *rel-fun* (*rel-fun* (=) *pcr-ennreal*) *pcr-ennreal* (λ*x*. *sup 0* (*liminf x*)) *liminf*
unfolding *liminf-SUP-INF*[*abs-def*] **by** (*transfer-prover-start*, *transfer-step*+; *simp*)
then show *?thesis*
apply (*subst* (*asm*) (*2*) *rel-fun-def*)
apply (*subst* (*2*) *rel-fun-def*)
apply (*auto simp*: *comp-def max.absorb2 Liminf-bounded rel-fun-eq-pcr-ennreal*)
done
qed

lemma *rel-fun-limsup*[*transfer-rule*]: *rel-fun* (*rel-fun* (=) *pcr-ennreal*) *pcr-ennreal limsup limsup*
proof −
have *rel-fun* (*rel-fun* (=) *pcr-ennreal*) *pcr-ennreal* (λ*x*. *INF n*. *sup 0* (*SUP i*∈{*n*..}. *x i*)) *limsup*
unfolding *limsup-INF-SUP*[*abs-def*] **by** (*transfer-prover-start*, *transfer-step*+; *simp*)
then show *?thesis*
unfolding *limsup-INF-SUP*[*abs-def*]
apply (*subst* (*asm*) (*2*) *rel-fun-def*)
apply (*subst* (*2*) *rel-fun-def*)
apply (*auto simp*: *comp-def max.absorb2 Sup-upper2 rel-fun-eq-pcr-ennreal*)
apply (*subst* (*asm*) *max.absorb2*)
apply (*auto intro*: *SUP-upper2*)
done
qed

lemma *sum-enn2ereal*[*simp*]: $(\bigwedge i.\ i \in I \Longrightarrow 0 \leq f\ i) \Longrightarrow (\sum i \in I.\ enn2ereal\ (f\ i)) = enn2ereal\ (sum\ f\ I)$
by (*induction I rule*: *infinite-finite-induct*) (*auto simp*: *sum-nonneg zero-ennreal.rep-eq plus-ennreal.rep-eq*)

lemma *transfer-e2ennreal-sum* [*transfer-rule*]:
rel-fun (*rel-fun* (=) *pcr-ennreal*) (*rel-fun* (=) *pcr-ennreal*) *sum sum*
by (*auto intro*!: *rel-funI simp*: *rel-fun-eq-pcr-ennreal comp-def*)

lemma *enn2ereal-of-nat*[*simp*]: *enn2ereal* (*of-nat n*) = *ereal n*
by (*induction n*) (*auto simp*: *zero-ennreal.rep-eq one-ennreal.rep-eq plus-ennreal.rep-eq*)

lemma *enn2ereal-numeral*[*simp*]: *enn2ereal* (*numeral a*) = *numeral a*

by (*metis enn2ereal-of-nat numeral-eq-ereal of-nat-numeral*)

lemma *transfer-numeral*[*transfer-rule*]: *pcr-ennreal* (*numeral a*) (*numeral a*)
unfolding *cr-ennreal-def pcr-ennreal-def* **by** *auto*

39.3 Cancellation simprocs

lemma *ennreal-add-left-cancel*: $a + b = a + c \longleftrightarrow a = (\infty::ennreal) \vee b = c$
unfolding *infinity-ennreal-def* **by** *transfer* (*simp add*: *top-ereal-def ereal-add-cancel-left*)

lemma *ennreal-add-left-cancel-le*: $a + b \leq a + c \longleftrightarrow a = (\infty::ennreal) \vee b \leq c$
unfolding *infinity-ennreal-def* **by** *transfer* (*simp add*: *ereal-add-le-add-iff top-ereal-def disj-commute*)

lemma *ereal-add-left-cancel-less*:
fixes *a b c* :: *ereal*
shows $0 \leq a \Longrightarrow 0 \leq b \Longrightarrow a + b < a + c \longleftrightarrow a \neq \infty \wedge b < c$
by (*cases a b c rule*: *ereal3-cases*) *auto*

lemma *ennreal-add-left-cancel-less*: $a + b < a + c \longleftrightarrow a \neq (\infty::ennreal) \wedge b < c$
unfolding *infinity-ennreal-def*
by *transfer* (*simp add*: *top-ereal-def ereal-add-left-cancel-less*)

ML ‹
structure Cancel-Ennreal-Common =
struct
(∗ *copied from src/HOL/Tools/nat-numeral-simprocs.ML* ∗)
fun find-first-t - - [] = *raise TERM*(*find-first-t*, [])
| *find-first-t past u* (*t*::*terms*) =
if u aconv t then (*rev past* @ *terms*)
else find-first-t (*t*::*past*) *u terms*

fun dest-summing (*Const* (***const-name*** ‹*Groups.plus*›, -) $ *t* $ *u*, *ts*) =
dest-summing (*t*, *dest-summing* (*u*, *ts*))
| *dest-summing* (*t*, *ts*) = *t* :: *ts*

val mk-sum = Arith-Data.long-mk-sum
fun dest-sum t = dest-summing (*t*, [])
val find-first = find-first-t []
val trans-tac = Numeral-Simprocs.trans-tac
val norm-ss =
simpset-of (*put-simpset HOL-basic-ss* ***context***
addsimps @{*thms ac-simps add-0-left add-0-right*})
fun norm-tac ctxt = ALLGOALS (*simp-tac* (*put-simpset norm-ss ctxt*))
fun simplify-meta-eq ctxt cancel-th th =
Arith-Data.simplify-meta-eq [] *ctxt*
([*th*, *cancel-th*] *MRS trans*)
fun mk-eq (*a*, *b*) = *HOLogic.mk-Trueprop* (*HOLogic.mk-eq* (*a*, *b*))
end

```
structure Eq-Ennreal-Cancel = ExtractCommonTermFun
(open Cancel-Ennreal-Common
  val mk-bal = HOLogic.mk-eq
  val dest-bal = HOLogic.dest-bin const-name ‹HOL.eq› typ ‹ennreal›
  fun simp-conv - - = SOME @{thm ennreal-add-left-cancel}
)

structure Le-Ennreal-Cancel = ExtractCommonTermFun
(open Cancel-Ennreal-Common
  val mk-bal = HOLogic.mk-binrel const-name ‹Orderings.less-eq›
  val dest-bal = HOLogic.dest-bin const-name ‹Orderings.less-eq› typ ‹ennreal›
  fun simp-conv - - = SOME @{thm ennreal-add-left-cancel-le}
)

structure Less-Ennreal-Cancel = ExtractCommonTermFun
(open Cancel-Ennreal-Common
  val mk-bal = HOLogic.mk-binrel const-name ‹Orderings.less›
  val dest-bal = HOLogic.dest-bin const-name ‹Orderings.less› typ ‹ennreal›
  fun simp-conv - - = SOME @{thm ennreal-add-left-cancel-less}
)
›

simproc-setup ennreal-eq-cancel
  ((l::ennreal) + m = n | (l::ennreal) = m + n) =
  ‹fn phi => fn ctxt => fn ct => Eq-Ennreal-Cancel.proc ctxt (Thm.term-of ct)›

simproc-setup ennreal-le-cancel
  ((l::ennreal) + m ≤ n | (l::ennreal) ≤ m + n) =
  ‹fn phi => fn ctxt => fn ct => Le-Ennreal-Cancel.proc ctxt (Thm.term-of ct)›

simproc-setup ennreal-less-cancel
  ((l::ennreal) + m < n | (l::ennreal) < m + n) =
  ‹fn phi => fn ctxt => fn ct => Less-Ennreal-Cancel.proc ctxt (Thm.term-of
ct)›
```

39.4 Order with top

lemma *ennreal-zero-less-top*[*simp*]: *0* < (*top*::*ennreal*)
  by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-one-less-top*[*simp*]: *1* < (*top*::*ennreal*)
  by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-zero-neq-top*[*simp*]: *0* ≠ (*top*::*ennreal*)
  by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-top-neq-zero*[*simp*]: (*top*::*ennreal*) ≠ *0*
  by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-top-neq-one*[*simp*]: $top \neq (1::ennreal)$
by *transfer* (*simp add*: *top-ereal-def one-ereal-def flip*: *ereal-max*)

lemma *ennreal-one-neq-top*[*simp*]: $1 \neq (top::ennreal)$
by *transfer* (*simp add*: *top-ereal-def one-ereal-def flip*: *ereal-max*)

lemma *ennreal-add-less-top*[*simp*]:
fixes $a\ b :: ennreal$
shows $a + b < top \longleftrightarrow a < top \wedge b < top$
by *transfer* (*auto simp*: *top-ereal-def*)

lemma *ennreal-add-eq-top*[*simp*]:
fixes $a\ b :: ennreal$
shows $a + b = top \longleftrightarrow a = top \vee b = top$
by *transfer* (*auto simp*: *top-ereal-def*)

lemma *ennreal-sum-less-top*[*simp*]:
fixes $f :: 'a \Rightarrow ennreal$
shows *finite* $I \Longrightarrow (\sum i{\in}I.\ f\ i) < top \longleftrightarrow (\forall\, i{\in}I.\ f\ i < top)$
by (*induction I rule*: *finite-induct*) *auto*

lemma *ennreal-sum-eq-top*[*simp*]:
fixes $f :: 'a \Rightarrow ennreal$
shows *finite* $I \Longrightarrow (\sum i{\in}I.\ f\ i) = top \longleftrightarrow (\exists\, i{\in}I.\ f\ i = top)$
by (*induction I rule*: *finite-induct*) *auto*

lemma *ennreal-mult-eq-top-iff*:
fixes $a\ b :: ennreal$
shows $a * b = top \longleftrightarrow (a = top \wedge b \neq 0) \vee (b = top \wedge a \neq 0)$
by *transfer* (*auto simp*: *top-ereal-def*)

lemma *ennreal-top-eq-mult-iff*:
fixes $a\ b :: ennreal$
shows $top = a * b \longleftrightarrow (a = top \wedge b \neq 0) \vee (b = top \wedge a \neq 0)$
using *ennreal-mult-eq-top-iff*[*of a b*] **by** *auto*

lemma *ennreal-mult-less-top*:
fixes $a\ b :: ennreal$
shows $a * b < top \longleftrightarrow (a = 0 \vee b = 0 \vee (a < top \wedge b < top))$
by *transfer* (*auto simp add*: *top-ereal-def*)

lemma *top-power-ennreal*: $top\ \hat{}\ n = (if\ n = 0\ then\ 1\ else\ top :: ennreal)$
by (*induction n*) (*simp-all add*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-prod-eq-0*[*simp*]:
fixes $f :: 'a \Rightarrow ennreal$
shows $(prod\ f\ A = 0) = (finite\ A \wedge (\exists\, i{\in}A.\ f\ i = 0))$
by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *ennreal-prod-eq-top*:
fixes $f :: 'a \Rightarrow ennreal$
shows $(\prod i{\in}I.\ f\ i) = top \longleftrightarrow (finite\ I \wedge ((\forall i{\in}I.\ f\ i \neq 0) \wedge (\exists i{\in}I.\ f\ i = top)))$
by (*induction I rule*: *infinite-finite-induct*) (*auto simp*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-top-mult*: $top * a = (if\ a = 0\ then\ 0\ else\ top :: ennreal)$
by (*simp add*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-mult-top*: $a * top = (if\ a = 0\ then\ 0\ else\ top :: ennreal)$
by (*simp add*: *ennreal-mult-eq-top-iff*)

lemma *enn2ereal-eq-top-iff*[*simp*]: $enn2ereal\ x = \infty \longleftrightarrow x = top$
by *transfer* (*simp add*: *top-ereal-def*)

lemma *enn2ereal-top*[*simp*]: $enn2ereal\ top = \infty$
by *transfer* (*simp add*: *top-ereal-def*)

lemma *e2ennreal-infty*[*simp*]: $e2ennreal\ \infty = top$
by (*simp add*: *top-ennreal.abs-eq top-ereal-def*)

lemma *ennreal-top-minus*[*simp*]: $top - x = (top::ennreal)$
by *transfer* (*auto simp*: *top-ereal-def max-def*)

lemma *minus-top-ennreal*: $x - top = (if\ x = top\ then\ top\ else\ 0:: ennreal)$
by *transfer* (*use ereal-eq-minus-iff top-ereal-def* **in** *force*)

lemma *bot-ennreal*: $bot = (0::ennreal)$
by *transfer rule*

lemma *ennreal-of-nat-neq-top*[*simp*]: $of\text{-}nat\ i \neq (top::ennreal)$
by (*induction i*) *auto*

lemma *numeral-eq-of-nat*: $(numeral\ a::ennreal) = of\text{-}nat\ (numeral\ a)$
by *simp*

lemma *of-nat-less-top*: $of\text{-}nat\ i < (top::ennreal)$
using *less-le-trans*[*of of-nat i of-nat* (*Suc i*) *top*::*ennreal*]
by *simp*

lemma *top-neq-numeral*[*simp*]: $top \neq (numeral\ i::ennreal)$
using *of-nat-less-top*[*of numeral i*] **by** *simp*

lemma *ennreal-numeral-less-top*[*simp*]: $numeral\ i < (top::ennreal)$
using *of-nat-less-top*[*of numeral i*] **by** *simp*

lemma *ennreal-add-bot*[*simp*]: $bot + x = (x::ennreal)$
by *transfer simp*

lemma *add-top-right-ennreal* [*simp*]: $x + top = (top :: ennreal)$
by (*cases x*) *auto*

lemma *add-top-left-ennreal* [*simp*]: $top + x = (top :: ennreal)$
by (*cases x*) *auto*

lemma *ennreal-top-mult-left* [*simp*]: $x \neq 0 \Longrightarrow x * top = (top :: ennreal)$
by (*subst ennreal-mult-eq-top-iff*) *auto*

lemma *ennreal-top-mult-right* [*simp*]: $x \neq 0 \Longrightarrow top * x = (top :: ennreal)$
by (*subst ennreal-mult-eq-top-iff*) *auto*

lemma *power-top-ennreal* [*simp*]: $n > 0 \Longrightarrow top \hat{} n = (top :: ennreal)$
by (*induction n*) *auto*

lemma *power-eq-top-ennreal-iff*: $x \hat{} n = top \longleftrightarrow x = (top :: ennreal) \wedge n > 0$
by (*induction n*) (*auto simp*: *ennreal-mult-eq-top-iff*)

lemma *ennreal-mult-le-mult-iff*: $c \neq 0 \Longrightarrow c \neq top \Longrightarrow c * a \leq c * b \longleftrightarrow a \leq (b :: ennreal)$
including *ennreal.lifting*
by (*transfer*, *subst ereal-mult-le-mult-iff*) (*auto simp*: *top-ereal-def*)

lemma *power-mono-ennreal*: $x \leq y \Longrightarrow x \hat{} n \leq (y \hat{} n :: ennreal)$
by (*induction n*) (*auto intro*!: *mult-mono*)

instance *ennreal* :: *semiring-char-0*
proof (*standard*, *safe intro*!: *linorder-injI*)
have $*$: *1 + of-nat k* $\neq$ (*0*::*ennreal*) **for** *k*
using *add-pos-nonneg*[*OF zero-less-one*, *of of-nat k* :: *ennreal*] **by** *auto*
fix *x y* :: *nat* **assume** *x < y of-nat x* = (*of-nat y*::*ennreal*) **then show** *False*
by (*auto simp add*: *less-iff-Suc-add* $*$)
qed

39.5 Arithmetic

lemma *ennreal-minus-zero*[*simp*]: $a - (0::ennreal) = a$
by *transfer* (*auto simp*: *max-def*)

lemma *ennreal-add-diff-cancel-right*[*simp*]:
fixes *x y z* :: *ennreal* **shows** $y \neq top \Longrightarrow (x + y) - y = x$
by *transfer* (*metis ereal-eq-minus-iff max-absorb2 not-MInfty-nonneg top-ereal-def*)

lemma *ennreal-add-diff-cancel-left*[*simp*]:
fixes *x y z* :: *ennreal* **shows** $y \neq top \Longrightarrow (y + x) - y = x$
by (*simp add*: *add.commute*)

lemma

fixes *a b* :: *ennreal*
shows $a - b = 0 \Longrightarrow a \leq b$
by *transfer* (*metis ereal-diff-gr0 le-cases max.absorb2 not-less*)

lemma *ennreal-minus-cancel*:
fixes *a b c* :: *ennreal*
shows $c \neq top \Longrightarrow a \leq c \Longrightarrow b \leq c \Longrightarrow c - a = c - b \Longrightarrow a = b$
apply *transfer*
subgoal for *a b c*
by (*cases a b c rule*: *ereal3-cases*)
(*auto simp*: *top-ereal-def max-def split*: *if-splits*)
done

lemma *sup-const-add-ennreal*:
fixes *a b c* :: *ennreal*
shows *sup* $(c + a)$ $(c + b) = c +$ *sup a b*
by *transfer* (*metis add-left-mono le-cases sup.absorb2 sup.orderE*)

lemma *ennreal-diff-add-assoc*:
fixes *a b c* :: *ennreal*
shows $a \leq b \Longrightarrow c + b - a = c + (b - a)$
apply *transfer*
subgoal for *a b c*
by (*cases a b c rule*: *ereal3-cases*) (*auto simp*: *field-simps max-absorb2*)
done

lemma *mult-divide-eq-ennreal*:
fixes *a b* :: *ennreal*
shows $b \neq 0 \Longrightarrow b \neq top \Longrightarrow (a * b) / b = a$
unfolding *divide-ennreal-def*
apply *transfer*
apply (*subst mult.assoc*)
apply (*simp add*: *top-ereal-def flip*: *divide-ereal-def*)
done

lemma *divide-mult-eq*: $a \neq 0 \Longrightarrow a \neq \infty \Longrightarrow x * a / (b * a) = x / (b::ennreal)$
unfolding *divide-ennreal-def infinity-ennreal-def*
apply *transfer*
subgoal for *a b c*
apply (*cases a b c rule*: *ereal3-cases*)
apply (*auto simp*: *top-ereal-def*)
done
done

lemma *ennreal-mult-divide-eq*:
fixes *a b* :: *ennreal*
shows $b \neq 0 \Longrightarrow b \neq top \Longrightarrow (a * b) / b = a$
unfolding *divide-ennreal-def*
apply *transfer*

apply (*subst mult.assoc*)
apply (*simp add*: *top-ereal-def flip*: *divide-ereal-def*)
done

lemma *ennreal-add-diff-cancel*:
fixes *a b* :: *ennreal*
shows $b \neq \infty \Longrightarrow (a + b) - b = a$
unfolding *infinity-ennreal-def*
by *transfer* (*simp add*: *max-absorb2 top-ereal-def ereal-add-diff-cancel*)

lemma *ennreal-minus-eq-0*:
$a - b = 0 \Longrightarrow a \leq (b::ennreal)$
by *transfer* (*metis ereal-diff-gr0 le-cases max.absorb2 not-less*)

lemma *ennreal-mono-minus-cancel*:
fixes *a b c* :: *ennreal*
shows $a - b \leq a - c \Longrightarrow a < top \Longrightarrow b \leq a \Longrightarrow c \leq a \Longrightarrow c \leq b$
by *transfer*
(*auto simp add*: *max.absorb2 ereal-diff-positive top-ereal-def dest*: *ereal-mono-minus-cancel*)

lemma *ennreal-mono-minus*:
fixes *a b c* :: *ennreal*
shows $c \leq b \Longrightarrow a - b \leq a - c$
by *transfer* (*meson ereal-minus-mono max.mono order-refl*)

lemma *ennreal-minus-pos-iff*:
fixes *a b* :: *ennreal*
shows $a < top \vee b < top \Longrightarrow 0 < a - b \Longrightarrow b < a$
by *transfer* (*use add.left-neutral ereal-minus-le-iff less-irrefl not-less* **in** *fastforce*)

lemma *ennreal-inverse-top*[*simp*]: *inverse top* = (*0*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def ereal-inverse-eq-0*)

lemma *ennreal-inverse-zero*[*simp*]: *inverse 0* = (*top*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def ereal-inverse-eq-0*)

lemma *ennreal-top-divide*: *top* / (*x*::*ennreal*) = (*if x* = *top then 0 else top*)
unfolding *divide-ennreal-def*
by *transfer* (*simp add*: *top-ereal-def ereal-inverse-eq-0 ereal-0-gt-inverse*)

lemma *ennreal-zero-divide*[*simp*]: *0* / (*x*::*ennreal*) = *0*
by (*simp add*: *divide-ennreal-def*)

lemma *ennreal-divide-zero*[*simp*]: *x* / (*0*::*ennreal*) = (*if x* = *0 then 0 else top*)
by (*simp add*: *divide-ennreal-def ennreal-mult-top*)

lemma *ennreal-divide-top*[*simp*]: *x* / (*top*::*ennreal*) = *0*
by (*simp add*: *divide-ennreal-def ennreal-top-mult*)

lemma *ennreal-times-divide*: $a * (b / c) = a * b / (c::ennreal)$
unfolding *divide-ennreal-def*
by *transfer* (*simp add*: *divide-ereal-def*[*symmetric*] *ereal-times-divide-eq*)

lemma *ennreal-zero-less-divide*: $0 < a / b \longleftrightarrow (0 < a \wedge b < (top::ennreal))$
unfolding *divide-ennreal-def*
by *transfer* (*auto simp*: *ereal-zero-less-0-iff top-ereal-def ereal-0-gt-inverse*)

lemma *add-divide-distrib-ennreal*: $(a + b) / c = a / c + b / (c :: ennreal)$
by (*simp add*: *divide-ennreal-def ring-distribs*)

lemma *divide-right-mono-ennreal*:
fixes *a b c* :: *ennreal*
shows $a \leq b \Longrightarrow a / c \leq b / c$
unfolding *divide-ennreal-def* **by** (*intro mult-mono*) *auto*

lemma *ennreal-mult-strict-right-mono*: $(a::ennreal) < c \Longrightarrow 0 < b \Longrightarrow b < top \Longrightarrow a * b < c * b$
by *transfer* (*auto intro*!: *ereal-mult-strict-right-mono*)

lemma *ennreal-indicator-less*[*simp*]:
$indicator\ A\ x \leq (indicator\ B\ x::ennreal) \longleftrightarrow (x \in A \longrightarrow x \in B)$
by (*simp add*: *indicator-def not-le*)

lemma *ennreal-inverse-positive*: $0 < inverse\ x \longleftrightarrow (x::ennreal) \neq top$
by *transfer* (*simp add*: *ereal-0-gt-inverse top-ereal-def*)

lemma *ennreal-inverse-mult'*: $((0 < b \vee a < top) \wedge (0 < a \vee b < top)) \Longrightarrow inverse\ (a * b::ennreal) = inverse\ a * inverse\ b$
apply *transfer*
subgoal for *a b*
by (*cases a b rule*: *ereal2-cases*) (*auto simp*: *top-ereal-def*)
done

lemma *ennreal-inverse-mult*: $a < top \Longrightarrow b < top \Longrightarrow inverse\ (a * b::ennreal) = inverse\ a * inverse\ b$
apply *transfer*
subgoal for *a b*
by (*cases a b rule*: *ereal2-cases*) (*auto simp*: *top-ereal-def*)
done

lemma *ennreal-inverse-1*[*simp*]: $inverse\ (1::ennreal) = 1$
by *transfer simp*

lemma *ennreal-inverse-eq-0-iff*[*simp*]: $inverse\ (a::ennreal) = 0 \longleftrightarrow a = top$
by *transfer* (*simp add*: *ereal-inverse-eq-0 top-ereal-def*)

lemma *ennreal-inverse-eq-top-iff*[*simp*]: $inverse\ (a::ennreal) = top \longleftrightarrow a = 0$
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-divide-eq-0-iff*[*simp*]: (*a*::*ennreal*) / *b* = *0* ⟷ (*a* = *0* ∨ *b* = *top*)
by (*simp add*: *divide-ennreal-def*)

lemma *ennreal-divide-eq-top-iff*: (*a*::*ennreal*) / *b* = *top* ⟷ ((*a* ≠ *0* ∧ *b* = *0*) ∨ (*a* = *top* ∧ *b* ≠ *top*))
by (*auto simp add*: *divide-ennreal-def ennreal-mult-eq-top-iff*)

lemma *one-divide-one-divide-ennreal*[*simp*]: *1* / (*1* / *c*) = (*c*::*ennreal*)
including *ennreal.lifting*
unfolding *divide-ennreal-def*
by *transfer auto*

lemma *ennreal-mult-left-cong*:
((*a*::*ennreal*) ≠ *0* ⟹ *b* = *c*) ⟹ *a* * *b* = *a* * *c*
by (*cases a* = *0*) *simp-all*

lemma *ennreal-mult-right-cong*:
((*a*::*ennreal*) ≠ *0* ⟹ *b* = *c*) ⟹ *b* * *a* = *c* * *a*
by (*cases a* = *0*) *simp-all*

lemma *ennreal-zero-less-mult-iff*: *0* < *a* * *b* ⟷ *0* < *a* ∧ *0* < (*b*::*ennreal*)
by *transfer* (*auto simp add*: *ereal-zero-less-0-iff le-less*)

lemma *less-diff-eq-ennreal*:
fixes *a b c* :: *ennreal*
shows *b* < *top* ∨ *c* < *top* ⟹ *a* < *b* − *c* ⟷ *a* + *c* < *b*
apply *transfer*
subgoal for *a b c*
by (*cases a b c rule*: *ereal3-cases*) (*auto split*: *split-max*)
done

lemma *diff-add-cancel-ennreal*:
fixes *a b* :: *ennreal* **shows** *a* ≤ *b* ⟹ *b* − *a* + *a* = *b*
unfolding *infinity-ennreal-def*
by *transfer* (*metis* (*no-types*) *add.commute ereal-diff-positive ereal-ineq-diff-add max-def not-MInfty-nonneg*)

lemma *ennreal-diff-self*[*simp*]: *a* ≠ *top* ⟹ *a* − *a* = (*0*::*ennreal*)
by *transfer* (*simp add*: *top-ereal-def*)

lemma *ennreal-minus-mono*:
fixes *a b c* :: *ennreal*
shows *a* ≤ *c* ⟹ *d* ≤ *b* ⟹ *a* − *b* ≤ *c* − *d*
by *transfer* (*meson ereal-minus-mono max.mono order-refl*)

lemma *ennreal-minus-eq-top*[*simp*]: *a* − (*b*::*ennreal*) = *top* ⟷ *a* = *top*
by *transfer* (*auto simp*: *top-ereal-def max.absorb2 ereal-minus-eq-PInfty-iff split*: *split-max*)

lemma *ennreal-divide-self*[*simp*]: $a \neq 0 \Longrightarrow a < top \Longrightarrow a \,/\, a = (1::ennreal)$
unfolding *divide-ennreal-def*
apply *transfer*
subgoal for *a*
by (*cases a*) (*auto simp*: *top-ereal-def*)
done

39.6 Coercion from *real* to *ennreal*

lift-definition *ennreal* :: *real* ⇒ *ennreal* **is** *sup 0* ∘ *ereal*
by *simp*

declare [[*coercion ennreal*]]

lemma *ennreal-cong*: $x = y \Longrightarrow ennreal\ x = ennreal\ y$ **by** *simp*

lemma *ennreal-cases*[*cases type*: *ennreal*]:
fixes *x* :: *ennreal*
obtains (*real*) *r* :: *real* **where** $0 \leq r\ x = ennreal\ r \mid (top)\ x = top$
apply *transfer*
subgoal for *x thesis*
by (*cases x*) (*auto simp*: *max.absorb2 top-ereal-def*)
done

lemmas *ennreal2-cases* = *ennreal-cases*[*case-product ennreal-cases*]
lemmas *ennreal3-cases* = *ennreal-cases*[*case-product ennreal2-cases*]

lemma *ennreal-neq-top*[*simp*]: $ennreal\ r \neq top$
by *transfer* (*simp add*: *top-ereal-def zero-ereal-def flip*: *ereal-max*)

lemma *top-neq-ennreal*[*simp*]: $top \neq ennreal\ r$
using *ennreal-neq-top*[*of r*] **by** (*auto simp del*: *ennreal-neq-top*)

lemma *ennreal-less-top*[*simp*]: $ennreal\ x < top$
by *transfer* (*simp add*: *top-ereal-def max-def*)

lemma *ennreal-neg*: $x \leq 0 \Longrightarrow ennreal\ x = 0$
by *transfer* (*simp add*: *max.absorb1*)

lemma *ennreal-inj*[*simp*]:
$0 \leq a \Longrightarrow 0 \leq b \Longrightarrow ennreal\ a = ennreal\ b \longleftrightarrow a = b$
by (*transfer fixing*: *a b*) (*auto simp*: *max-absorb2*)

lemma *ennreal-le-iff*[*simp*]: $0 \leq y \Longrightarrow ennreal\ x \leq ennreal\ y \longleftrightarrow x \leq y$
by (*auto simp*: *ennreal-def zero-ereal-def less-eq-ennreal.abs-eq eq-onp-def split*: *split-max*)

lemma *le-ennreal-iff*: $0 \leq r \Longrightarrow x \leq ennreal\ r \longleftrightarrow (\exists\, q{\geq}0.\ x = ennreal\ q \wedge q \leq$

r)
by (*cases x*) (*auto simp*: *top-unique*)

lemma *ennreal-less-iff*: $0 \leq r \Longrightarrow ennreal\ r < ennreal\ q \longleftrightarrow r < q$
unfolding *not-le*[*symmetric*] **by** *auto*

lemma *ennreal-eq-zero-iff*[*simp*]: $0 \leq x \Longrightarrow ennreal\ x = 0 \longleftrightarrow x = 0$
by *transfer* (*auto simp*: *max-absorb2*)

lemma *ennreal-less-zero-iff*[*simp*]: $0 < ennreal\ x \longleftrightarrow 0 < x$
by *transfer* (*auto simp*: *max-def*)

lemma *ennreal-lessI*: $0 < q \Longrightarrow r < q \Longrightarrow ennreal\ r < ennreal\ q$
by (*cases* $0 \leq r$) (*auto simp*: *ennreal-less-iff ennreal-neg*)

lemma *ennreal-leI*: $x \leq y \Longrightarrow ennreal\ x \leq ennreal\ y$
by (*cases* $0 \leq y$) (*auto simp*: *ennreal-neg*)

lemma *enn2ereal-ennreal*[*simp*]: $0 \leq x \Longrightarrow enn2ereal\ (ennreal\ x) = x$
by *transfer* (*simp add*: *max-absorb2*)

lemma *e2ennreal-enn2ereal*[*simp*]: $e2ennreal\ (enn2ereal\ x) = x$
by (*simp add*: *e2ennreal-def max-absorb2 ennreal.enn2ereal-inverse*)

lemma *enn2ereal-e2ennreal*: $x \geq 0 \Longrightarrow enn2ereal\ (e2ennreal\ x) = x$
by (*metis e2ennreal-enn2ereal ereal-ennreal-cases not-le*)

lemma *e2ennreal-ereal* [*simp*]: $e2ennreal\ (ereal\ x) = ennreal\ x$
by (*metis e2ennreal-def enn2ereal-inverse ennreal.rep-eq sup-ereal-def*)

lemma *ennreal-0*[*simp*]: $ennreal\ 0 = 0$
by (*simp add*: *ennreal-def max.absorb1 zero-ennreal.abs-eq*)

lemma *ennreal-1*[*simp*]: $ennreal\ 1 = 1$
by *transfer* (*simp add*: *max-absorb2*)

lemma *ennreal-eq-0-iff*: $ennreal\ x = 0 \longleftrightarrow x \leq 0$
by (*cases* $0 \leq x$) (*auto simp*: *ennreal-neg*)

lemma *ennreal-le-iff2*: $ennreal\ x \leq ennreal\ y \longleftrightarrow ((0 \leq y \wedge x \leq y) \vee (x \leq 0 \wedge y \leq 0))$
by (*cases* $0 \leq y$) (*auto simp*: *ennreal-eq-0-iff ennreal-neg*)

lemma *ennreal-eq-1*[*simp*]: $ennreal\ x = 1 \longleftrightarrow x = 1$
by (*cases* $0 \leq x$) (*auto simp*: *ennreal-neg simp flip*: *ennreal-1*)

lemma *ennreal-le-1*[*simp*]: $ennreal\ x \leq 1 \longleftrightarrow x \leq 1$
by (*cases* $0 \leq x$) (*auto simp*: *ennreal-neg simp flip*: *ennreal-1*)

lemma *ennreal-ge-1*[*simp*]: *ennreal x* ≥ *1* ⟷ *x* ≥ *1*
by (*cases 0* ≤ *x*) (*auto simp*: *ennreal-neg simp flip*: *ennreal-1*)

lemma *one-less-ennreal*[*simp*]: *1* < *ennreal x* ⟷ *1* < *x*
by *transfer* (*auto simp*: *max.absorb2 less-max-iff-disj*)

lemma *ennreal-plus*[*simp*]:
0 ≤ *a* ⟹ *0* ≤ *b* ⟹ *ennreal* (*a* + *b*) = *ennreal a* + *ennreal b*
by (*transfer fixing*: *a b*) (*auto simp*: *max-absorb2*)

lemma *add-mono-ennreal*: *x* < *ennreal y* ⟹ *x'* < *ennreal y'* ⟹ *x* + *x'* < *ennreal* (*y* + *y'*)
by (*metis* (*full-types*) *add-strict-mono ennreal-less-zero-iff ennreal-plus less-le not-less zero-le*)

lemma *sum-ennreal*[*simp*]: (⋀*i*. *i* ∈ *I* ⟹ *0* ≤ *f i*) ⟹ (∑ *i*∈*I*. *ennreal* (*f i*)) = *ennreal* (*sum f I*)
by (*induction I rule*: *infinite-finite-induct*) (*auto simp*: *sum-nonneg*)

lemma *sum-list-ennreal*[*simp*]:
assumes ⋀*x*. *x* ∈ *set xs* ⟹ *f x* ≥ *0*
shows *sum-list* (*map* (λ*x*. *ennreal* (*f x*)) *xs*) = *ennreal* (*sum-list* (*map f xs*))
using *assms*
proof (*induction xs*)
case (*Cons x xs*)
from *Cons* **have** (∑ *x*←*x* # *xs*. *ennreal* (*f x*)) = *ennreal* (*f x*) + *ennreal* (*sum-list* (*map f xs*))
by *simp*
also from *Cons.prems* **have** ... = *ennreal* (*f x* + *sum-list* (*map f xs*))
by (*intro ennreal-plus* [*symmetric*] *sum-list-nonneg*) *auto*
finally show *?case* **by** *simp*
qed *simp-all*

lemma *ennreal-of-nat-eq-real-of-nat*: *of-nat i* = *ennreal* (*of-nat i*)
by (*induction i*) *simp-all*

lemma *of-nat-le-ennreal-iff*[*simp*]: *0* ≤ *r* ⟹ *of-nat i* ≤ *ennreal r* ⟷ *of-nat i* ≤ *r*
by (*simp add*: *ennreal-of-nat-eq-real-of-nat*)

lemma *ennreal-le-of-nat-iff*[*simp*]: *ennreal r* ≤ *of-nat i* ⟷ *r* ≤ *of-nat i*
by (*simp add*: *ennreal-of-nat-eq-real-of-nat*)

lemma *ennreal-indicator*: *ennreal* (*indicator A x*) = *indicator A x*
by (*auto split*: *split-indicator*)

lemma *ennreal-numeral*[*simp*]: *ennreal* (*numeral n*) = *numeral n*
using *ennreal-of-nat-eq-real-of-nat*[*of numeral n*] **by** *simp*

lemma *ennreal-less-numeral-iff* [*simp*]: *ennreal n < numeral w ⟷ n < numeral w*
by (*metis ennreal-less-iff ennreal-numeral less-le not-less zero-less-numeral*)

lemma *numeral-less-ennreal-iff* [*simp*]: *numeral w < ennreal n ⟷ numeral w < n*
using *ennreal-less-iff zero-le-numeral* **by** *fastforce*

lemma *numeral-le-ennreal-iff* [*simp*]: *numeral n ≤ ennreal m ⟷ numeral n ≤ m*
by (*metis not-le ennreal-less-numeral-iff*)

lemma *min-ennreal*: *0 ≤ x ⟹ 0 ≤ y ⟹ min* (*ennreal x*) (*ennreal y*) = *ennreal* (*min x y*)
by (*auto split*: *split-min*)

lemma *ennreal-half*[*simp*]: *ennreal* (*1 / 2*) = *inverse 2*
by *transfer* (*simp add*: *max.absorb2*)

lemma *ennreal-minus*: *0 ≤ q ⟹ ennreal r − ennreal q* = *ennreal* (*r − q*)
by *transfer*
(*simp add*: *max.absorb2 zero-ereal-def flip*: *ereal-max*)

lemma *ennreal-minus-top*[*simp*]: *ennreal a − top = 0*
by (*simp add*: *minus-top-ennreal*)

lemma *e2eenreal-enn2ereal-diff* [*simp*]:
e2ennreal(*enn2ereal x − enn2ereal y*) = *x − y* **for** *x y*
by (*cases x*, *cases y*, *auto simp add*: *ennreal-minus e2ennreal-neg*)

lemma *ennreal-mult*: *0 ≤ a ⟹ 0 ≤ b ⟹ ennreal* (*a ∗ b*) = *ennreal a ∗ ennreal b*
by *transfer* (*simp add*: *max-absorb2*)

lemma *ennreal-mult′*: *0 ≤ a ⟹ ennreal* (*a ∗ b*) = *ennreal a ∗ ennreal b*
by (*cases 0 ≤ b*) (*auto simp*: *ennreal-mult ennreal-neg mult-nonneg-nonpos*)

lemma *indicator-mult-ennreal*: *indicator A x ∗ ennreal r* = *ennreal* (*indicator A x ∗ r*)
by (*simp split*: *split-indicator*)

lemma *ennreal-mult′′*: *0 ≤ b ⟹ ennreal* (*a ∗ b*) = *ennreal a ∗ ennreal b*
by (*cases 0 ≤ a*) (*auto simp*: *ennreal-mult ennreal-neg mult-nonpos-nonneg*)

lemma *numeral-mult-ennreal*: *0 ≤ x ⟹ numeral b ∗ ennreal x* = *ennreal* (*numeral b ∗ x*)
by (*simp add*: *ennreal-mult*)

lemma *ennreal-power*: *0 ≤ r ⟹ ennreal r ^ n* = *ennreal* (*r ^ n*)
by (*induction n*) (*auto simp*: *ennreal-mult*)

lemma *power-eq-top-ennreal*: $x \mathbin{\hat{}} n = top \longleftrightarrow (n \neq 0 \wedge (x::ennreal) = top)$
by (*cases x rule*: *ennreal-cases*)
(*auto simp*: *ennreal-power top-power-ennreal*)

lemma *inverse-ennreal*: $0 < r \Longrightarrow inverse\ (ennreal\ r) = ennreal\ (inverse\ r)$
by *transfer* (*simp add*: *max.absorb2*)

lemma *divide-ennreal*: $0 \leq r \Longrightarrow 0 < q \Longrightarrow ennreal\ r\ /\ ennreal\ q = ennreal\ (r\ /\ q)$
by (*simp add*: *divide-ennreal-def inverse-ennreal ennreal-mult*[*symmetric*] *inverse-eq-divide*)

lemma *ennreal-inverse-power*: $inverse\ (x \mathbin{\hat{}} n :: ennreal) = inverse\ x \mathbin{\hat{}} n$
proof (*cases x rule*: *ennreal-cases*)
case *top* **with** *power-eq-top-ennreal*[*of x n*] **show** *?thesis*
by (*cases n* = *0*) *auto*
next
case (*real r*) **then show** *?thesis*
proof *cases*
assume $x = 0$ **then show** *?thesis*
using *power-eq-top-ennreal*[*of top n* − *1*]
by (*cases n*) (*auto simp*: *ennreal-top-mult*)
next
assume $x \neq 0$
with *real* **have** $0 < r$ **by** *auto*
with *real* **show** *?thesis*
by (*induction n*)
(*auto simp add*: *ennreal-power ennreal-mult*[*symmetric*] *inverse-ennreal*)
qed
qed

lemma *power-divide-distrib-ennreal* [*algebra-simps*]:
$(x\ /\ y) \mathbin{\hat{}} n = x \mathbin{\hat{}} n\ /\ (y \mathbin{\hat{}} n :: ennreal)$
by (*simp add*: *divide-ennreal-def algebra-simps ennreal-inverse-power*)

lemma *ennreal-divide-numeral*: $0 \leq x \Longrightarrow ennreal\ x\ /\ numeral\ b = ennreal\ (x\ /\ numeral\ b)$
by (*subst divide-ennreal*[*symmetric*]) *auto*

lemma *prod-ennreal*: $(\bigwedge i.\ i \in A \Longrightarrow 0 \leq f\ i) \Longrightarrow (\prod i{\in}A.\ ennreal\ (f\ i)) = ennreal\ (prod\ f\ A)$
by (*induction A rule*: *infinite-finite-induct*)
(*auto simp*: *ennreal-mult prod-nonneg*)

lemma *prod-mono-ennreal*:
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \leq (g\ x :: ennreal)$
shows $prod\ f\ A \leq prod\ g\ A$
using *assms* **by** (*induction A rule*: *infinite-finite-induct*) (*auto intro*!: *mult-mono*)

lemma *mult-right-ennreal-cancel*: $a * ennreal\ c = b * ennreal\ c \longleftrightarrow (a = b \vee c \leq 0)$
proof (*cases* $0 \leq c$)
  case *True*
  then show *?thesis*
   by (*metis ennreal-eq-0-iff ennreal-mult-right-cong ennreal-neq-top mult-divide-eq-ennreal*)
qed (*use ennreal-neg* **in** *auto*)

lemma *ennreal-le-epsilon*:
  $(\bigwedge e{::}real.\ y < top \Longrightarrow 0 < e \Longrightarrow x \leq y + ennreal\ e) \Longrightarrow x \leq y$
  apply (*cases y rule*: *ennreal-cases*)
  apply (*cases x rule*: *ennreal-cases*)
   apply (*auto simp flip*: *ennreal-plus simp add*: *top-unique intro*: *zero-less-one field-le-epsilon*)
  done

lemma *ennreal-rat-dense*:
  fixes $x\ y :: ennreal$
  shows $x < y \Longrightarrow \exists r{::}rat.\ x < real\text{-}of\text{-}rat\ r \wedge real\text{-}of\text{-}rat\ r < y$
proof *transfer*
  fix $x\ y :: ereal$ **assume** *xy*: $0 \leq x\ 0 \leq y\ x < y$
  moreover
  from *ereal-dense3*[*OF* ‹$x < y$›]
  obtain r **where** r: $x < ereal\ (real\text{-}of\text{-}rat\ r)\ ereal\ (real\text{-}of\text{-}rat\ r) < y$
    by *auto*
  then have $0 \leq r$
    using *le-less-trans*[*OF* ‹$0 \leq x$› ‹$x < ereal\ (real\text{-}of\text{-}rat\ r)$›] **by** *auto*
  with r **show** $\exists r.\ x < (sup\ 0 \circ ereal)\ (real\text{-}of\text{-}rat\ r) \wedge (sup\ 0 \circ ereal)\ (real\text{-}of\text{-}rat\ r) < y$
    by (*intro exI*[*of - r*]) (*auto simp*: *max-absorb2*)
qed

lemma *ennreal-Ex-less-of-nat*: $(x{::}ennreal) < top \Longrightarrow \exists n.\ x < of\text{-}nat\ n$
  by (*cases x rule*: *ennreal-cases*)
     (*auto simp*: *ennreal-of-nat-eq-real-of-nat ennreal-less-iff reals-Archimedean2*)

39.7 Coercion from *ennreal* to *real*

definition $ennreal2real\ x = real\text{-}of\text{-}ereal\ (ennreal2ereal\ x)$

lemma *ennreal2real-nonneg*[*simp*]: $0 \leq ennreal2real\ x$
  by (*auto simp*: *ennreal2real-def intro*!: *real-of-ereal-pos ennreal2ereal-nonneg*)

lemma *ennreal2real-mono*: $a \leq b \Longrightarrow b < top \Longrightarrow ennreal2real\ a \leq ennreal2real\ b$
  by (*auto simp add*: *ennreal2real-def less-eq-ennreal.rep-eq intro*!: *real-of-ereal-positive-mono ennreal2ereal-nonneg*)

lemma *ennreal2real-of-nat*[*simp*]: $ennreal2real\ (of\text{-}nat\ n) = n$

by (*auto simp*: *enn2real-def*)

lemma *enn2real-ennreal*[*simp*]: $0 \leq r \Longrightarrow$ *enn2real* (*ennreal r*) = *r*
by (*simp add*: *enn2real-def*)

lemma *ennreal-enn2real*[*simp*]: $r < top \Longrightarrow$ *ennreal* (*enn2real r*) = *r*
by (*cases r rule*: *ennreal-cases*) *auto*

lemma *real-of-ereal-enn2ereal*[*simp*]: *real-of-ereal* (*enn2ereal x*) = *enn2real x*
by (*simp add*: *enn2real-def*)

lemma *enn2real-top*[*simp*]: *enn2real top* = *0*
unfolding *enn2real-def top-ennreal.rep-eq top-ereal-def* **by** *simp*

lemma *enn2real-0*[*simp*]: *enn2real 0* = *0*
unfolding *enn2real-def zero-ennreal.rep-eq* **by** *simp*

lemma *enn2real-1*[*simp*]: *enn2real 1* = *1*
unfolding *enn2real-def one-ennreal.rep-eq* **by** *simp*

lemma *enn2real-numeral*[*simp*]: *enn2real* (*numeral n*) = (*numeral n*)
unfolding *enn2real-def* **by** *simp*

lemma *enn2real-mult*: *enn2real* (*a* $*$ *b*) = *enn2real a* $*$ *enn2real b*
unfolding *enn2real-def*
by (*simp del*: *real-of-ereal-enn2ereal add*: *times-ennreal.rep-eq*)

lemma *enn2real-leI*: $0 \leq B \Longrightarrow x \leq$ *ennreal B* $\Longrightarrow$ *enn2real x* $\leq B$
by (*cases x rule*: *ennreal-cases*) (*auto simp*: *top-unique*)

lemma *enn2real-positive-iff*: $0 <$ *enn2real x* $\longleftrightarrow$ $(0 < x \wedge x < top)$
by (*cases x rule*: *ennreal-cases*) *auto*

lemma *enn2real-eq-posreal-iff*[*simp*]: $c > 0 \Longrightarrow$ *enn2real x* = *c* $\longleftrightarrow$ *x* = *c*
by (*cases x*) *auto*

lemma *ennreal-enn2real-if*: *ennreal* (*enn2real r*) = (*if r* = *top then 0 else r*)
by(*auto intro*!: *ennreal-enn2real simp add*: *less-top*)

39.8 Coercion from *enat* to *ennreal*

definition *ennreal-of-enat* :: *enat* $\Rightarrow$ *ennreal*
where
ennreal-of-enat n = (*case n of* $\infty \Rightarrow$ *top* | *enat n* $\Rightarrow$ *of-nat n*)

declare [[*coercion ennreal-of-enat*]]
declare [[*coercion of-nat* :: *nat* $\Rightarrow$ *ennreal*]]

lemma *ennreal-of-enat-infty*[*simp*]: *ennreal-of-enat* $\infty = \infty$

by (*simp add*: *ennreal-of-enat-def*)

lemma *ennreal-of-enat-enat*[*simp*]: *ennreal-of-enat* (*enat n*) = *of-nat n*
by (*simp add*: *ennreal-of-enat-def*)

lemma *ennreal-of-enat-0*[*simp*]: *ennreal-of-enat 0* = *0*
using *ennreal-of-enat-enat*[*of 0*] **unfolding** *enat-0* **by** *simp*

lemma *ennreal-of-enat-1*[*simp*]: *ennreal-of-enat 1* = *1*
using *ennreal-of-enat-enat*[*of 1*] **unfolding** *enat-1* **by** *simp*

lemma *ennreal-top-neq-of-nat*[*simp*]: (*top*::*ennreal*) ≠ *of-nat i*
using *ennreal-of-nat-neq-top*[*of i*] **by** *metis*

lemma *ennreal-of-enat-inj*[*simp*]: *ennreal-of-enat i* = *ennreal-of-enat j* ⟷ *i* = *j*
by (*cases i j rule*: *enat.exhaust*[*case-product enat.exhaust*]) *auto*

lemma *ennreal-of-enat-le-iff*[*simp*]: *ennreal-of-enat m* ≤ *ennreal-of-enat n* ⟷ *m* ≤ *n*
by (*auto simp*: *ennreal-of-enat-def top-unique split*: *enat.split*)

lemma *of-nat-less-ennreal-of-nat*[*simp*]: *of-nat n* ≤ *ennreal-of-enat x* ⟷ *of-nat n* ≤ *x*
by (*cases x*) (*auto simp*: *of-nat-eq-enat*)

lemma *ennreal-of-enat-Sup*: *ennreal-of-enat* (*Sup X*) = (*SUP x*∈*X*. *ennreal-of-enat x*)
proof −
have *ennreal-of-enat* (*Sup X*) ≤ (*SUP x* ∈ *X*. *ennreal-of-enat x*)
unfolding *Sup-enat-def*
proof (*clarsimp*, *intro conjI impI*)
fix *x* **assume** *finite X X* ≠ {}
then show *ennreal-of-enat* (*Max X*) ≤ (*SUP x* ∈ *X*. *ennreal-of-enat x*)
by (*intro SUP-upper Max-in*)
next
assume *infinite X X* ≠ {}
have ∃ *y*∈*X*. *r* < *ennreal-of-enat y* **if** *r*: *r* < *top* **for** *r*
proof −
obtain *n* **where** *n*: *r* < *of-nat n*
using *ennreal-Ex-less-of-nat*[*OF r*] **..**
have ¬ (*X* ⊆ *enat* ' {.. *n*})
using ‹*infinite X*› **by** (*auto dest*: *finite-subset*)
then obtain *x* **where** *x*: *x* ∈ *X x* ∉ *enat* ' {..*n*}
by *blast*
then have *of-nat n* ≤ *x*
by (*cases x*) (*auto simp*: *of-nat-eq-enat*)
with *x* **show** *?thesis*
by (*auto intro*!: *bexI*[*of* - *x*] *less-le-trans*[*OF n*])
qed

then have (*SUP x ∈ X. ennreal-of-enat x*) = *top*
by *simp*
then show *top* ≤ (*SUP x ∈ X. ennreal-of-enat x*)
unfolding *top-unique* **by** *simp*
qed
then show *?thesis*
by (*auto intro*!: *antisym Sup-least intro*: *Sup-upper*)
qed

lemma *ennreal-of-enat-eSuc*[*simp*]: *ennreal-of-enat* (*eSuc x*) = *1* + *ennreal-of-enat x*
by (*cases x*) (*auto simp*: *eSuc-enat*)

lemma *ennreal-of-enat-plus*[*simp*]: ‹*ennreal-of-enat* (*a*+*b*) = *ennreal-of-enat a* + *ennreal-of-enat b*›
apply (*induct a*)
apply (*metis enat.exhaust ennreal-add-eq-top ennreal-of-enat-enat ennreal-of-enat-infty infinity-ennreal-def of-nat-add plus-enat-simps*(*1*) *plus-eq-infty-iff-enat*)
apply *simp*
done

lemma *sum-ennreal-of-enat*[*simp*]: (∑ *i*∈*I*. *ennreal-of-enat* (*f i*)) = *ennreal-of-enat* (*sum f I*)
by (*induct I rule*: *infinite-finite-induct*) (*auto simp*: *sum-nonneg*)

39.9 Topology on *ennreal*

lemma *enn2ereal-Iio*: *enn2ereal* −‘ {..<*a*} = (*if 0* ≤ *a then* {..< *e2ennreal a*} *else* {})
using *enn2ereal-nonneg*
by (*cases a rule*: *ereal-ennreal-cases*)
(*auto simp add*: *vimage-def set-eq-iff ennreal.enn2ereal-inverse less-ennreal.rep-eq e2ennreal-def max-absorb2*
simp del: *enn2ereal-nonneg*
intro: *le-less-trans less-imp-le*)

lemma *enn2ereal-Ioi*: *enn2ereal* −‘ {*a* <..} = (*if 0* ≤ *a then* {*e2ennreal a* <..} *else UNIV*)
by (*cases a rule*: *ereal-ennreal-cases*)
(*auto simp add*: *vimage-def set-eq-iff ennreal.enn2ereal-inverse less-ennreal.rep-eq e2ennreal-def max-absorb2*
intro: *less-le-trans*)

instantiation *ennreal* :: *linear-continuum-topology*
begin

definition *open-ennreal* :: *ennreal set* ⇒ *bool*

where (*open* :: *ennreal set* ⇒ *bool*) = *generate-topology* (*range lessThan* ∪ *range greaterThan*)

instance
proof
show ∃ *a b*::*ennreal*. *a* ≠ *b*
using *zero-neq-one* **by** (*intro exI*)
show ⋀*x y*::*ennreal*. *x* < *y* ⟹ ∃ *z*>*x*. *z* < *y*
proof *transfer*
fix *x y* :: *ereal*
assume ∗: *0* ≤ *x*
assume *x* < *y*
from *dense*[*OF this*] **obtain** *z* **where** *x* < *z* ∧ *z* < *y* **..**
with ∗ **show** ∃ *z*∈*Collect* ((≤) *0*). *x* < *z* ∧ *z* < *y*
by (*intro bexI*[*of - z*]) *auto*
qed
qed (*rule open-ennreal-def*)

end

lemma *continuous-on-e2ennreal*: *continuous-on A e2ennreal*
proof (*rule continuous-on-subset*)
show *continuous-on* ({*0..*} ∪ {*..0*}) *e2ennreal*
proof (*rule continuous-on-closed-Un*)
show *continuous-on* {*0 ..*} *e2ennreal*
by (*rule continuous-onI-mono*)
(*auto simp add*: *less-eq-ennreal.abs-eq eq-onp-def enn2ereal-range*)
show *continuous-on* {*.. 0*} *e2ennreal*
by (*subst continuous-on-cong*[*OF refl*, *of - - λ-. 0*])
(*auto simp add*: *e2ennreal-neg continuous-on-const*)
qed *auto*
show *A* ⊆ {*0..*} ∪ {*..0*::*ereal*}
by *auto*
qed

lemma *continuous-at-e2ennreal*: *continuous* (*at x within A*) *e2ennreal*
by (*rule continuous-on-imp-continuous-within*[*OF continuous-on-e2ennreal*, *of - UNIV*]) *auto*

lemma *continuous-on-enn2ereal*: *continuous-on UNIV enn2ereal*
by (*rule continuous-on-generate-topology*[*OF open-generated-order*])
(*auto simp add*: *enn2ereal-Iio enn2ereal-Ioi*)

lemma *continuous-at-enn2ereal*: *continuous* (*at x within A*) *enn2ereal*
by (*rule continuous-on-imp-continuous-within*[*OF continuous-on-enn2ereal*]) *auto*

lemma *sup-continuous-e2ennreal*[*order-continuous-intros*]:
assumes *f*: *sup-continuous f* **shows** *sup-continuous* (*λx. e2ennreal* (*f x*))
proof (*rule sup-continuous-compose*[*OF - f*])

show *sup-continuous e2ennreal*
by (*simp add*: *continuous-at-e2ennreal continuous-at-left-imp-sup-continuous e2ennreal-mono mono-def*)
qed

lemma *sup-continuous-enn2ereal*[*order-continuous-intros*]:
assumes *f*: *sup-continuous f* **shows** *sup-continuous* (λ*x*. *enn2ereal* (*f x*))
proof (*rule sup-continuous-compose*[*OF - f*])
show *sup-continuous enn2ereal*
by (*simp add*: *continuous-at-enn2ereal continuous-at-left-imp-sup-continuous less-eq-ennreal.rep-eq mono-def*)
qed

lemma *sup-continuous-mult-left-ennreal'*:
fixes *c* :: *ennreal*
shows *sup-continuous* (λ*x*. *c* ∗ *x*)
unfolding *sup-continuous-def*
by *transfer* (*auto simp*: *SUP-ereal-mult-left max.absorb2 SUP-upper2*)

lemma *sup-continuous-mult-left-ennreal*[*order-continuous-intros*]:
sup-continuous f ⟹ *sup-continuous* (λ*x*. *c* ∗ *f x* :: *ennreal*)
by (*rule sup-continuous-compose*[*OF sup-continuous-mult-left-ennreal'*])

lemma *sup-continuous-mult-right-ennreal*[*order-continuous-intros*]:
sup-continuous f ⟹ *sup-continuous* (λ*x*. *f x* ∗ *c* :: *ennreal*)
using *sup-continuous-mult-left-ennreal*[*of f c*] **by** (*simp add*: *mult.commute*)

lemma *sup-continuous-divide-ennreal*[*order-continuous-intros*]:
fixes *f g* :: *'a*::*complete-lattice* ⇒ *ennreal*
shows *sup-continuous f* ⟹ *sup-continuous* (λ*x*. *f x* / *c*)
unfolding *divide-ennreal-def* **by** (*rule sup-continuous-mult-right-ennreal*)

lemma *transfer-enn2ereal-continuous-on* [*transfer-rule*]:
rel-fun (=) (*rel-fun* (*rel-fun* (=) *pcr-ennreal*) (=)) *continuous-on continuous-on*
proof −
have *continuous-on A f* **if** *continuous-on A* (λ*x*. *enn2ereal* (*f x*)) **for** *A* **and** *f* :: *'a* ⇒ *ennreal*
using *continuous-on-compose2*[*OF continuous-on-e2ennreal*[*of* {*0*..}] *that*]
by (*auto simp*: *ennreal.enn2ereal-inverse subset-eq e2ennreal-def max-absorb2*)
moreover
have *continuous-on A* (λ*x*. *enn2ereal* (*f x*)) **if** *continuous-on A f* **for** *A* **and** *f* :: *'a* ⇒ *ennreal*
using *continuous-on-compose2*[*OF continuous-on-enn2ereal that*] **by** *auto*
ultimately
show *?thesis*
by (*auto simp add*: *rel-fun-def ennreal.pcr-cr-eq cr-ennreal-def*)
qed

lemma *transfer-sup-continuous*[*transfer-rule*]:

(*rel-fun* (*rel-fun* (=) *pcr-ennreal*) (=)) *sup-continuous sup-continuous*
proof (*safe intro*!: *rel-funI dest*!: *rel-fun-eq-pcr-ennreal*[*THEN iffD1*])
show *sup-continuous* (*enn2ereal* ∘ *f*) ⟹ *sup-continuous f* **for** *f* :: *'a* ⇒ -
using *sup-continuous-e2ennreal*[*of enn2ereal* ∘ *f*] **by** *simp*
show *sup-continuous f* ⟹ *sup-continuous* (*enn2ereal* ∘ *f*) **for** *f* :: *'a* ⇒ -
using *sup-continuous-enn2ereal*[*of f*] **by** (*simp add*: *comp-def*)
qed

lemma *continuous-on-ennreal*[*tendsto-intros*]:
continuous-on A f ⟹ *continuous-on A* (λ*x*. *ennreal* (*f x*))
by *transfer* (*auto intro*!: *continuous-on-max continuous-on-const continuous-on-ereal*)

lemma *tendsto-ennrealD*:
assumes *lim*: ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal x*) *F*
assumes ∗: ∀ *x in F*. *0* ≤ *f x* **and** *x*: *0* ≤ *x*
shows (*f* ⟶ *x*) *F*
proof −
have ((λ*x*. *enn2ereal* (*ennreal* (*f x*))) ⟶ *enn2ereal* (*ennreal x*)) *F*
⟷ (*f* ⟶ *enn2ereal* (*ennreal x*)) *F*
using ∗ *eventually-mono*
by (*intro tendsto-cong*) *fastforce*
then show *?thesis*
using *assms*(*1*) *continuous-at-enn2ereal isCont-tendsto-compose x* **by** *fastforce*
qed

lemma *tendsto-ennreal-iff* [*simp*]:
‹((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal x*) *F* ⟷ (*f* ⟶ *x*) *F*› (**is** ‹*?P* ⟷ *?Q*›)
if ‹∀ *x in F*. *0* ≤ *f x*› ‹*0* ≤ *x*›
proof
assume ‹*?P*›
then show ‹*?Q*›
using *that* **by** (*rule tendsto-ennrealD*)
next
assume ‹*?Q*›
have ‹*continuous-on UNIV ereal*›
using *continuous-on-ereal* [*of* - *id*] **by** *simp*
then have ‹*continuous-on UNIV* (*e2ennreal* ∘ *ereal*)›
by (*rule continuous-on-compose*) (*simp-all add*: *continuous-on-e2ennreal*)
then have ‹((λ*x*. (*e2ennreal* ∘ *ereal*) (*f x*)) ⟶ (*e2ennreal* ∘ *ereal*) *x*) *F*›
using ‹*?Q*› **by** (*rule continuous-on-tendsto-compose*) *simp-all*
then show ‹*?P*›
by (*simp flip*: *e2ennreal-ereal*)
qed

lemma *tendsto-enn2ereal-iff*[*simp*]: ((λ*i*. *enn2ereal* (*f i*)) ⟶ *enn2ereal x*) *F* ⟷
(*f* ⟶ *x*) *F*
using *continuous-on-enn2ereal*[*THEN continuous-on-tendsto-compose*, *of f x F*]
continuous-on-e2ennreal[*THEN continuous-on-tendsto-compose*, *of* λ*x*. *enn2ereal*
(*f x*) *enn2ereal x F UNIV*]

by *auto*

lemma *ennreal-tendsto-0-iff*: $(\bigwedge n.\ f\ n \geq 0) \Longrightarrow ((\lambda n.\ ennreal\ (f\ n)) \longrightarrow 0) \longleftrightarrow (f \longrightarrow 0)$
by (*metis* (*mono-tags*) *ennreal-0 eventuallyI order-refl tendsto-ennreal-iff*)

lemma *continuous-on-add-ennreal*:
fixes $f\ g :: 'a::topological\text{-}space \Rightarrow ennreal$
shows *continuous-on A f* $\Longrightarrow$ *continuous-on A g* $\Longrightarrow$ *continuous-on A* $(\lambda x.\ f\ x + g\ x)$
by (*transfer fixing*: *A*) (*auto intro*!: *tendsto-add-ereal-nonneg simp*: *continuous-on-def*)

lemma *continuous-on-inverse-ennreal*[*continuous-intros*]:
fixes $f :: 'a::topological\text{-}space \Rightarrow ennreal$
shows *continuous-on A f* $\Longrightarrow$ *continuous-on A* $(\lambda x.\ inverse\ (f\ x))$
proof (*transfer fixing*: *A*)
show *pred-fun top* $((\leq)\ 0)\ f \Longrightarrow$ *continuous-on A* $(\lambda x.\ inverse\ (f\ x))$ **if** *continuous-on A f*
for $f :: 'a \Rightarrow ereal$
using *continuous-on-compose2*[*OF continuous-on-inverse-ereal that*] **by** (*auto simp*: *subset-eq*)
qed

instance *ennreal* :: *topological-comm-monoid-add*
proof
show $((\lambda x.\ fst\ x + snd\ x) \longrightarrow a + b)\ (nhds\ a \times_F nhds\ b)$ **for** *a b* :: *ennreal*
using *continuous-on-add-ennreal*[*of UNIV fst snd*]
using *tendsto-at-iff-tendsto-nhds*[*symmetric*, *of* $\lambda x::(ennreal \times ennreal).\ fst\ x + snd\ x$]
by (*auto simp*: *continuous-on-eq-continuous-at*)
(*simp add*: *isCont-def nhds-prod*[*symmetric*])
qed

lemma *sup-continuous-add-ennreal*[*order-continuous-intros*]:
fixes $f\ g :: 'a::complete\text{-}lattice \Rightarrow ennreal$
shows *sup-continuous f* $\Longrightarrow$ *sup-continuous g* $\Longrightarrow$ *sup-continuous* $(\lambda x.\ f\ x + g\ x)$
by *transfer* (*auto intro*!: *sup-continuous-add*)

lemma *ennreal-suminf-lessD*: $(\sum i.\ f\ i :: ennreal) < x \Longrightarrow f\ i < x$
using *le-less-trans*[*OF sum-le-suminf*[*OF summableI*, *of* $\{i\}$ *f*]] **by** *simp*

lemma *sums-ennreal*[*simp*]: $(\bigwedge i.\ 0 \leq f\ i) \Longrightarrow 0 \leq x \Longrightarrow (\lambda i.\ ennreal\ (f\ i))\ sums\ ennreal\ x \longleftrightarrow f\ sums\ x$
unfolding *sums-def* **by** (*simp add*: *always-eventually sum-nonneg*)

lemma *summable-suminf-not-top*: $(\bigwedge i.\ 0 \leq f\ i) \Longrightarrow (\sum i.\ ennreal\ (f\ i)) \neq top \Longrightarrow summable\ f$
using *summable-sums*[*OF summableI*, *of* $\lambda i.\ ennreal\ (f\ i)$]

by (*cases* $\sum i.\ ennreal\ (f\ i)$ *rule*: *ennreal-cases*)
(*auto simp*: *summable-def*)

lemma *suminf-ennreal*[*simp*]:
$(\bigwedge i.\ 0 \le f\ i) \Longrightarrow (\sum i.\ ennreal\ (f\ i)) \neq top \Longrightarrow (\sum i.\ ennreal\ (f\ i)) = ennreal\ (\sum i.\ f\ i)$
by (*rule sums-unique*[*symmetric*]) (*simp add*: *summable-suminf-not-top suminf-nonneg summable-sums*)

lemma *sums-enn2ereal*[*simp*]: $(\lambda i.\ enn2ereal\ (f\ i))\ sums\ enn2ereal\ x \longleftrightarrow f\ sums\ x$
unfolding *sums-def* **by** (*simp add*: *always-eventually sum-nonneg*)

lemma *suminf-enn2ereal*[*simp*]: $(\sum i.\ enn2ereal\ (f\ i)) = enn2ereal\ (suminf\ f)$
by (*rule sums-unique*[*symmetric*]) (*simp add*: *summable-sums*)

lemma *transfer-e2ennreal-suminf* [*transfer-rule*]: *rel-fun* (*rel-fun* (=) *pcr-ennreal*) *pcr-ennreal suminf suminf*
by (*auto simp*: *rel-funI rel-fun-eq-pcr-ennreal comp-def*)

lemma *ennreal-suminf-cmult*[*simp*]: $(\sum i.\ r * f\ i) = r * (\sum i.\ f\ i :: ennreal)$
by *transfer* (*auto intro*!: *suminf-cmult-ereal*)

lemma *ennreal-suminf-multc*[*simp*]: $(\sum i.\ f\ i * r) = (\sum i.\ f\ i :: ennreal) * r$
using *ennreal-suminf-cmult*[*of r f*] **by** (*simp add*: *ac-simps*)

lemma *ennreal-suminf-divide*[*simp*]: $(\sum i.\ f\ i\ /\ r) = (\sum i.\ f\ i :: ennreal)\ /\ r$
by (*simp add*: *divide-ennreal-def*)

lemma *ennreal-suminf-neq-top*: *summable f* $\Longrightarrow (\bigwedge i.\ 0 \le f\ i) \Longrightarrow (\sum i.\ ennreal\ (f\ i)) \neq top$
using *sums-ennreal*[*of f suminf f*]
by (*simp add*: *suminf-nonneg flip*: *sums-unique summable-sums-iff del*: *sums-ennreal*)

lemma *suminf-ennreal-eq*:
$(\bigwedge i.\ 0 \le f\ i) \Longrightarrow f\ sums\ x \Longrightarrow (\sum i.\ ennreal\ (f\ i)) = ennreal\ x$
using *suminf-nonneg*[*of f*] *sums-unique*[*of f x*]
by (*intro sums-unique*[*symmetric*]) (*auto simp*: *summable-sums-iff*)

lemma *ennreal-suminf-bound-add*:
fixes $f :: nat \Rightarrow ennreal$
shows $(\bigwedge N.\ (\sum n{<}N.\ f\ n) + y \le x) \Longrightarrow suminf\ f + y \le x$
by *transfer* (*auto intro*!: *suminf-bound-add*)

lemma *ennreal-suminf-SUP-eq-directed*:
fixes $f :: 'a \Rightarrow nat \Rightarrow ennreal$
assumes $*$: $\bigwedge N\ i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow finite\ N \Longrightarrow \exists k{\in}I.\ \forall n{\in}N.\ f\ i\ n \le f\ k\ n \wedge f\ j\ n \le f\ k\ n$
shows $(\sum n.\ SUP\ i{\in}I.\ f\ i\ n) = (SUP\ i{\in}I.\ \sum n.\ f\ i\ n)$

proof *cases*
assume $I \neq \{\}$
then obtain i **where** $i \in I$ **by** *auto*
from $*$ **show** *?thesis*
by (*transfer fixing*: I)
(*auto simp*: *max-absorb2 SUP-upper2*[*OF* ‹$i \in I$›] *suminf-nonneg summable-ereal-pos* ‹$I \neq \{\}$›
intro!: *suminf-SUP-eq-directed*)
qed (*simp add*: *bot-ennreal*)

lemma *INF-ennreal-add-const*:
fixes $f\ g :: nat \Rightarrow ennreal$
shows $(INF\ i.\ f\ i + c) = (INF\ i.\ f\ i) + c$
using *continuous-at-Inf-mono*[*of* $\lambda x.\ x + c$ *f'UNIV*]
using *continuous-add*[*of at-right* (*Inf* (*range f*)), *of* $\lambda x.\ x$ $\lambda x.\ c$]
by (*auto simp*: *mono-def image-comp*)

lemma *INF-ennreal-const-add*:
fixes $f\ g :: nat \Rightarrow ennreal$
shows $(INF\ i.\ c + f\ i) = c + (INF\ i.\ f\ i)$
using *INF-ennreal-add-const*[*of f c*] **by** (*simp add*: *ac-simps*)

lemma *SUP-mult-left-ennreal*: $c * (SUP\ i{\in}I.\ f\ i) = (SUP\ i{\in}I.\ c * f\ i :: ennreal)$
proof *cases*
assume $I \neq \{\}$ **then show** *?thesis*
by *transfer* (*auto simp add*: *SUP-ereal-mult-left max-absorb2 SUP-upper2*)
qed (*simp add*: *bot-ennreal*)

lemma *SUP-mult-right-ennreal*: $(SUP\ i{\in}I.\ f\ i) * c = (SUP\ i{\in}I.\ f\ i * c :: ennreal)$
using *SUP-mult-left-ennreal* **by** (*simp add*: *mult.commute*)

lemma *SUP-divide-ennreal*: $(SUP\ i{\in}I.\ f\ i) / c = (SUP\ i{\in}I.\ f\ i / c :: ennreal)$
using *SUP-mult-right-ennreal* **by** (*simp add*: *divide-ennreal-def*)

lemma *ennreal-SUP-of-nat-eq-top*: $(SUP\ x.\ of\text{-}nat\ x :: ennreal) = top$
proof (*intro antisym top-greatest le-SUP-iff*[*THEN iffD2*] *allI impI*)
fix $y :: ennreal$ **assume** $y < top$
then obtain r **where** $y = ennreal\ r$
by (*cases y rule*: *ennreal-cases*) *auto*
then show $\exists\, i{\in}UNIV.\ y < of\text{-}nat\ i$
using *reals-Archimedean2*[*of max 1 r*] *zero-less-one*
by (*simp add*: *ennreal-Ex-less-of-nat*)
qed

lemma *ennreal-SUP-eq-top*:
fixes $f :: 'a \Rightarrow ennreal$
assumes $\bigwedge n.\ \exists\, i{\in}I.\ of\text{-}nat\ n \leq f\ i$
shows $(SUP\ i \in I.\ f\ i) = top$
proof −

have (*SUP x. of-nat x :: ennreal*) ≤ (*SUP i ∈ I. f i*)
using *assms* **by** (*auto intro*!: *SUP-least intro*: *SUP-upper2*)
then show *?thesis*
by (*auto simp*: *ennreal-SUP-of-nat-eq-top top-unique*)
qed

lemma *ennreal-INF-const-minus*:
fixes *f* :: *'a* ⇒ *ennreal*
shows *I* ≠ {} ⟹ (*SUP x∈I. c* − *f x*) = *c* − (*INF x∈I. f x*)
by (*transfer fixing*: *I*)
(*simp add*: *sup-max*[*symmetric*] *SUP-sup-const1 SUP-ereal-minus-right del*: *sup-ereal-def*)

lemma *of-nat-Sup-ennreal*:
assumes *A* ≠ {} *bdd-above A*
shows *of-nat* (*Sup A*) = (*SUP a∈A. of-nat a* :: *ennreal*)
proof (*intro antisym*)
show (*SUP a∈A. of-nat a*::*ennreal*) ≤ *of-nat* (*Sup A*)
by (*intro SUP-least of-nat-mono*) (*auto intro*: *cSup-upper assms*)
have *Sup A* ∈ *A*
using *assms* **by** (*auto simp*: *Sup-nat-def bdd-above-nat*)
then show *of-nat* (*Sup A*) ≤ (*SUP a∈A. of-nat a*::*ennreal*)
by (*intro SUP-upper*)
qed

lemma *ennreal-tendsto-const-minus*:
fixes *g* :: *'a* ⇒ *ennreal*
assumes *ae*: ∀ *F x in F. g x* ≤ *c*
assumes *g*: ((λ*x. c* − *g x*) ⟶ *0*) *F*
shows (*g* ⟶ *c*) *F*
proof (*cases c rule*: *ennreal-cases*)
case *top* **with** *tendsto-unique*[*OF - g, of top*] **show** *?thesis*
by (*cases F* = *bot*) *auto*
next
case (*real r*)
then have ∀ *x*. ∃ *q*≥*0. g x* ≤ *c* ⟶ (*g x* = *ennreal q* ∧ *q* ≤ *r*)
by (*auto simp*: *le-ennreal-iff*)
then obtain *f* **where** ∗: *0* ≤ *f x g x* = *ennreal* (*f x*) *f x* ≤ *r* **if** *g x* ≤ *c* **for** *x*
by *metis*
from *ae* **have** *ae2*: ∀ *F x in F. c* − *g x* = *ennreal* (*r* − *f x*) ∧ *f x* ≤ *r* ∧ *g x* = *ennreal* (*f x*) ∧ *0* ≤ *f x*
proof *eventually-elim*
fix *x* **assume** *g x* ≤ *c* **with** ∗[*of x*] ‹*0* ≤ *r*› **show** *c* − *g x* = *ennreal* (*r* − *f x*) ∧ *f x* ≤ *r* ∧ *g x* = *ennreal* (*f x*) ∧ *0* ≤ *f x*
by (*auto simp*: *real ennreal-minus*)
qed
with *g* **have** ((λ*x. ennreal* (*r* − *f x*)) ⟶ *ennreal 0*) *F*
by (*auto simp add*: *tendsto-cong eventually-conj-iff*)
with *ae2* **have** ((λ*x. r* − *f x*) ⟶ *0*) *F*

by (*subst* (*asm*) *tendsto-ennreal-iff*) (*auto elim*: *eventually-mono*)
then have (*f* ⟶ *r*) *F*
by (*rule Lim-transform2*[*OF tendsto-const*])
with *ae2* **have** ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal r*) *F*
by (*subst tendsto-ennreal-iff*) (*auto elim*: *eventually-mono simp*: *real*)
with *ae2* **show** *?thesis*
by (*auto simp*: *real tendsto-cong eventually-conj-iff*)
qed

lemma *ennreal-SUP-add*:
fixes *f g* :: *nat* ⇒ *ennreal*
shows *incseq f* ⟹ *incseq g* ⟹ (*SUP i*. *f i* + *g i*) = *Sup* (*f* ‘ *UNIV*) + *Sup* (*g* ‘ *UNIV*)
unfolding *incseq-def le-fun-def*
by *transfer*
(*simp add*: *SUP-ereal-add incseq-def le-fun-def max-absorb2 SUP-upper2*)

lemma *ennreal-SUP-sum*:
fixes *f* :: *'a* ⇒ *nat* ⇒ *ennreal*
shows (⋀*i*. *i* ∈ *I* ⟹ *incseq* (*f i*)) ⟹ (*SUP n*. ∑*i*∈*I*. *f i n*) = (∑*i*∈*I*. *SUP n*. *f i n*)
unfolding *incseq-def*
by *transfer*
(*simp add*: *SUP-ereal-sum incseq-def SUP-upper2 max-absorb2 sum-nonneg*)

lemma *ennreal-liminf-minus*:
fixes *f* :: *nat* ⇒ *ennreal*
shows (⋀*n*. *f n* ≤ *c*) ⟹ *liminf* (λ*n*. *c* − *f n*) = *c* − *limsup f*
apply *transfer*
apply (*simp add*: *ereal-diff-positive max.absorb2 liminf-ereal-cminus*)
apply (*subst max.absorb2*)
apply (*rule ereal-diff-positive*)
apply (*rule Limsup-bounded*)
apply *auto*
done

lemma *ennreal-continuous-on-cmult*:
(*c*::*ennreal*) < *top* ⟹ *continuous-on A f* ⟹ *continuous-on A* (λ*x*. *c* ∗ *f x*)
by (*transfer fixing*: *A*) (*auto intro*: *continuous-on-cmult-ereal*)

lemma *ennreal-tendsto-cmult*:
(*c*::*ennreal*) < *top* ⟹ (*f* ⟶ *x*) *F* ⟹ ((λ*x*. *c* ∗ *f x*) ⟶ *c* ∗ *x*) *F*
by (*rule continuous-on-tendsto-compose*[**where** *g*=*f*, *OF ennreal-continuous-on-cmult*, **where** *s*=*UNIV*])
(*auto simp*: *continuous-on-id*)

lemma *tendsto-ennrealI*[*intro*, *simp*, *tendsto-intros*]:
(*f* ⟶ *x*) *F* ⟹ ((λ*x*. *ennreal* (*f x*)) ⟶ *ennreal x*) *F*
by (*auto simp*: *ennreal-def*

intro!: *continuous-on-tendsto-compose*[*OF continuous-on-e2ennreal*[*of UNIV*]] *tendsto-max*)

lemma *tendsto-enn2erealI* [*tendsto-intros*]:
assumes $(f \longrightarrow l)\ F$
shows $((\lambda i.\ enn2ereal(f\ i)) \longrightarrow enn2ereal\ l)\ F$
using *tendsto-enn2ereal-iff assms* **by** *auto*

lemma *tendsto-e2ennrealI* [*tendsto-intros*]:
assumes $(f \longrightarrow l)\ F$
shows $((\lambda i.\ e2ennreal(f\ i)) \longrightarrow e2ennreal\ l)\ F$
proof −
have ∗: $e2ennreal\ (max\ x\ 0) = e2ennreal\ x$ **for** x
by (*simp add*: *e2ennreal-def max.commute*)
have $((\lambda i.\ max\ (f\ i)\ 0) \longrightarrow max\ l\ 0)\ F$
apply (*intro tendsto-intros*) **using** *assms* **by** *auto*
then have $((\lambda i.\ enn2ereal(e2ennreal\ (max\ (f\ i)\ 0))) \longrightarrow enn2ereal\ (e2ennreal\ (max\ l\ 0)))\ F$
by (*subst enn2ereal-e2ennreal*, *auto*)+
then have $((\lambda i.\ e2ennreal\ (max\ (f\ i)\ 0)) \longrightarrow e2ennreal\ (max\ l\ 0))\ F$
using *tendsto-enn2ereal-iff* **by** *auto*
then show *?thesis*
unfolding ∗ **by** *auto*
qed

lemma *ennreal-suminf-minus*:
fixes $f\ g :: nat \Rightarrow ennreal$
shows $(\bigwedge i.\ g\ i \leq f\ i) \Longrightarrow suminf\ f \neq top \Longrightarrow suminf\ g \neq top \Longrightarrow (\sum i.\ f\ i - g\ i) = suminf\ f - suminf\ g$
by *transfer*
(*auto simp add*: *max.absorb2 ereal-diff-positive suminf-le-pos top-ereal-def intro*!: *suminf-ereal-minus*)

lemma *ennreal-Sup-countable-SUP*:
$A \neq \{\} \Longrightarrow \exists f{::}nat \Rightarrow ennreal.\ incseq\ f \wedge range\ f \subseteq A \wedge Sup\ A = (SUP\ i.\ f\ i)$
unfolding *incseq-def*
apply *transfer*
subgoal for A
using *Sup-countable-SUP*[*of A*]
by (*force simp add*: *incseq-def*[*symmetric*] *SUP-upper2 max.absorb2 image-subset-iff Sup-upper2 cong*: *conj-cong*)
done

lemma *ennreal-Inf-countable-INF*:
$A \neq \{\} \Longrightarrow \exists f{::}nat \Rightarrow ennreal.\ decseq\ f \wedge range\ f \subseteq A \wedge Inf\ A = (INF\ i.\ f\ i)$
including *ennreal.lifting*
unfolding *decseq-def*
apply *transfer*
subgoal for A

```
    using Inf-countable-INF[of A]
    apply (clarsimp simp flip: decseq-def)
    subgoal for f
      by (intro exI[of - f]) auto
    done
  done

lemma ennreal-SUP-countable-SUP:
  A ≠ {} ⟹ ∃f::nat ⇒ ennreal. range f ⊆ g`A ∧ Sup (g ` A) = Sup (f ` UNIV)
  using ennreal-Sup-countable-SUP [of g`A] by auto

lemma of-nat-tendsto-top-ennreal: (λn::nat. of-nat n :: ennreal) ⟶ top
  using LIMSEQ-SUP[of of-nat :: nat ⇒ ennreal]
  by (simp add: ennreal-SUP-of-nat-eq-top incseq-def)

lemma SUP-sup-continuous-ennreal:
  fixes f :: ennreal ⇒ 'a::complete-lattice
  assumes f: sup-continuous f and I ≠ {}
  shows (SUP i∈I. f (g i)) = f (SUP i∈I. g i)
proof (rule antisym)
  show (SUP i∈I. f (g i)) ≤ f (SUP i∈I. g i)
    by (rule mono-SUP[OF sup-continuous-mono[OF f]])
  from ennreal-Sup-countable-SUP[of g`I] ‹I ≠ {}›
  obtain M :: nat ⇒ ennreal where incseq M and M: range M ⊆ g ` I and eq:
(SUP i ∈ I. g i) = (SUP i. M i)
    by auto
  have f (SUP i ∈ I. g i) = (SUP i ∈ range M. f i)
    unfolding eq sup-continuousD[OF f ‹mono M›] by (simp add: image-comp)
  also have ... ≤ (SUP i ∈ I. f (g i))
    by (insert M, drule SUP-subset-mono) (auto simp add: image-comp)
  finally show f (SUP i ∈ I. g i) ≤ (SUP i ∈ I. f (g i)) .
qed

lemma ennreal-suminf-SUP-eq:
  fixes f :: nat ⇒ nat ⇒ ennreal
  shows (⋀i. incseq (λn. f n i)) ⟹ (∑ i. SUP n. f n i) = (SUP n. ∑ i. f n i)
  apply (rule ennreal-suminf-SUP-eq-directed)
  subgoal for N n j
    by (auto simp: incseq-def intro!:exI[of - max n j])
  done

lemma ennreal-SUP-add-left:
  fixes c :: ennreal
  shows I ≠ {} ⟹ (SUP i∈I. f i + c) = (SUP i∈I. f i) + c
  apply transfer
  apply (simp add: SUP-ereal-add-left)
  apply (subst (1 2) max.absorb2)
  apply (auto intro: SUP-upper2 add-nonneg-nonneg)
  done
```

lemma *ennreal-SUP-const-minus*:
fixes $f :: 'a \Rightarrow ennreal$
shows $I \neq \{\} \Longrightarrow c < top \Longrightarrow (INF\ x{\in}I.\ c - f\ x) = c - (SUP\ x{\in}I.\ f\ x)$
apply (*transfer fixing*: *I*)
unfolding *ex-in-conv*[*symmetric*]
apply (*auto simp add*: *SUP-upper2 sup-absorb2 simp flip*: *sup-ereal-def*)
apply (*subst INF-ereal-minus-right*[*symmetric*])
apply (*auto simp del*: *sup-ereal-def simp add*: *sup-INF*)
done

lemma *isCont-ennreal*[*simp*]: ‹*isCont ennreal x*›
apply (*auto intro*!: *sequentially-imp-eventually-within simp*: *continuous-within tendsto-def*)
by (*metis tendsto-def tendsto-ennrealI*)

lemma *isCont-ennreal-of-enat*[*simp*]: ‹*isCont ennreal-of-enat x*›
proof −
have *continuous-at-open*:
— Copied lemma from `HOL-Analysis` to avoid dependency.
continuous (*at x*) *f* ⟷ (∀ *t*. *open t* ∧ *f x* ∈ *t* −−> (∃ *s*. *open s* ∧ *x* ∈ *s* ∧ (∀ *x*′ ∈ *s*. (*f x*′) ∈ *t*))) **for** *f* :: ‹*enat* ⇒ '*z*::*topological-space*›
unfolding *continuous-within-topological* [*of x UNIV f*]
unfolding *imp-conjL*
by (*intro all-cong imp-cong ex-cong conj-cong refl*) *auto*
show *?thesis*
proof (*subst continuous-at-open*, *intro allI impI*, *cases* ‹*x* = ∞›)
case *True*

fix *t* **assume** ‹*open t* ∧ *ennreal-of-enat x* ∈ *t*›
then have ‹∃ *y*<∞. {*y* <.. ∞} ⊆ *t*›
apply (*rule-tac open-left*[**where** *y*=*0*])
by (*auto simp*: *True*)
then obtain *y* **where** ‹{*y*<..} ⊆ *t*› **and** ‹*y* ≠ ∞›
apply *atomize-elim*
apply (*auto simp*: *greaterThanAtMost-def*)
by (*metis atMost-iff inf.orderE subsetI top.not-eq-extremum top-greatest*)

from ‹*y* ≠ ∞›
obtain *x*′ **where** *x*′*y*: ‹*ennreal-of-enat x*′ > *y*› **and** ‹*x*′ ≠ ∞›
by (*metis enat.simps*(*3*) *ennreal-Ex-less-of-nat ennreal-of-enat-enat infinity-ennreal-def top.not-eq-extremum*)
define *s* **where** ‹*s* = {*x*′<..}›
have ‹*open s*›
by (*simp add*: *s-def*)
moreover have ‹*x* ∈ *s*›
by (*simp add*: ‹*x*′ ≠ ∞› *s-def True*)

```
    moreover have ‹ennreal-of-enat z ∈ t› if ‹z ∈ s› for z
      by (metis x'y ‹{y<..} ⊆ t› ennreal-of-enat-le-iff greaterThan-iff le-less-trans
less-imp-le not-less s-def subsetD that)
    ultimately show ‹∃ s. open s ∧ x ∈ s ∧ (∀ z∈s. ennreal-of-enat z ∈ t)›
      by auto
  next
    case False
    fix t assume asm: ‹open t ∧ ennreal-of-enat x ∈ t›
    define s where ‹s = {x}›
    have ‹open s›
      using False open-enat-iff s-def by blast
    moreover have ‹x ∈ s›
      using s-def by auto
    moreover have ‹ennreal-of-enat z ∈ t› if ‹z ∈ s› for z
      using asm s-def that by blast
    ultimately show ‹∃ s. open s ∧ x ∈ s ∧ (∀ z∈s. ennreal-of-enat z ∈ t)›
      by auto
  qed
qed
```

39.10 Approximation lemmas

```
lemma INF-approx-ennreal:
  fixes x::ennreal and e::real
  assumes e > 0
  assumes INF: x = (INF i ∈ A. f i)
  assumes x ≠ ∞
  shows ∃ i ∈ A. f i < x + e
proof −
  have (INF i ∈ A. f i) < x + e
    unfolding INF[symmetric] using ‹0<e› ‹x ≠ ∞› by (cases x) auto
  then show ?thesis
    unfolding INF-less-iff .
qed

lemma SUP-approx-ennreal:
  fixes x::ennreal and e::real
  assumes e > 0 A ≠ {}
  assumes SUP: x = (SUP i ∈ A. f i)
  assumes x ≠ ∞
  shows ∃ i ∈ A. x < f i + e
proof −
  have x < x + e
    using ‹0<e› ‹x ≠ ∞› by (cases x) auto
  also have x + e = (SUP i ∈ A. f i + e)
    unfolding SUP ennreal-SUP-add-left[OF ‹A ≠ {}›] ..
  finally show ?thesis
    unfolding less-SUP-iff .
qed
```

lemma *ennreal-approx-SUP*:
fixes *x*::*ennreal*
assumes *f-bound*: $\bigwedge i.\ i \in A \Longrightarrow f\ i \leq x$
assumes *approx*: $\bigwedge e.\ (e{::}real) > 0 \Longrightarrow \exists\, i \in A.\ x \leq f\ i + e$
shows $x = (SUP\ i \in A.\ f\ i)$
proof (*rule antisym*)
show $x \leq (SUP\ i{\in}A.\ f\ i)$
proof (*rule ennreal-le-epsilon*)
fix e :: *real* **assume** $0 < e$
from *approx*[*OF this*] **obtain** i **where** $i \in A$ **and** $*$: $x \leq f\ i + ennreal\ e$
by *blast*
from $*$ **have** $x \leq f\ i + e$
by *simp*
also have $\ldots \leq (SUP\ i{\in}A.\ f\ i) + e$
by (*intro add-mono* ‹$i \in A$› *SUP-upper order-refl*)
finally show $x \leq (SUP\ i{\in}A.\ f\ i) + e$ **.**
qed
qed (*intro SUP-least f-bound*)

lemma *ennreal-approx-INF*:
fixes *x*::*ennreal*
assumes *f-bound*: $\bigwedge i.\ i \in A \Longrightarrow x \leq f\ i$
assumes *approx*: $\bigwedge e.\ (e{::}real) > 0 \Longrightarrow \exists\, i \in A.\ f\ i \leq x + e$
shows $x = (INF\ i \in A.\ f\ i)$
proof (*rule antisym*)
show $(INF\ i{\in}A.\ f\ i) \leq x$
proof (*rule ennreal-le-epsilon*)
fix e :: *real* **assume** $0 < e$
from *approx*[*OF this*] **obtain** i **where** $i{\in}A\ f\ i \leq x + ennreal\ e$
by *blast*
then have $(INF\ i{\in}A.\ f\ i) \leq f\ i$
by (*intro INF-lower*)
also have $\ldots \leq x + e$
by *fact*
finally show $(INF\ i{\in}A.\ f\ i) \leq x + e$ **.**
qed
qed (*intro INF-greatest f-bound*)

lemma *ennreal-approx-unit*:
$(\bigwedge a{::}ennreal.\ 0 < a \Longrightarrow a < 1 \Longrightarrow a * z \leq y) \Longrightarrow z \leq y$
apply (*subst SUP-mult-right-ennreal*[*of* $\lambda x.\ x\ \{0 <..< 1\}\ z$, *simplified*])
apply (*auto intro*: *SUP-least*)
done

lemma *suminf-ennreal2*:
$(\bigwedge i.\ 0 \leq f\ i) \Longrightarrow summable\ f \Longrightarrow (\sum i.\ ennreal\ (f\ i)) = ennreal\ (\sum i.\ f\ i)$
using *suminf-ennreal-eq* **by** *blast*

lemma *less-top-ennreal*: $x < top \longleftrightarrow (\exists r{\geq}0.\ x = ennreal\ r)$
by (*cases x*) *auto*

lemma *enn2real-less-iff*[*simp*]: $x < top \Longrightarrow enn2real\ x < c \longleftrightarrow x < c$
using *ennreal-less-iff less-top-ennreal* **by** *auto*

lemma *enn2real-le-iff*[*simp*]: $[\![x < top;\ c > 0]\!] \Longrightarrow enn2real\ x \leq c \longleftrightarrow x \leq c$
by (*cases x*) *auto*

lemma *enn2real-less*:
assumes $enn2real\ e < r$ $e \neq top$ **shows** $e < ennreal\ r$
using *enn2real-less-iff assms top.not-eq-extremum* **by** *blast*

lemma *enn2real-le*:
assumes $enn2real\ e \leq r$ $e \neq top$ **shows** $e \leq ennreal\ r$
by (*metis assms enn2real-less ennreal-enn2real-if eq-iff less-le*)

lemma *tendsto-top-iff-ennreal*:
fixes $f :: 'a \Rightarrow ennreal$
shows $(f \longrightarrow top)\ F \longleftrightarrow (\forall l{\geq}0.\ eventually\ (\lambda x.\ ennreal\ l < f\ x)\ F)$
by (*auto simp*: *less-top-ennreal order-tendsto-iff*)

lemma *ennreal-tendsto-top-eq-at-top*:
$((\lambda z.\ ennreal\ (f\ z)) \longrightarrow top)\ F \longleftrightarrow (LIM\ z\ F.\ f\ z :> at\text{-}top)$
unfolding *filterlim-at-top-dense tendsto-top-iff-ennreal*
apply (*auto simp*: *ennreal-less-iff*)
subgoal for *y*
by (*auto elim*!: *eventually-mono allE*[*of - max 0 y*])
done

lemma *tendsto-0-if-Limsup-eq-0-ennreal*:
fixes $f :: - \Rightarrow ennreal$
shows $Limsup\ F\ f = 0 \Longrightarrow (f \longrightarrow 0)\ F$
using *Liminf-le-Limsup*[*of F f*] *tendsto-iff-Liminf-eq-Limsup*[*of F f 0*]
by (*cases F = bot*) *auto*

lemma *diff-le-self-ennreal*[*simp*]: $a - b \leq (a{::}ennreal)$
by (*cases a b rule*: *ennreal2-cases*) (*auto simp*: *ennreal-minus*)

lemma *ennreal-ineq-diff-add*: $b \leq a \Longrightarrow a = b + (a - b{::}ennreal)$
by *transfer* (*auto simp*: *ereal-diff-positive max.absorb2 ereal-ineq-diff-add*)

lemma *ennreal-mult-strict-left-mono*: $(a{::}ennreal) < c \Longrightarrow 0 < b \Longrightarrow b < top \Longrightarrow b * a < b * c$
by *transfer* (*auto intro*!: *ereal-mult-strict-left-mono*)

lemma *ennreal-between*: $0 < e \Longrightarrow 0 < x \Longrightarrow x < top \Longrightarrow x - e < (x{::}ennreal)$
by *transfer* (*auto intro*!: *ereal-between*)

lemma *minus-less-iff-ennreal*: $b < top \Longrightarrow b \leq a \Longrightarrow a - b < c \longleftrightarrow a < c + (b::ennreal)$
by *transfer*
(*auto simp*: *top-ereal-def ereal-minus-less le-less*)

lemma *tendsto-zero-ennreal*:
assumes *ev*: $\bigwedge r.\ 0 < r \Longrightarrow \forall_F\ x\ in\ F.\ f\ x < ennreal\ r$
shows $(f \longrightarrow 0)\ F$
proof (*rule order-tendstoI*)
fix *e*::*ennreal* **assume** $e > 0$
obtain e'::*real* **where** $e' > 0\ ennreal\ e' < e$
using ‹$0 < e$› *dense*[*of 0 if e = top then 1 else* (*enn2real e*)]
by (*cases e*) (*auto simp*: *ennreal-less-iff*)
from *ev*[*OF* ‹$e' > 0$›] **show** $\forall_F\ x\ in\ F.\ f\ x < e$
by *eventually-elim* (*insert* ‹*ennreal* $e' < e$›, *auto*)
qed *simp*

lifting-update *ennreal.lifting*
lifting-forget *ennreal.lifting*

39.11 *ennreal* theorems

lemma *neq-top-trans*: **fixes** $x\ y$ :: *ennreal* **shows** $[\![\ y \neq top;\ x \leq y\ ]\!] \Longrightarrow x \neq top$
by (*auto simp*: *top-unique*)

lemma *diff-diff-ennreal*: **fixes** $a\ b$ :: *ennreal* **shows** $a \leq b \Longrightarrow b \neq \infty \Longrightarrow b - (b - a) = a$
by (*cases a b rule*: *ennreal2-cases*) (*auto simp*: *ennreal-minus top-unique*)

lemma *ennreal-less-one-iff*[*simp*]: $ennreal\ x < 1 \longleftrightarrow x < 1$
by (*cases* $0 \leq x$) (*auto simp*: *ennreal-neg ennreal-less-iff simp flip*: *ennreal-1*)

lemma *SUP-const-minus-ennreal*:
fixes $f :: 'a \Rightarrow ennreal$ **shows** $I \neq \{\} \Longrightarrow (SUP\ x{\in}I.\ c - f\ x) = c - (INF\ x{\in}I.\ f\ x)$
including *ennreal.lifting*
by (*transfer fixing*: *I*)
(*simp add*: *SUP-sup-distrib*[*symmetric*] *SUP-ereal-minus-right*
flip: *sup-ereal-def*)

lemma *zero-minus-ennreal*[*simp*]: $0 - (a::ennreal) = 0$
including *ennreal.lifting*
by *transfer* (*simp split*: *split-max*)

lemma *diff-diff-commute-ennreal*:
fixes $a\ b\ c$ :: *ennreal* **shows** $a - b - c = a - c - b$
by (*cases a b c rule*: *ennreal3-cases*) (*simp-all add*: *ennreal-minus field-simps*)

lemma *diff-gr0-ennreal*: $b < (a::ennreal) \Longrightarrow 0 < a - b$

including *ennreal.lifting* **by** *transfer* (*auto simp*: *ereal-diff-gr0 ereal-diff-positive split*: *split-max*)

lemma *divide-le-posI-ennreal*:
fixes $x\ y\ z$ :: *ennreal*
shows $x > 0 \Longrightarrow z \leq x * y \Longrightarrow z\ /\ x \leq y$
by (*cases x y z rule*: *ennreal3-cases*)
(*auto simp*: *divide-ennreal ennreal-mult*[*symmetric*] *field-simps top-unique*)

lemma *add-diff-eq-ennreal*:
fixes $x\ y\ z$ :: *ennreal*
shows $z \leq y \Longrightarrow x + (y - z) = x + y - z$
including *ennreal.lifting*
by *transfer*
(*insert add-mono*[*of 0*::*ereal*], *auto simp add*: *ereal-diff-positive max.absorb2 add-diff-eq-ereal*)

lemma *add-diff-inverse-ennreal*:
fixes $x\ y$ :: *ennreal* **shows** $x \leq y \Longrightarrow x + (y - x) = y$
by (*cases x*) (*simp-all add*: *top-unique add-diff-eq-ennreal*)

lemma *add-diff-eq-iff-ennreal*[*simp*]:
fixes $x\ y$ :: *ennreal* **shows** $x + (y - x) = y \longleftrightarrow x \leq y$
proof
assume $*$: $x + (y - x) = y$ **show** $x \leq y$
by (*subst* $*$[*symmetric*]) *simp*
qed (*simp add*: *add-diff-inverse-ennreal*)

lemma *add-diff-le-ennreal*: $a + b - c \leq a + (b - c$::*ennreal*)
apply (*cases a b c rule*: *ennreal3-cases*)
subgoal for $a'\ b'\ c'$
by (*cases* $0 \leq b' - c'$) (*simp-all add*: *ennreal-minus top-add ennreal-neg flip*: *ennreal-plus*)
apply (*simp-all add*: *top-add flip*: *ennreal-plus*)
done

lemma *diff-eq-0-ennreal*: $a < top \Longrightarrow a \leq b \Longrightarrow a - b = (0$::*ennreal*)
using *ennreal-minus-pos-iff gr-zeroI not-less* **by** *blast*

lemma *diff-diff-ennreal'*: **fixes** $x\ y\ z$ :: *ennreal* **shows** $z \leq y \Longrightarrow y - z \leq x \Longrightarrow x - (y - z) = x + z - y$
by (*cases x*; *cases y*; *cases z*)
(*auto simp add*: *top-add add-top minus-top-ennreal ennreal-minus top-unique simp flip*: *ennreal-plus*)

lemma *diff-diff-ennreal''*: **fixes** $x\ y\ z$ :: *ennreal*
shows $z \leq y \Longrightarrow x - (y - z) = (if\ y - z \leq x\ then\ x + z - y\ else\ 0)$
by (*cases x*; *cases y*; *cases z*)
(*auto simp add*: *top-add add-top minus-top-ennreal ennreal-minus top-unique*

ennreal-neg
simp flip: *ennreal-plus*)

lemma *power-less-top-ennreal*: **fixes** x :: *ennreal* **shows** $x \hat{} n < top \longleftrightarrow x < top \vee n = 0$
using *power-eq-top-ennreal*[*of x n*] **by** (*auto simp*: *less-top*)

lemma *ennreal-divide-times*: $(a / b) * c = a * (c / b :: ennreal)$
by (*simp add*: *mult.commute ennreal-times-divide*)

lemma *diff-less-top-ennreal*: $a - b < top \longleftrightarrow a < (top :: ennreal)$
by (*cases a*; *cases b*) (*auto simp*: *ennreal-minus*)

lemma *divide-less-ennreal*: $b \neq 0 \Longrightarrow b < top \Longrightarrow a / b < c \longleftrightarrow a < (c * b :: ennreal)$
by (*cases a*; *cases b*; *cases c*)
(*auto simp*: *divide-ennreal ennreal-mult*[*symmetric*] *ennreal-less-iff field-simps ennreal-top-mult ennreal-top-divide*)

lemma *one-less-numeral*[*simp*]: $1 < (numeral\ n :: ennreal) \longleftrightarrow (num.One < n)$
by (*simp flip*: *ennreal-1 ennreal-numeral add*: *ennreal-less-iff*)

lemma *divide-eq-1-ennreal*: $a / b = (1 :: ennreal) \longleftrightarrow (b \neq top \wedge b \neq 0 \wedge b = a)$
by (*cases a* ; *cases b*; *cases b = 0*) (*auto simp*: *ennreal-top-divide divide-ennreal split*: *if-split-asm*)

lemma *ennreal-mult-cancel-left*: $(a * b = a * c) = (a = top \wedge b \neq 0 \wedge c \neq 0 \vee a = 0 \vee b = (c :: ennreal))$
by (*cases a*; *cases b*; *cases c*) (*auto simp*: *ennreal-mult*[*symmetric*] *ennreal-mult-top ennreal-top-mult*)

lemma *ennreal-minus-if*: *ennreal a − ennreal b = ennreal* (*if 0 ≤ b then* (*if b ≤ a then a − b else 0*) *else a*)
by (*auto simp*: *ennreal-minus ennreal-neg*)

lemma *ennreal-plus-if*: *ennreal a + ennreal b = ennreal* (*if 0 ≤ a then* (*if 0 ≤ b then a + b else a*) *else b*)
by (*auto simp*: *ennreal-neg*)

lemma *power-le-one-iff*: $0 \leq (a :: real) \Longrightarrow a \hat{} n \leq 1 \longleftrightarrow (n = 0 \vee a \leq 1)$
by (*metis* (*mono-tags*, *opaque-lifting*) *le-less neq0-conv not-le one-le-power power-0 power-eq-imp-eq-base power-le-one zero-le-one*)

lemma *ennreal-diff-le-mono-left*: $a \leq b \Longrightarrow a - c \leq (b :: ennreal)$
using *ennreal-mono-minus*[*of 0 c a*, *THEN order-trans*, *of b*] **by** *simp*

lemma *ennreal-minus-le-iff*: $a - b \leq c \longleftrightarrow (a \leq b + (c :: ennreal) \wedge (a = top \wedge b = top \longrightarrow c = top))$
by (*cases a*; *cases b*; *cases c*)

(*auto simp*: *top-unique top-add add-top ennreal-minus simp flip*: *ennreal-plus*)

lemma *ennreal-le-minus-iff*: $a \leq b - c \longleftrightarrow (a + c \leq (b::ennreal) \vee (a = 0 \wedge b \leq c))$
by (*cases a*; *cases b*; *cases c*)
(*auto simp*: *top-unique top-add add-top ennreal-minus ennreal-le-iff2*
simp flip: *ennreal-plus*)

lemma *diff-add-eq-diff-diff-swap-ennreal*: $x - (y + z :: ennreal) = x - y - z$
by (*cases x*; *cases y*; *cases z*)
(*auto simp*: *ennreal-minus-if add-top top-add simp flip*: *ennreal-plus*)

lemma *diff-add-assoc2-ennreal*: $b \leq a \Longrightarrow (a - b + c::ennreal) = a + c - b$
by (*cases a*; *cases b*; *cases c*)
(*auto simp add*: *ennreal-minus-if ennreal-plus-if add-top top-add top-unique simp del*: *ennreal-plus*)

lemma *diff-gt-0-iff-gt-ennreal*: $0 < a - b \longleftrightarrow (a = top \wedge b = top \vee b < (a::ennreal))$
by (*cases a*; *cases b*) (*auto simp*: *ennreal-minus-if ennreal-less-iff*)

lemma *diff-eq-0-iff-ennreal*: $(a - b::ennreal) = 0 \longleftrightarrow (a < top \wedge a \leq b)$
by (*cases a*) (*auto simp*: *ennreal-minus-eq-0 diff-eq-0-ennreal*)

lemma *add-diff-self-ennreal*: $a + (b - a::ennreal) = (if\ a \leq b\ then\ b\ else\ a)$
by (*auto simp*: *diff-eq-0-iff-ennreal less-top*)

lemma *diff-add-self-ennreal*: $(b - a + a::ennreal) = (if\ a \leq b\ then\ b\ else\ a)$
by (*auto simp*: *diff-add-cancel-ennreal diff-eq-0-iff-ennreal less-top*)

lemma *ennreal-minus-cancel-iff*:
fixes *a b c* :: *ennreal*
shows $a - b = a - c \longleftrightarrow (b = c \vee (a \leq b \wedge a \leq c) \vee a = top)$
by (*cases a*; *cases b*; *cases c*) (*auto simp*: *ennreal-minus-if*)

The next lemma is wrong for $a = top$, for $b = c = 1$ for instance.

lemma *ennreal-right-diff-distrib*:
fixes *a b c* :: *ennreal*
assumes $a \neq top$
shows $a * (b - c) = a * b - a * c$
apply (*cases a*; *cases b*; *cases c*)
apply (*use assms* **in** ‹*auto simp add*: *ennreal-mult-top ennreal-minus ennreal-mult'* [*symmetric*]›)
apply (*simp add*: *algebra-simps*)
done

lemma *SUP-diff-ennreal*:
$c < top \Longrightarrow (SUP\ i{\in}I.\ f\ i - c :: ennreal) = (SUP\ i{\in}I.\ f\ i) - c$
by (*auto intro*!: *SUP-eqI ennreal-minus-mono SUP-least intro*: *SUP-upper*
simp: *ennreal-minus-cancel-iff ennreal-minus-le-iff less-top*[*symmetric*])

lemma *ennreal-SUP-add-right*:
fixes c :: *ennreal* **shows** $I \neq \{\} \Longrightarrow c + (SUP\ i{\in}I.\ f\ i) = (SUP\ i{\in}I.\ c + f\ i)$
using *ennreal-SUP-add-left*[*of* I f c] **by** (*simp add*: *add.commute*)

lemma *SUP-add-directed-ennreal*:
fixes $f\ g$ :: - $\Rightarrow$ *ennreal*
assumes *directed*: $\bigwedge i\ j.\ i \in I \Longrightarrow j \in I \Longrightarrow \exists k{\in}I.\ f\ i + g\ j \leq f\ k + g\ k$
shows $(SUP\ i{\in}I.\ f\ i + g\ i) = (SUP\ i{\in}I.\ f\ i) + (SUP\ i{\in}I.\ g\ i)$
proof *cases*
assume $I = \{\}$ **then show** *?thesis*
by (*simp add*: *bot-ereal-def*)
next
assume $I \neq \{\}$
show *?thesis*
proof (*rule antisym*)
show $(SUP\ i{\in}I.\ f\ i + g\ i) \leq (SUP\ i{\in}I.\ f\ i) + (SUP\ i{\in}I.\ g\ i)$
by (*rule SUP-least*; *intro add-mono SUP-upper*)
next
have $(SUP\ i{\in}I.\ f\ i) + (SUP\ i{\in}I.\ g\ i) = (SUP\ i{\in}I.\ f\ i + (SUP\ i{\in}I.\ g\ i))$
by (*intro ennreal-SUP-add-left*[*symmetric*] ‹$I \neq \{\}$›)
also have $\ldots = (SUP\ i{\in}I.\ (SUP\ j{\in}I.\ f\ i + g\ j))$
using ‹$I \neq \{\}$› **by** (*simp add*: *ennreal-SUP-add-right*)
also have $\ldots \leq (SUP\ i{\in}I.\ f\ i + g\ i)$
using *directed* **by** (*intro SUP-least*) (*blast intro*: *SUP-upper2*)
finally show $(SUP\ i{\in}I.\ f\ i) + (SUP\ i{\in}I.\ g\ i) \leq (SUP\ i{\in}I.\ f\ i + g\ i)$ **.**
qed
qed

lemma *enn2real-eq-0-iff*: *enn2real* $x = 0 \longleftrightarrow x = 0 \vee x = top$
by (*cases* x) *auto*

lemma *continuous-on-diff-ennreal*:
continuous-on $A\ f \Longrightarrow$ *continuous-on* $A\ g \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \neq top) \Longrightarrow$ $(\bigwedge x.\ x \in A \Longrightarrow g\ x \neq top) \Longrightarrow$ *continuous-on* $A\ (\lambda z.\ f\ z - g\ z{::}ennreal)$
including *ennreal.lifting*
proof (*transfer fixing*: A, *simp add*: *top-ereal-def*)
fix $f\ g$ :: $'a \Rightarrow$ *ereal* **assume** $\forall x.\ 0 \leq f\ x\ \forall x.\ 0 \leq g\ x$ *continuous-on* $A\ f$ *continuous-on* $A\ g$
moreover assume $f\ x \neq \infty\ g\ x \neq \infty$ **if** $x \in A$ **for** x
ultimately show *continuous-on* $A\ (\lambda z.\ max\ 0\ (f\ z - g\ z))$
by (*intro continuous-on-max continuous-on-const continuous-on-diff-ereal*) *auto*
qed

lemma *tendsto-diff-ennreal*:
$(f \longrightarrow x)\ F \Longrightarrow (g \longrightarrow y)\ F \Longrightarrow x \neq top \Longrightarrow y \neq top \Longrightarrow ((\lambda z.\ f\ z - g\ z{::}ennreal) \longrightarrow x - y)\ F$
using *continuous-on-tendsto-compose*[**where** $f{=}\lambda x.\ fst\ x - snd\ x{::}ennreal$ **and** $s{=}\{(x,\ y).\ x \neq top \wedge y \neq top\}$ **and** $g{=}\lambda x.\ (f\ x,\ g\ x)$ **and** $l{=}(x,\ y)$ **and** $F{=}F$,

OF continuous-on-diff-ennreal]
by (*auto simp*: *tendsto-Pair eventually-conj-iff less-top order-tendstoD continuous-on-fst continuous-on-snd continuous-on-id*)

declare *lim-real-of-ereal* [*tendsto-intros*]

lemma *tendsto-enn2real* [*tendsto-intros*]:
assumes ($u \longrightarrow$ *ennreal l*) *F l* $\geq$ *0*
shows ((λn. *enn2real* (*u n*)) $\longrightarrow$ *l*) *F*
unfolding *enn2real-def*
by (*metis assms enn2ereal-ennreal lim-real-of-ereal tendsto-enn2erealI*)

end

40 Logarithm of Natural Numbers

theory *Log-Nat*
imports *Complex-Main*
begin

40.1 Preliminaries

lemma *divide-nat-diff-div-nat-less-one*:
real x / *real b* − *real* (*x div b*) < *1* **for** *x b* :: *nat*
proof (*cases b* = *0*)
case *True*
then show *?thesis*
by *simp*
next
case *False*
then have *real* (*x div b*) + *real* (*x mod b*) / *real b* − *real* (*x div b*) < *1*
by (*simp add*: *field-simps*)
then show *?thesis*
by (*simp add*: *real-of-nat-div-aux* [*symmetric*])
qed

40.2 Floorlog

definition *floorlog* :: *nat* ⇒ *nat* ⇒ *nat*
where *floorlog b a* = (*if a* > *0* ∧ *b* > *1 then nat* ⌊*log b a*⌋ + *1 else 0*)

lemma *floorlog-mono*: *x* ≤ *y* ⟹ *floorlog b x* ≤ *floorlog b y*
by (*auto simp*: *floorlog-def floor-mono nat-mono*)

lemma *floorlog-bounds*:
b ^ (*floorlog b x* − *1*) ≤ *x* ∧ *x* < *b* ^ (*floorlog b x*) **if** *x* > *0 b* > *1*
proof
show *b* ^ (*floorlog b x* − *1*) ≤ *x*
proof −

```
    have b ^ nat ⌊log b x⌋ = b powr ⌊log b x⌋
      using powr-realpow[symmetric, of b nat ⌊log b x⌋] ‹x > 0› ‹b > 1›
      by simp
    also have ... ≤ b powr log b x using ‹b > 1› by simp
    also have ... = real-of-int x using ‹0 < x› ‹b > 1› by simp
    finally have b ^ nat ⌊log b x⌋ ≤ real-of-int x by simp
    then show ?thesis
      using ‹0 < x› ‹b > 1› of-nat-le-iff
      by (fastforce simp add: floorlog-def)
  qed
  show x < b ^ (floorlog b x)
  proof −
    have x ≤ b powr (log b x) using ‹x > 0› ‹b > 1› by simp
    also have ... < b powr (⌊log b x⌋ + 1)
      using that by (intro powr-less-mono) auto
    also have ... = b ^ nat (⌊log b (real-of-int x)⌋ + 1)
      using that by (simp flip: powr-realpow)
    finally
    have x < b ^ nat (⌊log b (int x)⌋ + 1)
      by (rule of-nat-less-imp-less)
    then show ?thesis
      using ‹x > 0› ‹b > 1› by (simp add: floorlog-def nat-add-distrib)
  qed
qed

lemma floorlog-power [simp]:
  floorlog b (a * b ^ c) = floorlog b a + c if a > 0 b > 1
proof −
  have ⌊log b a + real c⌋ = ⌊log b a⌋ + c by arith
  then show ?thesis using that
    by (auto simp: floorlog-def log-mult powr-realpow[symmetric] nat-add-distrib)
qed

lemma floor-log-add-eqI:
  ⌊log b (a + r)⌋ = ⌊log b a⌋ if b > 1 a ≥ 1 0 ≤ r r < 1
    for a b :: nat and r :: real
proof (rule floor-eq2)
  have log b a ≤ log b (a + r) using that by force
  then show ⌊log b a⌋ ≤ log b (a + r) by arith
next
  define l::int where l = int b ^ (nat ⌊log b a⌋ + 1)
  have l-def-real: l = b powr (⌊log b a⌋ + 1)
    using that by (simp add: l-def powr-add powr-real-of-int)
  have a < l
  proof −
    have a = b powr (log b a) using that by simp
    also have ... < b powr floor ((log b a) + 1)
      using that(1) by auto
    also have ... = l
```

```
    using that by (simp add: l-def powr-real-of-int powr-add)
   finally show ?thesis by simp
 qed
 then have a + r < l using that by simp
 then have log b (a + r) < log b l using that by simp
 also have ... = real-of-int ⌊log b a⌋ + 1
   using that by (simp add: l-def-real)
 finally show log b (a + r) < real-of-int ⌊log b a⌋ + 1 .
qed

lemma floor-log-div:
 ⌊log b x⌋ = ⌊log b (x div b)⌋ + 1 if b > 1 x > 0 x div b > 0
   for b x :: nat
proof−
 have ⌊log b x⌋ = ⌊log b (x / b * b)⌋ using that by simp
 also have ... = ⌊log b (x / b) + log b b⌋
   using that by (subst log-mult) auto
 also have ... = ⌊log b (x / b)⌋ + 1 using that by simp
 also have ⌊log b (x / b)⌋ = ⌊log b (x div b + (x / b − x div b))⌋ by simp
 also have ... = ⌊log b (x div b)⌋
   using that real-of-nat-div4 divide-nat-diff-div-nat-less-one
   by (intro floor-log-add-eqI) auto
 finally show ?thesis .
qed

lemma compute-floorlog [code]:
 floorlog b x = (if x > 0 ∧ b > 1 then floorlog b (x div b) + 1 else 0)
 by (auto simp: floorlog-def floor-log-div[of b x] div-eq-0-iff nat-add-distrib
   intro!: floor-eq2)

lemma floor-log-eq-if:
 ⌊log b x⌋ = ⌊log b y⌋ if x div b = y div b b > 1 x > 0 x div b ≥ 1
   for b x y :: nat
proof −
 have y > 0 using that by (auto intro: ccontr)
 thus ?thesis using that by (simp add: floor-log-div)
qed

lemma floorlog-eq-if:
 floorlog b x = floorlog b y if x div b = y div b b > 1 x > 0 x div b ≥ 1
   for b x y :: nat
proof −
 have y > 0 using that by (auto intro: ccontr)
 then show ?thesis using that
   by (auto simp add: floorlog-def eq-nat-nat-iff intro: floor-log-eq-if)
qed

lemma floorlog-leD:
 floorlog b x ≤ w ⟹ b > 1 ⟹ x < b ^ w
```

by (*metis floorlog-bounds leD linorder-neqE-nat order.strict-trans power-strict-increasing-iff zero-less-one zero-less-power*)

lemma *floorlog-leI*:
x < *b* ^ *w* ⟹ *0* ≤ *w* ⟹ *b* > *1* ⟹ *floorlog b x* ≤ *w*
by (*drule less-imp-of-nat-less*[**where** *'a=real*])
(*auto simp*: *floorlog-def Suc-le-eq nat-less-iff floor-less-iff log-of-power-less*)

lemma *floorlog-eq-zero-iff*:
floorlog b x = *0* ⟷ *b* ≤ *1* ∨ *x* ≤ *0*
by (*auto simp*: *floorlog-def*)

lemma *floorlog-le-iff*:
floorlog b x ≤ *w* ⟷ *b* ≤ *1* ∨ *b* > *1* ∧ *0* ≤ *w* ∧ *x* < *b* ^ *w*
using *floorlog-leD*[*of b x w*] *floorlog-leI*[*of x b w*]
by (*auto simp*: *floorlog-eq-zero-iff*[*THEN iffD2*])

lemma *floorlog-ge-SucI*:
Suc w ≤ *floorlog b x* **if** *b* ^ *w* ≤ *x b* > *1*
using *that le-log-of-power*[*of b w x*] *power-not-zero*
by (*force simp*: *floorlog-def Suc-le-eq powr-realpow not-less Suc-nat-eq-nat-zadd1 zless-nat-eq-int-zless int-add-floor less-floor-iff simp del*: *floor-add2*)

lemma *floorlog-geI*:
w ≤ *floorlog b x* **if** *b* ^ (*w* − *1*) ≤ *x b* > *1*
using *floorlog-ge-SucI*[*of b w* − *1 x*] *that*
by *auto*

lemma *floorlog-geD*:
b ^ (*w* − *1*) ≤ *x* **if** *w* ≤ *floorlog b x w* > *0*
proof −
have *b* > *1 0* < *x*
using *that* **by** (*auto simp*: *floorlog-def split*: *if-splits*)
have *b* ^ (*w* − *1*) ≤ *x* **if** *b* ^ *w* ≤ *x*
proof −
have *b* ^ (*w* − *1*) ≤ *b* ^ *w*
using ‹*b* > *1*›
by (*auto intro*!: *power-increasing*)
also note *that*
finally show *?thesis* **.**
qed
moreover have *b* ^ *nat* ⌊*log* (*real b*) (*real x*)⌋ ≤ *x* (**is** *?l* ≤ -)
proof −
have *0* ≤ *log* (*real b*) (*real x*)
using ‹*b* > *1*› ‹*0* < *x*›
by (*auto simp*:)
then have *?l* ≤ *b powr log* (*real b*) (*real x*)
using ‹*b* > *1*›

```
    by (auto simp flip: powr-realpow intro!: powr-mono of-nat-floor)
  also have ... = x using ‹b > 1› ‹0 < x›
    by auto
  finally show ?thesis
    unfolding of-nat-le-iff .
 qed
 ultimately show ?thesis
  using that
  by (auto simp: floorlog-def le-nat-iff le-floor-iff le-log-iff powr-realpow
      split: if-splits elim!: le-SucE)
qed
```

40.3 Bitlen

```
definition bitlen :: int ⇒ int
 where bitlen a = floorlog 2 (nat a)

lemma bitlen-alt-def:
 bitlen a = (if a > 0 then ⌊log 2 a⌋ + 1 else 0)
 by (simp add: bitlen-def floorlog-def)

lemma bitlen-zero [simp]:
 bitlen 0 = 0
 by (auto simp: bitlen-def floorlog-def)

lemma bitlen-nonneg:
 0 ≤ bitlen x
 by (simp add: bitlen-def)

lemma bitlen-bounds:
 2 ^ nat (bitlen x − 1) ≤ x ∧ x < 2 ^ nat (bitlen x) if x > 0
proof −
 from that have bitlen x ≥ 1 by (auto simp: bitlen-alt-def)
 with that floorlog-bounds[of nat x 2] show ?thesis
  by (auto simp add: bitlen-def le-nat-iff nat-less-iff nat-diff-distrib)
qed

lemma bitlen-pow2 [simp]:
 bitlen (b * 2 ^ c) = bitlen b + c if b > 0
 using that by (simp add: bitlen-def nat-mult-distrib nat-power-eq)

lemma compute-bitlen [code]:
 bitlen x = (if x > 0 then bitlen (x div 2) + 1 else 0)
 by (simp add: bitlen-def nat-div-distrib compute-floorlog)

lemma bitlen-eq-zero-iff:
 bitlen x = 0 ⟷ x ≤ 0
 by (auto simp add: bitlen-alt-def)
 (metis compute-bitlen add.commute bitlen-alt-def bitlen-nonneg less-add-same-cancel2
```

not-less zero-less-one)

lemma *bitlen-div*:
1 ≤ real-of-int m / 2^nat (bitlen m − 1)
and *real-of-int m / 2^nat (bitlen m − 1) < 2* **if** *0 < m*
proof −
let *?B = 2^nat (bitlen m − 1)*

have *?B ≤ m* **using** *bitlen-bounds*[*OF ‹0 <m›*] **..**
then have *1 * ?B ≤ real-of-int m*
unfolding *of-int-le-iff*[*symmetric*] **by** *auto*
then show *1 ≤ real-of-int m / ?B* **by** *auto*

from *that* **have** *0 ≤ bitlen m − 1* **by** (*auto simp*: *bitlen-alt-def*)

have *m < 2^nat(bitlen m)* **using** *bitlen-bounds*[*OF that*] **..**
also from *that* **have** *... = 2^nat(bitlen m − 1 + 1)*
by (*auto simp*: *bitlen-def*)
also have *... = ?B * 2*
unfolding *nat-add-distrib*[*OF ‹0 ≤ bitlen m − 1› zero-le-one*] **by** *auto*
finally have *real-of-int m < 2 * ?B*
by (*metis* (*full-types*) *mult.commute power.simps*(*2*) *of-int-less-numeral-power-cancel-iff*)
then have *real-of-int m / ?B < 2 * ?B / ?B*
by (*rule divide-strict-right-mono*) *auto*
then show *real-of-int m / ?B < 2* **by** *auto*
qed

lemma *bitlen-le-iff-floorlog*:
bitlen x ≤ w ⟷ w ≥ 0 ∧ floorlog 2 (nat x) ≤ nat w
by (*auto simp*: *bitlen-def*)

lemma *bitlen-le-iff-power*:
bitlen x ≤ w ⟷ w ≥ 0 ∧ x < 2 ^ nat w
by (*auto simp*: *bitlen-le-iff-floorlog floorlog-le-iff*)

lemma *less-power-nat-iff-bitlen*:
x < 2 ^ w ⟷ bitlen (int x) ≤ w
using *bitlen-le-iff-power*[*of x w*]
by *auto*

lemma *bitlen-ge-iff-power*:
w ≤ bitlen x ⟷ w ≤ 0 ∨ 2 ^ (nat w − 1) ≤ x
unfolding *bitlen-def*
by (*auto simp flip*: *nat-le-iff intro*: *floorlog-geI dest*: *floorlog-geD*)

lemma *bitlen-twopow-add-eq*:
bitlen (2 ^ w + b) = w + 1 **if** *0 ≤ b b < 2 ^ w*
by (*auto simp*: *that nat-add-distrib bitlen-le-iff-power bitlen-ge-iff-power intro*!:
antisym)

end

41 Various algebraic structures combined with a lattice

theory *Lattice-Algebras*
imports *Complex-Main*
begin

class *semilattice-inf-ab-group-add* = *ordered-ab-group-add* + *semilattice-inf*
begin

lemma *add-inf-distrib-left*: *a* + *inf b c* = *inf* (*a* + *b*) (*a* + *c*)
apply (*rule order.antisym*)
apply (*simp-all add*: *le-infI*)
apply (*rule add-le-imp-le-left* [*of uminus a*])
apply (*simp only*: *add.assoc* [*symmetric*], *simp add*: *diff-le-eq add.commute*)
done

lemma *add-inf-distrib-right*: *inf a b* + *c* = *inf* (*a* + *c*) (*b* + *c*)
proof −
have *c* + *inf a b* = *inf* (*c* + *a*) (*c* + *b*)
by (*simp add*: *add-inf-distrib-left*)
then show *?thesis*
by (*simp add*: *add.commute*)
qed

end

class *semilattice-sup-ab-group-add* = *ordered-ab-group-add* + *semilattice-sup*
begin

lemma *add-sup-distrib-left*: *a* + *sup b c* = *sup* (*a* + *b*) (*a* + *c*)
apply (*rule order.antisym*)
apply (*rule add-le-imp-le-left* [*of uminus a*])
apply (*simp only*: *add.assoc* [*symmetric*], *simp*)
apply (*simp add*: *le-diff-eq add.commute*)
apply (*rule le-supI*)
apply (*rule add-le-imp-le-left* [*of a*], *simp only*: *add.assoc*[*symmetric*], *simp*)+
done

lemma *add-sup-distrib-right*: *sup a b* + *c* = *sup* (*a* + *c*) (*b* + *c*)
proof −
have *c* + *sup a b* = *sup* (*c*+*a*) (*c*+*b*)
by (*simp add*: *add-sup-distrib-left*)
then show *?thesis*
by (*simp add*: *add.commute*)

qed

end

class *lattice-ab-group-add* = *ordered-ab-group-add* + *lattice*
begin

subclass *semilattice-inf-ab-group-add* **..**
subclass *semilattice-sup-ab-group-add* **..**

lemmas *add-sup-inf-distribs* =
add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left

lemma *inf-eq-neg-sup*: *inf a b* = − *sup* (− *a*) (− *b*)
proof (*rule inf-unique*)
fix *a b c* :: *'a*
show − *sup* (− *a*) (− *b*) ≤ *a*
by (*rule add-le-imp-le-right* [*of* - *sup* (*uminus a*) (*uminus b*)])
(*simp*, *simp add*: *add-sup-distrib-left*)
show − *sup* (−*a*) (−*b*) ≤ *b*
by (*rule add-le-imp-le-right* [*of* - *sup* (*uminus a*) (*uminus b*)])
(*simp*, *simp add*: *add-sup-distrib-left*)
assume *a* ≤ *b a* ≤ *c*
then show *a* ≤ − *sup* (−*b*) (−*c*)
by (*subst neg-le-iff-le* [*symmetric*]) (*simp add*: *le-supI*)
qed

lemma *sup-eq-neg-inf*: *sup a b* = − *inf* (− *a*) (− *b*)
proof (*rule sup-unique*)
fix *a b c* :: *'a*
show *a* ≤ − *inf* (− *a*) (− *b*)
by (*rule add-le-imp-le-right* [*of* - *inf* (*uminus a*) (*uminus b*)])
(*simp*, *simp add*: *add-inf-distrib-left*)
show *b* ≤ − *inf* (− *a*) (− *b*)
by (*rule add-le-imp-le-right* [*of* - *inf* (*uminus a*) (*uminus b*)])
(*simp*, *simp add*: *add-inf-distrib-left*)
show − *inf* (− *a*) (− *b*) ≤ *c* **if** *a* ≤ *c b* ≤ *c*
using *that* **by** (*subst neg-le-iff-le* [*symmetric*]) (*simp add*: *le-infI*)
qed

lemma *neg-inf-eq-sup*: − *inf a b* = *sup* (− *a*) (− *b*)
by (*simp add*: *inf-eq-neg-sup*)

lemma *diff-inf-eq-sup*: *a* − *inf b c* = *a* + *sup* (− *b*) (− *c*)
using *neg-inf-eq-sup* [*of b c*, *symmetric*] **by** *simp*

lemma *neg-sup-eq-inf*: − *sup a b* = *inf* (− *a*) (− *b*)
by (*simp add*: *sup-eq-neg-inf*)

lemma *diff-sup-eq-inf*: $a - sup\ b\ c = a + inf\ (-b)\ (-c)$
using *neg-sup-eq-inf* [*of b c*, *symmetric*] **by** *simp*

lemma *add-eq-inf-sup*: $a + b = sup\ a\ b + inf\ a\ b$
proof −
have $0 = -\ inf\ 0\ (a - b) + inf\ (a - b)\ 0$
by (*simp add*: *inf-commute*)
then have $0 = sup\ 0\ (b - a) + inf\ (a - b)\ 0$
by (*simp add*: *inf-eq-neg-sup*)
then have $0 = (-\ a + sup\ a\ b) + (inf\ a\ b + (-\ b))$
by (*simp only*: *add-sup-distrib-left add-inf-distrib-right*) *simp*
then show *?thesis*
by (*simp add*: *algebra-simps*)
qed

41.1 Positive Part, Negative Part, Absolute Value

definition *nprt* :: $'a \Rightarrow 'a$
where $nprt\ x = inf\ x\ 0$

definition *pprt* :: $'a \Rightarrow 'a$
where $pprt\ x = sup\ x\ 0$

lemma *pprt-neg*: $pprt\ (-x) = -\ nprt\ x$
proof −
have $sup\ (-x)\ 0 = sup\ (-x)\ (-0)$
by (*simp only*: *minus-zero*)
also have $\ldots = -\ inf\ x\ 0$
by (*simp only*: *neg-inf-eq-sup*)
finally have $sup\ (-x)\ 0 = -\ inf\ x\ 0$ **.**
then show *?thesis*
by (*simp only*: *pprt-def nprt-def*)
qed

lemma *nprt-neg*: $nprt\ (-x) = -\ pprt\ x$
proof −
from *pprt-neg* **have** $pprt\ (-\ (-x)) = -\ nprt\ (-x)$ **.**
then have $pprt\ x = -\ nprt\ (-x)$ **by** *simp*
then show *?thesis* **by** *simp*
qed

lemma *prts*: $a = pprt\ a + nprt\ a$
by (*simp add*: *pprt-def nprt-def flip*: *add-eq-inf-sup*)

lemma *zero-le-pprt*[*simp*]: $0 \leq pprt\ a$
by (*simp add*: *pprt-def*)

lemma *nprt-le-zero*[*simp*]: $nprt\ a \leq 0$
by (*simp add*: *nprt-def*)

lemma *le-eq-neg*: $a \leq - b \longleftrightarrow a + b \leq 0$
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
show *?rhs*
by (*rule add-le-imp-le-right*[*of - uminus b -*]) (*simp add*: *add.assoc* ‹*?lhs*›)
next
assume *?rhs*
show *?lhs*
by (*rule add-le-imp-le-right*[*of - b -*]) (*simp add*: ‹*?rhs*›)
qed

lemma *pprt-0*[*simp*]: *pprt 0 = 0* **by** (*simp add*: *pprt-def*)
lemma *nprt-0*[*simp*]: *nprt 0 = 0* **by** (*simp add*: *nprt-def*)

lemma *pprt-eq-id* [*simp*, *no-atp*]: $0 \leq x \Longrightarrow pprt\ x = x$
by (*simp add*: *pprt-def sup-absorb1*)

lemma *nprt-eq-id* [*simp*, *no-atp*]: $x \leq 0 \Longrightarrow nprt\ x = x$
by (*simp add*: *nprt-def inf-absorb1*)

lemma *pprt-eq-0* [*simp*, *no-atp*]: $x \leq 0 \Longrightarrow pprt\ x = 0$
by (*simp add*: *pprt-def sup-absorb2*)

lemma *nprt-eq-0* [*simp*, *no-atp*]: $0 \leq x \Longrightarrow nprt\ x = 0$
by (*simp add*: *nprt-def inf-absorb2*)

lemma *sup-0-imp-0*:
assumes *sup a* $(-\ a) = 0$
shows $a = 0$
proof −
have *pos*: $0 \leq a$ **if** *sup a* $(-\ a) = 0$ **for** $a :: 'a$
proof −
from *that* **have** *sup a* $(-\ a) + a = a$
by *simp*
then have *sup* $(a + a)\ 0 = a$
by (*simp add*: *add-sup-distrib-right*)
then have *sup* $(a + a)\ 0 \leq a$
by *simp*
then show *?thesis*
by (*blast intro*: *order-trans inf-sup-ord*)
qed
from *assms* **have** $**$: *sup* $(-a)\ (-(-a)) = 0$
by (*simp add*: *sup-commute*)
from *pos*[*OF assms*] *pos*[*OF* $**$] **show** $a = 0$
by *simp*
qed

lemma *inf-0-imp-0*: *inf a (− a) = 0 ⟹ a = 0*
apply (*simp add*: *inf-eq-neg-sup*)
apply (*simp add*: *sup-commute*)
apply (*erule sup-0-imp-0*)
done

lemma *inf-0-eq-0* [*simp*, *no-atp*]: *inf a (− a) = 0 ⟷ a = 0*
apply (*rule iffI*)
apply (*erule inf-0-imp-0*)
apply *simp*
done

lemma *sup-0-eq-0* [*simp*, *no-atp*]: *sup a (− a) = 0 ⟷ a = 0*
apply (*rule iffI*)
apply (*erule sup-0-imp-0*)
apply *simp*
done

lemma *zero-le-double-add-iff-zero-le-single-add* [*simp*]: *0 ≤ a + a ⟷ 0 ≤ a*
(**is** *?lhs ⟷ ?rhs*)
proof
show *?rhs* **if** *?lhs*
proof −
from *that* **have** *a*: *inf (a + a) 0 = 0*
by (*simp add*: *inf-commute inf-absorb1*)
have *inf a 0 + inf a 0 = inf (inf (a + a) 0) a* (**is** *?l = -*)
by (*simp add*: *add-sup-inf-distribs inf-aci*)
then have *?l = 0 + inf a 0*
by (*simp add*: *a*, *simp add*: *inf-commute*)
then have *inf a 0 = 0*
by (*simp only*: *add-right-cancel*)
then show *?thesis*
unfolding *le-iff-inf* **by** (*simp add*: *inf-commute*)
qed
show *?lhs* **if** *?rhs*
by (*simp add*: *add-mono*[*OF that that*, *simplified*])
qed

lemma *double-zero* [*simp*]: *a + a = 0 ⟷ a = 0*
using *add-nonneg-eq-0-iff order.eq-iff* **by** *auto*

lemma *zero-less-double-add-iff-zero-less-single-add* [*simp*]: *0 < a + a ⟷ 0 < a*
by (*meson le-less-trans less-add-same-cancel2 less-le-not-le*
zero-le-double-add-iff-zero-le-single-add)

lemma *double-add-le-zero-iff-single-add-le-zero* [*simp*]: *a + a ≤ 0 ⟷ a ≤ 0*
proof −
have *a + a ≤ 0 ⟷ 0 ≤ − (a + a)*
by (*subst le-minus-iff*) *simp*

moreover have ... $\longleftrightarrow$ $a \leq 0$
by (*simp only*: *minus-add-distrib zero-le-double-add-iff-zero-le-single-add*) *simp*
ultimately show *?thesis*
by *blast*
qed

lemma *double-add-less-zero-iff-single-less-zero* [*simp*]: $a + a < 0 \longleftrightarrow a < 0$
proof −
have $a + a < 0 \longleftrightarrow 0 < -(a + a)$
by (*subst less-minus-iff*) *simp*
moreover have ... $\longleftrightarrow$ $a < 0$
by (*simp only*: *minus-add-distrib zero-less-double-add-iff-zero-less-single-add*) *simp*
ultimately show *?thesis*
by *blast*
qed

declare *neg-inf-eq-sup* [*simp*]
and *neg-sup-eq-inf* [*simp*]
and *diff-inf-eq-sup* [*simp*]
and *diff-sup-eq-inf* [*simp*]

lemma *le-minus-self-iff*: $a \leq -a \longleftrightarrow a \leq 0$
proof −
from *add-le-cancel-left* [*of uminus a plus a a zero*]
have $a \leq -a \longleftrightarrow a + a \leq 0$
by (*simp flip*: *add.assoc*)
then show *?thesis*
by *simp*
qed

lemma *minus-le-self-iff*: $-a \leq a \longleftrightarrow 0 \leq a$
proof −
have $-a \leq a \longleftrightarrow 0 \leq a + a$
using *add-le-cancel-left* [*of uminus a zero plus a a*]
by (*simp flip*: *add.assoc*)
then show *?thesis*
by *simp*
qed

lemma *zero-le-iff-zero-nprt*: $0 \leq a \longleftrightarrow nprt\ a = 0$
unfolding *le-iff-inf* **by** (*simp add*: *nprt-def inf-commute*)

lemma *le-zero-iff-zero-pprt*: $a \leq 0 \longleftrightarrow pprt\ a = 0$
unfolding *le-iff-sup* **by** (*simp add*: *pprt-def sup-commute*)

lemma *le-zero-iff-pprt-id*: $0 \leq a \longleftrightarrow pprt\ a = a$
unfolding *le-iff-sup* **by** (*simp add*: *pprt-def sup-commute*)

lemma *zero-le-iff-nprt-id*: $a \leq 0 \longleftrightarrow nprt\ a = a$
unfolding *le-iff-inf* **by** (*simp add*: *nprt-def inf-commute*)

lemma *pprt-mono* [*simp*, *no-atp*]: $a \leq b \Longrightarrow pprt\ a \leq pprt\ b$
unfolding *le-iff-sup* **by** (*simp add*: *pprt-def sup-aci sup-assoc* [*symmetric*, *of a*])

lemma *nprt-mono* [*simp*, *no-atp*]: $a \leq b \Longrightarrow nprt\ a \leq nprt\ b$
unfolding *le-iff-inf* **by** (*simp add*: *nprt-def inf-aci inf-assoc* [*symmetric*, *of a*])

end

lemmas *add-sup-inf-distribs* =
add-inf-distrib-right add-inf-distrib-left add-sup-distrib-right add-sup-distrib-left

class *lattice-ab-group-add-abs* = *lattice-ab-group-add* + *abs* +
assumes *abs-lattice*: $|a| = sup\ a\ (-\ a)$
begin

lemma *abs-prts*: $|a| = pprt\ a - nprt\ a$
proof −
have $0 \leq |a|$
proof −
have *a*: $a \leq |a|$ **and** *b*: $-\ a \leq |a|$
by (*auto simp add*: *abs-lattice*)
show *?thesis*
by (*rule add-mono* [*OF a b*, *simplified*])
qed
then have $0 \leq sup\ a\ (-\ a)$
unfolding *abs-lattice* **.**
then have $sup\ (sup\ a\ (-\ a))\ 0 = sup\ a\ (-\ a)$
by (*rule sup-absorb1*)
then show *?thesis*
by (*simp add*: *add-sup-inf-distribs ac-simps pprt-def nprt-def abs-lattice*)
qed

subclass *ordered-ab-group-add-abs*
proof
have *abs-ge-zero* [*simp*]: $0 \leq |a|$ **for** *a*
proof −
have *a*: $a \leq |a|$ **and** *b*: $-\ a \leq |a|$
by (*auto simp add*: *abs-lattice*)
show $0 \leq |a|$
by (*rule add-mono* [*OF a b*, *simplified*])
qed
have *abs-leI*: $a \leq b \Longrightarrow -\ a \leq b \Longrightarrow |a| \leq b$ **for** *a b*
by (*simp add*: *abs-lattice le-supI*)
fix *a b*
show $0 \leq |a|$

```
    by simp
  show a ≤ |a|
    by (auto simp add: abs-lattice)
  show |−a| = |a|
    by (simp add: abs-lattice sup-commute)
  show − a ≤ b ⟹ |a| ≤ b if a ≤ b
    using that by (rule abs-leI)
  show |a + b| ≤ |a| + |b|
  proof −
    have g: |a| + |b| = sup (a + b) (sup (− a − b) (sup (− a + b) (a + (− b))))
      (is - = sup ?m ?n)
      by (simp add: abs-lattice add-sup-inf-distribs ac-simps)
    have a: a + b ≤ sup ?m ?n
      by simp
    have b: − a − b ≤ ?n
      by simp
    have c: ?n ≤ sup ?m ?n
      by simp
    from b c have d: − a − b ≤ sup ?m ?n
      by (rule order-trans)
    have e: − a − b = − (a + b)
      by simp
    from a d e have |a + b| ≤ sup ?m ?n
      apply −
      apply (drule abs-leI)
       apply (simp-all only: algebra-simps minus-add)
      apply (metis add-uminus-conv-diff d sup-commute uminus-add-conv-diff)
      done
    with g[symmetric] show ?thesis by simp
  qed
qed

end

lemma sup-eq-if:
  fixes a :: 'a::{lattice-ab-group-add,linorder}
  shows sup a (− a) = (if a < 0 then − a else a)
  using add-le-cancel-right [of a a − a, symmetric, simplified]
    and add-le-cancel-right [of −a a a, symmetric, simplified]
  by (auto simp: sup-max max.absorb1 max.absorb2)

lemma abs-if-lattice:
  fixes a :: 'a::{lattice-ab-group-add-abs,linorder}
  shows |a| = (if a < 0 then − a else a)
  by auto

lemma estimate-by-abs:
  fixes a b c :: 'a::lattice-ab-group-add-abs
  assumes a + b ≤ c
```

shows $a \leq c + |b|$
proof −
from *assms* **have** $a \leq c + (-\ b)$
by (*simp add*: *algebra-simps*)
have $-\ b \leq |b|$
by (*rule abs-ge-minus-self*)
then have $c + (-\ b) \leq c + |b|$
by (*rule add-left-mono*)
with ‹$a \leq c + (-\ b)$› **show** *?thesis*
by (*rule order-trans*)
qed

class *lattice-ring* = *ordered-ring* + *lattice-ab-group-add-abs*
begin

subclass *semilattice-inf-ab-group-add* **..**
subclass *semilattice-sup-ab-group-add* **..**

end

lemma *abs-le-mult*:
fixes *a b* :: *'a*::*lattice-ring*
shows $|a * b| \leq |a| * |b|$
proof −
let *?x* = *pprt a* ∗ *pprt b* − *pprt a* ∗ *nprt b* − *nprt a* ∗ *pprt b* + *nprt a* ∗ *nprt b*
let *?y* = *pprt a* ∗ *pprt b* + *pprt a* ∗ *nprt b* + *nprt a* ∗ *pprt b* + *nprt a* ∗ *nprt b*
have *a*: $|a| * |b| =$ *?x*
by (*simp only*: *abs-prts*[*of a*] *abs-prts*[*of b*] *algebra-simps*)
have *bh*: $u = a \Longrightarrow v = b \Longrightarrow$
$u * v =$ *pprt a* ∗ *pprt b* + *pprt a* ∗ *nprt b* +
nprt a ∗ *pprt b* + *nprt a* ∗ *nprt b* **for** *u v* :: *'a*
apply (*subst prts*[*of u*], *subst prts*[*of v*])
apply (*simp add*: *algebra-simps*)
done
note *b* = *this*[*OF refl*[*of a*] *refl*[*of b*]]
have *xy*: $-\ ?x \leq ?y$
apply *simp*
apply (*metis* (*full-types*) *add-increasing add-uminus-conv-diff*
lattice-ab-group-add-class.minus-le-self-iff minus-add-distrib mult-nonneg-nonneg
mult-nonpos-nonpos nprt-le-zero zero-le-pprt)
done
have *yx*: $?y \leq ?x$
apply *simp*
apply (*metis* (*full-types*) *add-nonpos-nonpos add-uminus-conv-diff*
lattice-ab-group-add-class.le-minus-self-iff minus-add-distrib mult-nonneg-nonpos
mult-nonpos-nonneg nprt-le-zero zero-le-pprt)
done
have *i1*: $a * b \leq |a| * |b|$
by (*simp only*: *a b yx*)

```
  have i2: − (|a| * |b|) ≤ a * b
    by (simp only: a b xy)
  show ?thesis
    apply (rule abs-leI)
    apply (simp add: i1)
    apply (simp add: i2[simplified minus-le-iff])
    done
qed

instance lattice-ring ⊆ ordered-ring-abs
proof
  fix a b :: 'a::lattice-ring
  assume a: (0 ≤ a ∨ a ≤ 0) ∧ (0 ≤ b ∨ b ≤ 0)
  show |a * b| = |a| * |b|
  proof −
    have s: (0 ≤ a * b) ∨ (a * b ≤ 0)
      apply auto
      apply (rule-tac split-mult-pos-le)
      apply (rule-tac contrapos-np[of a * b ≤ 0])
      apply simp
      apply (rule-tac split-mult-neg-le)
      using a
      apply blast
      done
    have mulprts: a * b = (pprt a + nprt a) * (pprt b + nprt b)
      by (simp flip: prts)
    show ?thesis
    proof (cases 0 ≤ a * b)
      case True
      then show ?thesis
        apply (simp-all add: mulprts abs-prts)
        using a
        apply (auto simp add:
          algebra-simps
          iffD1[OF zero-le-iff-zero-nprt] iffD1[OF le-zero-iff-zero-pprt]
          iffD1[OF le-zero-iff-pprt-id] iffD1[OF zero-le-iff-nprt-id])
        apply(drule (1) mult-nonneg-nonpos[of a b], simp)
        apply(drule (1) mult-nonneg-nonpos2[of b a], simp)
        done
    next
      case False
      with s have a * b ≤ 0
        by simp
      then show ?thesis
        apply (simp-all add: mulprts abs-prts)
        apply (insert a)
        apply (auto simp add: algebra-simps)
        apply(drule (1) mult-nonneg-nonneg[of a b],simp)
        apply(drule (1) mult-nonpos-nonpos[of a b],simp)
```

```
      done
   qed
 qed
qed

lemma mult-le-prts:
 fixes a b :: 'a::lattice-ring
 assumes a1 ≤ a
   and a ≤ a2
   and b1 ≤ b
   and b ≤ b2
 shows a * b ≤
   pprt a2 * pprt b2 + pprt a1 * nprt b2 + nprt a2 * pprt b1 + nprt a1 * nprt
b1
proof −
 have a * b = (pprt a + nprt a) * (pprt b + nprt b)
   by (subst prts[symmetric])+ simp
 then have a * b = pprt a * pprt b + pprt a * nprt b + nprt a * pprt b + nprt
a * nprt b
   by (simp add: algebra-simps)
 moreover have pprt a * pprt b ≤ pprt a2 * pprt b2
   by (simp-all add: assms mult-mono)
 moreover have pprt a * nprt b ≤ pprt a1 * nprt b2
 proof −
   have pprt a * nprt b ≤ pprt a * nprt b2
     by (simp add: mult-left-mono assms)
   moreover have pprt a * nprt b2 ≤ pprt a1 * nprt b2
     by (simp add: mult-right-mono-neg assms)
   ultimately show ?thesis
     by simp
 qed
 moreover have nprt a * pprt b ≤ nprt a2 * pprt b1
 proof −
   have nprt a * pprt b ≤ nprt a2 * pprt b
     by (simp add: mult-right-mono assms)
   moreover have nprt a2 * pprt b ≤ nprt a2 * pprt b1
     by (simp add: mult-left-mono-neg assms)
   ultimately show ?thesis
     by simp
 qed
 moreover have nprt a * nprt b ≤ nprt a1 * nprt b1
 proof −
   have nprt a * nprt b ≤ nprt a * nprt b1
     by (simp add: mult-left-mono-neg assms)
   moreover have nprt a * nprt b1 ≤ nprt a1 * nprt b1
     by (simp add: mult-right-mono-neg assms)
   ultimately show ?thesis
     by simp
 qed
```

```
  ultimately show ?thesis
    by − (rule add-mono | simp)+
qed

lemma mult-ge-prts:
  fixes a b :: 'a::lattice-ring
  assumes a1 ≤ a
    and a ≤ a2
    and b1 ≤ b
    and b ≤ b2
  shows a * b ≥
    nprt a1 * pprt b2 + nprt a2 * nprt b2 + pprt a1 * pprt b1 + pprt a2 * nprt
b1
proof −
  from assms have a1: − a2 ≤ −a
    by auto
  from assms have a2: − a ≤ −a1
    by auto
  from mult-le-prts[of − a2 − a − a1 b1 b b2,
    OF a1 a2 assms(3) assms(4), simplified nprt-neg pprt-neg]
  have le: − (a * b) ≤
    − nprt a1 * pprt b2 + − nprt a2 * nprt b2 +
    − pprt a1 * pprt b1 + − pprt a2 * nprt b1
    by simp
  then have − (− nprt a1 * pprt b2 + − nprt a2 * nprt b2 +
      − pprt a1 * pprt b1 + − pprt a2 * nprt b1) ≤ a * b
    by (simp only: minus-le-iff)
  then show ?thesis
    by (simp add: algebra-simps)
qed

instance int :: lattice-ring
proof
  show |k| = sup k (− k) for k :: int
    by (auto simp add: sup-int-def)
qed

instance real :: lattice-ring
proof
  show |a| = sup a (− a) for a :: real
    by (auto simp add: sup-real-def)
qed

end
```

42 Floating-Point Numbers

```
theory Float
imports Log-Nat Lattice-Algebras
```

begin

definition *float* = {*m* ∗ *2 powr e* | (*m* :: *int*) (*e* :: *int*). *True*}

typedef *float* = *float*
  morphisms *real-of-float float-of*
  unfolding *float-def* **by** *auto*

setup-lifting *type-definition-float*

declare *real-of-float* [*code-unfold*]

lemmas *float-of-inject*[*simp*]

declare [[*coercion real-of-float* :: *float* ⇒ *real*]]

lemma *real-of-float-eq*: *f1* = *f2* ⟷ *real-of-float f1* = *real-of-float f2* **for** *f1 f2* :: *float*
  unfolding *real-of-float-inject* **..**

declare *real-of-float-inverse*[*simp*] *float-of-inverse* [*simp*]
declare *real-of-float* [*simp*]

42.1 Real operations preserving the representation as floating point number

lemma *floatI*: *m* ∗ *2 powr e* = *x* ⟹ *x* ∈ *float* **for** *m e* :: *int*
  by (*auto simp*: *float-def*)

lemma *zero-float*[*simp*]: *0* ∈ *float*
  by (*auto simp*: *float-def*)

lemma *one-float*[*simp*]: *1* ∈ *float*
  by (*intro floatI*[*of 1 0*]) *simp*

lemma *numeral-float*[*simp*]: *numeral i* ∈ *float*
  by (*intro floatI*[*of numeral i 0*]) *simp*

lemma *neg-numeral-float*[*simp*]: − *numeral i* ∈ *float*
  by (*intro floatI*[*of* − *numeral i 0*]) *simp*

lemma *real-of-int-float*[*simp*]: *real-of-int x* ∈ *float* **for** *x* :: *int*
  by (*intro floatI*[*of x 0*]) *simp*

lemma *real-of-nat-float*[*simp*]: *real x* ∈ *float* **for** *x* :: *nat*
  by (*intro floatI*[*of x 0*]) *simp*

lemma *two-powr-int-float*[*simp*]: *2 powr* (*real-of-int i*) ∈ *float* **for** *i* :: *int*
  by (*intro floatI*[*of 1 i*]) *simp*

lemma *two-powr-nat-float*[*simp*]: *2 powr* (*real i*) ∈ *float* **for** *i* :: *nat*
by (*intro floatI*[*of 1 i*]) *simp*

lemma *two-powr-minus-int-float*[*simp*]: *2 powr* − (*real-of-int i*) ∈ *float* **for** *i* :: *int*
by (*intro floatI*[*of 1* −*i*]) *simp*

lemma *two-powr-minus-nat-float*[*simp*]: *2 powr* − (*real i*) ∈ *float* **for** *i* :: *nat*
by (*intro floatI*[*of 1* −*i*]) *simp*

lemma *two-powr-numeral-float*[*simp*]: *2 powr numeral i* ∈ *float*
by (*intro floatI*[*of 1 numeral i*]) *simp*

lemma *two-powr-neg-numeral-float*[*simp*]: *2 powr* − *numeral i* ∈ *float*
by (*intro floatI*[*of 1* − *numeral i*]) *simp*

lemma *two-pow-float*[*simp*]: *2* ^ *n* ∈ *float*
by (*intro floatI*[*of 1 n*]) (*simp add*: *powr-realpow*)

lemma *plus-float*[*simp*]: *r* ∈ *float* ⟹ *p* ∈ *float* ⟹ *r* + *p* ∈ *float*
unfolding *float-def*
proof (*safe*, *simp*)
have ∗: ∃(*m*::*int*) (*e*::*int*). *m1* ∗ *2 powr e1* + *m2* ∗ *2 powr e2* = *m* ∗ *2 powr e*
if *e1* ≤ *e2* **for** *e1 m1 e2 m2* :: *int*
proof −
from *that* **have** *m1* ∗ *2 powr e1* + *m2* ∗ *2 powr e2* = (*m1* + *m2* ∗ *2* ^ *nat* (*e2* − *e1*)) ∗ *2 powr e1*
by (*simp add*: *powr-diff field-simps flip*: *powr-realpow*)
then show *?thesis*
by *blast*
qed
fix *e1 m1 e2 m2* :: *int*
consider *e2* ≤ *e1* | *e1* ≤ *e2* **by** (*rule linorder-le-cases*)
then show ∃(*m*::*int*) (*e*::*int*). *m1* ∗ *2 powr e1* + *m2* ∗ *2 powr e2* = *m* ∗ *2 powr e*
proof *cases*
case *1*
from ∗[*OF this*, *of m2 m1*] **show** *?thesis*
by (*simp add*: *ac-simps*)
next
case *2*
then show *?thesis* **by** (*rule* ∗)
qed
qed

lemma *uminus-float*[*simp*]: *x* ∈ *float* ⟹ −*x* ∈ *float*
apply (*auto simp*: *float-def*)
apply *hypsubst-thin*

apply (*rename-tac m e*)
apply (*rule-tac x=−m* **in** *exI*)
apply (*rule-tac x=e* **in** *exI*)
apply (*simp add*: *field-simps*)
done

lemma *times-float*[*simp*]: $x \in \mathit{float} \Longrightarrow y \in \mathit{float} \Longrightarrow x * y \in \mathit{float}$
apply (*auto simp*: *float-def*)
apply *hypsubst-thin*
apply (*rename-tac mx my ex ey*)
apply (*rule-tac x=mx * my* **in** *exI*)
apply (*rule-tac x=ex + ey* **in** *exI*)
apply (*simp add*: *powr-add*)
done

lemma *minus-float*[*simp*]: $x \in \mathit{float} \Longrightarrow y \in \mathit{float} \Longrightarrow x - y \in \mathit{float}$
using *plus-float* [*of x − y*] **by** *simp*

lemma *abs-float*[*simp*]: $x \in \mathit{float} \Longrightarrow |x| \in \mathit{float}$
by (*cases x rule*: *linorder-cases*[*of 0*]) *auto*

lemma *sgn-of-float*[*simp*]: $x \in \mathit{float} \Longrightarrow \mathit{sgn}\ x \in \mathit{float}$
by (*cases x rule*: *linorder-cases*[*of 0*]) (*auto intro*!: *uminus-float*)

lemma *div-power-2-float*[*simp*]: $x \in \mathit{float} \Longrightarrow x\ /\ 2\hat{}d \in \mathit{float}$
apply (*auto simp add*: *float-def*)
apply *hypsubst-thin*
apply (*rename-tac m e*)
apply (*rule-tac x=m* **in** *exI*)
apply (*rule-tac x=e − d* **in** *exI*)
apply (*simp flip*: *powr-realpow powr-add add*: *field-simps*)
done

lemma *div-power-2-int-float*[*simp*]: $x \in \mathit{float} \Longrightarrow x\ /\ (2::\mathit{int})\hat{}d \in \mathit{float}$
apply (*auto simp add*: *float-def*)
apply *hypsubst-thin*
apply (*rename-tac m e*)
apply (*rule-tac x=m* **in** *exI*)
apply (*rule-tac x=e − d* **in** *exI*)
apply (*simp flip*: *powr-realpow powr-add add*: *field-simps*)
done

lemma *div-numeral-Bit0-float*[*simp*]:
assumes $x\ /\ \mathit{numeral}\ n \in \mathit{float}$
shows $x\ /\ (\mathit{numeral}\ (\mathit{Num.Bit0}\ n)) \in \mathit{float}$
proof −
have $(x\ /\ \mathit{numeral}\ n)\ /\ 2\hat{}1 \in \mathit{float}$
by (*intro assms div-power-2-float*)
also have $(x\ /\ \mathit{numeral}\ n)\ /\ 2\hat{}1 = x\ /\ (\mathit{numeral}\ (\mathit{Num.Bit0}\ n))$

by (*induct n*) *auto*
finally show *?thesis* **.**
qed

lemma *div-neg-numeral-Bit0-float*[*simp*]:
assumes x / *numeral* $n \in$ *float*
shows x / ($-$ *numeral* (*Num.Bit0* n)) $\in$ *float*
proof $-$
have $-$ (x / *numeral* (*Num.Bit0* n)) $\in$ *float*
using *assms* **by** *simp*
also have $-$ (x / *numeral* (*Num.Bit0* n)) = x / $-$ *numeral* (*Num.Bit0* n)
by *simp*
finally show *?thesis* **.**
qed

lemma *power-float*[*simp*]:
assumes $a \in$ *float*
shows $a \hat{} b \in$ *float*
proof $-$
from *assms* **obtain** m e :: *int* **where** $a = m * 2$ *powr* e
by (*auto simp*: *float-def*)
then show *?thesis*
by (*auto intro*!: *floatI*[**where** m=$m\hat{}b$ **and** $e = e*b$]
simp: *power-mult-distrib powr-realpow*[*symmetric*] *powr-powr*)
qed

lift-definition *Float* :: *int* $\Rightarrow$ *int* $\Rightarrow$ *float* **is** $\lambda(m{::}int)\ (e{::}int).\ m * 2$ *powr* e
by *simp*
declare *Float.rep-eq*[*simp*]

code-datatype *Float*

lemma *compute-real-of-float*[*code*]:
real-of-float (*Float* m e) = (*if* $e \geq 0$ *then* $m * 2 \hat{} nat\ e$ *else* m / $2 \hat{} (nat\ (-e))$)
by (*simp add*: *powr-int*)

42.2 Arithmetic operations on floating point numbers

instantiation *float* :: {*ring-1*,*linorder*,*linordered-ring*,*linordered-idom*,*numeral*,*equal*}
begin

lift-definition *zero-float* :: *float* **is** *0* **by** *simp*
declare *zero-float.rep-eq*[*simp*]

lift-definition *one-float* :: *float* **is** *1* **by** *simp*
declare *one-float.rep-eq*[*simp*]

lift-definition *plus-float* :: *float* $\Rightarrow$ *float* $\Rightarrow$ *float* **is** (+) **by** *simp*
declare *plus-float.rep-eq*[*simp*]

lift-definition *times-float* :: *float* ⇒ *float* ⇒ *float* **is** (∗) **by** *simp*
declare *times-float.rep-eq*[*simp*]

lift-definition *minus-float* :: *float* ⇒ *float* ⇒ *float* **is** (−) **by** *simp*
declare *minus-float.rep-eq*[*simp*]

lift-definition *uminus-float* :: *float* ⇒ *float* **is** *uminus* **by** *simp*
declare *uminus-float.rep-eq*[*simp*]

lift-definition *abs-float* :: *float* ⇒ *float* **is** *abs* **by** *simp*
declare *abs-float.rep-eq*[*simp*]

lift-definition *sgn-float* :: *float* ⇒ *float* **is** *sgn* **by** *simp*
declare *sgn-float.rep-eq*[*simp*]

lift-definition *equal-float* :: *float* ⇒ *float* ⇒ *bool* **is** (=) :: *real* ⇒ *real* ⇒ *bool* **.**

lift-definition *less-eq-float* :: *float* ⇒ *float* ⇒ *bool* **is** (≤) **.**
declare *less-eq-float.rep-eq*[*simp*]

lift-definition *less-float* :: *float* ⇒ *float* ⇒ *bool* **is** (<) **.**
declare *less-float.rep-eq*[*simp*]

instance
by *standard* (*transfer*; *fastforce simp add*: *field-simps intro*: *mult-left-mono mult-right-mono*)+

end

lemma *real-of-float* [*simp*]: *real-of-float* (*of-nat n*) = *of-nat n*
by (*induct n*) *simp-all*

lemma *real-of-float-of-int-eq* [*simp*]: *real-of-float* (*of-int z*) = *of-int z*
by (*cases z rule*: *int-diff-cases*) (*simp-all add*: *of-rat-diff*)

lemma *Float-0-eq-0*[*simp*]: *Float 0 e* = *0*
by *transfer simp*

lemma *real-of-float-power*[*simp*]: *real-of-float* (*f*^*n*) = *real-of-float f*^*n* **for** *f* :: *float*
by (*induct n*) *simp-all*

lemma *real-of-float-min*: *real-of-float* (*min x y*) = *min* (*real-of-float x*) (*real-of-float y*)
and *real-of-float-max*: *real-of-float* (*max x y*) = *max* (*real-of-float x*) (*real-of-float y*)
for *x y* :: *float*
by (*simp-all add*: *min-def max-def*)

instance *float* :: *unbounded-dense-linorder*

proof
  fix *a b* :: *float*
  show ∃ *c*. *a* < *c*
    apply (*intro exI*[*of* - *a* + *1*])
    apply *transfer*
    apply *simp*
    done
  show ∃ *c*. *c* < *a*
    apply (*intro exI*[*of* - *a* − *1*])
    apply *transfer*
    apply *simp*
    done
  show ∃ *c*. *a* < *c* ∧ *c* < *b* **if** *a* < *b*
    apply (*rule exI*[*of* - (*a* + *b*) ∗ *Float 1* (− *1*)])
    using *that*
    apply *transfer*
    apply (*simp add*: *powr-minus*)
    done
qed

instantiation *float* :: *lattice-ab-group-add*
begin

definition *inf-float* :: *float* ⇒ *float* ⇒ *float*
  where *inf-float a b* = *min a b*

definition *sup-float* :: *float* ⇒ *float* ⇒ *float*
  where *sup-float a b* = *max a b*

instance
 by *standard* (*transfer*; *simp add*: *inf-float-def sup-float-def real-of-float-min real-of-float-max*)+

end

lemma *float-numeral*[*simp*]: *real-of-float* (*numeral x* :: *float*) = *numeral x*
  apply (*induct x*)
  apply *simp*
  apply (*simp-all only*: *numeral-Bit0 numeral-Bit1 real-of-float-eq float-of-inverse*
        plus-float.rep-eq one-float.rep-eq plus-float numeral-float one-float)
  done

lemma *transfer-numeral* [*transfer-rule*]:
  rel-fun (=) *pcr-float* (*numeral* :: - ⇒ *real*) (*numeral* :: - ⇒ *float*)
  by (*simp add*: *rel-fun-def float.pcr-cr-eq cr-float-def*)

lemma *float-neg-numeral*[*simp*]: *real-of-float* (− *numeral x* :: *float*) = − *numeral*
x
  by *simp*

lemma *transfer-neg-numeral* [*transfer-rule*]:
rel-fun (=) *pcr-float* (− *numeral* :: - ⇒ *real*) (− *numeral* :: - ⇒ *float*)
by (*simp add*: *rel-fun-def float.pcr-cr-eq cr-float-def*)

lemma *float-of-numeral*: *numeral k* = *float-of* (*numeral k*)
and *float-of-neg-numeral*: − *numeral k* = *float-of* (− *numeral k*)
unfolding *real-of-float-eq* **by** *simp-all*

42.3 Quickcheck

instantiation *float* :: *exhaustive*
begin

definition *exhaustive-float* **where**
exhaustive-float f d =
Quickcheck-Exhaustive.exhaustive (λ*x*. *Quickcheck-Exhaustive.exhaustive* (λ*y*. *f* (*Float x y*)) *d*) *d*

instance ..

end

context
includes *term-syntax*
begin

definition [*code-unfold*]:
valtermify-float x y = *Code-Evaluation.valtermify Float* {·} *x* {·} *y*

end

instantiation *float* :: *full-exhaustive*
begin

definition
full-exhaustive-float f d =
Quickcheck-Exhaustive.full-exhaustive
(λ*x*. *Quickcheck-Exhaustive.full-exhaustive* (λ*y*. *f* (*valtermify-float x y*)) *d*) *d*

instance ..

end

instantiation *float* :: *random*
begin

definition *Quickcheck-Random.random i* =
scomp (*Quickcheck-Random.random* (2 ^ *nat-of-natural i*))
(λ*man*. *scomp* (*Quickcheck-Random.random i*) (λ*exp*. *Pair* (*valtermify-float*

man exp)))

instance ..

end

42.4 Represent floats as unique mantissa and exponent

lemma *int-induct-abs*[*case-names less*]:
fixes *j* :: *int*
assumes *H*: $\bigwedge n.\ (\bigwedge i.\ |i| < |n| \Longrightarrow P\ i) \Longrightarrow P\ n$
shows *P j*
proof (*induct nat* |*j*| *arbitrary*: *j rule*: *less-induct*)
case *less*
show *?case* **by** (*rule H*[*OF less*]) *simp*
qed

lemma *int-cancel-factors*:
fixes *n* :: *int*
assumes *1 < r*
shows $n = 0 \vee (\exists k\ i.\ n = k * r \mathbin{\hat{}} i \wedge \neg\ r\ dvd\ k)$
proof (*induct n rule*: *int-induct-abs*)
case (*less n*)
have $\exists k\ i.\ n = k * r \mathbin{\hat{}} Suc\ i \wedge \neg\ r\ dvd\ k$ **if** $n \neq 0$ $n = m * r$ **for** *m*
proof −
from *that* **have** $|m| < |n|$
using ‹*1 < r*› **by** (*simp add*: *abs-mult*)
from *less*[*OF this*] *that* **show** *?thesis* **by** *auto*
qed
then show *?case*
by (*metis dvd-def monoid-mult-class.mult.right-neutral mult.commute power-0*)
qed

lemma *mult-powr-eq-mult-powr-iff-asym*:
fixes *m1 m2 e1 e2* :: *int*
assumes *m1*: ¬ *2 dvd m1*
and *e1* ≤ *e2*
shows *m1 * 2 powr e1 = m2 * 2 powr e2* ⟷ *m1 = m2* ∧ *e1 = e2*
(**is** *?lhs* ⟷ *?rhs*)
proof
show *?rhs* **if** *eq*: *?lhs*
proof −
have *m1* ≠ *0*
using *m1* **unfolding** *dvd-def* **by** *auto*
from ‹*e1* ≤ *e2*› *eq* **have** *m1 = m2 * 2 powr nat* (*e2* − *e1*)
by (*simp add*: *powr-diff field-simps*)
also have ... = *m2 * 2*^*nat* (*e2* − *e1*)
by (*simp add*: *powr-realpow*)
finally have *m1-eq*: *m1 = m2 * 2*^*nat* (*e2* − *e1*)

```
    by linarith
  with m1 have m1 = m2
    by (cases nat (e2 − e1)) (auto simp add: dvd-def)
  then show ?thesis
    using eq ‹m1 ≠ 0› by (simp add: powr-inj)
 qed
 show ?lhs if ?rhs
  using that by simp
qed

lemma mult-powr-eq-mult-powr-iff:
 ¬ 2 dvd m1 ⟹ ¬ 2 dvd m2 ⟹ m1 * 2 powr e1 = m2 * 2 powr e2 ⟷ m1 =
m2 ∧ e1 = e2
 for m1 m2 e1 e2 :: int
 using mult-powr-eq-mult-powr-iff-asym[of m1 e1 e2 m2]
 using mult-powr-eq-mult-powr-iff-asym[of m2 e2 e1 m1]
 by (cases e1 e2 rule: linorder-le-cases) auto

lemma floatE-normed:
 assumes x: x ∈ float
 obtains (zero) x = 0
  | (powr) m e :: int where x = m * 2 powr e ¬ 2 dvd m x ≠ 0
proof −
 have ∃(m::int) (e::int). x = m * 2 powr e ∧ ¬ (2::int) dvd m if x ≠ 0
 proof −
  from x obtain m e :: int where x: x = m * 2 powr e
    by (auto simp: float-def)
  with ‹x ≠ 0› int-cancel-factors[of 2 m] obtain k i where m = k * 2 ^ i ¬ 2
dvd k
    by auto
  with ‹¬ 2 dvd k› x show ?thesis
    apply (rule-tac exI[of - k])
    apply (rule-tac exI[of - e + int i])
    apply (simp add: powr-add powr-realpow)
    done
 qed
 with that show thesis by blast
qed

lemma float-normed-cases:
 fixes f :: float
 obtains (zero) f = 0
  | (powr) m e :: int where real-of-float f = m * 2 powr e ¬ 2 dvd m f ≠ 0
proof (atomize-elim, induct f)
 case (float-of y)
 then show ?case
  by (cases rule: floatE-normed) (auto simp: zero-float-def)
qed
```

definition *mantissa* :: *float* ⇒ *int*
where *mantissa f* =
fst (*SOME p*::*int* × *int*. (*f* = *0* ∧ *fst p* = *0* ∧ *snd p* = *0*) ∨
(*f* ≠ *0* ∧ *real-of-float f* = *real-of-int* (*fst p*) ∗ *2 powr real-of-int* (*snd p*) ∧ ¬ *2 dvd fst p*))

definition *exponent* :: *float* ⇒ *int*
where *exponent f* =
snd (*SOME p*::*int* × *int*. (*f* = *0* ∧ *fst p* = *0* ∧ *snd p* = *0*) ∨
(*f* ≠ *0* ∧ *real-of-float f* = *real-of-int* (*fst p*) ∗ *2 powr real-of-int* (*snd p*) ∧ ¬ *2 dvd fst p*))

lemma *exponent-0*[*simp*]: *exponent 0* = *0* (**is** *?E*)
and *mantissa-0*[*simp*]: *mantissa 0* = *0* (**is** *?M*)
proof −
have ⋀*p*::*int* × *int*. *fst p* = *0* ∧ *snd p* = *0* ⟷ *p* = (*0*, *0*)
by *auto*
then show *?E ?M*
by (*auto simp add*: *mantissa-def exponent-def zero-float-def*)
qed

lemma *mantissa-exponent*: *real-of-float f* = *mantissa f* ∗ *2 powr exponent f* (**is** *?E*)
and *mantissa-not-dvd*: *f* ≠ *0* ⟹ ¬ *2 dvd mantissa f* (**is** - ⟹ *?D*)
proof *cases*
assume [*simp*]: *f* ≠ *0*
have *f* = *mantissa f* ∗ *2 powr exponent f* ∧ ¬ *2 dvd mantissa f*
proof (*cases f rule*: *float-normed-cases*)
case *zero*
then show *?thesis* **by** *simp*
next
case (*powr m e*)
then have ∃ *p*::*int* × *int*. (*f* = *0* ∧ *fst p* = *0* ∧ *snd p* = *0*) ∨
(*f* ≠ *0* ∧ *real-of-float f* = *real-of-int* (*fst p*) ∗ *2 powr real-of-int* (*snd p*) ∧ ¬ *2 dvd fst p*)
by *auto*
then show *?thesis*
unfolding *exponent-def mantissa-def*
by (*rule someI2-ex*) *simp*
qed
then show *?E ?D* **by** *auto*
qed *simp*

lemma *mantissa-noteq-0*: *f* ≠ *0* ⟹ *mantissa f* ≠ *0*
using *mantissa-not-dvd*[*of f*] **by** *auto*

lemma *mantissa-eq-zero-iff*: *mantissa x* = *0* ⟷ *x* = *0*
(**is** *?lhs* ⟷ *?rhs*)
proof

```
  show ?rhs if ?lhs
  proof −
    from that have z: 0 = real-of-float x
      using mantissa-exponent by simp
    show ?thesis
      by (simp add: zero-float-def z)
  qed
  show ?lhs if ?rhs
    using that by simp
qed

lemma mantissa-pos-iff: 0 < mantissa x ⟷ 0 < x
  by (auto simp: mantissa-exponent algebra-split-simps)

lemma mantissa-nonneg-iff: 0 ≤ mantissa x ⟷ 0 ≤ x
  by (auto simp: mantissa-exponent algebra-split-simps)

lemma mantissa-neg-iff: 0 > mantissa x ⟷ 0 > x
  by (auto simp: mantissa-exponent algebra-split-simps)

lemma
  fixes m e :: int
  defines f ≡ float-of (m * 2 powr e)
  assumes dvd: ¬ 2 dvd m
  shows mantissa-float: mantissa f = m (is ?M)
    and exponent-float: m ≠ 0 ⟹ exponent f = e (is - ⟹ ?E)
proof cases
  assume m = 0
  with dvd show mantissa f = m by auto
next
  assume m ≠ 0
  then have f-not-0: f ≠ 0 by (simp add: f-def zero-float-def)
  from mantissa-exponent[of f] have m * 2 powr e = mantissa f * 2 powr exponent
f
    by (auto simp add: f-def)
  then show ?M ?E
    using mantissa-not-dvd[OF f-not-0] dvd
    by (auto simp: mult-powr-eq-mult-powr-iff)
qed
```

42.5 Compute arithmetic operations

```
lemma Float-mantissa-exponent: Float (mantissa f) (exponent f) = f
  unfolding real-of-float-eq mantissa-exponent[of f] by simp

lemma Float-cases [cases type: float]:
  fixes f :: float
  obtains (Float) m e :: int where f = Float m e
  using Float-mantissa-exponent[symmetric]
```

```
  by (atomize-elim) auto

lemma denormalize-shift:
  assumes f-def: f = Float m e
    and not-0: f ≠ 0
  obtains i where m = mantissa f * 2 ^ i e = exponent f - i
proof
  from mantissa-exponent[of f] f-def
  have m * 2 powr e = mantissa f * 2 powr exponent f
    by simp
  then have eq: m = mantissa f * 2 powr (exponent f - e)
    by (simp add: powr-diff field-simps)
  moreover
  have e ≤ exponent f
  proof (rule ccontr)
    assume ¬ e ≤ exponent f
    then have pos: exponent f < e by simp
    then have 2 powr (exponent f - e) = 2 powr - real-of-int (e - exponent f)
      by simp
    also have ... = 1 / 2^nat (e - exponent f)
      using pos by (simp flip: powr-realpow add: powr-diff)
    finally have m * 2^nat (e - exponent f) = real-of-int (mantissa f)
      using eq by simp
    then have mantissa f = m * 2^nat (e - exponent f)
      by linarith
    with ‹exponent f < e› have 2 dvd mantissa f
      apply (intro dvdI[where k=m * 2^(nat (e-exponent f)) div 2])
      apply (cases nat (e - exponent f))
      apply auto
      done
    then show False using mantissa-not-dvd[OF not-0] by simp
  qed
  ultimately have real-of-int m = mantissa f * 2^nat (exponent f - e)
    by (simp flip: powr-realpow)
  with ‹e ≤ exponent f›
  show m = mantissa f * 2 ^ nat (exponent f - e)
    by linarith
  show e = exponent f - nat (exponent f - e)
    using ‹e ≤ exponent f› by auto
qed

context
begin

qualified lemma compute-float-zero[code-unfold, code]: 0 = Float 0 0
  by transfer simp

qualified lemma compute-float-one[code-unfold, code]: 1 = Float 1 0
  by transfer simp
```

lift-definition *normfloat* :: *float* ⇒ *float* **is** λ*x*. *x* **.**
lemma *normloat-id*[*simp*]: *normfloat x* = *x* **by** *transfer rule*

qualified lemma *compute-normfloat*[*code*]:
normfloat (*Float m e*) =
(*if m mod 2* = *0* ∧ *m* ≠ *0 then normfloat* (*Float* (*m div 2*) (*e* + *1*))
else if m = *0 then 0 else Float m e*)
by *transfer* (*auto simp add*: *powr-add zmod-eq-0-iff*)

qualified lemma *compute-float-numeral*[*code-abbrev*]: *Float* (*numeral k*) *0* = *numeral k*
by *transfer simp*

qualified lemma *compute-float-neg-numeral*[*code-abbrev*]: *Float* (− *numeral k*) *0* = − *numeral k*
by *transfer simp*

qualified lemma *compute-float-uminus*[*code*]: − *Float m1 e1* = *Float* (− *m1*) *e1*
by *transfer simp*

qualified lemma *compute-float-times*[*code*]: *Float m1 e1* ∗ *Float m2 e2* = *Float* (*m1* ∗ *m2*) (*e1* + *e2*)
by *transfer* (*simp add*: *field-simps powr-add*)

qualified lemma *compute-float-plus*[*code*]:
Float m1 e1 + *Float m2 e2* =
(*if m1* = *0 then Float m2 e2*
else if m2 = *0 then Float m1 e1*
else if e1 ≤ *e2 then Float* (*m1* + *m2* ∗ *2^nat* (*e2* − *e1*)) *e1*
else Float (*m2* + *m1* ∗ *2^nat* (*e1* − *e2*)) *e2*)
by *transfer* (*simp add*: *field-simps powr-realpow*[*symmetric*] *powr-diff*)

qualified lemma *compute-float-minus*[*code*]: *f* − *g* = *f* + (−*g*) **for** *f g* :: *float*
by *simp*

qualified lemma *compute-float-sgn*[*code*]:
sgn (*Float m1 e1*) = (*if 0* < *m1 then 1 else if m1* < *0 then* −*1 else 0*)
by *transfer* (*simp add*: *sgn-mult*)

lift-definition *is-float-pos* :: *float* ⇒ *bool* **is** (<) *0* :: *real* ⇒ *bool* **.**

qualified lemma *compute-is-float-pos*[*code*]: *is-float-pos* (*Float m e*) ⟷ *0* < *m*
by *transfer* (*auto simp add*: *zero-less-mult-iff not-le*[*symmetric*, *of - 0*])

lift-definition *is-float-nonneg* :: *float* ⇒ *bool* **is** (≤) *0* :: *real* ⇒ *bool* **.**

qualified lemma *compute-is-float-nonneg*[*code*]: *is-float-nonneg* (*Float m e*) ⟷ *0* ≤ *m*

by *transfer* (*auto simp add*: *zero-le-mult-iff not-less*[*symmetric*, *of - 0*])

lift-definition *is-float-zero* :: *float* ⇒ *bool* **is** (=) *0* :: *real* ⇒ *bool* **.**

qualified lemma *compute-is-float-zero*[*code*]: *is-float-zero* (*Float m e*) ⟷ *0* = *m*
by *transfer* (*auto simp add*: *is-float-zero-def*)

qualified lemma *compute-float-abs*[*code*]: |*Float m e*| = *Float* |*m*| *e*
by *transfer* (*simp add*: *abs-mult*)

qualified lemma *compute-float-eq*[*code*]: *equal-class.equal f g* = *is-float-zero* (*f* − *g*)
by *transfer simp*

end

42.6 Lemmas for types *real*, *nat*, *int*

lemmas *real-of-ints* =
of-int-add
of-int-minus
of-int-diff
of-int-mult
of-int-power
of-int-numeral of-int-neg-numeral

lemmas *int-of-reals* = *real-of-ints*[*symmetric*]

42.7 Rounding Real Numbers

definition *round-down* :: *int* ⇒ *real* ⇒ *real*
where *round-down prec x* = ⌊*x* ∗ *2 powr prec*⌋ ∗ *2 powr* −*prec*

definition *round-up* :: *int* ⇒ *real* ⇒ *real*
where *round-up prec x* = ⌈*x* ∗ *2 powr prec*⌉ ∗ *2 powr* −*prec*

lemma *round-down-float*[*simp*]: *round-down prec x* ∈ *float*
unfolding *round-down-def*
by (*auto intro*!: *times-float simp flip*: *of-int-minus*)

lemma *round-up-float*[*simp*]: *round-up prec x* ∈ *float*
unfolding *round-up-def*
by (*auto intro*!: *times-float simp flip*: *of-int-minus*)

lemma *round-up*: *x* ≤ *round-up prec x*
by (*simp add*: *powr-minus-divide le-divide-eq round-up-def ceiling-correct*)

lemma *round-down*: *round-down prec x* ≤ *x*
by (*simp add*: *powr-minus-divide divide-le-eq round-down-def*)

lemma *round-up-0*[*simp*]: *round-up p 0 = 0*
unfolding *round-up-def* **by** *simp*

lemma *round-down-0*[*simp*]: *round-down p 0 = 0*
unfolding *round-down-def* **by** *simp*

lemma *round-up-diff-round-down*: *round-up prec x − round-down prec x ≤ 2 powr −prec*
proof −
have *round-up prec x − round-down prec x = (⌈x * 2 powr prec⌉ − ⌊x * 2 powr prec⌋) * 2 powr −prec*
by (*simp add*: *round-up-def round-down-def field-simps*)
also have ... ≤ *1 * 2 powr −prec*
by (*rule mult-mono*)
(*auto simp flip*: *of-int-diff simp*: *ceiling-diff-floor-le-1*)
finally show *?thesis* **by** *simp*
qed

lemma *round-down-shift*: *round-down p (x * 2 powr k) = 2 powr k * round-down (p + k) x*
unfolding *round-down-def*
by (*simp add*: *powr-add powr-mult field-simps powr-diff*)
(*simp flip*: *powr-add*)

lemma *round-up-shift*: *round-up p (x * 2 powr k) = 2 powr k * round-up (p + k) x*
unfolding *round-up-def*
by (*simp add*: *powr-add powr-mult field-simps powr-diff*)
(*simp flip*: *powr-add*)

lemma *round-up-uminus-eq*: *round-up p (−x) = − round-down p x*
and *round-down-uminus-eq*: *round-down p (−x) = − round-up p x*
by (*auto simp*: *round-up-def round-down-def ceiling-def*)

lemma *round-up-mono*: *x ≤ y ⟹ round-up p x ≤ round-up p y*
by (*auto intro*!: *ceiling-mono simp*: *round-up-def*)

lemma *round-up-le1*:
assumes *x ≤ 1 prec ≥ 0*
shows *round-up prec x ≤ 1*
proof −
have *real-of-int ⌈x * 2 powr prec⌉ ≤ real-of-int ⌈2 powr real-of-int prec⌉*
using *assms* **by** (*auto intro*!: *ceiling-mono*)
also have ... = *2 powr prec* **using** *assms* **by** (*auto simp*: *powr-int intro*!: *exI*[**where** *x=2^nat prec*])
finally show *?thesis*
by (*simp add*: *round-up-def*) (*simp add*: *powr-minus inverse-eq-divide*)
qed

lemma *round-up-less1*:
assumes $x < 1 / 2\ p > 0$
shows *round-up p x* < 1
proof −
have $x * 2\ powr\ p < 1 / 2 * 2\ powr\ p$
using *assms* **by** *simp*
also have $\ldots \leq 2\ powr\ p - 1$ **using** ‹$p > 0$›
by (*auto simp*: *powr-diff powr-int field-simps self-le-power*)
finally show *?thesis* **using** ‹$p > 0$›
by (*simp add*: *round-up-def field-simps powr-minus powr-int ceiling-less-iff*)
qed

lemma *round-down-ge1*:
assumes *x*: $x \geq 1$
assumes *prec*: $p \geq - log\ 2\ x$
shows $1 \leq$ *round-down p x*
proof *cases*
assume *nonneg*: $0 \leq p$
have *2 powr p* = *real-of-int* $\lfloor 2\ powr\ real\text{-}of\text{-}int\ p \rfloor$
using *nonneg* **by** (*auto simp*: *powr-int*)
also have $\ldots \leq$ *real-of-int* $\lfloor x * 2\ powr\ p \rfloor$
using *assms* **by** (*auto intro*!: *floor-mono*)
finally show *?thesis*
by (*simp add*: *round-down-def*) (*simp add*: *powr-minus inverse-eq-divide*)
next
assume *neg*: $\neg\ 0 \leq p$
have $x =$ *2 powr* (*log 2 x*)
using *x* **by** *simp*
also have *2 powr* (*log 2 x*) $\geq$ *2 powr* $- p$
using *prec* **by** *auto*
finally have *x-le*: $x \geq$ *2 powr* $-p$ **.**

from *neg* **have** *2 powr real-of-int p* $\leq$ *2 powr 0*
by (*intro powr-mono*) *auto*
also have $\ldots \leq \lfloor$*2 powr 0*::*real*$\rfloor$ **by** *simp*
also have $\ldots \leq \lfloor x * 2\ powr$ (*real-of-int p*)$\rfloor$
unfolding *of-int-le-iff*
using *x x-le* **by** (*intro floor-mono*) (*simp add*: *powr-minus-divide field-simps*)
finally show *?thesis*
using *prec x*
by (*simp add*: *round-down-def powr-minus-divide pos-le-divide-eq*)
qed

lemma *round-up-le0*: $x \leq 0 \Longrightarrow$ *round-up p x* ≤ 0
unfolding *round-up-def*
by (*auto simp*: *field-simps mult-le-0-iff zero-le-mult-iff*)

42.8 Rounding Floats

definition *div-twopow* :: *int* ⇒ *nat* ⇒ *int*
  where [*simp*]: *div-twopow x n* = *x div* (*2* ^ *n*)

definition *mod-twopow* :: *int* ⇒ *nat* ⇒ *int*
  where [*simp*]: *mod-twopow x n* = *x mod* (*2* ^ *n*)

lemma *compute-div-twopow*[*code*]:
  div-twopow x n = (*if x* = *0* ∨ *x* = −*1* ∨ *n* = *0 then x else div-twopow* (*x div 2*) (*n* − *1*))
  by (*cases n*) (*auto simp*: *zdiv-zmult2-eq div-eq-minus1*)

lemma *compute-mod-twopow*[*code*]:
  mod-twopow x n = (*if n* = *0 then 0 else x mod 2* + *2* ∗ *mod-twopow* (*x div 2*) (*n* − *1*))
  by (*cases n*) (*auto simp*: *zmod-zmult2-eq*)

lift-definition *float-up* :: *int* ⇒ *float* ⇒ *float* **is** *round-up* **by** *simp*
declare *float-up.rep-eq*[*simp*]

lemma *round-up-correct*: *round-up e f* − *f* ∈ {*0..2 powr* −*e*}
  unfolding *atLeastAtMost-iff*
proof
  have *round-up e f* − *f* ≤ *round-up e f* − *round-down e f*
    using *round-down* **by** *simp*
  also have ... ≤ *2 powr* −*e*
    using *round-up-diff-round-down* **by** *simp*
  finally show *round-up e f* − *f* ≤ *2 powr* − (*real-of-int e*)
    by *simp*
qed (*simp add*: *algebra-simps round-up*)

lemma *float-up-correct*: *real-of-float* (*float-up e f*) − *real-of-float f* ∈ {*0..2 powr* −*e*}
  by *transfer* (*rule round-up-correct*)

lift-definition *float-down* :: *int* ⇒ *float* ⇒ *float* **is** *round-down* **by** *simp*
declare *float-down.rep-eq*[*simp*]

lemma *round-down-correct*: *f* − (*round-down e f*) ∈ {*0..2 powr* −*e*}
  unfolding *atLeastAtMost-iff*
proof
  have *f* − *round-down e f* ≤ *round-up e f* − *round-down e f*
    using *round-up* **by** *simp*
  also have ... ≤ *2 powr* −*e*
    using *round-up-diff-round-down* **by** *simp*
  finally show *f* − *round-down e f* ≤ *2 powr* − (*real-of-int e*)
    by *simp*
qed (*simp add*: *algebra-simps round-down*)

lemma *float-down-correct*: *real-of-float f − real-of-float (float-down e f) ∈ {0..2 powr −e}*
by *transfer (rule round-down-correct)*

context
begin

qualified lemma *compute-float-down[code]*:
float-down p (Float m e) =
(if p + e < 0 then Float (div-twopow m (nat (−(p + e)))) (−p) else Float m e)
proof *(cases p + e < 0)*
case *True*
then have *real-of-int ((2::int) ^ nat (−(p + e))) = 2 powr (−(p + e))*
using *powr-realpow[of 2 nat (−(p + e))]* **by** *simp*
also have *... = 1 / 2 powr p / 2 powr e*
unfolding *powr-minus-divide of-int-minus* **by** *(simp add: powr-add)*
finally show *?thesis*
using *‹p + e < 0›*
apply *transfer*
apply *(simp add: round-down-def field-simps flip: floor-divide-of-int-eq)*
apply *(metis (no-types, opaque-lifting) Float.rep-eq*
add.inverse-inverse compute-real-of-float diff-minus-eq-add
floor-divide-of-int-eq int-of-reals(1) linorder-not-le
minus-add-distrib of-int-eq-numeral-power-cancel-iff powr-add)
done
next
case *False*
then have *r: real-of-int e + real-of-int p = real (nat (e + p))*
by *simp*
have *r: ⌊(m * 2 powr e) * 2 powr real-of-int p⌋ = (m * 2 powr e) * 2 powr real-of-int p*
by *(auto intro: exI[**where** x=m*2^nat (e+p)]*
simp add: ac-simps powr-add[symmetric] r powr-realpow)
with *‹¬ p + e < 0›* **show** *?thesis*
by *transfer (auto simp add: round-down-def field-simps powr-add powr-minus)*
qed

lemma *abs-round-down-le*: *|f − (round-down e f)| ≤ 2 powr −e*
using *round-down-correct[of f e]* **by** *simp*

lemma *abs-round-up-le*: *|f − (round-up e f)| ≤ 2 powr −e*
using *round-up-correct[of e f]* **by** *simp*

lemma *round-down-nonneg*: *0 ≤ s ⟹ 0 ≤ round-down p s*
by *(auto simp: round-down-def)*

lemma *ceil-divide-floor-conv*:
assumes *b ≠ 0*
shows *⌈real-of-int a / real-of-int b⌉ =*

```
    (if b dvd a then a div b else ⌊real-of-int a / real-of-int b⌋ + 1)
proof (cases b dvd a)
  case True
  then show ?thesis
    by (simp add: ceiling-def floor-divide-of-int-eq dvd-neg-div
          flip: of-int-minus divide-minus-left)
next
  case False
  then have a mod b ≠ 0
    by auto
  then have ne: real-of-int (a mod b) / real-of-int b ≠ 0
    using ‹b ≠ 0› by auto
  have ⌈real-of-int a / real-of-int b⌉ = ⌊real-of-int a / real-of-int b⌋ + 1
    apply (rule ceiling-eq)
    apply (auto simp flip: floor-divide-of-int-eq)
  proof −
    have real-of-int ⌊real-of-int a / real-of-int b⌋ ≤ real-of-int a / real-of-int b
      by simp
     moreover have real-of-int ⌊real-of-int a / real-of-int b⌋ ≠ real-of-int a /
real-of-int b
      apply (subst (2) real-of-int-div-aux)
      unfolding floor-divide-of-int-eq
      using ne ‹b ≠ 0› apply auto
      done
    ultimately show real-of-int ⌊real-of-int a / real-of-int b⌋ < real-of-int a /
real-of-int b by arith
  qed
  then show ?thesis
    using ‹¬ b dvd a› by simp
qed

qualified lemma compute-float-up[code]: float-up p x = − float-down p (−x)
  by transfer (simp add: round-down-uminus-eq)

end


lemma bitlen-Float:
  fixes m e
  defines [THEN meta-eq-to-obj-eq]: f ≡ Float m e
  shows bitlen |mantissa f| + exponent f = (if m = 0 then 0 else bitlen |m| + e)
proof (cases m = 0)
  case True
  then show ?thesis by (simp add: f-def bitlen-alt-def)
next
  case False
  then have f ≠ 0
    unfolding real-of-float-eq by (simp add: f-def)
  then have mantissa f ≠ 0
```

```
    by (simp add: mantissa-eq-zero-iff)
  moreover
  obtain i where m = mantissa f * 2 ^ i e = exponent f - int i
    by (rule f-def[THEN denormalize-shift, OF ‹f ≠ 0›])
  ultimately show ?thesis by (simp add: abs-mult)
qed

lemma float-gt1-scale:
  assumes 1 ≤ Float m e
  shows 0 ≤ e + (bitlen m - 1)
proof -
  have 0 < Float m e using assms by auto
  then have 0 < m using powr-gt-zero[of 2 e]
    by (auto simp: zero-less-mult-iff)
  then have m ≠ 0 by auto
  show ?thesis
  proof (cases 0 ≤ e)
    case True
    then show ?thesis
      using ‹0 < m› by (simp add: bitlen-alt-def)
  next
    case False
    have (1::int) < 2 by simp
    let ?S = 2^(nat (-e))
    have inverse (2 ^ nat (- e)) = 2 powr e
      using assms False powr-realpow[of 2 nat (-e)]
      by (auto simp: powr-minus field-simps)
    then have 1 ≤ real-of-int m * inverse ?S
      using assms False powr-realpow[of 2 nat (-e)]
      by (auto simp: powr-minus)
    then have 1 * ?S ≤ real-of-int m * inverse ?S * ?S
      by (rule mult-right-mono) auto
    then have ?S ≤ real-of-int m
      unfolding mult.assoc by auto
    then have ?S ≤ m
      unfolding of-int-le-iff[symmetric] by auto
    from this bitlen-bounds[OF ‹0 < m›, THEN conjunct2]
    have nat (-e) < (nat (bitlen m))
      unfolding power-strict-increasing-iff[OF ‹1 < 2›, symmetric]
      by (rule order-le-less-trans)
    then have -e < bitlen m
      using False by auto
    then show ?thesis
      by auto
  qed
qed
```

42.9 Truncating Real Numbers

definition *truncate-down*::*nat* ⇒ *real* ⇒ *real*
where *truncate-down prec x* = *round-down* (*prec* − ⌊*log 2* |*x*|⌋) *x*

lemma *truncate-down*: *truncate-down prec x* ≤ *x*
using *round-down* **by** (*simp add*: *truncate-down-def*)

lemma *truncate-down-le*: *x* ≤ *y* ⟹ *truncate-down prec x* ≤ *y*
by (*rule order-trans*[*OF truncate-down*])

lemma *truncate-down-zero*[*simp*]: *truncate-down prec 0* = *0*
by (*simp add*: *truncate-down-def*)

lemma *truncate-down-float*[*simp*]: *truncate-down p x* ∈ *float*
by (*auto simp*: *truncate-down-def*)

definition *truncate-up*::*nat* ⇒ *real* ⇒ *real*
where *truncate-up prec x* = *round-up* (*prec* − ⌊*log 2* |*x*|⌋) *x*

lemma *truncate-up*: *x* ≤ *truncate-up prec x*
using *round-up* **by** (*simp add*: *truncate-up-def*)

lemma *truncate-up-le*: *x* ≤ *y* ⟹ *x* ≤ *truncate-up prec y*
by (*rule order-trans*[*OF* - *truncate-up*])

lemma *truncate-up-zero*[*simp*]: *truncate-up prec 0* = *0*
by (*simp add*: *truncate-up-def*)

lemma *truncate-up-uminus-eq*: *truncate-up prec* (−*x*) = − *truncate-down prec x*
and *truncate-down-uminus-eq*: *truncate-down prec* (−*x*) = − *truncate-up prec x*
by (*auto simp*: *truncate-up-def round-up-def truncate-down-def round-down-def ceiling-def*)

lemma *truncate-up-float*[*simp*]: *truncate-up p x* ∈ *float*
by (*auto simp*: *truncate-up-def*)

lemma *mult-powr-eq*: *0* < *b* ⟹ *b* ≠ *1* ⟹ *0* < *x* ⟹ *x* ∗ *b powr y* = *b powr* (*y* + *log b x*)
by (*simp-all add*: *powr-add*)

lemma *truncate-down-pos*:
assumes *x* > *0*
shows *truncate-down p x* > *0*
proof −
have *0* ≤ *log 2 x* − *real-of-int* ⌊*log 2 x*⌋
by (*simp add*: *algebra-simps*)
with *assms*
show *?thesis*
apply (*auto simp*: *truncate-down-def round-down-def mult-powr-eq*

```
    intro!: ge-one-powr-ge-zero mult-pos-pos)
  by linarith
qed

lemma truncate-down-nonneg: 0 ≤ y ⟹ 0 ≤ truncate-down prec y
  by (auto simp: truncate-down-def round-down-def)

lemma truncate-down-ge1: 1 ≤ x ⟹ 1 ≤ truncate-down p x
  apply (auto simp: truncate-down-def algebra-simps intro!: round-down-ge1)
  apply linarith
  done

lemma truncate-up-nonpos: x ≤ 0 ⟹ truncate-up prec x ≤ 0
  by (auto simp: truncate-up-def round-up-def intro!: mult-nonpos-nonneg)

lemma truncate-up-le1:
  assumes x ≤ 1
  shows truncate-up p x ≤ 1
proof −
  consider x ≤ 0 | x > 0
    by arith
  then show ?thesis
  proof cases
    case 1
    with truncate-up-nonpos[OF this, of p] show ?thesis
      by simp
  next
    case 2
    then have le: ⌊log 2 |x|⌋ ≤ 0
      using assms by (auto simp: log-less-iff)
    from assms have 0 ≤ int p by simp
    from add-mono[OF this le]
    show ?thesis
      using assms by (simp add: truncate-up-def round-up-le1 add-mono)
  qed
qed

lemma truncate-down-shift-int:
  truncate-down p (x * 2 powr real-of-int k) = truncate-down p x * 2 powr k
  by (cases x = 0)
    (simp-all add: algebra-simps abs-mult log-mult truncate-down-def
      round-down-shift[of - - k, simplified])

lemma truncate-down-shift-nat: truncate-down p (x * 2 powr real k) = truncate-down p x * 2 powr k
  by (metis of-int-of-nat-eq truncate-down-shift-int)

lemma truncate-up-shift-int: truncate-up p (x * 2 powr real-of-int k) = truncate-up p x * 2 powr k
```

by (*cases x = 0*)
(*simp-all add*: *algebra-simps abs-mult log-mult truncate-up-def*
round-up-shift[*of - - k*, *simplified*])

lemma *truncate-up-shift-nat*: *truncate-up p* (*x* ∗ *2 powr real k*) = *truncate-up p x* ∗ *2 powr k*
by (*metis of-int-of-nat-eq truncate-up-shift-int*)

42.10 Truncating Floats

lift-definition *float-round-up* :: *nat* ⇒ *float* ⇒ *float* **is** *truncate-up*
by (*simp add*: *truncate-up-def*)

lemma *float-round-up*: *real-of-float x* ≤ *real-of-float* (*float-round-up prec x*)
using *truncate-up* **by** *transfer simp*

lemma *float-round-up-zero*[*simp*]: *float-round-up prec 0 = 0*
by *transfer simp*

lift-definition *float-round-down* :: *nat* ⇒ *float* ⇒ *float* **is** *truncate-down*
by (*simp add*: *truncate-down-def*)

lemma *float-round-down*: *real-of-float* (*float-round-down prec x*) ≤ *real-of-float x*
using *truncate-down* **by** *transfer simp*

lemma *float-round-down-zero*[*simp*]: *float-round-down prec 0 = 0*
by *transfer simp*

lemmas *float-round-up-le* = *order-trans*[*OF - float-round-up*]
and *float-round-down-le* = *order-trans*[*OF float-round-down*]

lemma *minus-float-round-up-eq*: − *float-round-up prec x* = *float-round-down prec* (− *x*)
and *minus-float-round-down-eq*: − *float-round-down prec x* = *float-round-up prec* (− *x*)
by (*transfer*; *simp add*: *truncate-down-uminus-eq truncate-up-uminus-eq*)+

context
begin

qualified lemma *compute-float-round-down*[*code*]:
float-round-down prec (*Float m e*) =
(*let d* = *bitlen* |*m*| − *int prec* − *1 in*
if 0 < *d then Float* (*div-twopow m* (*nat d*)) (*e* + *d*)
else Float m e)
using *Float.compute-float-down*[*of Suc prec* − *bitlen* |*m*| − *e m e*, *symmetric*]
by *transfer*
(*simp add*: *field-simps abs-mult log-mult bitlen-alt-def truncate-down-def*
cong del: *if-weak-cong*)

qualified lemma *compute-float-round-up*[*code*]:
float-round-up prec x = − *float-round-down prec* (−*x*)
by *transfer* (*simp add*: *truncate-down-uminus-eq*)

end

lemma *truncate-up-nonneg-mono*:
assumes *0* ≤ *x x* ≤ *y*
shows *truncate-up prec x* ≤ *truncate-up prec y*
proof −
consider ⌊*log 2 x*⌋ = ⌊*log 2 y*⌋ | ⌊*log 2 x*⌋ ≠ ⌊*log 2 y*⌋ *0* < *x* | *x* ≤ *0*
by *arith*
then show *?thesis*
proof *cases*
case *1*
then show *?thesis*
using *assms*
by (*auto simp*: *truncate-up-def round-up-def intro*!: *ceiling-mono*)
next
case *2*
from *assms* ‹*0* < *x*› **have** *log 2 x* ≤ *log 2 y*
by *auto*
with ‹⌊*log 2 x*⌋ ≠ ⌊*log 2 y*⌋›
have *logless*: *log 2 x* < *log 2 y*
by *linarith*
have *flogless*: ⌊*log 2 x*⌋ < ⌊*log 2 y*⌋
using ‹⌊*log 2 x*⌋ ≠ ⌊*log 2 y*⌋› ‹*log 2 x* ≤ *log 2 y*› **by** *linarith*
have *truncate-up prec x* =
real-of-int ⌈*x* ∗ *2 powr real-of-int* (*int prec* − ⌊*log 2 x*⌋)⌉ ∗ *2 powr* − *real-of-int* (*int prec* − ⌊*log 2 x*⌋)
using *assms* **by** (*simp add*: *truncate-up-def round-up-def*)
also have ⌈*x* ∗ *2 powr real-of-int* (*int prec* − ⌊*log 2 x*⌋)⌉ ≤ (*2* ^ (*Suc prec*))
proof (*simp only*: *ceiling-le-iff*)
have *x* ∗ *2 powr real-of-int* (*int prec* − ⌊*log 2 x*⌋) ≤
x ∗ (*2 powr real* (*Suc prec*) / (*2 powr log 2 x*))
using *real-of-int-floor-add-one-ge*[*of log 2 x*] *assms*
by (*auto simp*: *algebra-simps simp flip*: *powr-diff intro*!: *mult-left-mono*)
then show *x* ∗ *2 powr real-of-int* (*int prec* − ⌊*log 2 x*⌋) ≤ *real-of-int* ((*2*::*int*) ^ (*Suc prec*))
using ‹*0* < *x*› **by** (*simp add*: *powr-realpow powr-add*)
qed
then have *real-of-int* ⌈*x* ∗ *2 powr real-of-int* (*int prec* − ⌊*log 2 x*⌋)⌉ ≤ *2 powr int* (*Suc prec*)
by (*auto simp*: *powr-realpow powr-add*)
(*metis power-Suc of-int-le-numeral-power-cancel-iff*)
also
have *2 powr* − *real-of-int* (*int prec* − ⌊*log 2 x*⌋) ≤ *2 powr* − *real-of-int* (*int prec* − ⌊*log 2 y*⌋ + *1*)

```
      using logless flogless by (auto intro!: floor-mono)
    also have 2 powr real-of-int (int (Suc prec)) ≤
        2 powr (log 2 y + real-of-int (int prec − ⌊log 2 y⌋ + 1))
      using assms ‹0 < x›
      by (auto simp: algebra-simps)
    finally have truncate-up prec x ≤
      2 powr (log 2 y + real-of-int (int prec − ⌊log 2 y⌋ + 1)) * 2 powr − real-of-int
(int prec − ⌊log 2 y⌋ + 1)
      by simp
   also have ... = 2 powr (log 2 y + real-of-int (int prec − ⌊log 2 y⌋) − real-of-int
(int prec − ⌊log 2 y⌋))
      by (subst powr-add[symmetric]) simp
    also have ... = y
      using ‹0 < x› assms
      by (simp add: powr-add)
    also have ... ≤ truncate-up prec y
      by (rule truncate-up)
    finally show ?thesis .
  next
    case 3
    then show ?thesis
      using assms
      by (auto intro!: truncate-up-le)
  qed
qed

lemma truncate-up-switch-sign-mono:
  assumes x ≤ 0 0 ≤ y
  shows truncate-up prec x ≤ truncate-up prec y
proof −
  note truncate-up-nonpos[OF ‹x ≤ 0›]
  also note truncate-up-le[OF ‹0 ≤ y›]
  finally show ?thesis .
qed

lemma truncate-down-switch-sign-mono:
  assumes x ≤ 0
    and 0 ≤ y
    and x ≤ y
  shows truncate-down prec x ≤ truncate-down prec y
proof −
  note truncate-down-le[OF ‹x ≤ 0›]
  also note truncate-down-nonneg[OF ‹0 ≤ y›]
  finally show ?thesis .
qed

lemma truncate-down-nonneg-mono:
  assumes 0 ≤ x x ≤ y
  shows truncate-down prec x ≤ truncate-down prec y
```

```
proof −
  consider x ≤ 0 | ⌊log 2 |x|⌋ = ⌊log 2 |y|⌋ |
    0 < x ⌊log 2 |x|⌋ ≠ ⌊log 2 |y|⌋
    by arith
  then show ?thesis
  proof cases
    case 1
    with assms have x = 0 0 ≤ y by simp-all
    then show ?thesis
      by (auto intro!: truncate-down-nonneg)
  next
    case 2
    then show ?thesis
      using assms
      by (auto simp: truncate-down-def round-down-def intro!: floor-mono)
  next
    case 3
    from ‹0 < x› have log 2 x ≤ log 2 y 0 < y 0 ≤ y
      using assms by auto
    with ‹⌊log 2 |x|⌋ ≠ ⌊log 2 |y|⌋›
    have logless: log 2 x < log 2 y and flogless: ⌊log 2 x⌋ < ⌊log 2 y⌋
      unfolding atomize-conj abs-of-pos[OF ‹0 < x›] abs-of-pos[OF ‹0 < y›]
      by (metis floor-less-cancel linorder-cases not-le)
    have 2 powr prec ≤ y * 2 powr real prec / (2 powr log 2 y)
      using ‹0 < y› by simp
    also have ... ≤ y * 2 powr real (Suc prec) / (2 powr (real-of-int ⌊log 2 y⌋ +
1))
      using ‹0 ≤ y› ‹0 ≤ x› assms(2)
      by (auto intro!: powr-mono divide-left-mono
          simp: of-nat-diff powr-add powr-diff)
    also have ... = y * 2 powr real (Suc prec) / (2 powr real-of-int ⌊log 2 y⌋ * 2)
      by (auto simp: powr-add)
    finally have (2 ^ prec) ≤ ⌊y * 2 powr real-of-int (int (Suc prec) − ⌊log 2 |y|⌋
− 1)⌋
      using ‹0 ≤ y›
      by (auto simp: powr-diff le-floor-iff powr-realpow powr-add)
    then have (2 ^ (prec)) * 2 powr − real-of-int (int prec − ⌊log 2 |y|⌋) ≤
truncate-down prec y
      by (auto simp: truncate-down-def round-down-def)
    moreover have x ≤ (2 ^ prec) * 2 powr − real-of-int (int prec − ⌊log 2 |y|⌋)
    proof −
      have x = 2 powr (log 2 |x|) using ‹0 < x› by simp
      also have ... ≤ (2 ^ (Suc prec )) * 2 powr − real-of-int (int prec − ⌊log 2
|x|⌋)
        using real-of-int-floor-add-one-ge[of log 2 |x|] ‹0 < x›
      by (auto simp flip: powr-realpow powr-add simp: algebra-simps powr-mult-base
le-powr-iff)
      also
      have 2 powr − real-of-int (int prec − ⌊log 2 |x|⌋) ≤ 2 powr − real-of-int (int
```

prec − ⌊*log* 2 |*y*|⌋ + 1)
using *logless flogless* ‹*x* > 0› ‹*y* > 0›
by (*auto intro*!: *floor-mono*)
finally show *?thesis*
by (*auto simp flip*: *powr-realpow simp*: *powr-diff assms of-nat-diff*)
qed
ultimately show *?thesis*
by (*metis dual-order.trans truncate-down*)
qed
qed

lemma *truncate-down-eq-truncate-up*: *truncate-down p x* = − *truncate-up p* (−*x*)
and *truncate-up-eq-truncate-down*: *truncate-up p x* = − *truncate-down p* (−*x*)
by (*auto simp*: *truncate-up-uminus-eq truncate-down-uminus-eq*)

lemma *truncate-down-mono*: *x* ≤ *y* ⟹ *truncate-down p x* ≤ *truncate-down p y*
apply (*cases* 0 ≤ *x*)
apply (*rule truncate-down-nonneg-mono*, *assumption*+)
apply (*simp add*: *truncate-down-eq-truncate-up*)
apply (*cases* 0 ≤ *y*)
apply (*auto intro*: *truncate-up-nonneg-mono truncate-up-switch-sign-mono*)
done

lemma *truncate-up-mono*: *x* ≤ *y* ⟹ *truncate-up p x* ≤ *truncate-up p y*
by (*simp add*: *truncate-up-eq-truncate-down truncate-down-mono*)

lemma *truncate-up-nonneg*: 0 ≤ *truncate-up p x* **if** 0 ≤ *x*
by (*simp add*: *that truncate-up-le*)

lemma *truncate-up-pos*: 0 < *truncate-up p x* **if** 0 < *x*
by (*meson less-le-trans that truncate-up*)

lemma *truncate-up-less-zero-iff*[*simp*]: *truncate-up p x* < 0 ⟷ *x* < 0
proof −
have *f1*: ∀ *n r*. *truncate-up n r* + *truncate-down n* (− 1 ∗ *r*) = 0
by (*simp add*: *truncate-down-uminus-eq*)
have *f2*: (∀ *v0 v1*. *truncate-up v0 v1* + *truncate-down v0* (− 1 ∗ *v1*) = 0) =
(∀ *v0 v1*. *truncate-up v0 v1* = − 1 ∗ *truncate-down v0* (− 1 ∗ *v1*))
by (*auto simp*: *truncate-up-eq-truncate-down*)
have *f3*: ∀ *x1*. ((0::*real*) < *x1*) = (¬ *x1* ≤ 0)
by *fastforce*
have (− 1 ∗ *x* ≤ 0) = (0 ≤ *x*)
by *force*
then have 0 ≤ *x* ∨ ¬ *truncate-down p* (− 1 ∗ *x*) ≤ 0
using *f3* **by** (*meson truncate-down-pos*)
then have (0 ≤ *truncate-up p x*) ≠ (¬ 0 ≤ *x*)
using *f2 f1 truncate-up-nonneg* **by** *force*
then show *?thesis*
by *linarith*

qed

lemma *truncate-up-nonneg-iff*[*simp*]: *truncate-up p x* $\geq$ *0* $\longleftrightarrow$ *x* $\geq$ *0*
using *truncate-up-less-zero-iff*[*of p x*] *truncate-up-nonneg*[*of x*]
by *linarith*

lemma *truncate-down-less-zero-iff*[*simp*]: *truncate-down p x* $<$ *0* $\longleftrightarrow$ *x* $<$ *0*
by (*metis le-less-trans not-less-iff-gr-or-eq truncate-down truncate-down-pos truncate-down-zero*)

lemma *truncate-down-nonneg-iff*[*simp*]: *truncate-down p x* $\geq$ *0* $\longleftrightarrow$ *x* $\geq$ *0*
using *truncate-down-less-zero-iff*[*of p x*] *truncate-down-nonneg*[*of x p*]
by *linarith*

lemma *truncate-down-eq-zero-iff*[*simp*]: *truncate-down prec x* = *0* $\longleftrightarrow$ *x* = *0*
by (*metis not-less-iff-gr-or-eq truncate-down-less-zero-iff truncate-down-pos truncate-down-zero*)

lemma *truncate-up-eq-zero-iff*[*simp*]: *truncate-up prec x* = *0* $\longleftrightarrow$ *x* = *0*
by (*metis not-less-iff-gr-or-eq truncate-up-less-zero-iff truncate-up-pos truncate-up-zero*)

42.11 Approximation of positive rationals

lemma *div-mult-twopow-eq*: *a div* ((*2*::*nat*) ^ *n*) *div b* = *a div* (*b* $*$ *2* ^ *n*) **for** *a b* :: *nat*
by (*cases b* = *0*) (*simp-all add*: *div-mult2-eq*[*symmetric*] *ac-simps*)

lemma *real-div-nat-eq-floor-of-divide*: *a div b* = *real-of-int* $\lfloor$*a* / *b*$\rfloor$ **for** *a b* :: *nat*
by (*simp add*: *floor-divide-of-nat-eq* [*of a b*])

definition *rat-precision prec x y* =
(*let d* = *bitlen x* − *bitlen y*
in int prec − *d* + (*if Float* (*abs x*) *0* < *Float* (*abs y*) *d then 1 else 0*))

lemma *floor-log-divide-eq*:
assumes *i* > *0 j* > *0 p* > *1*
shows $\lfloor$*log p* (*i* / *j*)$\rfloor$ = *floor* (*log p i*) − *floor* (*log p j*) −
(*if i* $\geq$ *j* $*$ *p powr* (*floor* (*log p i*) − *floor* (*log p j*)) *then 0 else 1*)
proof −
let *?l* = *log p*
let *?fl* = λ*x*. *floor* (*?l x*)
have $\lfloor$*?l* (*i* / *j*)$\rfloor$ = $\lfloor$*?l i* − *?l j*$\rfloor$ **using** *assms*
by (*auto simp*: *log-divide*)
also have ... = *floor* (*real-of-int* (*?fl i* − *?fl j*) + (*?l i* − *?fl i* − (*?l j* − *?fl j*)))
(**is** - = *floor* (- + *?r*))
by (*simp add*: *algebra-simps*)
also note *floor-add2*
also note ‹*p* > *1*›
note *powr* = *powr-le-cancel-iff*[*symmetric*, *OF* ‹*1* < *p*›, *THEN iffD2*]

```
  note powr-strict = powr-less-cancel-iff[symmetric, OF ‹1 < p›, THEN iffD2]
  have floor ?r = (if i ≥ j * p powr (?fl i − ?fl j) then 0 else −1) (is - = ?if)
    using assms
    by (linarith |
      auto
        intro!: floor-eq2
        intro: powr-strict powr
        simp: powr-diff powr-add field-split-simps algebra-simps)+
  finally
  show ?thesis by simp
qed

lemma truncate-down-rat-precision:
    truncate-down prec (real x / real y) = round-down (rat-precision prec x y) (real
x / real y)
  and truncate-up-rat-precision:
    truncate-up prec (real x / real y) = round-up (rat-precision prec x y) (real x /
real y)
  unfolding truncate-down-def truncate-up-def rat-precision-def
  by (cases x; cases y) (auto simp: floor-log-divide-eq algebra-simps bitlen-alt-def)

lift-definition lapprox-posrat :: nat ⇒ nat ⇒ nat ⇒ float
  is λprec (x::nat) (y::nat). truncate-down prec (x / y)
  by simp

context
begin

qualified lemma compute-lapprox-posrat[code]:
  lapprox-posrat prec x y =
   (let
      l = rat-precision prec x y;
      d = if 0 ≤ l then x * 2^nat l div y else x div 2^nat (− l) div y
    in normfloat (Float d (− l)))
    unfolding div-mult-twopow-eq
    by transfer
      (simp add: round-down-def powr-int real-div-nat-eq-floor-of-divide field-simps
Let-def
        truncate-down-rat-precision del: two-powr-minus-int-float)

end

lift-definition rapprox-posrat :: nat ⇒ nat ⇒ nat ⇒ float
  is λprec (x::nat) (y::nat). truncate-up prec (x / y)
  by simp

context
begin
```

qualified lemma *compute-rapprox-posrat*[*code*]:
fixes *prec x y*
defines $l \equiv$ *rat-precision prec x y*
shows *rapprox-posrat prec x y* =
(*let*
$l = l$;
$(r, s) =$ *if* $0 \leq l$ *then* $(x * 2\hat{}nat\ l, y)$ *else* $(x, y * 2\hat{}nat(-l))$;
$d = r$ *div* s;
$m = r$ *mod* s
in normfloat (*Float* $(d +$ (*if* $m = 0 \vee y = 0$ *then* 0 *else* 1)) $(-l)$))
proof (*cases* $y = 0$)
assume $y = 0$
then show *?thesis* **by** *transfer simp*
next
assume $y \neq 0$
show *?thesis*
proof (*cases* $0 \leq l$)
case *True*
define x' **where** $x' = x * 2 \hat{} nat\ l$
have *int* $x * 2 \hat{} nat\ l = x'$
by (*simp add*: x'*-def*)
moreover have *real* $x * 2$ *powr* $l =$ *real* x'
by (*simp flip*: *powr-realpow add*: ‹$0 \leq l$› x'*-def*)
ultimately show *?thesis*
using *ceil-divide-floor-conv*[*of y* x'] *powr-realpow*[*of 2 nat l*] ‹$0 \leq l$› ‹$y \neq 0$›
l-def[*symmetric*, *THEN meta-eq-to-obj-eq*]
apply *transfer*
apply (*auto simp add*: *round-up-def truncate-up-rat-precision*)
apply (*metis dvd-triv-left of-nat-dvd-iff*)
apply (*metis floor-divide-of-int-eq of-int-of-nat-eq*)
done
next
case *False*
define y' **where** $y' = y * 2 \hat{} nat\ (-l)$
from ‹$y \neq 0$› **have** $y' \neq 0$ **by** (*simp add*: y'*-def*)
have *int* $y * 2 \hat{} nat\ (-l) = y'$
by (*simp add*: y'*-def*)
moreover have *real* $x *$ *real-of-int* (*2*::*int*) *powr real-of-int l* / *real* $y = x$ / *real* y'
using ‹$\neg\ 0 \leq l$› **by** (*simp flip*: *powr-realpow add*: *powr-minus* y'*-def field-simps*)
ultimately show *?thesis*
using *ceil-divide-floor-conv*[*of* y' *x*] ‹$\neg\ 0 \leq l$› ‹$y' \neq 0$› ‹$y \neq 0$›
l-def[*symmetric*, *THEN meta-eq-to-obj-eq*]
apply *transfer*
apply (*auto simp add*: *round-up-def ceil-divide-floor-conv truncate-up-rat-precision*)
apply (*metis dvd-triv-left of-nat-dvd-iff*)
apply (*metis floor-divide-of-int-eq of-int-of-nat-eq*)
done
qed

qed

end

lemma *rat-precision-pos*:
  assumes $0 \leq x$
    and $0 < y$
    and $2 * x < y$
  shows *rat-precision n* (*int x*) (*int y*) > 0
proof −
  have $0 < x \Longrightarrow$ *log 2 x* + *1* = *log 2* (*2* ∗ *x*)
    by (*simp add*: *log-mult*)
  then have *bitlen* (*int x*) < *bitlen* (*int y*)
    using *assms*
    by (*simp add*: *bitlen-alt-def*)
      (*auto intro*!: *floor-mono simp add*: *one-add-floor*)
  then show *?thesis*
    using *assms*
    by (*auto intro*!: *pos-add-strict simp add*: *field-simps rat-precision-def*)
qed

lemma *rapprox-posrat-less1*:
  $0 \leq x \Longrightarrow 0 < y \Longrightarrow 2 * x < y \Longrightarrow$ *real-of-float* (*rapprox-posrat n x y*) < *1*
 by *transfer* (*simp add*: *rat-precision-pos round-up-less1 truncate-up-rat-precision*)

lift-definition *lapprox-rat* :: *nat* ⇒ *int* ⇒ *int* ⇒ *float* **is**
  λ*prec* (*x*::*int*) (*y*::*int*). *truncate-down prec* (*x* / *y*)
  by *simp*

context
begin

qualified lemma *compute-lapprox-rat*[*code*]:
  lapprox-rat prec x y =
   (*if y* = *0 then 0*
    else if 0 ≤ *x then*
     (*if 0* < *y then lapprox-posrat prec* (*nat x*) (*nat y*)
      else − (*rapprox-posrat prec* (*nat x*) (*nat* (−*y*))))
      else
       (*if 0* < *y*
        then − (*rapprox-posrat prec* (*nat* (−*x*)) (*nat y*))
        else lapprox-posrat prec (*nat* (−*x*)) (*nat* (−*y*))))
  by *transfer* (*simp add*: *truncate-up-uminus-eq*)

lift-definition *rapprox-rat* :: *nat* ⇒ *int* ⇒ *int* ⇒ *float* **is**
  λ*prec* (*x*::*int*) (*y*::*int*). *truncate-up prec* (*x* / *y*)
  by *simp*

lemma *rapprox-rat* = *rapprox-posrat*

by *transfer auto*

lemma *lapprox-rat = lapprox-posrat*
by *transfer auto*

qualified lemma *compute-rapprox-rat*[*code*]:
rapprox-rat prec x y = − lapprox-rat prec (−x) y
by *transfer* (*simp add*: *truncate-down-uminus-eq*)

qualified lemma *compute-truncate-down*[*code*]:
truncate-down p (*Ratreal r*) = (*let* (*a*, *b*) = *quotient-of r in lapprox-rat p a b*)
by *transfer* (*auto split*: *prod.split simp*: *of-rat-divide dest*!: *quotient-of-div*)

qualified lemma *compute-truncate-up*[*code*]:
truncate-up p (*Ratreal r*) = (*let* (*a*, *b*) = *quotient-of r in rapprox-rat p a b*)
by *transfer* (*auto split*: *prod.split simp*: *of-rat-divide dest*!: *quotient-of-div*)

end

42.12 Division

definition *real-divl prec a b = truncate-down prec* (*a* / *b*)

definition *real-divr prec a b = truncate-up prec* (*a* / *b*)

lift-definition *float-divl* :: *nat* ⇒ *float* ⇒ *float* ⇒ *float* **is** *real-divl*
by (*simp add*: *real-divl-def*)

context
begin

qualified lemma *compute-float-divl*[*code*]:
float-divl prec (*Float m1 s1*) (*Float m2 s2*) = *lapprox-rat prec m1 m2* ∗ *Float 1* (*s1* − *s2*)
apply *transfer*
unfolding *real-divl-def of-int-1 mult-1 truncate-down-shift-int*[*symmetric*]
apply (*simp add*: *powr-diff powr-minus*)
done

lift-definition *float-divr* :: *nat* ⇒ *float* ⇒ *float* ⇒ *float* **is** *real-divr*
by (*simp add*: *real-divr-def*)

qualified lemma *compute-float-divr*[*code*]:
float-divr prec x y = − float-divl prec (−x) y
by *transfer* (*simp add*: *real-divr-def real-divl-def truncate-down-uminus-eq*)

end

42.13 Approximate Addition

definition *plus-down prec x y* = *truncate-down prec* $(x + y)$

definition *plus-up prec x y* = *truncate-up prec* $(x + y)$

lemma *float-plus-down-float*[*intro*, *simp*]: $x \in$ *float* $\Longrightarrow$ $y \in$ *float* $\Longrightarrow$ *plus-down p x y* $\in$ *float*
by (*simp add*: *plus-down-def*)

lemma *float-plus-up-float*[*intro*, *simp*]: $x \in$ *float* $\Longrightarrow$ $y \in$ *float* $\Longrightarrow$ *plus-up p x y* $\in$ *float*
by (*simp add*: *plus-up-def*)

lift-definition *float-plus-down* :: *nat* $\Rightarrow$ *float* $\Rightarrow$ *float* $\Rightarrow$ *float* **is** *plus-down* **..**

lift-definition *float-plus-up* :: *nat* $\Rightarrow$ *float* $\Rightarrow$ *float* $\Rightarrow$ *float* **is** *plus-up* **..**

lemma *plus-down*: *plus-down prec x y* $\leq x + y$
and *plus-up*: $x + y \leq$ *plus-up prec x y*
by (*auto simp*: *plus-down-def truncate-down plus-up-def truncate-up*)

lemma *float-plus-down*: *real-of-float* (*float-plus-down prec x y*) $\leq x + y$
and *float-plus-up*: $x + y \leq$ *real-of-float* (*float-plus-up prec x y*)
by (*transfer*; *rule plus-down plus-up*)+

lemmas *plus-down-le* = *order-trans*[*OF plus-down*]
and *plus-up-le* = *order-trans*[*OF - plus-up*]
and *float-plus-down-le* = *order-trans*[*OF float-plus-down*]
and *float-plus-up-le* = *order-trans*[*OF - float-plus-up*]

lemma *compute-plus-up*[*code*]: *plus-up p x y* = $-$ *plus-down p* $(-x)$ $(-y)$
using *truncate-down-uminus-eq*[*of p x* + *y*]
by (*auto simp*: *plus-down-def plus-up-def*)

lemma *truncate-down-log2-eqI*:
assumes $\lfloor$*log 2* $|x|\rfloor = \lfloor$*log 2* $|y|\rfloor$
assumes $\lfloor x * 2$ *powr* $(p - \lfloor$*log 2* $|x|\rfloor)\rfloor = \lfloor y * 2$ *powr* $(p - \lfloor$*log 2* $|x|\rfloor)\rfloor$
shows *truncate-down p x* = *truncate-down p y*
using *assms* **by** (*auto simp*: *truncate-down-def round-down-def*)

lemma *sum-neq-zeroI*:
$|a| \geq k \Longrightarrow |b| < k \Longrightarrow a + b \neq 0$
$|a| > k \Longrightarrow |b| \leq k \Longrightarrow a + b \neq 0$
for *a k* :: *real*
by *auto*

lemma *abs-real-le-2-powr-bitlen*[*simp*]: $|$*real-of-int m2*$| <$ *2 powr real-of-int* (*bitlen* $|m2|$)
proof (*cases m2* = *0*)

```
  case True
  then show ?thesis by simp
next
  case False
  then have |m2| < 2 ^ nat (bitlen |m2|)
    using bitlen-bounds[of |m2|]
    by (auto simp: powr-add bitlen-nonneg)
  then show ?thesis
    by (metis bitlen-nonneg powr-int of-int-abs of-int-less-numeral-power-cancel-iff
        zero-less-numeral)
qed

lemma floor-sum-times-2-powr-sgn-eq:
  fixes ai p q :: int
    and a b :: real
  assumes a * 2 powr p = ai
    and b-le-1: |b * 2 powr (p + 1)| ≤ 1
    and leqp: q ≤ p
  shows ⌊(a + b) * 2 powr q⌋ = ⌊(2 * ai + sgn b) * 2 powr (q - p - 1)⌋
proof -
  consider b = 0 | b > 0 | b < 0 by arith
  then show ?thesis
  proof cases
    case 1
    then show ?thesis
      by (simp flip: assms(1) powr-add add: algebra-simps powr-mult-base)
  next
    case 2
    then have b * 2 powr p < |b * 2 powr (p + 1)|
      by simp
    also note b-le-1
    finally have b-less-1: b * 2 powr real-of-int p < 1 .

    from b-less-1 ‹b > 0› have floor-eq: ⌊b * 2 powr real-of-int p⌋ = 0 ⌊sgn b /
2⌋ = 0
      by (simp-all add: floor-eq-iff)

    have ⌊(a + b) * 2 powr q⌋ = ⌊(a + b) * 2 powr p * 2 powr (q - p)⌋
      by (simp add: algebra-simps flip: powr-realpow powr-add)
    also have ... = ⌊(ai + b * 2 powr p) * 2 powr (q - p)⌋
      by (simp add: assms algebra-simps)
    also have ... = ⌊(ai + b * 2 powr p) / real-of-int ((2::int) ^ nat (p - q))⌋
      using assms
      by (simp add: algebra-simps divide-powr-uminus flip: powr-realpow powr-add)
    also have ... = ⌊ai / real-of-int ((2::int) ^ nat (p - q))⌋
      by (simp del: of-int-power add: floor-divide-real-eq-div floor-eq)
    finally have ⌊(a + b) * 2 powr real-of-int q⌋ = ⌊real-of-int ai / real-of-int
((2::int) ^ nat (p - q))⌋ .
    moreover
```

```
    have ⌊(2 * ai + (sgn b)) * 2 powr (real-of-int (q - p) - 1)⌋ =
        ⌊real-of-int ai / real-of-int ((2::int) ^ nat (p - q))⌋
    proof -
      have ⌊(2 * ai + sgn b) * 2 powr (real-of-int (q - p) - 1)⌋ = ⌊(ai + sgn b /
2) * 2 powr (q - p)⌋
        by (subst powr-diff) (simp add: field-simps)
      also have ... = ⌊(ai + sgn b / 2) / real-of-int ((2::int) ^ nat (p - q))⌋
        using leqp by (simp flip: powr-realpow add: powr-diff)
      also have ... = ⌊ai / real-of-int ((2::int) ^ nat (p - q))⌋
        by (simp del: of-int-power add: floor-divide-real-eq-div floor-eq)
      finally show ?thesis .
    qed
    ultimately show ?thesis by simp
  next
    case 3
    then have floor-eq: ⌊b * 2 powr (real-of-int p + 1)⌋ = -1
      using b-le-1
        by (auto simp: floor-eq-iff algebra-simps pos-divide-le-eq[symmetric] abs-if
divide-powr-uminus
        intro!: mult-neg-pos split: if-split-asm)
    have ⌊(a + b) * 2 powr q⌋ = ⌊(2*a + 2*b) * 2 powr p * 2 powr (q - p - 1)⌋
      by (simp add: algebra-simps powr-mult-base flip: powr-realpow powr-add)
    also have ... = ⌊(2 * (a * 2 powr p) + 2 * b * 2 powr p) * 2 powr (q - p -
1)⌋
      by (simp add: algebra-simps)
    also have ... = ⌊(2 * ai + b * 2 powr (p + 1)) / 2 powr (1 - q + p)⌋
     using assms by (simp add: algebra-simps powr-mult-base divide-powr-uminus)
    also have ... = ⌊(2 * ai + b * 2 powr (p + 1)) / real-of-int ((2::int) ^ nat
(p - q + 1))⌋
      using assms by (simp add: algebra-simps flip: powr-realpow)
    also have ... = ⌊(2 * ai - 1) / real-of-int ((2::int) ^ nat (p - q + 1))⌋
      using ‹b < 0› assms
      by (simp add: floor-divide-of-int-eq floor-eq floor-divide-real-eq-div
        del: of-int-mult of-int-power of-int-diff)
    also have ... = ⌊(2 * ai - 1) * 2 powr (q - p - 1)⌋
    using assms by (simp add: algebra-simps divide-powr-uminus flip: powr-realpow)
    finally show ?thesis
      using ‹b < 0› by simp
  qed
qed

lemma log2-abs-int-add-less-half-sgn-eq:
  fixes ai :: int
    and b :: real
  assumes |b| ≤ 1/2
    and ai ≠ 0
  shows ⌊log 2 |real-of-int ai + b|⌋ = ⌊log 2 |ai + sgn b / 2|⌋
proof (cases b = 0)
  case True
```

```
  then show ?thesis by simp
next
  case False
  define k where k = ⌊log 2 |ai|⌋
  then have ⌊log 2 |ai|⌋ = k
    by simp
  then have k: 2 powr k ≤ |ai| |ai| < 2 powr (k + 1)
    by (simp-all add: floor-log-eq-powr-iff ‹ai ≠ 0›)
  have k ≥ 0
    using assms by (auto simp: k-def)
  define r where r = |ai| − 2 ^ nat k
  have r: 0 ≤ r r < 2 powr k
    using ‹k ≥ 0› k
    by (auto simp: r-def k-def algebra-simps powr-add abs-if powr-int)
  then have r ≤ (2::int) ^ nat k − 1
    using ‹k ≥ 0› by (auto simp: powr-int)
  from this[simplified of-int-le-iff[symmetric]] ‹0 ≤ k›
  have r-le: r ≤ 2 powr k − 1
    by (auto simp: algebra-simps powr-int)
      (metis of-int-1 of-int-add of-int-le-numeral-power-cancel-iff)

  have |ai| = 2 powr k + r
    using ‹k ≥ 0› by (auto simp: k-def r-def simp flip: powr-realpow)

  have pos: |b| < 1 ⟹ 0 < 2 powr k + (r + b) for b :: real
    using ‹0 ≤ k› ‹ai ≠ 0›
    by (auto simp add: r-def powr-realpow[symmetric] abs-if sgn-if algebra-simps
      split: if-split-asm)
  have less: |sgn ai * b| < 1
    and less': |sgn (sgn ai * b) / 2| < 1
    using ‹|b| ≤ -› by (auto simp: abs-if sgn-if split: if-split-asm)

  have floor-eq: ⋀b::real. |b| ≤ 1 / 2 ⟹
      ⌊log 2 (1 + (r + b) / 2 powr k)⌋ = (if r = 0 ∧ b < 0 then −1 else 0)
    using ‹k ≥ 0› r r-le
    by (auto simp: floor-log-eq-powr-iff powr-minus-divide field-simps sgn-if)

  from ‹real-of-int |ai| = -› have |ai + b| = 2 powr k + (r + sgn ai * b)
    using ‹|b| ≤ -› ‹0 ≤ k› r
    by (auto simp add: sgn-if abs-if)
  also have ⌊log 2 ...⌋ = ⌊log 2 (2 powr k + r + sgn (sgn ai * b) / 2)⌋
  proof −
    have 2 powr k + (r + (sgn ai) * b) = 2 powr k * (1 + (r + sgn ai * b) / 2
powr k)
      by (simp add: field-simps)
    also have ⌊log 2 ...⌋ = k + ⌊log 2 (1 + (r + sgn ai * b) / 2 powr k)⌋
      using pos[OF less]
      by (subst log-mult) (simp-all add: log-mult powr-mult field-simps)
    also
```

```
    let ?if = if r = 0 ∧ sgn ai * b < 0 then −1 else 0
    have ⌊log 2 (1 + (r + sgn ai * b) / 2 powr k)⌋ = ?if
      using ‹|b| ≤ -›
      by (intro floor-eq) (auto simp: abs-mult sgn-if)
    also
    have ... = ⌊log 2 (1 + (r + sgn (sgn ai * b) / 2) / 2 powr k)⌋
      by (subst floor-eq) (auto simp: sgn-if)
    also have k + ... = ⌊log 2 (2 powr k * (1 + (r + sgn (sgn ai * b) / 2) / 2
powr k))⌋
      unfolding int-add-floor
      using pos[OF less'] ‹|b| ≤ -›
      by (simp add: field-simps add-log-eq-powr del: floor-add2)
    also have 2 powr k * (1 + (r + sgn (sgn ai * b) / 2) / 2 powr k) =
        2 powr k + r + sgn (sgn ai * b) / 2
      by (simp add: sgn-if field-simps)
    finally show ?thesis .
  qed
  also have 2 powr k + r + sgn (sgn ai * b) / 2 = |ai + sgn b / 2|
    unfolding ‹real-of-int |ai| = -›[symmetric] using ‹ai ≠ 0›
    by (auto simp: abs-if sgn-if algebra-simps)
  finally show ?thesis .
qed

context
begin

qualified lemma compute-far-float-plus-down:
  fixes m1 e1 m2 e2 :: int
    and p :: nat
  defines k1 ≡ Suc p − nat (bitlen |m1|)
  assumes H: bitlen |m2| ≤ e1 − e2 − k1 − 2 m1 ≠ 0 m2 ≠ 0 e1 ≥ e2
  shows float-plus-down p (Float m1 e1) (Float m2 e2) =
    float-round-down p (Float (m1 * 2 ^ (Suc (Suc k1)) + sgn m2) (e1 − int k1
− 2))
proof −
  let ?a = real-of-float (Float m1 e1)
  let ?b = real-of-float (Float m2 e2)
  let ?sum = ?a + ?b
  let ?shift = real-of-int e2 − real-of-int e1 + real k1 + 1
  let ?m1 = m1 * 2 ^ Suc k1
  let ?m2 = m2 * 2 powr ?shift
  let ?m2' = sgn m2 / 2
  let ?e = e1 − int k1 − 1

  have sum-eq: ?sum = (?m1 + ?m2) * 2 powr ?e
    by (auto simp flip: powr-add powr-mult powr-realpow simp: powr-mult-base
algebra-simps)

  have |?m2| * 2 < 2 powr (bitlen |m2| + ?shift + 1)
```

by (*auto simp*: *field-simps powr-add powr-mult-base powr-diff abs-mult*)
also have ... ≤ *2 powr 0*
using *H* **by** (*intro powr-mono*) *auto*
finally have *abs-m2-less-half*: |*?m2*| < *1 / 2*
by *simp*

then have |*real-of-int m2*| < *2 powr* −(*?shift* + *1*)
unfolding *powr-minus-divide* **by** (*auto simp*: *bitlen-alt-def field-simps powr-mult-base abs-mult*)
also have ... ≤ *2 powr real-of-int* (*e1* − *e2* − *2*)
by *simp*
finally have *b-less-quarter*: |*?b*| < *1/4* ∗ *2 powr real-of-int e1*
by (*simp add*: *powr-add field-simps powr-diff abs-mult*)
also have *1/4* < |*real-of-int m1*| / *2* **using** ‹*m1* ≠ *0*› **by** *simp*
finally have *b-less-half-a*: |*?b*| < *1/2* ∗ |*?a*|
by (*simp add*: *algebra-simps powr-mult-base abs-mult*)
then have *a-half-less-sum*: |*?a*| / *2* < |*?sum*|
by (*auto simp*: *field-simps abs-if split*: *if-split-asm*)

from *b-less-half-a* **have** |*?b*| < |*?a*| |*?b*| ≤ |*?a*|
by *simp-all*

have |*real-of-float* (*Float m1 e1*)| ≥ *1/4* ∗ *2 powr real-of-int e1*
using ‹*m1* ≠ *0*›
by (*auto simp*: *powr-add powr-int bitlen-nonneg divide-right-mono abs-mult*)
then have *?sum* ≠ *0* **using** *b-less-quarter*
by (*rule sum-neq-zeroI*)
then have *?m1* + *?m2* ≠ *0*
unfolding *sum-eq* **by** (*simp add*: *abs-mult zero-less-mult-iff*)

have |*real-of-int ?m1*| ≥ *2* ^ *Suc k1* |*?m2'*| < *2* ^ *Suc k1*
using ‹*m1* ≠ *0*› ‹*m2* ≠ *0*› **by** (*auto simp*: *sgn-if less-1-mult abs-mult simp del*: *power.simps*)
then have *sum'-nz*: *?m1* + *?m2'* ≠ *0*
by (*intro sum-neq-zeroI*)

have ⌊*log 2* |*real-of-float* (*Float m1 e1*) + *real-of-float* (*Float m2 e2*)|⌋ = ⌊*log 2* |*?m1* + *?m2*|⌋ + *?e*
using ‹*?m1* + *?m2* ≠ *0*›
unfolding *floor-add*[*symmetric*] *sum-eq*
by (*simp add*: *abs-mult log-mult*) *linarith*
also have ⌊*log 2* |*?m1* + *?m2*|⌋ = ⌊*log 2* |*?m1* + *sgn* (*real-of-int m2* ∗ *2 powr ?shift*) / *2*|⌋
using *abs-m2-less-half* ‹*m1* ≠ *0*›
by (*intro log2-abs-int-add-less-half-sgn-eq*) (*auto simp*: *abs-mult*)
also have *sgn* (*real-of-int m2* ∗ *2 powr ?shift*) = *sgn m2*
by (*auto simp*: *sgn-if zero-less-mult-iff less-not-sym*)
also
have |*?m1* + *?m2'*| ∗ *2 powr ?e* = |*?m1* ∗ *2* + *sgn m2*| ∗ *2 powr* (*?e* − *1*)

by (*auto simp*: *field-simps powr-minus*[*symmetric*] *powr-diff powr-mult-base*)
then have $\lfloor log\ 2\ |?m1 + ?m2'|\rfloor + ?e = \lfloor log\ 2\ |real\text{-}of\text{-}float\ (Float\ (?m1 * 2 + sgn\ m2)\ (?e - 1))|\rfloor$
using ‹$?m1 + ?m2' \neq 0$›
unfolding *floor-add-int*
by (*simp add*: *log-add-eq-powr abs-mult-pos del*: *floor-add2*)
finally
have $\lfloor log\ 2\ |?sum|\rfloor = \lfloor log\ 2\ |real\text{-}of\text{-}float\ (Float\ (?m1*2 + sgn\ m2)\ (?e - 1))|\rfloor$
.
then have *plus-down p* (*Float m1 e1*) (*Float m2 e2*) =
truncate-down p (*Float* (*?m1*2 + sgn m2*) (*?e − 1*))
unfolding *plus-down-def*
proof (*rule truncate-down-log2-eqI*)
let $?f = (int\ p - \lfloor log\ 2\ |real\text{-}of\text{-}float\ (Float\ m1\ e1) + real\text{-}of\text{-}float\ (Float\ m2\ e2)|\rfloor)$
let $?ai = m1 * 2\ \widehat{}\ (Suc\ k1)$
have $\lfloor(?a + ?b) * 2\ powr\ real\text{-}of\text{-}int\ ?f\rfloor = \lfloor(real\text{-}of\text{-}int\ (2 * ?ai) + sgn\ ?b) * 2\ powr\ real\text{-}of\text{-}int\ (?f - - ?e - 1)\rfloor$
proof (*rule floor-sum-times-2-powr-sgn-eq*)
show $?a * 2\ powr\ real\text{-}of\text{-}int\ (-?e) = real\text{-}of\text{-}int\ ?ai$
by (*simp add*: *powr-add powr-realpow*[*symmetric*] *powr-diff*)
show $|?b * 2\ powr\ real\text{-}of\text{-}int\ (-?e + 1)| \leq 1$
using *abs-m2-less-half*
by (*simp add*: *abs-mult powr-add*[*symmetric*] *algebra-simps powr-mult-base*)
next
have $e1 + \lfloor log\ 2\ |real\text{-}of\text{-}int\ m1|\rfloor - 1 = \lfloor log\ 2\ |?a|\rfloor - 1$
using ‹$m1 \neq 0$›
by (*simp add*: *int-add-floor algebra-simps log-mult abs-mult del*: *floor-add2*)
also have $\ldots \leq \lfloor log\ 2\ |?a + ?b|\rfloor$
using *a-half-less-sum* ‹$m1 \neq 0$› ‹$?sum \neq 0$›
unfolding *floor-diff-of-int*[*symmetric*]
by (*auto simp add*: *log-minus-eq-powr powr-minus-divide intro*!: *floor-mono*)
finally
have $int\ p - \lfloor log\ 2\ |?a + ?b|\rfloor \leq p - (bitlen\ |m1|) - e1 + 2$
by (*auto simp*: *algebra-simps bitlen-alt-def* ‹$m1 \neq 0$›)
also have $\ldots \leq - ?e$
using *bitlen-nonneg*[*of* $|m1|$] **by** (*simp add*: *k1-def*)
finally show $?f \leq - ?e$ **by** *simp*
qed
also have $sgn\ ?b = sgn\ m2$
using *powr-gt-zero*[*of 2 e2*]
by (*auto simp add*: *sgn-if zero-less-mult-iff simp del*: *powr-gt-zero*)
also have $\lfloor(real\text{-}of\text{-}int\ (2 * ?m1) + real\text{-}of\text{-}int\ (sgn\ m2)) * 2\ powr\ real\text{-}of\text{-}int\ (?f - - ?e - 1)\rfloor =$
$\lfloor Float\ (?m1 * 2 + sgn\ m2)\ (?e - 1) * 2\ powr\ ?f\rfloor$
by (*simp flip*: *powr-add powr-realpow add*: *algebra-simps*)
finally
show $\lfloor(?a + ?b) * 2\ powr\ ?f\rfloor = \lfloor real\text{-}of\text{-}float\ (Float\ (?m1 * 2 + sgn\ m2)\ (?e - 1)) * 2\ powr\ ?f\rfloor$.

```
  qed
  then show ?thesis
    by transfer (simp add: plus-down-def ac-simps Let-def)
qed

lemma compute-float-plus-down-naive[code]: float-plus-down p x y = float-round-down
p (x + y)
  by transfer (auto simp: plus-down-def)

qualified lemma compute-float-plus-down[code]:
  fixes p::nat and m1 e1 m2 e2::int
  shows float-plus-down p (Float m1 e1) (Float m2 e2) =
    (if m1 = 0 then float-round-down p (Float m2 e2)
    else if m2 = 0 then float-round-down p (Float m1 e1)
    else
      (if e1 ≥ e2 then
        (let k1 = Suc p − nat (bitlen |m1|) in
          if bitlen |m2| > e1 − e2 − k1 − 2
          then float-round-down p ((Float m1 e1) + (Float m2 e2))
           else float-round-down p (Float (m1 * 2 ^ (Suc (Suc k1)) + sgn m2) (e1
− int k1 − 2)))
    else float-plus-down p (Float m2 e2) (Float m1 e1)))
proof −
  {
    assume bitlen |m2| ≤ e1 − e2 − (Suc p − nat (bitlen |m1|)) − 2 m1 ≠ 0 m2
≠ 0 e1 ≥ e2
    note compute-far-float-plus-down[OF this]
  }
  then show ?thesis
    by transfer (simp add: Let-def plus-down-def ac-simps)
qed

qualified lemma compute-float-plus-up[code]: float-plus-up p x y = − float-plus-down
p (−x) (−y)
  using truncate-down-uminus-eq[of p x + y]
  by transfer (simp add: plus-down-def plus-up-def ac-simps)

lemma mantissa-zero: mantissa 0 = 0
  by (fact mantissa-0)

qualified lemma compute-float-less[code]: a < b ⟷ is-float-pos (float-plus-down
0 b (− a))
  using truncate-down[of 0 b − a] truncate-down-pos[of b − a 0]
  by transfer (auto simp: plus-down-def)

qualified lemma compute-float-le[code]: a ≤ b ⟷ is-float-nonneg (float-plus-down
0 b (− a))
  using truncate-down[of 0 b − a] truncate-down-nonneg[of b − a 0]
  by transfer (auto simp: plus-down-def)
```

end

lemma *plus-down-mono*: *plus-down p a b* ≤ *plus-down p c d* **if** *a* + *b* ≤ *c* + *d*
by (*auto simp*: *plus-down-def intro*!: *truncate-down-mono that*)

lemma *plus-up-mono*: *plus-up p a b* ≤ *plus-up p c d* **if** *a* + *b* ≤ *c* + *d*
by (*auto simp*: *plus-up-def intro*!: *truncate-up-mono that*)

42.14 Approximate Multiplication

lemma *mult-mono-nonpos-nonneg*: *a* ∗ *b* ≤ *c* ∗ *d*
if *a* ≤ *c a* ≤ *0 0* ≤ *d d* ≤ *b* **for** *a b c d*::′*a*::*ordered-ring*
by (*meson dual-order.trans mult-left-mono-neg mult-right-mono that*)

lemma *mult-mono-nonneg-nonpos*: *b* ∗ *a* ≤ *d* ∗ *c*
if *a* ≤ *c c* ≤ *0 0* ≤ *d d* ≤ *b* **for** *a b c d*::′*a*::*ordered-ring*
by (*meson dual-order.trans mult-right-mono-neg mult-left-mono that*)

lemma *mult-mono-nonpos-nonpos*: *a* ∗ *b* ≤ *c* ∗ *d*
if *a* ≥ *c a* ≤ *0 b* ≥ *d d* ≤ *0* **for** *a b c d*::*real*
by (*meson dual-order.trans mult-left-mono-neg mult-right-mono-neg that*)

lemma *mult-float-mono1*:
notes *mono-rules* = *plus-down-mono add-mono nprt-mono nprt-le-zero zero-le-pprt pprt-mono*
shows *a* ≤ *b* ⟹ *ab* ≤ *bb* ⟹
aa ≤ *a* ⟹
b ≤ *ba* ⟹
ac ≤ *ab* ⟹
bb ≤ *bc* ⟹
plus-down prec (*nprt aa* ∗ *pprt bc*)
(*plus-down prec* (*nprt ba* ∗ *nprt bc*)
(*plus-down prec* (*pprt aa* ∗ *pprt ac*)
(*pprt ba* ∗ *nprt ac*)))
≤ *plus-down prec* (*nprt a* ∗ *pprt bb*)
(*plus-down prec* (*nprt b* ∗ *nprt bb*)
(*plus-down prec* (*pprt a* ∗ *pprt ab*)
(*pprt b* ∗ *nprt ab*)))
apply (*rule order-trans*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono-nonpos-nonneg*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono-nonpos-nonpos*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono-nonneg-nonpos*)
apply (*rule mono-rules* | *assumption*)+

by (*rule order-refl*)+

lemma *mult-float-mono2*:
notes *mono-rules* = *plus-up-mono add-mono nprt-mono nprt-le-zero zero-le-pprt pprt-mono*
shows $a \leq b \Longrightarrow$
$ab \leq bb \Longrightarrow$
$aa \leq a \Longrightarrow$
$b \leq ba \Longrightarrow$
$ac \leq ab \Longrightarrow$
$bb \leq bc \Longrightarrow$
plus-up prec (*pprt b* $*$ *pprt bb*)
(*plus-up prec* (*pprt a* $*$ *nprt bb*)
(*plus-up prec* (*nprt b* $*$ *pprt ab*)
(*nprt a* $*$ *nprt ab*)))
$\leq$ *plus-up prec* (*pprt ba* $*$ *pprt bc*)
(*plus-up prec* (*pprt aa* $*$ *nprt bc*)
(*plus-up prec* (*nprt ba* $*$ *pprt ac*)
(*nprt aa* $*$ *nprt ac*)))
apply (*rule order-trans*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono-nonneg-nonpos*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono-nonpos-nonneg*)
apply (*rule mono-rules* | *assumption*)+
apply (*rule mult-mono-nonpos-nonpos*)
apply (*rule mono-rules* | *assumption*)+
by (*rule order-refl*)+

42.15 Approximate Power

lemma *div2-less-self*[*termination-simp*]: *odd n* $\Longrightarrow$ *n div 2* $<$ *n* **for** *n* :: *nat*
by (*simp add*: *odd-pos*)

fun *power-down* :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *nat* $\Rightarrow$ *real*
where
power-down p x 0 = *1*
| *power-down p x* (*Suc n*) =
(*if odd n then truncate-down* (*Suc p*) ((*power-down p x* (*Suc n div 2*))^2)
else truncate-down (*Suc p*) (*x* $*$ *power-down p x n*))

fun *power-up* :: *nat* $\Rightarrow$ *real* $\Rightarrow$ *nat* $\Rightarrow$ *real*
where
power-up p x 0 = *1*
| *power-up p x* (*Suc n*) =
(*if odd n then truncate-up p* ((*power-up p x* (*Suc n div 2*))^2)
else truncate-up p (*x* $*$ *power-up p x n*))

lift-definition *power-up-fl* :: *nat* ⇒ *float* ⇒ *nat* ⇒ *float* **is** *power-up*
by (*induct-tac rule*: *power-up.induct*) *simp-all*

lift-definition *power-down-fl* :: *nat* ⇒ *float* ⇒ *nat* ⇒ *float* **is** *power-down*
by (*induct-tac rule*: *power-down.induct*) *simp-all*

lemma *power-float-transfer*[*transfer-rule*]:
(*rel-fun pcr-float* (*rel-fun* (=) *pcr-float*)) (^) (^)
unfolding *power-def*
by *transfer-prover*

lemma *compute-power-up-fl*[*code*]:
power-up-fl p x 0 = *1*
power-up-fl p x (*Suc n*) =
(*if odd n then float-round-up p* ((*power-up-fl p x* (*Suc n div 2*))^2)
else float-round-up p (*x* ∗ *power-up-fl p x n*))
and *compute-power-down-fl*[*code*]:
power-down-fl p x 0 = *1*
power-down-fl p x (*Suc n*) =
(*if odd n then float-round-down* (*Suc p*) ((*power-down-fl p x* (*Suc n div 2*))^2)
else float-round-down (*Suc p*) (*x* ∗ *power-down-fl p x n*))
unfolding *atomize-conj* **by** *transfer simp*

lemma *power-down-pos*: *0* < *x* ⟹ *0* < *power-down p x n*
by (*induct p x n rule*: *power-down.induct*)
(*auto simp del*: *odd-Suc-div-two intro*!: *truncate-down-pos*)

lemma *power-down-nonneg*: *0* ≤ *x* ⟹ *0* ≤ *power-down p x n*
by (*induct p x n rule*: *power-down.induct*)
(*auto simp del*: *odd-Suc-div-two intro*!: *truncate-down-nonneg mult-nonneg-nonneg*)

lemma *power-down*: *0* ≤ *x* ⟹ *power-down p x n* ≤ *x* ^ *n*
proof (*induct p x n rule*: *power-down.induct*)
case (*2 p x n*)
have *?case* **if** *odd n*
proof −
from *that 2* **have** (*power-down p x* (*Suc n div 2*)) ^ *2* ≤ (*x* ^ (*Suc n div 2*)) ^ *2*
by (*auto intro*: *power-mono power-down-nonneg simp del*: *odd-Suc-div-two*)
also have ... = *x* ^ (*Suc n div 2* ∗ *2*)
by (*simp flip*: *power-mult*)
also have *Suc n div 2* ∗ *2* = *Suc n*
using ‹*odd n*› **by** *presburger*
finally show *?thesis*
using *that* **by** (*auto intro*!: *truncate-down-le simp del*: *odd-Suc-div-two*)
qed
then show *?case*
by (*auto intro*!: *truncate-down-le mult-left-mono 2 mult-nonneg-nonneg power-down-nonneg*)

qed *simp*

lemma *power-up*: $0 \leq x \Longrightarrow x \mathbin{\hat{}} n \leq$ *power-up p x n*
proof (*induct p x n rule*: *power-up.induct*)
  case (*2 p x n*)
  have *?case* **if** *odd n*
  proof −
    from *that even-Suc* **have** *Suc n* = *Suc n div 2* ∗ *2*
      by *presburger*
    then have $x \mathbin{\hat{}} Suc\ n \leq (x \mathbin{\hat{}} (Suc\ n\ div\ 2))^2$
      by (*simp flip*: *power-mult*)
    also from *that 2* **have** ... ≤ $(power\text{-}up\ p\ x\ (Suc\ n\ div\ 2))^2$
      by (*auto intro*: *power-mono simp del*: *odd-Suc-div-two*)
    finally show *?thesis*
      using *that* **by** (*auto intro*!: *truncate-up-le simp del*: *odd-Suc-div-two*)
  qed
  then show *?case*
    by (*auto intro*!: *truncate-up-le mult-left-mono 2*)
qed *simp*

lemmas *power-up-le* = *order-trans*[*OF* - *power-up*]
  and *power-up-less* = *less-le-trans*[*OF* - *power-up*]
  and *power-down-le* = *order-trans*[*OF power-down*]

lemma *power-down-fl*: $0 \leq x \Longrightarrow$ *power-down-fl p x n* ≤ $x \mathbin{\hat{}} n$
  by *transfer* (*rule power-down*)

lemma *power-up-fl*: $0 \leq x \Longrightarrow x \mathbin{\hat{}} n \leq$ *power-up-fl p x n*
  by *transfer* (*rule power-up*)

lemma *real-power-up-fl*: *real-of-float* (*power-up-fl p x n*) = *power-up p x n*
  by *transfer simp*

lemma *real-power-down-fl*: *real-of-float* (*power-down-fl p x n*) = *power-down p x n*
  by *transfer simp*

lemmas [*simp del*] = *power-down.simps*(*2*) *power-up.simps*(*2*)

lemmas *power-down-simp* = *power-down.simps*(*2*)
lemmas *power-up-simp* = *power-up.simps*(*2*)

lemma *power-down-even-nonneg*: *even n* ⟹ *0* ≤ *power-down p x n*
  by (*induct p x n rule*: *power-down.induct*)
   (*auto simp*: *power-down-simp simp del*: *odd-Suc-div-two intro*!: *truncate-down-nonneg*
)

lemma *power-down-eq-zero-iff*[*simp*]: *power-down prec b n* = *0* ⟷ *b* = *0* ∧ *n* ≠ *0*

proof (*induction n arbitrary*: *b rule*: *less-induct*)
case (*less x*)
then show *?case*
using *power-down-simp*[*of - - x − 1*]
by (*cases x*) (*auto simp add*: *div2-less-self*)
qed

lemma *power-down-nonneg-iff*[*simp*]:
power-down prec b n ≥ *0* ⟷ *even n* ∨ *b* ≥ *0*
proof (*induction n arbitrary*: *b rule*: *less-induct*)
case (*less x*)
show *?case*
using *less*(*1*)[*of x − 1 b*] *power-down-simp*[*of - - x − 1*]
by (*cases x*) (*auto simp*: *algebra-split-simps zero-le-mult-iff*)
qed

lemma *power-down-neg-iff*[*simp*]:
power-down prec b n < *0* ⟷
b < *0* ∧ *odd n*
using *power-down-nonneg-iff*[*of prec b n*] **by** (*auto simp del*: *power-down-nonneg-iff*)

lemma *power-down-nonpos-iff*[*simp*]:
notes [*simp del*] = *power-down-neg-iff power-down-eq-zero-iff*
shows *power-down prec b n* ≤ *0* ⟷ *b* < *0* ∧ *odd n* ∨ *b* = *0* ∧ *n* ≠ *0*
using *power-down-neg-iff*[*of prec b n*] *power-down-eq-zero-iff*[*of prec b n*]
by *auto*

lemma *power-down-mono*:
power-down prec a n ≤ *power-down prec b n*
if ((*0* ≤ *a* ∧ *a* ≤ *b*)∨(*odd n* ∧ *a* ≤ *b*) ∨ (*even n* ∧ *a* ≤ *0* ∧ *b* ≤ *a*))
using *that*
proof (*induction n arbitrary*: *a b rule*: *less-induct*)
case (*less i*)
show *?case*
proof (*cases i*)
case *j*: (*Suc j*)
note *IH* = *less*[*unfolded j even-Suc not-not*]
note [*simp del*] = *power-down.simps*
show *?thesis*
proof *cases*
assume [*simp*]: *even j*
have *a* ∗ *power-down prec a j* ≤ *b* ∗ *power-down prec b j*
by (*metis IH*(*1*) *IH*(*2*) ‹*even j*› *lessI linear mult-mono mult-mono′ mult-mono-nonpos-nonneg power-down-even-nonneg*)
then have *truncate-down* (*Suc prec*) (*a* ∗ *power-down prec a j*) ≤ *truncate-down* (*Suc prec*) (*b* ∗ *power-down prec b j*)
by (*auto intro*!: *truncate-down-mono simp*: *abs-le-square-iff*[*symmetric*] *abs-real-def*)
then show *?thesis*

```
      unfolding j
      by (simp add: power-down-simp)
    next
     assume [simp]: odd j
     have power-down prec 0 (Suc (j div 2)) ≤ - power-down prec b (Suc (j div
2))
      if b < 0 even (j div 2)
      apply (rule order-trans[where y=0])
      using IH that by (auto simp: div2-less-self)
     then have truncate-down (Suc prec) ((power-down prec a (Suc (j div 2)))^2)
      ≤ truncate-down (Suc prec) ((power-down prec b (Suc (j div 2)))^2)
      using IH
      by (auto intro!: truncate-down-mono intro: order-trans[where y=0]
          simp: abs-le-square-iff[symmetric] abs-real-def
          div2-less-self)
     then show ?thesis
      unfolding j
      by (simp add: power-down-simp)
    qed
  qed simp
qed

lemma power-up-even-nonneg: even n ⟹ 0 ≤ power-up p x n
  by (induct p x n rule: power-up.induct)
    (auto simp: power-up.simps simp del: odd-Suc-div-two intro!: )

lemma power-up-eq-zero-iff[simp]: power-up prec b n = 0 ⟷ b = 0 ∧ n ≠ 0
proof (induction n arbitrary: b rule: less-induct)
  case (less x)
  then show ?case
    using power-up-simp[of - - x - 1]
    by (cases x) (auto simp: algebra-split-simps zero-le-mult-iff div2-less-self)
qed

lemma power-up-nonneg-iff[simp]:
  power-up prec b n ≥ 0 ⟷ even n ∨ b ≥ 0
proof (induction n arbitrary: b rule: less-induct)
  case (less x)
  show ?case
    using less(1)[of x - 1 b] power-up-simp[of - - x - 1]
    by (cases x) (auto simp: algebra-split-simps zero-le-mult-iff)
qed

lemma power-up-neg-iff[simp]:
  power-up prec b n < 0 ⟷ b < 0 ∧ odd n
  using power-up-nonneg-iff[of prec b n] by (auto simp del: power-up-nonneg-iff)

lemma power-up-nonpos-iff[simp]:
  notes [simp del] = power-up-neg-iff power-up-eq-zero-iff
```

```
  shows power-up prec b n ≤ 0 ⟷ b < 0 ∧ odd n ∨ b = 0 ∧ n ≠ 0
  using power-up-neg-iff[of prec b n] power-up-eq-zero-iff[of prec b n]
  by auto

lemma power-up-mono:
  power-up prec a n ≤ power-up prec b n
  if ((0 ≤ a ∧ a ≤ b)∨(odd n ∧ a ≤ b) ∨ (even n ∧ a ≤ 0 ∧ b ≤ a))
  using that
proof (induction n arbitrary: a b rule: less-induct)
  case (less i)
  show ?case
  proof (cases i)
    case j: (Suc j)
    note IH = less[unfolded j even-Suc not-not]
    note [simp del] = power-up.simps
    show ?thesis
    proof cases
      assume [simp]: even j
      have a * power-up prec a j ≤ b * power-up prec b j
      by (metis IH(1) IH(2) ‹even j› lessI linear mult-mono mult-mono' mult-mono-nonpos-nonneg
power-up-even-nonneg)
      then have truncate-up prec (a * power-up prec a j) ≤ truncate-up prec (b *
power-up prec b j)
      by (auto intro!: truncate-up-mono simp: abs-le-square-iff[symmetric] abs-real-def)
      then show ?thesis
        unfolding j
        by (simp add: power-up-simp)
    next
      assume [simp]: odd j
      have power-up prec 0 (Suc (j div 2)) ≤ − power-up prec b (Suc (j div 2))
        if b < 0 even (j div 2)
        apply (rule order-trans[where y=0])
        using IH that by (auto simp: div2-less-self)
      then have truncate-up prec ((power-up prec a (Suc (j div 2)))^2)
        ≤ truncate-up prec ((power-up prec b (Suc (j div 2)))^2)
        using IH
        by (auto intro!: truncate-up-mono intro: order-trans[where y=0]
            simp: abs-le-square-iff[symmetric] abs-real-def
            div2-less-self)
      then show ?thesis
        unfolding j
        by (simp add: power-up-simp)
    qed
  qed simp
qed
```

42.16 Lemmas needed by Approximate

```
lemma Float-num[simp]:
```

real-of-float (Float 1 0) = 1
real-of-float (Float 1 1) = 2
real-of-float (Float 1 2) = 4
real-of-float (Float 1 (− 1)) = 1/2
real-of-float (Float 1 (− 2)) = 1/4
real-of-float (Float 1 (− 3)) = 1/8
real-of-float (Float (− 1) 0) = −1
real-of-float (Float (numeral n) 0) = numeral n
real-of-float (Float (− numeral n) 0) = − numeral n
using *two-powr-int-float*[*of 2*] *two-powr-int-float*[*of −1*] *two-powr-int-float*[*of −2*] *two-powr-int-float*[*of −3*]
using *powr-realpow*[*of 2 2*] *powr-realpow*[*of 2 3*]
using *powr-minus*[*of 2*::*real 1*] *powr-minus*[*of 2*::*real 2*] *powr-minus*[*of 2*::*real 3*]
by *auto*

lemma *real-of-Float-int*[*simp*]: *real-of-float (Float n 0) = real n*
by *simp*

lemma *float-zero*[*simp*]: *real-of-float (Float 0 e) = 0*
by *simp*

lemma *abs-div-2-less*: *a ≠ 0 ⟹ a ≠ −1 ⟹ |(a*::*int) div 2| < |a|*
by *arith*

lemma *lapprox-rat*: *real-of-float (lapprox-rat prec x y) ≤ real-of-int x / real-of-int y*
by (*simp add*: *lapprox-rat.rep-eq truncate-down*)

lemma *mult-div-le*:
fixes *a b* :: *int*
assumes *b > 0*
shows *a ≥ b ∗ (a div b)*
proof −
from *minus-div-mult-eq-mod* [*symmetric*, *of a b*] **have** *a = b ∗ (a div b) + a mod b*
by *simp*
also have *... ≥ b ∗ (a div b) + 0*
apply (*rule add-left-mono*)
apply (*rule pos-mod-sign*)
using *assms*
apply *simp*
done
finally show *?thesis*
by *simp*
qed

lemma *lapprox-rat-nonneg*:
assumes *0 ≤ x* **and** *0 ≤ y*
shows *0 ≤ real-of-float (lapprox-rat n x y)*

using *assms*
by *transfer* (*simp add*: *truncate-down-nonneg*)

lemma *rapprox-rat*: *real-of-int x / real-of-int y ≤ real-of-float* (*rapprox-rat prec x y*)
by *transfer* (*simp add*: *truncate-up*)

lemma *rapprox-rat-le1*:
assumes *0 ≤ x 0 < y x ≤ y*
shows *real-of-float* (*rapprox-rat n x y*) *≤ 1*
using *assms*
by *transfer* (*simp add*: *truncate-up-le1*)

lemma *rapprox-rat-nonneg-nonpos*: *0 ≤ x ⟹ y ≤ 0 ⟹ real-of-float* (*rapprox-rat n x y*) *≤ 0*
by *transfer* (*simp add*: *truncate-up-nonpos divide-nonneg-nonpos*)

lemma *rapprox-rat-nonpos-nonneg*: *x ≤ 0 ⟹ 0 ≤ y ⟹ real-of-float* (*rapprox-rat n x y*) *≤ 0*
by *transfer* (*simp add*: *truncate-up-nonpos divide-nonpos-nonneg*)

lemma *real-divl*: *real-divl prec x y ≤ x / y*
by (*simp add*: *real-divl-def truncate-down*)

lemma *real-divr*: *x / y ≤ real-divr prec x y*
by (*simp add*: *real-divr-def truncate-up*)

lemma *float-divl*: *real-of-float* (*float-divl prec x y*) *≤ x / y*
by *transfer* (*rule real-divl*)

lemma *real-divl-lower-bound*: *0 ≤ x ⟹ 0 ≤ y ⟹ 0 ≤ real-divl prec x y*
by (*simp add*: *real-divl-def truncate-down-nonneg*)

lemma *float-divl-lower-bound*: *0 ≤ x ⟹ 0 ≤ y ⟹ 0 ≤ real-of-float* (*float-divl prec x y*)
by *transfer* (*rule real-divl-lower-bound*)

lemma *exponent-1*: *exponent 1 = 0*
using *exponent-float*[*of 1 0*] **by** (*simp add*: *one-float-def*)

lemma *mantissa-1*: *mantissa 1 = 1*
using *mantissa-float*[*of 1 0*] **by** (*simp add*: *one-float-def*)

lemma *bitlen-1*: *bitlen 1 = 1*
by (*simp add*: *bitlen-alt-def*)

lemma *float-upper-bound*: *x ≤ 2 powr* (*bitlen |mantissa x| + exponent x*)
proof (*cases x = 0*)
case *True*

then show *?thesis* **by** *simp*
next
case *False*
then have *mantissa x ≠ 0*
using *mantissa-eq-zero-iff* **by** *auto*
have *x = mantissa x * 2 powr (exponent x)*
by (*rule mantissa-exponent*)
also have *mantissa x ≤ |mantissa x|*
by *simp*
also have *... ≤ 2 powr (bitlen |mantissa x|)*
using *bitlen-bounds*[*of |mantissa x|*] *bitlen-nonneg* ‹*mantissa x ≠ 0*›
by (*auto simp del*: *of-int-abs simp add*: *powr-int*)
finally show *?thesis* **by** (*simp add*: *powr-add*)
qed

lemma *real-divl-pos-less1-bound*:
assumes *0 < x x ≤ 1*
shows *1 ≤ real-divl prec 1 x*
using *assms*
by (*auto intro*!: *truncate-down-ge1 simp*: *real-divl-def*)

lemma *float-divl-pos-less1-bound*:
0 < real-of-float x ⟹ real-of-float x ≤ 1 ⟹ prec ≥ 1 ⟹
1 ≤ real-of-float (*float-divl prec 1 x*)
by *transfer* (*rule real-divl-pos-less1-bound*)

lemma *float-divr*: *real-of-float x / real-of-float y ≤ real-of-float* (*float-divr prec x y*)
by *transfer* (*rule real-divr*)

lemma *real-divr-pos-less1-lower-bound*:
assumes *0 < x*
and *x ≤ 1*
shows *1 ≤ real-divr prec 1 x*
proof −
have *1 ≤ 1 / x*
using ‹*0 < x*› **and** ‹*x ≤ 1*› **by** *auto*
also have *... ≤ real-divr prec 1 x*
using *real-divr*[**where** *x = 1* **and** *y = x*] **by** *auto*
finally show *?thesis* **by** *auto*
qed

lemma *float-divr-pos-less1-lower-bound*: *0 < x ⟹ x ≤ 1 ⟹ 1 ≤ float-divr prec 1 x*
by *transfer* (*rule real-divr-pos-less1-lower-bound*)

lemma *real-divr-nonpos-pos-upper-bound*: *x ≤ 0 ⟹ 0 ≤ y ⟹ real-divr prec x y ≤ 0*
by (*simp add*: *real-divr-def truncate-up-nonpos divide-le-0-iff*)

lemma *float-divr-nonpos-pos-upper-bound*:
real-of-float x ≤ 0 ⟹ 0 ≤ real-of-float y ⟹ real-of-float (float-divr prec x y) ≤ 0
by *transfer* (*rule real-divr-nonpos-pos-upper-bound*)

lemma *real-divr-nonneg-neg-upper-bound*: *0 ≤ x ⟹ y ≤ 0 ⟹ real-divr prec x y ≤ 0*
by (*simp add*: *real-divr-def truncate-up-nonpos divide-le-0-iff*)

lemma *float-divr-nonneg-neg-upper-bound*:
0 ≤ real-of-float x ⟹ real-of-float y ≤ 0 ⟹ real-of-float (float-divr prec x y) ≤ 0
by *transfer* (*rule real-divr-nonneg-neg-upper-bound*)

lemma *Float-le-zero-iff*: *Float a b ≤ 0 ⟷ a ≤ 0*
by (*auto simp*: *zero-float-def mult-le-0-iff*)

lemma *real-of-float-pprt*[*simp*]:
fixes *a* :: *float*
shows *real-of-float* (*pprt a*) = *pprt* (*real-of-float a*)
unfolding *pprt-def sup-float-def max-def sup-real-def* **by** *auto*

lemma *real-of-float-nprt*[*simp*]:
fixes *a* :: *float*
shows *real-of-float* (*nprt a*) = *nprt* (*real-of-float a*)
unfolding *nprt-def inf-float-def min-def inf-real-def* **by** *auto*

context
begin

lift-definition *int-floor-fl* :: *float* ⇒ *int* **is** *floor* **.**

qualified lemma *compute-int-floor-fl*[*code*]:
int-floor-fl (*Float m e*) = (*if 0 ≤ e then m * 2 ^ nat e else m div* (*2 ^* (*nat* (*−e*))))
apply *transfer*
apply (*simp add*: *powr-int floor-divide-of-int-eq*)
apply (*metis* (*no-types*, *opaque-lifting*)*floor-divide-of-int-eq of-int-numeral of-int-power floor-of-int of-int-mult*)
done

lift-definition *floor-fl* :: *float* ⇒ *float* **is** *λx*. *real-of-int* ⌊*x*⌋
by *simp*

qualified lemma *compute-floor-fl*[*code*]:
floor-fl (*Float m e*) = (*if 0 ≤ e then Float m e else Float* (*m div* (*2 ^*(*nat* (*−e*)))) *0*)
apply *transfer*

apply (*simp add*: *powr-int floor-divide-of-int-eq*)
apply (*metis* (*no-types*, *opaque-lifting*)*floor-divide-of-int-eq of-int-numeral of-int-power of-int-mult*)
done

end

lemma *floor-fl*: *real-of-float* (*floor-fl x*) ≤ *real-of-float x*
by *transfer simp*

lemma *int-floor-fl*: *real-of-int* (*int-floor-fl x*) ≤ *real-of-float x*
by *transfer simp*

lemma *floor-pos-exp*: *exponent* (*floor-fl x*) ≥ *0*
proof (*cases floor-fl x = 0*)
case *True*
then show *?thesis*
by (*simp add*: *floor-fl-def*)
next
case *False*
have *eq*: *floor-fl x = Float* ⌊*real-of-float x*⌋ *0*
by *transfer simp*
obtain *i* **where** ⌊*real-of-float x*⌋ = *mantissa* (*floor-fl x*) * *2* ^ *i 0* = *exponent* (*floor-fl x*) − *int i*
by (*rule denormalize-shift*[*OF eq False*])
then show *?thesis*
by *simp*
qed

lemma *compute-mantissa*[*code*]:
mantissa (*Float m e*) =
(*if m = 0 then 0 else if 2 dvd m then mantissa* (*normfloat* (*Float m e*)) *else m*)
by (*auto simp*: *mantissa-float Float.abs-eq simp flip*: *zero-float-def*)

lemma *compute-exponent*[*code*]:
exponent (*Float m e*) =
(*if m = 0 then 0 else if 2 dvd m then exponent* (*normfloat* (*Float m e*)) *else e*)
by (*auto simp*: *exponent-float Float.abs-eq simp flip*: *zero-float-def*)

lifting-update *Float.float.lifting*
lifting-forget *Float.float.lifting*

end

43 Pointwise instantiation of functions to division

theory *Function-Division*
imports *Function-Algebras*
begin

43.1 Syntactic with division

instantiation *fun* :: (*type*, *inverse*) *inverse*
begin

definition *inverse f* = *inverse* ∘ *f*

definition *f div g* = (λ*x*. *f x* / *g x*)

instance ..

end

lemma *inverse-fun-apply* [*simp*]:
inverse f x = *inverse* (*f x*)
by (*simp add*: *inverse-fun-def*)

lemma *divide-fun-apply* [*simp*]:
(*f* / *g*) *x* = *f x* / *g x*
by (*simp add*: *divide-fun-def*)

Unfortunately, we cannot lift this operations to algebraic type classes for division: being different from the constant zero function $f \neq (0::'a)$ is too weak as precondition. So we must introduce our own set of lemmas.

abbreviation *zero-free* :: ('*b* ⇒ '*a*::*field*) ⇒ *bool* **where**
zero-free f ≡ ¬ (∃ *x*. *f x* = *0*)

lemma *fun-left-inverse*:
fixes *f* :: '*b* ⇒ '*a*::*field*
shows *zero-free f* ⟹ *inverse f* ∗ *f* = *1*
by (*simp add*: *fun-eq-iff*)

lemma *fun-right-inverse*:
fixes *f* :: '*b* ⇒ '*a*::*field*
shows *zero-free f* ⟹ *f* ∗ *inverse f* = *1*
by (*simp add*: *fun-eq-iff*)

lemma *fun-divide-inverse*:
fixes *f g* :: '*b* ⇒ '*a*::*field*
shows *f* / *g* = *f* ∗ *inverse g*
by (*simp add*: *fun-eq-iff divide-inverse*)

Feel free to extend this.

Another possibility would be a reformulation of the division type classes to user a *zero-free* predicate rather than a direct $a \neq (0::'a)$ condition.

end

44 Lexicographic order on functions

theory *Fun-Lexorder*
imports *Main*
begin

definition *less-fun* :: ('*a*::*linorder* ⇒ '*b*::*linorder*) ⇒ ('*a* ⇒ '*b*) ⇒ *bool*
where
  less-fun f g ⟷ (∃ *k*. *f k* < *g k* ∧ (∀ *k'* < *k*. *f k'* = *g k'*))

lemma *less-funI*:
  assumes ∃ *k*. *f k* < *g k* ∧ (∀ *k'* < *k*. *f k'* = *g k'*)
  shows *less-fun f g*
  using *assms* **by** (*simp add*: *less-fun-def*)

lemma *less-funE*:
  assumes *less-fun f g*
  obtains *k* **where** *f k* < *g k* **and** ⋀*k'*. *k'* < *k* ⟹ *f k'* = *g k'*
  using *assms* **unfolding** *less-fun-def* **by** *blast*

lemma *less-fun-asym*:
  assumes *less-fun f g*
  shows ¬ *less-fun g f*
proof
  from *assms* **obtain** *k1* **where** *k1*: *f k1* < *g k1* *k'* < *k1* ⟹ *f k'* = *g k'* **for** *k'*
    by (*blast elim*!: *less-funE*)
  assume *less-fun g f* **then obtain** *k2* **where** *k2*: *g k2* < *f k2* *k'* < *k2* ⟹ *g k'*
= *f k'* **for** *k'*
    by (*blast elim*!: *less-funE*)
  show *False* **proof** (*cases k1 k2 rule*: *linorder-cases*)
    case *equal* **with** *k1 k2* **show** *False* **by** *simp*
  next
    case *less* **with** *k2* **have** *g k1* = *f k1* **by** *simp*
    with *k1* **show** *False* **by** *simp*
  next
    case *greater* **with** *k1* **have** *f k2* = *g k2* **by** *simp*
    with *k2* **show** *False* **by** *simp*
  qed
qed

lemma *less-fun-irrefl*:
  ¬ *less-fun f f*
proof
  assume *less-fun f f*
  then obtain *k* **where** *k*: *f k* < *f k*
    by (*blast elim*!: *less-funE*)
  then show *False* **by** *simp*
qed

lemma *less-fun-trans*:
assumes *less-fun f g* **and** *less-fun g h*
shows *less-fun f h*
proof (*rule less-funI*)
from ‹*less-fun f g*› **obtain** *k1* **where** *k1*: $f\ k1 < g\ k1\ k' < k1 \Longrightarrow f\ k' = g\ k'$ **for** k'
by (*blast elim*!: *less-funE*)
from ‹*less-fun g h*› **obtain** *k2* **where** *k2*: $g\ k2 < h\ k2\ k' < k2 \Longrightarrow g\ k' = h\ k'$ **for** k'
by (*blast elim*!: *less-funE*)
show $\exists k.\ f\ k < h\ k \wedge (\forall k'<k.\ f\ k' = h\ k')$
proof (*cases k1 k2 rule*: *linorder-cases*)
case *equal* **with** *k1 k2* **show** *?thesis* **by** (*auto simp add*: *exI* [*of* - *k2*])
next
case *less* **with** *k2* **have** $g\ k1 = h\ k1\ \bigwedge k'.\ k' < k1 \Longrightarrow g\ k' = h\ k'$ **by** *simp-all*
with *k1* **show** *?thesis* **by** (*auto intro*: *exI* [*of* - *k1*])
next
case *greater* **with** *k1* **have** $f\ k2 = g\ k2\ \bigwedge k'.\ k' < k2 \Longrightarrow f\ k' = g\ k'$ **by** *simp-all*
with *k2* **show** *?thesis* **by** (*auto intro*: *exI* [*of* - *k2*])
qed
qed

lemma *order-less-fun*:
class.order ($\lambda f\ g.$ *less-fun f g* $\vee$ $f = g$) *less-fun*
by (*rule order-strictI*) (*auto intro*: *less-fun-trans intro*!: *less-fun-irrefl less-fun-asym*)

lemma *less-fun-trichotomy*:
assumes *finite* $\{k.\ f\ k \neq g\ k\}$
shows *less-fun f g* $\vee$ $f = g \vee$ *less-fun g f*
proof −
{ define K **where** $K = \{k.\ f\ k \neq g\ k\}$
assume $f \neq g$
then obtain k' **where** $f\ k' \neq g\ k'$ **by** *auto*
then have [*simp*]: $K \neq \{\}$ **by** (*auto simp add*: *K-def*)
with *assms* **have** [*simp*]: *finite K* **by** (*simp add*: *K-def*)
define q **where** $q = Min\ K$
then have $q \in K$ **and** $\bigwedge k.\ k \in K \Longrightarrow k \geq q$ **by** *auto*
then have $\bigwedge k.\ \neg\ k \geq q \Longrightarrow k \notin K$ **by** *blast*
then have $*$: $\bigwedge k.\ k < q \Longrightarrow f\ k = g\ k$ **by** (*simp add*: *K-def*)
from ‹$q \in K$› **have** $f\ q \neq g\ q$ **by** (*simp add*: *K-def*)
then have $f\ q < g\ q \vee f\ q > g\ q$ **by** *auto*
with $*$ **have** *less-fun f g* $\vee$ *less-fun g f*
by (*auto intro*!: *less-funI*)
} then show *?thesis* **by** *blast*
qed

end

45 The *going-to* filter

theory *Going-To-Filter*
imports *Complex-Main*
begin

definition *going-to-within* :: $('a \Rightarrow 'b) \Rightarrow 'b$ *filter* $\Rightarrow 'a$ *set* $\Rightarrow 'a$ *filter*
(‹(-)/ *going'-to* (-)/ *within* (-)› [*1000*,*60*,*60*] *60*) **where**
f going-to F within A = *inf* (*filtercomap f F*) (*principal A*)

abbreviation *going-to* :: $('a \Rightarrow 'b) \Rightarrow 'b$ *filter* $\Rightarrow 'a$ *filter*
(**infix** ‹*going'-to*› *60*)
where *f going-to F* ≡ *f going-to F within UNIV*

The *going-to* filter is, in a sense, the opposite of *filtermap*. It corresponds to the intuition of, given a function $f : A \to B$ and a filter F on the range of B, looking at such values of x that $f(x)$ approaches F. This can be written as *f going-to F*.

A classic example is the *at-infinity* filter, which describes the neigbourhood of infinity (i. e. all values sufficiently far away from the zero). This can also be written as *norm going-to at-top*.

Additionally, the *going-to* filter can be restricted with an optional 'within' parameter. For instance, if one would would want to consider the filter of complex numbers near infinity that do not lie on the negative real line, one could write *cmod going-to at-top within − complex-of-real ' {..0}*.

A third, less mathematical example lies in the complexity analysis of algorithms. Suppose we wanted to say that an algorithm on lists takes $O(n^2)$ time where n is the length of the input list. We can write this using the Landau symbols from the AFP, where the underlying filter is *length going-to sequentially*. If, on the other hand, we want to look the complexity of the algorithm on sorted lists, we could use the filter *length going-to sequentially within Collect sorted*.

lemma *going-to-def*: *f going-to F* = *filtercomap f F*
by (*simp add*: *going-to-within-def*)

lemma *eventually-going-toI* [*intro*]:
assumes *eventually P F*
shows *eventually* (λ*x*. *P* (*f x*)) (*f going-to F*)
using *assms* **by** (*auto simp*: *going-to-def*)

lemma *filterlim-going-toI-weak* [*intro*]: *filterlim f F* (*f going-to F within A*)
unfolding *going-to-within-def*
by (*meson filterlim-filtercomap filterlim-iff inf-le1 le-filter-def*)

lemma *going-to-mono*: *F* ≤ *G* ⟹ *A* ⊆ *B* ⟹ *f going-to F within A* ≤ *f going-to G within B*
unfolding *going-to-within-def* **by** (*intro inf-mono filtercomap-mono*) *simp-all*

lemma *going-to-inf*:
f going-to (*inf F G*) *within A* = *inf* (*f going-to F within A*) (*f going-to G within A*)
by (*simp add*: *going-to-within-def filtercomap-inf inf-assoc inf-commute inf-left-commute*)

lemma *going-to-sup*:
f going-to (*sup F G*) *within A* ≥ *sup* (*f going-to F within A*) (*f going-to G within A*)
by (*auto simp*: *going-to-within-def intro*!: *inf.coboundedI1 filtercomap-sup filtercomap-mono*)

lemma *going-to-top* [*simp*]: *f going-to top within A* = *principal A*
by (*simp add*: *going-to-within-def*)

lemma *going-to-bot* [*simp*]: *f going-to bot within A* = *bot*
by (*simp add*: *going-to-within-def*)

lemma *going-to-principal*:
f going-to principal A within B = *principal* (*f* −‘ *A* ∩ *B*)
by (*simp add*: *going-to-within-def*)

lemma *going-to-within-empty* [*simp*]: *f going-to F within* {} = *bot*
by (*simp add*: *going-to-within-def*)

lemma *going-to-within-union* [*simp*]:
f going-to F within (*A* ∪ *B*) = *sup* (*f going-to F within A*) (*f going-to F within B*)
by (*simp add*: *going-to-within-def flip*: *inf-sup-distrib1*)

lemma *eventually-going-to-at-top-linorder*:
fixes *f* :: *'a* ⇒ *'b* :: *linorder*
shows *eventually P* (*f going-to at-top within A*) ⟷ (∃ *C*. ∀ *x*∈*A*. *f x* ≥ *C* ⟶ *P x*)
unfolding *going-to-within-def eventually-filtercomap*
eventually-inf-principal eventually-at-top-linorder **by** *fast*

lemma *eventually-going-to-at-bot-linorder*:
fixes *f* :: *'a* ⇒ *'b* :: *linorder*
shows *eventually P* (*f going-to at-bot within A*) ⟷ (∃ *C*. ∀ *x*∈*A*. *f x* ≤ *C* ⟶ *P x*)
unfolding *going-to-within-def eventually-filtercomap*
eventually-inf-principal eventually-at-bot-linorder **by** *fast*

lemma *eventually-going-to-at-top-dense*:
fixes *f* :: *'a* ⇒ *'b* :: {*linorder*,*no-top*}
shows *eventually P* (*f going-to at-top within A*) ⟷ (∃ *C*. ∀ *x*∈*A*. *f x* > *C* ⟶ *P x*)
unfolding *going-to-within-def eventually-filtercomap*

eventually-inf-principal eventually-at-top-dense **by** *fast*

lemma *eventually-going-to-at-bot-dense*:
fixes $f :: 'a \Rightarrow 'b :: \{linorder, no\text{-}bot\}$
shows *eventually P* (*f going-to at-bot within A*) $\longleftrightarrow$ $(\exists\, C.\ \forall\, x{\in}A.\ f\ x < C \longrightarrow P\ x)$
unfolding *going-to-within-def eventually-filtercomap*
eventually-inf-principal eventually-at-bot-dense **by** *fast*

lemma *eventually-going-to-nhds*:
eventually P (*f going-to nhds a within A*) $\longleftrightarrow$
$(\exists\, S.\ open\ S \wedge a \in S \wedge (\forall\, x{\in}A.\ f\ x \in S \longrightarrow P\ x))$
unfolding *going-to-within-def eventually-filtercomap eventually-inf-principal*
eventually-nhds **by** *fast*

lemma *eventually-going-to-at*:
eventually P (*f going-to* (*at a within B*) *within A*) $\longleftrightarrow$
$(\exists\, S.\ open\ S \wedge a \in S \wedge (\forall\, x{\in}A.\ f\ x \in B \cap S - \{a\} \longrightarrow P\ x))$
unfolding *at-within-def going-to-inf eventually-inf-principal*
eventually-going-to-nhds going-to-principal **by** *fast*

lemma *norm-going-to-at-top-eq*: *norm going-to at-top* = *at-infinity*
by (*simp add*: *eventually-at-infinity eventually-going-to-at-top-linorder filter-eq-iff*)

lemmas *at-infinity-altdef* = *norm-going-to-at-top-eq* [*symmetric*]

end

46 Big sum and product over function bodies

theory *Groups-Big-Fun*
imports
Main
begin

46.1 Abstract product

locale *comm-monoid-fun* = *comm-monoid*
begin

definition $G :: ('b \Rightarrow 'a) \Rightarrow 'a$
where
expand-set: $G\ g$ = *comm-monoid-set.F f* **1** $g\ \{a.\ g\ a \neq \mathbf{1}\}$

interpretation *F*: *comm-monoid-set f* **1**
..

lemma *expand-superset*:
assumes *finite A* **and** $\{a.\ g\ a \neq \mathbf{1}\} \subseteq A$

shows $G\ g = F.F\ g\ A$
apply (*simp add*: *expand-set*)
apply (*rule F.same-carrierI* [*of A*])
apply (*simp-all add*: *assms*)
done

lemma *conditionalize*:
assumes *finite A*
shows $F.F\ g\ A = G\ (\lambda a.\ \textit{if}\ a \in A\ \textit{then}\ g\ a\ \textit{else}\ \mathbf{1})$
using *assms*
apply (*simp add*: *expand-set*)
apply (*rule F.same-carrierI* [*of A*])
apply *auto*
done

lemma *neutral* [*simp*]:
$G\ (\lambda a.\ \mathbf{1}) = \mathbf{1}$
by (*simp add*: *expand-set*)

lemma *update* [*simp*]:
assumes *finite* $\{a.\ g\ a \neq \mathbf{1}\}$
assumes $g\ a = \mathbf{1}$
shows $G\ (g(a := b)) = b * G\ g$
proof (*cases* $b = \mathbf{1}$)
case *True* **with** ‹$g\ a = \mathbf{1}$› **show** *?thesis*
by (*simp add*: *expand-set*) (*rule F.cong*, *auto*)
next
case *False*
moreover have $\{a'.\ a' \neq a \longrightarrow g\ a' \neq \mathbf{1}\} = \textit{insert}\ a\ \{a.\ g\ a \neq \mathbf{1}\}$
by *auto*
moreover from ‹$g\ a = \mathbf{1}$› **have** $a \notin \{a.\ g\ a \neq \mathbf{1}\}$
by *simp*
moreover have $F.F\ (\lambda a'.\ \textit{if}\ a' = a\ \textit{then}\ b\ \textit{else}\ g\ a')\ \{a.\ g\ a \neq \mathbf{1}\} = F.F\ g\ \{a.\ g\ a \neq \mathbf{1}\}$
by (*rule F.cong*) (*auto simp add*: ‹$g\ a = \mathbf{1}$›)
ultimately show *?thesis* **using** ‹*finite* $\{a.\ g\ a \neq \mathbf{1}\}$› **by** (*simp add*: *expand-set*)
qed

lemma *infinite* [*simp*]:
$\neg$ *finite* $\{a.\ g\ a \neq \mathbf{1}\} \Longrightarrow G\ g = \mathbf{1}$
by (*simp add*: *expand-set*)

lemma *cong* [*cong*]:
assumes $\bigwedge a.\ g\ a = h\ a$
shows $G\ g = G\ h$
using *assms* **by** (*simp add*: *expand-set*)

lemma *not-neutral-obtains-not-neutral*:
assumes $G\ g \neq \mathbf{1}$

obtains a **where** $g\ a \neq \mathbf{1}$
using *assms* **by** (*auto elim*: *F.not-neutral-contains-not-neutral simp add*: *expand-set*)

lemma *reindex-cong*:
assumes *bij l*
assumes $g \circ l = h$
shows $G\ g = G\ h$
proof −
from *assms* **have** *unfold*: $h = g \circ l$ **by** *simp*
from ‹*bij l*› **have** *inj l* **by** (*rule bij-is-inj*)
then have *inj-on l* $\{a.\ h\ a \neq \mathbf{1}\}$ **by** (*rule subset-inj-on*) *simp*
moreover from ‹*bij l*› **have** $\{a.\ g\ a \neq \mathbf{1}\} = l\ `\ \{a.\ h\ a \neq \mathbf{1}\}$
by (*auto simp add*: *image-Collect unfold elim*: *bij-pointE*)
moreover have $\bigwedge x.\ x \in \{a.\ h\ a \neq \mathbf{1}\} \Longrightarrow g\ (l\ x) = h\ x$
by (*simp add*: *unfold*)
ultimately have $F.F\ g\ \{a.\ g\ a \neq \mathbf{1}\} = F.F\ h\ \{a.\ h\ a \neq \mathbf{1}\}$
by (*rule F.reindex-cong*)
then show *?thesis* **by** (*simp add*: *expand-set*)
qed

lemma *distrib*:
assumes *finite* $\{a.\ g\ a \neq \mathbf{1}\}$ **and** *finite* $\{a.\ h\ a \neq \mathbf{1}\}$
shows $G\ (\lambda a.\ g\ a * h\ a) = G\ g * G\ h$
proof −
from *assms* **have** *finite* $(\{a.\ g\ a \neq \mathbf{1}\} \cup \{a.\ h\ a \neq \mathbf{1}\})$ **by** *simp*
moreover have $\{a.\ g\ a * h\ a \neq \mathbf{1}\} \subseteq \{a.\ g\ a \neq \mathbf{1}\} \cup \{a.\ h\ a \neq \mathbf{1}\}$
by *auto* (*drule sym*, *simp*)
ultimately show *?thesis*
using *assms*
by (*simp add*: *expand-superset* [*of* $\{a.\ g\ a \neq \mathbf{1}\} \cup \{a.\ h\ a \neq \mathbf{1}\}$] *F.distrib*)
qed

lemma *swap*:
assumes *finite C*
assumes *subset*: $\{a.\ \exists b.\ g\ a\ b \neq \mathbf{1}\} \times \{b.\ \exists a.\ g\ a\ b \neq \mathbf{1}\} \subseteq C$ (**is** *?A* × *?B* ⊆ *C*)
shows $G\ (\lambda a.\ G\ (g\ a)) = G\ (\lambda b.\ G\ (\lambda a.\ g\ a\ b))$
proof −
from ‹*finite C*› *subset*
have *finite* $(\{a.\ \exists b.\ g\ a\ b \neq \mathbf{1}\} \times \{b.\ \exists a.\ g\ a\ b \neq \mathbf{1}\})$
by (*rule rev-finite-subset*)
then have *fins*:
finite $\{b.\ \exists a.\ g\ a\ b \neq \mathbf{1}\}$ *finite* $\{a.\ \exists b.\ g\ a\ b \neq \mathbf{1}\}$
by (*auto simp add*: *finite-cartesian-product-iff*)
have *subsets*: $\bigwedge a.\ \{b.\ g\ a\ b \neq \mathbf{1}\} \subseteq \{b.\ \exists a.\ g\ a\ b \neq \mathbf{1}\}$
$\bigwedge b.\ \{a.\ g\ a\ b \neq \mathbf{1}\} \subseteq \{a.\ \exists b.\ g\ a\ b \neq \mathbf{1}\}$
$\{a.\ F.F\ (g\ a)\ \{b.\ \exists a.\ g\ a\ b \neq \mathbf{1}\} \neq \mathbf{1}\} \subseteq \{a.\ \exists b.\ g\ a\ b \neq \mathbf{1}\}$
$\{a.\ F.F\ (\lambda aa.\ g\ aa\ a)\ \{a.\ \exists b.\ g\ a\ b \neq \mathbf{1}\} \neq \mathbf{1}\} \subseteq \{b.\ \exists a.\ g\ a\ b \neq \mathbf{1}\}$

```
    by (auto elim: F.not-neutral-contains-not-neutral)
  from F.swap have
    F.F (λa. F.F (g a) {b. ∃ a. g a b ≠ 1}) {a. ∃ b. g a b ≠ 1} =
      F.F (λb. F.F (λa. g a b) {a. ∃ b. g a b ≠ 1}) {b. ∃ a. g a b ≠ 1} .
  with subsets fins have G (λa. F.F (g a) {b. ∃ a. g a b ≠ 1}) =
    G (λb. F.F (λa. g a b) {a. ∃ b. g a b ≠ 1})
    by (auto simp add: expand-superset [of {b. ∃ a. g a b ≠ 1}]
      expand-superset [of {a. ∃ b. g a b ≠ 1}])
  with subsets fins show ?thesis
    by (auto simp add: expand-superset [of {b. ∃ a. g a b ≠ 1}]
      expand-superset [of {a. ∃ b. g a b ≠ 1}])
qed

lemma cartesian-product:
  assumes finite C
  assumes subset: {a. ∃ b. g a b ≠ 1} × {b. ∃ a. g a b ≠ 1} ⊆ C (is ?A × ?B ⊆
C)
  shows G (λa. G (g a)) = G (λ(a, b). g a b)
proof −
  from subset ‹finite C› have fin-prod: finite (?A × ?B)
    by (rule finite-subset)
  from fin-prod have finite ?A and finite ?B
    by (auto simp add: finite-cartesian-product-iff)
  have ∗: G (λa. G (g a)) =
    (F.F (λa. F.F (g a) {b. ∃ a. g a b ≠ 1}) {a. ∃ b. g a b ≠ 1})
    apply (subst expand-superset [of ?B])
    apply (rule ‹finite ?B›)
    apply auto
    apply (subst expand-superset [of ?A])
    apply (rule ‹finite ?A›)
    apply auto
    apply (erule F.not-neutral-contains-not-neutral)
    apply auto
    done
  have {p. (case p of (a, b) ⇒ g a b) ≠ 1} ⊆ ?A × ?B
    by auto
  with subset have ∗∗: {p. (case p of (a, b) ⇒ g a b) ≠ 1} ⊆ C
    by blast
  show ?thesis
    apply (simp add: ∗)
    apply (simp add: F.cartesian-product)
    apply (subst expand-superset [of C])
    apply (rule ‹finite C›)
    apply (simp-all add: ∗∗)
    apply (rule F.same-carrierI [of C])
    apply (rule ‹finite C›)
    apply (simp-all add: subset)
    apply auto
    done
```

qed

lemma *cartesian-product2*:
assumes *fin*: *finite D*
assumes *subset*: $\{(a, b).\ \exists c.\ g\ a\ b\ c \neq \mathbf{1}\} \times \{c.\ \exists a\ b.\ g\ a\ b\ c \neq \mathbf{1}\} \subseteq D$ (**is** *?AB* × *?C* ⊆ *D*)
shows $G\ (\lambda(a, b).\ G\ (g\ a\ b)) = G\ (\lambda(a, b, c).\ g\ a\ b\ c)$
proof −
have *bij*: *bij* $(\lambda(a, b, c).\ ((a, b), c))$
by (*auto intro*!: *bijI injI simp add*: *image-def*)
have $\{p.\ \exists c.\ g\ (fst\ p)\ (snd\ p)\ c \neq \mathbf{1}\} \times \{c.\ \exists p.\ g\ (fst\ p)\ (snd\ p)\ c \neq \mathbf{1}\} \subseteq D$
by *auto* (*insert subset*, *blast*)
with *fin* **have** $G\ (\lambda p.\ G\ (g\ (fst\ p)\ (snd\ p))) = G\ (\lambda(p, c).\ g\ (fst\ p)\ (snd\ p)\ c)$
by (*rule cartesian-product*)
then have $G\ (\lambda(a, b).\ G\ (g\ a\ b)) = G\ (\lambda((a, b), c).\ g\ a\ b\ c)$
by (*auto simp add*: *split-def*)
also have $G\ (\lambda((a, b), c).\ g\ a\ b\ c) = G\ (\lambda(a, b, c).\ g\ a\ b\ c)$
using *bij* **by** (*rule reindex-cong* [*of* $\lambda(a, b, c).\ ((a, b), c)$]) (*simp add*: *fun-eq-iff*)
finally show *?thesis* **.**
qed

lemma *delta* [*simp*]:
$G\ (\lambda b.\ if\ b = a\ then\ g\ b\ else\ \mathbf{1}) = g\ a$
proof −
have $\{b.\ (if\ b = a\ then\ g\ b\ else\ \mathbf{1}) \neq \mathbf{1}\} \subseteq \{a\}$ **by** *auto*
then show *?thesis* **by** (*simp add*: *expand-superset* [*of* $\{a\}$])
qed

lemma *delta′* [*simp*]:
$G\ (\lambda b.\ if\ a = b\ then\ g\ b\ else\ \mathbf{1}) = g\ a$
proof −
have $(\lambda b.\ if\ a = b\ then\ g\ b\ else\ \mathbf{1}) = (\lambda b.\ if\ b = a\ then\ g\ b\ else\ \mathbf{1})$
by (*simp add*: *fun-eq-iff*)
then have $G\ (\lambda b.\ if\ a = b\ then\ g\ b\ else\ \mathbf{1}) = G\ (\lambda b.\ if\ b = a\ then\ g\ b\ else\ \mathbf{1})$
by (*simp cong del*: *cong*)
then show *?thesis* **by** *simp*
qed

end

46.2 Concrete sum

context *comm-monoid-add*
begin

sublocale *Sum-any*: *comm-monoid-fun plus 0*
rewrites *comm-monoid-set.F plus 0 = sum*
defines *Sum-any = Sum-any.G*
proof −

show *comm-monoid-fun plus 0* **..**
then interpret *Sum-any*: *comm-monoid-fun plus 0* **.**
from *sum-def* **show** *comm-monoid-set.F plus 0 = sum* **by** (*auto intro*: *sym*)
qed

end

syntax (*ASCII*)
-Sum-any :: *pttrn* ⇒ *'a* ⇒ *'a*::*comm-monoid-add* ((*3SUM -. -*) [*0*, *10*] *10*)
syntax
-Sum-any :: *pttrn* ⇒ *'a* ⇒ *'a*::*comm-monoid-add* ((*3*$\sum$ *-. -*) [*0*, *10*] *10*)
translations
$\sum a.\ b \rightleftharpoons$ *CONST Sum-any* ($\lambda a.\ b$)

lemma *Sum-any-left-distrib*:
fixes *r* :: *'a* :: *semiring-0*
assumes *finite* {*a. g a* ≠ *0*}
shows *Sum-any g* ∗ *r* = ($\sum n.\ g\ n$ ∗ *r*)
proof −
note *assms*
moreover have {*a. g a* ∗ *r* ≠ *0*} ⊆ {*a. g a* ≠ *0*} **by** *auto*
ultimately show *?thesis*
by (*simp add*: *sum-distrib-right Sum-any.expand-superset* [*of* {*a. g a* ≠ *0*}])
qed

lemma *Sum-any-right-distrib*:
fixes *r* :: *'a* :: *semiring-0*
assumes *finite* {*a. g a* ≠ *0*}
shows *r* ∗ *Sum-any g* = ($\sum n.\ r$ ∗ *g n*)
proof −
note *assms*
moreover have {*a. r* ∗ *g a* ≠ *0*} ⊆ {*a. g a* ≠ *0*} **by** *auto*
ultimately show *?thesis*
by (*simp add*: *sum-distrib-left Sum-any.expand-superset* [*of* {*a. g a* ≠ *0*}])
qed

lemma *Sum-any-product*:
fixes *f g* :: *'b* ⇒ *'a*::*semiring-0*
assumes *finite* {*a. f a* ≠ *0*} **and** *finite* {*b. g b* ≠ *0*}
shows *Sum-any f* ∗ *Sum-any g* = ($\sum a.\ \sum b.\ f\ a$ ∗ *g b*)
proof −
have *subset-f*: {*a.* ($\sum b.\ f\ a$ ∗ *g b*) ≠ *0*} ⊆ {*a. f a* ≠ *0*}
by *rule* (*simp*, *rule*, *auto*)
moreover have *subset-g*: $\bigwedge a.$ {*b. f a* ∗ *g b* ≠ *0*} ⊆ {*b. g b* ≠ *0*}
by *rule* (*simp*, *rule*, *auto*)
ultimately show *?thesis* **using** *assms*
by (*auto simp add*: *Sum-any.expand-set* [*of f*] *Sum-any.expand-set* [*of g*]
Sum-any.expand-superset [*of* {*a. f a* ≠ *0*}] *Sum-any.expand-superset* [*of* {*b. g b* ≠ *0*}]

sum-product)
qed

lemma *Sum-any-eq-zero-iff* [*simp*]:
fixes $f :: 'a \Rightarrow nat$
assumes *finite* $\{a.\ f\ a \neq 0\}$
shows *Sum-any* $f = 0 \longleftrightarrow f = (\lambda\text{-}.\ 0)$
using *assms* **by** (*simp add*: *Sum-any.expand-set fun-eq-iff*)

46.3 Concrete product

context *comm-monoid-mult*
begin

sublocale *Prod-any*: *comm-monoid-fun times 1*
rewrites *comm-monoid-set.F times 1 = prod*
defines *Prod-any = Prod-any.G*
proof −
show *comm-monoid-fun times 1* **..**
then interpret *Prod-any*: *comm-monoid-fun times 1* **.**
from *prod-def* **show** *comm-monoid-set.F times 1 = prod* **by** (*auto intro*: *sym*)
qed

end

syntax (*ASCII*)
-Prod-any :: *pttrn* ⇒ *'a* ⇒ *'a*::*comm-monoid-mult* ((*3PROD* -. -) [*0*, *10*] *10*)
syntax
-Prod-any :: *pttrn* ⇒ *'a* ⇒ *'a*::*comm-monoid-mult* ((*3*$\prod$ -. -) [*0*, *10*] *10*)
translations
$\prod a.\ b == CONST\ Prod\text{-}any\ (\lambda a.\ b)$

lemma *Prod-any-zero*:
fixes $f :: 'b \Rightarrow 'a$ :: *comm-semiring-1*
assumes *finite* $\{a.\ f\ a \neq 1\}$
assumes $f\ a = 0$
shows $(\prod a.\ f\ a) = 0$
proof −
from ‹$f\ a = 0$› **have** $f\ a \neq 1$ **by** *simp*
with ‹$f\ a = 0$› **have** $\exists a.\ f\ a \neq 1 \wedge f\ a = 0$ **by** *blast*
with ‹*finite* $\{a.\ f\ a \neq 1\}$› **show** *?thesis*
by (*simp add*: *Prod-any.expand-set prod-zero*)
qed

lemma *Prod-any-not-zero*:
fixes $f :: 'b \Rightarrow 'a$ :: *comm-semiring-1*
assumes *finite* $\{a.\ f\ a \neq 1\}$
assumes $(\prod a.\ f\ a) \neq 0$
shows $f\ a \neq 0$

using *assms Prod-any-zero* [*of f*] **by** *blast*

lemma *power-Sum-any*:
assumes *finite* {*a*. *f a* ≠ *0*}
shows *c* ^ (∑ *a*. *f a*) = (∏ *a*. *c* ^ *f a*)
proof −
have {*a*. *c* ^ *f a* ≠ *1*} ⊆ {*a*. *f a* ≠ *0*}
by (*auto intro*: *ccontr*)
with *assms* **show** *?thesis*
by (*simp add*: *Sum-any.expand-set Prod-any.expand-superset power-sum*)
qed

end

47 Interval Type

theory *Interval*
imports
Complex-Main
Lattice-Algebras
Set-Algebras
begin

A type of non-empty, closed intervals.

typedef (**overloaded**) *'a interval* =
{(*a*::*'a*::*preorder*, *b*). *a* ≤ *b*}
morphisms *bounds-of-interval Interval*
by *auto*

setup-lifting *type-definition-interval*

lift-definition *lower*::(*'a*::*preorder*) *interval* ⇒ *'a* **is** *fst* .

lift-definition *upper*::(*'a*::*preorder*) *interval* ⇒ *'a* **is** *snd* .

lemma *interval-eq-iff*: *a* = *b* ⟷ *lower a* = *lower b* ∧ *upper a* = *upper b*
by *transfer auto*

lemma *interval-eqI*: *lower a* = *lower b* ⟹ *upper a* = *upper b* ⟹ *a* = *b*
by (*auto simp*: *interval-eq-iff*)

lemma *lower-le-upper*[*simp*]: *lower i* ≤ *upper i*
by *transfer auto*

lift-definition *set-of* :: *'a*::*preorder interval* ⇒ *'a set* **is** λ*x*. {*fst x* .. *snd x*} .

lemma *set-of-eq*: *set-of x* = {*lower x* .. *upper x*}
by *transfer simp*

context notes [[*typedef-overloaded*]] **begin**

lift-definition(*code-dt*) *Interval'*::'*a*::*preorder* ⇒ '*a*::*preorder* ⇒ '*a interval option*
is λ*a b*. *if a* ≤ *b then Some* (*a*, *b*) *else None*
by *auto*

lemma *Interval'-split*:
P (*Interval' a b*) ⟷
(∀ *ivl*. *a* ≤ *b* ⟶ *lower ivl* = *a* ⟶ *upper ivl* = *b* ⟶ *P* (*Some ivl*)) ∧ (¬*a*≤*b* ⟶ *P None*)
by *transfer auto*

lemma *Interval'-split-asm*:
P (*Interval' a b*) ⟷
¬((∃ *ivl*. *a* ≤ *b* ∧ *lower ivl* = *a* ∧ *upper ivl* = *b* ∧ ¬*P* (*Some ivl*)) ∨ (¬*a*≤*b* ∧ ¬*P None*))
unfolding *Interval'-split*
by *auto*

lemmas *Interval'-splits* = *Interval'-split Interval'-split-asm*

lemma *Interval'-eq-Some*: *Interval' a b* = *Some i* ⟹ *lower i* = *a* ∧ *upper i* = *b*
by (*simp split*: *Interval'-splits*)

end

instantiation *interval* :: ({*preorder*,*equal*}) *equal*
begin

definition *equal-class.equal a b* ≡ (*lower a* = *lower b*) ∧ (*upper a* = *upper b*)

instance proof qed (*simp add*: *equal-interval-def interval-eq-iff*)
end

instantiation *interval* :: (*preorder*) *ord* **begin**

definition *less-eq-interval* :: '*a interval* ⇒ '*a interval* ⇒ *bool*
where *less-eq-interval a b* ⟷ *lower b* ≤ *lower a* ∧ *upper a* ≤ *upper b*

definition *less-interval* :: '*a interval* ⇒ '*a interval* ⇒ *bool*
where *less-interval x y* = (*x* ≤ *y* ∧ ¬ *y* ≤ *x*)

instance proof qed
end

instantiation *interval* :: (*lattice*) *semilattice-sup*
begin

lift-definition *sup-interval* :: '*a interval* ⇒ '*a interval* ⇒ '*a interval*

is $\lambda(a,\ b)\ (c,\ d).\ (inf\ a\ c,\ sup\ b\ d)$
by (*auto simp*: *le-infI1 le-supI1*)

lemma *lower-sup*[*simp*]: $lower\ (sup\ A\ B) = inf\ (lower\ A)\ (lower\ B)$
by *transfer auto*

lemma *upper-sup*[*simp*]: $upper\ (sup\ A\ B) = sup\ (upper\ A)\ (upper\ B)$
by *transfer auto*

instance proof qed (*auto simp*: *less-eq-interval-def less-interval-def interval-eq-iff*)
end

lemma *set-of-interval-union*: $set\text{-}of\ A \cup set\text{-}of\ B \subseteq set\text{-}of\ (sup\ A\ B)$ **for** A::$'a$::*lattice interval*
by (*auto simp*: *set-of-eq*)

lemma *interval-union-commute*: $sup\ A\ B = sup\ B\ A$ **for** A::$'a$::*lattice interval*
by (*auto simp add*: *interval-eq-iff inf.commute sup.commute*)

lemma *interval-union-mono1*: $set\text{-}of\ a \subseteq set\text{-}of\ (sup\ a\ A)$ **for** A :: $'a$::*lattice interval*
using *set-of-interval-union* **by** *blast*

lemma *interval-union-mono2*: $set\text{-}of\ A \subseteq set\text{-}of\ (sup\ a\ A)$ **for** A :: $'a$::*lattice interval*
using *set-of-interval-union* **by** *blast*

lift-definition *interval-of* :: $'a$::*preorder* $\Rightarrow$ $'a$ *interval* **is** $\lambda x.\ (x,\ x)$
by *auto*

lemma *lower-interval-of*[*simp*]: $lower\ (interval\text{-}of\ a) = a$
by *transfer auto*

lemma *upper-interval-of*[*simp*]: $upper\ (interval\text{-}of\ a) = a$
by *transfer auto*

definition *width* :: $'a$::{*preorder*,*minus*} *interval* $\Rightarrow$ $'a$
where $width\ i = upper\ i - lower\ i$

instantiation *interval* :: (*ordered-ab-semigroup-add*) *ab-semigroup-add*
begin

lift-definition *plus-interval*::$'a$ *interval* $\Rightarrow$ $'a$ *interval* $\Rightarrow$ $'a$ *interval*
is $\lambda(a,\ b).\ \lambda(c,\ d).\ (a + c,\ b + d)$
by (*auto intro*!: *add-mono*)
lemma *lower-plus*[*simp*]: $lower\ (plus\ A\ B) = plus\ (lower\ A)\ (lower\ B)$
by *transfer auto*
lemma *upper-plus*[*simp*]: $upper\ (plus\ A\ B) = plus\ (upper\ A)\ (upper\ B)$

by *transfer auto*

instance proof qed (*auto simp*: *interval-eq-iff less-eq-interval-def ac-simps*)
end

instance *interval* :: ({*ordered-ab-semigroup-add*, *lattice*}) *ordered-ab-semigroup-add*
proof qed (*auto simp*: *less-eq-interval-def intro*!: *add-mono*)

instantiation *interval* :: ({*preorder*,*zero*}) *zero*
begin

lift-definition *zero-interval*::'*a interval* **is** (*0*, *0*) **by** *auto*
lemma *lower-zero*[*simp*]: *lower 0 = 0*
by *transfer auto*
lemma *upper-zero*[*simp*]: *upper 0 = 0*
by *transfer auto*
instance proof qed
end

instance *interval* :: ({*ordered-comm-monoid-add*}) *comm-monoid-add*
proof qed (*auto simp*: *interval-eq-iff*)

instance *interval* :: ({*ordered-comm-monoid-add*,*lattice*}) *ordered-comm-monoid-add*
..

instantiation *interval* :: ({*ordered-ab-group-add*}) *uminus*
begin

lift-definition *uminus-interval*::'*a interval* ⇒ '*a interval* **is** λ(*a*, *b*). (−*b*, −*a*) **by** *auto*
lemma *lower-uminus*[*simp*]: *lower* (− *A*) = − *upper A*
by *transfer auto*
lemma *upper-uminus*[*simp*]: *upper* (− *A*) = − *lower A*
by *transfer auto*
instance ..
end

instantiation *interval* :: ({*ordered-ab-group-add*}) *minus*
begin

definition *minus-interval*::'*a interval* ⇒ '*a interval* ⇒ '*a interval*
where *minus-interval a b* = *a* + − *b*
lemma *lower-minus*[*simp*]: *lower* (*minus A B*) = *minus* (*lower A*) (*upper B*)
by (*auto simp*: *minus-interval-def*)
lemma *upper-minus*[*simp*]: *upper* (*minus A B*) = *minus* (*upper A*) (*lower B*)
by (*auto simp*: *minus-interval-def*)

instance ..
end

instantiation *interval* :: (*linordered-semiring*) *times*
begin

lift-definition *times-interval* :: *'a interval* ⇒ *'a interval* ⇒ *'a interval*
is λ(*a1*, *a2*). λ(*b1*, *b2*).
(*let x1* = *a1* ∗ *b1*; *x2* = *a1* ∗ *b2*; *x3* = *a2* ∗ *b1*; *x4* = *a2* ∗ *b2*
in (*min x1* (*min x2* (*min x3 x4*)), *max x1* (*max x2* (*max x3 x4*))))
by (*auto simp*: *Let-def intro*!: *min.coboundedI1 max.coboundedI1*)

lemma *lower-times*:
lower (*times A B*) = *Min* {*lower A* ∗ *lower B*, *lower A* ∗ *upper B*, *upper A* ∗ *lower B*, *upper A* ∗ *upper B*}
by *transfer* (*auto simp*: *Let-def*)

lemma *upper-times*:
upper (*times A B*) = *Max* {*lower A* ∗ *lower B*, *lower A* ∗ *upper B*, *upper A* ∗ *lower B*, *upper A* ∗ *upper B*}
by *transfer* (*auto simp*: *Let-def*)

instance ..
end

lemma *interval-eq-set-of-iff*: *X* = *Y* ⟷ *set-of X* = *set-of Y* **for** *X Y*::*'a*::*order interval*
by (*auto simp*: *set-of-eq interval-eq-iff*)

47.1 Membership

abbreviation (**in** *preorder*) *in-interval* ((-/ ∈$_i$ -) [*51*, *51*] *50*)
where *in-interval x X* ≡ *x* ∈ *set-of X*

lemma *in-interval-to-interval*[*intro*!]: *a* ∈$_i$ *interval-of a*
by (*auto simp*: *set-of-eq*)

lemma *plus-in-intervalI*:
fixes *x y* :: *'a* :: *ordered-ab-semigroup-add*
shows *x* ∈$_i$ *X* ⟹ *y* ∈$_i$ *Y* ⟹ *x* + *y* ∈$_i$ *X* + *Y*
by (*simp add*: *add-mono-thms-linordered-semiring*(*1*) *set-of-eq*)

lemma *connected-set-of*[*intro*, *simp*]:
connected (*set-of X*) **for** *X*::*'a*::*linear-continuum-topology interval*
by (*auto simp*: *set-of-eq*)

lemma *ex-sum-in-interval-lemma*: ∃ *xa*∈{*la* .. *ua*}. ∃ *xb*∈{*lb* .. *ub*}. *x* = *xa* + *xb*
if *la* ≤ *ua lb* ≤ *ub la* + *lb* ≤ *x x* ≤ *ua* + *ub*
ua − *la* ≤ *ub* − *lb*
for *la b c d*::*'a*::*linordered-ab-group-add*
proof −

```
  define wa where wa = ua − la
  define wb where wb = ub − lb
  define w where w = wa + wb
  define d where d = x − la − lb
  define da where da = max 0 (min wa (d − wa))
  define db where db = d − da
  from that have nonneg: 0 ≤ wa 0 ≤ wb 0 ≤ w 0 ≤ d d ≤ w
    by (auto simp add: wa-def wb-def w-def d-def add.commute le-diff-eq)
  have 0 ≤ db
    by (auto simp: da-def nonneg db-def intro!: min.coboundedI2)
  have x = (la + da) + (lb + db)
    by (simp add: da-def db-def d-def)
  moreover
  have x − la − ub ≤ da
    using that
    unfolding da-def
    by (intro max.coboundedI2) (auto simp: wa-def d-def diff-le-eq diff-add-eq)
  then have db ≤ wb
    by (auto simp: db-def d-def wb-def algebra-simps)
  with ‹0 ≤ db› that nonneg have lb + db ∈ {lb..ub}
    by (auto simp: wb-def algebra-simps)
  moreover
  have da ≤ wa
    by (auto simp: da-def nonneg)
  then have la + da ∈ {la..ua}
    by (auto simp: da-def wa-def algebra-simps)
  ultimately show ?thesis
    by force
qed


lemma ex-sum-in-interval: ∃xa≥la. xa ≤ ua ∧ (∃xb≥lb. xb ≤ ub ∧ x = xa + xb)
  if a: la ≤ ua and b: lb ≤ ub and x: la + lb ≤ x x ≤ ua + ub
  for la b c d::'a::linordered-ab-group-add
proof −
  from linear consider ua − la ≤ ub − lb | ub − lb ≤ ua − la
    by blast
  then show ?thesis
  proof cases
    case 1
    from ex-sum-in-interval-lemma[OF that 1]
    show ?thesis by auto
  next
    case 2
    from x have lb + la ≤ x x ≤ ub + ua by (simp-all add: ac-simps)
    from ex-sum-in-interval-lemma[OF b a this 2]
    show ?thesis by auto
  qed
qed
```

lemma *Icc-plus-Icc*:
if $a \leq b$ $c \leq d$
for a b c d::$'a$::*linordered-ab-group-add*
using *ex-sum-in-interval*[*OF that*]
by (*auto intro*: *add-mono simp*: *atLeastAtMost-iff Bex-def set-plus-def*)

lemma *set-of-plus*:
fixes A :: $'a$::*linordered-ab-group-add interval*
shows *set-of* $(A + B)$ = *set-of* A + *set-of* B
using *Icc-plus-Icc*[*of lower A upper A lower B upper B*]
by (*auto simp*: *set-of-eq*)

lemma *plus-in-intervalE*:
fixes xy :: $'a$ :: *linordered-ab-group-add*
assumes $xy \in_i X + Y$
obtains x y **where** $xy = x + y$ $x \in_i X$ $y \in_i Y$
using *assms*
unfolding *set-of-plus set-plus-def*
by *auto*

lemma *set-of-uminus*: *set-of* $(-X) = \{- x \mid x.\ x \in$ *set-of* $X\}$
for X :: $'a$ :: *ordered-ab-group-add interval*
by (*auto simp*: *set-of-eq simp*: *le-minus-iff minus-le-iff*
intro!: *exI*[**where** x=$-x$ **for** x])

lemma *uminus-in-intervalI*:
fixes x :: $'a$ :: *ordered-ab-group-add*
shows $x \in_i X \Longrightarrow -x \in_i -X$
by (*auto simp*: *set-of-uminus*)

lemma *uminus-in-intervalD*:
fixes x :: $'a$ :: *ordered-ab-group-add*
shows $x \in_i - X \Longrightarrow - x \in_i X$
by (*auto simp*: *set-of-uminus*)

lemma *minus-in-intervalI*:
fixes x y :: $'a$ :: *ordered-ab-group-add*
shows $x \in_i X \Longrightarrow y \in_i Y \Longrightarrow x - y \in_i X - Y$
by (*metis diff-conv-add-uminus minus-interval-def plus-in-intervalI uminus-in-intervalI*)

lemma *set-of-minus*: *set-of* $(X - Y) = \{x - y \mid x\ y\ .\ x \in$ *set-of* $X \wedge y \in$ *set-of* $Y\}$
for X Y :: $'a$ :: *linordered-ab-group-add interval*
unfolding *minus-interval-def set-of-plus set-of-uminus set-plus-def*
by *force*

lemma *times-in-intervalI*:

fixes $x\ y::'a::linordered\text{-}ring$
assumes $x \in_i X\ y \in_i Y$
shows $x * y \in_i X * Y$
proof −
define $X1$ **where** $X1 \equiv lower\ X$
define $X2$ **where** $X2 \equiv upper\ X$
define $Y1$ **where** $Y1 \equiv lower\ Y$
define $Y2$ **where** $Y2 \equiv upper\ Y$
from *assms* **have** *assms*: $X1 \leq x\ x \leq X2\ Y1 \leq y\ y \leq Y2$
by (*auto simp*: *X1-def X2-def Y1-def Y2-def set-of-eq*)
have $(X1 * Y1 \leq x * y \vee X1 * Y2 \leq x * y \vee X2 * Y1 \leq x * y \vee X2 * Y2 \leq x * y) \wedge$
$(X1 * Y1 \geq x * y \vee X1 * Y2 \geq x * y \vee X2 * Y1 \geq x * y \vee X2 * Y2 \geq x * y)$
proof (*cases x 0*::'*a rule*: *linorder-cases*)
case *x0*: *less*
show *?thesis*
proof (*cases* $y < 0$)
case *y0*: *True*
from *y0 x0 assms* **have** $x * y \leq X1 * y$ **by** (*intro mult-right-mono-neg*, *auto*)
also from *x0 y0 assms* **have** $X1 * y \leq X1 * Y1$ **by** (*intro mult-left-mono-neg*, *auto*)
finally have *1*: $x * y \leq X1 * Y1$**.**
show *?thesis* **proof**(*cases* $X2 \leq 0$)
case *True*
with *assms* **have** $X2 * Y2 \leq X2 * y$ **by** (*auto intro*: *mult-left-mono-neg*)
also from *assms y0* **have** $... \leq x * y$ **by** (*auto intro*: *mult-right-mono-neg*)
finally have $X2 * Y2 \leq x * y$**.**
with *1* **show** *?thesis* **by** *auto*
next
case *False*
with *assms* **have** $X2 * Y1 \leq X2 * y$ **by** (*auto intro*: *mult-left-mono*)
also from *assms y0* **have** $... \leq x * y$ **by** (*auto intro*: *mult-right-mono-neg*)
finally have $X2 * Y1 \leq x * y$**.**
with *1* **show** *?thesis* **by** *auto*
qed
next
case *False*
then have *y0*: $y \geq 0$ **by** *auto*
from *x0 y0 assms* **have** $X1 * Y2 \leq x * Y2$ **by** (*intro mult-right-mono*, *auto*)
also from *y0 x0 assms* **have** $... \leq x * y$ **by** (*intro mult-left-mono-neg*, *auto*)
finally have *1*: $X1 * Y2 \leq x * y$**.**
show *?thesis*
proof(*cases* $X2 \leq 0$)
case *X2*: *True*
from *assms y0* **have** $x * y \leq X2 * y$ **by** (*intro mult-right-mono*)
also from *assms X2* **have** $... \leq X2 * Y1$ **by** (*auto intro*: *mult-left-mono-neg*)
finally have $x * y \leq X2 * Y1$**.**
with *1* **show** *?thesis* **by** *auto*

```
      next
        case X2: False
        from assms y0 have x * y ≤ X2 * y by (intro mult-right-mono)
        also from assms X2 have ... ≤ X2 * Y2 by (auto intro: mult-left-mono)
        finally have x * y ≤ X2 * Y2.
        with 1 show ?thesis by auto
      qed
    qed
  next
    case [simp]: equal
    with assms show ?thesis by (cases Y2 ≤ 0, auto intro:mult-sign-intros)
  next
    case x0: greater
    show ?thesis
    proof (cases y < 0)
      case y0: True
      from x0 y0 assms have X2 * Y1 ≤ X2 * y by (intro mult-left-mono, auto)
      also from y0 x0 assms have X2 * y ≤ x * y by (intro mult-right-mono-neg,
auto)
      finally have 1: X2 * Y1 ≤ x * y.
      show ?thesis
      proof(cases Y2 ≤ 0)
        case Y2: True
        from x0 assms have x * y ≤ x * Y2 by (auto intro: mult-left-mono)
      also from assms Y2 have ... ≤ X1 * Y2 by (auto intro: mult-right-mono-neg)
        finally have x * y ≤ X1 * Y2.
        with 1 show ?thesis by auto
      next
        case Y2: False
        from x0 assms have x * y ≤ x * Y2 by (auto intro: mult-left-mono)
       also from assms Y2 have ... ≤ X2 * Y2 by (auto intro: mult-right-mono)
        finally have x * y ≤ X2 * Y2.
        with 1 show ?thesis by auto
      qed
    next
      case y0: False
      from x0 y0 assms have x * y ≤ X2 * y by (intro mult-right-mono, auto)
      also from y0 x0 assms have ... ≤ X2 * Y2 by (intro mult-left-mono, auto)
      finally have 1: x * y ≤ X2 * Y2.
      show ?thesis
      proof(cases X1 ≤ 0)
        case True
        with assms have X1 * Y2 ≤ X1 * y by (auto intro: mult-left-mono-neg)
        also from assms y0 have ... ≤ x * y by (auto intro: mult-right-mono)
        finally have X1 * Y2 ≤ x * y.
        with 1 show ?thesis by auto
      next
        case False
        with assms have X1 * Y1 ≤ X1 * y by (auto intro: mult-left-mono)
```

```
      also from assms y0 have ... ≤ x * y by (auto intro: mult-right-mono)
      finally have X1 * Y1 ≤ x * y.
      with 1 show ?thesis by auto
    qed
  qed
 qed
 hence min:min (X1 * Y1) (min (X1 * Y2) (min (X2 * Y1) (X2 * Y2))) ≤ x
* y
   and max:x * y ≤ max (X1 * Y1) (max (X1 * Y2) (max (X2 * Y1) (X2 *
Y2)))
   by (auto simp:min-le-iff-disj le-max-iff-disj)
 show ?thesis using min max
   by (auto simp: Let-def X1-def X2-def Y1-def Y2-def set-of-eq lower-times up-
per-times)
qed

lemma times-in-intervalE:
 fixes xy :: 'a :: {linordered-semiring, real-normed-algebra, linear-continuum-topology}
   — TODO: linear continuum topology is pretty strong
 assumes xy ∈i X * Y
 obtains x y where xy = x * y x ∈i X y ∈i Y
proof −
 let ?mult = λ(x, y). x * y
 let ?XY = set-of X × set-of Y
 have cont: continuous-on ?XY ?mult
   by (auto intro!: tendsto-eq-intros simp: continuous-on-def split-beta')
 have conn: connected (?mult ' ?XY)
   by (rule connected-continuous-image[OF cont]) auto
 have lower (X * Y) ∈ ?mult ' ?XY upper (X * Y) ∈ ?mult ' ?XY
  by (auto simp: set-of-eq lower-times upper-times min-def max-def split: if-splits)
 from connectedD-interval[OF conn this, of xy] assms
 obtain x y where xy = x * y x ∈i X y ∈i Y by (auto simp: set-of-eq)
 then show ?thesis ..
qed

lemma set-of-times: set-of (X * Y) = {x * y | x y. x ∈ set-of X ∧ y ∈ set-of Y}
 for X Y::'a :: {linordered-ring, real-normed-algebra, linear-continuum-topology}
interval
 by (auto intro!: times-in-intervalI elim!: times-in-intervalE)

instance interval :: (linordered-idom) cancel-semigroup-add
proof qed (auto simp: interval-eq-iff)

lemma interval-mul-commute: A * B = B * A for A B:: 'a::linordered-idom in-
terval
 by (simp add: interval-eq-iff lower-times upper-times ac-simps)

lemma interval-times-zero-right[simp]: A * 0 = 0 for A :: 'a::linordered-ring in-
terval
```

by (*simp add*: *interval-eq-iff lower-times upper-times ac-simps*)

lemma *interval-times-zero-left*[*simp*]:
*0 * A = 0* **for** *A* :: *'a*::*linordered-ring interval*
by (*simp add*: *interval-eq-iff lower-times upper-times ac-simps*)

instantiation *interval* :: ({*preorder*,*one*}) *one*
begin

lift-definition *one-interval*::*'a interval* **is** (*1*, *1*) **by** *auto*
lemma *lower-one*[*simp*]: *lower 1 = 1*
by *transfer auto*
lemma *upper-one*[*simp*]: *upper 1 = 1*
by *transfer auto*
instance proof qed
end

instance *interval* :: ({*one*, *preorder*, *linordered-semiring*}) *power*
proof qed

lemma *set-of-one*[*simp*]: *set-of* (*1*::*'a*::{*one*, *order*} *interval*) = {*1*}
by (*auto simp*: *set-of-eq*)

instance *interval* ::
({*linordered-idom*,*linordered-ring*, *real-normed-algebra*, *linear-continuum-topology*})
monoid-mult
apply *standard*
unfolding *interval-eq-set-of-iff set-of-times*
subgoal
by (*auto simp*: *interval-eq-set-of-iff set-of-times*; *metis mult.assoc*)
by *auto*

lemma *one-times-ivl-left*[*simp*]: *1 * A = A* **for** *A* :: *'a*::*linordered-idom interval*
by (*simp add*: *interval-eq-iff lower-times upper-times ac-simps min-def max-def*)

lemma *one-times-ivl-right*[*simp*]: *A * 1 = A* **for** *A* :: *'a*::*linordered-idom interval*
by (*metis interval-mul-commute one-times-ivl-left*)

lemma *set-of-power-mono*: *a^n* ∈ *set-of* (*A^n*) **if** *a* ∈ *set-of A*
for *a* :: *'a*::*linordered-idom*
using *that*
by (*induction n*) (*auto intro*!: *times-in-intervalI*)

lemma *set-of-add-cong*:
set-of (*A* + *B*) = *set-of* (*A'* + *B'*)
if *set-of A = set-of A' set-of B = set-of B'*
for *A* :: *'a*::*linordered-ab-group-add interval*
unfolding *set-of-plus that* **..**

lemma *set-of-add-inc-left*:
set-of $(A + B) \subseteq$ *set-of* $(A' + B)$
if *set-of* $A \subseteq$ *set-of* A'
for A :: *'a*::*linordered-ab-group-add interval*
unfolding *set-of-plus* **using** *that* **by** (*auto simp*: *set-plus-def*)

lemma *set-of-add-inc-right*:
set-of $(A + B) \subseteq$ *set-of* $(A + B')$
if *set-of* $B \subseteq$ *set-of* B'
for A :: *'a*::*linordered-ab-group-add interval*
using *set-of-add-inc-left*[*OF that*]
by (*simp add*: *add.commute*)

lemma *set-of-add-inc*:
set-of $(A + B) \subseteq$ *set-of* $(A' + B')$
if *set-of* $A \subseteq$ *set-of* A' *set-of* $B \subseteq$ *set-of* B'
for A :: *'a*::*linordered-ab-group-add interval*
using *set-of-add-inc-left*[*OF that*(*1*)] *set-of-add-inc-right*[*OF that*(*2*)]
by *auto*

lemma *set-of-neg-inc*:
set-of $(-A) \subseteq$ *set-of* $(-A')$
if *set-of* $A \subseteq$ *set-of* A'
for A :: *'a*::*ordered-ab-group-add interval*
using *that*
unfolding *set-of-uminus*
by *auto*

lemma *set-of-sub-inc-left*:
set-of $(A - B) \subseteq$ *set-of* $(A' - B)$
if *set-of* $A \subseteq$ *set-of* A'
for A :: *'a*::*linordered-ab-group-add interval*
using *that*
unfolding *set-of-minus*
by *auto*

lemma *set-of-sub-inc-right*:
set-of $(A - B) \subseteq$ *set-of* $(A - B')$
if *set-of* $B \subseteq$ *set-of* B'
for A :: *'a*::*linordered-ab-group-add interval*
using *that*
unfolding *set-of-minus*
by *auto*

lemma *set-of-sub-inc*:
set-of $(A - B) \subseteq$ *set-of* $(A' - B')$
if *set-of* $A \subseteq$ *set-of* A' *set-of* $B \subseteq$ *set-of* B'
for A :: *'a*::*linordered-idom interval*
using *set-of-sub-inc-left*[*OF that*(*1*)] *set-of-sub-inc-right*[*OF that*(*2*)]

by *auto*

lemma *set-of-mul-inc-right*:
set-of $(A * B) \subseteq$ *set-of* $(A * B')$
if *set-of* $B \subseteq$ *set-of* B'
for A :: *'a*::*linordered-ring interval*
using *that*
apply *transfer*
apply (*clarsimp simp add*: *Let-def*)
apply (*intro conjI*)
apply (*metis linear min.coboundedI1 min.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear min.coboundedI1 min.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear min.coboundedI1 min.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear min.coboundedI1 min.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear max.coboundedI1 max.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear max.coboundedI1 max.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear max.coboundedI1 max.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
apply (*metis linear max.coboundedI1 max.coboundedI2 mult-left-mono mult-left-mono-neg order-trans*)
done

lemma *set-of-distrib-left*:
set-of $(B * (A1 + A2)) \subseteq$ *set-of* $(B * A1 + B * A2)$
for $A1$ :: *'a*::*linordered-ring interval*
apply *transfer*
apply (*clarsimp simp*: *Let-def distrib-left distrib-right*)
apply (*intro conjI*)
apply (*metis add-mono min.cobounded1 min.left-commute*)
apply (*metis add-mono min.cobounded1 min.left-commute*)
apply (*metis add-mono min.cobounded1 min.left-commute*)
apply (*metis add-mono min.assoc min.cobounded2*)
apply (*meson add-mono order.trans max.cobounded1 max.cobounded2*)
apply (*meson add-mono order.trans max.cobounded1 max.cobounded2*)
apply (*meson add-mono order.trans max.cobounded1 max.cobounded2*)
apply (*meson add-mono order.trans max.cobounded1 max.cobounded2*)
done

lemma *set-of-distrib-right*:
set-of $((A1 + A2) * B) \subseteq$ *set-of* $(A1 * B + A2 * B)$
for $A1\ A2\ B$ :: *'a*::{*linordered-ring*, *real-normed-algebra*, *linear-continuum-topology*} *interval*
unfolding *set-of-times set-of-plus set-plus-def*

apply *clarsimp*
subgoal for *b a1 a2*
apply (*rule exI*[**where** *x*=*a1* ∗ *b*])
apply (*rule conjI*)
subgoal by *force*
subgoal
apply (*rule exI*[**where** *x*=*a2* ∗ *b*])
apply (*rule conjI*)
subgoal by *force*
subgoal by (*simp add*: *algebra-simps*)
done
done
done

lemma *set-of-mul-inc-left*:
set-of (*A* ∗ *B*) ⊆ *set-of* (*A'* ∗ *B*)
if *set-of A* ⊆ *set-of A'*
for *A* :: *'a*::{*linordered-ring*, *real-normed-algebra*, *linear-continuum-topology*} *interval*
using *that*
unfolding *set-of-times*
by *auto*

lemma *set-of-mul-inc*:
set-of (*A* ∗ *B*) ⊆ *set-of* (*A'* ∗ *B'*)
if *set-of A* ⊆ *set-of A' set-of B* ⊆ *set-of B'*
for *A* :: *'a*::{*linordered-ring*, *real-normed-algebra*, *linear-continuum-topology*} *interval*
using *that* **unfolding** *set-of-times* **by** *auto*

lemma *set-of-pow-inc*:
set-of (*A*^*n*) ⊆ *set-of* (*A'*^*n*)
if *set-of A* ⊆ *set-of A'*
for *A* :: *'a*::{*linordered-idom*, *real-normed-algebra*, *linear-continuum-topology*} *interval*
using *that*
by (*induction n*, *simp-all add*: *set-of-mul-inc*)

lemma *set-of-distrib-right-left*:
set-of ((*A1* + *A2*) ∗ (*B1* + *B2*)) ⊆ *set-of* (*A1* ∗ *B1* + *A1* ∗ *B2* + *A2* ∗ *B1* + *A2* ∗ *B2*)
for *A1* :: *'a*::{*linordered-idom*, *real-normed-algebra*, *linear-continuum-topology*} *interval*
proof−
have *set-of* ((*A1* + *A2*) ∗ (*B1* + *B2*)) ⊆ *set-of* (*A1* ∗ (*B1* + *B2*) + *A2* ∗ (*B1* + *B2*))
by (*rule set-of-distrib-right*)
also have ... ⊆ *set-of* ((*A1* ∗ *B1* + *A1* ∗ *B2*) + *A2* ∗ (*B1* + *B2*))
by (*rule set-of-add-inc-left*[*OF set-of-distrib-left*])

also have ... ⊆ *set-of* ((*A1* ∗ *B1* + *A1* ∗ *B2*) + (*A2* ∗ *B1* + *A2* ∗ *B2*))
by (*rule set-of-add-inc-right*[*OF set-of-distrib-left*])
finally show *?thesis*
by (*simp add*: *add.assoc*)
qed

lemma *mult-bounds-enclose-zero1*:
min (*la* ∗ *lb*) (*min* (*la* ∗ *ub*) (*min* (*lb* ∗ *ua*) (*ua* ∗ *ub*))) ≤ *0*
0 ≤ *max* (*la* ∗ *lb*) (*max* (*la* ∗ *ub*) (*max* (*lb* ∗ *ua*) (*ua* ∗ *ub*)))
if *la* ≤ *0 0* ≤ *ua*
for *la lb ua ub*:: *'a*::*linordered-idom*
subgoal by (*metis* (*no-types*, *opaque-lifting*) *that eq-iff min-le-iff-disj mult-zero-left mult-zero-right*
zero-le-mult-iff)
subgoal by (*metis that le-max-iff-disj mult-zero-right order-refl zero-le-mult-iff*)
done

lemma *mult-bounds-enclose-zero2*:
min (*la* ∗ *lb*) (*min* (*la* ∗ *ub*) (*min* (*lb* ∗ *ua*) (*ua* ∗ *ub*))) ≤ *0*
0 ≤ *max* (*la* ∗ *lb*) (*max* (*la* ∗ *ub*) (*max* (*lb* ∗ *ua*) (*ua* ∗ *ub*)))
if *lb* ≤ *0 0* ≤ *ub*
for *la lb ua ub*:: *'a*::*linordered-idom*
using *mult-bounds-enclose-zero1*[*OF that*, *of la ua*]
by (*simp-all add*: *ac-simps*)

lemma *set-of-mul-contains-zero*:
0 ∈ *set-of* (*A* ∗ *B*)
if *0* ∈ *set-of A* ∨ *0* ∈ *set-of B*
for *A* :: *'a*::*linordered-idom interval*
using *that*
by (*auto simp*: *set-of-eq lower-times upper-times algebra-simps mult-le-0-iff*
mult-bounds-enclose-zero1 mult-bounds-enclose-zero2)

instance *interval* :: (*linordered-semiring*) *mult-zero*
apply *standard*
subgoal by *transfer auto*
subgoal by *transfer auto*
done

lift-definition *min-interval*::*'a*::*linorder interval* ⇒ *'a interval* ⇒ *'a interval* **is**
λ(*l1*, *u1*). λ(*l2*, *u2*). (*min l1 l2*, *min u1 u2*)
by (*auto simp*: *min-def*)
lemma *lower-min-interval*[*simp*]: *lower* (*min-interval x y*) = *min* (*lower x*) (*lower y*)
by *transfer auto*
lemma *upper-min-interval*[*simp*]: *upper* (*min-interval x y*) = *min* (*upper x*) (*upper y*)
by *transfer auto*

lemma *min-intervalI*:
a $\in_i$ *A* $\Longrightarrow$ *b* $\in_i$ *B* $\Longrightarrow$ *min a b* $\in_i$ *min-interval A B*
by (*auto simp*: *set-of-eq min-def*)

lift-definition *max-interval*::'*a*::*linorder interval* $\Rightarrow$ '*a interval* $\Rightarrow$ '*a interval* **is**
λ(*l1*, *u1*). λ(*l2*, *u2*). (*max l1 l2*, *max u1 u2*)
by (*auto simp*: *max-def*)
lemma *lower-max-interval*[*simp*]: *lower* (*max-interval x y*) = *max* (*lower x*) (*lower y*)
by *transfer auto*
lemma *upper-max-interval*[*simp*]: *upper* (*max-interval x y*) = *max* (*upper x*) (*upper y*)
by *transfer auto*

lemma *max-intervalI*:
a $\in_i$ *A* $\Longrightarrow$ *b* $\in_i$ *B* $\Longrightarrow$ *max a b* $\in_i$ *max-interval A B*
by (*auto simp*: *set-of-eq max-def*)

lift-definition *abs-interval*::'*a*::*linordered-idom interval* $\Rightarrow$ '*a interval* **is**
(λ(*l*,*u*). (*if l* $<$ *0* $\wedge$ *0* $<$ *u then 0 else min* |*l*| |*u*|, *max* |*l*| |*u*|))
by *auto*

lemma *lower-abs-interval*[*simp*]:
lower (*abs-interval x*) = (*if lower x* $<$ *0* $\wedge$ *0* $<$ *upper x then 0 else min* |*lower x*| |*upper x*|)
by *transfer auto*
lemma *upper-abs-interval*[*simp*]: *upper* (*abs-interval x*) = *max* |*lower x*| |*upper x*|
by *transfer auto*

lemma *in-abs-intervalI1*:
lx $<$ *0* $\Longrightarrow$ *0* $<$ *ux* $\Longrightarrow$ *0* $\leq$ *xa* $\Longrightarrow$ *xa* $\leq$ *max* ($-$ *lx*) (*ux*) $\Longrightarrow$ *xa* $\in$ *abs* ' {*lx*..*ux*}
for *xa*::'*a*::*linordered-idom*
by (*metis abs-minus-cancel abs-of-nonneg atLeastAtMost-iff image-eqI le-less le-max-iff-disj le-minus-iff neg-le-0-iff-le order-trans*)

lemma *in-abs-intervalI2*:
min (|*lx*|) |*ux*| $\leq$ *xa* $\Longrightarrow$ *xa* $\leq$ *max* |*lx*| |*ux*| $\Longrightarrow$ *lx* $\leq$ *ux* $\Longrightarrow$ *0* $\leq$ *lx* $\vee$ *ux* $\leq$ *0* $\Longrightarrow$
xa $\in$ *abs* ' {*lx*..*ux*}
for *xa*::'*a*::*linordered-idom*
by (*force intro*: *image-eqI*[**where** *x*=$-$*xa*] *image-eqI*[**where** *x*=*xa*])

lemma *set-of-abs-interval*: *set-of* (*abs-interval x*) = *abs* ' *set-of x*
by (*auto simp*: *set-of-eq not-less intro*: *in-abs-intervalI1 in-abs-intervalI2 cong del*: *image-cong-simp*)

fun *split-domain* :: ('*a*::*preorder interval* $\Rightarrow$ '*a interval list*) $\Rightarrow$ '*a interval list* $\Rightarrow$ '*a interval list list*
where *split-domain split* [] = [[]]

```
  | split-domain split (I#Is) = (
        let S = split I;
            D = split-domain split Is
        in concat (map (λd. map (λs. s # d) S) D)
       )

context notes [[typedef-overloaded]] begin
lift-definition(code-dt) split-interval::'a::linorder interval ⇒ 'a ⇒ ('a interval ×
'a interval)
  is λ(l, u) x. ((min l x, max l x), (min u x, max u x))
  by (auto simp: min-def)
end

lemma split-domain-nonempty:
  assumes ⋀I. split I ≠ []
  shows split-domain split I ≠ []
  using last-in-set assms
  by (induction I, auto)

lemma lower-split-interval1: lower (fst (split-interval X m)) = min (lower X) m
  and lower-split-interval2: lower (snd (split-interval X m)) = min (upper X) m
  and upper-split-interval1: upper (fst (split-interval X m)) = max (lower X) m
  and upper-split-interval2: upper (snd (split-interval X m)) = max (upper X) m
  subgoal by transfer auto
  subgoal by transfer (auto simp: min.commute)
  subgoal by transfer (auto simp: )
  subgoal by transfer (auto simp: )
  done

lemma split-intervalD: split-interval X x = (A, B) ⟹ set-of X ⊆ set-of A ∪ set-of
B
  unfolding set-of-eq
  by transfer (auto simp: min-def max-def split: if-splits)

instantiation interval :: ({topological-space, preorder}) topological-space
begin

definition open-interval-def[code del]: open (X::'a interval set) =
  (∀ x∈X.
     ∃ A B.
        open A ∧
        open B ∧
        lower x ∈ A ∧ upper x ∈ B ∧ Interval ' (A × B) ⊆ X)

instance
proof
  show open (UNIV :: ('a interval) set)
    unfolding open-interval-def by auto
next
```

fix S T :: ($'a$ *interval*) *set*
assume *open* S *open* T
show *open* ($S \cap T$)
unfolding *open-interval-def*
proof (*safe*)
fix x **assume** $x \in S$ $x \in T$
from ‹$x \in S$› ‹*open* S› **obtain** *Sl Su* **where** S:
open Sl open Su lower $x \in$ *Sl upper* $x \in$ *Su Interval* ‘ ($Sl \times Su$) $\subseteq S$
by (*auto simp*: *open-interval-def*)
from ‹$x \in T$› ‹*open* T› **obtain** *Tl Tu* **where** T:
open Tl open Tu lower $x \in$ *Tl upper* $x \in$ *Tu Interval* ‘ ($Tl \times Tu$) $\subseteq T$
by (*auto simp*: *open-interval-def*)

let *?L* = $Sl \cap Tl$ **and** *?U* = $Su \cap Tu$
have *open ?L* ∧ *open ?U* ∧ *lower* $x \in$ *?L* ∧ *upper* $x \in$ *?U* ∧ *Interval* ‘ (*?L* × *?U*) $\subseteq S \cap T$
using S T **by** (*auto simp add*: *open-Int*)
then show ∃ A B. *open* A ∧ *open* B ∧ *lower* $x \in A$ ∧ *upper* $x \in B$ ∧ *Interval* ‘ ($A \times B$) $\subseteq S \cap T$
by *fast*
qed
qed (*unfold open-interval-def*, *fast*)

end

47.2 Quickcheck

lift-definition *Ivl*::$'a \Rightarrow$ $'a$::*preorder* ⇒ $'a$ *interval* **is** λa b. (*min* a b, b)
by (*auto simp*: *min-def*)

instantiation *interval* :: ({*exhaustive*,*preorder*}) *exhaustive*
begin

definition *exhaustive-interval*::($'a$ *interval* ⇒ (*bool* × *term list*) *option*)
⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
exhaustive-interval f d =
Quickcheck-Exhaustive.exhaustive (λx. *Quickcheck-Exhaustive.exhaustive* (λy. f (*Ivl* x y)) d) d

instance ..

end

context
includes *term-syntax*
begin

definition [*code-unfold*]:

valtermify-interval x y = Code-Evaluation.valtermify (Ivl::'a::{preorder,typerep}⇒-) {·} x {·} y

end

instantiation *interval* :: ({*full-exhaustive*,*preorder*,*typerep*}) *full-exhaustive*
begin

definition *full-exhaustive-interval*::
('a interval × (unit ⇒ term) ⇒ (bool × term list) option)
⇒ natural ⇒ (bool × term list) option **where**
full-exhaustive-interval f d =
Quickcheck-Exhaustive.full-exhaustive
(λx. Quickcheck-Exhaustive.full-exhaustive (λy. f (valtermify-interval x y)) d) d

instance ..

end

instantiation *interval* :: ({*random*,*preorder*,*typerep*}) *random*
begin

definition *random-interval* ::
natural
⇒ natural × natural
⇒ ('a interval × (unit ⇒ term)) × natural × natural **where**
random-interval i =
scomp (Quickcheck-Random.random i)
(λman. scomp (Quickcheck-Random.random i) (λexp. Pair (valtermify-interval man exp)))

instance ..

end

lifting-update *interval.lifting*
lifting-forget *interval.lifting*

end

48 Approximate Operations on Intervals of Floating Point Numbers

theory *Interval-Float*
imports
Interval
Float

begin

definition *mid* :: *float interval* ⇒ *float*
where *mid i* = (*lower i* + *upper i*) ∗ *Float 1* (−*1*)

lemma *mid-in-interval*: *mid i* ∈$_i$ *i*
using *lower-le-upper*[*of i*]
by (*auto simp*: *mid-def set-of-eq powr-minus*)

lemma *mid-le*: *lower i* ≤ *mid i mid i* ≤ *upper i*
using *mid-in-interval*
by (*auto simp*: *set-of-eq*)

definition *centered* :: *float interval* ⇒ *float interval*
where *centered i* = *i* − *interval-of* (*mid i*)

definition *split-float-interval x* = *split-interval x* ((*lower x* + *upper x*) ∗ *Float 1* (−*1*))

lemma *split-float-intervalD*: *split-float-interval X* = (*A*, *B*) ⟹ *set-of X* ⊆ *set-of A* ∪ *set-of B*
by (*auto dest*!: *split-intervalD simp*: *split-float-interval-def*)

lemma *split-float-interval-bounds*:
shows
lower-split-float-interval1: *lower* (*fst* (*split-float-interval X*)) = *lower X*
and *lower-split-float-interval2*: *lower* (*snd* (*split-float-interval X*)) = *mid X*
and *upper-split-float-interval1*: *upper* (*fst* (*split-float-interval X*)) = *mid X*
and *upper-split-float-interval2*: *upper* (*snd* (*split-float-interval X*)) = *upper X*
using *mid-le*[*of X*]
by (*auto simp*: *split-float-interval-def mid-def*[*symmetric*] *min-def max-def real-of-float-eq*
lower-split-interval1 lower-split-interval2
upper-split-interval1 upper-split-interval2)

lemmas *float-round-down-le*[*intro*] = *order-trans*[*OF float-round-down*]
and *float-round-up-ge*[*intro*] = *order-trans*[*OF* - *float-round-up*]

TODO: many of the lemmas should move to theories Float or Approximation (the latter should be based on type *interval*.

48.1 Intervals with Floating Point Bounds

context includes *interval.lifting* **begin**

lift-definition *round-interval* :: *nat* ⇒ *float interval* ⇒ *float interval*
is λ*p*. λ(*l*, *u*). (*float-round-down p l*, *float-round-up p u*)
by (*auto simp*: *intro*!: *float-round-down-le float-round-up-le*)

lemma *lower-round-ivl*[*simp*]: *lower* (*round-interval p x*) = *float-round-down p* (*lower x*)

by *transfer auto*
lemma *upper-round-ivl*[*simp*]: *upper* (*round-interval p x*) = *float-round-up p* (*upper x*)
by *transfer auto*

lemma *round-ivl-correct*: *set-of A* ⊆ *set-of* (*round-interval prec A*)
by (*auto simp*: *set-of-eq float-round-down-le float-round-up-le*)

lift-definition *truncate-ivl* :: *nat* ⇒ *real interval* ⇒ *real interval*
is λ*p*. λ(*l*, *u*). (*truncate-down p l*, *truncate-up p u*)
by (*auto intro*!: *truncate-down-le truncate-up-le*)

lemma *lower-truncate-ivl*[*simp*]: *lower* (*truncate-ivl p x*) = *truncate-down p* (*lower x*)
by *transfer auto*
lemma *upper-truncate-ivl*[*simp*]: *upper* (*truncate-ivl p x*) = *truncate-up p* (*upper x*)
by *transfer auto*

lemma *truncate-ivl-correct*: *set-of A* ⊆ *set-of* (*truncate-ivl prec A*)
by (*auto simp*: *set-of-eq intro*!: *truncate-down-le truncate-up-le*)

lift-definition *real-interval*::*float interval* ⇒ *real interval*
is λ(*l*, *u*). (*real-of-float l*, *real-of-float u*)
by *auto*

lemma *lower-real-interval*[*simp*]: *lower* (*real-interval x*) = *lower x*
by *transfer auto*
lemma *upper-real-interval*[*simp*]: *upper* (*real-interval x*) = *upper x*
by *transfer auto*

definition *set-of′ x* = (*case x of None* ⇒ *UNIV* | *Some i* ⇒ *set-of* (*real-interval i*))

lemma *real-interval-min-interval*[*simp*]:
real-interval (*min-interval a b*) = *min-interval* (*real-interval a*) (*real-interval b*)
by (*auto simp*: *interval-eq-set-of-iff set-of-eq real-of-float-min*)

lemma *real-interval-max-interval*[*simp*]:
real-interval (*max-interval a b*) = *max-interval* (*real-interval a*) (*real-interval b*)
by (*auto simp*: *interval-eq-set-of-iff set-of-eq real-of-float-max*)

lemma *in-intervalI*:
x ∈$_i$ *X* **if** *lower X* ≤ *x x* ≤ *upper X*
using *that* **by** (*auto simp*: *set-of-eq*)

abbreviation *in-real-interval* ((-/ ∈$_r$ -) [*51*, *51*] *50*) **where**
x ∈$_r$ *X* ≡ *x* ∈$_i$ *real-interval X*

lemma *in-real-intervalI*:
x $\in_r$ *X* **if** *lower X* $\leq$ *x x* $\leq$ *upper X* **for** *x*::*real* **and** *X*::*float interval*
using *that*
by (*intro in-intervalI*) *auto*

48.2 intros for *real-interval*

lemma *in-round-intervalI*: *x* $\in_r$ *A* $\Longrightarrow$ *x* $\in_r$ (*round-interval prec A*)
by (*auto simp*: *set-of-eq float-round-down-le float-round-up-le*)

lemma *zero-in-float-intervalI*: *0* $\in_r$ *0*
by (*auto simp*: *set-of-eq*)

lemma *plus-in-float-intervalI*: *a* + *b* $\in_r$ *A* + *B* **if** *a* $\in_r$ *A b* $\in_r$ *B*
using *that*
by (*auto simp*: *set-of-eq*)

lemma *minus-in-float-intervalI*: *a* − *b* $\in_r$ *A* − *B* **if** *a* $\in_r$ *A b* $\in_r$ *B*
using *that*
by (*auto simp*: *set-of-eq*)

lemma *uminus-in-float-intervalI*: −*a* $\in_r$ −*A* **if** *a* $\in_r$ *A*
using *that*
by (*auto simp*: *set-of-eq*)

lemma *real-interval-times*: *real-interval* (*A* ∗ *B*) = *real-interval A* ∗ *real-interval B*
by (*auto simp*: *interval-eq-iff lower-times upper-times min-def max-def*)

lemma *times-in-float-intervalI*: *a* ∗ *b* $\in_r$ *A* ∗ *B* **if** *a* $\in_r$ *A b* $\in_r$ *B*
using *times-in-intervalI*[*OF that*]
by (*auto simp*: *real-interval-times*)

lemma *real-interval-abs*: *real-interval* (*abs-interval A*) = *abs-interval* (*real-interval A*)
by (*auto simp*: *interval-eq-iff min-def max-def*)

lemma *abs-in-float-intervalI*: *abs a* $\in_r$ *abs-interval A* **if** *a* $\in_r$ *A*
by (*auto simp*: *set-of-abs-interval real-interval-abs intro*!: *imageI that*)

lemma *interval-of*[*intro*,*simp*]: *x* $\in_r$ *interval-of x*
by (*auto simp*: *set-of-eq*)

lemma *split-float-interval-realD*: *split-float-interval X* = (*A*, *B*) $\Longrightarrow$ *x* $\in_r$ *X* $\Longrightarrow$ *x* $\in_r$ *A* ∨ *x* $\in_r$ *B*
by (*auto simp*: *set-of-eq prod-eq-iff split-float-interval-bounds*)

48.3 bounds for lists

lemma *lower-Interval*: *lower* (*Interval x*) = *fst x*

and *upper-Interval*: *upper* (*Interval x*) = *snd x*
if *fst x* ≤ *snd x*
using *that*
by (*auto simp*: *lower-def upper-def Interval-inverse split-beta'*)

definition *all-in-i* :: *'a*::*preorder list* ⇒ *'a interval list* ⇒ *bool*
(**infix** (*all'-in*$_i$) *50*)
where *x all-in*$_i$ *I* = (*length x* = *length I* ∧ (∀ *i* < *length I*. *x* ! *i* ∈$_i$ *I* ! *i*))

definition *all-in* :: *real list* ⇒ *float interval list* ⇒ *bool*
(**infix** (*all'-in*) *50*)
where *x all-in I* = (*length x* = *length I* ∧ (∀ *i* < *length I*. *x* ! *i* ∈$_r$ *I* ! *i*))

definition *all-subset* :: *'a*::*order interval list* ⇒ *'a interval list* ⇒ *bool*
(**infix** (*all'-subset*) *50*)
where *I all-subset J* = (*length I* = *length J* ∧ (∀ *i* < *length I*. *set-of* (*I*!*i*) ⊆ *set-of* (*J*!*i*)))

lemmas [*simp*] = *all-in-def all-subset-def*

lemma *all-subsetD*:
assumes *I all-subset J*
assumes *x all-in I*
shows *x all-in J*
using *assms*
by (*auto simp*: *set-of-eq*; *fastforce*)

lemma *round-interval-mono*: *set-of* (*round-interval prec X*) ⊆ *set-of* (*round-interval prec Y*)
if *set-of X* ⊆ *set-of Y*
using *that*
by *transfer*
(*auto simp*: *float-round-down.rep-eq float-round-up.rep-eq truncate-down-mono truncate-up-mono*)

lemma *Ivl-simps*[*simp*]: *lower* (*Ivl a b*) = *min a b upper* (*Ivl a b*) = *b*
subgoal by *transfer simp*
subgoal by *transfer simp*
done

lemma *set-of-subset-iff*: *set-of X* ⊆ *set-of Y* ⟷ *lower Y* ≤ *lower X* ∧ *upper X* ≤ *upper Y*
for *X Y*::*'a*::*linorder interval*
by (*auto simp*: *set-of-eq subset-iff*)

lemma *set-of-subset-iff'*:
set-of a ⊆ *set-of* (*b* :: *'a* :: *linorder interval*) ⟷ *a* ≤ *b*
unfolding *less-eq-interval-def set-of-subset-iff* **..**

lemma *bounds-of-interval-eq-lower-upper*:
bounds-of-interval ivl = (*lower ivl*, *upper ivl*) **if** *lower ivl* ≤ *upper ivl*
using *that*
by (*auto simp*: *lower.rep-eq upper.rep-eq*)

lemma *real-interval-Ivl*: *real-interval* (*Ivl a b*) = *Ivl a b*
by *transfer* (*auto simp*: *min-def*)

lemma *set-of-mul-contains-real-zero*:
0 ∈$_r$ (*A* ∗ *B*) **if** *0* ∈$_r$ *A* ∨ *0* ∈$_r$ *B*
using *that set-of-mul-contains-zero*[*of A B*]
by (*auto simp*: *set-of-eq*)

fun *subdivide-interval* :: *nat* ⇒ *float interval* ⇒ *float interval list*
where *subdivide-interval 0 I* = [*I*]
| *subdivide-interval* (*Suc n*) *I* = (
let m = *mid I*
in (*subdivide-interval n* (*Ivl* (*lower I*) *m*)) @ (*subdivide-interval n* (*Ivl m* (*upper I*)))
)

lemma *subdivide-interval-length*:
shows *length* (*subdivide-interval n I*) = *2*^*n*
by(*induction n arbitrary*: *I*, *simp-all add*: *Let-def*)

lemma *lower-le-mid*: *lower x* ≤ *mid x real-of-float* (*lower x*) ≤ *mid x*
and *mid-le-upper*: *mid x* ≤ *upper x real-of-float* (*mid x*) ≤ *upper x*
unfolding *mid-def*
subgoal by *transfer* (*auto simp*: *powr-neg-one*)
subgoal by *transfer* (*auto simp*: *powr-neg-one*)
subgoal by *transfer* (*auto simp*: *powr-neg-one*)
subgoal by *transfer* (*auto simp*: *powr-neg-one*)
done

lemma *subdivide-interval-correct*:
list-ex (λ*i*. *x* ∈$_r$ *i*) (*subdivide-interval n I*) **if** *x* ∈$_r$ *I* **for** *x*::*real*
using *that*
proof(*induction n arbitrary*: *x I*)
case *0*
then show *?case* **by** *simp*
next
case (*Suc n*)
from ‹*x* ∈$_r$ *I*› **consider** *x* ∈$_r$ *Ivl* (*lower I*) (*mid I*) | *x* ∈$_r$ *Ivl* (*mid I*) (*upper I*)
by (*cases x* ≤ *real-of-float* (*mid I*))
(*auto simp*: *set-of-eq min-def lower-le-mid mid-le-upper*)
from *this*[*case-names lower upper*] **show** *?case*
by *cases* (*use Suc.IH* **in** ‹*auto simp*: *Let-def*›)
qed

fun *interval-list-union* :: *'a*::*lattice interval list* ⇒ *'a interval*
  where *interval-list-union* [] = *undefined*
  | *interval-list-union* [*I*] = *I*
  | *interval-list-union* (*I*#*Is*) = *sup I* (*interval-list-union Is*)

lemma *interval-list-union-correct*:
  assumes *S* ≠ []
  assumes *i* < *length S*
  shows *set-of* (*S*!*i*) ⊆ *set-of* (*interval-list-union S*)
  using *assms*
proof(*induction S arbitrary*: *i*)
  case (*Cons a S i*)
  thus *?case*
  proof(*cases S*)
    fix *b S'*
    assume *S* = *b* # *S'*
    hence *S* ≠ []
      by *simp*
    show *?thesis*
    proof(*cases i*)
      case *0*
      show *?thesis*
        apply(*cases S*)
        using *interval-union-mono1*
        by (*auto simp add*: *0*)
    next
      case (*Suc i-prev*)
      hence *i-prev* < *length S*
        using *Cons*(*3*) **by** *simp*

      from *Cons*(*1*)[*OF* ‹*S* ≠ []› *this*] *Cons*(*1*)
      have *set-of* ((*a* # *S*) ! *i*) ⊆ *set-of* (*interval-list-union S*)
        by (*simp add*: ‹*i* = *Suc i-prev*›)
      also have ... ⊆ *set-of* (*interval-list-union* (*a* # *S*))
        using ‹*S* ≠ []›
        apply(*cases S*)
        using *interval-union-mono2*
        by *auto*
      finally show *?thesis* **.**
    qed
  qed *simp*
qed *simp*

lemma *split-domain-correct*:
  fixes *x* :: *real list*
  assumes *x all-in I*
  assumes *split-correct*: ⋀*x a I*. *x* ∈*r I* ⟹ *list-ex* (λ*i*::*float interval*. *x* ∈*r i*) (*split I*)
  shows *list-ex* (λ*s*. *x all-in s*) (*split-domain split I*)

using *assms*(*1*)
proof(*induction I arbitrary*: *x*)
case (*Cons I Is x*)
have $x \neq []$
using *Cons*(*2*) **by** *auto*
obtain x' *xs* **where** *x-decomp*: $x = x' \# xs$
using ‹$x \neq []$› *list.exhaust* **by** *auto*
hence $x' \in_r I$ *xs all-in Is*
using *Cons*(*2*)
by *auto*
show *?case*
using *Cons*(*1*)[*OF* ‹*xs all-in Is*›]
split-correct[*OF* ‹$x' \in_r I$›]
apply (*auto simp add*: *list-ex-iff set-of-eq*)
by (*smt* (*verit*, *ccfv-SIG*) *One-nat-def Suc-pred* ‹$x \neq []$› *le-simps*(*3*) *length-greater-0-conv length-tl linorder-not-less list.sel*(*3*) *neq0-conv nth-Cons'* *x-decomp*)
qed *simp*

lift-definition(*code-dt*) *inverse-float-interval*::*nat* ⇒ *float interval* ⇒ *float interval option* **is**
λ*prec* (*l*, *u*). *if* (*0* < *l* ∨ *u* < *0*) *then Some* (*float-divl prec 1 u*, *float-divr prec 1 l*) *else None*
by (*auto intro*!: *order-trans*[*OF float-divl*] *order-trans*[*OF* - *float-divr*] *simp*: *divide-simps*)

lemma *inverse-float-interval-eq-Some-conv*:
defines *one* ≡ (*1*::*float*)
shows
inverse-float-interval p X = *Some R* ⟷
(*lower X* > *0* ∨ *upper X* < *0*) ∧
lower R = *float-divl p one* (*upper X*) ∧
upper R = *float-divr p one* (*lower X*)
by *clarsimp* (*transfer fixing*: *one*, *force simp*: *one-def split*: *if-splits*)

lemma *inverse-float-interval*:
inverse ‘ *set-of* (*real-interval X*) ⊆ *set-of* (*real-interval Y*)
if *inverse-float-interval p X* = *Some Y*
using *that*
apply (*clarsimp simp*: *set-of-eq inverse-float-interval-eq-Some-conv*)
by (*intro order-trans*[*OF float-divl*] *order-trans*[*OF* - *float-divr*] *conjI*)
(*auto simp*: *divide-simps*)

lemma *inverse-float-intervalI*:
$x \in_r X \Longrightarrow$ *inverse x* ∈ *set-of'* (*inverse-float-interval p X*)
using *inverse-float-interval*[*of p X*]
by (*auto simp*: *set-of'-def split*: *option.splits*)

lemma *inverse-float-interval-eqI*: *inverse-float-interval p X* = *Some IVL* ⟹ $x \in_r$

$X \Longrightarrow$ *inverse* $x \in_r$ *IVL*
using *inverse-float-intervalI*[*of* x X p]
by (*auto simp*: *set-of'-def*)

lemma *real-interval-abs-interval*[*simp*]:
real-interval (*abs-interval* x) = *abs-interval* (*real-interval* x)
by (*auto simp*: *interval-eq-set-of-iff set-of-eq real-of-float-max real-of-float-min*)

lift-definition *floor-float-interval*::*float interval* ⇒ *float interval* **is**
$\lambda(l, u)$. (*floor-fl* l, *floor-fl* u)
by (*auto intro*!: *floor-mono simp*: *floor-fl.rep-eq*)

lemma *lower-floor-float-interval*[*simp*]: *lower* (*floor-float-interval* x) = *floor-fl* (*lower* x)
by *transfer auto*
lemma *upper-floor-float-interval*[*simp*]: *upper* (*floor-float-interval* x) = *floor-fl* (*upper* x)
by *transfer auto*

lemma *floor-float-intervalI*: $\lfloor x \rfloor \in_r$ *floor-float-interval* X **if** $x \in_r X$
using *that* **by** (*auto simp*: *set-of-eq floor-fl-def floor-mono*)

end

48.4 constants for code generation

definition *lowerF*::*float interval* ⇒ *float* **where** *lowerF* = *lower*
definition *upperF*::*float interval* ⇒ *float* **where** *upperF* = *upper*

end

49 Immutable Arrays with Code Generation

theory *IArray*
imports *Main*
begin

49.1 Fundamental operations

Immutable arrays are lists wrapped up in an additional constructor. There are no update operations. Hence code generation can safely implement this type by efficient target language arrays. Currently only SML is provided. Could be extended to other target languages and more operations.

context
begin

datatype *'a iarray* = *IArray 'a list*

qualified primrec *list-of* :: *'a iarray* ⇒ *'a list* **where**
list-of (*IArray xs*) = *xs*

qualified definition *of-fun* :: (*nat* ⇒ *'a*) ⇒ *nat* ⇒ *'a iarray* **where**
[*simp*]: *of-fun f n* = *IArray* (*map f* [*0*..<*n*])

qualified definition *sub* :: *'a iarray* ⇒ *nat* ⇒ *'a* (**infixl** !! *100*) **where**
[*simp*]: *as* !! *n* = *IArray.list-of as* ! *n*

qualified definition *length* :: *'a iarray* ⇒ *nat* **where**
[*simp*]: *length as* = *List.length* (*IArray.list-of as*)

qualified definition *all* :: (*'a* ⇒ *bool*) ⇒ *'a iarray* ⇒ *bool* **where**
[*simp*]: *all p as* ⟷ (∀ *a* ∈ *set* (*list-of as*). *p a*)

qualified definition *exists* :: (*'a* ⇒ *bool*) ⇒ *'a iarray* ⇒ *bool* **where**
[*simp*]: *exists p as* ⟷ (∃ *a* ∈ *set* (*list-of as*). *p a*)

lemma *of-fun-nth*:
IArray.of-fun f n !! *i* = *f i* **if** *i* < *n*
using *that* **by** (*simp add*: *map-nth*)

end

49.2 Generic code equations

lemma [*code*]:
size (*as* :: *'a iarray*) = *Suc* (*IArray.length as*)
by (*cases as*) *simp*

lemma [*code*]:
size-iarray f as = *Suc* (*size-list f* (*IArray.list-of as*))
by (*cases as*) *simp*

lemma [*code*]:
rec-iarray f as = *f* (*IArray.list-of as*)
by (*cases as*) *simp*

lemma [*code*]:
case-iarray f as = *f* (*IArray.list-of as*)
by (*cases as*) *simp*

lemma [*code*]:
set-iarray as = *set* (*IArray.list-of as*)
by (*cases as*) *auto*

lemma [*code*]:
map-iarray f as = *IArray* (*map f* (*IArray.list-of as*))

by (*cases as*) *auto*

lemma [*code*]:
rel-iarray r as bs = list-all2 r (*IArray.list-of as*) (*IArray.list-of bs*)
by (*cases as*, *cases bs*) *auto*

lemma *list-of-code* [*code*]:
IArray.list-of as = map (λ*n. as* !! *n*) [*0* ..< *IArray.length as*]
by (*cases as*) (*simp add*: *map-nth*)

lemma [*code*]:
HOL.equal as bs ⟷ *HOL.equal* (*IArray.list-of as*) (*IArray.list-of bs*)
by (*cases as*, *cases bs*) (*simp add*: *equal*)

lemma [*code*]:
IArray.all p = Not ∘ *IArray.exists* (*Not* ∘ *p*)
by (*simp add*: *fun-eq-iff*)

context
includes *term-syntax*
begin

lemma [*code*]:
Code-Evaluation.term-of (*as* :: *'a*::*typerep iarray*) =
Code-Evaluation.Const (*STR ''IArray.iarray.IArray''*) (*TYPEREP*(*'a list* ⇒ *'a iarray*)) <·> (*Code-Evaluation.term-of* (*IArray.list-of as*))
by (*subst term-of-anything*) *rule*

end

49.3 Auxiliary operations for code generation

context
begin

qualified primrec *tabulate* :: *integer* × (*integer* ⇒ *'a*) ⇒ *'a iarray* **where**
tabulate (*n*, *f*) = *IArray* (*map* (*f* ∘ *integer-of-nat*) [*0*..<*nat-of-integer n*])

lemma [*code*]:
IArray.of-fun f n = IArray.tabulate (*integer-of-nat n*, *f* ∘ *nat-of-integer*)
by *simp*

qualified primrec *sub'* :: *'a iarray* × *integer* ⇒ *'a* **where**
sub' (*as*, *n*) = *as* !! *nat-of-integer n*

lemma [*code*]:
IArray.sub' (*IArray as*, *n*) = *as* ! *nat-of-integer n*
by *simp*

lemma [*code*]:
as !! *n* = *IArray.sub'* (*as*, *integer-of-nat n*)
by *simp*

qualified definition *length'* :: *'a iarray* ⇒ *integer* **where**
[*simp*]: *length' as* = *integer-of-nat* (*List.length* (*IArray.list-of as*))

lemma [*code*]:
IArray.length' (*IArray as*) = *integer-of-nat* (*List.length as*)
by *simp*

lemma [*code*]:
IArray.length as = *nat-of-integer* (*IArray.length' as*)
by *simp*

qualified definition *exists-upto* :: (*'a* ⇒ *bool*) ⇒ *integer* ⇒ *'a iarray* ⇒ *bool* **where**
[*simp*]: *exists-upto p k as* ⟷ (∃ *l*. *0* ≤ *l* ∧ *l* < *k* ∧ *p* (*sub'* (*as*, *l*)))

lemma *exists-upto-of-nat*:
exists-upto p (*of-nat n*) *as* ⟷ (∃ *m*<*n*. *p* (*as* !! *m*))
including *integer.lifting* **by** (*simp*, *transfer*)
(*metis nat-int nat-less-iff of-nat-0-le-iff*)

lemma [*code*]:
exists-upto p k as ⟷ (*if k* ≤ *0 then False else*
let l = *k* − *1 in p* (*sub'* (*as*, *l*)) ∨ *exists-upto p l as*)
proof (*cases k* ≥ *1*)
case *False*
then show *?thesis*
by (*auto simp add*: *not-le discrete*)
next
case *True*
then have *less*: *k* ≤ *0* ⟷ *False*
by *simp*
define *n* **where** *n* = *nat-of-integer* (*k* − *1*)
with *True* **have** *k*: *k* − *1* = *of-nat n k* = *of-nat* (*Suc n*)
by *simp-all*
show *?thesis* **unfolding** *less Let-def k*(*1*) **unfolding** *k*(*2*) *exists-upto-of-nat*
using *less-Suc-eq* **by** *auto*
qed

lemma [*code*]:
IArray.exists p as ⟷ *exists-upto p* (*length' as*) *as*
including *integer.lifting* **by** (*simp*, *transfer*)
(*auto*, *metis in-set-conv-nth less-imp-of-nat-less nat-int of-nat-0-le-iff*)

end

49.4 Code Generation for SML

Note that arrays cannot be printed directly but only by turning them into lists first. Arrays could be converted back into lists for printing if they were wrapped up in an additional constructor.

code-reserved *SML Vector*

code-printing
type-constructor *iarray* ⇀ (*SML*) - *Vector.vector*
| **constant** *IArray* ⇀ (*SML*) *Vector.fromList*
| **constant** *IArray.all* ⇀ (*SML*) *Vector.all*
| **constant** *IArray.exists* ⇀ (*SML*) *Vector.exists*
| **constant** *IArray.tabulate* ⇀ (*SML*) *Vector.tabulate*
| **constant** *IArray.sub'* ⇀ (*SML*) *Vector.sub*
| **constant** *IArray.length'* ⇀ (*SML*) *Vector.length*

49.5 Code Generation for Haskell

We map *'a iarray*s in Isabelle/HOL to *Data.Array.IArray.array* in Haskell. Performance mapping to *Data.Array.Unboxed.Array* and *Data.Array.Array* is similar.

code-printing
code-module *IArray* ⇀ (*Haskell*) ‹
module IArray(*IArray*, *tabulate*, *of-list*, *sub*, *length*) *where* {

import Prelude (*Bool*(*True*, *False*), *not*, *Maybe*(*Nothing*, *Just*),
Integer, (+), (−), (<), *fromInteger*, *toInteger*, *map*, *seq*, (.));
import qualified Prelude;
import qualified Data.Array.IArray;
import qualified Data.Array.Base;
import qualified Data.Ix;

newtype IArray e = *IArray* (*Data.Array.IArray.Array Integer e*);

tabulate :: (*Integer*, (*Integer* −> *e*)) −> *IArray e*;
tabulate (*k*, *f*) = *IArray* (*Data.Array.IArray.array* (*0*, *k* − *1*) (*map* (*i* −> *let fi* = *f i in fi* ‘*seq*‘ (*i*, *fi*)) [*0*..*k* − *1*]));

of-list :: [*e*] −> *IArray e*;
of-list l = *IArray* (*Data.Array.IArray.listArray* (*0*, (*toInteger* . *Prelude.length*) *l* − *1*) *l*);

sub :: (*IArray e*, *Integer*) −> *e*;
sub (*IArray v*, *i*) = *v* ‘*Data.Array.Base.unsafeAt*‘ *fromInteger i*;

length :: *IArray e* −> *Integer*;
length (*IArray v*) = *toInteger* (*Data.Ix.rangeSize* (*Data.Array.IArray.bounds v*));

}› **for type-constructor** *iarray* **constant** *IArray IArray.tabulate IArray.sub′ IArray.length′*

code-reserved *Haskell IArray-Impl*

code-printing
  type-constructor *iarray* ⇀ (*Haskell*) *IArray.IArray* -
| **constant** *IArray* ⇀ (*Haskell*) *IArray.of′-list*
| **constant** *IArray.tabulate* ⇀ (*Haskell*) *IArray.tabulate*
| **constant** *IArray.sub′* ⇀ (*Haskell*) *IArray.sub*
| **constant** *IArray.length′* ⇀ (*Haskell*) *IArray.length*

end

50 Definition of Landau symbols

theory *Landau-Symbols*
imports
  Complex-Main
begin

lemma *eventually-subst′*:
  eventually (λ*x*. *f x* = *g x*) *F* ⟹ *eventually* (λ*x*. *P x* (*f x*)) *F* = *eventually* (λ*x*. *P x* (*g x*)) *F*
  by (*rule eventually-subst*, *erule eventually-rev-mp*) *simp*

50.1 Definition of Landau symbols

Our Landau symbols are sign-oblivious, i.e. any function always has the same growth as its absolute. This has the advantage of making some cancelling rules for sums nicer, but introduces some problems in other places. Nevertheless, we found this definition more convenient to work with.

definition *bigo* :: *′a filter* ⇒ (*′a* ⇒ (*′b* :: *real-normed-field*)) ⇒ (*′a* ⇒ *′b*) *set*
    (‹(*1O*[-]*′*(-*′*))›)
  where *bigo F g* = {*f*. (∃ *c*>*0*. *eventually* (λ*x*. *norm* (*f x*) ≤ *c* ∗ *norm* (*g x*)) *F*)}

definition *smallo* :: *′a filter* ⇒ (*′a* ⇒ (*′b* :: *real-normed-field*)) ⇒ (*′a* ⇒ *′b*) *set*
    (‹(*1o*[-]*′*(-*′*))›)
  where *smallo F g* = {*f*. (∀ *c*>*0*. *eventually* (λ*x*. *norm* (*f x*) ≤ *c* ∗ *norm* (*g x*)) *F*)}

definition *bigomega* :: *′a filter* ⇒ (*′a* ⇒ (*′b* :: *real-normed-field*)) ⇒ (*′a* ⇒ *′b*) *set*
    (‹(*1Ω*[-]*′*(-*′*))›)
  where *bigomega F g* = {*f*. (∃ *c*>*0*. *eventually* (λ*x*. *norm* (*f x*) ≥ *c* ∗ *norm* (*g x*)) *F*)}

definition *smallomega* :: *′a filter* ⇒ (*′a* ⇒ (*′b* :: *real-normed-field*)) ⇒ (*′a* ⇒ *′b*) *set*

(‹(*1ω*[-]'(-'))›)
where *smallomega F g* = {*f*. (∀ *c*>*0*. *eventually* (λ*x*. *norm* (*f x*) ≥ *c* ∗ *norm* (*g x*)) *F*)}

definition *bigtheta* :: *'a filter* ⇒ (*'a* ⇒ (*'b* :: *real-normed-field*)) ⇒ (*'a* ⇒ *'b*) *set*
(‹(*1Θ*[-]'(-'))›)
where *bigtheta F g* = *bigo F g* ∩ *bigomega F g*

abbreviation *bigo-at-top* (‹(*2O'*(-'))›) **where**
O(*g*) ≡ *bigo at-top g*

abbreviation *smallo-at-top* (‹(*2o'*(-'))›) **where**
o(*g*) ≡ *smallo at-top g*

abbreviation *bigomega-at-top* (‹(*2Ω'*(-'))›) **where**
Ω(*g*) ≡ *bigomega at-top g*

abbreviation *smallomega-at-top* (‹(*2ω'*(-'))›) **where**
ω(*g*) ≡ *smallomega at-top g*

abbreviation *bigtheta-at-top* (‹(*2Θ'*(-'))›) **where**
Θ(*g*) ≡ *bigtheta at-top g*

The following is a set of properties that all Landau symbols satisfy.

named-theorems *landau-divide-simps*

locale *landau-symbol* =
fixes *L* :: *'a filter* ⇒ (*'a* ⇒ (*'b* :: *real-normed-field*)) ⇒ (*'a* ⇒ *'b*) *set*
and *L'* :: *'c filter* ⇒ (*'c* ⇒ (*'b* :: *real-normed-field*)) ⇒ (*'c* ⇒ *'b*) *set*
and *Lr* :: *'a filter* ⇒ (*'a* ⇒ *real*) ⇒ (*'a* ⇒ *real*) *set*
assumes *bot'*: *L bot f* = *UNIV*
assumes *filter-mono'*: *F1* ≤ *F2* ⟹ *L F2 f* ⊆ *L F1 f*
assumes *in-filtermap-iff*:
f' ∈ *L* (*filtermap h' F'*) *g'* ⟷ (λ*x*. *f'* (*h' x*)) ∈ *L' F'* (λ*x*. *g'* (*h' x*))
assumes *filtercomap*:
f' ∈ *L F''* *g'* ⟹ (λ*x*. *f'* (*h' x*)) ∈ *L'* (*filtercomap h' F''*) (λ*x*. *g'* (*h' x*))
assumes *sup*: *f* ∈ *L F1 g* ⟹ *f* ∈ *L F2 g* ⟹ *f* ∈ *L* (*sup F1 F2*) *g*
assumes *in-cong*: *eventually* (λ*x*. *f x* = *g x*) *F* ⟹ *f* ∈ *L F* (*h*) ⟷ *g* ∈ *L F* (*h*)
assumes *cong*: *eventually* (λ*x*. *f x* = *g x*) *F* ⟹ *L F* (*f*) = *L F* (*g*)
assumes *cong-bigtheta*: *f* ∈ Θ[*F*](*g*) ⟹ *L F* (*f*) = *L F* (*g*)
assumes *in-cong-bigtheta*: *f* ∈ Θ[*F*](*g*) ⟹ *f* ∈ *L F* (*h*) ⟷ *g* ∈ *L F* (*h*)
assumes *cmult* [*simp*]: *c* ≠ *0* ⟹ *L F* (λ*x*. *c* ∗ *f x*) = *L F* (*f*)
assumes *cmult-in-iff* [*simp*]: *c* ≠ *0* ⟹ (λ*x*. *c* ∗ *f x*) ∈ *L F* (*g*) ⟷ *f* ∈ *L F* (*g*)
assumes *mult-left* [*simp*]: *f* ∈ *L F* (*g*) ⟹ (λ*x*. *h x* ∗ *f x*) ∈ *L F* (λ*x*. *h x* ∗ *g x*)
assumes *inverse*: *eventually* (λ*x*. *f x* ≠ *0*) *F* ⟹ *eventually* (λ*x*. *g x* ≠ *0*) *F* ⟹
f ∈ *L F* (*g*) ⟹ (λ*x*. *inverse* (*g x*)) ∈ *L F* (λ*x*. *inverse* (*f x*))
assumes *subsetI*: *f* ∈ *L F* (*g*) ⟹ *L F* (*f*) ⊆ *L F* (*g*)

assumes *plus-subset1*: $f \in o[F](g) \Longrightarrow L\ F\ (g) \subseteq L\ F\ (\lambda x.\ f\ x + g\ x)$
assumes *trans*: $f \in L\ F\ (g) \Longrightarrow g \in L\ F\ (h) \Longrightarrow f \in L\ F\ (h)$
assumes *compose*: $f \in L\ F\ (g) \Longrightarrow \mathit{filterlim}\ h'\ F\ G \Longrightarrow (\lambda x.\ f\ (h'\ x)) \in L'\ G\ (\lambda x.\ g\ (h'\ x))$
assumes *norm-iff* [*simp*]: $(\lambda x.\ \mathit{norm}\ (f\ x)) \in Lr\ F\ (\lambda x.\ \mathit{norm}\ (g\ x)) \longleftrightarrow f \in L\ F\ (g)$
assumes *abs* [*simp*]: $Lr\ Fr\ (\lambda x.\ |fr\ x|) = Lr\ Fr\ fr$
assumes *abs-in-iff* [*simp*]: $(\lambda x.\ |fr\ x|) \in Lr\ Fr\ gr \longleftrightarrow fr \in Lr\ Fr\ gr$
begin

lemma *bot* [*simp*]: $f \in L\ bot\ g$ **by** (*simp add*: *bot'*)

lemma *filter-mono*: $F1 \leq F2 \Longrightarrow f \in L\ F2\ g \Longrightarrow f \in L\ F1\ g$
using *filter-mono'*[*of F1 F2*] **by** *blast*

lemma *cong-ex*:
$\mathit{eventually}\ (\lambda x.\ f1\ x = f2\ x)\ F \Longrightarrow \mathit{eventually}\ (\lambda x.\ g1\ x = g2\ x)\ F \Longrightarrow$
$f1 \in L\ F\ (g1) \longleftrightarrow f2 \in L\ F\ (g2)$
by (*subst cong*, *assumption*, *subst in-cong*, *assumption*, *rule refl*)

lemma *cong-ex-bigtheta*:
$f1 \in \Theta[F](f2) \Longrightarrow g1 \in \Theta[F](g2) \Longrightarrow f1 \in L\ F\ (g1) \longleftrightarrow f2 \in L\ F\ (g2)$
by (*subst cong-bigtheta*, *assumption*, *subst in-cong-bigtheta*, *assumption*, *rule refl*)

lemma *bigtheta-trans1*:
$f \in L\ F\ (g) \Longrightarrow g \in \Theta[F](h) \Longrightarrow f \in L\ F\ (h)$
by (*subst cong-bigtheta*[*symmetric*])

lemma *bigtheta-trans2*:
$f \in \Theta[F](g) \Longrightarrow g \in L\ F\ (h) \Longrightarrow f \in L\ F\ (h)$
by (*subst in-cong-bigtheta*)

lemma *cmult'* [*simp*]: $c \neq 0 \Longrightarrow L\ F\ (\lambda x.\ f\ x * c) = L\ F\ (f)$
by (*subst mult.commute*) (*rule cmult*)

lemma *cmult-in-iff'* [*simp*]: $c \neq 0 \Longrightarrow (\lambda x.\ f\ x * c) \in L\ F\ (g) \longleftrightarrow f \in L\ F\ (g)$
by (*subst mult.commute*) (*rule cmult-in-iff*)

lemma *cdiv* [*simp*]: $c \neq 0 \Longrightarrow L\ F\ (\lambda x.\ f\ x\ /\ c) = L\ F\ (f)$
using *cmult'*[*of inverse c F f*] **by** (*simp add*: *field-simps*)

lemma *cdiv-in-iff'* [*simp*]: $c \neq 0 \Longrightarrow (\lambda x.\ f\ x\ /\ c) \in L\ F\ (g) \longleftrightarrow f \in L\ F\ (g)$
using *cmult-in-iff'*[*of inverse c f*] **by** (*simp add*: *field-simps*)

lemma *uminus* [*simp*]: $L\ F\ (\lambda x.\ -g\ x) = L\ F\ (g)$ **using** *cmult*[*of −1*] **by** *simp*

lemma *uminus-in-iff* [*simp*]: $(\lambda x.\ -f\ x) \in L\ F\ (g) \longleftrightarrow f \in L\ F\ (g)$
using *cmult-in-iff*[*of −1*] **by** *simp*

lemma *const*: $c \neq 0 \Longrightarrow L\ F\ (\lambda\text{-}.\ c) = L\ F\ (\lambda\text{-}.\ 1)$
by (*subst* (*2*) *cmult*[*symmetric*]) *simp-all*

lemma *const'* [*simp*]: *NO-MATCH 1 c* $\Longrightarrow c \neq 0 \Longrightarrow L\ F\ (\lambda\text{-}.\ c) = L\ F\ (\lambda\text{-}.\ 1)$
by (*rule const*)

lemma *const-in-iff*: $c \neq 0 \Longrightarrow (\lambda\text{-}.\ c) \in L\ F\ (f) \longleftrightarrow (\lambda\text{-}.\ 1) \in L\ F\ (f)$
using *cmult-in-iff'*[*of c* λ-. *1*] **by** *simp*

lemma *const-in-iff'* [*simp*]: *NO-MATCH 1 c* $\Longrightarrow c \neq 0 \Longrightarrow (\lambda\text{-}.\ c) \in L\ F\ (f) \longleftrightarrow (\lambda\text{-}.\ 1) \in L\ F\ (f)$
by (*rule const-in-iff*)

lemma *plus-subset2*: $g \in o[F](f) \Longrightarrow L\ F\ (f) \subseteq L\ F\ (\lambda x.\ f\ x + g\ x)$
by (*subst add.commute*) (*rule plus-subset1*)

lemma *mult-right* [*simp*]: $f \in L\ F\ (g) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in L\ F\ (\lambda x.\ g\ x * h\ x)$
using *mult-left* **by** (*simp add*: *mult.commute*)

lemma *mult*: *f1* $\in L\ F\ (g1) \Longrightarrow$ *f2* $\in L\ F\ (g2) \Longrightarrow (\lambda x.\ f1\ x * f2\ x) \in L\ F\ (\lambda x.\ g1\ x * g2\ x)$
by (*rule trans*, *erule mult-left*, *erule mult-right*)

lemma *inverse-cancel*:
assumes *eventually* $(\lambda x.\ f\ x \neq 0)\ F$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
shows $(\lambda x.\ inverse\ (f\ x)) \in L\ F\ (\lambda x.\ inverse\ (g\ x)) \longleftrightarrow g \in L\ F\ (f)$
proof
assume $(\lambda x.\ inverse\ (f\ x)) \in L\ F\ (\lambda x.\ inverse\ (g\ x))$
from *inverse*[*OF - - this*] *assms* **show** $g \in L\ F\ (f)$ **by** *simp*
qed (*intro inverse assms*)

lemma *divide-right*:
assumes *eventually* $(\lambda x.\ h\ x \neq 0)\ F$
assumes $f \in L\ F\ (g)$
shows $(\lambda x.\ f\ x\ /\ h\ x) \in L\ F\ (\lambda x.\ g\ x\ /\ h\ x)$
by (*subst* (*1 2*) *divide-inverse*) (*intro mult-right inverse assms*)

lemma *divide-right-iff*:
assumes *eventually* $(\lambda x.\ h\ x \neq 0)\ F$
shows $(\lambda x.\ f\ x\ /\ h\ x) \in L\ F\ (\lambda x.\ g\ x\ /\ h\ x) \longleftrightarrow f \in L\ F\ (g)$
proof
assume $(\lambda x.\ f\ x\ /\ h\ x) \in L\ F\ (\lambda x.\ g\ x\ /\ h\ x)$
from *mult-right*[*OF this*, *of h*] *assms* **show** $f \in L\ F\ (g)$
by (*subst* (*asm*) *cong-ex*[*of - f F - g*]) (*auto elim*!: *eventually-mono*)
qed (*simp add*: *divide-right assms*)

lemma *divide-left*:

assumes *eventually* $(\lambda x.\ f\ x \neq 0)\ F$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
assumes $g \in L\ F(f)$
shows $(\lambda x.\ h\ x\ /\ f\ x) \in L\ F\ (\lambda x.\ h\ x\ /\ g\ x)$
by (*subst* (*1 2*) *divide-inverse*) (*intro mult-left inverse assms*)

lemma *divide-left-iff*:
assumes *eventually* $(\lambda x.\ f\ x \neq 0)\ F$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
assumes *eventually* $(\lambda x.\ h\ x \neq 0)\ F$
shows $(\lambda x.\ h\ x\ /\ f\ x) \in L\ F\ (\lambda x.\ h\ x\ /\ g\ x) \longleftrightarrow g \in L\ F\ (f)$
proof
assume *A*: $(\lambda x.\ h\ x\ /\ f\ x) \in L\ F\ (\lambda x.\ h\ x\ /\ g\ x)$
from *assms* **have** *B*: *eventually* $(\lambda x.\ h\ x\ /\ f\ x\ /\ h\ x = inverse\ (f\ x))\ F$
by *eventually-elim* (*simp add*: *divide-inverse*)
from *assms* **have** *C*: *eventually* $(\lambda x.\ h\ x\ /\ g\ x\ /\ h\ x = inverse\ (g\ x))\ F$
by *eventually-elim* (*simp add*: *divide-inverse*)
from *divide-right*[*OF assms*(*3*) *A*] *assms* **show** $g \in L\ F\ (f)$
by (*subst* (*asm*) *cong-ex*[*OF B C*]) (*simp add*: *inverse-cancel*)
qed (*simp add*: *divide-left assms*)

lemma *divide*:
assumes *eventually* $(\lambda x.\ g1\ x \neq 0)\ F$
assumes *eventually* $(\lambda x.\ g2\ x \neq 0)\ F$
assumes $f1 \in L\ F\ (f2)$ $g2 \in L\ F\ (g1)$
shows $(\lambda x.\ f1\ x\ /\ g1\ x) \in L\ F\ (\lambda x.\ f2\ x\ /\ g2\ x)$
by (*subst* (*1 2*) *divide-inverse*) (*intro mult inverse assms*)

lemma *divide-eq1*:
assumes *eventually* $(\lambda x.\ h\ x \neq 0)\ F$
shows $f \in L\ F\ (\lambda x.\ g\ x\ /\ h\ x) \longleftrightarrow (\lambda x.\ f\ x * h\ x) \in L\ F\ (g)$
proof−
have $f \in L\ F\ (\lambda x.\ g\ x\ /\ h\ x) \longleftrightarrow (\lambda x.\ f\ x * h\ x\ /\ h\ x) \in L\ F\ (\lambda x.\ g\ x\ /\ h\ x)$
using *assms* **by** (*intro in-cong*) (*auto elim*: *eventually-mono*)
thus *?thesis* **by** (*simp only*: *divide-right-iff assms*)
qed

lemma *divide-eq2*:
assumes *eventually* $(\lambda x.\ h\ x \neq 0)\ F$
shows $(\lambda x.\ f\ x\ /\ h\ x) \in L\ F\ (\lambda x.\ g\ x) \longleftrightarrow f \in L\ F\ (\lambda x.\ g\ x * h\ x)$
proof−
have $L\ F\ (\lambda x.\ g\ x) = L\ F\ (\lambda x.\ g\ x * h\ x\ /\ h\ x)$
using *assms* **by** (*intro cong*) (*auto elim*: *eventually-mono*)
thus *?thesis* **by** (*simp only*: *divide-right-iff assms*)
qed

lemma *inverse-eq1*:
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
shows $f \in L\ F\ (\lambda x.\ inverse\ (g\ x)) \longleftrightarrow (\lambda x.\ f\ x * g\ x) \in L\ F\ (\lambda\text{-}.\ 1)$

using *divide-eq1*[*of g F f* λ-. *1*] **by** (*simp add*: *divide-inverse assms*)

lemma *inverse-eq2*:
assumes *eventually* (λ*x*. *f x* ≠ *0*) *F*
shows (λ*x*. *inverse* (*f x*)) ∈ *L F* (*g*) ⟷ (λ*x*. *1*) ∈ *L F* (λ*x*. *f x* ∗ *g x*)
using *divide-eq2*[*of f F* λ-. *1 g*] **by** (*simp add*: *divide-inverse assms mult-ac*)

lemma *inverse-flip*:
assumes *eventually* (λ*x*. *g x* ≠ *0*) *F*
assumes *eventually* (λ*x*. *h x* ≠ *0*) *F*
assumes (λ*x*. *inverse* (*g x*)) ∈ *L F* (*h*)
shows (λ*x*. *inverse* (*h x*)) ∈ *L F* (*g*)
using *assms* **by** (*simp add*: *divide-eq1 divide-eq2 inverse-eq-divide mult.commute*)

lemma *lift-trans*:
assumes *f* ∈ *L F* (*g*)
assumes (λ*x*. *t x* (*g x*)) ∈ *L F* (*h*)
assumes ⋀*f g*. *f* ∈ *L F* (*g*) ⟹ (λ*x*. *t x* (*f x*)) ∈ *L F* (λ*x*. *t x* (*g x*))
shows (λ*x*. *t x* (*f x*)) ∈ *L F* (*h*)
by (*rule trans*[*OF assms*(*3*)[*OF assms*(*1*)] *assms*(*2*)])

lemma *lift-trans′*:
assumes *f* ∈ *L F* (λ*x*. *t x* (*g x*))
assumes *g* ∈ *L F* (*h*)
assumes ⋀*g h*. *g* ∈ *L F* (*h*) ⟹ (λ*x*. *t x* (*g x*)) ∈ *L F* (λ*x*. *t x* (*h x*))
shows *f* ∈ *L F* (λ*x*. *t x* (*h x*))
by (*rule trans*[*OF assms*(*1*) *assms*(*3*)[*OF assms*(*2*)]])

lemma *lift-trans-bigtheta*:
assumes *f* ∈ *L F* (*g*)
assumes (λ*x*. *t x* (*g x*)) ∈ Θ[*F*](*h*)
assumes ⋀*f g*. *f* ∈ *L F* (*g*) ⟹ (λ*x*. *t x* (*f x*)) ∈ *L F* (λ*x*. *t x* (*g x*))
shows (λ*x*. *t x* (*f x*)) ∈ *L F* (*h*)
using *cong-bigtheta*[*OF assms*(*2*)] *assms*(*3*)[*OF assms*(*1*)] **by** *simp*

lemma *lift-trans-bigtheta′*:
assumes *f* ∈ *L F* (λ*x*. *t x* (*g x*))
assumes *g* ∈ Θ[*F*](*h*)
assumes ⋀*g h*. *g* ∈ Θ[*F*](*h*) ⟹ (λ*x*. *t x* (*g x*)) ∈ Θ[*F*](λ*x*. *t x* (*h x*))
shows *f* ∈ *L F* (λ*x*. *t x* (*h x*))
using *cong-bigtheta*[*OF assms*(*3*)[*OF assms*(*2*)]] *assms*(*1*) **by** *simp*

lemma (**in** *landau-symbol*) *mult-in-1*:
assumes *f* ∈ *L F* (λ-. *1*) *g* ∈ *L F* (λ-. *1*)
shows (λ*x*. *f x* ∗ *g x*) ∈ *L F* (λ-. *1*)
using *mult*[*OF assms*] **by** *simp*

lemma (**in** *landau-symbol*) *of-real-cancel*:
(λ*x*. *of-real* (*f x*)) ∈ *L F* (λ*x*. *of-real* (*g x*)) ⟹ *f* ∈ *Lr F g*

by (*subst* (*asm*) *norm-iff* [*symmetric*], *subst* (*asm*) (*1 2*) *norm-of-real*) *simp-all*

lemma (**in** *landau-symbol*) *of-real-iff*:
(λ*x*. *of-real* (*f x*)) ∈ *L F* (λ*x*. *of-real* (*g x*)) ⟷ *f* ∈ *Lr F g*
by (*subst norm-iff* [*symmetric*], *subst* (*1 2*) *norm-of-real*) *simp-all*

lemmas [*landau-divide-simps*] =
inverse-cancel divide-left-iff divide-eq1 divide-eq2 inverse-eq1 inverse-eq2

end

The symbols O and o and Ω and ω are dual, so for many rules, replacing O with Ω, o with ω, and $\leq$ with $\geq$ in a theorem yields another valid theorem. The following locale captures this fact.

locale *landau-pair* =
fixes *L l* :: *'a filter* ⇒ (*'a* ⇒ (*'b* :: *real-normed-field*)) ⇒ (*'a* ⇒ *'b*) *set*
fixes *L' l'* :: *'c filter* ⇒ (*'c* ⇒ (*'b* :: *real-normed-field*)) ⇒ (*'c* ⇒ *'b*) *set*
fixes *Lr lr* :: *'a filter* ⇒ (*'a* ⇒ *real*) ⇒ (*'a* ⇒ *real*) *set*
and *R* :: *real* ⇒ *real* ⇒ *bool*
assumes *L-def*: *L F g* = {*f*. ∃ *c*>*0*. *eventually* (λ*x*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g x*))) *F*}
and *l-def*: *l F g* = {*f*. ∀ *c*>*0*. *eventually* (λ*x*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g x*))) *F*}
and *L'-def*: *L' F' g'* = {*f*. ∃ *c*>*0*. *eventually* (λ*x*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g' x*))) *F'*}
and *l'-def*: *l' F' g'* = {*f*. ∀ *c*>*0*. *eventually* (λ*x*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g' x*))) *F'*}
and *Lr-def*: *Lr F'' g''* = {*f*. ∃ *c*>*0*. *eventually* (λ*x*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g'' x*))) *F''*}
and *lr-def*: *lr F'' g''* = {*f*. ∀ *c*>*0*. *eventually* (λ*x*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g'' x*))) *F''*}
and *R*: *R* = (≤) ∨ *R* = (≥)

interpretation *landau-o*:
landau-pair bigo smallo bigo smallo bigo smallo (≤)
by *unfold-locales* (*auto simp*: *bigo-def smallo-def intro*!: *ext*)

interpretation *landau-omega*:
landau-pair bigomega smallomega bigomega smallomega bigomega smallomega (≥)
by *unfold-locales* (*auto simp*: *bigomega-def smallomega-def intro*!: *ext*)

context *landau-pair*
begin

lemmas *R-E* = *disjE* [*OF R*, *case-names le ge*]

lemma *bigI*:

$c > 0 \Longrightarrow$ *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * norm\ (g\ x)))\ F \Longrightarrow f \in L\ F\ (g)$
unfolding *L-def* **by** *blast*

lemma *bigE*:
assumes $f \in L\ F\ (g)$
obtains c **where** $c > 0$ *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * (norm\ (g\ x))))\ F$
using *assms* **unfolding** *L-def* **by** *blast*

lemma *smallI*:
$(\bigwedge c.\ c > 0 \Longrightarrow$ *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * (norm\ (g\ x))))\ F) \Longrightarrow f \in l\ F\ (g)$
unfolding *l-def* **by** *blast*

lemma *smallD*:
$f \in l\ F\ (g) \Longrightarrow c > 0 \Longrightarrow$ *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * (norm\ (g\ x))))\ F$
unfolding *l-def* **by** *blast*

lemma *bigE-nonneg-real*:
assumes $f \in Lr\ F\ (g)$ *eventually* $(\lambda x.\ f\ x \geq 0)\ F$
obtains c **where** $c > 0$ *eventually* $(\lambda x.\ R\ (f\ x)\ (c * |g\ x|))\ F$
proof−
from *assms*(*1*) **obtain** c **where** *c*: $c > 0$ *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * norm\ (g\ x)))\ F$
by (*auto simp*: *Lr-def*)
from *c*(*2*) *assms*(*2*) **have** *eventually* $(\lambda x.\ R\ (f\ x)\ (c * |g\ x|))\ F$
by *eventually-elim simp*
from *c*(*1*) **and** *this* **show** *?thesis* **by** (*rule that*)
qed

lemma *smallD-nonneg-real*:
assumes $f \in lr\ F\ (g)$ *eventually* $(\lambda x.\ f\ x \geq 0)\ F\ c > 0$
shows *eventually* $(\lambda x.\ R\ (f\ x)\ (c * |g\ x|))\ F$
using *assms* **by** (*auto simp*: *lr-def dest*!: *spec*[*of* - *c*] *elim*: *eventually-elim2*)

lemma *small-imp-big*: $f \in l\ F\ (g) \Longrightarrow f \in L\ F\ (g)$
by (*rule bigI*[*OF* - *smallD*, *of 1*]) *simp-all*

lemma *small-subset-big*: $l\ F\ (g) \subseteq L\ F\ (g)$
using *small-imp-big* **by** *blast*

lemma *R-refl* [*simp*]: $R\ x\ x$ **using** *R* **by** *auto*

lemma *R-linear*: $\neg R\ x\ y \Longrightarrow R\ y\ x$
using *R* **by** *auto*

lemma *R-trans* [*trans*]: $R\ a\ b \Longrightarrow R\ b\ c \Longrightarrow R\ a\ c$
using *R* **by** *auto*

lemma *R-mult-left-mono*: $R\ a\ b \Longrightarrow c \geq 0 \Longrightarrow R\ (c*a)\ (c*b)$

using R **by** (*auto simp*: *mult-left-mono*)

lemma *R-mult-right-mono*: $R\ a\ b \Longrightarrow c \geq 0 \Longrightarrow R\ (a*c)\ (b*c)$
using R **by** (*auto simp*: *mult-right-mono*)

lemma *big-trans*:
assumes $f \in L\ F\ (g)$ $g \in L\ F\ (h)$
shows $f \in L\ F\ (h)$
proof−
from *assms* **obtain** c d **where** $*$: $0 < c\ 0 < d$
and $**$: $\forall_F\ x\ in\ F.\ R\ (norm\ (f\ x))\ (c * norm\ (g\ x))$
$\forall_F\ x\ in\ F.\ R\ (norm\ (g\ x))\ (d * norm\ (h\ x))$
by (*elim bigE*)
from $**$ **have** *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * d * (norm\ (h\ x))))\ F$
proof *eventually-elim*
fix x **assume** $R\ (norm\ (f\ x))\ (c * (norm\ (g\ x)))$
also assume $R\ (norm\ (g\ x))\ (d * (norm\ (h\ x)))$
with ‹$0 < c$› **have** $R\ (c * (norm\ (g\ x)))\ (c * (d * (norm\ (h\ x))))$
by (*intro R-mult-left-mono*) *simp-all*
finally show $R\ (norm\ (f\ x))\ (c * d * (norm\ (h\ x)))$
by (*simp add*: *algebra-simps*)
qed
with $*$ **show** *?thesis* **by** (*intro bigI*[*of c*∗*d*]) *simp-all*
qed

lemma *big-small-trans*:
assumes $f \in L\ F\ (g)$ $g \in l\ F\ (h)$
shows $f \in l\ F\ (h)$
proof (*rule smallI*)
fix c :: *real* **assume** c: $c > 0$
from *assms*(*1*) **obtain** d **where** d: $d > 0$ **and** $*$: $\forall_F\ x\ in\ F.\ R\ (norm\ (f\ x))\ (d$
$*\ norm\ (g\ x))$
by (*elim bigE*)
from *assms*(*2*) c d **have** *eventually* $(\lambda x.\ R\ (norm\ (g\ x))\ (c * inverse\ d * norm$
$(h\ x)))\ F$
by (*intro smallD*) *simp-all*
with $*$ **show** *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * (norm\ (h\ x))))\ F$
proof *eventually-elim*
case (*elim x*)
show *?case*
by (*use elim*(*1*) **in** ‹*rule R-trans*›) (*use elim*(*2*) R d **in** ‹*auto simp*: *field-simps*›)
qed
qed

lemma *small-big-trans*:
assumes $f \in l\ F\ (g)$ $g \in L\ F\ (h)$
shows $f \in l\ F\ (h)$
proof (*rule smallI*)
fix c :: *real* **assume** c: $c > 0$

from *assms*(*2*) **obtain** *d* **where** *d*: *d* > *0* **and** ∗: ∀ *F* *x* *in* *F*. *R* (*norm* (*g* *x*)) (*d* ∗ *norm* (*h* *x*))
by (*elim* *bigE*)
from *assms*(*1*) *c* *d* **have** *eventually* (λ*x*. *R* (*norm* (*f* *x*)) (*c* ∗ *inverse* *d* ∗ *norm* (*g* *x*))) *F*
by (*intro* *smallD*) *simp-all*
with ∗ **show** *eventually* (λ*x*. *R* (*norm* (*f* *x*)) (*c* ∗ *norm* (*h* *x*))) *F*
by *eventually-elim* (*rotate-tac* *2*, *erule* *R-trans*, *insert* *R* *c* *d*, *auto* *simp*: *field-simps*)
qed

lemma *small-trans*:
f ∈ *l* *F* (*g*) ⟹ *g* ∈ *l* *F* (*h*) ⟹ *f* ∈ *l* *F* (*h*)
by (*rule* *big-small-trans*[*OF* *small-imp-big*])

lemma *small-big-trans'*:
f ∈ *l* *F* (*g*) ⟹ *g* ∈ *L* *F* (*h*) ⟹ *f* ∈ *L* *F* (*h*)
by (*rule* *small-imp-big*[*OF* *small-big-trans*])

lemma *big-small-trans'*:
f ∈ *L* *F* (*g*) ⟹ *g* ∈ *l* *F* (*h*) ⟹ *f* ∈ *L* *F* (*h*)
by (*rule* *small-imp-big*[*OF* *big-small-trans*])

lemma *big-subsetI* [*intro*]: *f* ∈ *L* *F* (*g*) ⟹ *L* *F* (*f*) ⊆ *L* *F* (*g*)
by (*intro* *subsetI*) (*drule* (*1*) *big-trans*)

lemma *small-subsetI* [*intro*]: *f* ∈ *L* *F* (*g*) ⟹ *l* *F* (*f*) ⊆ *l* *F* (*g*)
by (*intro* *subsetI*) (*drule* (*1*) *small-big-trans*)

lemma *big-refl* [*simp*]: *f* ∈ *L* *F* (*f*)
by (*rule* *bigI*[*of* *1*]) *simp-all*

lemma *small-refl-iff*: *f* ∈ *l* *F* (*f*) ⟷ *eventually* (λ*x*. *f* *x* = *0*) *F*
proof (*rule* *iffI*[*OF* - *smallI*])
assume *f*: *f* ∈ *l* *F* *f*
have (*1*/*2*::*real*) > *0* (*2*::*real*) > *0* **by** *simp-all*
from *smallD*[*OF* *f* *this*(*1*)] *smallD*[*OF* *f* *this*(*2*)]
show *eventually* (λ*x*. *f* *x* = *0*) *F* **by** *eventually-elim* (*insert* *R*, *auto*)
next
fix *c* :: *real* **assume** *c* > *0* *eventually* (λ*x*. *f* *x* = *0*) *F*
from *this*(*2*) **show** *eventually* (λ*x*. *R* (*norm* (*f* *x*)) (*c* ∗ *norm* (*f* *x*))) *F*
by *eventually-elim* *simp-all*
qed

lemma *big-small-asymmetric*: *f* ∈ *L* *F* (*g*) ⟹ *g* ∈ *l* *F* (*f*) ⟹ *eventually* (λ*x*. *f* *x* = *0*) *F*
by (*drule* (*1*) *big-small-trans*) (*simp* *add*: *small-refl-iff*)

lemma *small-big-asymmetric*: *f* ∈ *l* *F* (*g*) ⟹ *g* ∈ *L* *F* (*f*) ⟹ *eventually* (λ*x*. *f*

$x = 0)$ F
by (*drule* (*1*) *small-big-trans*) (*simp add*: *small-refl-iff*)

lemma *small-asymmetric*: $f \in l\ F\ (g) \Longrightarrow g \in l\ F\ (f) \Longrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)\ F$
by (*drule* (*1*) *small-trans*) (*simp add*: *small-refl-iff*)

lemma *plus-aux*:
assumes $f \in o[F](g)$
shows $g \in L\ F\ (\lambda x.\ f\ x + g\ x)$
proof (*rule R-E*)
assume *R*: $R = (\leq)$
have *A*: $1/2 > (0::real)$ **by** *simp*
have *B*: $1/2 * (norm\ (g\ x)) \leq norm\ (f\ x + g\ x)$
if $norm\ (f\ x) \leq 1/2 * norm\ (g\ x)$ **for** x
proof −
from *that* **have** $1/2 * (norm\ (g\ x)) \leq (norm\ (g\ x)) - (norm\ (f\ x))$
by *simp*
also have $norm\ (g\ x) - norm\ (f\ x) \leq norm\ (f\ x + g\ x)$
by (*subst add.commute*) (*rule norm-diff-ineq*)
finally show *?thesis* **by** *simp*
qed
show $g \in L\ F\ (\lambda x.\ f\ x + g\ x)$
apply (*rule bigI*[*of 2*])
apply *simp*
apply (*use landau-o.smallD*[*OF assms A*] **in** *eventually-elim*)
apply (*use B* **in** ‹*simp add*: *R algebra-simps*›)
done
next
assume *R*: $R = (\lambda x\ y.\ x \geq y)$
show $g \in L\ F\ (\lambda x.\ f\ x + g\ x)$
proof (*rule bigI*[*of 1/2*])
show *eventually* $(\lambda x.\ R\ (norm\ (g\ x))\ (1/2 * norm\ (f\ x + g\ x)))\ F$
using *landau-o.smallD*[*OF assms zero-less-one*]
proof *eventually-elim*
case (*elim x*)
have $norm\ (f\ x + g\ x) \leq norm\ (f\ x) + norm\ (g\ x)$
by (*rule norm-triangle-ineq*)
also note *elim*
finally show *?case* **by** (*simp add*: *R*)
qed
qed *simp-all*
qed

end

lemma *bigomega-iff-bigo*: $g \in \Omega[F](f) \longleftrightarrow f \in O[F](g)$

proof
assume $f \in O[F](g)$
then obtain c **where** $0 < c \; \forall_F \; x \; in \; F. \; norm \; (f \; x) \leq c * norm \; (g \; x)$
by (*rule landau-o.bigE*)
then show $g \in \Omega[F](f)$
by (*intro landau-omega.bigI*[*of inverse c*]) (*simp-all add*: *field-simps*)
next
assume $g \in \Omega[F](f)$
then obtain c **where** $0 < c \; \forall_F \; x \; in \; F. \; c * norm \; (f \; x) \leq norm \; (g \; x)$
by (*rule landau-omega.bigE*)
then show $f \in O[F](g)$
by (*intro landau-o.bigI*[*of inverse c*]) (*simp-all add*: *field-simps*)
qed

lemma *smallomega-iff-smallo*: $g \in \omega[F](f) \longleftrightarrow f \in o[F](g)$
proof
assume $f \in o[F](g)$
from *landau-o.smallD*[*OF this*, *of inverse c* **for** *c*]
show $g \in \omega[F](f)$ **by** (*intro landau-omega.smallI*) (*simp-all add*: *field-simps*)
next
assume $g \in \omega[F](f)$
from *landau-omega.smallD*[*OF this*, *of inverse c* **for** *c*]
show $f \in o[F](g)$ **by** (*intro landau-o.smallI*) (*simp-all add*: *field-simps*)
qed

context *landau-pair*
begin

lemma *big-mono*:
eventually ($\lambda x. \; R \; (norm \; (f \; x)) \; (norm \; (g \; x))$) $F \Longrightarrow f \in L \; F \; (g)$
by (*rule bigI*[*OF zero-less-one*]) *simp*

lemma *big-mult*:
assumes $f1 \in L \; F \; (g1) \; f2 \in L \; F \; (g2)$
shows $(\lambda x. \; f1 \; x * f2 \; x) \in L \; F \; (\lambda x. \; g1 \; x * g2 \; x)$
proof−
from *assms* **obtain** $c1 \; c2$ **where** $*$: $c1 > 0 \; c2 > 0$
and $**$: $\forall_F \; x \; in \; F. \; R \; (norm \; (f1 \; x)) \; (c1 * norm \; (g1 \; x))$
$\forall_F \; x \; in \; F. \; R \; (norm \; (f2 \; x)) \; (c2 * norm \; (g2 \; x))$
by (*elim bigE*)
from $*$ **have** $c1 * c2 > 0$ **by** *simp*
moreover have *eventually* ($\lambda x. \; R \; (norm \; (f1 \; x * f2 \; x)) \; (c1 * c2 * norm \; (g1 \; x$
$* \; g2 \; x))) \; F$
using $**$
proof *eventually-elim*
case (*elim x*)
show *?case*
proof (*cases rule*: *R-E*)

```
    case le
    have norm (f1 x) * norm (f2 x) ≤ (c1 * norm (g1 x)) * (c2 * norm (g2 x))
      using elim le * by (intro mult-mono mult-nonneg-nonneg) auto
    with le show ?thesis by (simp add: le norm-mult mult-ac)
  next
    case ge
    have (c1 * norm (g1 x)) * (c2 * norm (g2 x)) ≤ norm (f1 x) * norm (f2 x)
      using elim ge * by (intro mult-mono mult-nonneg-nonneg) auto
    with ge show ?thesis by (simp-all add: norm-mult mult-ac)
  qed
 qed
 ultimately show ?thesis by (rule bigI)
qed

lemma small-big-mult:
 assumes f1 ∈ l F (g1) f2 ∈ L F (g2)
 shows   (λx. f1 x * f2 x) ∈ l F (λx. g1 x * g2 x)
proof (rule smallI)
 fix c1 :: real assume c1: c1 > 0
 from assms(2) obtain c2 where c2: c2 > 0
   and *: ∀ F x in F. R (norm (f2 x)) (c2 * norm (g2 x)) by (elim bigE)
 from assms(1) c1 c2 have eventually (λx. R (norm (f1 x)) (c1 * inverse c2 *
norm (g1 x))) F
   by (auto intro!: smallD)
 with * show eventually (λx. R (norm (f1 x * f2 x)) (c1 * norm (g1 x * g2 x)))
F
 proof eventually-elim
   case (elim x)
   show ?case
   proof (cases rule: R-E)
     case le
     have norm (f1 x) * norm (f2 x) ≤ (c1 * inverse c2 * norm (g1 x)) * (c2 *
norm (g2 x))
       using elim le c1 c2 by (intro mult-mono mult-nonneg-nonneg) auto
     with le c2 show ?thesis by (simp add: le norm-mult field-simps)
   next
     case ge
     have norm (f1 x) * norm (f2 x) ≥ (c1 * inverse c2 * norm (g1 x)) * (c2 *
norm (g2 x))
       using elim ge c1 c2 by (intro mult-mono mult-nonneg-nonneg) auto
     with ge c2 show ?thesis by (simp add: ge norm-mult field-simps)
   qed
 qed
qed

lemma big-small-mult:
 f1 ∈ L F (g1) ⟹ f2 ∈ l F (g2) ⟹ (λx. f1 x * f2 x) ∈ l F (λx. g1 x * g2 x)
 by (subst (1 2) mult.commute) (rule small-big-mult)
```

lemma *small-mult*: *f1* ∈ *l F* (*g1*) ⟹ *f2* ∈ *l F* (*g2*) ⟹ (λ*x*. *f1 x* ∗ *f2 x*) ∈ *l F* (λ*x*. *g1 x* ∗ *g2 x*)
by (*rule small-big-mult*, *assumption*, *rule small-imp-big*)

lemmas *mult* = *big-mult small-big-mult big-small-mult small-mult*

sublocale *big*: *landau-symbol L L' Lr*
proof
have *L*: *L* = *bigo* ∨ *L* = *bigomega*
by (*rule R-E*) (*auto simp*: *bigo-def L-def bigomega-def fun-eq-iff*)
have *A*: (λ*x*. *c* ∗ *f x*) ∈ *L F f* **if** *c* ≠ *0* **for** *c* :: *'b* **and** *F* **and** *f* :: *'a* ⇒ *'b*
using *that* **by** (*intro bigI*[*of norm c*]) (*simp-all add*: *norm-mult*)
show *L F* (λ*x*. *c* ∗ *f x*) = *L F f* **if** *c* ≠ *0* **for** *c* :: *'b* **and** *F* **and** *f* :: *'a* ⇒ *'b*
using ‹*c* ≠ *0*› **and** *A*[*of c f*] **and** *A*[*of inverse c* λ*x*. *c* ∗ *f x*]
by (*intro equalityI big-subsetI*) (*simp-all add*: *field-simps*)
show ((λ*x*. *c* ∗ *f x*) ∈ *L F g*) = (*f* ∈ *L F g*) **if** *c* ≠ *0*
for *c* :: *'b* **and** *F* **and** *f g* :: *'a* ⇒ *'b*
proof −
from ‹*c* ≠ *0*› **and** *A*[*of c f*] **and** *A*[*of inverse c* λ*x*. *c* ∗ *f x*]
have (λ*x*. *c* ∗ *f x*) ∈ *L F f f* ∈ *L F* (λ*x*. *c* ∗ *f x*)
by (*simp-all add*: *field-simps*)
then show *?thesis* **by** (*intro iffI*) (*erule* (*1*) *big-trans*)+
qed
show (λ*x*. *inverse* (*g x*)) ∈ *L F* (λ*x*. *inverse* (*f x*))
if ∗: *f* ∈ *L F* (*g*) **and** ∗∗: *eventually* (λ*x*. *f x* ≠ *0*) *F eventually* (λ*x*. *g x* ≠ *0*) *F*
for *f g* :: *'a* ⇒ *'b* **and** *F*
proof −
from ∗ **obtain** *c* **where** *c*: *c* > *0* **and** ∗∗∗: ∀ *F x in F*. *R* (*norm* (*f x*)) (*c* ∗ *norm* (*g x*))
by (*elim bigE*)
from ∗∗ ∗∗∗ **have** *eventually* (λ*x*. *R* (*norm* (*inverse* (*g x*))) (*c* ∗ *norm* (*inverse* (*f x*)))) *F*
by *eventually-elim* (*rule R-E*, *simp-all add*: *field-simps norm-inverse norm-divide c*)
with *c* **show** *?thesis* **by** (*rule bigI*)
qed
show *L F g* ⊆ *L F* (λ*x*. *f x* + *g x*) **if** *f* ∈ *o*[*F*](*g*) **for** *f g* :: *'a* ⇒ *'b* **and** *F*
using *plus-aux that* **by** (*blast intro*!: *big-subsetI*)
show *L F* (*f*) = *L F* (*g*) **if** *eventually* (λ*x*. *f x* = *g x*) *F* **for** *f g* :: *'a* ⇒ *'b* **and** *F*
unfolding *L-def* **by** (*subst eventually-subst'*[*OF that*]) (*rule refl*)
show *f* ∈ *L F* (*h*) ⟷ *g* ∈ *L F* (*h*) **if** *eventually* (λ*x*. *f x* = *g x*) *F*
for *f g h* :: *'a* ⇒ *'b* **and** *F*
unfolding *L-def mem-Collect-eq*
by (*subst* (*1*) *eventually-subst'*[*OF that*]) (*rule refl*)
show *L F f* ⊆ *L F g* **if** *f* ∈ *L F g* **for** *f g* :: *'a* ⇒ *'b* **and** *F*
using *that* **by** (*rule big-subsetI*)
show *L F* (*f*) = *L F* (*g*) **if** *f* ∈ Θ[*F*](*g*) **for** *f g* :: *'a* ⇒ *'b* **and** *F*

using *L that* **unfolding** *bigtheta-def*
by (*intro equalityI big-subsetI*) (*auto simp*: *bigomega-iff-bigo*)
show $f \in L\ F\ (h) \longleftrightarrow g \in L\ F\ (h)$ **if** $f \in \Theta[F](g)$ **for** $f\ g\ h :: 'a \Rightarrow 'b$ **and** F
by (*rule disjE*[*OF L*])
(*use that* **in** ‹*auto simp*: *bigtheta-def bigomega-iff-bigo intro*: *landau-o.big-trans*›)
show $(\lambda x.\ h\ x * f\ x) \in L\ F\ (\lambda x.\ h\ x * g\ x)$ **if** $f \in L\ F\ g$ **for** $f\ g\ h :: 'a \Rightarrow 'b$ **and** F
using *that* **by** (*intro big-mult*) *simp*
show $f \in L\ F\ (h)$ **if** $f \in L\ F\ g\ g \in L\ F\ h$ **for** $f\ g\ h :: 'a \Rightarrow 'b$ **and** F
using *that* **by** (*rule big-trans*)
show $(\lambda x.\ f\ (h\ x)) \in L'\ G\ (\lambda x.\ g\ (h\ x))$
if $f \in L\ F\ g$ **and** *filterlim h F G*
for $f\ g :: 'a \Rightarrow 'b$ **and** $h :: 'c \Rightarrow 'a$ **and** $F\ G$
using *that* **by** (*auto simp*: *L-def L'-def filterlim-iff*)
show $f \in L\ (sup\ F\ G)\ g$ **if** $f \in L\ F\ g\ f \in L\ G\ g$
for $f\ g :: 'a \Rightarrow 'b$ **and** $F\ G :: 'a\ filter$
proof −
from *that* [*THEN bigE*] **obtain** *c1 c2*
where $*$: *c1* > 0 *c2* > 0
and $**$: $\forall_F\ x\ in\ F.\ R\ (norm\ (f\ x))\ (c1 * norm\ (g\ x))$
$\forall_F\ x\ in\ G.\ R\ (norm\ (f\ x))\ (c2 * norm\ (g\ x))$ **.**
define *c* **where** *c* = (*if R c1 c2 then c2 else c1*)
from $*$ **have** *c*: *R c1 c R c2 c c* > 0
by (*auto simp*: *c-def dest*: *R-linear*)
with $**$ **have** *eventually* $(\lambda x.\ R\ (norm\ (f\ x))\ (c * norm\ (g\ x)))\ F$
eventually $(\lambda x.\ R\ (norm\ (f\ x))\ (c * norm\ (g\ x)))\ G$
by (*force elim*: *eventually-mono intro*: *R-trans*[*OF - R-mult-right-mono*])+
with *c* **show** $f \in L\ (sup\ F\ G)\ g$
by (*auto simp*: *L-def eventually-sup*)
qed
show $((\lambda x.\ f\ (h\ x)) \in L'\ (filtercomap\ h\ F)\ (\lambda x.\ g\ (h\ x)))$ **if** $(f \in L\ F\ g)$
for $f\ g :: 'a \Rightarrow 'b$ **and** $h :: 'c \Rightarrow 'a$ **and** $F\ G :: 'a\ filter$
using *that* **unfolding** *L-def L'-def* **by** *auto*
qed (*auto simp*: *L-def Lr-def eventually-filtermap L'-def*
intro: *filter-leD exI*[*of - 1*::*real*])

sublocale *small*: *landau-symbol l l' lr*
proof
have *A*: $(\lambda x.\ c * f\ x) \in L\ F\ f$ **if** $c \neq 0$ **for** $c :: 'b$ **and** $f :: 'a \Rightarrow 'b$ **and** F
using *that* **by** (*intro bigI*[*of norm c*]) (*simp-all add*: *norm-mult*)
show $l\ F\ (\lambda x.\ c * f\ x) = l\ F\ f$ **if** $c \neq 0$ **for** $c :: 'b$ **and** $f :: 'a \Rightarrow 'b$ **and** F
using *that* **and** *A*[*of c f*] **and** *A*[*of inverse c* $\lambda x.\ c * f\ x$]
by (*intro equalityI small-subsetI*) (*simp-all add*: *field-simps*)
show $((\lambda x.\ c * f\ x) \in l\ F\ g) = (f \in l\ F\ g)$ **if** $c \neq 0$ **for** $c :: 'b$ **and** $f\ g :: 'a \Rightarrow 'b$ **and** F
proof −
from *that* **and** *A*[*of c f*] **and** *A*[*of inverse c* $\lambda x.\ c * f\ x$]
have $(\lambda x.\ c * f\ x) \in L\ F\ f\ f \in L\ F\ (\lambda x.\ c * f\ x)$
by (*simp-all add*: *field-simps*)

```
    then show ?thesis
      by (intro iffI) (erule (1) big-small-trans)+
  qed
  show l F g ⊆ l F (λx. f x + g x) if f ∈ o[F](g) for f g :: 'a ⇒ 'b and F
    using plus-aux that by (blast intro!: small-subsetI)
  show (λx. inverse (g x)) ∈ l F (λx. inverse (f x))
    if A: f ∈ l F (g) and B: eventually (λx. f x ≠ 0) F eventually (λx. g x ≠ 0) F
    for f g :: 'a ⇒ 'b and F
  proof (rule smallI)
    fix c :: real assume c: c > 0
    from B smallD[OF A c]
    show eventually (λx. R (norm (inverse (g x))) (c * norm (inverse (f x)))) F
    by eventually-elim (rule R-E, simp-all add: field-simps norm-inverse norm-divide)
  qed
  show l F (f) = l F (g) if eventually (λx. f x = g x) F for f g :: 'a ⇒ 'b and F
    unfolding l-def by (subst eventually-subst'[OF that]) (rule refl)
  show f ∈ l F (h) ⟷ g ∈ l F (h) if eventually (λx. f x = g x) F for f g h :: 'a
⇒ 'b and F
    unfolding l-def mem-Collect-eq by (subst (1) eventually-subst'[OF that]) (rule
refl)
  show l F f ⊆ l F g if f ∈ l F g for f g :: 'a ⇒ 'b and F
    using that by (intro small-subsetI small-imp-big)
  show l F (f) = l F (g) if f ∈ Θ[F](g) for f g :: 'a ⇒ 'b and F
  proof −
    have L: L = bigo ∨ L = bigomega
      by (rule R-E) (auto simp: bigo-def L-def bigomega-def fun-eq-iff)
    with that show ?thesis unfolding bigtheta-def
      by (intro equalityI small-subsetI) (auto simp: bigomega-iff-bigo)
  qed
  show f ∈ l F (h) ⟷ g ∈ l F (h) if f ∈ Θ[F](g) for f g h :: 'a ⇒ 'b and F
  proof −
    have l: l = smallo ∨ l = smallomega
      by (rule R-E) (auto simp: smallo-def l-def smallomega-def fun-eq-iff)
    show ?thesis
      by (rule disjE[OF l])
        (use that in ‹auto simp: bigtheta-def bigomega-iff-bigo smallomega-iff-smallo
          intro: landau-o.big-small-trans landau-o.small-big-trans›)
  qed
  show (λx. h x * f x) ∈ l F (λx. h x * g x) if f ∈ l F g for f g h :: 'a ⇒ 'b and F
    using that by (intro big-small-mult) simp
  show f ∈ l F (h) if f ∈ l F g g ∈ l F h for f g h :: 'a ⇒ 'b and F
    using that by (rule small-trans)
  show (λx. f (h x)) ∈ l' G (λx. g (h x)) if f ∈ l F g and filterlim h F G
    for f g :: 'a ⇒ 'b and h :: 'c ⇒ 'a and F G
    using that by (auto simp: l-def l'-def filterlim-iff)
  show ((λx. f (h x)) ∈ l' (filtercomap h F) (λx. g (h x))) if f ∈ l F g
    for f g :: 'a ⇒ 'b and h :: 'c ⇒ 'a and F G :: 'a filter
    using that unfolding l-def l'-def by auto
qed (auto simp: l-def lr-def eventually-filtermap l'-def eventually-sup intro: fil-
```

ter-leD)

These rules allow chaining of Landau symbol propositions in Isar with "also".

lemma *big-mult-1*: $f \in L\ F\ (g) \Longrightarrow (\lambda\text{-}.\ 1) \in L\ F\ (h) \Longrightarrow f \in L\ F\ (\lambda x.\ g\ x * h\ x)$
and *big-mult-1'*: $(\lambda\text{-}.\ 1) \in L\ F\ (g) \Longrightarrow f \in L\ F\ (h) \Longrightarrow f \in L\ F\ (\lambda x.\ g\ x * h\ x)$
and *small-mult-1*: $f \in l\ F\ (g) \Longrightarrow (\lambda\text{-}.\ 1) \in L\ F\ (h) \Longrightarrow f \in l\ F\ (\lambda x.\ g\ x * h\ x)$
and *small-mult-1'*: $(\lambda\text{-}.\ 1) \in L\ F\ (g) \Longrightarrow f \in l\ F\ (h) \Longrightarrow f \in l\ F\ (\lambda x.\ g\ x * h\ x)$
and *small-mult-1''*: $f \in L\ F\ (g) \Longrightarrow (\lambda\text{-}.\ 1) \in l\ F\ (h) \Longrightarrow f \in l\ F\ (\lambda x.\ g\ x * h\ x)$
and *small-mult-1'''*: $(\lambda\text{-}.\ 1) \in l\ F\ (g) \Longrightarrow f \in L\ F\ (h) \Longrightarrow f \in l\ F\ (\lambda x.\ g\ x * h\ x)$
by (*drule* (*1*) *big.mult big-small-mult small-big-mult*, *simp*)+

lemma *big-1-mult*: $f \in L\ F\ (g) \Longrightarrow h \in L\ F\ (\lambda\text{-}.\ 1) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in L\ F\ (g)$
and *big-1-mult'*: $h \in L\ F\ (\lambda\text{-}.\ 1) \Longrightarrow f \in L\ F\ (g) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in L\ F\ (g)$
and *small-1-mult*: $f \in l\ F\ (g) \Longrightarrow h \in L\ F\ (\lambda\text{-}.\ 1) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in l\ F\ (g)$
and *small-1-mult'*: $h \in L\ F\ (\lambda\text{-}.\ 1) \Longrightarrow f \in l\ F\ (g) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in l\ F\ (g)$
and *small-1-mult''*: $f \in L\ F\ (g) \Longrightarrow h \in l\ F\ (\lambda\text{-}.\ 1) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in l\ F\ (g)$
and *small-1-mult'''*: $h \in l\ F\ (\lambda\text{-}.\ 1) \Longrightarrow f \in L\ F\ (g) \Longrightarrow (\lambda x.\ f\ x * h\ x) \in l\ F\ (g)$
by (*drule* (*1*) *big.mult big-small-mult small-big-mult*, *simp*)+

lemmas *mult-1-trans* =
big-mult-1 big-mult-1' small-mult-1 small-mult-1' small-mult-1'' small-mult-1'''
big-1-mult big-1-mult' small-1-mult small-1-mult' small-1-mult'' small-1-mult'''

lemma *big-equal-iff-bigtheta*: $L\ F\ (f) = L\ F\ (g) \longleftrightarrow f \in \Theta[F](g)$
proof
have *L*: $L = bigo \vee L = bigomega$
by (*rule R-E*) (*auto simp*: *fun-eq-iff L-def bigo-def bigomega-def*)
fix $f\ g :: 'a \Rightarrow 'b$ **assume** $L\ F\ (f) = L\ F\ (g)$
with *big-refl*[*of f F*] *big-refl*[*of g F*] **have** $f \in L\ F\ (g)\ g \in L\ F\ (f)$ **by** *simp-all*
thus $f \in \Theta[F](g)$ **using** *L* **unfolding** *bigtheta-def* **by** (*auto simp*: *bigomega-iff-bigo*)
qed (*rule big.cong-bigtheta*)

lemma *big-prod*:
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \in L\ F\ (g\ x)$
shows $(\lambda y.\ \prod x{\in}A.\ f\ x\ y) \in L\ F\ (\lambda y.\ \prod x{\in}A.\ g\ x\ y)$
using *assms* **by** (*induction A rule*: *infinite-finite-induct*) (*auto intro*!: *big.mult*)

lemma *big-prod-in-1*:
assumes $\bigwedge x.\ x \in A \Longrightarrow f\ x \in L\ F\ (\lambda\text{-}.\ 1)$

shows $(\lambda y.\ \prod x{\in}A.\ f\ x\ y) \in L\ F\ (\lambda\text{-}.\ 1)$
using *assms* **by** (*induction A rule*: *infinite-finite-induct*) (*auto intro*!: *big.mult-in-1*)

end

context *landau-symbol*
begin

lemma *plus-absorb1*:
assumes $f \in o[F](g)$
shows $L\ F\ (\lambda x.\ f\ x + g\ x) = L\ F\ (g)$
proof (*intro equalityI*)
from *plus-subset1* **and** *assms* **show** $L\ F\ g \subseteq L\ F\ (\lambda x.\ f\ x + g\ x)$ **.**
from *landau-o.small.plus-subset1*[*OF assms*] **and** *assms* **have** $(\lambda x.\ -f\ x) \in o[F](\lambda x.\ f\ x + g\ x)$
by (*auto simp*: *landau-o.small.uminus-in-iff*)
from *plus-subset1*[*OF this*] **show** $L\ F\ (\lambda x.\ f\ x + g\ x) \subseteq L\ F\ (g)$ **by** *simp*
qed

lemma *plus-absorb2*: $g \in o[F](f) \Longrightarrow L\ F\ (\lambda x.\ f\ x + g\ x) = L\ F\ (f)$
using *plus-absorb1*[*of g F f*] **by** (*simp add*: *add.commute*)

lemma *diff-absorb1*: $f \in o[F](g) \Longrightarrow L\ F\ (\lambda x.\ f\ x - g\ x) = L\ F\ (g)$
by (*simp only*: *diff-conv-add-uminus plus-absorb1 landau-o.small.uminus uminus*)

lemma *diff-absorb2*: $g \in o[F](f) \Longrightarrow L\ F\ (\lambda x.\ f\ x - g\ x) = L\ F\ (f)$
by (*simp only*: *diff-conv-add-uminus plus-absorb2 landau-o.small.uminus-in-iff*)

lemmas *absorb* = *plus-absorb1 plus-absorb2 diff-absorb1 diff-absorb2*

end

lemma *bigthetaI* [*intro*]: $f \in O[F](g) \Longrightarrow f \in \Omega[F](g) \Longrightarrow f \in \Theta[F](g)$
unfolding *bigtheta-def bigomega-def* **by** *blast*

lemma *bigthetaD1* [*dest*]: $f \in \Theta[F](g) \Longrightarrow f \in O[F](g)$
and *bigthetaD2* [*dest*]: $f \in \Theta[F](g) \Longrightarrow f \in \Omega[F](g)$
unfolding *bigtheta-def bigo-def bigomega-def* **by** *blast*+

lemma *bigtheta-refl* [*simp*]: $f \in \Theta[F](f)$
unfolding *bigtheta-def* **by** *simp*

lemma *bigtheta-sym*: $f \in \Theta[F](g) \longleftrightarrow g \in \Theta[F](f)$
unfolding *bigtheta-def* **by** (*auto simp*: *bigomega-iff-bigo*)

lemmas *landau-flip* =
bigomega-iff-bigo[*symmetric*] *smallomega-iff-smallo*[*symmetric*]

bigomega-iff-bigo smallomega-iff-smallo bigtheta-sym

interpretation *landau-theta*: *landau-symbol bigtheta bigtheta bigtheta*
proof
fix $f\ g :: 'a \Rightarrow 'b$ **and** F
assume $f \in o[F](g)$
hence $O[F](g) \subseteq O[F](\lambda x.\ f\ x + g\ x)$ $\Omega[F](g) \subseteq \Omega[F](\lambda x.\ f\ x + g\ x)$
by (*rule landau-o.big.plus-subset1 landau-omega.big.plus-subset1*)+
thus $\Theta[F](g) \subseteq \Theta[F](\lambda x.\ f\ x + g\ x)$ **unfolding** *bigtheta-def* **by** *blast*
next
fix $f\ g :: 'a \Rightarrow 'b$ **and** F
assume $f \in \Theta[F](g)$
thus *A*: $\Theta[F](f) = \Theta[F](g)$
apply (*subst* (*1 2*) *bigtheta-def*)
apply (*subst landau-o.big.cong-bigtheta landau-omega.big.cong-bigtheta*, *assumption*)+
apply (*rule refl*)
done
thus $\Theta[F](f) \subseteq \Theta[F](g)$ **by** *simp*
fix $h :: 'a \Rightarrow 'b$
show $f \in \Theta[F](h) \longleftrightarrow g \in \Theta[F](h)$ **by** (*subst* (*1 2*) *bigtheta-sym*) (*simp add*: *A*)
next
fix $f\ g\ h :: 'a \Rightarrow 'b$ **and** F
assume $f \in \Theta[F](g)$ $g \in \Theta[F](h)$
thus $f \in \Theta[F](h)$ **unfolding** *bigtheta-def*
by (*blast intro*: *landau-o.big.trans landau-omega.big.trans*)
next
fix $f :: 'a \Rightarrow 'b$ **and** *F1 F2* :: *'a filter*
assume *F1* $\leq$ *F2*
thus $\Theta[F2](f) \subseteq \Theta[F1](f)$
by (*auto simp*: *bigtheta-def intro*: *landau-o.big.filter-mono landau-omega.big.filter-mono*)
qed (*auto simp*: *bigtheta-def landau-o.big.norm-iff*
landau-o.big.cmult landau-omega.big.cmult
landau-o.big.cmult-in-iff landau-omega.big.cmult-in-iff
landau-o.big.in-cong landau-omega.big.in-cong
landau-o.big.mult landau-omega.big.mult
landau-o.big.inverse landau-omega.big.inverse
landau-o.big.compose landau-omega.big.compose
landau-o.big.bot' landau-omega.big.bot'
landau-o.big.in-filtermap-iff landau-omega.big.in-filtermap-iff
landau-o.big.sup landau-omega.big.sup
landau-o.big.filtercomap landau-omega.big.filtercomap
dest: *landau-o.big.cong landau-omega.big.cong*)

lemmas *landau-symbols* =
landau-o.big.landau-symbol-axioms landau-o.small.landau-symbol-axioms
landau-omega.big.landau-symbol-axioms landau-omega.small.landau-symbol-axioms
landau-theta.landau-symbol-axioms

lemma *bigoI* [*intro*]:
assumes *eventually* (λx. (*norm* (*f x*)) ≤ *c* ∗ (*norm* (*g x*))) *F*
shows *f* ∈ *O*[*F*](*g*)
proof (*rule landau-o.bigI*)
show *max 1 c* > *0* **by** *simp*
have *c* ∗ (*norm* (*g x*)) ≤ *max 1 c* ∗ (*norm* (*g x*)) **for** *x*
by (*simp add*: *mult-right-mono*)
with *assms* **show** *eventually* (λx. (*norm* (*f x*)) ≤ *max 1 c* ∗ (*norm* (*g x*))) *F*
by (*auto elim*!: *eventually-mono dest*: *order.trans*)
qed

lemma *smallomegaD* [*dest*]:
assumes *f* ∈ ω[*F*](*g*)
shows *eventually* (λx. (*norm* (*f x*)) ≥ *c* ∗ (*norm* (*g x*))) *F*
proof (*cases c* > *0*)
case *False*
show *?thesis*
by (*intro always-eventually allI*, *rule order.trans*[*of - 0*])
(*insert False*, *auto intro*!: *mult-nonpos-nonneg*)
qed (*blast dest*: *landau-omega.smallD*[*OF assms*, *of c*])

lemma *bigthetaI'*:
assumes *c1* > *0 c2* > *0*
assumes *eventually* (λx. *c1* ∗ (*norm* (*g x*)) ≤ (*norm* (*f x*)) ∧ (*norm* (*f x*)) ≤ *c2* ∗ (*norm* (*g x*))) *F*
shows *f* ∈ Θ[*F*](*g*)
apply (*rule bigthetaI*)
apply (*rule landau-o.bigI*[*OF assms*(*2*)]) **using** *assms*(*3*) **apply** (*eventually-elim*, *simp*)
apply (*rule landau-omega.bigI*[*OF assms*(*1*)]) **using** *assms*(*3*) **apply** (*eventually-elim*, *simp*)
done

lemma *bigthetaI-cong*: *eventually* (λx. *f x* = *g x*) *F* ⟹ *f* ∈ Θ[*F*](*g*)
by (*intro bigthetaI'*[*of 1 1*]) (*auto elim*!: *eventually-mono*)

lemma (**in** *landau-symbol*) *ev-eq-trans1*:
f ∈ *L F* (λx. *g x* (*h x*)) ⟹ *eventually* (λx. *h x* = *h'* *x*) *F* ⟹ *f* ∈ *L F* (λx. *g x* (*h'* *x*))
by (*rule bigtheta-trans1*[*OF - bigthetaI-cong*]) (*auto elim*!: *eventually-mono*)

lemma (**in** *landau-symbol*) *ev-eq-trans2*:
eventually (λx. *f x* = *f'* *x*) *F* ⟹ (λx. *g x* (*f'* *x*)) ∈ *L F* (*h*) ⟹ (λx. *g x* (*f x*)) ∈ *L F* (*h*)
by (*rule bigtheta-trans2*[*OF bigthetaI-cong*]) (*auto elim*!: *eventually-mono*)

declare *landau-o.smallI landau-omega.bigI landau-omega.smallI* [*intro*]

declare *landau-o.bigE landau-omega.bigE* [*elim*]
declare *landau-o.smallD*

lemma (**in** *landau-symbol*) *bigtheta-trans1'*:
$f \in L\ F\ (g) \Longrightarrow h \in \Theta[F](g) \Longrightarrow f \in L\ F\ (h)$
by (*subst cong-bigtheta*[*symmetric*]) (*simp add*: *bigtheta-sym*)

lemma (**in** *landau-symbol*) *bigtheta-trans2'*:
$g \in \Theta[F](f) \Longrightarrow g \in L\ F\ (h) \Longrightarrow f \in L\ F\ (h)$
by (*rule bigtheta-trans2*, *subst bigtheta-sym*)

lemma *bigo-bigomega-trans*: $f \in O[F](g) \Longrightarrow h \in \Omega[F](g) \Longrightarrow f \in O[F](h)$
and *bigo-smallomega-trans*: $f \in O[F](g) \Longrightarrow h \in \omega[F](g) \Longrightarrow f \in o[F](h)$
and *smallo-bigomega-trans*: $f \in o[F](g) \Longrightarrow h \in \Omega[F](g) \Longrightarrow f \in o[F](h)$
and *smallo-smallomega-trans*: $f \in o[F](g) \Longrightarrow h \in \omega[F](g) \Longrightarrow f \in o[F](h)$
and *bigomega-bigo-trans*: $f \in \Omega[F](g) \Longrightarrow h \in O[F](g) \Longrightarrow f \in \Omega[F](h)$
and *bigomega-smallo-trans*: $f \in \Omega[F](g) \Longrightarrow h \in o[F](g) \Longrightarrow f \in \omega[F](h)$
and *smallomega-bigo-trans*: $f \in \omega[F](g) \Longrightarrow h \in O[F](g) \Longrightarrow f \in \omega[F](h)$
and *smallomega-smallo-trans*: $f \in \omega[F](g) \Longrightarrow h \in o[F](g) \Longrightarrow f \in \omega[F](h)$
by (*unfold bigomega-iff-bigo smallomega-iff-smallo*)
(*erule* (*1*) *landau-o.big-trans landau-o.big-small-trans landau-o.small-big-trans*
landau-o.big-trans landau-o.small-trans)+

lemmas *landau-trans-lift* [*trans*] =
landau-symbols[*THEN landau-symbol.lift-trans*]
landau-symbols[*THEN landau-symbol.lift-trans'*]
landau-symbols[*THEN landau-symbol.lift-trans-bigtheta*]
landau-symbols[*THEN landau-symbol.lift-trans-bigtheta'*]

lemmas *landau-mult-1-trans* [*trans*] =
landau-o.mult-1-trans landau-omega.mult-1-trans

lemmas *landau-trans* [*trans*] =
landau-symbols[*THEN landau-symbol.bigtheta-trans1*]
landau-symbols[*THEN landau-symbol.bigtheta-trans2*]
landau-symbols[*THEN landau-symbol.bigtheta-trans1'*]
landau-symbols[*THEN landau-symbol.bigtheta-trans2'*]
landau-symbols[*THEN landau-symbol.ev-eq-trans1*]
landau-symbols[*THEN landau-symbol.ev-eq-trans2*]

landau-o.big-trans landau-o.small-trans landau-o.small-big-trans landau-o.big-small-trans
landau-omega.big-trans landau-omega.small-trans
landau-omega.small-big-trans landau-omega.big-small-trans

bigo-bigomega-trans bigo-smallomega-trans smallo-bigomega-trans smallo-smallomega-trans
bigomega-bigo-trans bigomega-smallo-trans smallomega-bigo-trans smallomega-smallo-trans

lemma *bigtheta-inverse* [*simp*]:
shows $(\lambda x.\ inverse\ (f\ x)) \in \Theta[F](\lambda x.\ inverse\ (g\ x)) \longleftrightarrow f \in \Theta[F](g)$

proof −
have $(\lambda x.\ inverse\ (f\ x)) \in O[F](\lambda x.\ inverse\ (g\ x))$
if *A*: $f \in \Theta[F](g)$
for $f\ g :: 'a \Rightarrow 'b$ **and** *F*
proof −
from *A* **obtain** *c1 c2* :: *real* **where** ∗: *c1* > *0 c2* > *0*
and ∗∗: $\forall_F\ x\ in\ F.\ norm\ (f\ x) \leq c1 * norm\ (g\ x)$
$\forall_F\ x\ in\ F.\ c2 * norm\ (g\ x) \leq norm\ (f\ x)$
unfolding *bigtheta-def* **by** (*elim landau-o.bigE landau-omega.bigE IntE*)
from ‹*c2* > *0*› **have** *c2*: *inverse c2* > *0* **by** *simp*
from ∗∗ **have** *eventually* $(\lambda x.\ norm\ (inverse\ (f\ x)) \leq inverse\ c2 * norm\ (inverse\ (g\ x)))\ F$
proof *eventually-elim*
fix *x* **assume** *A*: $norm\ (f\ x) \leq c1 * norm\ (g\ x)\ c2 * norm\ (g\ x) \leq norm\ (f\ x)$
from *A* ∗ **have** $f\ x = 0 \longleftrightarrow g\ x = 0$
by (*auto simp*: *field-simps mult-le-0-iff*)
with *A* ∗ **show** $norm\ (inverse\ (f\ x)) \leq inverse\ c2 * norm\ (inverse\ (g\ x))$
by (*force simp*: *field-simps norm-inverse norm-divide*)
qed
with *c2* **show** *?thesis* **by** (*rule landau-o.bigI*)
qed
then show *?thesis*
unfolding *bigtheta-def*
by (*force simp*: *bigomega-iff-bigo bigtheta-sym*)
qed

lemma *bigtheta-divide*:
assumes *f1* ∈ Θ(*f2*) *g1* ∈ Θ(*g2*)
shows $(\lambda x.\ f1\ x\ /\ g1\ x) \in \Theta(\lambda x.\ f2\ x\ /\ g2\ x)$
by (*subst* (*1 2*) *divide-inverse*, *intro landau-theta.mult*) (*simp-all add*: *bigtheta-inverse assms*)

lemma *eventually-nonzero-bigtheta*:
assumes $f \in \Theta[F](g)$
shows $eventually\ (\lambda x.\ f\ x \neq 0)\ F \longleftrightarrow eventually\ (\lambda x.\ g\ x \neq 0)\ F$
proof −
have $eventually\ (\lambda x.\ g\ x \neq 0)\ F$
if *A*: $f \in \Theta[F](g)$ **and** *B*: $eventually\ (\lambda x.\ f\ x \neq 0)\ F$
for $f\ g :: 'a \Rightarrow 'b$
proof −
from *A* **obtain** *c1 c2* **where**
$\forall_F\ x\ in\ F.\ norm\ (f\ x) \leq c1 * norm\ (g\ x)$
$\forall_F\ x\ in\ F.\ c2 * norm\ (g\ x) \leq norm\ (f\ x)$
unfolding *bigtheta-def* **by** (*elim landau-o.bigE landau-omega.bigE IntE*)
with *B* **show** *?thesis* **by** *eventually-elim auto*
qed
with *assms* **show** *?thesis* **by** (*force simp*: *bigtheta-sym*)
qed

50.2 Landau symbols and limits

lemma *bigoI-tendsto-norm*:
fixes *f g*
assumes $((\lambda x.\ norm\ (f\ x\ /\ g\ x)) \longrightarrow c)\ F$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
shows $f \in O[F](g)$
proof (*rule bigoI*)
from *assms* **have** *eventually* $(\lambda x.\ dist\ (norm\ (f\ x\ /\ g\ x))\ c < 1)\ F$
using *tendstoD* **by** *force*
thus *eventually* $(\lambda x.\ (norm\ (f\ x)) \leq (norm\ c + 1) * (norm\ (g\ x)))\ F$
unfolding *dist-real-def* **using** *assms*(*2*)
proof *eventually-elim*
case (*elim x*)
have $(norm\ (f\ x)) - norm\ c * (norm\ (g\ x)) \leq norm\ ((norm\ (f\ x)) - c * (norm\ (g\ x)))$
unfolding *norm-mult* [*symmetric*] **using** *norm-triangle-ineq2*[*of norm* $(f\ x)$ $c * norm\ (g\ x)$]
by (*simp add*: *norm-mult abs-mult*)
also from *elim* **have** $\ldots = norm\ (norm\ (g\ x)) * norm\ (norm\ (f\ x\ /\ g\ x) - c)$
unfolding *norm-mult* [*symmetric*] **by** (*simp add*: *algebra-simps norm-divide*)
also from *elim* **have** $norm\ (norm\ (f\ x\ /\ g\ x) - c) \leq 1$ **by** *simp*
hence $norm\ (norm\ (g\ x)) * norm\ (norm\ (f\ x\ /\ g\ x) - c) \leq norm\ (norm\ (g\ x)) * 1$
by (*rule mult-left-mono*) *simp-all*
finally show *?case* **by** (*simp add*: *algebra-simps*)
qed
qed

lemma *bigoI-tendsto*:
assumes $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow c)\ F$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
shows $f \in O[F](g)$
using *assms* **by** (*rule bigoI-tendsto-norm*[*OF tendsto-norm*])

lemma *bigomegaI-tendsto-norm*:
assumes *c-not-0*: $(c::real) \neq 0$
assumes *lim*: $((\lambda x.\ norm\ (f\ x\ /\ g\ x)) \longrightarrow c)\ F$
shows $f \in \Omega[F](g)$
proof (*cases* $F = bot$)
case *False*
show *?thesis*
proof (*rule landau-omega.bigI*)
from *lim* **have** $c \geq 0$ **by** (*rule tendsto-lowerbound*) (*insert False*, *simp-all*)
with *c-not-0* **have** $c > 0$ **by** *simp*
with *c-not-0* **show** $c/2 > 0$ **by** *simp*
from *lim* **have** *ev*: $\bigwedge\varepsilon.\ \varepsilon > 0 \Longrightarrow$ *eventually* $(\lambda x.\ norm\ (norm\ (f\ x\ /\ g\ x) - c) < \varepsilon)\ F$
by (*subst* (*asm*) *tendsto-iff*) (*simp add*: *dist-real-def*)
from *ev*[*OF* ‹$c/2 > 0$›] **show** *eventually* $(\lambda x.\ (norm\ (f\ x)) \geq c/2 * (norm\ (g$

```
x))) F
    proof (eventually-elim)
      fix x assume B: norm (norm (f x / g x) - c) < c / 2
      from B have g: g x ≠ 0 by auto
      from B have -c/2 < -norm (norm (f x / g x) - c) by simp
      also have ... ≤ norm (f x / g x) - c by simp
      finally show (norm (f x)) ≥ c/2 * (norm (g x)) using g
        by (simp add: field-simps norm-mult norm-divide)
    qed
  qed
qed simp

lemma bigomegaI-tendsto:
  assumes c-not-0:  (c::real) ≠ 0
  assumes lim:      ((λx. f x / g x) ⟶ c) F
  shows   f ∈ Ω[F](g)
 by (rule bigomegaI-tendsto-norm[OF - tendsto-norm, of c]) (insert assms, simp-all)

lemma smallomegaI-filterlim-at-top-norm:
  assumes lim: filterlim (λx. norm (f x / g x)) at-top F
  shows   f ∈ ω[F](g)
proof (rule landau-omega.smallI)
  fix c :: real assume c-pos: c > 0
  from lim have ev: eventually (λx. norm (f x / g x) ≥ c) F
    by (subst (asm) filterlim-at-top) simp
  thus eventually (λx. (norm (f x)) ≥ c * (norm (g x))) F
  proof eventually-elim
    fix x assume A: norm (f x / g x) ≥ c
    from A c-pos have g x ≠ 0 by auto
     with A show (norm (f x)) ≥ c * (norm (g x)) by (simp add: field-simps
norm-divide)
  qed
qed

lemma smallomegaI-filterlim-at-infinity:
  assumes lim: filterlim (λx. f x / g x) at-infinity F
  shows   f ∈ ω[F](g)
proof (rule smallomegaI-filterlim-at-top-norm)
  from lim show filterlim (λx. norm (f x / g x)) at-top F
    by (rule filterlim-at-infinity-imp-norm-at-top)
qed

lemma smallomegaD-filterlim-at-top-norm:
  assumes f ∈ ω[F](g)
  assumes eventually (λx. g x ≠ 0) F
  shows   LIM x F. norm (f x / g x) :> at-top
proof (subst filterlim-at-top-gt, clarify)
  fix c :: real assume c: c > 0
  from landau-omega.smallD[OF assms(1) this] assms(2)
```

show *eventually* $(\lambda x.\ norm\ (f\ x\ /\ g\ x) \geq c)\ F$
by *eventually-elim* (*simp add*: *field-simps norm-divide*)
qed

lemma *smallomegaD-filterlim-at-infinity*:
assumes $f \in \omega[F](g)$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
shows $LIM\ x\ F.\ f\ x\ /\ g\ x :> at\text{-}infinity$
using *assms* **by** (*intro filterlim-norm-at-top-imp-at-infinity smallomegaD-filterlim-at-top-norm*)

lemma *smallomega-1-conv-filterlim*: $f \in \omega[F](\lambda\text{-}.\ 1) \longleftrightarrow filterlim\ f\ at\text{-}infinity\ F$
by (*auto intro*: *smallomegaI-filterlim-at-infinity dest*: *smallomegaD-filterlim-at-infinity*)

lemma *smalloI-tendsto*:
assumes *lim*: $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow 0)\ F$
assumes *eventually* $(\lambda x.\ g\ x \neq 0)\ F$
shows $f \in o[F](g)$
proof (*rule landau-o.smallI*)
fix $c :: real$ **assume** *c-pos*: $c > 0$
from *c-pos* **and** *lim* **have** *ev*: *eventually* $(\lambda x.\ norm\ (f\ x\ /\ g\ x) < c)\ F$
by (*subst* (*asm*) *tendsto-iff*) (*simp add*: *dist-real-def*)
with *assms*(*2*) **show** *eventually* $(\lambda x.\ (norm\ (f\ x)) \leq c * (norm\ (g\ x)))\ F$
by *eventually-elim* (*simp add*: *field-simps norm-divide*)
qed

lemma *smalloD-tendsto*:
assumes $f \in o[F](g)$
shows $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow 0)\ F$
unfolding *tendsto-iff*
proof *clarify*
fix $e :: real$ **assume** *e*: $e > 0$
hence $e/2 > 0$ **by** *simp*
from *landau-o.smallD*[*OF assms this*] **show** *eventually* $(\lambda x.\ dist\ (f\ x\ /\ g\ x)\ 0 < e)\ F$
proof *eventually-elim*
fix *x* **assume** $(norm\ (f\ x)) \leq e/2 * (norm\ (g\ x))$
with *e* **have** $dist\ (f\ x\ /\ g\ x)\ 0 \leq e/2$
by (*cases* $g\ x = 0$) (*simp-all add*: *dist-real-def norm-divide field-simps*)
also from *e* **have** $... < e$ **by** *simp*
finally show $dist\ (f\ x\ /\ g\ x)\ 0 < e$ **by** *simp*
qed
qed

lemma *bigthetaI-tendsto-norm*:
assumes *c-not-0*: $(c::real) \neq 0$
assumes *lim*: $((\lambda x.\ norm\ (f\ x\ /\ g\ x)) \longrightarrow c)\ F$
shows $f \in \Theta[F](g)$
proof (*rule bigthetaI*)
from *c-not-0* **have** $|c| > 0$ **by** *simp*

with *lim* **have** *eventually* $(\lambda x.\ norm\ (norm\ (f\ x\ /\ g\ x) - c) < |c|)\ F$
by (*subst* (*asm*) *tendsto-iff*) (*simp add*: *dist-real-def*)
hence *g*: *eventually* $(\lambda x.\ g\ x \neq 0)\ F$ **by** *eventually-elim* (*auto simp add*: *field-simps*)

from *lim g* **show** $f \in O[F](g)$ **by** (*rule bigoI-tendsto-norm*)
from *c-not-0* **and** *lim* **show** $f \in \Omega[F](g)$ **by** (*rule bigomegaI-tendsto-norm*)
qed

lemma *bigthetaI-tendsto*:
assumes *c-not-0*: $(c::real) \neq 0$
assumes *lim*: $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow c)\ F$
shows $f \in \Theta[F](g)$
using *assms* **by** (*intro bigthetaI-tendsto-norm*[*OF - tendsto-norm, of c*]) *simp-all*

lemma *tendsto-add-smallo*:
assumes $(f1 \longrightarrow a)\ F$
assumes $f2 \in o[F](f1)$
shows $((\lambda x.\ f1\ x + f2\ x) \longrightarrow a)\ F$
proof (*subst filterlim-cong*[*OF refl refl*])
from *landau-o.smallD*[*OF assms*(*2*) *zero-less-one*]
have *eventually* $(\lambda x.\ norm\ (f2\ x) \leq norm\ (f1\ x))\ F$ **by** *simp*
thus *eventually* $(\lambda x.\ f1\ x + f2\ x = f1\ x * (1 + f2\ x\ /\ f1\ x))\ F$
by *eventually-elim* (*auto simp*: *field-simps*)
next
from *assms*(*1*) **show** $((\lambda x.\ f1\ x * (1 + f2\ x\ /\ f1\ x)) \longrightarrow a)\ F$
by (*force intro*: *tendsto-eq-intros smalloD-tendsto*[*OF assms*(*2*)])
qed

lemma *tendsto-diff-smallo*:
shows $(f1 \longrightarrow a)\ F \Longrightarrow f2 \in o[F](f1) \Longrightarrow ((\lambda x.\ f1\ x - f2\ x) \longrightarrow a)\ F$
using *tendsto-add-smallo*[*of f1 a F* $\lambda x.\ -f2\ x$] **by** *simp*

lemma *tendsto-add-smallo-iff*:
assumes $f2 \in o[F](f1)$
shows $(f1 \longrightarrow a)\ F \longleftrightarrow ((\lambda x.\ f1\ x + f2\ x) \longrightarrow a)\ F$
proof
assume $((\lambda x.\ f1\ x + f2\ x) \longrightarrow a)\ F$
hence $((\lambda x.\ f1\ x + f2\ x - f2\ x) \longrightarrow a)\ F$
by (*rule tendsto-diff-smallo*) (*simp add*: *landau-o.small.plus-absorb2 assms*)
thus $(f1 \longrightarrow a)\ F$ **by** *simp*
qed (*rule tendsto-add-smallo*[*OF - assms*])

lemma *tendsto-diff-smallo-iff*:
shows $f2 \in o[F](f1) \Longrightarrow (f1 \longrightarrow a)\ F \longleftrightarrow ((\lambda x.\ f1\ x - f2\ x) \longrightarrow a)\ F$
using *tendsto-add-smallo-iff*[*of* $\lambda x.\ -f2\ x\ F\ f1\ a$] **by** *simp*

lemma *tendsto-divide-smallo*:
assumes $((\lambda x.\ f1\ x\ /\ g1\ x) \longrightarrow a)\ F$
assumes $f2 \in o[F](f1)\ g2 \in o[F](g1)$

assumes *eventually* (λx. *g1 x* ≠ *0*) *F*
shows ((λx. (*f1 x* + *f2 x*) / (*g1 x* + *g2 x*)) ⟶ *a*) *F* (**is** (*?f* ⟶ -) -)
proof (*subst tendsto-cong*)
let *?f'* = λx. (*f1 x* / *g1 x*) ∗ (*1* + *f2 x* / *f1 x*) / (*1* + *g2 x* / *g1 x*)

have (*?f'* ⟶ *a* ∗ (*1* + *0*) / (*1* + *0*)) *F*
by (*rule tendsto-mult tendsto-divide tendsto-add assms tendsto-const*
smalloD-tendsto[*OF assms*(*2*)] *smalloD-tendsto*[*OF assms*(*3*)])+ *simp-all*
thus (*?f'* ⟶ *a*) *F* **by** *simp*

have (*1/2*::*real*) > *0* **by** *simp*
from *landau-o.smallD*[*OF assms*(*2*) *this*] *landau-o.smallD*[*OF assms*(*3*) *this*]
have *eventually* (λx. *norm* (*f2 x*) ≤ *norm* (*f1 x*)/*2*) *F*
eventually (λx. *norm* (*g2 x*) ≤ *norm* (*g1 x*)/*2*) *F* **by** *simp-all*
with *assms*(*4*) **show** *eventually* (λx. *?f x* = *?f' x*) *F*
proof *eventually-elim*
fix *x* **assume** *A*: *norm* (*f2 x*) ≤ *norm* (*f1 x*)/*2* **and**
B: *norm* (*g2 x*) ≤ *norm* (*g1 x*)/*2* **and** *C*: *g1 x* ≠ *0*
show *?f x* = *?f' x*
proof (*cases f1 x* = *0*)
assume *D*: *f1 x* ≠ *0*
from *D* **have** *f1 x* + *f2 x* = *f1 x* ∗ (*1* + *f2 x*/*f1 x*) **by** (*simp add*: *field-simps*)
moreover from *C* **have** *g1 x* + *g2 x* = *g1 x* ∗ (*1* + *g2 x*/*g1 x*) **by** (*simp add*: *field-simps*)
ultimately have *?f x* = (*f1 x* ∗ (*1* + *f2 x*/*f1 x*)) / (*g1 x* ∗ (*1* + *g2 x*/*g1 x*))
by (*simp only*:)
also have ... = *?f' x* **by** *simp*
finally show *?thesis* **.**
qed (*insert A*, *simp*)
qed
qed

lemma *bigo-powr*:
fixes *f* :: *'a* ⇒ *real*
assumes *f* ∈ *O*[*F*](*g*) *p* ≥ *0*
shows (λx. |*f x*| *powr p*) ∈ *O*[*F*](λx. |*g x*| *powr p*)
proof−
from *assms*(*1*) **obtain** *c* **where** *c*: *c* > *0* **and** ∗: ∀ *F* *x in F*. *norm* (*f x*) ≤ *c* ∗ *norm* (*g x*)
by (*elim landau-o.bigE landau-omega.bigE IntE*)
from *assms*(*2*) ∗ **have** *eventually* (λx. (*norm* (*f x*)) *powr p* ≤ (*c* ∗ *norm* (*g x*)) *powr p*) *F*
by (*auto elim*!: *eventually-mono intro*!: *powr-mono2*)
with *c* **show** (λx. |*f x*| *powr p*) ∈ *O*[*F*](λx. |*g x*| *powr p*)
by (*intro bigoI*[*of* - *c powr p*]) (*simp-all add*: *powr-mult*)
qed

lemma *smallo-powr*:

fixes $f :: 'a \Rightarrow real$
assumes $f \in o[F](g)$ $p > 0$
shows $(\lambda x.\ |f\ x|\ powr\ p) \in o[F](\lambda x.\ |g\ x|\ powr\ p)$
proof (*rule landau-o.smallI*)
fix $c :: real$ **assume** c: $c > 0$
hence $c\ powr\ (1/p) > 0$ **by** *simp*
from *landau-o.smallD*[*OF assms*(*1*) *this*]
show *eventually* $(\lambda x.\ norm\ (|f\ x|\ powr\ p) \leq c * norm\ (|g\ x|\ powr\ p))\ F$
proof *eventually-elim*
fix x **assume** $(norm\ (f\ x)) \leq c\ powr\ (1\ /\ p) * (norm\ (g\ x))$
with *assms*(*2*) **have** $(norm\ (f\ x))\ powr\ p \leq (c\ powr\ (1\ /\ p) * (norm\ (g\ x)))$ *powr* p
by (*intro powr-mono2*) *simp-all*
also from *assms*(*2*) c **have** $... = c * (norm\ (g\ x))\ powr\ p$
by (*simp add*: *field-simps powr-mult powr-powr*)
finally show $norm\ (|f\ x|\ powr\ p) \leq c * norm\ (|g\ x|\ powr\ p)$ **by** *simp*
qed
qed

lemma *smallo-powr-nonneg*:
fixes $f :: 'a \Rightarrow real$
assumes $f \in o[F](g)$ $p > 0$ *eventually* $(\lambda x.\ f\ x \geq 0)\ F$ *eventually* $(\lambda x.\ g\ x \geq 0)$ F
shows $(\lambda x.\ f\ x\ powr\ p) \in o[F](\lambda x.\ g\ x\ powr\ p)$
proof −
from *assms*(*3*) **have** $(\lambda x.\ f\ x\ powr\ p) \in \Theta[F](\lambda x.\ |f\ x|\ powr\ p)$
by (*intro bigthetaI-cong*) (*auto elim*!: *eventually-mono*)
also have $(\lambda x.\ |f\ x|\ powr\ p) \in o[F](\lambda x.\ |g\ x|\ powr\ p)$ **by** (*intro smallo-powr*) *fact*+
also from *assms*(*4*) **have** $(\lambda x.\ |g\ x|\ powr\ p) \in \Theta[F](\lambda x.\ g\ x\ powr\ p)$
by (*intro bigthetaI-cong*) (*auto elim*!: *eventually-mono*)
finally show *?thesis* **.**
qed

lemma *bigtheta-powr*:
fixes $f :: 'a \Rightarrow real$
shows $f \in \Theta[F](g) \Longrightarrow (\lambda x.\ |f\ x|\ powr\ p) \in \Theta[F](\lambda x.\ |g\ x|\ powr\ p)$
apply (*cases* $p < 0$)
apply (*subst bigtheta-inverse*[*symmetric*], *subst* (*1 2*) *powr-minus*[*symmetric*])
unfolding *bigtheta-def* **apply** (*auto simp*: *bigomega-iff-bigo intro*!: *bigo-powr*)
done

lemma *bigo-powr-nonneg*:
fixes $f :: 'a \Rightarrow real$
assumes $f \in O[F](g)$ $p \geq 0$ *eventually* $(\lambda x.\ f\ x \geq 0)\ F$ *eventually* $(\lambda x.\ g\ x \geq 0)$ F
shows $(\lambda x.\ f\ x\ powr\ p) \in O[F](\lambda x.\ g\ x\ powr\ p)$
proof −
from *assms*(*3*) **have** $(\lambda x.\ f\ x\ powr\ p) \in \Theta[F](\lambda x.\ |f\ x|\ powr\ p)$

by (*intro bigthetaI-cong*) (*auto elim*!: *eventually-mono*)
also have $(\lambda x.\ |f\ x|\ powr\ p) \in O[F](\lambda x.\ |g\ x|\ powr\ p)$ **by** (*intro bigo-powr*) *fact+*
also from *assms*(*4*) **have** $(\lambda x.\ |g\ x|\ powr\ p) \in \Theta[F](\lambda x.\ g\ x\ powr\ p)$
by (*intro bigthetaI-cong*) (*auto elim*!: *eventually-mono*)
finally show *?thesis* **.**
qed

lemma *zero-in-smallo* [*simp*]: $(\lambda\text{-}.\ 0) \in o[F](f)$
by (*intro landau-o.smallI*) *simp-all*

lemma *zero-in-bigo* [*simp*]: $(\lambda\text{-}.\ 0) \in O[F](f)$
by (*intro landau-o.bigI*[*of 1*]) *simp-all*

lemma *in-bigomega-zero* [*simp*]: $f \in \Omega[F](\lambda x.\ 0)$
by (*rule landau-omega.bigI*[*of 1*]) *simp-all*

lemma *in-smallomega-zero* [*simp*]: $f \in \omega[F](\lambda x.\ 0)$
by (*simp add*: *smallomega-iff-smallo*)

lemma *in-smallo-zero-iff* [*simp*]: $f \in o[F](\lambda\text{-}.\ 0) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)\ F$
proof
assume $f \in o[F](\lambda\text{-}.\ 0)$
from *landau-o.smallD*[*OF this*, *of 1*] **show** *eventually* $(\lambda x.\ f\ x = 0)\ F$ **by** *simp*
next
assume *eventually* $(\lambda x.\ f\ x = 0)\ F$
hence $\forall\, c{>}0.$ *eventually* $(\lambda x.\ (norm\ (f\ x)) \leq c * |0|)\ F$ **by** *simp*
thus $f \in o[F](\lambda\text{-}.\ 0)$ **unfolding** *smallo-def* **by** *simp*
qed

lemma *in-bigo-zero-iff* [*simp*]: $f \in O[F](\lambda\text{-}.\ 0) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)\ F$
proof
assume $f \in O[F](\lambda\text{-}.\ 0)$
thus *eventually* $(\lambda x.\ f\ x = 0)\ F$ **by** (*elim landau-o.bigE*) *simp*
next
assume *eventually* $(\lambda x.\ f\ x = 0)\ F$
hence *eventually* $(\lambda x.\ (norm\ (f\ x)) \leq 1 * |0|)\ F$ **by** *simp*
thus $f \in O[F](\lambda\text{-}.\ 0)$ **by** (*intro landau-o.bigI*[*of 1*]) *simp-all*
qed

lemma *zero-in-smallomega-iff* [*simp*]: $(\lambda\text{-}.\ 0) \in \omega[F](f) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)\ F$
by (*simp add*: *smallomega-iff-smallo*)

lemma *zero-in-bigomega-iff* [*simp*]: $(\lambda\text{-}.\ 0) \in \Omega[F](f) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)\ F$
by (*simp add*: *bigomega-iff-bigo*)

lemma *zero-in-bigtheta-iff* [*simp*]: $(\lambda\text{-}.\ 0) \in \Theta[F](f) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)$

F
unfolding *bigtheta-def* **by** *simp*

lemma *in-bigtheta-zero-iff* [*simp*]: $f \in \Theta[F](\lambda x.\ 0) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)$ *F*
unfolding *bigtheta-def* **by** *simp*

lemma *cmult-in-bigo-iff* [*simp*]: $(\lambda x.\ c * f\ x) \in O[F](g) \longleftrightarrow c = 0 \vee f \in O[F](g)$
and *cmult-in-bigo-iff'* [*simp*]: $(\lambda x.\ f\ x * c) \in O[F](g) \longleftrightarrow c = 0 \vee f \in O[F](g)$
and *cmult-in-smallo-iff* [*simp*]: $(\lambda x.\ c * f\ x) \in o[F](g) \longleftrightarrow c = 0 \vee f \in o[F](g)$
and *cmult-in-smallo-iff'* [*simp*]: $(\lambda x.\ f\ x * c) \in o[F](g) \longleftrightarrow c = 0 \vee f \in o[F](g)$
by (*cases* $c = 0$, *simp*, *simp*)+

lemma *bigo-const* [*simp*]: $(\lambda\text{-}.\ c) \in O[F](\lambda\text{-}.\ 1)$ **by** (*rule bigoI*[*of - norm c*]) *simp*

lemma *bigo-const-iff* [*simp*]: $(\lambda\text{-}.\ c1) \in O[F](\lambda\text{-}.\ c2) \longleftrightarrow F = bot \vee c1 = 0 \vee c2 \neq 0$
by (*cases* $c1 = 0$; *cases* $c2 = 0$)
(*auto simp*: *bigo-def eventually-False intro*: *exI*[*of - 1*] *exI*[*of - norm c1 / norm c2*])

lemma *bigomega-const-iff* [*simp*]: $(\lambda\text{-}.\ c1) \in \Omega[F](\lambda\text{-}.\ c2) \longleftrightarrow F = bot \vee c1 \neq 0 \vee c2 = 0$
by (*cases* $c1 = 0$; *cases* $c2 = 0$)
(*auto simp*: *bigomega-def eventually-False mult-le-0-iff*
intro: *exI*[*of - 1*] *exI*[*of - norm c1 / norm c2*])

lemma *smallo-real-nat-transfer*:
$(f :: real \Rightarrow real) \in o(g) \Longrightarrow (\lambda x{::}nat.\ f\ (real\ x)) \in o(\lambda x.\ g\ (real\ x))$
by (*rule landau-o.small.compose*[*OF - filterlim-real-sequentially*])

lemma *bigo-real-nat-transfer*:
$(f :: real \Rightarrow real) \in O(g) \Longrightarrow (\lambda x{::}nat.\ f\ (real\ x)) \in O(\lambda x.\ g\ (real\ x))$
by (*rule landau-o.big.compose*[*OF - filterlim-real-sequentially*])

lemma *smallomega-real-nat-transfer*:
$(f :: real \Rightarrow real) \in \omega(g) \Longrightarrow (\lambda x{::}nat.\ f\ (real\ x)) \in \omega(\lambda x.\ g\ (real\ x))$
by (*rule landau-omega.small.compose*[*OF - filterlim-real-sequentially*])

lemma *bigomega-real-nat-transfer*:
$(f :: real \Rightarrow real) \in \Omega(g) \Longrightarrow (\lambda x{::}nat.\ f\ (real\ x)) \in \Omega(\lambda x.\ g\ (real\ x))$
by (*rule landau-omega.big.compose*[*OF - filterlim-real-sequentially*])

lemma *bigtheta-real-nat-transfer*:
$(f :: real \Rightarrow real) \in \Theta(g) \Longrightarrow (\lambda x{::}nat.\ f\ (real\ x)) \in \Theta(\lambda x.\ g\ (real\ x))$
unfolding *bigtheta-def* **using** *bigo-real-nat-transfer bigomega-real-nat-transfer*
by *blast*

lemmas *landau-real-nat-transfer* [*intro*] =
bigo-real-nat-transfer smallo-real-nat-transfer bigomega-real-nat-transfer
smallomega-real-nat-transfer bigtheta-real-nat-transfer

lemma *landau-symbol-if-at-top-eq* [*simp*]:
assumes *landau-symbol L L' Lr*
shows *L at-top* (λx::'a::*linordered-semidom. if x = a then f x else g x*) = *L at-top* (*g*)
apply (*rule landau-symbol.cong*[*OF assms*])
using *less-add-one*[*of a*] **apply** (*auto intro*: *eventually-mono eventually-ge-at-top*[*of a + 1*])
done

lemmas *landau-symbols-if-at-top-eq* [*simp*] = *landau-symbols*[*THEN landau-symbol-if-at-top-eq*]

lemma *sum-in-smallo*:
assumes *f* ∈ *o*[*F*](*h*) *g* ∈ *o*[*F*](*h*)
shows (λx. *f x + g x*) ∈ *o*[*F*](*h*) (λx. *f x − g x*) ∈ *o*[*F*](*h*)
proof −
have (λx. *f x + g x*) ∈ *o*[*F*](*h*) **if** *f* ∈ *o*[*F*](*h*) *g* ∈ *o*[*F*](*h*) **for** *f g*
proof (*rule landau-o.smallI*)
fix *c* :: *real* **assume** *c > 0*
hence *c/2 > 0* **by** *simp*
from *that*[*THEN landau-o.smallD*[*OF - this*]]
show *eventually* (λx. *norm* (*f x + g x*) ≤ *c* ∗ (*norm* (*h x*))) *F*
by *eventually-elim* (*auto intro*: *order.trans*[*OF norm-triangle-ineq*])
qed
from *this*[*of f g*] *this*[*of f* λx. −*g x*] *assms*
show (λx. *f x + g x*) ∈ *o*[*F*](*h*) (λx. *f x − g x*) ∈ *o*[*F*](*h*) **by** *simp-all*
qed

lemma *big-sum-in-smallo*:
assumes ⋀x. *x* ∈ *A* ⟹ *f x* ∈ *o*[*F*](*g*)
shows (λx. *sum* (λy. *f y x*) *A*) ∈ *o*[*F*](*g*)
using *assms* **by** (*induction A rule*: *infinite-finite-induct*) (*auto intro*: *sum-in-smallo*)

lemma *sum-in-bigo*:
assumes *f* ∈ *O*[*F*](*h*) *g* ∈ *O*[*F*](*h*)
shows (λx. *f x + g x*) ∈ *O*[*F*](*h*) (λx. *f x − g x*) ∈ *O*[*F*](*h*)
proof −
have (λx. *f x + g x*) ∈ *O*[*F*](*h*) **if** *f* ∈ *O*[*F*](*h*) *g* ∈ *O*[*F*](*h*) **for** *f g*
proof −
from *that* **obtain** *c1 c2* **where** ∗: *c1 > 0 c2 > 0*
and ∗∗: ∀ *F x in F. norm* (*f x*) ≤ *c1* ∗ *norm* (*h x*)
∀ *F x in F. norm* (*g x*) ≤ *c2* ∗ *norm* (*h x*)

```
    by (elim landau-o.bigE)
  from ** have eventually (λx. norm (f x + g x) ≤ (c1 + c2) * (norm (h x))) F
    by eventually-elim (auto simp: algebra-simps intro: order.trans[OF norm-triangle-ineq])
  then show ?thesis by (rule bigoI)
 qed
 from assms this[of f g] this[of f λx. − g x]
 show (λx. f x + g x) ∈ O[F](h) (λx. f x − g x) ∈ O[F](h) by simp-all
qed

lemma big-sum-in-bigo:
 assumes ⋀x. x ∈ A ⟹ f x ∈ O[F](g)
 shows   (λx. sum (λy. f y x) A) ∈ O[F](g)
 using assms by (induction A rule: infinite-finite-induct) (auto intro: sum-in-bigo)

context landau-symbol
begin

lemma mult-cancel-left:
 assumes f1 ∈ Θ[F](g1) and eventually (λx. g1 x ≠ 0) F
 notes   [trans] = bigtheta-trans1 bigtheta-trans2
 shows   (λx. f1 x * f2 x) ∈ L F (λx. g1 x * g2 x) ⟷ f2 ∈ L F (g2)
proof
 assume A: (λx. f1 x * f2 x) ∈ L F (λx. g1 x * g2 x)
  from assms have nz: eventually (λx. f1 x ≠ 0) F by (simp add: eventually-nonzero-bigtheta)
 hence f2 ∈ Θ[F](λx. f1 x * f2 x / f1 x)
   by (intro bigthetaI-cong) (auto elim!: eventually-mono)
 also from A assms nz have (λx. f1 x * f2 x / f1 x) ∈ L F (λx. g1 x * g2 x / f1 x)
   by (intro divide-right) simp-all
 also from assms nz have (λx. g1 x * g2 x / f1 x) ∈ Θ[F](λx. g1 x * g2 x / g1 x)
   by (intro landau-theta.mult landau-theta.divide) (simp-all add: bigtheta-sym)
 also from assms have (λx. g1 x * g2 x / g1 x) ∈ Θ[F](g2)
   by (intro bigthetaI-cong) (auto elim!: eventually-mono)
 finally show f2 ∈ L F (g2) .
next
 assume f2 ∈ L F (g2)
 hence (λx. f1 x * f2 x) ∈ L F (λx. f1 x * g2 x) by (rule mult-left)
 also have (λx. f1 x * g2 x) ∈ Θ[F](λx. g1 x * g2 x)
   by (intro landau-theta.mult-right assms)
 finally show (λx. f1 x * f2 x) ∈ L F (λx. g1 x * g2 x) .
qed

lemma mult-cancel-right:
 assumes f2 ∈ Θ[F](g2) and eventually (λx. g2 x ≠ 0) F
 shows   (λx. f1 x * f2 x) ∈ L F (λx. g1 x * g2 x) ⟷ f1 ∈ L F (g1)
 by (subst (1 2) mult.commute) (rule mult-cancel-left[OF assms])
```

lemma *divide-cancel-right*:
assumes *f2* ∈ Θ[*F*](*g2*) **and** *eventually* (λ*x*. *g2 x* ≠ *0*) *F*
shows (λ*x*. *f1 x* / *f2 x*) ∈ *L F* (λ*x*. *g1 x* / *g2 x*) ⟷ *f1* ∈ *L F* (*g1*)
by (*subst* (*1 2*) *divide-inverse*, *intro mult-cancel-right bigtheta-inverse*) (*simp-all add*: *assms*)

lemma *divide-cancel-left*:
assumes *f1* ∈ Θ[*F*](*g1*) **and** *eventually* (λ*x*. *g1 x* ≠ *0*) *F*
shows (λ*x*. *f1 x* / *f2 x*) ∈ *L F* (λ*x*. *g1 x* / *g2 x*) ⟷
(λ*x*. *inverse* (*f2 x*)) ∈ *L F* (λ*x*. *inverse* (*g2 x*))
by (*simp only*: *divide-inverse mult-cancel-left*[*OF assms*])

end

lemma *powr-smallo-iff*:
assumes *filterlim g at-top F F* ≠ *bot*
shows (λ*x*. *g x powr p* :: *real*) ∈ *o*[*F*](λ*x*. *g x powr q*) ⟷ *p* < *q*
proof−
from *assms* **have** *eventually* (λ*x*. *g x* ≥ *1*) *F* **by** (*force simp*: *filterlim-at-top*)
hence *A*: *eventually* (λ*x*. *g x* ≠ *0*) *F* **by** *eventually-elim simp*
have *B*: (λ*x*. *g x powr q*) ∈ *O*[*F*](λ*x*. *g x powr p*) ⟹ (λ*x*. *g x powr p*) ∉ *o*[*F*](λ*x*. *g x powr q*)
proof
assume (λ*x*. *g x powr q*) ∈ *O*[*F*](λ*x*. *g x powr p*) (λ*x*. *g x powr p*) ∈ *o*[*F*](λ*x*. *g x powr q*)
from *landau-o.big-small-asymmetric*[*OF this*] **have** *eventually* (λ*x*. *g x* = *0*) *F* **by** *simp*
with *A* **have** *eventually* (λ-::'*a*. *False*) *F* **by** *eventually-elim simp*
thus *False* **by** (*simp add*: *eventually-False assms*)
qed
show *?thesis*
proof (*cases p q rule*: *linorder-cases*)
assume *p* < *q*
hence (λ*x*. *g x powr p*) ∈ *o*[*F*](λ*x*. *g x powr q*) **using** *assms A*
by (*auto intro*!: *smalloI-tendsto tendsto-neg-powr simp flip*: *powr-diff*)
with ‹*p* < *q*› **show** *?thesis* **by** *auto*
next
assume *p* = *q*
hence (λ*x*. *g x powr q*) ∈ *O*[*F*](λ*x*. *g x powr p*) **by** (*auto intro*!: *bigthetaD1*)
with *B* ‹*p* = *q*› **show** *?thesis* **by** *auto*
next
assume *p* > *q*
hence (λ*x*. *g x powr q*) ∈ *O*[*F*](λ*x*. *g x powr p*) **using** *assms A*
by (*auto intro*!: *smalloI-tendsto tendsto-neg-powr landau-o.small-imp-big simp flip*: *powr-diff*)
with *B* ‹*p* > *q*› **show** *?thesis* **by** *auto*
qed

qed

lemma *powr-bigo-iff*:
assumes *filterlim g at-top F F* ≠ *bot*
shows (λ*x*. *g x powr p* :: *real*) ∈ *O*[*F*](λ*x*. *g x powr q*) ⟷ *p* ≤ *q*
proof−
from *assms* **have** *eventually* (λ*x*. *g x* ≥ *1*) *F* **by** (*force simp*: *filterlim-at-top*)
hence *A*: *eventually* (λ*x*. *g x* ≠ *0*) *F* **by** *eventually-elim simp*
have *B*: (λ*x*. *g x powr q*) ∈ *o*[*F*](λ*x*. *g x powr p*) ⟹ (λ*x*. *g x powr p*) ∉ *O*[*F*](λ*x*. *g x powr q*)
proof
assume (λ*x*. *g x powr q*) ∈ *o*[*F*](λ*x*. *g x powr p*) (λ*x*. *g x powr p*) ∈ *O*[*F*](λ*x*. *g x powr q*)
from *landau-o.small-big-asymmetric*[*OF this*] **have** *eventually* (λ*x*. *g x* = *0*) *F* **by** *simp*
with *A* **have** *eventually* (λ-::′*a*. *False*) *F* **by** *eventually-elim simp*
thus *False* **by** (*simp add*: *eventually-False assms*)
qed
show *?thesis*
proof (*cases p q rule*: *linorder-cases*)
assume *p* < *q*
hence (λ*x*. *g x powr p*) ∈ *o*[*F*](λ*x*. *g x powr q*) **using** *assms A*
by (*auto intro*!: *smalloI-tendsto tendsto-neg-powr simp flip*: *powr-diff*)
with ‹*p* < *q*› **show** *?thesis* **by** (*auto intro*: *landau-o.small-imp-big*)
next
assume *p* = *q*
hence (λ*x*. *g x powr q*) ∈ *O*[*F*](λ*x*. *g x powr p*) **by** (*auto intro*!: *bigthetaD1*)
with *B* ‹*p* = *q*› **show** *?thesis* **by** *auto*
next
assume *p* > *q*
hence (λ*x*. *g x powr q*) ∈ *o*[*F*](λ*x*. *g x powr p*) **using** *assms A*
by (*auto intro*!: *smalloI-tendsto tendsto-neg-powr simp flip*: *powr-diff*)
with *B* ‹*p* > *q*› **show** *?thesis* **by** (*auto intro*: *landau-o.small-imp-big*)
qed
qed

lemma *powr-bigtheta-iff*:
assumes *filterlim g at-top F F* ≠ *bot*
shows (λ*x*. *g x powr p* :: *real*) ∈ Θ[*F*](λ*x*. *g x powr q*) ⟷ *p* = *q*
using *assms* **unfolding** *bigtheta-def* **by** (*auto simp*: *bigomega-iff-bigo powr-bigo-iff*)

50.3 Flatness of real functions

Given two real-valued functions f and g, we say that f is flatter than g if any power of $f(x)$ is asymptotically dominated by any positive power of $g(x)$. This is a useful notion since, given two products of powers of functions sorted by flatness, we can compare them asymptotically by simply comparing the exponent lists lexicographically.

A simple sufficient criterion for flatness it that $\ln f(x) \in o(\ln g(x))$, which

we show now.

lemma *ln-smallo-imp-flat*:
fixes *f g* :: *real* ⇒ *real*
assumes *lim-f*: *filterlim f at-top at-top*
assumes *lim-g*: *filterlim g at-top at-top*
assumes *ln-o-ln*: $(\lambda x.\ ln\ (f\ x)) \in o(\lambda x.\ ln\ (g\ x))$
assumes *q*: *q* > *0*
shows $(\lambda x.\ f\ x\ powr\ p) \in o(\lambda x.\ g\ x\ powr\ q)$
proof (*rule smalloI-tendsto*)
from *lim-f* **have** *eventually* $(\lambda x.\ f\ x > 0)$ *at-top*
by (*simp add*: *filterlim-at-top-dense*)
hence *f-nz*: *eventually* $(\lambda x.\ f\ x \neq 0)$ *at-top* **by** *eventually-elim simp*

from *lim-g* **have** *g-gt-1*: *eventually* $(\lambda x.\ g\ x > 1)$ *at-top*
by (*simp add*: *filterlim-at-top-dense*)
hence *g-nz*: *eventually* $(\lambda x.\ g\ x \neq 0)$ *at-top* **by** *eventually-elim simp*
thus *eventually* $(\lambda x.\ g\ x\ powr\ q \neq 0)$ *at-top*
by *eventually-elim simp*

have *eq*: *eventually* $(\lambda x.\ q * (p/q * (ln\ (f\ x)\ /\ ln\ (g\ x)) - 1) * ln\ (g\ x) =$
$p * ln\ (f\ x) - q * ln\ (g\ x))$ *at-top*
using *g-gt-1* **by** *eventually-elim* (*insert q*, *simp-all add*: *field-simps*)
have *filterlim* $(\lambda x.\ q * (p/q * (ln\ (f\ x)\ /\ ln\ (g\ x)) - 1) * ln\ (g\ x))$ *at-bot at-top*
by (*insert q*)
(*rule filterlim-tendsto-neg-mult-at-bot tendsto-mult*
tendsto-const tendsto-diff smalloD-tendsto[*OF ln-o-ln*] *lim-g*
filterlim-compose[*OF ln-at-top*] | *simp*)+
hence *filterlim* $(\lambda x.\ p * ln\ (f\ x) - q * ln\ (g\ x))$ *at-bot at-top*
by (*subst* (*asm*) *filterlim-cong*[*OF refl refl eq*])
hence ∗: $((\lambda x.\ exp\ (p * ln\ (f\ x) - q * ln\ (g\ x))) \longrightarrow 0)$ *at-top*
by (*rule filterlim-compose*[*OF exp-at-bot*])
have *eq*: *eventually* $(\lambda x.\ exp\ (p * ln\ (f\ x) - q * ln\ (g\ x)) = f\ x\ powr\ p\ /\ g\ x\ powr\ q)$ *at-top*
using *f-nz g-nz* **by** *eventually-elim* (*simp add*: *powr-def exp-diff*)
show $((\lambda x.\ f\ x\ powr\ p\ /\ g\ x\ powr\ q) \longrightarrow 0)$ *at-top*
using ∗ **by** (*subst* (*asm*) *filterlim-cong*[*OF refl refl eq*])
qed

lemma *ln-smallo-imp-flat'*:
fixes *f g* :: *real* ⇒ *real*
assumes *lim-f*: *filterlim f at-top at-top*
assumes *lim-g*: *filterlim g at-top at-top*
assumes *ln-o-ln*: $(\lambda x.\ ln\ (f\ x)) \in o(\lambda x.\ ln\ (g\ x))$
assumes *q*: *q* < *0*
shows $(\lambda x.\ g\ x\ powr\ q) \in o(\lambda x.\ f\ x\ powr\ p)$
proof −
from *lim-f lim-g* **have** *eventually* $(\lambda x.\ f\ x > 0)$ *at-top eventually* $(\lambda x.\ g\ x > 0)$ *at-top*
by (*simp-all add*: *filterlim-at-top-dense*)

hence *eventually* $(\lambda x.\ f\ x \neq 0)$ *at-top eventually* $(\lambda x.\ g\ x \neq 0)$ *at-top*
by (*auto elim*: *eventually-mono*)
moreover from *assms* **have** $(\lambda x.\ f\ x\ powr\ -p) \in o(\lambda x.\ g\ x\ powr\ -q)$
by (*intro ln-smallo-imp-flat assms*) *simp-all*
ultimately show *?thesis* **unfolding** *powr-minus*
by (*simp add*: *landau-o.small.inverse-cancel*)
qed

50.4 Asymptotic Equivalence

named-theorems *asymp-equiv-intros*
named-theorems *asymp-equiv-simps*

definition *asymp-equiv* :: $('a \Rightarrow ('b :: real\text{-}normed\text{-}field)) \Rightarrow 'a\ filter \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool$
$(\langle\text{- } \sim[\text{-}]\text{ -}\rangle\ [51,\ 10,\ 51]\ 50)$
where $f \sim[F]\ g \longleftrightarrow ((\lambda x.\ if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x) \longrightarrow 1)\ F$

abbreviation (*input*) *asymp-equiv-at-top* **where**
$asymp\text{-}equiv\text{-}at\text{-}top\ f\ g \equiv f \sim[at\text{-}top]\ g$

bundle *asymp-equiv-notation*
begin
notation *asymp-equiv-at-top* (**infix** $\langle\sim\rangle$ *50*)
end

lemma *asymp-equivI*: $((\lambda x.\ if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x) \longrightarrow 1)\ F \Longrightarrow f \sim[F]\ g$
by (*simp add*: *asymp-equiv-def*)

lemma *asymp-equivD*: $f \sim[F]\ g \Longrightarrow ((\lambda x.\ if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x) \longrightarrow 1)\ F$
by (*simp add*: *asymp-equiv-def*)

lemma *asymp-equiv-filtermap-iff*:
$f \sim[filtermap\ h\ F]\ g \longleftrightarrow (\lambda x.\ f\ (h\ x)) \sim[F]\ (\lambda x.\ g\ (h\ x))$
by (*simp add*: *asymp-equiv-def filterlim-filtermap*)

lemma *asymp-equiv-refl* [*simp*, *asymp-equiv-intros*]: $f \sim[F]\ f$
proof (*intro asymp-equivI*)
have *eventually* $(\lambda x.\ 1 = (if\ f\ x = 0 \wedge f\ x = 0\ then\ 1\ else\ f\ x\ /\ f\ x))\ F$
by (*intro always-eventually*) *simp*
moreover have $((\lambda\text{-}.\ 1) \longrightarrow 1)\ F$ **by** *simp*
ultimately show $((\lambda x.\ if\ f\ x = 0 \wedge f\ x = 0\ then\ 1\ else\ f\ x\ /\ f\ x) \longrightarrow 1)\ F$
by (*simp add*: *tendsto-eventually*)
qed

lemma *asymp-equiv-symI*:
assumes $f \sim[F]\ g$

shows $g \sim[F] f$
using *tendsto-inverse*[*OF asymp-equivD*[*OF assms*]]
by (*auto intro*!: *asymp-equivI simp*: *if-distrib conj-commute cong*: *if-cong*)

lemma *asymp-equiv-sym*: $f \sim[F] g \longleftrightarrow g \sim[F] f$
by (*blast intro*: *asymp-equiv-symI*)

lemma *asymp-equivI'*:
assumes $((\lambda x.\ f\ x\ /\ g\ x) \longrightarrow 1)\ F$
shows $f \sim[F] g$
proof (*cases* $F = bot$)
case *False*
have *eventually* $(\lambda x.\ f\ x \neq 0)\ F$
proof (*rule ccontr*)
assume $\neg$*eventually* $(\lambda x.\ f\ x \neq 0)\ F$
hence *frequently* $(\lambda x.\ f\ x = 0)\ F$ **by** (*simp add*: *frequently-def*)
hence *frequently* $(\lambda x.\ f\ x\ /\ g\ x = 0)\ F$ **by** (*auto elim*!: *frequently-elim1*)
from *limit-frequently-eq*[*OF False this assms*] **show** *False* **by** *simp-all*
qed
hence *eventually* $(\lambda x.\ f\ x\ /\ g\ x = (if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x))\ F$
by *eventually-elim simp*
with *assms* **show** $f \sim[F] g$ **unfolding** *asymp-equiv-def*
by (*rule Lim-transform-eventually*)
qed (*simp-all add*: *asymp-equiv-def*)

lemma *asymp-equiv-cong*:
assumes *eventually* $(\lambda x.\ f1\ x = f2\ x)\ F$ *eventually* $(\lambda x.\ g1\ x = g2\ x)\ F$
shows $f1 \sim[F] g1 \longleftrightarrow f2 \sim[F] g2$
unfolding *asymp-equiv-def*
proof (*rule tendsto-cong*, *goal-cases*)
case *1*
from *assms* **show** *?case* **by** *eventually-elim simp*
qed

lemma *asymp-equiv-eventually-zeros*:
fixes $f\ g :: 'a \Rightarrow 'b :: real\text{-}normed\text{-}field$
assumes $f \sim[F] g$
shows *eventually* $(\lambda x.\ f\ x = 0 \longleftrightarrow g\ x = 0)\ F$
proof −
let *?h* $= \lambda x.\ if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x$
have *eventually* $(\lambda x.\ x \neq 0)$ $(nhds\ (1::'b))$
by (*rule t1-space-nhds*) *auto*
hence *eventually* $(\lambda x.\ x \neq 0)$ $(filtermap\ ?h\ F)$
using *assms* **unfolding** *asymp-equiv-def filterlim-def*
by (*rule filter-leD* [*rotated*])
hence *eventually* $(\lambda x.\ ?h\ x \neq 0)\ F$ **by** (*simp add*: *eventually-filtermap*)
thus *?thesis* **by** *eventually-elim* (*auto split*: *if-splits*)
qed

lemma *asymp-equiv-transfer*:
assumes *f1* $\sim[F]$ *g1 eventually* $(\lambda x.\ f1\ x = f2\ x)$ *F eventually* $(\lambda x.\ g1\ x = g2\ x)$ *F*
shows *f2* $\sim[F]$ *g2*
using *assms*(*1*) *asymp-equiv-cong*[*OF assms*(*2*,*3*)] **by** *simp*

lemma *asymp-equiv-transfer-trans* [*trans*]:
assumes $(\lambda x.\ f\ x\ (h1\ x)) \sim[F] (\lambda x.\ g\ x\ (h1\ x))$
assumes *eventually* $(\lambda x.\ h1\ x = h2\ x)$ *F*
shows $(\lambda x.\ f\ x\ (h2\ x)) \sim[F] (\lambda x.\ g\ x\ (h2\ x))$
by (*rule asymp-equiv-transfer*[*OF assms*(*1*)]) (*insert assms*(*2*), *auto elim*!: *eventually-mono*)

lemma *asymp-equiv-trans* [*trans*]:
fixes *f g h*
assumes $f \sim[F] g$ $g \sim[F] h$
shows $f \sim[F] h$
proof −
let *?T* $= \lambda f\ g\ x.$ *if* $f\ x = 0 \wedge g\ x = 0$ *then 1 else* $f\ x\ /\ g\ x$
from *tendsto-mult*[*OF assms*[*THEN asymp-equivD*]]
have $((\lambda x.\ ?T\ f\ g\ x * ?T\ g\ h\ x) \longrightarrow 1)\ F$ **by** *simp*
moreover from *assms*[*THEN asymp-equiv-eventually-zeros*]
have *eventually* $(\lambda x.\ ?T\ f\ g\ x * ?T\ g\ h\ x = ?T\ f\ h\ x)\ F$ **by** *eventually-elim simp*
ultimately show *?thesis* **unfolding** *asymp-equiv-def* **by** (*rule Lim-transform-eventually*)
qed

lemma *asymp-equiv-trans-lift1* [*trans*]:
assumes $a \sim[F] f\ b$ $b \sim[F] c \bigwedge c\ d.\ c \sim[F] d \Longrightarrow f\ c \sim[F] f\ d$
shows $a \sim[F] f\ c$
using *assms* **by** (*blast intro*: *asymp-equiv-trans*)

lemma *asymp-equiv-trans-lift2* [*trans*]:
assumes $f\ a \sim[F] b$ $a \sim[F] c \bigwedge c\ d.\ c \sim[F] d \Longrightarrow f\ c \sim[F] f\ d$
shows $f\ c \sim[F] b$
using *asymp-equiv-symI*[*OF assms*(*3*)[*OF assms*(*2*)]] *assms*(*1*)
by (*blast intro*: *asymp-equiv-trans*)

lemma *asymp-equivD-const*:
assumes $f \sim[F] (\lambda\text{-}.\ c)$
shows $(f \longrightarrow c)\ F$
proof (*cases* $c = 0$)
case *False*
with *tendsto-mult-right*[*OF asymp-equivD*[*OF assms*], *of c*] **show** *?thesis* **by** *simp*
next
case *True*
with *asymp-equiv-eventually-zeros*[*OF assms*] **show** *?thesis*
by (*simp add*: *tendsto-eventually*)
qed

lemma *asymp-equiv-refl-ev*:
assumes *eventually* (λ*x*. *f x* = *g x*) *F*
shows *f* ∼[*F*] *g*
by (*intro asymp-equivI tendsto-eventually*)
(*insert assms*, *auto elim*!: *eventually-mono*)

lemma *asymp-equiv-sandwich*:
fixes *f g h* :: *'a* ⇒ *'b* :: {*real-normed-field*, *order-topology*, *linordered-field*}
assumes *eventually* (λ*x*. *f x* ≥ *0*) *F*
assumes *eventually* (λ*x*. *f x* ≤ *g x*) *F*
assumes *eventually* (λ*x*. *g x* ≤ *h x*) *F*
assumes *f* ∼[*F*] *h*
shows *g* ∼[*F*] *f g* ∼[*F*] *h*
proof −
show *g* ∼[*F*] *f*
proof (*rule asymp-equivI*, *rule tendsto-sandwich*)
from *assms*(*1*−*3*) *asymp-equiv-eventually-zeros*[*OF assms*(*4*)]
show *eventually* (λ*n*. (*if h n* = *0* ∧ *f n* = *0 then 1 else h n* / *f n*) ≥
(*if g n* = *0* ∧ *f n* = *0 then 1 else g n* / *f n*)) *F*
by *eventually-elim* (*auto intro*!: *divide-right-mono*)
from *assms*(*1*−*3*) *asymp-equiv-eventually-zeros*[*OF assms*(*4*)]
show *eventually* (λ*n*. *1* ≤
(*if g n* = *0* ∧ *f n* = *0 then 1 else g n* / *f n*)) *F*
by *eventually-elim* (*auto intro*!: *divide-right-mono*)
qed (*insert asymp-equiv-symI*[*OF assms*(*4*)], *simp-all add*: *asymp-equiv-def*)
also note ‹*f* ∼[*F*] *h*›
finally show *g* ∼[*F*] *h* **.**
qed

lemma *asymp-equiv-imp-eventually-same-sign*:
fixes *f g* :: *real* ⇒ *real*
assumes *f* ∼[*F*] *g*
shows *eventually* (λ*x*. *sgn* (*f x*) = *sgn* (*g x*)) *F*
proof −
from *assms* **have** ((λ*x*. *sgn* (*if f x* = *0* ∧ *g x* = *0 then 1 else f x* / *g x*)) ⟶
sgn 1) *F*
unfolding *asymp-equiv-def* **by** (*rule tendsto-sgn*) *simp-all*
from *order-tendstoD*(*1*)[*OF this*, *of 1*/*2*]
have *eventually* (λ*x*. *sgn* (*if f x* = *0* ∧ *g x* = *0 then 1 else f x* / *g x*) > *1*/*2*) *F*
by *simp*
thus *eventually* (λ*x*. *sgn* (*f x*) = *sgn* (*g x*)) *F*
proof *eventually-elim*
case (*elim x*)
thus *?case*
by (*cases f x 0* :: *real rule*: *linorder-cases*;
cases g x 0 :: *real rule*: *linorder-cases*) *simp-all*
qed
qed

lemma
fixes $f\ g$ :: - ⇒ *real*
assumes $f \sim[F]\ g$
shows *asymp-equiv-eventually-same-sign*: *eventually* $(\lambda x.\ sgn\ (f\ x) = sgn\ (g\ x))\ F$ (**is** *?th1*)
and *asymp-equiv-eventually-neg-iff*: *eventually* $(\lambda x.\ f\ x < 0 \longleftrightarrow g\ x < 0)\ F$ (**is** *?th2*)
and *asymp-equiv-eventually-pos-iff*: *eventually* $(\lambda x.\ f\ x > 0 \longleftrightarrow g\ x > 0)\ F$ (**is** *?th3*)
proof −
from *assms* **have** *filterlim* $(\lambda x.\ if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x)$ $(nhds\ 1)\ F$
by (*rule asymp-equivD*)
from *order-tendstoD*(*1*)[*OF this zero-less-one*]
show *?th1 ?th2 ?th3*
by (*eventually-elim*; *force simp*: *sgn-if field-split-simps split*: *if-splits*)+
qed

lemma *asymp-equiv-tendsto-transfer*:
assumes $f \sim[F]\ g$ **and** $(f \longrightarrow c)\ F$
shows $(g \longrightarrow c)\ F$
proof −
let *?h* = $\lambda x.\ (if\ g\ x = 0 \wedge f\ x = 0\ then\ 1\ else\ g\ x\ /\ f\ x) * f\ x$
from *assms*(*1*) **have** $g \sim[F]\ f$ **by** (*rule asymp-equiv-symI*)
hence *filterlim* $(\lambda x.\ if\ g\ x = 0 \wedge f\ x = 0\ then\ 1\ else\ g\ x\ /\ f\ x)\ (nhds\ 1)\ F$
by (*rule asymp-equivD*)
from *tendsto-mult*[*OF this assms*(*2*)] **have** $(?h \longrightarrow c)\ F$ **by** *simp*
moreover
have *eventually* $(\lambda x.\ ?h\ x = g\ x)\ F$
using *asymp-equiv-eventually-zeros*[*OF assms*(*1*)] **by** *eventually-elim simp*
ultimately show *?thesis*
by (*rule Lim-transform-eventually*)
qed

lemma *tendsto-asymp-equiv-cong*:
assumes $f \sim[F]\ g$
shows $(f \longrightarrow c)\ F \longleftrightarrow (g \longrightarrow c)\ F$
proof −
have $(f \longrightarrow c * 1)\ F$ **if** *fg*: $f \sim[F]\ g$ **and** $(g \longrightarrow c)\ F$ **for** $f\ g$ :: $'a \Rightarrow 'b$
proof −
from *that* **have** ∗: $((\lambda x.\ g\ x * (if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x)) \longrightarrow c * 1)\ F$
by (*intro tendsto-intros asymp-equivD*)
have *eventually* $(\lambda x.\ g\ x * (if\ f\ x = 0 \wedge g\ x = 0\ then\ 1\ else\ f\ x\ /\ g\ x) = f\ x)\ F$
using *asymp-equiv-eventually-zeros*[*OF fg*] **by** *eventually-elim simp*
with ∗ **show** *?thesis* **by** (*rule Lim-transform-eventually*)
qed
from *this*[*of f g*] *this*[*of g f*] *assms* **show** *?thesis* **by** (*auto simp*: *asymp-equiv-sym*)

qed

lemma *smallo-imp-eventually-sgn*:
fixes $f\ g :: real \Rightarrow real$
assumes $g \in o(f)$
shows *eventually* $(\lambda x.\ sgn\ (f\ x + g\ x) = sgn\ (f\ x))$ *at-top*
proof −
have $0 < (1/2 :: real)$ **by** *simp*
from *landau-o.smallD*[*OF assms*, *OF this*]
have *eventually* $(\lambda x.\ |g\ x| \leq 1/2 * |f\ x|)$ *at-top* **by** *simp*
thus *?thesis*
proof *eventually-elim*
case (*elim x*)
thus *?case*
by (*cases f x 0*::*real rule*: *linorder-cases*;
cases f x + g x 0::*real rule*: *linorder-cases*) *simp-all*
qed
qed

context
begin

private lemma *asymp-equiv-add-rightI*:
assumes $f \sim[F]\ g$ $h \in o[F](g)$
shows $(\lambda x.\ f\ x + h\ x) \sim[F]\ g$
proof −
let $?T = \lambda f\ g\ x.$ *if* $f\ x = 0 \wedge g\ x = 0$ *then 1 else* $f\ x\ /\ g\ x$
from *landau-o.smallD*[*OF assms*(*2*) *zero-less-one*]
have *ev*: *eventually* $(\lambda x.\ g\ x = 0 \longrightarrow h\ x = 0)\ F$ **by** *eventually-elim auto*
have $(\lambda x.\ f\ x + h\ x) \sim[F]\ g \longleftrightarrow ((\lambda x.\ ?T\ f\ g\ x + h\ x\ /\ g\ x) \longrightarrow 1)\ F$
unfolding *asymp-equiv-def* **using** *ev*
by (*intro tendsto-cong*) (*auto elim*!: *eventually-mono simp*: *field-split-simps*)
also have ... $\longleftrightarrow ((\lambda x.\ ?T\ f\ g\ x + h\ x\ /\ g\ x) \longrightarrow 1 + 0)\ F$ **by** *simp*
also have ... **by** (*intro tendsto-intros asymp-equivD assms smalloD-tendsto*)
finally show $(\lambda x.\ f\ x + h\ x) \sim[F]\ g$ **.**
qed

lemma *asymp-equiv-add-right* [*asymp-equiv-simps*]:
assumes $h \in o[F](g)$
shows $(\lambda x.\ f\ x + h\ x) \sim[F]\ g \longleftrightarrow f \sim[F]\ g$
proof
assume $(\lambda x.\ f\ x + h\ x) \sim[F]\ g$
from *asymp-equiv-add-rightI*[*OF this*, *of* $\lambda x.\ -h\ x$] *assms* **show** $f \sim[F]\ g$
by *simp*
qed (*simp-all add*: *asymp-equiv-add-rightI assms*)

end

lemma *asymp-equiv-add-left* [*asymp-equiv-simps*]:
assumes $h \in o[F](g)$
shows $(\lambda x.\ h\ x + f\ x) \sim[F]\ g \longleftrightarrow f \sim[F]\ g$
using *asymp-equiv-add-right*[*OF assms*] **by** (*simp add*: *add.commute*)

lemma *asymp-equiv-add-right'* [*asymp-equiv-simps*]:
assumes $h \in o[F](g)$
shows $g \sim[F]\ (\lambda x.\ f\ x + h\ x) \longleftrightarrow g \sim[F]\ f$
using *asymp-equiv-add-right*[*OF assms*] **by** (*simp add*: *asymp-equiv-sym*)

lemma *asymp-equiv-add-left'* [*asymp-equiv-simps*]:
assumes $h \in o[F](g)$
shows $g \sim[F]\ (\lambda x.\ h\ x + f\ x) \longleftrightarrow g \sim[F]\ f$
using *asymp-equiv-add-left*[*OF assms*] **by** (*simp add*: *asymp-equiv-sym*)

lemma *smallo-imp-asymp-equiv*:
assumes $(\lambda x.\ f\ x - g\ x) \in o[F](g)$
shows $f \sim[F]\ g$
proof −
from *assms* **have** $(\lambda x.\ f\ x - g\ x + g\ x) \sim[F]\ g$
by (*subst asymp-equiv-add-left*) *simp-all*
thus *?thesis* **by** *simp*
qed

lemma *asymp-equiv-uminus* [*asymp-equiv-intros*]:
$f \sim[F]\ g \Longrightarrow (\lambda x.\ -f\ x) \sim[F]\ (\lambda x.\ -g\ x)$
by (*simp add*: *asymp-equiv-def cong*: *if-cong*)

lemma *asymp-equiv-uminus-iff* [*asymp-equiv-simps*]:
$(\lambda x.\ -f\ x) \sim[F]\ g \longleftrightarrow f \sim[F]\ (\lambda x.\ -g\ x)$
by (*simp add*: *asymp-equiv-def cong*: *if-cong*)

lemma *asymp-equiv-mult* [*asymp-equiv-intros*]:
fixes *f1 f2 g1 g2* :: *'a* ⇒ *'b* :: *real-normed-field*
assumes *f1* $\sim[F]$ *g1 f2* $\sim[F]$ *g2*
shows $(\lambda x.\ f1\ x * f2\ x) \sim[F]\ (\lambda x.\ g1\ x * g2\ x)$
proof −
let *?T* = $\lambda f\ g\ x.$ *if f x* = *0* ∧ *g x* = *0 then 1 else f x* / *g x*
let *?S* = $\lambda x.$ (*if f1 x* = *0* ∧ *g1 x* = *0 then 1* − *?T f2 g2 x*
else if f2 x = *0* ∧ *g2 x* = *0 then 1* − *?T f1 g1 x else 0*)
let *?S'* = $\lambda x.$ *?T* ($\lambda x.$ *f1 x* ∗ *f2 x*) ($\lambda x.$ *g1 x* ∗ *g2 x*) *x* − *?T f1 g1 x* ∗ *?T f2 g2 x*
have *A*: (($\lambda x.$ *1* − *?T f g x*) ⟶ *0*) *F* **if** *f* $\sim[F]$ *g* **for** *f g* :: *'a* ⇒ *'b*
by (*rule tendsto-eq-intros refl asymp-equivD*[*OF that*])+ *simp-all*

from *assms* **have** ∗: (($\lambda x.$ *?T f1 g1 x* ∗ *?T f2 g2 x*) ⟶ *1* ∗ *1*) *F*
by (*intro tendsto-mult asymp-equivD*)
{
have (*?S* ⟶ *0*) *F*
by (*intro filterlim-If assms*[*THEN A*, *THEN tendsto-mono*[*rotated*]])

```
      (auto intro: le-infI1 le-infI2)
   moreover have eventually (λx. ?S x = ?S' x) F
    using assms[THEN asymp-equiv-eventually-zeros] by eventually-elim auto
   ultimately have (?S' ⟶ 0) F by (rule Lim-transform-eventually)
  }
  with * have (?T (λx. f1 x * f2 x) (λx. g1 x * g2 x) ⟶ 1 * 1) F
   by (rule Lim-transform)
  then show ?thesis by (simp add: asymp-equiv-def)
qed

lemma asymp-equiv-power [asymp-equiv-intros]:
  f ∼[F] g ⟹ (λx. f x ^ n) ∼[F] (λx. g x ^ n)
  by (induction n) (simp-all add: asymp-equiv-mult)

lemma asymp-equiv-inverse [asymp-equiv-intros]:
  assumes f ∼[F] g
  shows   (λx. inverse (f x)) ∼[F] (λx. inverse (g x))
proof -
  from tendsto-inverse[OF asymp-equivD[OF assms]]
   have ((λx. if f x = 0 ∧ g x = 0 then 1 else g x / f x) ⟶ 1) F
   by (simp add: if-distrib cong: if-cong)
  also have (λx. if f x = 0 ∧ g x = 0 then 1 else g x / f x) =
              (λx. if f x = 0 ∧ g x = 0 then 1 else inverse (f x) / inverse (g x))
   by (intro ext) (simp add: field-simps)
  finally show ?thesis by (simp add: asymp-equiv-def)
qed

lemma asymp-equiv-inverse-iff [asymp-equiv-simps]:
  (λx. inverse (f x)) ∼[F] (λx. inverse (g x)) ⟷ f ∼[F] g
proof
  assume (λx. inverse (f x)) ∼[F] (λx. inverse (g x))
  hence (λx. inverse (inverse (f x))) ∼[F] (λx. inverse (inverse (g x))) (is ?P)
   by (rule asymp-equiv-inverse)
  also have ?P ⟷ f ∼[F] g by (intro asymp-equiv-cong) simp-all
  finally show f ∼[F] g .
qed (simp-all add: asymp-equiv-inverse)

lemma asymp-equiv-divide [asymp-equiv-intros]:
  assumes f1 ∼[F] g1 f2 ∼[F] g2
  shows   (λx. f1 x / f2 x) ∼[F] (λx. g1 x / g2 x)
   using asymp-equiv-mult[OF assms(1) asymp-equiv-inverse[OF assms(2)]] by
(simp add: field-simps)

lemma asymp-equiv-compose [asymp-equiv-intros]:
  assumes f ∼[G] g filterlim h G F
  shows   f ∘ h ∼[F] g ∘ h
proof -
  let ?T = λf g x. if f x = 0 ∧ g x = 0 then 1 else f x / g x
  have f ∘ h ∼[F] g ∘ h ⟷ ((?T f g ∘ h) ⟶ 1) F
```

by (*simp add*: *asymp-equiv-def o-def*)
also have ... $\longleftrightarrow$ (*?T f g* $\longrightarrow$ *1*) (*filtermap h F*)
by (*rule tendsto-compose-filtermap*)
also have ...
by (*rule tendsto-mono*[*of - G*]) (*insert assms*, *simp-all add*: *asymp-equiv-def filterlim-def*)
finally show *?thesis* **.**
qed

lemma *asymp-equiv-compose'*:
assumes $f \sim[G]\ g$ *filterlim h G F*
shows $(\lambda x.\ f\ (h\ x)) \sim[F]\ (\lambda x.\ g\ (h\ x))$
using *asymp-equiv-compose*[*OF assms*] **by** (*simp add*: *o-def*)

lemma *asymp-equiv-powr-real* [*asymp-equiv-intros*]:
fixes *f g* :: *'a* $\Rightarrow$ *real*
assumes $f \sim[F]\ g$ *eventually* $(\lambda x.\ f\ x \geq 0)\ F$ *eventually* $(\lambda x.\ g\ x \geq 0)\ F$
shows $(\lambda x.\ f\ x\ powr\ y) \sim[F]\ (\lambda x.\ g\ x\ powr\ y)$
proof −
let *?T* = $\lambda f\ g\ x.$ *if f x* = *0* $\wedge$ *g x* = *0 then 1 else f x / g x*
have $((\lambda x.\ ?T\ f\ g\ x\ powr\ y) \longrightarrow 1\ powr\ y)\ F$
by (*intro tendsto-intros asymp-equivD*[*OF assms*(*1*)]) *simp-all*
hence $((\lambda x.\ ?T\ f\ g\ x\ powr\ y) \longrightarrow 1)\ F$ **by** *simp*
moreover have *eventually* $(\lambda x.\ ?T\ f\ g\ x\ powr\ y = ?T\ (\lambda x.\ f\ x\ powr\ y)\ (\lambda x.\ g\ x\ powr\ y)\ x)\ F$
using *asymp-equiv-eventually-zeros*[*OF assms*(*1*)] *assms*(*2*,*3*)
by *eventually-elim* (*auto simp*: *powr-divide*)
ultimately show *?thesis* **unfolding** *asymp-equiv-def* **by** (*rule Lim-transform-eventually*)
qed

lemma *asymp-equiv-norm* [*asymp-equiv-intros*]:
fixes *f g* :: *'a* $\Rightarrow$ *'b* :: *real-normed-field*
assumes $f \sim[F]\ g$
shows $(\lambda x.\ norm\ (f\ x)) \sim[F]\ (\lambda x.\ norm\ (g\ x))$
using *tendsto-norm*[*OF asymp-equivD*[*OF assms*]]
by (*simp add*: *if-distrib asymp-equiv-def norm-divide cong*: *if-cong*)

lemma *asymp-equiv-abs-real* [*asymp-equiv-intros*]:
fixes *f g* :: *'a* $\Rightarrow$ *real*
assumes $f \sim[F]\ g$
shows $(\lambda x.\ |f\ x|) \sim[F]\ (\lambda x.\ |g\ x|)$
using *tendsto-rabs*[*OF asymp-equivD*[*OF assms*]]
by (*simp add*: *if-distrib asymp-equiv-def cong*: *if-cong*)

lemma *asymp-equiv-imp-eventually-le*:
assumes $f \sim[F]\ g$ *c* > *1*
shows *eventually* $(\lambda x.\ norm\ (f\ x) \leq c * norm\ (g\ x))\ F$
proof −
from *order-tendstoD*(*2*)[*OF asymp-equivD*[*OF asymp-equiv-norm*[*OF assms*(*1*)]]]

assms(*2*)]
asymp-equiv-eventually-zeros[*OF assms*(*1*)]
show *?thesis* **by** *eventually-elim* (*auto split*: *if-splits simp*: *field-simps*)
qed

lemma *asymp-equiv-imp-eventually-ge*:
assumes $f \sim[F]\ g$ $c < 1$
shows *eventually* $(\lambda x.\ norm\ (f\ x) \geq c * norm\ (g\ x))\ F$
proof −
from *order-tendstoD*(*1*)[*OF asymp-equivD*[*OF asymp-equiv-norm*[*OF assms*(*1*)]]
assms(*2*)]
asymp-equiv-eventually-zeros[*OF assms*(*1*)]
show *?thesis* **by** *eventually-elim* (*auto split*: *if-splits simp*: *field-simps*)
qed

lemma *asymp-equiv-imp-bigo*:
assumes $f \sim[F]\ g$
shows $f \in O[F](g)$
proof (*rule bigoI*)
have (*3*/*2*::*real*) > *1* **by** *simp*
from *asymp-equiv-imp-eventually-le*[*OF assms this*]
show *eventually* $(\lambda x.\ norm\ (f\ x) \leq 3/2 * norm\ (g\ x))\ F$
by *eventually-elim simp*
qed

lemma *asymp-equiv-imp-bigomega*:
$f \sim[F]\ g \Longrightarrow f \in \Omega[F](g)$
using *asymp-equiv-imp-bigo*[*of g F f*] **by** (*simp add*: *asymp-equiv-sym bigomega-iff-bigo*)

lemma *asymp-equiv-imp-bigtheta*:
$f \sim[F]\ g \Longrightarrow f \in \Theta[F](g)$
by (*intro bigthetaI asymp-equiv-imp-bigo asymp-equiv-imp-bigomega*)

lemma *asymp-equiv-at-infinity-transfer*:
assumes $f \sim[F]\ g$ *filterlim f at-infinity F*
shows *filterlim g at-infinity F*
proof −
from *assms*(*1*) **have** $g \in \Theta[F](f)$ **by** (*rule asymp-equiv-imp-bigtheta*[*OF asymp-equiv-symI*])
also from *assms* **have** $f \in \omega[F](\lambda\text{-}.\ 1)$ **by** (*simp add*: *smallomega-1-conv-filterlim*)
finally show *?thesis* **by** (*simp add*: *smallomega-1-conv-filterlim*)
qed

lemma *asymp-equiv-at-top-transfer*:
fixes *f g* :: - ⇒ *real*
assumes $f \sim[F]\ g$ *filterlim f at-top F*
shows *filterlim g at-top F*
proof (*rule filterlim-at-infinity-imp-filterlim-at-top*)
show *filterlim g at-infinity F*
by (*rule asymp-equiv-at-infinity-transfer*[*OF assms*(*1*) *filterlim-mono*[*OF assms*(*2*)]])

(*auto simp*: *at-top-le-at-infinity*)
from *assms*(*2*) **have** *eventually* (λ*x*. *f x* > *0*) *F*
using *filterlim-at-top-dense* **by** *blast*
with *asymp-equiv-eventually-pos-iff*[*OF assms*(*1*)] **show** *eventually* (λ*x*. *g x* > *0*) *F*
by *eventually-elim blast*
qed

lemma *asymp-equiv-at-bot-transfer*:
fixes *f g* :: - ⇒ *real*
assumes *f* ∼[*F*] *g filterlim f at-bot F*
shows *filterlim g at-bot F*
unfolding *filterlim-uminus-at-bot*
by (*rule asymp-equiv-at-top-transfer*[*of* λ*x*. −*f x F* λ*x*. −*g x*])
(*insert assms*, *auto simp*: *filterlim-uminus-at-bot asymp-equiv-uminus*)

lemma *asymp-equivI′-const*:
assumes ((λ*x*. *f x* / *g x*) ⟶ *c*) *F c* ≠ *0*
shows *f* ∼[*F*] (λ*x*. *c* ∗ *g x*)
using *tendsto-mult*[*OF assms*(*1*) *tendsto-const*[*of inverse c*]] *assms*(*2*)
by (*intro asymp-equivI′*) (*simp add*: *field-simps*)

lemma *asymp-equivI′-inverse-const*:
assumes ((λ*x*. *f x* / *g x*) ⟶ *inverse c*) *F c* ≠ *0*
shows (λ*x*. *c* ∗ *f x*) ∼[*F*] *g*
using *tendsto-mult*[*OF assms*(*1*) *tendsto-const*[*of c*]] *assms*(*2*)
by (*intro asymp-equivI′*) (*simp add*: *field-simps*)

lemma *filterlim-at-bot-imp-at-infinity*: *filterlim f at-bot F* ⟹ *filterlim f at-infinity F*
for *f* :: - ⇒ *real* **using** *at-bot-le-at-infinity filterlim-mono* **by** *blast*

lemma *asymp-equiv-imp-diff-smallo*:
assumes *f* ∼[*F*] *g*
shows (λ*x*. *f x* − *g x*) ∈ *o*[*F*](*g*)
proof (*rule landau-o.smallI*)
fix *c* :: *real* **assume** *c* > *0*
hence *c*: *min c 1* > *0* **by** *simp*
let *?h* = λ*x*. *if f x* = *0* ∧ *g x* = *0 then 1 else f x* / *g x*
from *assms* **have** ((λ*x*. *?h x* − *1*) ⟶ *1* − *1*) *F*
by (*intro tendsto-diff asymp-equivD tendsto-const*)
from *tendstoD*[*OF this c*] **show** *eventually* (λ*x*. *norm* (*f x* − *g x*) ≤ *c* ∗ *norm* (*g x*)) *F*
proof *eventually-elim*
case (*elim x*)
from *elim* **have** *norm* (*f x* − *g x*) ≤ *norm* (*f x* / *g x* − *1*) ∗ *norm* (*g x*)
by (*subst norm-mult* [*symmetric*]) (*auto split*: *if-splits simp add*: *algebra-simps*)
also have *norm* (*f x* / *g x* − *1*) ∗ *norm* (*g x*) ≤ *c* ∗ *norm* (*g x*) **using** *elim*
by (*auto split*: *if-splits simp*: *mult-right-mono*)

finally show *?case* **.**
qed
qed

lemma *asymp-equiv-altdef*:
$f \sim[F]\ g \longleftrightarrow (\lambda x.\ f\ x - g\ x) \in o[F](g)$
by (*rule iffI*[*OF asymp-equiv-imp-diff-smallo smallo-imp-asymp-equiv*])

lemma *asymp-equiv-0-left-iff* [*simp*]: $(\lambda\text{-}.\ 0) \sim[F]\ f \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)$ F
and *asymp-equiv-0-right-iff* [*simp*]: $f \sim[F]\ (\lambda\text{-}.\ 0) \longleftrightarrow$ *eventually* $(\lambda x.\ f\ x = 0)$ F
by (*simp-all add*: *asymp-equiv-altdef landau-o.small-refl-iff*)

lemma *asymp-equiv-sandwich-real*:
fixes $f\ g\ l\ u :: 'a \Rightarrow real$
assumes $l \sim[F]\ g\ u \sim[F]\ g$ *eventually* $(\lambda x.\ f\ x \in \{l\ x..u\ x\})\ F$
shows $f \sim[F]\ g$
unfolding *asymp-equiv-altdef*
proof (*rule landau-o.smallI*)
fix $c :: real$ **assume** c: $c > 0$
have *eventually* $(\lambda x.\ norm\ (f\ x - g\ x) \leq max\ (norm\ (l\ x - g\ x))\ (norm\ (u\ x - g\ x)))\ F$
using *assms*(*3*) **by** *eventually-elim auto*
moreover have *eventually* $(\lambda x.\ norm\ (l\ x - g\ x) \leq c * norm\ (g\ x))\ F$
eventually $(\lambda x.\ norm\ (u\ x - g\ x) \leq c * norm\ (g\ x))\ F$
using *assms*(*1*,*2*) **by** (*auto simp*: *asymp-equiv-altdef dest*: *landau-o.smallD*[*OF - c*])
hence *eventually* $(\lambda x.\ max\ (norm\ (l\ x - g\ x))\ (norm\ (u\ x - g\ x)) \leq c * norm\ (g\ x))\ F$
by *eventually-elim simp*
ultimately show *eventually* $(\lambda x.\ norm\ (f\ x - g\ x) \leq c * norm\ (g\ x))\ F$
by *eventually-elim* (*rule order.trans*)
qed

lemma *asymp-equiv-sandwich-real'*:
fixes $f\ g\ l\ u :: 'a \Rightarrow real$
assumes $f \sim[F]\ l\ f \sim[F]\ u$ *eventually* $(\lambda x.\ g\ x \in \{l\ x..u\ x\})\ F$
shows $f \sim[F]\ g$
using *asymp-equiv-sandwich-real*[*of l F f u g*] *assms* **by** (*simp add*: *asymp-equiv-sym*)

lemma *asymp-equiv-sandwich-real''*:
fixes $f\ g\ l\ u :: 'a \Rightarrow real$
assumes $l1 \sim[F]\ u1\ u1 \sim[F]\ l2\ l2 \sim[F]\ u2$
eventually $(\lambda x.\ f\ x \in \{l1\ x..u1\ x\})\ F$ *eventually* $(\lambda x.\ g\ x \in \{l2\ x..u2\ x\})\ F$
shows $f \sim[F]\ g$
by (*rule asymp-equiv-sandwich-real*[*OF asymp-equiv-sandwich-real'*[*OF - - assms*(*5*)] *asymp-equiv-sandwich-real'*[*OF - - assms*(*5*)] *assms*(*4*)];
blast intro: *asymp-equiv-trans assms*(*1*,*2*,*3*))+

end

theory *Lattice-Constructions*
imports *Main*
begin

50.5 Values extended by a bottom element

datatype *'a bot* = *Value 'a* | *Bot*

instantiation *bot* :: (*preorder*) *preorder*
begin

definition *less-eq-bot* **where**
$x \leq y \longleftrightarrow$ (*case x of Bot* $\Rightarrow$ *True* | *Value x* $\Rightarrow$ (*case y of Bot* $\Rightarrow$ *False* | *Value y* $\Rightarrow$ $x \leq y$))

definition *less-bot* **where**
$x < y \longleftrightarrow$ (*case y of Bot* $\Rightarrow$ *False* | *Value y* $\Rightarrow$ (*case x of Bot* $\Rightarrow$ *True* | *Value x* $\Rightarrow$ $x < y$))

lemma *less-eq-bot-Bot* [*simp*]: *Bot* $\leq x$
by (*simp add*: *less-eq-bot-def*)

lemma *less-eq-bot-Bot-code* [*code*]: *Bot* $\leq x \longleftrightarrow$ *True*
by *simp*

lemma *less-eq-bot-Bot-is-Bot*: $x \leq$ *Bot* $\Longrightarrow x =$ *Bot*
by (*cases x*) (*simp-all add*: *less-eq-bot-def*)

lemma *less-eq-bot-Value-Bot* [*simp*, *code*]: *Value x* $\leq$ *Bot* $\longleftrightarrow$ *False*
by (*simp add*: *less-eq-bot-def*)

lemma *less-eq-bot-Value* [*simp*, *code*]: *Value x* $\leq$ *Value y* $\longleftrightarrow x \leq y$
by (*simp add*: *less-eq-bot-def*)

lemma *less-bot-Bot* [*simp*, *code*]: $x <$ *Bot* $\longleftrightarrow$ *False*
by (*simp add*: *less-bot-def*)

lemma *less-bot-Bot-is-Value*: *Bot* $< x \Longrightarrow \exists z.\ x =$ *Value z*
by (*cases x*) (*simp-all add*: *less-bot-def*)

lemma *less-bot-Bot-Value* [*simp*]: *Bot* $<$ *Value x*
by (*simp add*: *less-bot-def*)

lemma *less-bot-Bot-Value-code* [*code*]: *Bot* $<$ *Value x* $\longleftrightarrow$ *True*
by *simp*

lemma *less-bot-Value* [*simp*, *code*]: *Value x* < *Value y* ⟷ *x* < *y*
by (*simp add*: *less-bot-def*)

instance
by *standard*
(*auto simp add*: *less-eq-bot-def less-bot-def less-le-not-le elim*: *order-trans split*: *bot.splits*)

end

instance *bot* :: (*order*) *order*
by *standard* (*auto simp add*: *less-eq-bot-def less-bot-def split*: *bot.splits*)

instance *bot* :: (*linorder*) *linorder*
by *standard* (*auto simp add*: *less-eq-bot-def less-bot-def split*: *bot.splits*)

instantiation *bot* :: (*order*) *bot*
begin
definition *bot* = *Bot*
instance ..
end

instantiation *bot* :: (*top*) *top*
begin
definition *top* = *Value top*
instance ..
end

instantiation *bot* :: (*semilattice-inf*) *semilattice-inf*
begin

definition *inf-bot*
where
inf x y =
(*case x of*
Bot ⇒ *Bot*
| *Value v* ⇒
(*case y of*
Bot ⇒ *Bot*
| *Value v'* ⇒ *Value* (*inf v v'*)))

instance
by *standard* (*auto simp add*: *inf-bot-def less-eq-bot-def split*: *bot.splits*)

end

instantiation *bot* :: (*semilattice-sup*) *semilattice-sup*
begin

definition *sup-bot*
where
sup x y =
(case x of
Bot ⇒ y
| Value v ⇒
(case y of
Bot ⇒ x
| Value v′ ⇒ Value (sup v v′)))

instance
by *standard* (*auto simp add*: *sup-bot-def less-eq-bot-def split*: *bot.splits*)

end

instance *bot* :: (*lattice*) *bounded-lattice-bot*
by *intro-classes* (*simp add*: *bot-bot-def*)

50.6 Values extended by a top element

datatype *'a top = Value 'a | Top*

instantiation *top* :: (*preorder*) *preorder*
begin

definition *less-eq-top* **where**
x ≤ y ⟷ (case y of Top ⇒ True | Value y ⇒ (case x of Top ⇒ False | Value x ⇒ x ≤ y))

definition *less-top* **where**
x < y ⟷ (case x of Top ⇒ False | Value x ⇒ (case y of Top ⇒ True | Value y ⇒ x < y))

lemma *less-eq-top-Top* [*simp*]: *x ≤ Top*
by (*simp add*: *less-eq-top-def*)

lemma *less-eq-top-Top-code* [*code*]: *x ≤ Top ⟷ True*
by *simp*

lemma *less-eq-top-is-Top*: *Top ≤ x ⟹ x = Top*
by (*cases x*) (*simp-all add*: *less-eq-top-def*)

lemma *less-eq-top-Top-Value* [*simp*, *code*]: *Top ≤ Value x ⟷ False*
by (*simp add*: *less-eq-top-def*)

lemma *less-eq-top-Value-Value* [*simp*, *code*]: *Value x ≤ Value y ⟷ x ≤ y*
by (*simp add*: *less-eq-top-def*)

lemma *less-top-Top* [*simp*, *code*]: *Top* < *x* ⟷ *False*
by (*simp add*: *less-top-def*)

lemma *less-top-Top-is-Value*: *x* < *Top* ⟹ ∃ *z*. *x* = *Value z*
by (*cases x*) (*simp-all add*: *less-top-def*)

lemma *less-top-Value-Top* [*simp*]: *Value x* < *Top*
by (*simp add*: *less-top-def*)

lemma *less-top-Value-Top-code* [*code*]: *Value x* < *Top* ⟷ *True*
by *simp*

lemma *less-top-Value* [*simp*, *code*]: *Value x* < *Value y* ⟷ *x* < *y*
by (*simp add*: *less-top-def*)

instance
by *standard*
(*auto simp add*: *less-eq-top-def less-top-def less-le-not-le elim*: *order-trans split*: *top.splits*)

end

instance *top* :: (*order*) *order*
by *standard* (*auto simp add*: *less-eq-top-def less-top-def split*: *top.splits*)

instance *top* :: (*linorder*) *linorder*
by *standard* (*auto simp add*: *less-eq-top-def less-top-def split*: *top.splits*)

instantiation *top* :: (*order*) *top*
begin
definition *top* = *Top*
instance ..
end

instantiation *top* :: (*bot*) *bot*
begin
definition *bot* = *Value bot*
instance ..
end

instantiation *top* :: (*semilattice-inf*) *semilattice-inf*
begin

definition *inf-top*
where
inf x y =
(*case x of*
Top ⇒ *y*
| *Value v* ⇒

```
    (case y of
      Top ⇒ x
    | Value v' ⇒ Value (inf v v')))

instance
  by standard (auto simp add: inf-top-def less-eq-top-def split: top.splits)

end

instantiation top :: (semilattice-sup) semilattice-sup
begin

definition sup-top
where
  sup x y =
    (case x of
      Top ⇒ Top
    | Value v ⇒
        (case y of
          Top ⇒ Top
        | Value v' ⇒ Value (sup v v')))

instance
  by standard (auto simp add: sup-top-def less-eq-top-def split: top.splits)

end

instance top :: (lattice) bounded-lattice-top
  by standard (simp add: top-top-def)
```

50.7 Values extended by a top and a bottom element

```
datatype 'a flat-complete-lattice = Value 'a | Bot | Top

instantiation flat-complete-lattice :: (type) order
begin

definition less-eq-flat-complete-lattice
where
  x ≤ y ≡
    (case x of
      Bot ⇒ True
    | Value v1 ⇒
        (case y of
          Bot ⇒ False
        | Value v2 ⇒ v1 = v2
        | Top ⇒ True)
    | Top ⇒ y = Top)
```

definition *less-flat-complete-lattice*
where
  $x < y =$
    (*case x of*
      Bot $\Rightarrow$ $y \neq$ *Bot*
    | *Value v1* $\Rightarrow$ $y =$ *Top*
    | *Top* $\Rightarrow$ *False*)

lemma [*simp*]: *Bot* $\leq$ *y*
  unfolding *less-eq-flat-complete-lattice-def* **by** *auto*

lemma [*simp*]: *y* $\leq$ *Top*
  unfolding *less-eq-flat-complete-lattice-def* **by** (*auto split*: *flat-complete-lattice.splits*)

lemma *greater-than-two-values*:
  assumes $a \neq b$ *Value a* $\leq z$ *Value b* $\leq z$
  shows $z =$ *Top*
  using *assms*
  by (*cases z*) (*auto simp add*: *less-eq-flat-complete-lattice-def*)

lemma *lesser-than-two-values*:
  assumes $a \neq b$ $z \leq$ *Value a* $z \leq$ *Value b*
  shows $z =$ *Bot*
  using *assms*
  by (*cases z*) (*auto simp add*: *less-eq-flat-complete-lattice-def*)

instance
  by *standard*
    (*auto simp add*: *less-eq-flat-complete-lattice-def less-flat-complete-lattice-def*
      split: *flat-complete-lattice.splits*)

end

instantiation *flat-complete-lattice* :: (*type*) *bot*
begin
  definition *bot* = *Bot*
  instance ..
end

instantiation *flat-complete-lattice* :: (*type*) *top*
begin
  definition *top* = *Top*
  instance ..
end

instantiation *flat-complete-lattice* :: (*type*) *lattice*
begin

definition *inf-flat-complete-lattice*

where
inf x y =
(*case x of*
Bot ⇒ *Bot*
| *Value v1* ⇒
(*case y of*
Bot ⇒ *Bot*
| *Value v2* ⇒ *if v1* = *v2 then x else Bot*
| *Top* ⇒ *x*)
| *Top* ⇒ *y*)

definition *sup-flat-complete-lattice*
where
sup x y =
(*case x of*
Bot ⇒ *y*
| *Value v1* ⇒
(*case y of*
Bot ⇒ *x*
| *Value v2* ⇒ *if v1* = *v2 then x else Top*
| *Top* ⇒ *Top*)
| *Top* ⇒ *Top*)

instance
by *standard*
(*auto simp add*: *inf-flat-complete-lattice-def sup-flat-complete-lattice-def*
less-eq-flat-complete-lattice-def split: *flat-complete-lattice.splits*)

end

instantiation *flat-complete-lattice* :: (*type*) *complete-lattice*
begin

definition *Sup-flat-complete-lattice*
where
Sup A =
(*if A* = {} ∨ *A* = {*Bot*} *then Bot*
else if ∃ *v*. *A* − {*Bot*} = {*Value v*} *then Value* (*THE v*. *A* − {*Bot*} = {*Value v*})
else Top)

definition *Inf-flat-complete-lattice*
where
Inf A =
(*if A* = {} ∨ *A* = {*Top*} *then Top*
else if ∃ *v*. *A* − {*Top*} = {*Value v*} *then Value* (*THE v*. *A* − {*Top*} = {*Value v*})
else Bot)

instance
proof
  fix $x :: 'a$ *flat-complete-lattice*
  fix A
  assume $x \in A$
  {
    fix v
    assume $A - \{Top\} = \{Value\ v\}$
    then have $(THE\ v.\ A - \{Top\} = \{Value\ v\}) = v$
      by (*auto intro*!: *the1-equality*)
    moreover
    from ‹$x \in A$› ‹$A - \{Top\} = \{Value\ v\}$› **have** $x = Top \vee x = Value\ v$
      by *auto*
    ultimately have $Value\ (THE\ v.\ A - \{Top\} = \{Value\ v\}) \leq x$
      by *auto*
  }
  with ‹$x \in A$› **show** $Inf\ A \leq x$
    unfolding *Inf-flat-complete-lattice-def*
    by *fastforce*
next
  fix $z :: 'a$ *flat-complete-lattice*
  fix A
  show $z \leq Inf\ A$ **if** z: $\bigwedge x.\ x \in A \Longrightarrow z \leq x$
  proof −
    consider $A = \{\} \vee A = \{Top\}$
      | $A \neq \{\}\ A \neq \{Top\}\ \exists v.\ A - \{Top\} = \{Value\ v\}$
      | $A \neq \{\}\ A \neq \{Top\}\ \neg\ (\exists v.\ A - \{Top\} = \{Value\ v\})$
      by *blast*
    then show *?thesis*
    proof *cases*
      case *1*
      then have $Inf\ A = Top$
        unfolding *Inf-flat-complete-lattice-def* **by** *auto*
      then show *?thesis* **by** *simp*
    next
      case *2*
      then obtain v **where** *v1*: $A - \{Top\} = \{Value\ v\}$
        by *auto*
      then have *v2*: $(THE\ v.\ A - \{Top\} = \{Value\ v\}) = v$
        by (*auto intro*!: *the1-equality*)
      from *2 v2* **have** *Inf*: $Inf\ A = Value\ v$
        unfolding *Inf-flat-complete-lattice-def* **by** *simp*
      from *v1* **have** $Value\ v \in A$ **by** *blast*
      then have $z \leq Value\ v$ **by** (*rule z*)
      with *Inf* **show** *?thesis* **by** *simp*
    next
      case *3*
      then have *Inf*: $Inf\ A = Bot$
        unfolding *Inf-flat-complete-lattice-def* **by** *auto*

```
      have z ≤ Bot
      proof (cases A − {Top} = {Bot})
        case True
        then have Bot ∈ A by blast
        then show ?thesis by (rule z)
      next
        case False
        from 3 obtain a1 where a1: a1 ∈ A − {Top}
          by auto
        from 3 False a1 obtain a2 where a2 ∈ A − {Top} ∧ a1 ≠ a2
          by (cases a1) auto
        with a1 z[of a1] z[of a2] show ?thesis
          apply (cases a1)
          apply auto
          apply (cases a2)
          apply auto
          apply (auto dest!: lesser-than-two-values)
          done
      qed
      with Inf show ?thesis by simp
    qed
  qed
next
  fix x :: 'a flat-complete-lattice
  fix A
  assume x ∈ A
  {
    fix v
    assume A − {Bot} = {Value v}
    then have (THE v. A − {Bot} = {Value v}) = v
      by (auto intro!: the1-equality)
    moreover
    from ‹x ∈ A› ‹A − {Bot} = {Value v}› have x = Bot ∨ x = Value v
      by auto
    ultimately have x ≤ Value (THE v. A − {Bot} = {Value v})
      by auto
  }
  with ‹x ∈ A› show x ≤ Sup A
    unfolding Sup-flat-complete-lattice-def
    by fastforce
next
  fix z :: 'a flat-complete-lattice
  fix A
  show Sup A ≤ z if z: ⋀x. x ∈ A ⟹ x ≤ z
  proof −
    consider A = {} ∨ A = {Bot}
      | A ≠ {} A ≠ {Bot} ∃ v. A − {Bot} = {Value v}
      | A ≠ {} A ≠ {Bot} ¬ (∃ v. A − {Bot} = {Value v})
      by blast
```

```
    then show ?thesis
    proof cases
      case 1
      then have Sup A = Bot
        unfolding Sup-flat-complete-lattice-def by auto
      then show ?thesis by simp
    next
      case 2
      then obtain v where v1: A − {Bot} = {Value v}
        by auto
      then have v2: (THE v. A − {Bot} = {Value v}) = v
        by (auto intro!: the1-equality)
      from 2 v2 have Sup: Sup A = Value v
        unfolding Sup-flat-complete-lattice-def by simp
      from v1 have Value v ∈ A by blast
      then have Value v ≤ z by (rule z)
      with Sup show ?thesis by simp
    next
      case 3
      then have Sup: Sup A = Top
        unfolding Sup-flat-complete-lattice-def by auto
      have Top ≤ z
      proof (cases A − {Bot} = {Top})
        case True
        then have Top ∈ A by blast
        then show ?thesis by (rule z)
      next
        case False
        from 3 obtain a1 where a1: a1 ∈ A − {Bot}
          by auto
        from 3 False a1 obtain a2 where a2 ∈ A − {Bot} ∧ a1 ≠ a2
          by (cases a1) auto
        with a1 z[of a1] z[of a2] show ?thesis
          apply (cases a1)
          apply auto
          apply (cases a2)
          apply (auto dest!: greater-than-two-values)
          done
      qed
      with Sup show ?thesis by simp
    qed
  qed
next
  show Inf {} = (top :: 'a flat-complete-lattice)
    by (simp add: Inf-flat-complete-lattice-def top-flat-complete-lattice-def)
  show Sup {} = (bot :: 'a flat-complete-lattice)
    by (simp add: Sup-flat-complete-lattice-def bot-flat-complete-lattice-def)
qed
```

end

end

51 Infinite Streams

theory *Stream*
imports *Nat-Bijection*
begin

codatatype (*sset*: $'a$) *stream* =
SCons (*shd*: $'a$) (*stl*: $'a$ *stream*) (**infixr** ‹##› *65*)
for
map: *smap*
rel: *stream-all2*

context
begin

— for code generation only
qualified definition *smember* :: $'a \Rightarrow 'a$ *stream* $\Rightarrow$ *bool* **where**
[*code-abbrev*]: *smember x s* $\longleftrightarrow$ *x* $\in$ *sset s*

lemma *smember-code*[*code*, *simp*]: *smember x* (*y* ## *s*) = (*if x* = *y then True else smember x s*)
unfolding *smember-def* **by** *auto*

end

lemmas *smap-simps*[*simp*] = *stream.map-sel*
lemmas *shd-sset* = *stream.set-sel*(*1*)
lemmas *stl-sset* = *stream.set-sel*(*2*)

theorem *sset-induct*[*consumes 1*, *case-names shd stl*, *induct set*: *sset*]:
assumes *y* $\in$ *sset s* **and** $\bigwedge$*s*. *P* (*shd s*) *s* **and** $\bigwedge$*s y*. ⟦*y* $\in$ *sset* (*stl s*); *P y* (*stl s*)⟧ $\Longrightarrow$ *P y s*
shows *P y s*
using *assms* **by** *induct* (*metis stream.sel*(*1*), *auto*)

lemma *smap-ctr*: *smap f s* = *x* ## *s'* $\longleftrightarrow$ *f* (*shd s*) = *x* $\wedge$ *smap f* (*stl s*) = *s'*
by (*cases s*) *simp*

51.1 prepend list to stream

primrec *shift* :: $'a$ *list* $\Rightarrow$ $'a$ *stream* $\Rightarrow$ $'a$ *stream* (**infixr** ‹@−› *65*) **where**
shift [] *s* = *s*
| *shift* (*x* # *xs*) *s* = *x* ## *shift xs s*

lemma *smap-shift*[*simp*]: *smap f* (*xs* @− *s*) = *map f xs* @− *smap f s*

by (*induct xs*) *auto*

lemma *shift-append*[*simp*]: (*xs* @ *ys*) @− *s* = *xs* @− *ys* @− *s*
by (*induct xs*) *auto*

lemma *shift-simps*[*simp*]:
shd (*xs* @− *s*) = (*if xs* = [] *then shd s else hd xs*)
stl (*xs* @− *s*) = (*if xs* = [] *then stl s else tl xs* @− *s*)
by (*induct xs*) *auto*

lemma *sset-shift*[*simp*]: *sset* (*xs* @− *s*) = *set xs* ∪ *sset s*
by (*induct xs*) *auto*

lemma *shift-left-inj*[*simp*]: *xs* @− *s1* = *xs* @− *s2* ⟷ *s1* = *s2*
by (*induct xs*) *auto*

51.2 set of streams with elements in some fixed set

context
notes [[*inductive-internals*]]
begin

coinductive-set
streams :: *'a set* ⇒ *'a stream set*
for *A* :: *'a set*
where
Stream[*intro*!, *simp*, *no-atp*]: ⟦*a* ∈ *A*; *s* ∈ *streams A*⟧ ⟹ *a* ## *s* ∈ *streams A*

end

lemma *in-streams*: *stl s* ∈ *streams S* ⟹ *shd s* ∈ *S* ⟹ *s* ∈ *streams S*
by (*cases s*) *auto*

lemma *streamsE*: *s* ∈ *streams A* ⟹ (*shd s* ∈ *A* ⟹ *stl s* ∈ *streams A* ⟹ *P*)
⟹ *P*
by (*erule streams.cases*) *simp-all*

lemma *Stream-image*: *x* ## *y* ∈ ((##) *x'*) ‘ *Y* ⟷ *x* = *x'* ∧ *y* ∈ *Y*
by *auto*

lemma *shift-streams*: ⟦*w* ∈ *lists A*; *s* ∈ *streams A*⟧ ⟹ *w* @− *s* ∈ *streams A*
by (*induct w*) *auto*

lemma *streams-Stream*: *x* ## *s* ∈ *streams A* ⟷ *x* ∈ *A* ∧ *s* ∈ *streams A*
by (*auto elim*: *streams.cases*)

lemma *streams-stl*: *s* ∈ *streams A* ⟹ *stl s* ∈ *streams A*
by (*cases s*) (*auto simp*: *streams-Stream*)

lemma *streams-shd*: $s \in$ *streams* $A \Longrightarrow$ *shd* $s \in A$
by (*cases s*) (*auto simp*: *streams-Stream*)

lemma *sset-streams*:
assumes *sset* $s \subseteq A$
shows $s \in$ *streams* A
using *assms* **proof** (*coinduction arbitrary*: *s*)
case *streams* **then show** *?case* **by** (*cases s*) *simp*
qed

lemma *streams-sset*:
assumes $s \in$ *streams* A
shows *sset* $s \subseteq A$
proof
fix x **assume** $x \in$ *sset* s **from** *this* ‹$s \in$ *streams* A› **show** $x \in A$
by (*induct s*) (*auto intro*: *streams-shd streams-stl*)
qed

lemma *streams-iff-sset*: $s \in$ *streams* $A \longleftrightarrow$ *sset* $s \subseteq A$
by (*metis sset-streams streams-sset*)

lemma *streams-mono*: $s \in$ *streams* $A \Longrightarrow A \subseteq B \Longrightarrow s \in$ *streams* B
unfolding *streams-iff-sset* **by** *auto*

lemma *streams-mono2*: $S \subseteq T \Longrightarrow$ *streams* $S \subseteq$ *streams* T
by (*auto intro*: *streams-mono*)

lemma *smap-streams*: $s \in$ *streams* $A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \in B) \Longrightarrow$ *smap* $f\ s \in$ *streams* B
unfolding *streams-iff-sset stream.set-map* **by** *auto*

lemma *streams-empty*: *streams* $\{\} = \{\}$
by (*auto elim*: *streams.cases*)

lemma *streams-UNIV*[*simp*]: *streams UNIV* = *UNIV*
by (*auto simp*: *streams-iff-sset*)

51.3 nth, take, drop for streams

primrec *snth* :: *'a stream* ⇒ *nat* ⇒ *'a* (**infixl** ‹!!› *100*) **where**
s !! *0* = *shd s*
| *s* !! *Suc n* = *stl s* !! *n*

lemma *snth-Stream*: (*x* ## *s*) !! *Suc i* = *s* !! *i*
by *simp*

lemma *snth-smap*[*simp*]: *smap f s* !! *n* = *f* (*s* !! *n*)
by (*induct n arbitrary*: *s*) *auto*

lemma *shift-snth-less*[*simp*]: $p < \mathit{length}\ \mathit{xs} \Longrightarrow (\mathit{xs}\ @{-}\ s)\ !!\ p = \mathit{xs}\ !\ p$
by (*induct p arbitrary*: *xs*) (*auto simp*: *hd-conv-nth nth-tl*)

lemma *shift-snth-ge*[*simp*]: $p \geq \mathit{length}\ \mathit{xs} \Longrightarrow (\mathit{xs}\ @{-}\ s)\ !!\ p = s\ !!\ (p - \mathit{length}\ \mathit{xs})$
by (*induct p arbitrary*: *xs*) (*auto simp*: *Suc-diff-eq-diff-pred*)

lemma *shift-snth*: $(\mathit{xs}\ @{-}\ s)\ !!\ n = (\mathit{if}\ n < \mathit{length}\ \mathit{xs}\ \mathit{then}\ \mathit{xs}\ !\ n\ \mathit{else}\ s\ !!\ (n - \mathit{length}\ \mathit{xs}))$
by *auto*

lemma *snth-sset*[*simp*]: $s\ !!\ n \in \mathit{sset}\ s$
by (*induct n arbitrary*: *s*) (*auto intro*: *shd-sset stl-sset*)

lemma *sset-range*: $\mathit{sset}\ s = \mathit{range}\ (\mathit{snth}\ s)$
proof (*intro equalityI subsetI*)
fix *x* **assume** $x \in \mathit{sset}\ s$
thus $x \in \mathit{range}\ (\mathit{snth}\ s)$
proof (*induct s*)
case (*stl s x*)
then obtain *n* **where** $x = \mathit{stl}\ s\ !!\ n$ **by** *auto*
thus *?case* **by** (*auto intro*: *range-eqI*[*of - - Suc n*])
qed (*auto intro*: *range-eqI*[*of - - 0*])
qed *auto*

lemma *streams-iff-snth*: $s \in \mathit{streams}\ X \longleftrightarrow (\forall n.\ s\ !!\ n \in X)$
by (*force simp*: *streams-iff-sset sset-range*)

lemma *snth-in*: $s \in \mathit{streams}\ X \Longrightarrow s\ !!\ n \in X$
by (*simp add*: *streams-iff-snth*)

primrec *stake* :: $\mathit{nat} \Rightarrow 'a\ \mathit{stream} \Rightarrow 'a\ \mathit{list}$ **where**
$\mathit{stake}\ 0\ s = []$
| $\mathit{stake}\ (\mathit{Suc}\ n)\ s = \mathit{shd}\ s\ \#\ \mathit{stake}\ n\ (\mathit{stl}\ s)$

lemma *length-stake*[*simp*]: $\mathit{length}\ (\mathit{stake}\ n\ s) = n$
by (*induct n arbitrary*: *s*) *auto*

lemma *stake-smap*[*simp*]: $\mathit{stake}\ n\ (\mathit{smap}\ f\ s) = \mathit{map}\ f\ (\mathit{stake}\ n\ s)$
by (*induct n arbitrary*: *s*) *auto*

lemma *take-stake*: $\mathit{take}\ n\ (\mathit{stake}\ m\ s) = \mathit{stake}\ (\mathit{min}\ n\ m)\ s$
proof (*induct m arbitrary*: *s n*)
case (*Suc m*) **thus** *?case* **by** (*cases n*) *auto*
qed *simp*

primrec *sdrop* :: $\mathit{nat} \Rightarrow 'a\ \mathit{stream} \Rightarrow 'a\ \mathit{stream}$ **where**
$\mathit{sdrop}\ 0\ s = s$
| $\mathit{sdrop}\ (\mathit{Suc}\ n)\ s = \mathit{sdrop}\ n\ (\mathit{stl}\ s)$

lemma *sdrop-simps*[*simp*]:
shd (*sdrop n s*) = *s* !! *n stl* (*sdrop n s*) = *sdrop* (*Suc n*) *s*
by (*induct n arbitrary*: *s*) *auto*

lemma *sdrop-smap*[*simp*]: *sdrop n* (*smap f s*) = *smap f* (*sdrop n s*)
by (*induct n arbitrary*: *s*) *auto*

lemma *sdrop-stl*: *sdrop n* (*stl s*) = *stl* (*sdrop n s*)
by (*induct n*) *auto*

lemma *drop-stake*: *drop n* (*stake m s*) = *stake* (*m* − *n*) (*sdrop n s*)
proof (*induct m arbitrary*: *s n*)
case (*Suc m*) **thus** *?case* **by** (*cases n*) *auto*
qed *simp*

lemma *stake-sdrop*: *stake n s* @− *sdrop n s* = *s*
by (*induct n arbitrary*: *s*) *auto*

lemma *id-stake-snth-sdrop*:
s = *stake i s* @− *s* !! *i* ## *sdrop* (*Suc i*) *s*
by (*subst stake-sdrop*[*symmetric*, *of - i*]) (*metis sdrop-simps stream.collapse*)

lemma *smap-alt*: *smap f s* = *s′* ⟷ (∀ *n*. *f* (*s* !! *n*) = *s′* !! *n*) (**is** *?L* = *?R*)
proof
assume *?R*
then have ⋀*n*. *smap f* (*sdrop n s*) = *sdrop n s′*
by *coinduction* (*auto intro*: *exI*[*of - 0*] *simp del*: *sdrop.simps*(*2*))
then show *?L* **using** *sdrop.simps*(*1*) **by** *metis*
qed *auto*

lemma *stake-invert-Nil*[*iff*]: *stake n s* = [] ⟷ *n* = *0*
by (*induct n*) *auto*

lemma *sdrop-shift*: *sdrop i* (*w* @− *s*) = *drop i w* @− *sdrop* (*i* − *length w*) *s*
by (*induct i arbitrary*: *w s*) (*auto simp*: *drop-tl drop-Suc neq-Nil-conv*)

lemma *stake-shift*: *stake i* (*w* @− *s*) = *take i w* @ *stake* (*i* − *length w*) *s*
by (*induct i arbitrary*: *w s*) (*auto simp*: *neq-Nil-conv*)

lemma *stake-add*[*simp*]: *stake m s* @ *stake n* (*sdrop m s*) = *stake* (*m* + *n*) *s*
by (*induct m arbitrary*: *s*) *auto*

lemma *sdrop-add*[*simp*]: *sdrop n* (*sdrop m s*) = *sdrop* (*m* + *n*) *s*
by (*induct m arbitrary*: *s*) *auto*

lemma *sdrop-snth*: *sdrop n s* !! *m* = *s* !! (*n* + *m*)
by (*induct n arbitrary*: *m s*) *auto*

partial-function (*tailrec*) *sdrop-while* :: (*′a* ⇒ *bool*) ⇒ *′a stream* ⇒ *′a stream*

where
sdrop-while P s = (*if P* (*shd s*) *then sdrop-while P* (*stl s*) *else s*)

lemma *sdrop-while-SCons*[*code*]:
sdrop-while P (*a ## s*) = (*if P a then sdrop-while P s else a ## s*)
by (*subst sdrop-while.simps*) *simp*

lemma *sdrop-while-sdrop-LEAST*:
assumes ∃ *n*. *P* (*s !! n*)
shows *sdrop-while* (*Not* ∘ *P*) *s* = *sdrop* (*LEAST n*. *P* (*s !! n*)) *s*
proof −
from *assms* **obtain** *m* **where** *P* (*s !! m*) ⋀*n*. *P* (*s !! n*) ⟹ *m* ≤ *n*
and ∗: (*LEAST n*. *P* (*s !! n*)) = *m* **by** *atomize-elim* (*auto intro*: *LeastI Least-le*)
thus *?thesis* **unfolding** ∗
proof (*induct m arbitrary*: *s*)
case (*Suc m*)
hence *sdrop-while* (*Not* ∘ *P*) (*stl s*) = *sdrop m* (*stl s*)
by (*metis* (*full-types*) *not-less-eq-eq snth.simps*(*2*))
moreover from *Suc*(*3*) **have** ¬ (*P* (*s !! 0*)) **by** *blast*
ultimately show *?case* **by** (*subst sdrop-while.simps*) *simp*
qed (*metis comp-apply sdrop.simps*(*1*) *sdrop-while.simps snth.simps*(*1*))
qed

primcorec *sfilter* **where**
shd (*sfilter P s*) = *shd* (*sdrop-while* (*Not* ∘ *P*) *s*)
| *stl* (*sfilter P s*) = *sfilter P* (*stl* (*sdrop-while* (*Not* ∘ *P*) *s*))

lemma *sfilter-Stream*: *sfilter P* (*x ## s*) = (*if P x then x ## sfilter P s else sfilter P s*)
proof (*cases P x*)
case *True* **thus** *?thesis* **by** (*subst sfilter.ctr*) (*simp add*: *sdrop-while-SCons*)
next
case *False* **thus** *?thesis* **by** (*subst* (*1 2*) *sfilter.ctr*) (*simp add*: *sdrop-while-SCons*)
qed

51.4 unary predicates lifted to streams

definition *stream-all P s* = (∀ *p*. *P* (*s !! p*))

lemma *stream-all-iff*[*iff*]: *stream-all P s* ⟷ *Ball* (*sset s*) *P*
unfolding *stream-all-def sset-range* **by** *auto*

lemma *stream-all-shift*[*simp*]: *stream-all P* (*xs* @− *s*) = (*list-all P xs* ∧ *stream-all P s*)
unfolding *stream-all-iff list-all-iff* **by** *auto*

lemma *stream-all-Stream*: *stream-all P* (*x ## X*) ⟷ *P x* ∧ *stream-all P X*
by *simp*

51.5 recurring stream out of a list

primcorec *cycle* :: *'a list* ⇒ *'a stream* **where**
  shd (*cycle xs*) = *hd xs*
| *stl* (*cycle xs*) = *cycle* (*tl xs* @ [*hd xs*])

lemma *cycle-decomp*: *u* ≠ [] ⟹ *cycle u* = *u* @− *cycle u*
proof (*coinduction arbitrary*: *u*)
  case *Eq-stream* **then show** *?case* **using** *stream.collapse*[*of cycle u*]
    by (*auto intro*!: *exI*[*of - tl u* @ [*hd u*]])
qed

lemma *cycle-Cons*[*code*]: *cycle* (*x* # *xs*) = *x* ## *cycle* (*xs* @ [*x*])
  by (*subst cycle.ctr*) *simp*

lemma *cycle-rotated*: ⟦*v* ≠ []; *cycle u* = *v* @− *s*⟧ ⟹ *cycle* (*tl u* @ [*hd u*]) = *tl v* @− *s*
  by (*auto dest*: *arg-cong*[*of - - stl*])

lemma *stake-append*: *stake n* (*u* @− *s*) = *take* (*min* (*length u*) *n*) *u* @ *stake* (*n* − *length u*) *s*
proof (*induct n arbitrary*: *u*)
  case (*Suc n*) **thus** *?case* **by** (*cases u*) *auto*
qed *auto*

lemma *stake-cycle-le*[*simp*]:
  assumes *u* ≠ [] *n* < *length u*
  shows *stake n* (*cycle u*) = *take n u*
using *min-absorb2*[*OF less-imp-le-nat*[*OF assms*(*2*)]]
  by (*subst cycle-decomp*[*OF assms*(*1*)], *subst stake-append*) *auto*

lemma *stake-cycle-eq*[*simp*]: *u* ≠ [] ⟹ *stake* (*length u*) (*cycle u*) = *u*
  by (*subst cycle-decomp*) (*auto simp*: *stake-shift*)

lemma *sdrop-cycle-eq*[*simp*]: *u* ≠ [] ⟹ *sdrop* (*length u*) (*cycle u*) = *cycle u*
  by (*subst cycle-decomp*) (*auto simp*: *sdrop-shift*)

lemma *stake-cycle-eq-mod-0*[*simp*]: ⟦*u* ≠ []; *n mod length u* = *0*⟧ ⟹
    stake n (*cycle u*) = *concat* (*replicate* (*n div length u*) *u*)
  by (*induct n div length u arbitrary*: *n u*)
    (*auto simp*: *stake-add* [*symmetric*] *mod-eq-0-iff-dvd elim*!: *dvdE*)

lemma *sdrop-cycle-eq-mod-0*[*simp*]: ⟦*u* ≠ []; *n mod length u* = *0*⟧ ⟹
    sdrop n (*cycle u*) = *cycle u*
  by (*induct n div length u arbitrary*: *n u*)
    (*auto simp*: *sdrop-add* [*symmetric*] *mod-eq-0-iff-dvd elim*!: *dvdE*)

lemma *stake-cycle*: *u* ≠ [] ⟹
    stake n (*cycle u*) = *concat* (*replicate* (*n div length u*) *u*) @ *take* (*n mod length u*) *u*

by (*subst div-mult-mod-eq*[*of n length u*, *symmetric*], *unfold stake-add*[*symmetric*])
auto

lemma *sdrop-cycle*: $u \neq [] \Longrightarrow$ *sdrop n* (*cycle u*) = *cycle* (*rotate* (*n mod length u*)
u)
by (*induct n arbitrary*: *u*) (*auto simp*: *rotate1-rotate-swap rotate1-hd-tl rotate-conv-mod*[*symmetric*])

lemma *sset-cycle*[*simp*]:
assumes *xs* $\neq$ []
shows *sset* (*cycle xs*) = *set xs*
proof (*intro set-eqI iffI*)
fix *x*
assume *x* $\in$ *sset* (*cycle xs*)
then show *x* $\in$ *set xs* **using** *assms*
by (*induction cycle xs arbitrary*: *xs rule*: *sset-induct*) (*fastforce simp*: *neq-Nil-conv*)+
qed (*metis assms UnI1 cycle-decomp sset-shift*)

51.6 iterated application of a function

primcorec *siterate* **where**
shd (*siterate f x*) = *x*
| *stl* (*siterate f x*) = *siterate f* (*f x*)

lemma *stake-Suc*: *stake* (*Suc n*) *s* = *stake n s* @ [*s* !! *n*]
by (*induct n arbitrary*: *s*) *auto*

lemma *snth-siterate*[*simp*]: *siterate f x* !! *n* = (*f*^^*n*) *x*
by (*induct n arbitrary*: *x*) (*auto simp*: *funpow-swap1*)

lemma *sdrop-siterate*[*simp*]: *sdrop n* (*siterate f x*) = *siterate f* ((*f*^^*n*) *x*)
by (*induct n arbitrary*: *x*) (*auto simp*: *funpow-swap1*)

lemma *stake-siterate*[*simp*]: *stake n* (*siterate f x*) = *map* (λ*n*. (*f*^^*n*) *x*) [*0* ..< *n*]
by (*induct n arbitrary*: *x*) (*auto simp del*: *stake.simps*(*2*) *simp*: *stake-Suc*)

lemma *sset-siterate*: *sset* (*siterate f x*) = {(*f*^^*n*) *x* | *n*. *True*}
by (*auto simp*: *sset-range*)

lemma *smap-siterate*: *smap f* (*siterate f x*) = *siterate f* (*f x*)
by (*coinduction arbitrary*: *x*) *auto*

51.7 stream repeating a single element

abbreviation *sconst* ≡ *siterate id*

lemma *shift-replicate-sconst*[*simp*]: *replicate n x* @− *sconst x* = *sconst x*
by (*subst* (*3*) *stake-sdrop*[*symmetric*]) (*simp add*: *map-replicate-trivial*)

lemma *sset-sconst*[*simp*]: *sset* (*sconst x*) = {*x*}
by (*simp add*: *sset-siterate*)

lemma *sconst-alt*: $s = sconst\ x \longleftrightarrow sset\ s = \{x\}$
proof
  assume $sset\ s = \{x\}$
  then show $s = sconst\ x$
  proof (*coinduction arbitrary*: *s*)
    case *Eq-stream*
    then have $shd\ s = x\ sset\ (stl\ s) \subseteq \{x\}$ **by** (*cases s*; *auto*)+
    then have $sset\ (stl\ s) = \{x\}$ **by** (*cases stl s*) *auto*
    with ‹$shd\ s = x$› **show** *?case* **by** *auto*
  qed
qed *simp*

lemma *sconst-cycle*: $sconst\ x = cycle\ [x]$
  by *coinduction auto*

lemma *smap-sconst*: $smap\ f\ (sconst\ x) = sconst\ (f\ x)$
  by *coinduction auto*

lemma *sconst-streams*: $x \in A \Longrightarrow sconst\ x \in streams\ A$
  by (*simp add*: *streams-iff-sset*)

lemma *streams-empty-iff*: $streams\ S = \{\} \longleftrightarrow S = \{\}$
proof *safe*
  fix *x* **assume** $x \in S\ streams\ S = \{\}$
  then have $sconst\ x \in streams\ S$
    by (*intro sconst-streams*)
  then show $x \in \{\}$
    unfolding ‹$streams\ S = \{\}$› **by** *simp*
qed (*auto simp*: *streams-empty*)

51.8 stream of natural numbers

abbreviation $fromN \equiv siterate\ Suc$

abbreviation $nats \equiv fromN\ 0$

lemma *sset-fromN*[*simp*]: $sset\ (fromN\ n) = \{n\ ..\}$
  by (*auto simp add*: *sset-siterate le-iff-add*)

lemma *stream-smap-fromN*: $s = smap\ (\lambda j.\ let\ i = j - n\ in\ s\ !!\ i)\ (fromN\ n)$
  by (*coinduction arbitrary*: *s n*)
    (*force simp*: *neq-Nil-conv Let-def Suc-diff-Suc simp flip*: *snth.simps*(*2*)
      intro: *stream.map-cong split*: *if-splits*)

lemma *stream-smap-nats*: $s = smap\ (snth\ s)\ nats$
  using *stream-smap-fromN*[**where** $n = 0$] **by** *simp*

51.9 flatten a stream of lists

primcorec *flat* **where**
shd (*flat ws*) = *hd* (*shd ws*)
| *stl* (*flat ws*) = *flat* (*if tl* (*shd ws*) = [] *then stl ws else tl* (*shd ws*) ## *stl ws*)

lemma *flat-Cons*[*simp*, *code*]: *flat* ((*x* # *xs*) ## *ws*) = *x* ## *flat* (*if xs* = [] *then ws else xs* ## *ws*)
by (*subst flat.ctr*) *simp*

lemma *flat-Stream*[*simp*]: *xs* ≠ [] ⟹ *flat* (*xs* ## *ws*) = *xs* @− *flat ws*
by (*induct xs*) *auto*

lemma *flat-unfold*: *shd ws* ≠ [] ⟹ *flat ws* = *shd ws* @− *flat* (*stl ws*)
by (*cases ws*) *auto*

lemma *flat-snth*: ∀ *xs* ∈ *sset s*. *xs* ≠ [] ⟹ *flat s* !! *n* = (*if n* < *length* (*shd s*) *then shd s* ! *n else flat* (*stl s*) !! (*n* − *length* (*shd s*)))
by (*metis flat-unfold not-less shd-sset shift-snth-ge shift-snth-less*)

lemma *sset-flat*[*simp*]: ∀ *xs* ∈ *sset s*. *xs* ≠ [] ⟹
sset (*flat s*) = (⋃ *xs* ∈ *sset s*. *set xs*) (**is** *?P* ⟹ *?L* = *?R*)
proof *safe*
fix *x* **assume** *?P x* ∈ *?L*
then obtain *m* **where** *x* = *flat s* !! *m* **by** (*metis image-iff sset-range*)
with ‹*?P*› **obtain** *n m′* **where** *x* = *s* !! *n* ! *m′ m′* < *length* (*s* !! *n*)
proof (*atomize-elim*, *induct m arbitrary*: *s rule*: *less-induct*)
case (*less y*)
thus *?case*
proof (*cases y* < *length* (*shd s*))
case *True* **thus** *?thesis* **by** (*metis flat-snth less*(*2*,*3*) *snth.simps*(*1*))
next
case *False*
hence *x* = *flat* (*stl s*) !! (*y* − *length* (*shd s*)) **by** (*metis less*(*2*,*3*) *flat-snth*)
moreover
{ from *less*(*2*) **have** ∗: *length* (*shd s*) > *0* **by** (*cases s*) *simp-all*
with *False* **have** *y* > *0* **by** (*cases y*) *simp-all*
with ∗ **have** *y* − *length* (*shd s*) < *y* **by** *simp*
}
moreover have ∀ *xs* ∈ *sset* (*stl s*). *xs* ≠ [] **using** *less*(*2*) **by** (*cases s*) *auto*
ultimately have ∃ *n m′*. *x* = *stl s* !! *n* ! *m′* ∧ *m′* < *length* (*stl s* !! *n*) **by** (*intro less*(*1*)) *auto*
thus *?thesis* **by** (*metis snth.simps*(*2*))
qed
qed
thus *x* ∈ *?R* **by** (*auto simp*: *sset-range dest*!: *nth-mem*)
next
fix *x xs* **assume** *xs* ∈ *sset s ?P x* ∈ *set xs* **thus** *x* ∈ *?L*
by (*induct rule*: *sset-induct*)
(*metis UnI1 flat-unfold shift.simps*(*1*) *sset-shift*,

metis UnI2 flat-unfold shd-sset stl-sset sset-shift)
qed

51.10 merge a stream of streams

definition *smerge* :: *'a stream stream* ⇒ *'a stream* **where**
smerge ss = *flat* (*smap* (λ*n*. *map* (λ*s*. *s* !! *n*) (*stake* (*Suc n*) *ss*) @ *stake n* (*ss* !! *n*)) *nats*)

lemma *stake-nth*[*simp*]: *m* < *n* ⟹ *stake n s* ! *m* = *s* !! *m*
by (*induct n arbitrary*: *s m*) (*auto simp*: *nth-Cons'*, *metis Suc-pred snth.simps*(*2*))

lemma *snth-sset-smerge*: *ss* !! *n* !! *m* ∈ *sset* (*smerge ss*)
proof (*cases n* ≤ *m*)
case *False* **thus** *?thesis* **unfolding** *smerge-def*
by (*subst sset-flat*)
(*auto simp*: *stream.set-map in-set-conv-nth simp del*: *stake.simps*
intro!: *exI*[*of* - *n*, *OF disjI2*] *exI*[*of* - *m*, *OF mp*])
next
case *True* **thus** *?thesis* **unfolding** *smerge-def*
by (*subst sset-flat*)
(*auto simp*: *stream.set-map in-set-conv-nth image-iff simp del*: *stake.simps snth.simps*
intro!: *exI*[*of* - *m*, *OF disjI1*] *bexI*[*of* - *ss* !! *n*] *exI*[*of* - *n*, *OF mp*])
qed

lemma *sset-smerge*: *sset* (*smerge ss*) = ⋃(*sset* ' (*sset ss*))
proof *safe*
fix *x* **assume** *x* ∈ *sset* (*smerge ss*)
thus *x* ∈ ⋃(*sset* ' (*sset ss*))
unfolding *smerge-def* **by** (*subst* (*asm*) *sset-flat*)
(*auto simp*: *stream.set-map in-set-conv-nth sset-range simp del*: *stake.simps*, *fast*+)
next
fix *s x* **assume** *s* ∈ *sset ss x* ∈ *sset s*
thus *x* ∈ *sset* (*smerge ss*) **using** *snth-sset-smerge* **by** (*auto simp*: *sset-range*)
qed

51.11 product of two streams

definition *sproduct* :: *'a stream* ⇒ *'b stream* ⇒ (*'a* × *'b*) *stream* **where**
sproduct s1 s2 = *smerge* (*smap* (λ*x*. *smap* (*Pair x*) *s2*) *s1*)

lemma *sset-sproduct*: *sset* (*sproduct s1 s2*) = *sset s1* × *sset s2*
unfolding *sproduct-def sset-smerge* **by** (*auto simp*: *stream.set-map*)

51.12 interleave two streams

primcorec *sinterleave* **where**
shd (*sinterleave s1 s2*) = *shd s1*

| *stl* (*sinterleave s1 s2*) = *sinterleave s2* (*stl s1*)

lemma *sinterleave-code*[*code*]:
sinterleave (*x ## s1*) *s2* = *x ## sinterleave s2 s1*
by (*subst sinterleave.ctr*) *simp*

lemma *sinterleave-snth*[*simp*]:
even n ⟹ *sinterleave s1 s2* !! *n* = *s1* !! (*n div 2*)
odd n ⟹ *sinterleave s1 s2* !! *n* = *s2* !! (*n div 2*)
by (*induct n arbitrary*: *s1 s2*) *simp-all*

lemma *sset-sinterleave*: *sset* (*sinterleave s1 s2*) = *sset s1* ∪ *sset s2*
proof (*intro equalityI subsetI*)
fix *x* **assume** *x* ∈ *sset* (*sinterleave s1 s2*)
then obtain *n* **where** *x* = *sinterleave s1 s2* !! *n* **unfolding** *sset-range* **by** *blast*
thus *x* ∈ *sset s1* ∪ *sset s2* **by** (*cases even n*) *auto*
next
fix *x* **assume** *x* ∈ *sset s1* ∪ *sset s2*
thus *x* ∈ *sset* (*sinterleave s1 s2*)
proof
assume *x* ∈ *sset s1*
then obtain *n* **where** *x* = *s1* !! *n* **unfolding** *sset-range* **by** *blast*
hence *sinterleave s1 s2* !! (*2* ∗ *n*) = *x* **by** *simp*
thus *?thesis* **unfolding** *sset-range* **by** *blast*
next
assume *x* ∈ *sset s2*
then obtain *n* **where** *x* = *s2* !! *n* **unfolding** *sset-range* **by** *blast*
hence *sinterleave s1 s2* !! (*2* ∗ *n* + *1*) = *x* **by** *simp*
thus *?thesis* **unfolding** *sset-range* **by** *blast*
qed
qed

51.13 zip

primcorec *szip* **where**
shd (*szip s1 s2*) = (*shd s1*, *shd s2*)
| *stl* (*szip s1 s2*) = *szip* (*stl s1*) (*stl s2*)

lemma *szip-unfold*[*code*]: *szip* (*a ## s1*) (*b ## s2*) = (*a*, *b*) *##* (*szip s1 s2*)
by (*subst szip.ctr*) *simp*

lemma *snth-szip*[*simp*]: *szip s1 s2* !! *n* = (*s1* !! *n*, *s2* !! *n*)
by (*induct n arbitrary*: *s1 s2*) *auto*

lemma *stake-szip*[*simp*]:
stake n (*szip s1 s2*) = *zip* (*stake n s1*) (*stake n s2*)
by (*induct n arbitrary*: *s1 s2*) *auto*

lemma *sdrop-szip*[*simp*]: *sdrop n* (*szip s1 s2*) = *szip* (*sdrop n s1*) (*sdrop n s2*)

by (*induct n arbitrary*: *s1 s2*) *auto*

lemma *smap-szip-fst*:
smap (λ*x*. *f* (*fst x*)) (*szip s1 s2*) = *smap f s1*
by (*coinduction arbitrary*: *s1 s2*) *auto*

lemma *smap-szip-snd*:
smap (λ*x*. *g* (*snd x*)) (*szip s1 s2*) = *smap g s2*
by (*coinduction arbitrary*: *s1 s2*) *auto*

51.14 zip via function

primcorec *smap2* **where**
shd (*smap2 f s1 s2*) = *f* (*shd s1*) (*shd s2*)
| *stl* (*smap2 f s1 s2*) = *smap2 f* (*stl s1*) (*stl s2*)

lemma *smap2-unfold*[*code*]:
smap2 f (*a ## s1*) (*b ## s2*) = *f a b ##* (*smap2 f s1 s2*)
by (*subst smap2.ctr*) *simp*

lemma *smap2-szip*:
smap2 f s1 s2 = *smap* (*case-prod f*) (*szip s1 s2*)
by (*coinduction arbitrary*: *s1 s2*) *auto*

lemma *smap-smap2*[*simp*]:
smap f (*smap2 g s1 s2*) = *smap2* (λ*x y*. *f* (*g x y*)) *s1 s2*
unfolding *smap2-szip stream.map-comp o-def split-def* **..**

lemma *smap2-alt*:
(*smap2 f s1 s2* = *s*) = (∀ *n*. *f* (*s1* !! *n*) (*s2* !! *n*) = *s* !! *n*)
unfolding *smap2-szip smap-alt* **by** *auto*

lemma *snth-smap2*[*simp*]:
smap2 f s1 s2 !! *n* = *f* (*s1* !! *n*) (*s2* !! *n*)
by (*induct n arbitrary*: *s1 s2*) *auto*

lemma *stake-smap2*[*simp*]:
stake n (*smap2 f s1 s2*) = *map* (*case-prod f*) (*zip* (*stake n s1*) (*stake n s2*))
by (*induct n arbitrary*: *s1 s2*) *auto*

lemma *sdrop-smap2*[*simp*]:
sdrop n (*smap2 f s1 s2*) = *smap2 f* (*sdrop n s1*) (*sdrop n s2*)
by (*induct n arbitrary*: *s1 s2*) *auto*

end

52 List prefixes, suffixes, and homeomorphic embedding

theory *Sublist*
imports *Main*
begin

52.1 Prefix order on lists

definition *prefix* :: *'a list* ⇒ *'a list* ⇒ *bool*
  where *prefix xs ys* ⟷ (∃*zs*. *ys* = *xs* @ *zs*)

definition *strict-prefix* :: *'a list* ⇒ *'a list* ⇒ *bool*
  where *strict-prefix xs ys* ⟷ *prefix xs ys* ∧ *xs* ≠ *ys*

global-interpretation *prefix-order*: *ordering prefix strict-prefix*
  by *standard* (*auto simp add*: *prefix-def strict-prefix-def*)

interpretation *prefix-order*: *order prefix strict-prefix*
  by *standard* (*auto simp*: *prefix-def strict-prefix-def*)

global-interpretation *prefix-bot*: *ordering-top* ‹λ*xs ys*. *prefix ys xs*› ‹λ*xs ys*. *strict-prefix ys xs*› ‹[]›
  by *standard* (*simp add*: *prefix-def*)

interpretation *prefix-bot*: *order-bot Nil prefix strict-prefix*
  by *standard* (*simp add*: *prefix-def*)

lemma *prefixI* [*intro?*]: *ys* = *xs* @ *zs* ⟹ *prefix xs ys*
  unfolding *prefix-def* **by** *blast*

lemma *prefixE* [*elim?*]:
  assumes *prefix xs ys*
  obtains *zs* **where** *ys* = *xs* @ *zs*
  using *assms* **unfolding** *prefix-def* **by** *blast*

lemma *strict-prefixI'* [*intro?*]: *ys* = *xs* @ *z* # *zs* ⟹ *strict-prefix xs ys*
  unfolding *strict-prefix-def prefix-def* **by** *blast*

lemma *strict-prefixE'* [*elim?*]:
  assumes *strict-prefix xs ys*
  obtains *z zs* **where** *ys* = *xs* @ *z* # *zs*
proof −
  from ‹*strict-prefix xs ys*› **obtain** *us* **where** *ys* = *xs* @ *us* **and** *xs* ≠ *ys*
    unfolding *strict-prefix-def prefix-def* **by** *blast*
  with *that* **show** *?thesis* **by** (*auto simp add*: *neq-Nil-conv*)
qed

lemma *strict-prefixI* [*intro?*]: *prefix xs ys* $\Longrightarrow$ *xs* $\neq$ *ys* $\Longrightarrow$ *strict-prefix xs ys*
by(*fact prefix-order.le-neq-trans*)

lemma *strict-prefixE* [*elim?*]:
fixes *xs ys* :: *'a list*
assumes *strict-prefix xs ys*
obtains *prefix xs ys* **and** *xs* $\neq$ *ys*
using *assms* **unfolding** *strict-prefix-def* **by** *blast*

52.2 Basic properties of prefixes

theorem *Nil-prefix* [*simp*]: *prefix* [] *xs*
by (*fact prefix-bot.bot-least*)

theorem *prefix-Nil* [*simp*]: (*prefix xs* []) = (*xs* = [])
by (*fact prefix-bot.bot-unique*)

lemma *prefix-snoc* [*simp*]: *prefix xs* (*ys* @ [*y*]) $\longleftrightarrow$ *xs* = *ys* @ [*y*] $\vee$ *prefix xs ys*
proof
assume *prefix xs* (*ys* @ [*y*])
then obtain *zs* **where** *zs*: *ys* @ [*y*] = *xs* @ *zs* **..**
show *xs* = *ys* @ [*y*] $\vee$ *prefix xs ys*
by (*metis append-Nil2 butlast-append butlast-snoc prefixI zs*)
next
assume *xs* = *ys* @ [*y*] $\vee$ *prefix xs ys*
then show *prefix xs* (*ys* @ [*y*])
by *auto* (*metis append.assoc prefix-def*)
qed

lemma *Cons-prefix-Cons* [*simp*]: *prefix* (*x* # *xs*) (*y* # *ys*) = (*x* = *y* $\wedge$ *prefix xs ys*)
by (*auto simp add*: *prefix-def*)

lemma *prefix-code* [*code*]:
prefix [] *xs* $\longleftrightarrow$ *True*
prefix (*x* # *xs*) [] $\longleftrightarrow$ *False*
prefix (*x* # *xs*) (*y* # *ys*) $\longleftrightarrow$ *x* = *y* $\wedge$ *prefix xs ys*
by *simp-all*

lemma *same-prefix-prefix* [*simp*]: *prefix* (*xs* @ *ys*) (*xs* @ *zs*) = *prefix ys zs*
by (*induct xs*) *simp-all*

lemma *same-prefix-nil* [*simp*]: *prefix* (*xs* @ *ys*) *xs* = (*ys* = [])
by (*simp add*: *prefix-def*)

lemma *prefix-prefix* [*simp*]: *prefix xs ys* $\Longrightarrow$ *prefix xs* (*ys* @ *zs*)
unfolding *prefix-def* **by** *fastforce*

lemma *append-prefixD*: *prefix* (*xs* @ *ys*) *zs* $\Longrightarrow$ *prefix xs zs*

by (*auto simp add*: *prefix-def*)

theorem *prefix-Cons*: *prefix xs* (*y # ys*) = (*xs* = [] ∨ (∃ *zs*. *xs* = *y # zs* ∧ *prefix zs ys*))
by (*cases xs*) (*auto simp add*: *prefix-def*)

theorem *prefix-append*:
prefix xs (*ys* @ *zs*) = (*prefix xs ys* ∨ (∃ *us*. *xs* = *ys* @ *us* ∧ *prefix us zs*))
apply (*induct zs rule*: *rev-induct*)
apply *force*
apply (*simp flip*: *append-assoc*)
apply (*metis append-eq-appendI*)
done

lemma *append-one-prefix*:
prefix xs ys ⟹ *length xs* < *length ys* ⟹ *prefix* (*xs* @ [*ys* ! *length xs*]) *ys*
proof (*unfold prefix-def*)
assume *a1*: ∃ *zs*. *ys* = *xs* @ *zs*
then obtain *sk* :: *'a list* **where** *sk*: *ys* = *xs* @ *sk* **by** *fastforce*
assume *a2*: *length xs* < *length ys*
have *f1*: ⋀*v*. ([]::*'a list*) @ *v* = *v* **using** *append-Nil2* **by** *simp*
have [] ≠ *sk* **using** *a1 a2 sk less-not-refl* **by** *force*
hence ∃ *v*. *xs* @ *hd sk # v* = *ys* **using** *sk* **by** (*metis hd-Cons-tl*)
thus ∃ *zs*. *ys* = (*xs* @ [*ys* ! *length xs*]) @ *zs* **using** *f1* **by** *fastforce*
qed

theorem *prefix-length-le*: *prefix xs ys* ⟹ *length xs* ≤ *length ys*
by (*auto simp add*: *prefix-def*)

lemma *prefix-same-cases*:
prefix (*xs*$_1$::*'a list*) *ys* ⟹ *prefix xs*$_2$ *ys* ⟹ *prefix xs*$_1$ *xs*$_2$ ∨ *prefix xs*$_2$ *xs*$_1$
unfolding *prefix-def* **by** (*force simp*: *append-eq-append-conv2*)

lemma *prefix-length-prefix*:
prefix ps xs ⟹ *prefix qs xs* ⟹ *length ps* ≤ *length qs* ⟹ *prefix ps qs*
by (*auto simp*: *prefix-def*) (*metis append-Nil2 append-eq-append-conv-if*)

lemma *set-mono-prefix*: *prefix xs ys* ⟹ *set xs* ⊆ *set ys*
by (*auto simp add*: *prefix-def*)

lemma *take-is-prefix*: *prefix* (*take n xs*) *xs*
unfolding *prefix-def* **by** (*metis append-take-drop-id*)

lemma *takeWhile-is-prefix*: *prefix* (*takeWhile P xs*) *xs*
unfolding *prefix-def* **by** (*metis takeWhile-dropWhile-id*)

lemma *prefixeq-butlast*: *prefix* (*butlast xs*) *xs*
by (*simp add*: *butlast-conv-take take-is-prefix*)

lemma *prefix-map-rightE*:
  assumes *prefix xs* (*map f ys*)
  shows ∃ *xs'*. *prefix xs' ys* ∧ *xs* = *map f xs'*
proof −
  define *n* **where** *n* = *length xs*
  have *xs* = *take n* (*map f ys*)
    using *assms* **by** (*auto simp*: *prefix-def n-def*)
  thus *?thesis*
    by (*intro exI*[*of* - *take n ys*]) (*auto simp*: *take-map take-is-prefix*)
qed

lemma *map-mono-prefix*: *prefix xs ys* ⟹ *prefix* (*map f xs*) (*map f ys*)
by (*auto simp*: *prefix-def*)

lemma *filter-mono-prefix*: *prefix xs ys* ⟹ *prefix* (*filter P xs*) (*filter P ys*)
by (*auto simp*: *prefix-def*)

lemma *sorted-antimono-prefix*: *prefix xs ys* ⟹ *sorted ys* ⟹ *sorted xs*
by (*metis sorted-append prefix-def*)

lemma *prefix-length-less*: *strict-prefix xs ys* ⟹ *length xs* < *length ys*
  by (*auto simp*: *strict-prefix-def prefix-def*)

lemma *prefix-snocD*: *prefix* (*xs*@[*x*]) *ys* ⟹ *strict-prefix xs ys*
  by (*simp add*: *strict-prefixI' prefix-order.dual-order.strict-trans1*)

lemma *strict-prefix-simps* [*simp*, *code*]:
  strict-prefix xs [] ⟷ *False*
  strict-prefix [] (*x* # *xs*) ⟷ *True*
  strict-prefix (*x* # *xs*) (*y* # *ys*) ⟷ *x* = *y* ∧ *strict-prefix xs ys*
  by (*simp-all add*: *strict-prefix-def cong*: *conj-cong*)

lemma *take-strict-prefix*: *strict-prefix xs ys* ⟹ *strict-prefix* (*take n xs*) *ys*
proof (*induct n arbitrary*: *xs ys*)
  case *0*
  then show *?case* **by** (*cases ys*) *simp-all*
next
  case (*Suc n*)
  then show *?case* **by** (*metis prefix-order.less-trans strict-prefixI take-is-prefix*)
qed

lemma *prefix-takeWhile*:
  assumes *prefix xs ys*
  shows *prefix* (*takeWhile P xs*) (*takeWhile P ys*)
proof −
  from *assms* **obtain** *zs* **where** *ys*: *ys* = *xs* @ *zs*
    by (*auto simp*: *prefix-def*)
  have *prefix* (*takeWhile P xs*) (*takeWhile P* (*xs* @ *zs*))
    by (*induction xs*) *auto*

```
  thus ?thesis by (simp add: ys)
qed

lemma prefix-dropWhile:
  assumes prefix xs ys
  shows   prefix (dropWhile P xs) (dropWhile P ys)
proof −
  from assms obtain zs where ys: ys = xs @ zs
    by (auto simp: prefix-def)
  have prefix (dropWhile P xs) (dropWhile P (xs @ zs))
    by (induction xs) auto
  thus ?thesis by (simp add: ys)
qed

lemma prefix-remdups-adj:
  assumes prefix xs ys
  shows   prefix (remdups-adj xs) (remdups-adj ys)
  using assms
proof (induction length xs arbitrary: xs ys rule: less-induct)
  case (less xs)
  show ?case
  proof (cases xs)
    case [simp]: (Cons x xs')
    then obtain y ys' where [simp]: ys = y # ys'
      using ‹prefix xs ys› by (cases ys) auto
    from less show ?thesis
      by (auto simp: remdups-adj-Cons' less-Suc-eq-le length-dropWhile-le
            intro!: less prefix-dropWhile)
  qed auto
qed

lemma not-prefix-cases:
  assumes pfx: ¬ prefix ps ls
  obtains
    (c1) ps ≠ [] and ls = []
  | (c2) a as x xs where ps = a#as and ls = x#xs and x = a and ¬ prefix as xs
  | (c3) a as x xs where ps = a#as and ls = x#xs and x ≠ a
proof (cases ps)
  case Nil
  then show ?thesis using pfx by simp
next
  case (Cons a as)
  note c = ‹ps = a#as›
  show ?thesis
  proof (cases ls)
    case Nil then show ?thesis by (metis append-Nil2 pfx c1 same-prefix-nil)
  next
    case (Cons x xs)
    show ?thesis
```

```
    proof (cases x = a)
      case True
      have ¬ prefix as xs using pfx c Cons True by simp
      with c Cons True show ?thesis by (rule c2)
    next
      case False
      with c Cons show ?thesis by (rule c3)
    qed
  qed
qed

lemma not-prefix-induct [consumes 1, case-names Nil Neq Eq]:
  assumes np: ¬ prefix ps ls
    and base: ⋀x xs. P (x#xs) []
    and r1: ⋀x xs y ys. x ≠ y ⟹ P (x#xs) (y#ys)
    and r2: ⋀x xs y ys. ⟦ x = y; ¬ prefix xs ys; P xs ys ⟧ ⟹ P (x#xs) (y#ys)
  shows P ps ls using np
proof (induct ls arbitrary: ps)
  case Nil
  then show ?case
    by (auto simp: neq-Nil-conv elim!: not-prefix-cases intro!: base)
next
  case (Cons y ys)
  then have npfx: ¬ prefix ps (y # ys) by simp
  then obtain x xs where pv: ps = x # xs
    by (rule not-prefix-cases) auto
  show ?case by (metis Cons.hyps Cons-prefix-Cons npfx pv r1 r2)
qed
```

52.3 Prefixes

```
primrec prefixes where
prefixes [] = [[]] |
prefixes (x#xs) = [] # map ((#) x) (prefixes xs)

lemma in-set-prefixes[simp]: xs ∈ set (prefixes ys) ⟷ prefix xs ys
proof (induct xs arbitrary: ys)
  case Nil
  then show ?case by (cases ys) auto
next
  case (Cons a xs)
  then show ?case by (cases ys) auto
qed

lemma length-prefixes[simp]: length (prefixes xs) = length xs+1
  by (induction xs) auto

lemma distinct-prefixes [intro]: distinct (prefixes xs)
  by (induction xs) (auto simp: distinct-map)
```

lemma *prefixes-snoc* [*simp*]: *prefixes* (*xs*@[*x*]) = *prefixes xs* @ [*xs*@[*x*]]
by (*induction xs*) *auto*

lemma *prefixes-not-Nil* [*simp*]: *prefixes xs* ≠ []
by (*cases xs*) *auto*

lemma *hd-prefixes* [*simp*]: *hd* (*prefixes xs*) = []
by (*cases xs*) *simp-all*

lemma *last-prefixes* [*simp*]: *last* (*prefixes xs*) = *xs*
by (*induction xs*) (*simp-all add*: *last-map*)

lemma *prefixes-append*:
prefixes (*xs* @ *ys*) = *prefixes xs* @ *map* (λ*ys'*. *xs* @ *ys'*) (*tl* (*prefixes ys*))
proof (*induction xs*)
case *Nil*
thus *?case* **by** (*cases ys*) *auto*
qed *simp-all*

lemma *prefixes-eq-snoc*:
prefixes ys = *xs* @ [*x*] ⟷
(*ys* = [] ∧ *xs* = [] ∨ (∃ *z zs*. *ys* = *zs*@[*z*] ∧ *xs* = *prefixes zs*)) ∧ *x* = *ys*
by (*cases ys rule*: *rev-cases*) *auto*

lemma *prefixes-tailrec* [*code*]:
prefixes xs = *rev* (*snd* (*foldl* (λ(*acc1*, *acc2*) *x*. (*x*#*acc1*, *rev* (*x*#*acc1*)#*acc2*))
([],[[]]) *xs*))
proof −
have *foldl* (λ(*acc1*, *acc2*) *x*. (*x*#*acc1*, *rev* (*x*#*acc1*)#*acc2*)) (*ys*, *rev ys* # *zs*)
xs =
(*rev xs* @ *ys*, *rev* (*map* (λ*as*. *rev ys* @ *as*) (*prefixes xs*)) @ *zs*) **for** *ys zs*
proof (*induction xs arbitrary*: *ys zs*)
case (*Cons x xs ys zs*)
from *Cons.IH*[*of x* # *ys rev ys* # *zs*]
show *?case* **by** (*simp add*: *o-def*)
qed *simp-all*
from *this* [*of* [] []] **show** *?thesis* **by** *simp*
qed

lemma *set-prefixes-eq*: *set* (*prefixes xs*) = {*ys*. *prefix ys xs*}
by *auto*

lemma *card-set-prefixes* [*simp*]: *card* (*set* (*prefixes xs*)) = *Suc* (*length xs*)
by (*subst distinct-card*) *auto*

lemma *set-prefixes-append*:
set (*prefixes* (*xs* @ *ys*)) = *set* (*prefixes xs*) ∪ {*xs* @ *ys'* |*ys'*. *ys'* ∈ *set* (*prefixes ys*)}

by (*subst prefixes-append*, *cases ys*) *auto*

52.4 Longest Common Prefix

definition *Longest-common-prefix* :: *'a list set* ⇒ *'a list* **where**
Longest-common-prefix L = (*ARG-MAX length ps*. ∀ *xs* ∈ *L*. *prefix ps xs*)

lemma *Longest-common-prefix-ex*: *L* ≠ {} ⟹
∃ *ps*. (∀ *xs* ∈ *L*. *prefix ps xs*) ∧ (∀ *qs*. (∀ *xs* ∈ *L*. *prefix qs xs*) ⟶ *size qs* ≤ *size ps*)
(**is** - ⟹ ∃ *ps*. *?P L ps*)
proof(*induction LEAST n*. ∃ *xs* ∈*L*. *n* = *length xs arbitrary*: *L*)
case *0*
have [] ∈ *L* **using** *0.hyps LeastI*[*of* λ*n*. ∃ *xs*∈*L*. *n* = *length xs*] ‹*L* ≠ {}›
by *auto*
hence *?P L* [] **by**(*auto*)
thus *?case* **..**
next
case (*Suc n*)
let *?EX* = λ*n*. ∃ *xs*∈*L*. *n* = *length xs*
obtain *x xs* **where** *xxs*: *x*#*xs* ∈ *L size xs* = *n* **using** *Suc.prems Suc.hyps*(*2*)
by(*metis LeastI-ex*[*of ?EX*] *Suc-length-conv ex-in-conv*)
hence [] ∉ *L* **using** *Suc.hyps*(*2*) **by** *auto*
show *?case*
proof (*cases* ∀ *xs* ∈ *L*. ∃ *ys*. *xs* = *x*#*ys*)
case *True*
let *?L* = {*ys*. *x*#*ys* ∈ *L*}
have *1*: (*LEAST n*. ∃ *xs* ∈ *?L*. *n* = *length xs*) = *n*
using *xxs Suc.prems Suc.hyps*(*2*) *Least-le*[*of ?EX*]
by − (*rule Least-equality*, *fastforce*+)
have *2*: *?L* ≠ {} **using** ‹*x* # *xs* ∈ *L*› **by** *auto*
from *Suc.hyps*(*1*)[*OF 1*[*symmetric*] *2*] **obtain** *ps* **where** *IH*: *?P ?L ps* **..**
{ fix *qs*
assume ∀ *qs*. (∀ *xa*. *x* # *xa* ∈ *L* ⟶ *prefix qs xa*) ⟶ *length qs* ≤ *length ps*
and ∀ *xs*∈*L*. *prefix qs xs*
hence *length* (*tl qs*) ≤ *length ps*
by (*metis Cons-prefix-Cons hd-Cons-tl list.sel*(*2*) *Nil-prefix*)
hence *length qs* ≤ *Suc* (*length ps*) **by** *auto*
}
hence *?P L* (*x*#*ps*) **using** *True IH* **by** *auto*
thus *?thesis* **..**
next
case *False*
then obtain *y ys* **where** *yys*: *x*≠*y y*#*ys* ∈ *L* **using** ‹[] ∉ *L*›
by (*auto*) (*metis list.exhaust*)
have ∀ *qs*. (∀ *xs*∈*L*. *prefix qs xs*) ⟶ *qs* = [] **using** *yys* ‹*x*#*xs* ∈ *L*›
by *auto* (*metis Cons-prefix-Cons prefix-Cons*)
hence *?P L* [] **by** *auto*
thus *?thesis* **..**

qed
qed

lemma *Longest-common-prefix-unique*:
‹∃! *ps*. (∀ *xs* ∈ *L*. *prefix ps xs*) ∧ (∀ *qs*. (∀ *xs* ∈ *L*. *prefix qs xs*) ⟶ *length qs* ≤ *length ps*)›
if ‹*L* ≠ {}›
using *that* **apply** (*rule ex-ex1I*[*OF Longest-common-prefix-ex*])
using *that* **apply** (*auto simp add*: *prefix-def*)
apply (*metis append-eq-append-conv-if order.antisym*)
done

lemma *Longest-common-prefix-eq*:
⟦ *L* ≠ {}; ∀ *xs* ∈ *L*. *prefix ps xs*;
∀ *qs*. (∀ *xs* ∈ *L*. *prefix qs xs*) ⟶ *size qs* ≤ *size ps* ⟧
⟹ *Longest-common-prefix L* = *ps*
unfolding *Longest-common-prefix-def arg-max-def is-arg-max-linorder*
by(*rule some1-equality*[*OF Longest-common-prefix-unique*]) *auto*

lemma *Longest-common-prefix-prefix*:
xs ∈ *L* ⟹ *prefix* (*Longest-common-prefix L*) *xs*
unfolding *Longest-common-prefix-def arg-max-def is-arg-max-linorder*
by(*rule someI2-ex*[*OF Longest-common-prefix-ex*]) *auto*

lemma *Longest-common-prefix-longest*:
L ≠ {} ⟹ ∀ *xs*∈*L*. *prefix ps xs* ⟹ *length ps* ≤ *length*(*Longest-common-prefix L*)
unfolding *Longest-common-prefix-def arg-max-def is-arg-max-linorder*
by(*rule someI2-ex*[*OF Longest-common-prefix-ex*]) *auto*

lemma *Longest-common-prefix-max-prefix*:
L ≠ {} ⟹ ∀ *xs*∈*L*. *prefix ps xs* ⟹ *prefix ps* (*Longest-common-prefix L*)
by(*metis Longest-common-prefix-prefix Longest-common-prefix-longest*
prefix-length-prefix ex-in-conv)

lemma *Longest-common-prefix-Nil*: [] ∈ *L* ⟹ *Longest-common-prefix L* = []
using *Longest-common-prefix-prefix prefix-Nil* **by** *blast*

lemma *Longest-common-prefix-image-Cons*: *L* ≠ {} ⟹
Longest-common-prefix ((#) *x* ‘ *L*) = *x* # *Longest-common-prefix L*
apply(*rule Longest-common-prefix-eq*)
apply(*simp*)
apply (*simp add*: *Longest-common-prefix-prefix*)
apply *simp*
by(*metis Longest-common-prefix-longest*[*of L*] *Cons-prefix-Cons Nitpick.size-list-simp*(*2*)
Suc-le-mono hd-Cons-tl order.strict-implies-order zero-less-Suc)

lemma *Longest-common-prefix-eq-Cons*: **assumes** *L* ≠ {} [] ∉ *L* ∀ *xs*∈*L*. *hd xs* = *x*

shows *Longest-common-prefix L = x # Longest-common-prefix {ys. x#ys ∈ L}*
proof −
have *L = (#) x ' {ys. x#ys ∈ L}* **using** *assms(2,3)*
by (*auto simp*: *image-def*)(*metis hd-Cons-tl*)
thus *?thesis*
by (*metis Longest-common-prefix-image-Cons image-is-empty assms(1)*)
qed

lemma *Longest-common-prefix-eq-Nil*:
⟦*x#ys ∈ L*; *y#zs ∈ L*; *x ≠ y* ⟧ ⟹ *Longest-common-prefix L = []*
by (*metis Longest-common-prefix-prefix list.inject prefix-Cons*)

fun *longest-common-prefix* :: *'a list ⇒ 'a list ⇒ 'a list* **where**
longest-common-prefix (x#xs) (y#ys) =
(*if x=y then x # longest-common-prefix xs ys else []*) |
longest-common-prefix - - = []

lemma *longest-common-prefix-prefix1*:
prefix (longest-common-prefix xs ys) xs
by(*induction xs ys rule*: *longest-common-prefix.induct*) *auto*

lemma *longest-common-prefix-prefix2*:
prefix (longest-common-prefix xs ys) ys
by(*induction xs ys rule*: *longest-common-prefix.induct*) *auto*

lemma *longest-common-prefix-max-prefix*:
⟦ *prefix ps xs*; *prefix ps ys* ⟧
⟹ *prefix ps (longest-common-prefix xs ys)*
by(*induction xs ys arbitrary*: *ps rule*: *longest-common-prefix.induct*)
(*auto simp*: *prefix-Cons*)

52.5 Parallel lists

definition *parallel* :: *'a list ⇒ 'a list ⇒ bool* (**infixl** ∥ *50*)
where (*xs* ∥ *ys*) = (¬ *prefix xs ys* ∧ ¬ *prefix ys xs*)

lemma *parallelI* [*intro*]: ¬ *prefix xs ys* ⟹ ¬ *prefix ys xs* ⟹ *xs* ∥ *ys*
unfolding *parallel-def* **by** *blast*

lemma *parallelE* [*elim*]:
assumes *xs* ∥ *ys*
obtains ¬ *prefix xs ys* ∧ ¬ *prefix ys xs*
using *assms* **unfolding** *parallel-def* **by** *blast*

theorem *prefix-cases*:
obtains *prefix xs ys* | *strict-prefix ys xs* | *xs* ∥ *ys*
unfolding *parallel-def strict-prefix-def* **by** *blast*

lemma *parallel-cancel*: *a#xs* ∥ *a#ys* ⟹ *xs* ∥ *ys*

by (*simp add*: *parallel-def*)

theorem *parallel-decomp*:
xs ‖ *ys* ⟹ ∃ *as b bs c cs*. *b* ≠ *c* ∧ *xs* = *as* @ *b* # *bs* ∧ *ys* = *as* @ *c* # *cs*
proof (*induct rule*: *list-induct2'*, *blast*, *force*, *force*)
case (*4 x xs y ys*)
then show *?case*
proof (*cases x* ≠ *y*, *blast*)
assume ¬ *x* ≠ *y* **hence** *x* = *y* **by** *blast*
then show *?thesis*
using *4.hyps*[*OF parallel-cancel*[*OF 4.prems*[*folded* ‹*x* = *y*›]]]
by (*meson Cons-eq-appendI*)
qed
qed

lemma *parallel-append*: *a* ‖ *b* ⟹ *a* @ *c* ‖ *b* @ *d*
apply (*rule parallelI*)
apply (*erule parallelE*, *erule conjE*,
induct rule: *not-prefix-induct*, *simp*+)+
done

lemma *parallel-appendI*: *xs* ‖ *ys* ⟹ *x* = *xs* @ *xs'* ⟹ *y* = *ys* @ *ys'* ⟹ *x* ‖ *y*
by (*simp add*: *parallel-append*)

lemma *parallel-commute*: *a* ‖ *b* ⟷ *b* ‖ *a*
unfolding *parallel-def* **by** *auto*

52.6 Suffix order on lists

definition *suffix* :: *'a list* ⇒ *'a list* ⇒ *bool*
where *suffix xs ys* = (∃ *zs*. *ys* = *zs* @ *xs*)

definition *strict-suffix* :: *'a list* ⇒ *'a list* ⇒ *bool*
where *strict-suffix xs ys* ⟷ *suffix xs ys* ∧ *xs* ≠ *ys*

global-interpretation *suffix-order*: *ordering suffix strict-suffix*
by *standard* (*auto simp*: *suffix-def strict-suffix-def*)

interpretation *suffix-order*: *order suffix strict-suffix*
by *standard* (*auto simp*: *suffix-def strict-suffix-def*)

global-interpretation *suffix-bot*: *ordering-top* ‹λ*xs ys*. *suffix ys xs*› ‹λ*xs ys*. *strict-suffix ys xs*› ‹[]›
by *standard* (*simp add*: *suffix-def*)

interpretation *suffix-bot*: *order-bot Nil suffix strict-suffix*
by *standard* (*simp add*: *suffix-def*)

lemma *suffixI* [*intro?*]: *ys* = *zs* @ *xs* ⟹ *suffix xs ys*

unfolding *suffix-def* **by** *blast*

lemma *suffixE* [*elim?*]:
assumes *suffix xs ys*
obtains *zs* **where** *ys* = *zs* @ *xs*
using *assms* **unfolding** *suffix-def* **by** *blast*

lemma *suffix-tl* [*simp*]: *suffix* (*tl xs*) *xs*
by (*induct xs*) (*auto simp*: *suffix-def*)

lemma *strict-suffix-tl* [*simp*]: *xs* ≠ [] ⟹ *strict-suffix* (*tl xs*) *xs*
by (*induct xs*) (*auto simp*: *strict-suffix-def suffix-def*)

lemma *Nil-suffix* [*simp*]: *suffix* [] *xs*
by (*simp add*: *suffix-def*)

lemma *suffix-Nil* [*simp*]: (*suffix xs* []) = (*xs* = [])
by (*auto simp add*: *suffix-def*)

lemma *suffix-ConsI*: *suffix xs ys* ⟹ *suffix xs* (*y* # *ys*)
by (*auto simp add*: *suffix-def*)

lemma *suffix-ConsD*: *suffix* (*x* # *xs*) *ys* ⟹ *suffix xs ys*
by (*auto simp add*: *suffix-def*)

lemma *suffix-appendI*: *suffix xs ys* ⟹ *suffix xs* (*zs* @ *ys*)
by (*auto simp add*: *suffix-def*)

lemma *suffix-appendD*: *suffix* (*zs* @ *xs*) *ys* ⟹ *suffix xs ys*
by (*auto simp add*: *suffix-def*)

lemma *strict-suffix-set-subset*: *strict-suffix xs ys* ⟹ *set xs* ⊆ *set ys*
by (*auto simp*: *strict-suffix-def suffix-def*)

lemma *set-mono-suffix*: *suffix xs ys* ⟹ *set xs* ⊆ *set ys*
by (*auto simp*: *suffix-def*)

lemma *sorted-antimono-suffix*: *suffix xs ys* ⟹ *sorted ys* ⟹ *sorted xs*
by (*metis sorted-append suffix-def*)

lemma *suffix-ConsD2*: *suffix* (*x* # *xs*) (*y* # *ys*) ⟹ *suffix xs ys*
proof −
assume *suffix* (*x* # *xs*) (*y* # *ys*)
then obtain *zs* **where** *y* # *ys* = *zs* @ *x* # *xs* **..**
then show *?thesis*
by (*induct zs*) (*auto intro*!: *suffix-appendI suffix-ConsI*)
qed

lemma *suffix-to-prefix* [*code*]: *suffix xs ys* ⟷ *prefix* (*rev xs*) (*rev ys*)

proof
assume *suffix xs ys*
then obtain *zs* **where** *ys* = *zs* @ *xs* **..**
then have *rev ys* = *rev xs* @ *rev zs* **by** *simp*
then show *prefix* (*rev xs*) (*rev ys*) **..**
next
assume *prefix* (*rev xs*) (*rev ys*)
then obtain *zs* **where** *rev ys* = *rev xs* @ *zs* **..**
then have *rev* (*rev ys*) = *rev zs* @ *rev* (*rev xs*) **by** *simp*
then have *ys* = *rev zs* @ *xs* **by** *simp*
then show *suffix xs ys* **..**
qed

lemma *strict-suffix-to-prefix* [*code*]: *strict-suffix xs ys* ⟷ *strict-prefix* (*rev xs*) (*rev ys*)
by (*auto simp*: *suffix-to-prefix strict-suffix-def strict-prefix-def*)

lemma *distinct-suffix*: *distinct ys* ⟹ *suffix xs ys* ⟹ *distinct xs*
by (*clarsimp elim*!: *suffixE*)

lemma *map-mono-suffix*: *suffix xs ys* ⟹ *suffix* (*map f xs*) (*map f ys*)
by (*auto elim*!: *suffixE intro*: *suffixI*)

lemma *filter-mono-suffix*: *suffix xs ys* ⟹ *suffix* (*filter P xs*) (*filter P ys*)
by (*auto simp*: *suffix-def*)

lemma *suffix-drop*: *suffix* (*drop n as*) *as*
unfolding *suffix-def* **by** (*metis append-take-drop-id*)

lemma *suffix-dropWhile*: *suffix* (*dropWhile P xs*) *xs*
unfolding *suffix-def* **by** (*metis takeWhile-dropWhile-id*)

lemma *suffix-take*: *suffix xs ys* ⟹ *ys* = *take* (*length ys* − *length xs*) *ys* @ *xs*
by (*auto elim*!: *suffixE*)

lemma *strict-suffix-reflclp-conv*: *strict-suffix*$^{==}$ = *suffix*
by (*intro ext*) (*auto simp*: *suffix-def strict-suffix-def*)

lemma *suffix-lists*: *suffix xs ys* ⟹ *ys* ∈ *lists A* ⟹ *xs* ∈ *lists A*
unfolding *suffix-def* **by** *auto*

lemma *suffix-snoc* [*simp*]: *suffix xs* (*ys* @ [*y*]) ⟷ *xs* = [] ∨ (∃ *zs*. *xs* = *zs* @ [*y*] ∧ *suffix zs ys*)
by (*cases xs rule*: *rev-cases*) (*auto simp*: *suffix-def*)

lemma *snoc-suffix-snoc* [*simp*]: *suffix* (*xs* @ [*x*]) (*ys* @ [*y*]) = (*x* = *y* ∧ *suffix xs ys*)
by (*auto simp add*: *suffix-def*)

lemma *same-suffix-suffix* [*simp*]: *suffix* (*ys* @ *xs*) (*zs* @ *xs*) = *suffix ys zs*
by (*simp add*: *suffix-to-prefix*)

lemma *same-suffix-nil* [*simp*]: *suffix* (*ys* @ *xs*) *xs* = (*ys* = [])
by (*simp add*: *suffix-to-prefix*)

theorem *suffix-Cons*: *suffix xs* (*y* # *ys*) ⟷ *xs* = *y* # *ys* ∨ *suffix xs ys*
unfolding *suffix-def* **by** (*auto simp*: *Cons-eq-append-conv*)

theorem *suffix-append*:
suffix xs (*ys* @ *zs*) ⟷ *suffix xs zs* ∨ (∃ *xs'*. *xs* = *xs'* @ *zs* ∧ *suffix xs' ys*)
by (*auto simp*: *suffix-def append-eq-append-conv2*)

theorem *suffix-length-le*: *suffix xs ys* ⟹ *length xs* ≤ *length ys*
by (*auto simp add*: *suffix-def*)

lemma *suffix-same-cases*:
suffix (*xs*$_1$::'*a list*) *ys* ⟹ *suffix xs*$_2$ *ys* ⟹ *suffix xs*$_1$ *xs*$_2$ ∨ *suffix xs*$_2$ *xs*$_1$
unfolding *suffix-def* **by** (*force simp*: *append-eq-append-conv2*)

lemma *suffix-length-suffix*:
suffix ps xs ⟹ *suffix qs xs* ⟹ *length ps* ≤ *length qs* ⟹ *suffix ps qs*
by (*auto simp*: *suffix-to-prefix intro*: *prefix-length-prefix*)

lemma *suffix-length-less*: *strict-suffix xs ys* ⟹ *length xs* < *length ys*
by (*auto simp*: *strict-suffix-def suffix-def*)

lemma *suffix-ConsD'*: *suffix* (*x*#*xs*) *ys* ⟹ *strict-suffix xs ys*
by (*auto simp*: *strict-suffix-def suffix-def*)

lemma *drop-strict-suffix*: *strict-suffix xs ys* ⟹ *strict-suffix* (*drop n xs*) *ys*
proof (*induct n arbitrary*: *xs ys*)
case *0*
then show *?case* **by** (*cases ys*) *simp-all*
next
case (*Suc n*)
then show *?case*
by (*cases xs*) (*auto intro*: *Suc dest*: *suffix-ConsD' suffix-order.less-imp-le*)
qed

lemma *suffix-map-rightE*:
assumes *suffix xs* (*map f ys*)
shows ∃ *xs'*. *suffix xs' ys* ∧ *xs* = *map f xs'*
proof −
from *assms* **obtain** *xs'* **where** *xs'*: *map f ys* = *xs'* @ *xs*
by (*auto simp*: *suffix-def*)
define *n* **where** *n* = *length xs'*
have *xs* = *drop n* (*map f ys*)
by (*simp add*: *xs' n-def*)

thus *?thesis*
by (*intro exI*[*of - drop n ys*]) (*auto simp*: *drop-map suffix-drop*)
qed

lemma *suffix-remdups-adj*: *suffix xs ys* $\Longrightarrow$ *suffix* (*remdups-adj xs*) (*remdups-adj ys*)
using *prefix-remdups-adj*[*of rev xs rev ys*]
by (*simp add*: *suffix-to-prefix*)

lemma *not-suffix-cases*:
assumes *pfx*: $\neg$ *suffix ps ls*
obtains
(*c1*) *ps* $\neq$ [] **and** *ls* = []
| (*c2*) *a as x xs* **where** *ps* = *as*@[*a*] **and** *ls* = *xs*@[*x*] **and** *x* = *a* **and** $\neg$ *suffix as xs*
| (*c3*) *a as x xs* **where** *ps* = *as*@[*a*] **and** *ls* = *xs*@[*x*] **and** *x* $\neq$ *a*
proof (*cases ps rule*: *rev-cases*)
case *Nil*
then show *?thesis* **using** *pfx* **by** *simp*
next
case (*snoc as a*)
note *c* = ‹*ps* = *as*@[*a*]›
show *?thesis*
proof (*cases ls rule*: *rev-cases*)
case *Nil* **then show** *?thesis* **by** (*metis append-Nil2 pfx c1 same-suffix-nil*)
next
case (*snoc xs x*)
show *?thesis*
proof (*cases x* = *a*)
case *True*
have $\neg$ *suffix as xs* **using** *pfx c snoc True* **by** *simp*
with *c snoc True* **show** *?thesis* **by** (*rule c2*)
next
case *False*
with *c snoc* **show** *?thesis* **by** (*rule c3*)
qed
qed
qed

lemma *not-suffix-induct* [*consumes 1*, *case-names Nil Neq Eq*]:
assumes *np*: $\neg$ *suffix ps ls*
and *base*: $\bigwedge x\ xs.\ P\ (xs@[x])\ []$
and *r1*: $\bigwedge x\ xs\ y\ ys.\ x \neq y \Longrightarrow P\ (xs@[x])\ (ys@[y])$
and *r2*: $\bigwedge x\ xs\ y\ ys.\ [\![\ x = y;\ \neg\ suffix\ xs\ ys;\ P\ xs\ ys\ ]\!] \Longrightarrow P\ (xs@[x])\ (ys@[y])$
shows *P ps ls* **using** *np*
proof (*induct ls arbitrary*: *ps rule*: *rev-induct*)
case *Nil*
then show *?case* **by** (*cases ps rule*: *rev-cases*) (*auto intro*: *base*)
next

case (*snoc y ys ps*)
then have *npfx*: ¬ *suffix ps* (*ys* @ [*y*]) **by** *simp*
then obtain *x xs* **where** *pv*: *ps* = *xs* @ [*x*]
by (*rule not-suffix-cases*) *auto*
show *?case* **by** (*metis snoc.hyps snoc-suffix-snoc npfx pv r1 r2*)
qed

lemma *parallelD1*: *x* ∥ *y* ⟹ ¬ *prefix x y*
by *blast*

lemma *parallelD2*: *x* ∥ *y* ⟹ ¬ *prefix y x*
by *blast*

lemma *parallel-Nil1* [*simp*]: ¬ *x* ∥ []
unfolding *parallel-def* **by** *simp*

lemma *parallel-Nil2* [*simp*]: ¬ [] ∥ *x*
unfolding *parallel-def* **by** *simp*

lemma *Cons-parallelI1*: *a* ≠ *b* ⟹ *a* # *as* ∥ *b* # *bs*
by *auto*

lemma *Cons-parallelI2*: ⟦ *a* = *b*; *as* ∥ *bs* ⟧ ⟹ *a* # *as* ∥ *b* # *bs*
by (*metis Cons-prefix-Cons parallelE parallelI*)

lemma *not-equal-is-parallel*:
assumes *neq*: *xs* ≠ *ys*
and *len*: *length xs* = *length ys*
shows *xs* ∥ *ys*
using *len neq*
proof (*induct rule*: *list-induct2*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a as b bs*)
have *ih*: *as* ≠ *bs* ⟹ *as* ∥ *bs* **by** *fact*
show *?case*
proof (*cases a* = *b*)
case *True*
then have *as* ≠ *bs* **using** *Cons* **by** *simp*
then show *?thesis* **by** (*rule Cons-parallelI2* [*OF True ih*])
next
case *False*
then show *?thesis* **by** (*rule Cons-parallelI1*)
qed
qed

52.7 Suffixes

primrec *suffixes* **where**
  suffixes [] = [[]]
| *suffixes* (*x*#*xs*) = *suffixes xs* @ [*x* # *xs*]

lemma *in-set-suffixes* [*simp*]: *xs* ∈ *set* (*suffixes ys*) ⟷ *suffix xs ys*
  by (*induction ys*) (*auto simp*: *suffix-def Cons-eq-append-conv*)

lemma *distinct-suffixes* [*intro*]: *distinct* (*suffixes xs*)
  by (*induction xs*) (*auto simp*: *suffix-def*)

lemma *length-suffixes* [*simp*]: *length* (*suffixes xs*) = *Suc* (*length xs*)
  by (*induction xs*) *auto*

lemma *suffixes-snoc* [*simp*]: *suffixes* (*xs* @ [*x*]) = [] # *map* (λ*ys*. *ys* @ [*x*]) (*suffixes xs*)
  by (*induction xs*) *auto*

lemma *suffixes-not-Nil* [*simp*]: *suffixes xs* ≠ []
  by (*cases xs*) *auto*

lemma *hd-suffixes* [*simp*]: *hd* (*suffixes xs*) = []
  by (*induction xs*) *simp-all*

lemma *last-suffixes* [*simp*]: *last* (*suffixes xs*) = *xs*
  by (*cases xs*) *simp-all*

lemma *suffixes-append*:
  suffixes (*xs* @ *ys*) = *suffixes ys* @ *map* (λ*xs*′. *xs*′ @ *ys*) (*tl* (*suffixes xs*))
proof (*induction ys rule*: *rev-induct*)
  case *Nil*
  thus *?case* **by** (*cases xs rule*: *rev-cases*) *auto*
next
  case (*snoc y ys*)
  show *?case*
    by (*simp only*: *append.assoc* [*symmetric*] *suffixes-snoc snoc.IH*) *simp*
qed

lemma *suffixes-eq-snoc*:
  suffixes ys = *xs* @ [*x*] ⟷
    (*ys* = [] ∧ *xs* = [] ∨ (∃ *z zs*. *ys* = *z*#*zs* ∧ *xs* = *suffixes zs*)) ∧ *x* = *ys*
  by (*cases ys*) *auto*

lemma *suffixes-tailrec* [*code*]:
  suffixes xs = *rev* (*snd* (*foldl* (λ(*acc1*, *acc2*) *x*. (*x*#*acc1*, (*x*#*acc1*)#*acc2*)) ([],[[]]) (*rev xs*)))
proof −
  have *foldl* (λ(*acc1*, *acc2*) *x*. (*x*#*acc1*, (*x*#*acc1*)#*acc2*)) (*ys*, *ys* # *zs*) (*rev xs*) =

```
      (xs @ ys, rev (map (λas. as @ ys) (suffixes xs)) @ zs) for ys zs
  proof (induction xs arbitrary: ys zs)
    case (Cons x xs ys zs)
    from Cons.IH[of ys zs]
      show ?case by (simp add: o-def case-prod-unfold)
  qed simp-all
  from this [of [] []] show ?thesis by simp
qed

lemma set-suffixes-eq: set (suffixes xs) = {ys. suffix ys xs}
  by auto

lemma card-set-suffixes [simp]: card (set (suffixes xs)) = Suc (length xs)
  by (subst distinct-card) auto

lemma set-suffixes-append:
  set (suffixes (xs @ ys)) = set (suffixes ys) ∪ {xs' @ ys |xs'. xs' ∈ set (suffixes
xs)}
  by (subst suffixes-append, cases xs rule: rev-cases) auto


lemma suffixes-conv-prefixes: suffixes xs = map rev (prefixes (rev xs))
  by (induction xs) auto

lemma prefixes-conv-suffixes: prefixes xs = map rev (suffixes (rev xs))
  by (induction xs) auto

lemma prefixes-rev: prefixes (rev xs) = map rev (suffixes xs)
  by (induction xs) auto

lemma suffixes-rev: suffixes (rev xs) = map rev (prefixes xs)
  by (induction xs) auto
```

52.8 Homeomorphic embedding on lists

```
inductive list-emb :: ('a ⇒ 'a ⇒ bool) ⇒ 'a list ⇒ 'a list ⇒ bool
  for P :: ('a ⇒ 'a ⇒ bool)
where
  list-emb-Nil [intro, simp]: list-emb P [] ys
| list-emb-Cons [intro] : list-emb P xs ys ⟹ list-emb P xs (y#ys)
| list-emb-Cons2 [intro]: P x y ⟹ list-emb P xs ys ⟹ list-emb P (x#xs) (y#ys)

lemma list-emb-mono:
  assumes ⋀x y. P x y ⟶ Q x y
  shows list-emb P xs ys ⟶ list-emb Q xs ys
proof
  assume list-emb P xs ys
  then show list-emb Q xs ys by (induct) (auto simp: assms)
qed
```

lemma *list-emb-Nil2* [*simp*]:
assumes *list-emb P xs* [] **shows** *xs* = []
using *assms* **by** (*cases rule*: *list-emb.cases*) *auto*

lemma *list-emb-refl*:
assumes $\bigwedge x.\ x \in set\ xs \Longrightarrow P\ x\ x$
shows *list-emb P xs xs*
using *assms* **by** (*induct xs*) *auto*

lemma *list-emb-Cons-Nil* [*simp*]: *list-emb P* (*x*#*xs*) [] = *False*
proof −
{ **assume** *list-emb P* (*x*#*xs*) []
from *list-emb-Nil2* [*OF this*] **have** *False* **by** *simp*
} **moreover** {
assume *False*
then have *list-emb P* (*x*#*xs*) [] **by** *simp*
} **ultimately show** *?thesis* **by** *blast*
qed

lemma *list-emb-append2* [*intro*]: *list-emb P xs ys* $\Longrightarrow$ *list-emb P xs* (*zs* @ *ys*)
by (*induct zs*) *auto*

lemma *list-emb-prefix* [*intro*]:
assumes *list-emb P xs ys* **shows** *list-emb P xs* (*ys* @ *zs*)
using *assms*
by (*induct arbitrary*: *zs*) *auto*

lemma *list-emb-ConsD*:
assumes *list-emb P* (*x*#*xs*) *ys*
shows $\exists us\ v\ vs.\ ys = us\ @\ v\ \#\ vs \wedge P\ x\ v \wedge$ *list-emb P xs vs*
using *assms*
proof (*induct x* ≡ *x* # *xs ys arbitrary*: *x xs*)
case *list-emb-Cons*
then show *?case* **by** (*metis append-Cons*)
next
case (*list-emb-Cons2 x y xs ys*)
then show *?case* **by** *blast*
qed

lemma *list-emb-appendD*:
assumes *list-emb P* (*xs* @ *ys*) *zs*
shows $\exists us\ vs.\ zs = us\ @\ vs \wedge$ *list-emb P xs us* $\wedge$ *list-emb P ys vs*
using *assms*
proof (*induction xs arbitrary*: *ys zs*)
case *Nil* **then show** *?case* **by** *auto*
next
case (*Cons x xs*)
then obtain *us v vs* **where**

zs: $zs = us \mathbin{@} v \# vs$ **and** *p*: $P\ x\ v$ **and** *lh*: *list-emb* $P\ (xs \mathbin{@} ys)\ vs$
by (*auto dest*: *list-emb-ConsD*)
obtain $sk_0 :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list$ **and** $sk_1 :: 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list$
where
sk: $\forall x_0\ x_1.\ \neg$ *list-emb* $P\ (xs \mathbin{@} x_0)\ x_1 \vee sk_0\ x_0\ x_1 \mathbin{@} sk_1\ x_0\ x_1 = x_1 \wedge$ *list-emb* $P\ xs\ (sk_0\ x_0\ x_1) \wedge$ *list-emb* $P\ x_0\ (sk_1\ x_0\ x_1)$
using *Cons*(*1*) **by** (*metis* (*no-types*))
hence $\forall x_2.$ *list-emb* $P\ (x \# xs)\ (x_2 \mathbin{@} v \# sk_0\ ys\ vs)$ **using** *p lh* **by** *auto*
thus *?case* **using** *lh zs sk* **by** (*metis* (*no-types*) *append-Cons append-assoc*)
qed

lemma *list-emb-strict-suffix*:
assumes *list-emb P xs ys* **and** *strict-suffix ys zs*
shows *list-emb P xs zs*
using *assms*(*2*) **and** *list-emb-append2* [*OF assms*(*1*)] **by** (*auto simp*: *strict-suffix-def suffix-def*)

lemma *list-emb-suffix*:
assumes *list-emb P xs ys* **and** *suffix ys zs*
shows *list-emb P xs zs*
using *assms* **and** *list-emb-strict-suffix*
unfolding *strict-suffix-reflclp-conv*[*symmetric*] **by** *auto*

lemma *list-emb-length*: *list-emb P xs ys* $\Longrightarrow$ *length xs* $\leq$ *length ys*
by (*induct rule*: *list-emb.induct*) *auto*

lemma *list-emb-trans*:
assumes $\bigwedge x\ y\ z.\ [\![x \in set\ xs;\ y \in set\ ys;\ z \in set\ zs;\ P\ x\ y;\ P\ y\ z]\!] \Longrightarrow P\ x\ z$
shows $[\![$*list-emb P xs ys*; *list-emb P ys zs*$]\!] \Longrightarrow$ *list-emb P xs zs*
proof −
assume *list-emb P xs ys* **and** *list-emb P ys zs*
then show *list-emb P xs zs* **using** *assms*
proof (*induction arbitrary*: *zs*)
case *list-emb-Nil* **show** *?case* **by** *blast*
next
case (*list-emb-Cons xs ys y*)
from *list-emb-ConsD* [*OF* ‹*list-emb P* (*y*#*ys*) *zs*›] **obtain** *us v vs*
where *zs*: $zs = us \mathbin{@} v \# vs$ **and** $P^{==}\ y\ v$ **and** *list-emb P ys vs* **by** *blast*
then have *list-emb P ys* (*v*#*vs*) **by** *blast*
then have *list-emb P ys zs* **unfolding** *zs* **by** (*rule list-emb-append2*)
from *list-emb-Cons.IH* [*OF this*] **and** *list-emb-Cons.prems* **show** *?case* **by** *auto*
next
case (*list-emb-Cons2 x y xs ys*)
from *list-emb-ConsD* [*OF* ‹*list-emb P* (*y*#*ys*) *zs*›] **obtain** *us v vs*
where *zs*: $zs = us \mathbin{@} v \# vs$ **and** *P y v* **and** *list-emb P ys vs* **by** *blast*
with *list-emb-Cons2* **have** *list-emb P xs vs* **by** *auto*
moreover have *P x v*
proof −
from *zs* **have** $v \in set\ zs$ **by** *auto*

moreover have $x \in set\ (x\#xs)$ **and** $y \in set\ (y\#ys)$ **by** *simp-all*
ultimately show *?thesis*
using ‹$P\ x\ y$› **and** ‹$P\ y\ v$› **and** *list-emb-Cons2*
by *blast*
qed
ultimately have *list-emb P* $(x\#xs)$ $(v\#vs)$ **by** *blast*
then show *?case* **unfolding** *zs* **by** (*rule list-emb-append2*)
qed
qed

lemma *list-emb-set*:
assumes *list-emb P xs ys* **and** $x \in set\ xs$
obtains *y* **where** $y \in set\ ys$ **and** *P x y*
using *assms* **by** (*induct*) *auto*

lemma *list-emb-Cons-iff1* [*simp*]:
assumes *P x y*
shows *list-emb P* $(x\#xs)$ $(y\#ys) \longleftrightarrow$ *list-emb P xs ys*
using *assms* **by** (*subst list-emb.simps*) (*auto dest*: *list-emb-ConsD*)

lemma *list-emb-Cons-iff2* [*simp*]:
assumes $\neg P\ x\ y$
shows *list-emb P* $(x\#xs)$ $(y\#ys) \longleftrightarrow$ *list-emb P* $(x\#xs)$ *ys*
using *assms* **by** (*subst list-emb.simps*) *auto*

lemma *list-emb-code* [*code*]:
list-emb P [] *ys* $\longleftrightarrow$ *True*
list-emb P $(x\#xs)$ [] $\longleftrightarrow$ *False*
list-emb P $(x\#xs)$ $(y\#ys) \longleftrightarrow$ (*if P x y then list-emb P xs ys else list-emb P* $(x\#xs)$ *ys*)
by *simp-all*

52.9 Subsequences (special case of homeomorphic embedding)

abbreviation *subseq* :: *'a list* ⇒ *'a list* ⇒ *bool*
where *subseq xs ys* ≡ *list-emb* (=) *xs ys*

definition *strict-subseq* **where** *strict-subseq xs ys* $\longleftrightarrow$ $xs \neq ys \wedge$ *subseq xs ys*

lemma *subseq-Cons2*: *subseq xs ys* $\Longrightarrow$ *subseq* $(x\#xs)$ $(x\#ys)$ **by** *auto*

lemma *subseq-same-length*:
assumes *subseq xs ys* **and** *length xs* = *length ys* **shows** *xs* = *ys*
using *assms* **by** (*induct*) (*auto dest*: *list-emb-length*)

lemma *not-subseq-length* [*simp*]: *length ys* < *length xs* $\Longrightarrow \neg$ *subseq xs ys*
by (*metis list-emb-length linorder-not-less*)

lemma *subseq-Cons'*: *subseq* (*x*#*xs*) *ys* ⟹ *subseq xs ys*
by (*induct xs*, *simp*, *blast dest*: *list-emb-ConsD*)

lemma *subseq-Cons2'*:
assumes *subseq* (*x*#*xs*) (*x*#*ys*) **shows** *subseq xs ys*
using *assms* **by** (*cases*) (*rule subseq-Cons'*)

lemma *subseq-Cons2-neq*:
assumes *subseq* (*x*#*xs*) (*y*#*ys*)
shows *x* ≠ *y* ⟹ *subseq* (*x*#*xs*) *ys*
using *assms* **by** (*cases*) *auto*

lemma *subseq-Cons2-iff* [*simp*]:
subseq (*x*#*xs*) (*y*#*ys*) = (*if x* = *y then subseq xs ys else subseq* (*x*#*xs*) *ys*)
by *simp*

lemma *subseq-append'*: *subseq* (*zs* @ *xs*) (*zs* @ *ys*) ⟷ *subseq xs ys*
by (*induct zs*) *simp-all*

global-interpretation *subseq-order*: *ordering subseq strict-subseq*
proof
show ‹*subseq xs xs*› **for** *xs* :: ‹'*a list*›
using *refl* **by** (*rule list-emb-refl*)
show ‹*subseq xs zs*› **if** ‹*subseq xs ys*› **and** ‹*subseq ys zs*›
for *xs ys zs* :: ‹'*a list*›
using *trans* [*OF refl*] *that* **by** (*rule list-emb-trans*) *simp*
show ‹*xs* = *ys*› **if** ‹*subseq xs ys*› **and** ‹*subseq ys xs*›
for *xs ys* :: ‹'*a list*›
using *that* **proof** *induction*
case *list-emb-Nil*
from *list-emb-Nil2* [*OF this*] **show** *?case* **by** *simp*
next
case *list-emb-Cons2*
then show *?case* **by** *simp*
next
case *list-emb-Cons*
hence *False* **using** *subseq-Cons'* **by** *fastforce*
then show *?case* **..**
qed
show ‹*strict-subseq xs ys* ⟷ *subseq xs ys* ∧ *xs* ≠ *ys*›
for *xs ys* :: ‹'*a list*›
by (*auto simp*: *strict-subseq-def*)
qed

interpretation *subseq-order*: *order subseq strict-subseq*
by (*rule ordering-orderI*) *standard*

lemma *in-set-subseqs* [*simp*]: *xs* ∈ *set* (*subseqs ys*) ⟷ *subseq xs ys*
proof

```
  assume xs ∈ set (subseqs ys)
  thus subseq xs ys
    by (induction ys arbitrary: xs) (auto simp: Let-def)
next
  have [simp]: [] ∈ set (subseqs ys) for ys :: 'a list
    by (induction ys) (auto simp: Let-def)
  assume subseq xs ys
  thus xs ∈ set (subseqs ys)
    by (induction xs ys rule: list-emb.induct) (auto simp: Let-def)
qed

lemma set-subseqs-eq: set (subseqs ys) = {xs. subseq xs ys}
  by auto

lemma subseq-append-le-same-iff: subseq (xs @ ys) ys ⟷ xs = []
  by (auto dest: list-emb-length)

lemma subseq-singleton-left: subseq [x] ys ⟷ x ∈ set ys
  by (fastforce dest: list-emb-ConsD split-list-last)

lemma list-emb-append-mono:
  ⟦ list-emb P xs xs'; list-emb P ys ys' ⟧ ⟹ list-emb P (xs@ys) (xs'@ys')
  by (induct rule: list-emb.induct) auto

lemma prefix-imp-subseq [intro]: prefix xs ys ⟹ subseq xs ys
  by (auto simp: prefix-def)

lemma suffix-imp-subseq [intro]: suffix xs ys ⟹ subseq xs ys
  by (auto simp: suffix-def)
```

52.10 Appending elements

```
lemma subseq-append [simp]:
  subseq (xs @ zs) (ys @ zs) ⟷ subseq xs ys (is ?l = ?r)
proof
  { fix xs' ys' xs ys zs :: 'a list assume subseq xs' ys'
    then have xs' = xs @ zs ∧ ys' = ys @ zs ⟶ subseq xs ys
    proof (induct arbitrary: xs ys zs)
      case list-emb-Nil show ?case by simp
    next
      case (list-emb-Cons xs' ys' x)
      { assume ys=[] then have ?case using list-emb-Cons(1) by auto }
      moreover
      { fix us assume ys = x#us
      then have ?case using list-emb-Cons(2) by(simp add: list-emb.list-emb-Cons)
}
      ultimately show ?case by (auto simp:Cons-eq-append-conv)
    next
      case (list-emb-Cons2 x y xs' ys')
```

{ assume *xs*=[] **then have** *?case* **using** *list-emb-Cons2*(*1*) **by** *auto* **}**
moreover
{ fix *us vs* **assume** *xs*=*x*#*us ys*=*x*#*vs* **then have** *?case* **using** *list-emb-Cons2*
by *auto***}**
moreover
{ fix *us* **assume** *xs*=*x*#*us ys*=[] **then have** *?case* **using** *list-emb-Cons2*(*2*)
by *bestsimp* **}**
ultimately show *?case* **using** ‹(=) *x y*› **by** (*auto simp*: *Cons-eq-append-conv*)
qed }
moreover assume *?l*
ultimately show *?r* **by** *blast*
next
assume *?r* **then show** *?l* **by** (*metis list-emb-append-mono subseq-order.order-refl*)
qed

lemma *subseq-append-iff*:
subseq xs (*ys* @ *zs*) ⟷ (∃ *xs1 xs2*. *xs* = *xs1* @ *xs2* ∧ *subseq xs1 ys* ∧ *subseq xs2 zs*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs* **thus** *?rhs*
proof (*induction xs ys* @ *zs arbitrary*: *ys zs rule*: *list-emb.induct*)
case (*list-emb-Cons xs ws y ys zs*)
from *list-emb-Cons*(*2*)[*of tl ys zs*] **and** *list-emb-Cons*(*2*)[*of* [] *tl zs*] **and** *list-emb-Cons*(*1*,*3*)
show *?case* **by** (*cases ys*) *auto*
next
case (*list-emb-Cons2 x y xs ws ys zs*)
from *list-emb-Cons2*(*3*)[*of tl ys zs*] **and** *list-emb-Cons2*(*3*)[*of* [] *tl zs*]
and *list-emb-Cons2*(*1*,*2*,*4*)
show *?case* **by** (*cases ys*) (*auto simp*: *Cons-eq-append-conv*)
qed *auto*
qed (*auto intro*: *list-emb-append-mono*)

lemma *subseq-appendE* [*case-names append*]:
assumes *subseq xs* (*ys* @ *zs*)
obtains *xs1 xs2* **where** *xs* = *xs1* @ *xs2 subseq xs1 ys subseq xs2 zs*
using *assms* **by** (*subst* (*asm*) *subseq-append-iff*) *auto*

lemma *subseq-drop-many*: *subseq xs ys* ⟹ *subseq xs* (*zs* @ *ys*)
by (*induct zs*) *auto*

lemma *subseq-rev-drop-many*: *subseq xs ys* ⟹ *subseq xs* (*ys* @ *zs*)
by (*metis append-Nil2 list-emb-Nil list-emb-append-mono*)

52.11 Relation to standard list operations

lemma *subseq-map*:
assumes *subseq xs ys* **shows** *subseq* (*map f xs*) (*map f ys*)
using *assms* **by** (*induct*) *auto*

lemma *subseq-filter-left* [*simp*]: *subseq* (*filter P xs*) *xs*
by (*induct xs*) *auto*

lemma *subseq-filter* [*simp*]:
assumes *subseq xs ys* **shows** *subseq* (*filter P xs*) (*filter P ys*)
using *assms* **by** *induct auto*

lemma *subseq-conv-nths*:
subseq xs ys ⟷ (∃ *N*. *xs* = *nths ys N*) (**is** *?L* = *?R*)
proof
assume *?L*
then show *?R*
proof (*induct*)
case *list-emb-Nil* **show** *?case* **by** (*metis nths-empty*)
next
case (*list-emb-Cons xs ys x*)
then obtain *N* **where** *xs* = *nths ys N* **by** *blast*
then have *xs* = *nths* (*x*#*ys*) (*Suc* ‘ *N*)
by (*clarsimp simp add*: *nths-Cons inj-image-mem-iff*)
then show *?case* **by** *blast*
next
case (*list-emb-Cons2 x y xs ys*)
then obtain *N* **where** *xs* = *nths ys N* **by** *blast*
then have *x*#*xs* = *nths* (*x*#*ys*) (*insert 0* (*Suc* ‘ *N*))
by (*clarsimp simp add*: *nths-Cons inj-image-mem-iff*)
moreover from *list-emb-Cons2* **have** *x* = *y* **by** *simp*
ultimately show *?case* **by** *blast*
qed
next
assume *?R*
then obtain *N* **where** *xs* = *nths ys N* **..**
moreover have *subseq* (*nths ys N*) *ys*
proof (*induct ys arbitrary*: *N*)
case *Nil* **show** *?case* **by** *simp*
next
case *Cons* **then show** *?case* **by** (*auto simp*: *nths-Cons*)
qed
ultimately show *?L* **by** *simp*
qed

52.12 Contiguous sublists

52.12.1 *sublist*

definition *sublist* :: *'a list* ⇒ *'a list* ⇒ *bool* **where**
sublist xs ys = (∃ *ps ss*. *ys* = *ps* @ *xs* @ *ss*)

definition *strict-sublist* :: *'a list* ⇒ *'a list* ⇒ *bool* **where**
strict-sublist xs ys ⟷ *sublist xs ys* ∧ *xs* ≠ *ys*

interpretation *sublist-order*: *order sublist strict-sublist*
proof
fix *xs ys zs* :: *'a list*
assume *sublist xs ys sublist ys zs*
then obtain *xs1 xs2 ys1 ys2* **where** *ys = xs1 @ xs @ xs2 zs = ys1 @ ys @ ys2*
by (*auto simp*: *sublist-def*)
hence *zs = (ys1 @ xs1) @ xs @ (xs2 @ ys2)* **by** *simp*
thus *sublist xs zs* **unfolding** *sublist-def* **by** *blast*
next
fix *xs ys* :: *'a list*
{
assume *sublist xs ys sublist ys xs*
then obtain *as bs cs ds*
where *xs*: *xs = as @ ys @ bs* **and** *ys*: *ys = cs @ xs @ ds*
by (*auto simp*: *sublist-def*)
have *xs = as @ cs @ xs @ ds @ bs* **by** (*subst xs*, *subst ys*) *auto*
also have *length ... = length as + length cs + length xs + length bs + length ds*
by *simp*
finally have *as = []* *bs = []* **by** *simp-all*
with *xs* **show** *xs = ys* **by** *simp*
}
thus *strict-sublist xs ys* ⟷ (*sublist xs ys* ∧ ¬*sublist ys xs*)
by (*auto simp*: *strict-sublist-def*)
qed (*auto simp*: *strict-sublist-def sublist-def intro*: *exI*[*of* - []])

lemma *sublist-Nil-left* [*simp*, *intro*]: *sublist [] ys*
by (*auto simp*: *sublist-def*)

lemma *sublist-Cons-Nil* [*simp*]: ¬*sublist* (*x*#*xs*) []
by (*auto simp*: *sublist-def*)

lemma *sublist-Nil-right* [*simp*]: *sublist xs []* ⟷ *xs = []*
by (*cases xs*) *auto*

lemma *sublist-appendI* [*simp*, *intro*]: *sublist xs* (*ps @ xs @ ss*)
by (*auto simp*: *sublist-def*)

lemma *sublist-append-leftI* [*simp*, *intro*]: *sublist xs* (*ps @ xs*)
by (*auto simp*: *sublist-def intro*: *exI*[*of* - []])

lemma *sublist-append-rightI* [*simp*, *intro*]: *sublist xs* (*xs @ ss*)
by (*auto simp*: *sublist-def intro*: *exI*[*of* - []])

lemma *sublist-altdef*: *sublist xs ys* ⟷ (∃ *ys'*. *prefix ys' ys* ∧ *suffix xs ys'*)
proof *safe*
assume *sublist xs ys*
then obtain *ps ss* **where** *ys = ps @ xs @ ss* **by** (*auto simp*: *sublist-def*)

thus $\exists ys'.\ prefix\ ys'\ ys \wedge suffix\ xs\ ys'$
by (*intro exI*[*of* - *ps* @ *xs*] *conjI suffix-appendI*) *auto*
next
fix *ys'*
assume *prefix ys' ys suffix xs ys'*
thus *sublist xs ys* **by** (*auto simp*: *prefix-def suffix-def*)
qed

lemma *sublist-altdef'*: $sublist\ xs\ ys \longleftrightarrow (\exists ys'.\ suffix\ ys'\ ys \wedge prefix\ xs\ ys')$
proof *safe*
assume *sublist xs ys*
then obtain *ps ss* **where** *ys* = *ps* @ *xs* @ *ss* **by** (*auto simp*: *sublist-def*)
thus $\exists ys'.\ suffix\ ys'\ ys \wedge prefix\ xs\ ys'$
by (*intro exI*[*of* - *xs* @ *ss*] *conjI suffixI*) *auto*
next
fix *ys'*
assume *suffix ys' ys prefix xs ys'*
thus *sublist xs ys* **by** (*auto simp*: *prefix-def suffix-def*)
qed

lemma *sublist-Cons-right*: $sublist\ xs\ (y \# ys) \longleftrightarrow prefix\ xs\ (y \# ys) \vee sublist\ xs\ ys$
by (*auto simp*: *sublist-def prefix-def Cons-eq-append-conv*)

lemma *sublist-code* [*code*]:
$sublist\ []\ ys \longleftrightarrow True$
$sublist\ (x \# xs)\ [] \longleftrightarrow False$
$sublist\ (x \# xs)\ (y \# ys) \longleftrightarrow prefix\ (x \# xs)\ (y \# ys) \vee sublist\ (x \# xs)\ ys$
by (*simp-all add*: *sublist-Cons-right*)

lemma *sublist-append*:
$sublist\ xs\ (ys\ @\ zs) \longleftrightarrow$
$sublist\ xs\ ys \vee sublist\ xs\ zs \vee (\exists xs1\ xs2.\ xs = xs1\ @\ xs2 \wedge suffix\ xs1\ ys \wedge prefix\ xs2\ zs)$
by (*auto simp*: *sublist-altdef prefix-append suffix-append*)

lemma *map-mono-sublist*:
assumes *sublist xs ys*
shows *sublist* (*map f xs*) (*map f ys*)
proof −
from *assms* **obtain** *xs1 xs2* **where** *ys*: *ys* = *xs1* @ *xs* @ *xs2*
by (*auto simp*: *sublist-def*)
have *map f ys* = *map f xs1* @ *map f xs* @ *map f xs2*
by (*auto simp*: *ys*)
thus *?thesis*
by (*auto simp*: *sublist-def*)
qed

lemma *sublist-length-le*: $sublist\ xs\ ys \Longrightarrow length\ xs \leq length\ ys$

by (*auto simp add*: *sublist-def*)

lemma *set-mono-sublist*: *sublist xs ys* $\Longrightarrow$ *set xs* $\subseteq$ *set ys*
by (*auto simp add*: *sublist-def*)

lemma *prefix-imp-sublist* [*simp*, *intro*]: *prefix xs ys* $\Longrightarrow$ *sublist xs ys*
by (*auto simp*: *sublist-def prefix-def intro*: *exI*[*of* - []])

lemma *suffix-imp-sublist* [*simp*, *intro*]: *suffix xs ys* $\Longrightarrow$ *sublist xs ys*
by (*auto simp*: *sublist-def suffix-def intro*: *exI*[*of* - []])

lemma *sublist-take* [*simp*, *intro*]: *sublist* (*take n xs*) *xs*
by (*rule prefix-imp-sublist*[*OF take-is-prefix*])

lemma *sublist-takeWhile* [*simp*, *intro*]: *sublist* (*takeWhile P xs*) *xs*
by (*rule prefix-imp-sublist*[*OF takeWhile-is-prefix*])

lemma *sublist-drop* [*simp*, *intro*]: *sublist* (*drop n xs*) *xs*
by (*rule suffix-imp-sublist*[*OF suffix-drop*])

lemma *sublist-dropWhile* [*simp*, *intro*]: *sublist* (*dropWhile P xs*) *xs*
by (*rule suffix-imp-sublist*[*OF suffix-dropWhile*])

lemma *sublist-tl* [*simp*, *intro*]: *sublist* (*tl xs*) *xs*
by (*rule suffix-imp-sublist*) (*simp-all add*: *suffix-drop*)

lemma *sublist-butlast* [*simp*, *intro*]: *sublist* (*butlast xs*) *xs*
by (*rule prefix-imp-sublist*) (*simp-all add*: *prefixeq-butlast*)

lemma *sublist-rev* [*simp*]: *sublist* (*rev xs*) (*rev ys*) = *sublist xs ys*
proof
assume *sublist* (*rev xs*) (*rev ys*)
then obtain *as bs* **where** *rev ys* = *as* @ *rev xs* @ *bs*
by (*auto simp*: *sublist-def*)
also have *rev* ... = *rev bs* @ *xs* @ *rev as* **by** *simp*
finally show *sublist xs ys* **by** *simp*
next
assume *sublist xs ys*
then obtain *as bs* **where** *ys* = *as* @ *xs* @ *bs*
by (*auto simp*: *sublist-def*)
also have *rev* ... = *rev bs* @ *rev xs* @ *rev as* **by** *simp*
finally show *sublist* (*rev xs*) (*rev ys*) **by** *simp*
qed

lemma *sublist-rev-left*: *sublist* (*rev xs*) *ys* = *sublist xs* (*rev ys*)
by (*subst sublist-rev* [*symmetric*]) (*simp only*: *rev-rev-ident*)

lemma *sublist-rev-right*: *sublist xs* (*rev ys*) = *sublist* (*rev xs*) *ys*
by (*subst sublist-rev* [*symmetric*]) (*simp only*: *rev-rev-ident*)

lemma *snoc-sublist-snoc*:
sublist (*xs* @ [*x*]) (*ys* @ [*y*]) ⟷
(*x* = *y* ∧ *suffix xs ys* ∨ *sublist* (*xs* @ [*x*]) *ys*)
by (*subst* (*1 2*) *sublist-rev* [*symmetric*])
(*simp del*: *sublist-rev add*: *sublist-Cons-right suffix-to-prefix*)

lemma *sublist-snoc*:
sublist xs (*ys* @ [*y*]) ⟷ *suffix xs* (*ys* @ [*y*]) ∨ *sublist xs ys*
by (*subst* (*1 2*) *sublist-rev* [*symmetric*])
(*simp del*: *sublist-rev add*: *sublist-Cons-right suffix-to-prefix*)

lemma *sublist-imp-subseq* [*intro*]: *sublist xs ys* ⟹ *subseq xs ys*
by (*auto simp*: *sublist-def*)

lemma *sublist-map-rightE*:
assumes *sublist xs* (*map f ys*)
shows ∃*xs'*. *sublist xs' ys* ∧ *xs* = *map f xs'*
proof −
note *takedrop* = *sublist-take sublist-drop*
define *n* **where** *n* = (*length ys* − *length xs*)
from *assms* **obtain** *xs1 xs2* **where** *xs12*: *map f ys* = *xs1* @ *xs* @ *xs2*
by (*auto simp*: *sublist-def*)
define *n* **where** *n* = *length xs1*
have *xs* = *take* (*length xs*) (*drop n* (*map f ys*))
by (*simp add*: *xs12 n-def*)
thus *?thesis*
by (*intro exI*[*of - take* (*length xs*) (*drop n ys*)])
(*auto simp*: *take-map drop-map intro*!: *takedrop*[*THEN sublist-order.order.trans*])
qed

lemma *sublist-remdups-adj*:
assumes *sublist xs ys*
shows *sublist* (*remdups-adj xs*) (*remdups-adj ys*)
proof −
from *assms* **obtain** *xs1 xs2* **where** *ys*: *ys* = *xs1* @ *xs* @ *xs2*
by (*auto simp*: *sublist-def*)
have *suffix* (*remdups-adj* (*xs* @ *xs2*)) (*remdups-adj* (*xs1* @ *xs* @ *xs2*))
by (*rule suffix-remdups-adj*, *rule suffix-appendI*) *auto*
then obtain *zs1* **where** *zs1*: *remdups-adj* (*xs1* @ *xs* @ *xs2*) = *zs1* @ *remdups-adj* (*xs* @ *xs2*)
by (*auto simp*: *suffix-def*)
have *prefix* (*remdups-adj xs*) (*remdups-adj* (*xs* @ *xs2*))
by (*intro prefix-remdups-adj*) *auto*
then obtain *zs2* **where** *zs2*: *remdups-adj* (*xs* @ *xs2*) = *remdups-adj xs* @ *zs2*
by (*auto simp*: *prefix-def*)
show *?thesis*
by (*simp add*: *ys zs1 zs2*)
qed

52.12.2 *sublists*

primrec *sublists* :: *'a list* ⇒ *'a list list* **where**
  sublists [] = [[]]
| *sublists* (*x* # *xs*) = *sublists xs* @ *map* ((#) *x*) (*prefixes xs*)

lemma *in-set-sublists* [*simp*]: *xs* ∈ *set* (*sublists ys*) ⟷ *sublist xs ys*
  by (*induction ys arbitrary*: *xs*) (*auto simp*: *sublist-Cons-right prefix-Cons*)

lemma *set-sublists-eq*: *set* (*sublists xs*) = {*ys*. *sublist ys xs*}
  by *auto*

lemma *length-sublists* [*simp*]: *length* (*sublists xs*) = *Suc* (*length xs* ∗ *Suc* (*length xs*) *div 2*)
  by (*induction xs*) *simp-all*

52.13 Parametricity

context includes *lifting-syntax*
begin

private lemma *prefix-primrec*:
  prefix = *rec-list* (λ*xs*. *True*) (λ*x xs xsa ys*.
            case ys of [] ⇒ *False* | *y* # *ys* ⇒ *x* = *y* ∧ *xsa ys*)
proof (*intro ext*, *goal-cases*)
  case (*1 xs ys*)
 show *?case* **by** (*induction xs arbitrary*: *ys*) (*auto simp*: *prefix-Cons split*: *list.splits*)
qed

private lemma *sublist-primrec*:
  sublist = (λ*xs ys*. *rec-list* (λ*xs*. *xs* = []) (λ*y ys ysa xs*. *prefix xs* (*y* # *ys*) ∨ *ysa xs*) *ys xs*)
proof (*intro ext*, *goal-cases*)
  case (*1 xs ys*)
  show *?case* **by** (*induction ys*) (*auto simp*: *sublist-Cons-right*)
qed

private lemma *list-emb-primrec*:
  list-emb = (λ*uu uua uuaa*. *rec-list* (λ*P xs*. *List.null xs*) (λ*y ys ysa P xs*. *case xs of* [] ⇒ *True*
    | *x* # *xs* ⇒ *if P x y then ysa P xs else ysa P* (*x* # *xs*)) *uuaa uu uua*)
proof (*intro ext*, *goal-cases*)
  case (*1 P xs ys*)
  show *?case*
    by (*induction ys arbitrary*: *xs*)
      (*auto simp*: *list-emb-code List.null-def split*: *list.splits*)
qed

lemma *prefix-transfer* [*transfer-rule*]:
  assumes [*transfer-rule*]: *bi-unique A*

shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *prefix prefix*
unfolding *prefix-primrec* **by** *transfer-prover*

lemma *suffix-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *suffix suffix*
unfolding *suffix-to-prefix* [*abs-def*] **by** *transfer-prover*

lemma *sublist-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *sublist sublist*
unfolding *sublist-primrec* **by** *transfer-prover*

lemma *parallel-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *parallel parallel*
unfolding *parallel-def* **by** *transfer-prover*

lemma *list-emb-transfer* [*transfer-rule*]:
((*A* ===> *A* ===> (=)) ===> *list-all2 A* ===> *list-all2 A* ===> (=)) *list-emb list-emb*
unfolding *list-emb-primrec* **by** *transfer-prover*

lemma *strict-prefix-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *strict-prefix strict-prefix*
unfolding *strict-prefix-def* **by** *transfer-prover*

lemma *strict-suffix-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *strict-suffix strict-suffix*
unfolding *strict-suffix-def* **by** *transfer-prover*

lemma *strict-subseq-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *strict-subseq strict-subseq*
unfolding *strict-subseq-def* **by** *transfer-prover*

lemma *strict-sublist-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2 A* ===> (=)) *strict-sublist strict-sublist*
unfolding *strict-sublist-def* **by** *transfer-prover*

lemma *prefixes-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2* (*list-all2 A*)) *prefixes prefixes*
unfolding *prefixes-def* **by** *transfer-prover*

lemma *suffixes-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2* (*list-all2 A*)) *suffixes suffixes*
unfolding *suffixes-def* **by** *transfer-prover*

lemma *sublists-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> *list-all2* (*list-all2 A*)) *sublists sublists*
unfolding *sublists-def* **by** *transfer-prover*

end

end

53 Linear Temporal Logic on Streams

theory *Linear-Temporal-Logic-on-Streams*
imports *Stream Sublist Extended-Nat Infinite-Set*
begin

54 Preliminaries

lemma *shift-prefix*:
assumes *xl* @− *xs* = *yl* @− *ys* **and** *length xl* ≤ *length yl*
shows *prefix xl yl*
using *assms* **proof**(*induct xl arbitrary*: *yl xs ys*)
case (*Cons x xl yl xs ys*)
thus *?case* **by** (*cases yl*) *auto*
qed *auto*

lemma *shift-prefix-cases*:
assumes *xl* @− *xs* = *yl* @− *ys*
shows *prefix xl yl* ∨ *prefix yl xl*
using *shift-prefix*[*OF assms*]
by (*cases length xl* ≤ *length yl*) (*metis*, *metis assms nat-le-linear shift-prefix*)

55 Linear temporal logic

Propositional connectives:

abbreviation (*input*) *IMPL* (**infix** *impl 60*)
where *φ impl ψ* ≡ λ *xs*. *φ xs* ⟶ *ψ xs*

abbreviation (*input*) *OR* (**infix** *or 60*)
where *φ or ψ* ≡ λ *xs*. *φ xs* ∨ *ψ xs*

abbreviation (*input*) *AND* (**infix** *aand 60*)

where φ *aand* $\psi \equiv \lambda$ *xs*. φ *xs* $\wedge$ ψ *xs*

abbreviation (*input*) *not* **where** *not* $\varphi \equiv \lambda$ *xs*. $\neg$ φ *xs*

abbreviation (*input*) *true* $\equiv \lambda$ *xs*. *True*

abbreviation (*input*) *false* $\equiv \lambda$ *xs*. *False*

lemma *impl-not-or*: φ *impl* ψ = (*not* φ) *or* ψ
by *blast*

lemma *not-or*: *not* (φ *or* ψ) = (*not* φ) *aand* (*not* ψ)
by *blast*

lemma *not-aand*: *not* (φ *aand* ψ) = (*not* φ) *or* (*not* ψ)
by *blast*

lemma *non-not*[*simp*]: *not* (*not* φ) = φ **by** *simp*

Temporal (LTL) connectives:

fun *holds* **where** *holds P xs* $\longleftrightarrow$ *P* (*shd xs*)
fun *nxt* **where** *nxt* φ *xs* = φ (*stl xs*)

definition *HLD s* = *holds* (λx. $x \in s$)

abbreviation *HLD-nxt* (**infixr** $\cdot$ *65*) **where**
s $\cdot$ *P* $\equiv$ *HLD s aand nxt P*

context
notes [[*inductive-internals*]]
begin

inductive *ev* **for** φ **where**
base: φ *xs* $\Longrightarrow$ *ev* φ *xs*
|
step: *ev* φ (*stl xs*) $\Longrightarrow$ *ev* φ *xs*

coinductive *alw* **for** φ **where**
alw: $[\![\varphi$ *xs*; *alw* φ (*stl xs*)$]\!] \Longrightarrow$ *alw* φ *xs*

— weak until:
coinductive *UNTIL* (**infix** *until 60*) **for** φ ψ **where**
base: ψ *xs* $\Longrightarrow$ (φ *until* ψ) *xs*
|
step: $[\![\varphi$ *xs*; (φ *until* ψ) (*stl xs*)$]\!] \Longrightarrow$ (φ *until* ψ) *xs*

end

lemma *holds-mono*:

assumes *holds*: *holds P xs* **and** *0*: $\bigwedge$ *x*. *P x* $\Longrightarrow$ *Q x*
shows *holds Q xs*
using *assms* **by** *auto*

lemma *holds-aand*:
(*holds P aand holds Q*) *steps* $\longleftrightarrow$ *holds* (λ *step*. *P step* $\wedge$ *Q step*) *steps* **by** *auto*

lemma *HLD-iff*: *HLD s* ω $\longleftrightarrow$ *shd* ω $\in$ *s*
by (*simp add*: *HLD-def*)

lemma *HLD-Stream*[*simp*]: *HLD X* (*x* ## ω) $\longleftrightarrow$ *x* $\in$ *X*
by (*simp add*: *HLD-iff*)

lemma *nxt-mono*:
assumes *nxt*: *nxt* φ *xs* **and** *0*: $\bigwedge$ *xs*. φ *xs* $\Longrightarrow$ ψ *xs*
shows *nxt* ψ *xs*
using *assms* **by** *auto*

declare *ev.intros*[*intro*]
declare *alw.cases*[*elim*]

lemma *ev-induct-strong*[*consumes 1*, *case-names base step*]:
ev φ *x* $\Longrightarrow$ ($\bigwedge$*xs*. φ *xs* $\Longrightarrow$ *P xs*) $\Longrightarrow$ ($\bigwedge$*xs*. *ev* φ (*stl xs*) $\Longrightarrow$ $\neg$ φ *xs* $\Longrightarrow$ *P* (*stl xs*) $\Longrightarrow$ *P xs*) $\Longrightarrow$ *P x*
by (*induct rule*: *ev.induct*) *auto*

lemma *alw-coinduct*[*consumes 1*, *case-names alw stl*]:
X x $\Longrightarrow$ ($\bigwedge$*x*. *X x* $\Longrightarrow$ φ *x*) $\Longrightarrow$ ($\bigwedge$*x*. *X x* $\Longrightarrow$ $\neg$ *alw* φ (*stl x*) $\Longrightarrow$ *X* (*stl x*)) $\Longrightarrow$ *alw* φ *x*
using *alw.coinduct*[*of X x* φ] **by** *auto*

lemma *ev-mono*:
assumes *ev*: *ev* φ *xs* **and** *0*: $\bigwedge$ *xs*. φ *xs* $\Longrightarrow$ ψ *xs*
shows *ev* ψ *xs*
using *ev* **by** *induct* (*auto simp*: *0*)

lemma *alw-mono*:
assumes *alw*: *alw* φ *xs* **and** *0*: $\bigwedge$ *xs*. φ *xs* $\Longrightarrow$ ψ *xs*
shows *alw* ψ *xs*
using *alw* **by** *coinduct* (*auto simp*: *0*)

lemma *until-monoL*:
assumes *until*: (φ*1 until* ψ) *xs* **and** *0*: $\bigwedge$ *xs*. φ*1 xs* $\Longrightarrow$ φ*2 xs*
shows (φ*2 until* ψ) *xs*
using *until* **by** *coinduct* (*auto elim*: *UNTIL.cases simp*: *0*)

lemma *until-monoR*:
assumes *until*: (φ *until* ψ*1*) *xs* **and** *0*: $\bigwedge$ *xs*. ψ*1 xs* $\Longrightarrow$ ψ*2 xs*
shows (φ *until* ψ*2*) *xs*

using *until* **by** *coinduct* (*auto elim*: *UNTIL.cases simp*: *0*)

lemma *until-mono*:
assumes *until*: (*φ1 until ψ1*) *xs* **and**
0: ⋀ *xs*. *φ1 xs* ⟹ *φ2 xs* ⋀ *xs*. *ψ1 xs* ⟹ *ψ2 xs*
shows (*φ2 until ψ2*) *xs*
using *until* **by** *coinduct* (*auto elim*: *UNTIL.cases simp*: *0*)

lemma *until-false*: *φ until false* = *alw φ*
proof−
{fix *xs* **assume** (*φ until false*) *xs* **hence** *alw φ xs*
by *coinduct* (*auto elim*: *UNTIL.cases*)
}
moreover
{fix *xs* **assume** *alw φ xs* **hence** (*φ until false*) *xs*
by *coinduct auto*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *ev-nxt*: *ev φ* = (*φ or nxt* (*ev φ*))
by (*rule ext*) (*metis ev.simps nxt.simps*)

lemma *alw-nxt*: *alw φ* = (*φ aand nxt* (*alw φ*))
by (*rule ext*) (*metis alw.simps nxt.simps*)

lemma *ev-ev*[*simp*]: *ev* (*ev φ*) = *ev φ*
proof−
{fix *xs*
assume *ev* (*ev φ*) *xs* **hence** *ev φ xs*
by *induct auto*
}
thus *?thesis* **by** *auto*
qed

lemma *alw-alw*[*simp*]: *alw* (*alw φ*) = *alw φ*
proof−
{fix *xs*
assume *alw φ xs* **hence** *alw* (*alw φ*) *xs*
by *coinduct auto*
}
thus *?thesis* **by** *auto*
qed

lemma *ev-shift*:
assumes *ev φ xs*
shows *ev φ* (*xl* @− *xs*)
using *assms* **by** (*induct xl*) *auto*

lemma *ev-imp-shift*:
assumes *ev* φ *xs* **shows** $\exists$ *xl xs2*. *xs* = *xl* @− *xs2* $\wedge$ φ *xs2*
using *assms* **by** *induct* (*metis shift.simps*(*1*), *metis shift.simps*(*2*) *stream.collapse*)+

lemma *alw-ev-shift*: *alw* φ *xs1* $\Longrightarrow$ *ev* (*alw* φ) (*xl* @− *xs1*)
by (*auto intro*: *ev-shift*)

lemma *alw-shift*:
assumes *alw* φ (*xl* @− *xs*)
shows *alw* φ *xs*
using *assms* **by** (*induct xl*) *auto*

lemma *ev-ex-nxt*:
assumes *ev* φ *xs*
shows $\exists$ *n*. (*nxt* ^^ *n*) φ *xs*
using *assms* **proof** *induct*
  case (*base xs*) **thus** *?case* **by** (*intro exI*[*of* - *0*]) *auto*
next
  case (*step xs*)
  then obtain *n* **where** (*nxt* ^^ *n*) φ (*stl xs*) **by** *blast*
  thus *?case* **by** (*intro exI*[*of* - *Suc n*]) (*metis funpow.simps*(*2*) *nxt.simps o-def*)
qed

lemma *alw-sdrop*:
assumes *alw* φ *xs* **shows** *alw* φ (*sdrop n xs*)
by (*metis alw-shift assms stake-sdrop*)

lemma *nxt-sdrop*: (*nxt* ^^ *n*) φ *xs* $\longleftrightarrow$ φ (*sdrop n xs*)
by (*induct n arbitrary*: *xs*) *auto*

definition *wait* φ *xs* $\equiv$ *LEAST n*. (*nxt* ^^ *n*) φ *xs*

lemma *nxt-wait*:
assumes *ev* φ *xs* **shows** (*nxt* ^^ (*wait* φ *xs*)) φ *xs*
unfolding *wait-def* **using** *ev-ex-nxt*[*OF assms*] **by**(*rule LeastI-ex*)

lemma *nxt-wait-least*:
assumes *ev*: *ev* φ *xs* **and** *nxt*: (*nxt* ^^ *n*) φ *xs* **shows** *wait* φ *xs* $\leq$ *n*
unfolding *wait-def* **using** *ev-ex-nxt*[*OF ev*] **by** (*metis Least-le nxt*)

lemma *sdrop-wait*:
assumes *ev* φ *xs* **shows** φ (*sdrop* (*wait* φ *xs*) *xs*)
using *nxt-wait*[*OF assms*] **unfolding** *nxt-sdrop* **.**

lemma *sdrop-wait-least*:
assumes *ev*: *ev* φ *xs* **and** *nxt*: φ (*sdrop n xs*) **shows** *wait* φ *xs* $\leq$ *n*
using *assms nxt-wait-least* **unfolding** *nxt-sdrop* **by** *auto*

lemma *nxt-ev*: (*nxt* ^^ *n*) φ *xs* $\Longrightarrow$ *ev* φ *xs*

by (*induct n arbitrary*: *xs*) *auto*

lemma *not-ev*: *not* (*ev* φ) = *alw* (*not* φ)
proof(*rule ext*, *safe*)
fix *xs* **assume** *not* (*ev* φ) *xs* **thus** *alw* (*not* φ) *xs*
by (*coinduct*) *auto*
next
fix *xs* **assume** *ev* φ *xs* **and** *alw* (*not* φ) *xs* **thus** *False*
by (*induct*) *auto*
qed

lemma *not-alw*: *not* (*alw* φ) = *ev* (*not* φ)
proof−
have *not* (*alw* φ) = *not* (*alw* (*not* (*not* φ))) **by** *simp*
also have ... = *ev* (*not* φ) **unfolding** *not-ev*[*symmetric*] **by** *simp*
finally show *?thesis* **.**
qed

lemma *not-ev-not*[*simp*]: *not* (*ev* (*not* φ)) = *alw* φ
unfolding *not-ev* **by** *simp*

lemma *not-alw-not*[*simp*]: *not* (*alw* (*not* φ)) = *ev* φ
unfolding *not-alw* **by** *simp*

lemma *alw-ev-sdrop*:
assumes *alw* (*ev* φ) (*sdrop m xs*)
shows *alw* (*ev* φ) *xs*
using *assms*
by *coinduct* (*metis alw-nxt ev-shift funpow-swap1 nxt.simps nxt-sdrop stake-sdrop*)

lemma *ev-alw-imp-alw-ev*:
assumes *ev* (*alw* φ) *xs* **shows** *alw* (*ev* φ) *xs*
using *assms* **by** *induct* (*metis* (*full-types*) *alw-mono ev.base*, *metis alw alw-nxt ev.step*)

lemma *alw-aand*: *alw* (φ *aand* ψ) = *alw* φ *aand alw* ψ
proof−
{fix *xs* **assume** *alw* (φ *aand* ψ) *xs* **hence** (*alw* φ *aand alw* ψ) *xs*
by (*auto elim*: *alw-mono*)
}
moreover
{fix *xs* **assume** (*alw* φ *aand alw* ψ) *xs* **hence** *alw* (φ *aand* ψ) *xs*
by *coinduct auto*
}
ultimately show *?thesis* **by** *blast*
qed

lemma *ev-or*: *ev* (φ *or* ψ) = *ev* φ *or ev* ψ
proof−

```
  {fix xs assume (ev φ or ev ψ) xs hence ev (φ or ψ) xs
   by (auto elim: ev-mono)
  }
  moreover
  {fix xs assume ev (φ or ψ) xs hence (ev φ or ev ψ) xs
   by induct auto
  }
  ultimately show ?thesis by blast
qed

lemma ev-alw-aand:
assumes φ: ev (alw φ) xs and ψ: ev (alw ψ) xs
shows ev (alw (φ aand ψ)) xs
proof−
  obtain xl xs1 where xs1: xs = xl @− xs1 and φφ: alw φ xs1
  using φ by (metis ev-imp-shift)
  moreover obtain yl ys1 where xs2: xs = yl @− ys1 and ψψ: alw ψ ys1
  using ψ by (metis ev-imp-shift)
  ultimately have 0: xl @− xs1 = yl @− ys1 by auto
  hence prefix xl yl ∨ prefix yl xl using shift-prefix-cases by auto
  thus ?thesis proof
    assume prefix xl yl
    then obtain yl1 where yl: yl = xl @ yl1 by (elim prefixE)
    have xs1': xs1 = yl1 @− ys1 using 0 unfolding yl by simp
    have alw φ ys1 using φφ unfolding xs1' by (metis alw-shift)
    hence alw (φ aand ψ) ys1 using ψψ unfolding alw-aand by auto
    thus ?thesis unfolding xs2 by (auto intro: alw-ev-shift)
  next
    assume prefix yl xl
    then obtain xl1 where xl: xl = yl @ xl1 by (elim prefixE)
    have ys1': ys1 = xl1 @− xs1 using 0 unfolding xl by simp
    have alw ψ xs1 using ψψ unfolding ys1' by (metis alw-shift)
    hence alw (φ aand ψ) xs1 using φφ unfolding alw-aand by auto
    thus ?thesis unfolding xs1 by (auto intro: alw-ev-shift)
  qed
qed

lemma ev-alw-alw-impl:
assumes ev (alw φ) xs and alw (alw φ impl ev ψ) xs
shows ev ψ xs
using assms by induct auto

lemma ev-alw-stl[simp]: ev (alw φ) (stl x) ⟷ ev (alw φ) x
by (metis (full-types) alw-nxt ev-nxt nxt.simps)

lemma alw-alw-impl-ev:
alw (alw φ impl ev ψ) = (ev (alw φ) impl alw (ev ψ)) (is ?A = ?B)
proof−
  {fix xs assume ?A xs ∧ ev (alw φ) xs hence alw (ev ψ) xs
```

```
   by coinduct (auto elim: ev-alw-alw-impl)
  }
  moreover
  {fix xs assume ?B xs hence ?A xs
   by coinduct auto
  }
  ultimately show ?thesis by blast
qed

lemma ev-alw-impl:
assumes ev φ xs and alw (φ impl ψ) xs  shows ev ψ xs
using assms by induct auto

lemma ev-alw-impl-ev:
assumes ev φ xs and alw (φ impl ev ψ) xs  shows ev ψ xs
using ev-alw-impl[OF assms] by simp

lemma alw-mp:
assumes alw φ xs and alw (φ impl ψ) xs
shows alw ψ xs
proof−
  {assume alw φ xs ∧ alw (φ impl ψ) xs hence ?thesis
   by coinduct auto
  }
  thus ?thesis using assms by auto
qed

lemma all-imp-alw:
assumes ⋀ xs. φ xs  shows alw φ xs
proof−
  {assume ∀ xs. φ xs
   hence ?thesis by coinduct auto
  }
  thus ?thesis using assms by auto
qed

lemma alw-impl-ev-alw:
assumes alw (φ impl ev ψ) xs
shows alw (ev φ impl ev ψ) xs
using assms by coinduct (auto dest: ev-alw-impl)

lemma ev-holds-sset:
ev (holds P) xs ⟷ (∃ x ∈ sset xs. P x) (is ?L ⟷ ?R)
proof safe
 assume ?L thus ?R by induct (metis holds.simps stream.set-sel(1), metis stl-sset)
next
  fix x assume x ∈ sset xs P x
  thus ?L by (induct rule: sset-induct) (simp-all add: ev.base ev.step)
qed
```

LTL as a program logic:

lemma *alw-invar*:
assumes φ *xs* **and** *alw* (φ *impl nxt* φ) *xs*
shows *alw* φ *xs*
proof−
{assume φ *xs* $\wedge$ *alw* (φ *impl nxt* φ) *xs* **hence** *?thesis*
by *coinduct auto*
}
thus *?thesis* **using** *assms* **by** *auto*
qed

lemma *variance*:
assumes *1*: φ *xs* **and** *2*: *alw* (φ *impl* (ψ *or nxt* φ)) *xs*
shows (*alw* φ *or ev* ψ) *xs*
proof−
{assume ¬ *ev* ψ *xs* **hence** *alw* (*not* ψ) *xs* **unfolding** *not-ev*[*symmetric*] **.**
moreover have *alw* (*not* ψ *impl* (φ *impl nxt* φ)) *xs*
using *2* **by** *coinduct auto*
ultimately have *alw* (φ *impl nxt* φ) *xs* **by**(*auto dest*: *alw-mp*)
with *1* **have** *alw* φ *xs* **by**(*rule alw-invar*)
}
thus *?thesis* **by** *blast*
qed

lemma *ev-alw-imp-nxt*:
assumes *e*: *ev* φ *xs* **and** *a*: *alw* (φ *impl* (*nxt* φ)) *xs*
shows *ev* (*alw* φ) *xs*
proof−
obtain *xl xs1* **where** *xs*: *xs* = *xl* @− *xs1* **and** φ: φ *xs1*
using *e* **by** (*metis ev-imp-shift*)
have φ *xs1* $\wedge$ *alw* (φ *impl* (*nxt* φ)) *xs1* **using** *a* φ **unfolding** *xs* **by** (*metis alw-shift*)
hence *alw* φ *xs1* **by**(*coinduct xs1 rule*: *alw.coinduct*) *auto*
thus *?thesis* **unfolding** *xs* **by** (*auto intro*: *alw-ev-shift*)
qed

inductive *ev-at* :: ('*a stream* ⇒ *bool*) ⇒ *nat* ⇒ '*a stream* ⇒ *bool* **for** *P* :: '*a stream* ⇒ *bool* **where**
base: *P* ω ⟹ *ev-at P 0* ω
| *step*:¬ *P* ω ⟹ *ev-at P n* (*stl* ω) ⟹ *ev-at P* (*Suc n*) ω

inductive-simps *ev-at-0*[*simp*]: *ev-at P 0* ω
inductive-simps *ev-at-Suc*[*simp*]: *ev-at P* (*Suc n*) ω

lemma *ev-at-imp-snth*: *ev-at P n* ω ⟹ *P* (*sdrop n* ω)
by (*induction n arbitrary*: ω) *auto*

lemma *ev-at-HLD-imp-snth*: *ev-at* (*HLD X*) *n* ω ⟹ ω !! *n* ∈ *X*

by (*auto dest*!: *ev-at-imp-snth simp*: *HLD-iff*)

lemma *ev-at-HLD-single-imp-snth*: *ev-at* (*HLD* {*x*}) *n* ω ⟹ ω !! *n* = *x*
by (*drule ev-at-HLD-imp-snth*) *simp*

lemma *ev-at-unique*: *ev-at P n* ω ⟹ *ev-at P m* ω ⟹ *n* = *m*
proof (*induction arbitrary*: *m rule*: *ev-at.induct*)
case (*base* ω) **then show** *?case*
by (*simp add*: *ev-at.simps*[*of* - - ω])
next
case (*step* ω *n*) **from** *step.prems step.hyps step.IH*[*of m − 1*] **show** *?case*
by (*auto simp add*: *ev-at.simps*[*of* - - ω])
qed

lemma *ev-iff-ev-at*: *ev P* ω ⟷ (∃ *n*. *ev-at P n* ω)
proof
assume *ev P* ω **then show** ∃ *n*. *ev-at P n* ω
by (*induction rule*: *ev-induct-strong*) (*auto intro*: *ev-at.intros*)
next
assume ∃ *n*. *ev-at P n* ω
then obtain *n* **where** *ev-at P n* ω
by *auto*
then show *ev P* ω
by *induction auto*
qed

lemma *ev-at-shift*: *ev-at* (*HLD X*) *i* (*stake* (*Suc i*) ω @− ω′ :: *'s stream*) ⟷ *ev-at* (*HLD X*) *i* ω
by (*induction i arbitrary*: ω) (*auto simp*: *HLD-iff*)

lemma *ev-iff-ev-at-unique*: *ev P* ω ⟷ (∃!*n*. *ev-at P n* ω)
by (*auto intro*: *ev-at-unique simp*: *ev-iff-ev-at*)

lemma *alw-HLD-iff-streams*: *alw* (*HLD X*) ω ⟷ ω ∈ *streams X*
proof
assume *alw* (*HLD X*) ω **then show** ω ∈ *streams X*
proof (*coinduction arbitrary*: ω)
case (*streams* ω) **then show** *?case* **by** (*cases* ω) *auto*
qed
next
assume ω ∈ *streams X* **then show** *alw* (*HLD X*) ω
proof (*coinduction arbitrary*: ω)
case (*alw* ω) **then show** *?case* **by** (*cases* ω) *auto*
qed
qed

lemma *not-HLD*: *not* (*HLD X*) = *HLD* (− *X*)
by (*auto simp*: *HLD-iff*)

lemma *not-alw-iff*: $\neg$ (*alw P* ω) $\longleftrightarrow$ *ev* (*not P*) ω
using *not-alw*[*of P*] **by** (*simp add*: *fun-eq-iff*)

lemma *not-ev-iff*: $\neg$ (*ev P* ω) $\longleftrightarrow$ *alw* (*not P*) ω
using *not-alw-iff*[*of not P* ω, *symmetric*] **by** *simp*

lemma *ev-Stream*: *ev P* (*x ## s*) $\longleftrightarrow$ *P* (*x ## s*) $\vee$ *ev P s*
by (*auto elim*: *ev.cases*)

lemma *alw-ev-imp-ev-alw*:
assumes *alw* (*ev P*) ω **shows** *ev* (*P aand alw* (*ev P*)) ω
proof −
have *ev P* ω **using** *assms* **by** *auto*
from *this assms* **show** *?thesis*
by *induct auto*
qed

lemma *ev-False*: *ev* (λ*x*. *False*) ω $\longleftrightarrow$ *False*
proof
assume *ev* (λ*x*. *False*) ω **then show** *False*
by *induct auto*
qed *auto*

lemma *alw-False*: *alw* (λ*x*. *False*) ω $\longleftrightarrow$ *False*
by *auto*

lemma *ev-iff-sdrop*: *ev P* ω $\longleftrightarrow$ ($\exists$ *m*. *P* (*sdrop m* ω))
proof *safe*
assume *ev P* ω **then show** $\exists$ *m*. *P* (*sdrop m* ω)
by (*induct rule*: *ev-induct-strong*) (*auto intro*: *exI*[*of - 0*] *exI*[*of - Suc n* **for** *n*])
next
fix *m* **assume** *P* (*sdrop m* ω) **then show** *ev P* ω
by (*induct m arbitrary*: ω) *auto*
qed

lemma *alw-iff-sdrop*: *alw P* ω $\longleftrightarrow$ ($\forall$ *m*. *P* (*sdrop m* ω))
proof *safe*
fix *m* **assume** *alw P* ω **then show** *P* (*sdrop m* ω)
by (*induct m arbitrary*: ω) *auto*
next
assume $\forall$ *m*. *P* (*sdrop m* ω) **then show** *alw P* ω
by (*coinduction arbitrary*: ω) (*auto elim*: *allE*[*of - 0*] *allE*[*of - Suc n* **for** *n*])
qed

lemma *infinite-iff-alw-ev*: *infinite* {*m*. *P* (*sdrop m* ω)} $\longleftrightarrow$ *alw* (*ev P*) ω
unfolding *infinite-nat-iff-unbounded-le alw-iff-sdrop ev-iff-sdrop*
by *simp* (*metis le-Suc-ex le-add1*)

lemma *alw-inv*:

assumes *stl*: $\bigwedge s.\ f\ (stl\ s) = stl\ (f\ s)$
shows $alw\ P\ (f\ s) \longleftrightarrow alw\ (\lambda x.\ P\ (f\ x))\ s$
proof
assume $alw\ P\ (f\ s)$ **then show** $alw\ (\lambda x.\ P\ (f\ x))\ s$
by (*coinduction arbitrary*: *s rule*: *alw-coinduct*)
(*auto simp*: *stl*)
next
assume $alw\ (\lambda x.\ P\ (f\ x))\ s$ **then show** $alw\ P\ (f\ s)$
by (*coinduction arbitrary*: *s rule*: *alw-coinduct*) (*auto simp flip*: *stl*)
qed

lemma *ev-inv*:
assumes *stl*: $\bigwedge s.\ f\ (stl\ s) = stl\ (f\ s)$
shows $ev\ P\ (f\ s) \longleftrightarrow ev\ (\lambda x.\ P\ (f\ x))\ s$
proof
assume $ev\ P\ (f\ s)$ **then show** $ev\ (\lambda x.\ P\ (f\ x))\ s$
by (*induction f s arbitrary*: *s*) (*auto simp*: *stl*)
next
assume $ev\ (\lambda x.\ P\ (f\ x))\ s$ **then show** $ev\ P\ (f\ s)$
by *induction* (*auto simp flip*: *stl*)
qed

lemma *alw-smap*: $alw\ P\ (smap\ f\ s) \longleftrightarrow alw\ (\lambda x.\ P\ (smap\ f\ x))\ s$
by (*rule alw-inv*) *simp*

lemma *ev-smap*: $ev\ P\ (smap\ f\ s) \longleftrightarrow ev\ (\lambda x.\ P\ (smap\ f\ x))\ s$
by (*rule ev-inv*) *simp*

lemma *alw-cong*:
assumes *P*: $alw\ P\ \omega$ **and** *eq*: $\bigwedge \omega.\ P\ \omega \Longrightarrow Q1\ \omega \longleftrightarrow Q2\ \omega$
shows $alw\ Q1\ \omega \longleftrightarrow alw\ Q2\ \omega$
proof −
from *eq* **have** $(alw\ P\ aand\ Q1) = (alw\ P\ aand\ Q2)$ **by** *auto*
then have $alw\ (alw\ P\ aand\ Q1)\ \omega = alw\ (alw\ P\ aand\ Q2)\ \omega$ **by** *auto*
with *P* **show** $alw\ Q1\ \omega \longleftrightarrow alw\ Q2\ \omega$
by (*simp add*: *alw-aand*)
qed

lemma *ev-cong*:
assumes *P*: $alw\ P\ \omega$ **and** *eq*: $\bigwedge \omega.\ P\ \omega \Longrightarrow Q1\ \omega \longleftrightarrow Q2\ \omega$
shows $ev\ Q1\ \omega \longleftrightarrow ev\ Q2\ \omega$
proof −
from *P* **have** $alw\ (\lambda xs.\ Q1\ xs \longrightarrow Q2\ xs)\ \omega$ **by** (*rule alw-mono*) (*simp add*: *eq*)
moreover from *P* **have** $alw\ (\lambda xs.\ Q2\ xs \longrightarrow Q1\ xs)\ \omega$ **by** (*rule alw-mono*) (*simp add*: *eq*)
moreover note *ev-alw-impl*[*of Q1 ω Q2*] *ev-alw-impl*[*of Q2 ω Q1*]
ultimately show $ev\ Q1\ \omega \longleftrightarrow ev\ Q2\ \omega$
by *auto*
qed

lemma *alwD*: *alw P x* ⟹ *P x*
by *auto*

lemma *alw-alwD*: *alw P ω* ⟹ *alw* (*alw P*) *ω*
by *simp*

lemma *alw-ev-stl*: *alw* (*ev P*) (*stl ω*) ⟷ *alw* (*ev P*) *ω*
by (*auto intro*: *alw.intros*)

lemma *holds-Stream*: *holds P* (*x ## s*) ⟷ *P x*
by *simp*

lemma *holds-eq1*[*simp*]: *holds* ((=) *x*) = *HLD* {*x*}
by *rule* (*auto simp*: *HLD-iff*)

lemma *holds-eq2*[*simp*]: *holds* (λ*y*. *y* = *x*) = *HLD* {*x*}
by *rule* (*auto simp*: *HLD-iff*)

lemma *not-holds-eq*[*simp*]: *holds* (− (=) *x*) = *not* (*HLD* {*x*})
by *rule* (*auto simp*: *HLD-iff*)

Strong until

context
notes [[*inductive-internals*]]
begin

inductive *suntil* (**infix** *suntil 60*) **for** *φ ψ* **where**
base: *ψ ω* ⟹ (*φ suntil ψ*) *ω*
| *step*: *φ ω* ⟹ (*φ suntil ψ*) (*stl ω*) ⟹ (*φ suntil ψ*) *ω*

inductive-simps *suntil-Stream*: (*φ suntil ψ*) (*x ## s*)

end

lemma *suntil-induct-strong*[*consumes 1*, *case-names base step*]:
(*φ suntil ψ*) *x* ⟹
(⋀*ω*. *ψ ω* ⟹ *P ω*) ⟹
(⋀*ω*. *φ ω* ⟹ ¬ *ψ ω* ⟹ (*φ suntil ψ*) (*stl ω*) ⟹ *P* (*stl ω*) ⟹ *P ω*) ⟹ *P x*
using *suntil.induct*[*of φ ψ x P*] **by** *blast*

lemma *ev-suntil*: (*φ suntil ψ*) *ω* ⟹ *ev ψ ω*
by (*induct rule*: *suntil.induct*) *auto*

lemma *suntil-inv*:
assumes *stl*: ⋀*s*. *f* (*stl s*) = *stl* (*f s*)
shows (*P suntil Q*) (*f s*) ⟷ ((λ*x*. *P* (*f x*)) *suntil* (λ*x*. *Q* (*f x*))) *s*
proof
assume (*P suntil Q*) (*f s*) **then show** ((λ*x*. *P* (*f x*)) *suntil* (λ*x*. *Q* (*f x*))) *s*

by (*induction f s arbitrary*: *s*) (*auto simp*: *stl intro*: *suntil.intros*)
next
assume ((λ*x*. *P* (*f x*)) *suntil* (λ*x*. *Q* (*f x*))) *s* **then show** (*P suntil Q*) (*f s*)
by *induction* (*auto simp flip*: *stl intro*: *suntil.intros*)
qed

lemma *suntil-smap*: (*P suntil Q*) (*smap f s*) ⟷ ((λ*x*. *P* (*smap f x*)) *suntil* (λ*x*. *Q* (*smap f x*))) *s*
by (*rule suntil-inv*) *simp*

lemma *hld-smap*: *HLD x* (*smap f s*) = *holds* (λ*y*. *f y* ∈ *x*) *s*
by (*simp add*: *HLD-def*)

lemma *suntil-mono*:
assumes *eq*: ⋀ω. *P* ω ⟹ *Q1* ω ⟹ *Q2* ω ⋀ω. *P* ω ⟹ *R1* ω ⟹ *R2* ω
assumes ∗: (*Q1 suntil R1*) ω *alw P* ω **shows** (*Q2 suntil R2*) ω
using ∗ **by** *induct* (*auto intro*: *eq suntil.intros*)

lemma *suntil-cong*:
alw P ω ⟹ (⋀ω. *P* ω ⟹ *Q1* ω ⟷ *Q2* ω) ⟹ (⋀ω. *P* ω ⟹ *R1* ω ⟷ *R2* ω) ⟹
(*Q1 suntil R1*) ω ⟷ (*Q2 suntil R2*) ω
using *suntil-mono*[*of P Q1 Q2 R1 R2* ω] *suntil-mono*[*of P Q2 Q1 R2 R1* ω] **by** *auto*

lemma *ev-suntil-iff*: *ev* (*P suntil Q*) ω ⟷ *ev Q* ω
proof
assume *ev* (*P suntil Q*) ω **then show** *ev Q* ω
by *induct* (*auto dest*: *ev-suntil*)
next
assume *ev Q* ω **then show** *ev* (*P suntil Q*) ω
by *induct* (*auto intro*: *suntil.intros*)
qed

lemma *true-suntil*: ((λ-. *True*) *suntil P*) = *ev P*
by (*simp add*: *suntil-def ev-def*)

lemma *suntil-lfp*: (φ *suntil* ψ) = *lfp* (λ*P s*. ψ *s* ∨ (φ *s* ∧ *P* (*stl s*)))
by (*simp add*: *suntil-def*)

lemma *sfilter-P*[*simp*]: *P* (*shd s*) ⟹ *sfilter P s* = *shd s* ## *sfilter P* (*stl s*)
using *sfilter-Stream*[*of P shd s stl s*] **by** *simp*

lemma *sfilter-not-P*[*simp*]: ¬ *P* (*shd s*) ⟹ *sfilter P s* = *sfilter P* (*stl s*)
using *sfilter-Stream*[*of P shd s stl s*] **by** *simp*

lemma *sfilter-eq*:
assumes *ev* (*holds P*) *s*
shows *sfilter P s* = *x* ## *s*′ ⟷

```
    P x ∧ (not (holds P) suntil (HLD {x} aand nxt (λs. sfilter P s = s'))) s
  using assms
  by (induct rule: ev-induct-strong)
    (auto simp add: HLD-iff intro: suntil.intros elim: suntil.cases)

lemma sfilter-streams:
  alw (ev (holds P)) ω ⟹ ω ∈ streams A ⟹ sfilter P ω ∈ streams {x∈A. P x}
proof (coinduction arbitrary: ω)
  case (streams ω)
  then have ev (holds P) ω by blast
  from this streams show ?case
    by (induct rule: ev-induct-strong) (auto elim: streamsE)
qed

lemma alw-sfilter:
  assumes ∗: alw (ev (holds P)) s
  shows alw Q (sfilter P s) ⟷ alw (λx. Q (sfilter P x)) s
proof
  assume alw Q (sfilter P s) with ∗ show alw (λx. Q (sfilter P x)) s
  proof (coinduction arbitrary: s rule: alw-coinduct)
    case (stl s)
    then have ev (holds P) s
      by blast
    from this stl show ?case
      by (induct rule: ev-induct-strong) auto
  qed auto
next
  assume alw (λx. Q (sfilter P x)) s with ∗ show alw Q (sfilter P s)
  proof (coinduction arbitrary: s rule: alw-coinduct)
    case (stl s)
    then have ev (holds P) s
      by blast
    from this stl show ?case
      by (induct rule: ev-induct-strong) auto
  qed auto
qed

lemma ev-sfilter:
  assumes ∗: alw (ev (holds P)) s
  shows ev Q (sfilter P s) ⟷ ev (λx. Q (sfilter P x)) s
proof
  assume ev Q (sfilter P s) from this ∗ show ev (λx. Q (sfilter P x)) s
  proof (induction sfilter P s arbitrary: s rule: ev-induct-strong)
    case (step s)
    then have ev (holds P) s
      by blast
    from this step show ?case
      by (induct rule: ev-induct-strong) auto
  qed auto
```

next
assume *ev* (λ*x*. *Q* (*sfilter P x*)) *s* **then show** *ev Q* (*sfilter P s*)
proof (*induction rule*: *ev-induct-strong*)
case (*step s*) **then show** *?case*
by (*cases P* (*shd s*)) *auto*
qed *auto*
qed

lemma *holds-sfilter*:
assumes *ev* (*holds Q*) *s* **shows** *holds P* (*sfilter Q s*) ⟷ (*not* (*holds Q*) *suntil* (*holds* (*Q aand P*))) *s*
proof
assume *holds P* (*sfilter Q s*) **with** *assms* **show** (*not* (*holds Q*) *suntil* (*holds* (*Q aand P*))) *s*
by (*induct rule*: *ev-induct-strong*) (*auto intro*: *suntil.intros*)
next
assume (*not* (*holds Q*) *suntil* (*holds* (*Q aand P*))) *s* **then show** *holds P* (*sfilter Q s*)
by *induct auto*
qed

lemma *suntil-aand-nxt*:
(φ *suntil* (φ *aand nxt* ψ)) ω ⟷ (φ *aand nxt* (φ *suntil* ψ)) ω
proof
assume (φ *suntil* (φ *aand nxt* ψ)) ω **then show** (φ *aand nxt* (φ *suntil* ψ)) ω
by *induction* (*auto intro*: *suntil.intros*)
next
assume (φ *aand nxt* (φ *suntil* ψ)) ω
then have (φ *suntil* ψ) (*stl* ω) φ ω
by *auto*
then show (φ *suntil* (φ *aand nxt* ψ)) ω
by (*induction stl* ω *arbitrary*: ω)
(*auto elim*: *suntil.cases intro*: *suntil.intros*)
qed

lemma *alw-sconst*: *alw P* (*sconst x*) ⟷ *P* (*sconst x*)
proof
assume *P* (*sconst x*) **then show** *alw P* (*sconst x*)
by *coinduction auto*
qed *auto*

lemma *ev-sconst*: *ev P* (*sconst x*) ⟷ *P* (*sconst x*)
proof
assume *ev P* (*sconst x*) **then show** *P* (*sconst x*)
by (*induction sconst x*) *auto*
qed *auto*

lemma *suntil-sconst*: (φ *suntil* ψ) (*sconst x*) ⟷ ψ (*sconst x*)
proof

assume (φ *suntil* ψ) (*sconst x*) **then show** ψ (*sconst x*)
by (*induction sconst x*) *auto*
qed (*auto intro*: *suntil.intros*)

lemma *hld-smap'*: *HLD x* (*smap f s*) = *HLD* (*f* −‘ *x*) *s*
by (*simp add*: *HLD-def*)

lemma *pigeonhole-stream*:
assumes *alw* (*HLD s*) ω
assumes *finite s*
shows $\exists x{\in}s.$ *alw* (*ev* (*HLD* {*x*})) ω
proof −
have $\forall i{\in}UNIV.\ \exists x{\in}s.\ \omega$!! $i = x$
using ‹*alw* (*HLD s*) ω› **by** (*simp add*: *alw-iff-sdrop HLD-iff*)
from *pigeonhole-infinite-rel*[*OF infinite-UNIV-nat* ‹*finite s*› *this*]
show *?thesis*
by (*simp add*: *HLD-iff flip*: *infinite-iff-alw-ev*)
qed

lemma *ev-eq-suntil*: *ev P* $\omega \longleftrightarrow$ (*not P suntil P*) ω
proof
assume *ev P* ω **then show** ((λ*xs*. ¬ *P xs*) *suntil P*) ω
by (*induction rule*: *ev-induct-strong*) (*auto intro*: *suntil.intros*)
qed (*auto simp*: *ev-suntil*)

56 Weak vs. strong until (contributed by Michael Foster, University of Sheffield)

lemma *suntil-implies-until*: (φ *suntil* ψ) $\omega \Longrightarrow$ (φ *until* ψ) ω
by (*induct rule*: *suntil-induct-strong*) (*auto intro*: *UNTIL.intros*)

lemma *alw-implies-until*: *alw* φ $\omega \Longrightarrow$ (φ *until* ψ) ω
unfolding *until-false*[*symmetric*] **by** (*auto elim*: *until-mono*)

lemma *until-ev-suntil*: (φ *until* ψ) $\omega \Longrightarrow$ *ev* ψ $\omega \Longrightarrow$ (φ *suntil* ψ) ω
proof (*rotate-tac*, *induction rule*: *ev.induct*)
case (*base xs*)
then show *?case*
by (*simp add*: *suntil.base*)
next
case (*step xs*)
then show *?case*
by (*metis UNTIL.cases suntil.base suntil.step*)
qed

lemma *suntil-as-until*: (φ *suntil* ψ) ω = ((φ *until* ψ) ω ∧ *ev* ψ ω)
using *ev-suntil suntil-implies-until until-ev-suntil* **by** *blast*

lemma *until-not-relesased-now*: $(\varphi\ until\ \psi)\ \omega \Longrightarrow \neg\ \psi\ \omega \Longrightarrow \varphi\ \omega$
using *UNTIL.cases* **by** *auto*

lemma *until-must-release-ev*: $(\varphi\ until\ \psi)\ \omega \Longrightarrow ev\ (not\ \varphi)\ \omega \Longrightarrow ev\ \psi\ \omega$
proof (*rotate-tac*, *induction rule*: *ev.induct*)
case (*base xs*)
then show *?case*
using *until-not-relesased-now* **by** *blast*
next
case (*step xs*)
then show *?case*
using *UNTIL.cases* **by** *blast*
qed

lemma *until-as-suntil*: $(\varphi\ until\ \psi)\ \omega = ((\varphi\ suntil\ \psi)\ or\ (alw\ \varphi))\ \omega$
using *alw-implies-until not-alw-iff suntil-implies-until until-ev-suntil until-must-release-ev*
by *blast*

lemma *alw-holds*: $alw\ (holds\ P)\ (h\#\#t) = (P\ h \wedge alw\ (holds\ P)\ t)$
by (*metis alw.simps holds-Stream stream.sel*(*2*))

lemma *alw-holds2*: $alw\ (holds\ P)\ ss = (P\ (shd\ ss) \wedge alw\ (holds\ P)\ (stl\ ss))$
by (*meson alw.simps holds.elims*(*2*) *holds.elims*(*3*))

lemma *alw-eq-sconst*: $(alw\ (HLD\ \{h\})\ t) = (t = sconst\ h)$
unfolding *sconst-alt alw-HLD-iff-streams streams-iff-sset*
using *stream.set-sel*(*1*) **by** *force*

lemma *sdrop-if-suntil*: $(p\ suntil\ q)\ \omega \Longrightarrow \exists j.\ q\ (sdrop\ j\ \omega) \wedge (\forall k < j.\ p\ (sdrop\ k\ \omega))$
proof(*induction rule*: *suntil.induct*)
case (*base* ω)
then show *?case*
by *force*
next
case (*step* ω)
then obtain *j* **where** $q\ (sdrop\ j\ (stl\ \omega))\ \forall k{<}j.\ p\ (sdrop\ k\ (stl\ \omega))$ **by** *blast*
with *step*(*1*,*2*) **show** *?case*
using *ev-at-imp-snth less-Suc-eq-0-disj* **by** (*auto intro*!: *exI*[**where** *x*=*j*+*1*])
qed

lemma *not-suntil*: $(\neg\ (p\ suntil\ q)\ \omega) = (\neg\ (p\ until\ q)\ \omega \vee alw\ (not\ q)\ \omega)$
by (*simp add*: *suntil-as-until alw-iff-sdrop ev-iff-sdrop*)

lemma *sdrop-until*: $q\ (sdrop\ j\ \omega) \Longrightarrow \forall k{<}j.\ p\ (sdrop\ k\ \omega) \Longrightarrow (p\ until\ q)\ \omega$
proof(*induct j arbitrary*: ω)
case *0*
then show *?case*
by (*simp add*: *UNTIL.base*)

next
case (*Suc j*)
then show *?case*
by (*metis Suc-mono UNTIL.simps sdrop.simps*(*1*) *sdrop.simps*(*2*) *zero-less-Suc*)
qed

lemma *sdrop-suntil*: *q* (*sdrop j* ω) $\Longrightarrow$ ($\forall$ *k* < *j*. *p* (*sdrop k* ω)) $\Longrightarrow$ (*p suntil q*) ω
by (*metis ev-iff-sdrop sdrop-until suntil-as-until*)

lemma *suntil-iff-sdrop*: (*p suntil q*) ω = ($\exists$ *j*. *q* (*sdrop j* ω) $\wedge$ ($\forall$ *k* < *j*. *p* (*sdrop k* ω)))
using *sdrop-if-suntil sdrop-suntil* **by** *blast*

end

57 Lists as vectors

theory *ListVector*
imports *Main*
begin

A vector-space like structure of lists and arithmetic operations on them. Is only a vector space if restricted to lists of the same length.

Multiplication with a scalar:

abbreviation *scale* :: ('*a*::*times*) $\Rightarrow$ '*a list* $\Rightarrow$ '*a list* (**infix** $*_s$ *70*)
where *x* $*_s$ *xs* $\equiv$ *map* ((*) *x*) *xs*

lemma *scale1*[*simp*]: (*1*::'*a*::*monoid-mult*) $*_s$ *xs* = *xs*
by (*induct xs*) *simp-all*

57.1 + and −

fun *zipwith0* :: ('*a*::*zero* $\Rightarrow$ '*b*::*zero* $\Rightarrow$ '*c*) $\Rightarrow$ '*a list* $\Rightarrow$ '*b list* $\Rightarrow$ '*c list*
where
zipwith0 f [] [] = [] |
zipwith0 f (*x*#*xs*) (*y*#*ys*) = *f x y* # *zipwith0 f xs ys* |
zipwith0 f (*x*#*xs*) [] = *f x 0* # *zipwith0 f xs* [] |
zipwith0 f [] (*y*#*ys*) = *f 0 y* # *zipwith0 f* [] *ys*

instantiation *list* :: ({*zero*, *plus*}) *plus*
begin

definition
list-add-def: (+) = *zipwith0* (+)

instance ..

end

instantiation *list* :: ({*zero*, *uminus*}) *uminus*
begin

definition
list-uminus-def: *uminus* = *map uminus*

instance ..

end

instantiation *list* :: ({*zero*,*minus*}) *minus*
begin

definition
list-diff-def: (−) = *zipwith0* (−)

instance ..

end

lemma *zipwith0-Nil*[*simp*]: *zipwith0 f* [] *ys* = *map* (*f 0*) *ys*
by(*induct ys*) *simp-all*

lemma *list-add-Nil*[*simp*]: [] + *xs* = (*xs*::'*a*::*monoid-add list*)
by (*induct xs*) (*auto simp*:*list-add-def*)

lemma *list-add-Nil2*[*simp*]: *xs* + [] = (*xs*::'*a*::*monoid-add list*)
by (*induct xs*) (*auto simp*:*list-add-def*)

lemma *list-add-Cons*[*simp*]: (*x*#*xs*) + (*y*#*ys*) = (*x*+*y*)#(*xs*+*ys*)
by(*auto simp*:*list-add-def*)

lemma *list-diff-Nil*[*simp*]: [] − *xs* = −(*xs*::'*a*::*group-add list*)
by (*induct xs*) (*auto simp*:*list-diff-def list-uminus-def*)

lemma *list-diff-Nil2*[*simp*]: *xs* − [] = (*xs*::'*a*::*group-add list*)
by (*induct xs*) (*auto simp*:*list-diff-def*)

lemma *list-diff-Cons-Cons*[*simp*]: (*x*#*xs*) − (*y*#*ys*) = (*x*−*y*)#(*xs*−*ys*)
by (*induct xs*) (*auto simp*:*list-diff-def*)

lemma *list-uminus-Cons*[*simp*]: −(*x*#*xs*) = (−*x*)#(−*xs*)
by (*induct xs*) (*auto simp*:*list-uminus-def*)

lemma *self-list-diff*:
xs − *xs* = *replicate* (*length*(*xs*::'*a*::*group-add list*)) *0*
by(*induct xs*) *simp-all*

lemma *list-add-assoc*: **fixes** *xs* :: *'a::monoid-add list*
shows *(xs+ys)+zs = xs+(ys+zs)*
apply(*induct xs arbitrary*: *ys zs*)
apply *simp*
apply(*case-tac ys*)
apply(*simp*)
apply(*simp*)
apply(*case-tac zs*)
apply(*simp*)
apply(*simp add*: *add.assoc*)
done

57.2 Inner product

definition *iprod* :: *'a::ring list* ⇒ *'a list* ⇒ *'a* (⟨-,-⟩) **where**
⟨xs,ys⟩ = (∑(x,y) ← zip xs ys. x∗y)

lemma *iprod-Nil*[*simp*]: *⟨[],ys⟩ = 0*
by(*simp add*: *iprod-def*)

lemma *iprod-Nil2*[*simp*]: *⟨xs,[]⟩ = 0*
by(*simp add*: *iprod-def*)

lemma *iprod-Cons*[*simp*]: *⟨x#xs,y#ys⟩ = x∗y + ⟨xs,ys⟩*
by(*simp add*: *iprod-def*)

lemma *iprod0-if-coeffs0*: *∀ c∈set cs. c = 0 ⟹ ⟨cs,xs⟩ = 0*
apply(*induct cs arbitrary*:*xs*)
apply *simp*
apply(*case-tac xs*) **apply** *simp*
apply *auto*
done

lemma *iprod-uminus*[*simp*]: *⟨−xs,ys⟩ = −⟨xs,ys⟩*
by(*simp add*: *iprod-def uminus-sum-list-map o-def split-def map-zip-map list-uminus-def*)

lemma *iprod-left-add-distrib*: *⟨xs + ys,zs⟩ = ⟨xs,zs⟩ + ⟨ys,zs⟩*
apply(*induct xs arbitrary*: *ys zs*)
apply (*simp add*: *o-def split-def*)
apply(*case-tac ys*)
apply *simp*
apply(*case-tac zs*)
apply (*simp*)
apply(*simp add*: *distrib-right*)
done

lemma *iprod-left-diff-distrib*: *⟨xs − ys, zs⟩ = ⟨xs,zs⟩ − ⟨ys,zs⟩*
apply(*induct xs arbitrary*: *ys zs*)
apply (*simp add*: *o-def split-def*)

apply(*case-tac ys*)
apply *simp*
apply(*case-tac zs*)
apply (*simp*)
apply(*simp add*: *left-diff-distrib*)
done

lemma *iprod-assoc*: $\langle x *_s xs, ys\rangle = x * \langle xs,ys\rangle$
apply(*induct xs arbitrary*: *ys*)
apply *simp*
apply(*case-tac ys*)
apply (*simp*)
apply (*simp add*: *distrib-left mult.assoc*)
done

end

58 Definitions of Least Upper Bounds and Greatest Lower Bounds

theory *Lub-Glb*
imports *Complex-Main*
begin

Thanks to suggestions by James Margetson

definition *setle* :: *'a set* ⇒ *'a*::*ord* ⇒ *bool* (**infixl** ‹*<=› *70*)
where $S *<= x = (\forall y{\in}S.\ y \le x)$

definition *setge* :: *'a*::*ord* ⇒ *'a set* ⇒ *bool* (**infixl** ‹<=*› *70*)
where $x <=* S = (\forall y{\in}S.\ x \le y)$

58.1 Rules for the Relations *<= and <=*

lemma *setleI*: $\forall y{\in}S.\ y \le x \Longrightarrow S *<= x$
by (*simp add*: *setle-def*)

lemma *setleD*: $S *<= x \Longrightarrow y{\in}S \Longrightarrow y \le x$
by (*simp add*: *setle-def*)

lemma *setgeI*: $\forall y{\in}S.\ x \le y \Longrightarrow x <=* S$
by (*simp add*: *setge-def*)

lemma *setgeD*: $x <=* S \Longrightarrow y{\in}S \Longrightarrow x \le y$
by (*simp add*: *setge-def*)

definition *leastP* :: (*'a* ⇒ *bool*) ⇒ *'a*::*ord* ⇒ *bool*
where $leastP\ P\ x = (P\ x \wedge x <=* Collect\ P)$

definition *isUb* :: *'a set* ⇒ *'a set* ⇒ *'a*::*ord* ⇒ *bool*
where *isUb R S x* = (*S* *<= *x* ∧ *x* ∈ *R*)

definition *isLub* :: *'a set* ⇒ *'a set* ⇒ *'a*::*ord* ⇒ *bool*
where *isLub R S x* = *leastP* (*isUb R S*) *x*

definition *ubs* :: *'a set* ⇒ *'a*::*ord set* ⇒ *'a set*
where *ubs R S* = *Collect* (*isUb R S*)

58.2 Rules about the Operators *leastP*, *ub* and *lub*

lemma *leastPD1*: *leastP P x* ⟹ *P x*
by (*simp add*: *leastP-def*)

lemma *leastPD2*: *leastP P x* ⟹ *x* <=* *Collect P*
by (*simp add*: *leastP-def*)

lemma *leastPD3*: *leastP P x* ⟹ *y* ∈ *Collect P* ⟹ *x* ≤ *y*
by (*blast dest*!: *leastPD2 setgeD*)

lemma *isLubD1*: *isLub R S x* ⟹ *S* *<= *x*
by (*simp add*: *isLub-def isUb-def leastP-def*)

lemma *isLubD1a*: *isLub R S x* ⟹ *x* ∈ *R*
by (*simp add*: *isLub-def isUb-def leastP-def*)

lemma *isLub-isUb*: *isLub R S x* ⟹ *isUb R S x*
unfolding *isUb-def* **by** (*blast dest*: *isLubD1 isLubD1a*)

lemma *isLubD2*: *isLub R S x* ⟹ *y* ∈ *S* ⟹ *y* ≤ *x*
by (*blast dest*!: *isLubD1 setleD*)

lemma *isLubD3*: *isLub R S x* ⟹ *leastP* (*isUb R S*) *x*
by (*simp add*: *isLub-def*)

lemma *isLubI1*: *leastP*(*isUb R S*) *x* ⟹ *isLub R S x*
by (*simp add*: *isLub-def*)

lemma *isLubI2*: *isUb R S x* ⟹ *x* <=* *Collect* (*isUb R S*) ⟹ *isLub R S x*
by (*simp add*: *isLub-def leastP-def*)

lemma *isUbD*: *isUb R S x* ⟹ *y* ∈ *S* ⟹ *y* ≤ *x*
by (*simp add*: *isUb-def setle-def*)

lemma *isUbD2*: *isUb R S x* ⟹ *S* *<= *x*
by (*simp add*: *isUb-def*)

lemma *isUbD2a*: *isUb R S x* ⟹ *x* ∈ *R*

by (*simp add*: *isUb-def*)

lemma *isUbI*: $S *<= x \Longrightarrow x \in R \Longrightarrow isUb\ R\ S\ x$
by (*simp add*: *isUb-def*)

lemma *isLub-le-isUb*: $isLub\ R\ S\ x \Longrightarrow isUb\ R\ S\ y \Longrightarrow x \leq y$
unfolding *isLub-def* **by** (*blast intro*!: *leastPD3*)

lemma *isLub-ubs*: $isLub\ R\ S\ x \Longrightarrow x <=* ubs\ R\ S$
unfolding *ubs-def isLub-def* **by** (*rule leastPD2*)

lemma *isLub-unique*: $[|\ isLub\ R\ S\ x;\ isLub\ R\ S\ y\ |] ==> x = (y::'a::linorder)$
apply (*frule isLub-isUb*)
apply (*frule-tac* $x = y$ **in** *isLub-isUb*)
apply (*blast intro*!: *order-antisym dest*!: *isLub-le-isUb*)
done

lemma *isUb-UNIV-I*: $(\bigwedge y.\ y \in S \Longrightarrow y \leq u) \Longrightarrow isUb\ UNIV\ S\ u$
by (*simp add*: *isUbI setleI*)

definition *greatestP* :: $('a \Rightarrow bool) \Rightarrow 'a::ord \Rightarrow bool$
where $greatestP\ P\ x = (P\ x \wedge Collect\ P\ *<=\ x)$

definition *isLb* :: $'a\ set \Rightarrow 'a\ set \Rightarrow 'a::ord \Rightarrow bool$
where $isLb\ R\ S\ x = (x <=*\ S \wedge x \in R)$

definition *isGlb* :: $'a\ set \Rightarrow 'a\ set \Rightarrow 'a::ord \Rightarrow bool$
where $isGlb\ R\ S\ x = greatestP\ (isLb\ R\ S)\ x$

definition *lbs* :: $'a\ set \Rightarrow 'a::ord\ set \Rightarrow 'a\ set$
where $lbs\ R\ S = Collect\ (isLb\ R\ S)$

58.3 Rules about the Operators *greatestP*, *isLb* and *isGlb*

lemma *greatestPD1*: $greatestP\ P\ x \Longrightarrow P\ x$
by (*simp add*: *greatestP-def*)

lemma *greatestPD2*: $greatestP\ P\ x \Longrightarrow Collect\ P\ *<=\ x$
by (*simp add*: *greatestP-def*)

lemma *greatestPD3*: $greatestP\ P\ x \Longrightarrow y \in Collect\ P \Longrightarrow x \geq y$
by (*blast dest*!: *greatestPD2 setleD*)

lemma *isGlbD1*: $isGlb\ R\ S\ x \Longrightarrow x <=*\ S$
by (*simp add*: *isGlb-def isLb-def greatestP-def*)

lemma *isGlbD1a*: $isGlb\ R\ S\ x \Longrightarrow x \in R$
by (*simp add*: *isGlb-def isLb-def greatestP-def*)

lemma *isGlb-isLb*: *isGlb R S x* $\Longrightarrow$ *isLb R S x*
unfolding *isLb-def* **by** (*blast dest*: *isGlbD1 isGlbD1a*)

lemma *isGlbD2*: *isGlb R S x* $\Longrightarrow$ $y \in S \Longrightarrow y \geq x$
by (*blast dest*!: *isGlbD1 setgeD*)

lemma *isGlbD3*: *isGlb R S x* $\Longrightarrow$ *greatestP* (*isLb R S*) *x*
by (*simp add*: *isGlb-def*)

lemma *isGlbI1*: *greatestP* (*isLb R S*) *x* $\Longrightarrow$ *isGlb R S x*
by (*simp add*: *isGlb-def*)

lemma *isGlbI2*: *isLb R S x* $\Longrightarrow$ *Collect* (*isLb R S*) *∗<= x* $\Longrightarrow$ *isGlb R S x*
by (*simp add*: *isGlb-def greatestP-def*)

lemma *isLbD*: *isLb R S x* $\Longrightarrow$ $y \in S \Longrightarrow y \geq x$
by (*simp add*: *isLb-def setge-def*)

lemma *isLbD2*: *isLb R S x* $\Longrightarrow$ *x <=∗ S*
by (*simp add*: *isLb-def*)

lemma *isLbD2a*: *isLb R S x* $\Longrightarrow$ $x \in R$
by (*simp add*: *isLb-def*)

lemma *isLbI*: *x <=∗ S* $\Longrightarrow$ $x \in R \Longrightarrow$ *isLb R S x*
by (*simp add*: *isLb-def*)

lemma *isGlb-le-isLb*: *isGlb R S x* $\Longrightarrow$ *isLb R S y* $\Longrightarrow$ $x \geq y$
unfolding *isGlb-def* **by** (*blast intro*!: *greatestPD3*)

lemma *isGlb-ubs*: *isGlb R S x* $\Longrightarrow$ *lbs R S ∗<= x*
unfolding *lbs-def isGlb-def* **by** (*rule greatestPD2*)

lemma *isGlb-unique*: [| *isGlb R S x*; *isGlb R S y* |] ==> $x = (y::'a::linorder)$
apply (*frule isGlb-isLb*)
apply (*frule-tac x = y* **in** *isGlb-isLb*)
apply (*blast intro*!: *order-antisym dest*!: *isGlb-le-isLb*)
done

lemma *bdd-above-setle*: *bdd-above A* $\longleftrightarrow$ $(\exists\, a.\ A$ *∗<= a*$)$
by (*auto simp*: *bdd-above-def setle-def*)

lemma *bdd-below-setge*: *bdd-below A* $\longleftrightarrow$ $(\exists\, a.\ a$ *<=∗ A*$)$
by (*auto simp*: *bdd-below-def setge-def*)

lemma *isLub-cSup*:
$(S::'a$ :: *conditionally-complete-lattice set*$) \neq \{\} \Longrightarrow (\exists\, b.\ S$ *∗<= b*$) \Longrightarrow$ *isLub UNIV S* (*Sup S*)

by (*auto simp add*: *isLub-def setle-def leastP-def isUb-def*
intro!: *setgeI cSup-upper cSup-least*)

lemma *isGlb-cInf*:
(*S*::'*a* :: *conditionally-complete-lattice set*) ≠ {} ⟹ (∃ *b*. *b* <=∗ *S*) ⟹ *isGlb UNIV S* (*Inf S*)
by (*auto simp add*: *isGlb-def setge-def greatestP-def isLb-def*
intro!: *setleI cInf-lower cInf-greatest*)

lemma *cSup-le*: (*S*::'*a*::*conditionally-complete-lattice set*) ≠ {} ⟹ *S* ∗<= *b* ⟹ *Sup S* ≤ *b*
by (*metis cSup-least setle-def*)

lemma *cInf-ge*: (*S*::'*a* :: *conditionally-complete-lattice set*) ≠ {} ⟹ *b* <=∗ *S* ⟹ *Inf S* ≥ *b*
by (*metis cInf-greatest setge-def*)

lemma *cSup-bounds*:
fixes *S* :: '*a* :: *conditionally-complete-lattice set*
shows *S* ≠ {} ⟹ *a* <=∗ *S* ⟹ *S* ∗<= *b* ⟹ *a* ≤ *Sup S* ∧ *Sup S* ≤ *b*
using *cSup-least*[*of S b*] *cSup-upper2*[*of - S a*]
by (*auto simp*: *bdd-above-setle setge-def setle-def*)

lemma *cSup-unique*: (*S*::'*a* :: {*conditionally-complete-linorder*, *no-bot*} *set*) ∗<= *b* ⟹ (∀ *b*'<*b*. ∃ *x*∈*S*. *b*' < *x*) ⟹ *Sup S* = *b*
by (*rule cSup-eq*) (*auto simp*: *not-le*[*symmetric*] *setle-def*)

lemma *cInf-unique*: *b* <=∗ (*S*::'*a* :: {*conditionally-complete-linorder*, *no-top*} *set*) ⟹ (∀ *b*'>*b*. ∃ *x*∈*S*. *b*' > *x*) ⟹ *Inf S* = *b*
by (*rule cInf-eq*) (*auto simp*: *not-le*[*symmetric*] *setge-def*)

Use completeness of reals (supremum property) to show that any bounded sequence has a least upper bound

lemma *reals-complete*: ∃ *X*. *X* ∈ *S* ⟹ ∃ *Y*. *isUb* (*UNIV*::*real set*) *S Y* ⟹ ∃ *t*. *isLub* (*UNIV* :: *real set*) *S t*
by (*intro exI*[*of - Sup S*] *isLub-cSup*) (*auto simp*: *setle-def isUb-def intro*!: *cSup-upper*)

lemma *Bseq-isUb*: ⋀*X* :: *nat* ⇒ *real*. *Bseq X* ⟹ ∃ *U*. *isUb* (*UNIV*::*real set*) {*x*. ∃ *n*. *X n* = *x*} *U*
by (*auto intro*: *isUbI setleI simp add*: *Bseq-def abs-le-iff*)

lemma *Bseq-isLub*: ⋀*X* :: *nat* ⇒ *real*. *Bseq X* ⟹ ∃ *U*. *isLub* (*UNIV*::*real set*) {*x*. ∃ *n*. *X n* = *x*} *U*
by (*blast intro*: *reals-complete Bseq-isUb*)

lemma *isLub-mono-imp-LIMSEQ*:
fixes *X* :: *nat* ⇒ *real*
assumes *u*: *isLub UNIV* {*x*. ∃ *n*. *X n* = *x*} *u*
assumes *X*: ∀ *m n*. *m* ≤ *n* ⟶ *X m* ≤ *X n*

shows $X \longrightarrow u$
proof −
have $X \longrightarrow (SUP\ i.\ X\ i)$
using $u[THEN\ isLubD1]\ X$
by (*intro LIMSEQ-incseq-SUP*) (*auto simp*: *incseq-def image-def eq-commute bdd-above-setle*)
also have $(SUP\ i.\ X\ i) = u$
using $isLub\text{-}cSup[of\ range\ X]\ u[THEN\ isLubD1]$
by (*intro isLub-unique*[*OF - u*]) (*auto simp add*: *image-def eq-commute*)
finally show *?thesis* **.**
qed

lemmas *real-isGlb-unique* = *isGlb-unique*[**where** *'a=real*]

lemma *real-le-inf-subset*: $t \neq \{\} \Longrightarrow t \subseteq s \Longrightarrow \exists\, b.\ b <=* s \Longrightarrow Inf\ s \leq Inf\ (t::real\ set)$
by (*rule cInf-superset-mono*) (*auto simp*: *bdd-below-setge*)

lemma *real-ge-sup-subset*: $t \neq \{\} \Longrightarrow t \subseteq s \Longrightarrow \exists\, b.\ s\ *<=\ b \Longrightarrow Sup\ s \geq Sup\ (t::real\ set)$
by (*rule cSup-subset-mono*) (*auto simp*: *bdd-above-setle*)

end

59 An abstract view on maps for code generation.

theory *Mapping*
imports *Main AList*
begin

59.1 Parametricity transfer rules

lemma *map-of-foldr*: *map-of xs* = *foldr* $(\lambda(k,\ v)\ m.\ m(k \mapsto v))$ *xs Map.empty*
using *map-add-map-of-foldr* [*of Map.empty*] **by** *auto*

context includes *lifting-syntax*
begin

lemma *empty-parametric*: (*A* ===> *rel-option B*) *Map.empty Map.empty*
by *transfer-prover*

lemma *lookup-parametric*: ((*A* ===> *B*) ===> *A* ===> *B*) (λ*m k. m k*) (λ*m k. m k*)
by *transfer-prover*

lemma *update-parametric*:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> *B* ===> (*A* ===> *rel-option B*) ===> *A* ===> *rel-option B*)

(λ*k v m*. *m*(*k* ↦ *v*)) (λ*k v m*. *m*(*k* ↦ *v*))
by *transfer-prover*

lemma *delete-parametric*:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> (*A* ===> *rel-option B*) ===> *A* ===> *rel-option B*)
(λ*k m*. *m*(*k* := *None*)) (λ*k m*. *m*(*k* := *None*))
by *transfer-prover*

lemma *is-none-parametric* [*transfer-rule*]:
(*rel-option A* ===> *HOL.eq*) *Option.is-none Option.is-none*
by (*auto simp add*: *Option.is-none-def rel-fun-def rel-option-iff split*: *option.split*)

lemma *dom-parametric*:
assumes [*transfer-rule*]: *bi-total A*
shows ((*A* ===> *rel-option B*) ===> *rel-set A*) *dom dom*
unfolding *dom-def* [*abs-def*] *Option.is-none-def* [*symmetric*] **by** *transfer-prover*

lemma *graph-parametric*:
assumes *bi-total A*
shows ((*A* ===> *rel-option B*) ===> *rel-set* (*rel-prod A B*)) *Map.graph Map.graph*
proof
fix *f g* **assume** (*A* ===> *rel-option B*) *f g*
with *assms*[*unfolded bi-total-def*] **show** *rel-set* (*rel-prod A B*) (*Map.graph f*) (*Map.graph g*)
unfolding *graph-def rel-set-def rel-fun-def*
by *auto* (*metis option-rel-Some1 option-rel-Some2*)+
qed

lemma *map-of-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique R1*
shows (*list-all2* (*rel-prod R1 R2*) ===> *R1* ===> *rel-option R2*) *map-of map-of*
unfolding *map-of-def* **by** *transfer-prover*

lemma *map-entry-parametric* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique A*
shows (*A* ===> (*B* ===> *B*) ===> (*A* ===> *rel-option B*) ===> *A* ===> *rel-option B*)
(λ*k f m*. (*case m k of None* ⇒ *m*
| *Some v* ⇒ *m* (*k* ↦ (*f v*)))) (λ*k f m*. (*case m k of None* ⇒ *m*
| *Some v* ⇒ *m* (*k* ↦ (*f v*))))
by *transfer-prover*

lemma *tabulate-parametric*:
assumes [*transfer-rule*]: *bi-unique A*
shows (*list-all2 A* ===> (*A* ===> *B*) ===> *A* ===> *rel-option B*)
(λ*ks f*. (*map-of* (*map* (λ*k*. (*k*, *f k*)) *ks*))) (λ*ks f*. (*map-of* (*map* (λ*k*. (*k*, *f k*)) *ks*)))

by *transfer-prover*

lemma *bulkload-parametric*:
(*list-all2 A* ===> *HOL.eq* ===> *rel-option A*)
(λ*xs k. if k* < *length xs then Some* (*xs* ! *k*) *else None*)
(λ*xs k. if k* < *length xs then Some* (*xs* ! *k*) *else None*)
proof
fix *xs ys*
assume *list-all2 A xs ys*
then show
(*HOL.eq* ===> *rel-option A*)
(λ*k. if k* < *length xs then Some* (*xs* ! *k*) *else None*)
(λ*k. if k* < *length ys then Some* (*ys* ! *k*) *else None*)
apply *induct*
apply *auto*
unfolding *rel-fun-def*
apply *clarsimp*
apply (*case-tac xa*)
apply (*auto dest*: *list-all2-lengthD list-all2-nthD*)
done
qed

lemma *map-parametric*:
((*A* ===> *B*) ===> (*C* ===> *D*) ===> (*B* ===> *rel-option C*) ===> *A* ===> *rel-option D*)
(λ*f g m.* (*map-option g* ∘ *m* ∘ *f*)) (λ*f g m.* (*map-option g* ∘ *m* ∘ *f*))
by *transfer-prover*

lemma *combine-with-key-parametric*:
((*A* ===> *B* ===> *B* ===> *B*) ===> (*A* ===> *rel-option B*) ===> (*A* ===> *rel-option B*) ===>
(*A* ===> *rel-option B*)) (λ*f m1 m2 x. combine-options* (*f x*) (*m1 x*) (*m2 x*))
(λ*f m1 m2 x. combine-options* (*f x*) (*m1 x*) (*m2 x*))
unfolding *combine-options-def* **by** *transfer-prover*

lemma *combine-parametric*:
((*B* ===> *B* ===> *B*) ===> (*A* ===> *rel-option B*) ===> (*A* ===> *rel-option B*) ===>
(*A* ===> *rel-option B*)) (λ*f m1 m2 x. combine-options f* (*m1 x*) (*m2 x*))
(λ*f m1 m2 x. combine-options f* (*m1 x*) (*m2 x*))
unfolding *combine-options-def* **by** *transfer-prover*

end

59.2 Type definition and primitive operations

typedef ('*a*, '*b*) *mapping* = *UNIV* :: ('*a* ⇀ '*b*) *set*
morphisms *rep Mapping* **..**

setup-lifting *type-definition-mapping*

lift-definition *empty* :: (*'a*, *'b*) *mapping*
is *Map.empty* **parametric** *empty-parametric* .

lift-definition *lookup* :: (*'a*, *'b*) *mapping* ⇒ *'a* ⇒ *'b option*
is λ*m k*. *m k* **parametric** *lookup-parametric* .

definition *lookup-default d m k* = (*case Mapping.lookup m k of None* ⇒ *d* | *Some v* ⇒ *v*)

lift-definition *update* :: *'a* ⇒ *'b* ⇒ (*'a*, *'b*) *mapping* ⇒ (*'a*, *'b*) *mapping*
is λ*k v m*. *m*(*k* ↦ *v*) **parametric** *update-parametric* .

lift-definition *delete* :: *'a* ⇒ (*'a*, *'b*) *mapping* ⇒ (*'a*, *'b*) *mapping*
is λ*k m*. *m*(*k* := *None*) **parametric** *delete-parametric* .

lift-definition *filter* :: (*'a* ⇒ *'b* ⇒ *bool*) ⇒ (*'a*, *'b*) *mapping* ⇒ (*'a*, *'b*) *mapping*
is λ*P m k*. *case m k of None* ⇒ *None* | *Some v* ⇒ *if P k v then Some v else None*
.

lift-definition *keys* :: (*'a*, *'b*) *mapping* ⇒ *'a set*
is *dom* **parametric** *dom-parametric* .

lift-definition *entries* :: (*'a*, *'b*) *mapping* ⇒ (*'a* × *'b*) *set*
is *Map.graph* **parametric** *graph-parametric* .

lift-definition *tabulate* :: *'a list* ⇒ (*'a* ⇒ *'b*) ⇒ (*'a*, *'b*) *mapping*
is λ*ks f*. (*map-of* (*List.map* (λ*k*. (*k*, *f k*)) *ks*)) **parametric** *tabulate-parametric* .

lift-definition *bulkload* :: *'a list* ⇒ (*nat*, *'a*) *mapping*
is λ*xs k*. *if k* < *length xs then Some* (*xs* ! *k*) *else None* **parametric** *bulk-load-parametric* .

lift-definition *map* :: (*'c* ⇒ *'a*) ⇒ (*'b* ⇒ *'d*) ⇒ (*'a*, *'b*) *mapping* ⇒ (*'c*, *'d*) *mapping*
is λ*f g m*. (*map-option g* ∘ *m* ∘ *f*) **parametric** *map-parametric* .

lift-definition *map-values* :: (*'c* ⇒ *'a* ⇒ *'b*) ⇒ (*'c*, *'a*) *mapping* ⇒ (*'c*, *'b*) *mapping*
is λ*f m x*. *map-option* (*f x*) (*m x*) .

lift-definition *combine-with-key* ::
(*'a* ⇒ *'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a*,*'b*) *mapping* ⇒ (*'a*,*'b*) *mapping* ⇒ (*'a*,*'b*) *mapping*
is λ*f m1 m2 x*. *combine-options* (*f x*) (*m1 x*) (*m2 x*) **parametric** *combine-with-key-parametric*
.

lift-definition *combine* ::
(*'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a*,*'b*) *mapping* ⇒ (*'a*,*'b*) *mapping* ⇒ (*'a*,*'b*) *mapping*
is λ*f m1 m2 x*. *combine-options f* (*m1 x*) (*m2 x*) **parametric** *combine-parametric*
.

definition *All-mapping m P* $\longleftrightarrow$
(∀ *x*. *case Mapping.lookup m x of None* ⇒ *True* | *Some y* ⇒ *P x y*)

declare [[*code drop*: *map*]]

59.3 Functorial structure

functor *map*: *map*
by (*transfer*, *auto simp add*: *fun-eq-iff option.map-comp option.map-id*)+

59.4 Derived operations

definition *ordered-keys* :: ('*a*::*linorder*, '*b*) *mapping* ⇒ '*a list*
where *ordered-keys m* = (*if finite* (*keys m*) *then sorted-list-of-set* (*keys m*) *else* [])

definition *ordered-entries* :: ('*a*::*linorder*, '*b*) *mapping* ⇒ ('*a* × '*b*) *list*
where *ordered-entries m* = (*if finite* (*entries m*) *then sorted-key-list-of-set fst* (*entries m*)
else [])

definition *fold* :: ('*a*::*linorder* ⇒ '*b* ⇒ '*c* ⇒ '*c*) ⇒ ('*a*, '*b*) *mapping* ⇒ '*c* ⇒ '*c*
where *fold f m a* = *List.fold* (*case-prod f*) (*ordered-entries m*) *a*

definition *is-empty* :: ('*a*, '*b*) *mapping* ⇒ *bool*
where *is-empty m* $\longleftrightarrow$ *keys m* = {}

definition *size* :: ('*a*, '*b*) *mapping* ⇒ *nat*
where *size m* = (*if finite* (*keys m*) *then card* (*keys m*) *else 0*)

definition *replace* :: '*a* ⇒ '*b* ⇒ ('*a*, '*b*) *mapping* ⇒ ('*a*, '*b*) *mapping*
where *replace k v m* = (*if k* ∈ *keys m then update k v m else m*)

definition *default* :: '*a* ⇒ '*b* ⇒ ('*a*, '*b*) *mapping* ⇒ ('*a*, '*b*) *mapping*
where *default k v m* = (*if k* ∈ *keys m then m else update k v m*)

Manual derivation of transfer rule is non-trivial

lift-definition *map-entry* :: '*a* ⇒ ('*b* ⇒ '*b*) ⇒ ('*a*, '*b*) *mapping* ⇒ ('*a*, '*b*) *mapping*
is
λk f m.
(*case m k of*
None ⇒ *m*
| *Some v* ⇒ *m* (*k* ↦ (*f v*))) **parametric** *map-entry-parametric* .

lemma *map-entry-code* [*code*]:
map-entry k f m =
(*case lookup m k of*
None ⇒ *m*
| *Some v* ⇒ *update k* (*f v*) *m*)

by *transfer rule*

definition *map-default* :: *'a* ⇒ *'b* ⇒ (*'b* ⇒ *'b*) ⇒ (*'a*, *'b*) *mapping* ⇒ (*'a*, *'b*) *mapping*
where *map-default k v f m* = *map-entry k f* (*default k v m*)

definition *of-alist* :: (*'k* × *'v*) *list* ⇒ (*'k*, *'v*) *mapping*
where *of-alist xs* = *foldr* (λ(*k*, *v*) *m*. *update k v m*) *xs empty*

instantiation *mapping* :: (*type*, *type*) *equal*
begin

definition *HOL.equal m1 m2* ⟷ (∀ *k*. *lookup m1 k* = *lookup m2 k*)

instance
apply *standard*
unfolding *equal-mapping-def*
apply *transfer*
apply *auto*
done

end

context includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-total A*
and [*transfer-rule*]: *bi-unique B*
shows (*pcr-mapping A B* ===> *pcr-mapping A B* ===> (=)) *HOL.eq HOL.equal*
unfolding *equal* **by** *transfer-prover*

lemma *of-alist-transfer* [*transfer-rule*]:
assumes [*transfer-rule*]: *bi-unique R1*
shows (*list-all2* (*rel-prod R1 R2*) ===> *pcr-mapping R1 R2*) *map-of of-alist*
unfolding *of-alist-def* [*abs-def*] *map-of-foldr* [*abs-def*] **by** *transfer-prover*

end

59.5 Properties

lemma *mapping-eqI*: (⋀*x*. *lookup m x* = *lookup m' x*) ⟹ *m* = *m'*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *mapping-eqI'*:
assumes ⋀*x*. *x* ∈ *Mapping.keys m* ⟹ *Mapping.lookup-default d m x* = *Mapping.lookup-default d m' x*
and *Mapping.keys m* = *Mapping.keys m'*
shows *m* = *m'*

proof (*intro mapping-eqI*)
show *Mapping.lookup m x = Mapping.lookup m' x* **for** *x*
proof (*cases Mapping.lookup m x*)
case *None*
then have *x ∉ Mapping.keys m*
by *transfer* (*simp add*: *dom-def*)
then have *x ∉ Mapping.keys m'*
by (*simp add*: *assms*)
then have *Mapping.lookup m' x = None*
by *transfer* (*simp add*: *dom-def*)
with *None* **show** *?thesis*
by *simp*
next
case (*Some y*)
then have *A*: *x ∈ Mapping.keys m*
by *transfer* (*simp add*: *dom-def*)
then have *x ∈ Mapping.keys m'*
by (*simp add*: *assms*)
then have *∃ y'. Mapping.lookup m' x = Some y'*
by *transfer* (*simp add*: *dom-def*)
with *Some assms*(*1*)[*OF A*] **show** *?thesis*
by (*auto simp add*: *lookup-default-def*)
qed
qed

lemma *lookup-update*[*simp*]: *lookup* (*update k v m*) *k = Some v*
by *transfer simp*

lemma *lookup-update-neq*[*simp*]: *k ≠ k' ⟹ lookup* (*update k v m*) *k' = lookup m k'*
by *transfer simp*

lemma *lookup-update'*: *lookup* (*update k v m*) *k' = (if k = k' then Some v else lookup m k')*
by *transfer simp*

lemma *lookup-empty*[*simp*]: *lookup empty k = None*
by *transfer simp*

lemma *lookup-delete*[*simp*]: *lookup* (*delete k m*) *k = None*
by *transfer simp*

lemma *lookup-delete-neq*[*simp*]: *k ≠ k' ⟹ lookup* (*delete k m*) *k' = lookup m k'*
by *transfer simp*

lemma *lookup-filter*:
lookup (*filter P m*) *k =*
(*case lookup m k of*
None ⇒ None

| *Some v ⇒ if P k v then Some v else None*)
by *transfer simp-all*

lemma *lookup-map-values*: *lookup* (*map-values f m*) *k* = *map-option* (*f k*) (*lookup m k*)
by *transfer simp-all*

lemma *lookup-default-empty*: *lookup-default d empty k* = *d*
by (*simp add*: *lookup-default-def lookup-empty*)

lemma *lookup-default-update*: *lookup-default d* (*update k v m*) *k* = *v*
by (*simp add*: *lookup-default-def*)

lemma *lookup-default-update-neq*:
k ≠ *k'* ⟹ *lookup-default d* (*update k v m*) *k'* = *lookup-default d m k'*
by (*simp add*: *lookup-default-def*)

lemma *lookup-default-update'*:
lookup-default d (*update k v m*) *k'* = (*if k* = *k' then v else lookup-default d m k'*)
by (*auto simp*: *lookup-default-update lookup-default-update-neq*)

lemma *lookup-default-filter*:
lookup-default d (*filter P m*) *k* =
(*if P k* (*lookup-default d m k*) *then lookup-default d m k else d*)
by (*simp add*: *lookup-default-def lookup-filter split*: *option.splits*)

lemma *lookup-default-map-values*:
lookup-default (*f k d*) (*map-values f m*) *k* = *f k* (*lookup-default d m k*)
by (*simp add*: *lookup-default-def lookup-map-values split*: *option.splits*)

lemma *lookup-combine-with-key*:
Mapping.lookup (*combine-with-key f m1 m2*) *x* =
combine-options (*f x*) (*Mapping.lookup m1 x*) (*Mapping.lookup m2 x*)
by *transfer* (*auto split*: *option.splits*)

lemma *combine-altdef*: *combine f m1 m2* = *combine-with-key* (λ-. *f*) *m1 m2*
by *transfer'* (*rule refl*)

lemma *lookup-combine*:
Mapping.lookup (*combine f m1 m2*) *x* =
combine-options f (*Mapping.lookup m1 x*) (*Mapping.lookup m2 x*)
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-default-neutral-combine-with-key*:
assumes ⋀*x*. *f k d x* = *x* ⋀*x*. *f k x d* = *x*
shows *Mapping.lookup-default d* (*combine-with-key f m1 m2*) *k* =
f k (*Mapping.lookup-default d m1 k*) (*Mapping.lookup-default d m2 k*)
by (*auto simp*: *lookup-default-def lookup-combine-with-key assms split*: *option.splits*)

lemma *lookup-default-neutral-combine*:
assumes $\bigwedge x.\ f\ d\ x = x\ \bigwedge x.\ f\ x\ d = x$
shows *Mapping.lookup-default d* (*combine f m1 m2*) *x* =
f (*Mapping.lookup-default d m1 x*) (*Mapping.lookup-default d m2 x*)
by (*auto simp*: *lookup-default-def lookup-combine assms split*: *option.splits*)

lemma *lookup-map-entry*: *lookup* (*map-entry x f m*) *x* = *map-option f* (*lookup m x*)
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-map-entry-neq*: $x \neq y \Longrightarrow$ *lookup* (*map-entry x f m*) *y* = *lookup m y*
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-map-entry'*:
lookup (*map-entry x f m*) *y* =
(*if x* = *y then map-option f* (*lookup m y*) *else lookup m y*)
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-default*: *lookup* (*default x d m*) *x* = *Some* (*lookup-default d m x*)
unfolding *lookup-default-def default-def*
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-default-neq*: $x \neq y \Longrightarrow$ *lookup* (*default x d m*) *y* = *lookup m y*
unfolding *lookup-default-def default-def*
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-default'*:
lookup (*default x d m*) *y* =
(*if x* = *y then Some* (*lookup-default d m x*) *else lookup m y*)
unfolding *lookup-default-def default-def*
by *transfer* (*auto split*: *option.splits*)

lemma *lookup-map-default*: *lookup* (*map-default x d f m*) *x* = *Some* (*f* (*lookup-default d m x*))
unfolding *lookup-default-def default-def*
by (*simp add*: *map-default-def lookup-map-entry lookup-default lookup-default-def*)

lemma *lookup-map-default-neq*: $x \neq y \Longrightarrow$ *lookup* (*map-default x d f m*) *y* = *lookup m y*
unfolding *lookup-default-def default-def*
by (*simp add*: *map-default-def lookup-map-entry-neq lookup-default-neq*)

lemma *lookup-map-default'*:
lookup (*map-default x d f m*) *y* =
(*if x* = *y then Some* (*f* (*lookup-default d m x*)) *else lookup m y*)
unfolding *lookup-default-def default-def*
by (*simp add*: *map-default-def lookup-map-entry' lookup-default' lookup-default-def*)

lemma *lookup-tabulate*:
assumes *distinct xs*
shows *Mapping.lookup* (*Mapping.tabulate xs f*) *x* = (*if x* ∈ *set xs then Some* (*f x*) *else None*)
using *assms* **by** *transfer* (*auto simp*: *map-of-eq-None-iff o-def dest*!: *map-of-SomeD*)

lemma *lookup-of-alist*: *lookup* (*of-alist xs*) *k* = *map-of xs k*
by *transfer simp-all*

lemma *keys-is-none-rep* [*code-unfold*]: *k* ∈ *keys m* ⟷ ¬ (*Option.is-none* (*lookup m k*))
by *transfer* (*auto simp add*: *Option.is-none-def*)

lemma *update-update*:
update k v (*update k w m*) = *update k v m*
k ≠ *l* ⟹ *update k v* (*update l w m*) = *update l w* (*update k v m*)
by (*transfer*; *simp add*: *fun-upd-twist*)+

lemma *update-delete* [*simp*]: *update k v* (*delete k m*) = *update k v m*
by *transfer simp*

lemma *delete-update*:
delete k (*update k v m*) = *delete k m*
k ≠ *l* ⟹ *delete k* (*update l v m*) = *update l v* (*delete k m*)
by (*transfer*; *simp add*: *fun-upd-twist*)+

lemma *delete-empty* [*simp*]: *delete k empty* = *empty*
by *transfer simp*

lemma *Mapping-delete-if-notin-keys*[*simp*]:
k ∉ *keys m* ⟹ *delete k m* = *m*
by *transfer simp*

lemma *replace-update*:
k ∉ *keys m* ⟹ *replace k v m* = *m*
k ∈ *keys m* ⟹ *replace k v m* = *update k v m*
by (*transfer*; *auto simp add*: *replace-def fun-upd-twist*)+

lemma *map-values-update*: *map-values f* (*update k v m*) = *update k* (*f k v*) (*map-values f m*)
by *transfer* (*simp-all add*: *fun-eq-iff*)

lemma *size-mono*: *finite* (*keys m*′) ⟹ *keys m* ⊆ *keys m*′ ⟹ *size m* ≤ *size m*′
unfolding *size-def* **by** (*auto intro*: *card-mono*)

lemma *size-empty* [*simp*]: *size empty* = *0*
unfolding *size-def* **by** *transfer simp*

lemma *size-update*:

finite (*keys m*) ⟹ *size* (*update k v m*) =
(*if k* ∈ *keys m then size m else Suc* (*size m*))
unfolding *size-def* **by** *transfer* (*auto simp add*: *insert-dom*)

lemma *size-delete*: *size* (*delete k m*) = (*if k* ∈ *keys m then size m* − *1 else size m*)
unfolding *size-def* **by** *transfer simp*

lemma *size-tabulate* [*simp*]: *size* (*tabulate ks f*) = *length* (*remdups ks*)
unfolding *size-def* **by** *transfer* (*auto simp add*: *map-of-map-restrict card-set comp-def*)

lemma *keys-filter*: *keys* (*filter P m*) ⊆ *keys m*
by *transfer* (*auto split*: *option.splits*)

lemma *size-filter*: *finite* (*keys m*) ⟹ *size* (*filter P m*) ≤ *size m*
by (*intro size-mono keys-filter*)

lemma *bulkload-tabulate*: *bulkload xs* = *tabulate* [*0..*<*length xs*] (*nth xs*)
by *transfer* (*auto simp add*: *map-of-map-restrict*)

lemma *is-empty-empty* [*simp*]: *is-empty empty*
unfolding *is-empty-def* **by** *transfer simp*

lemma *is-empty-update* [*simp*]: ¬ *is-empty* (*update k v m*)
unfolding *is-empty-def* **by** *transfer simp*

lemma *is-empty-delete*: *is-empty* (*delete k m*) ⟷ *is-empty m* ∨ *keys m* = {*k*}
unfolding *is-empty-def* **by** *transfer* (*auto simp del*: *dom-eq-empty-conv*)

lemma *is-empty-replace* [*simp*]: *is-empty* (*replace k v m*) ⟷ *is-empty m*
unfolding *is-empty-def replace-def* **by** *transfer auto*

lemma *is-empty-default* [*simp*]: ¬ *is-empty* (*default k v m*)
unfolding *is-empty-def default-def* **by** *transfer auto*

lemma *is-empty-map-entry* [*simp*]: *is-empty* (*map-entry k f m*) ⟷ *is-empty m*
unfolding *is-empty-def* **by** *transfer* (*auto split*: *option.split*)

lemma *is-empty-map-values* [*simp*]: *is-empty* (*map-values f m*) ⟷ *is-empty m*
unfolding *is-empty-def* **by** *transfer* (*auto simp*: *fun-eq-iff*)

lemma *is-empty-map-default* [*simp*]: ¬ *is-empty* (*map-default k v f m*)
by (*simp add*: *map-default-def*)

lemma *keys-dom-lookup*: *keys m* = *dom* (*Mapping.lookup m*)
by *transfer rule*

lemma *keys-empty* [*simp*]: *keys empty* = {}
by *transfer* (*fact dom-empty*)

lemma *in-keysD*: $k \in keys\ m \Longrightarrow \exists v.\ lookup\ m\ k = Some\ v$
by *transfer* (*fact domD*)

lemma *keys-update* [*simp*]: $keys\ (update\ k\ v\ m) = insert\ k\ (keys\ m)$
by *transfer simp*

lemma *keys-delete* [*simp*]: $keys\ (delete\ k\ m) = keys\ m - \{k\}$
by *transfer simp*

lemma *keys-replace* [*simp*]: $keys\ (replace\ k\ v\ m) = keys\ m$
unfolding *replace-def* **by** *transfer* (*simp add*: *insert-absorb*)

lemma *keys-default* [*simp*]: $keys\ (default\ k\ v\ m) = insert\ k\ (keys\ m)$
unfolding *default-def* **by** *transfer* (*simp add*: *insert-absorb*)

lemma *keys-map-entry* [*simp*]: $keys\ (map\text{-}entry\ k\ f\ m) = keys\ m$
by *transfer* (*auto split*: *option.split*)

lemma *keys-map-default* [*simp*]: $keys\ (map\text{-}default\ k\ v\ f\ m) = insert\ k\ (keys\ m)$
by (*simp add*: *map-default-def*)

lemma *keys-map-values* [*simp*]: $keys\ (map\text{-}values\ f\ m) = keys\ m$
by *transfer* (*simp-all add*: *dom-def*)

lemma *keys-combine-with-key* [*simp*]:
$Mapping.keys\ (combine\text{-}with\text{-}key\ f\ m1\ m2) = Mapping.keys\ m1 \cup Mapping.keys\ m2$
by *transfer* (*auto simp*: *dom-def combine-options-def split*: *option.splits*)

lemma *keys-combine* [*simp*]: $Mapping.keys\ (combine\ f\ m1\ m2) = Mapping.keys\ m1 \cup Mapping.keys\ m2$
by (*simp add*: *combine-altdef*)

lemma *keys-tabulate* [*simp*]: $keys\ (tabulate\ ks\ f) = set\ ks$
by *transfer* (*simp add*: *map-of-map-restrict o-def*)

lemma *keys-of-alist* [*simp*]: $keys\ (of\text{-}alist\ xs) = set\ (List.map\ fst\ xs)$
by *transfer* (*simp-all add*: *dom-map-of-conv-image-fst*)

lemma *keys-bulkload* [*simp*]: $keys\ (bulkload\ xs) = \{0..<length\ xs\}$
by (*simp add*: *bulkload-tabulate*)

lemma *finite-keys-update*[*simp*]:
$finite\ (keys\ (update\ k\ v\ m)) = finite\ (keys\ m)$
by *transfer simp*

lemma *set-ordered-keys*[*simp*]:
$finite\ (Mapping.keys\ m) \Longrightarrow set\ (Mapping.ordered\text{-}keys\ m) = Mapping.keys\ m$

unfolding *ordered-keys-def* **by** *transfer auto*

lemma *distinct-ordered-keys* [*simp*]: *distinct* (*ordered-keys m*)
by (*simp add*: *ordered-keys-def*)

lemma *ordered-keys-infinite* [*simp*]: ¬ *finite* (*keys m*) ⟹ *ordered-keys m* = []
by (*simp add*: *ordered-keys-def*)

lemma *ordered-keys-empty* [*simp*]: *ordered-keys empty* = []
by (*simp add*: *ordered-keys-def*)

lemma *sorted-ordered-keys*[*simp*]: *sorted* (*ordered-keys m*)
unfolding *ordered-keys-def* **by** *simp*

lemma *ordered-keys-update* [*simp*]:
k ∈ *keys m* ⟹ *ordered-keys* (*update k v m*) = *ordered-keys m*
finite (*keys m*) ⟹ *k* ∉ *keys m* ⟹
ordered-keys (*update k v m*) = *insort k* (*ordered-keys m*)
by (*simp-all add*: *ordered-keys-def*)
(*auto simp only*: *sorted-list-of-set-insert-remove*[*symmetric*] *insert-absorb*)

lemma *ordered-keys-delete* [*simp*]: *ordered-keys* (*delete k m*) = *remove1 k* (*ordered-keys m*)
proof (*cases finite* (*keys m*))
case *False*
then show *?thesis* **by** *simp*
next
case *fin*: *True*
show *?thesis*
proof (*cases k* ∈ *keys m*)
case *False*
with *fin* **have** *k* ∉ *set* (*sorted-list-of-set* (*keys m*))
by *simp*
with *False* **show** *?thesis*
by (*simp add*: *ordered-keys-def remove1-idem*)
next
case *True*
with *fin* **show** *?thesis*
by (*simp add*: *ordered-keys-def sorted-list-of-set-remove*)
qed
qed

lemma *ordered-keys-replace* [*simp*]: *ordered-keys* (*replace k v m*) = *ordered-keys m*
by (*simp add*: *replace-def*)

lemma *ordered-keys-default* [*simp*]:
k ∈ *keys m* ⟹ *ordered-keys* (*default k v m*) = *ordered-keys m*
finite (*keys m*) ⟹ *k* ∉ *keys m* ⟹ *ordered-keys* (*default k v m*) = *insort k* (*ordered-keys m*)

by (*simp-all add*: *default-def*)

lemma *ordered-keys-map-entry* [*simp*]: *ordered-keys* (*map-entry k f m*) = *ordered-keys m*
by (*simp add*: *ordered-keys-def*)

lemma *ordered-keys-map-default* [*simp*]:
k ∈ *keys m* ⟹ *ordered-keys* (*map-default k v f m*) = *ordered-keys m*
finite (*keys m*) ⟹ *k* ∉ *keys m* ⟹ *ordered-keys* (*map-default k v f m*) = *insort k* (*ordered-keys m*)
by (*simp-all add*: *map-default-def*)

lemma *ordered-keys-tabulate* [*simp*]: *ordered-keys* (*tabulate ks f*) = *sort* (*remdups ks*)
by (*simp add*: *ordered-keys-def sorted-list-of-set-sort-remdups*)

lemma *ordered-keys-bulkload* [*simp*]: *ordered-keys* (*bulkload ks*) = [*0*..<*length ks*]
by (*simp add*: *ordered-keys-def*)

lemma *tabulate-fold*: *tabulate xs f* = *List.fold* (λ*k m*. *update k* (*f k*) *m*) *xs empty*
proof *transfer*
fix *f* :: *'a* ⇒ *'b* **and** *xs*
have *map-of* (*List.map* (λ*k*. (*k*, *f k*)) *xs*) = *foldr* (λ*k m*. *m*(*k* ↦ *f k*)) *xs Map.empty*
by (*simp add*: *foldr-map comp-def map-of-foldr*)
also have *foldr* (λ*k m*. *m*(*k* ↦ *f k*)) *xs* = *List.fold* (λ*k m*. *m*(*k* ↦ *f k*)) *xs*
by (*rule foldr-fold*) (*simp add*: *fun-eq-iff*)
ultimately show *map-of* (*List.map* (λ*k*. (*k*, *f k*)) *xs*) = *List.fold* (λ*k m*. *m*(*k* ↦ *f k*)) *xs Map.empty*
by *simp*
qed

lemma *All-mapping-mono*:
(⋀*k v*. *k* ∈ *keys m* ⟹ *P k v* ⟹ *Q k v*) ⟹ *All-mapping m P* ⟹ *All-mapping m Q*
unfolding *All-mapping-def* **by** *transfer* (*auto simp*: *All-mapping-def dom-def split*: *option.splits*)

lemma *All-mapping-empty* [*simp*]: *All-mapping Mapping.empty P*
by (*auto simp*: *All-mapping-def lookup-empty*)

lemma *All-mapping-update-iff*:
All-mapping (*Mapping.update k v m*) *P* ⟷ *P k v* ∧ *All-mapping m* (λ*k' v'*. *k* = *k'* ∨ *P k' v'*)
unfolding *All-mapping-def*
proof *safe*
assume ∀*x*. *case Mapping.lookup* (*Mapping.update k v m*) *x of None* ⇒ *True* | *Some y* ⇒ *P x y*
then have ∗: *case Mapping.lookup* (*Mapping.update k v m*) *x of None* ⇒ *True* |

Some y ⇒ P x y **for** *x*
by *blast*
from *∗[of k]* **show** *P k v*
by (*simp add*: *lookup-update*)
show *case Mapping.lookup m x of None ⇒ True | Some v′ ⇒ k = x ∨ P x v′* **for** *x*
using *∗[of x]* **by** (*auto simp add*: *lookup-update′ split*: *if-splits option.splits*)
next
assume *P k v*
assume *∀x. case Mapping.lookup m x of None ⇒ True | Some v′ ⇒ k = x ∨ P x v′*
then have *A*: *case Mapping.lookup m x of None ⇒ True | Some v′ ⇒ k = x ∨ P x v′* **for** *x*
by *blast*
show *case Mapping.lookup (Mapping.update k v m) x of None ⇒ True | Some xa ⇒ P x xa* **for** *x*
using *‹P k v› A[of x]* **by** (*auto simp*: *lookup-update′ split*: *option.splits*)
qed

lemma *All-mapping-update*:
P k v ⟹ All-mapping m (λk′ v′. k = k′ ∨ P k′ v′) ⟹ All-mapping (Mapping.update k v m) P
by (*simp add*: *All-mapping-update-iff*)

lemma *All-mapping-filter-iff*: *All-mapping (filter P m) Q ⟷ All-mapping m (λk v. P k v ⟶ Q k v)*
by (*auto simp*: *All-mapping-def lookup-filter split*: *option.splits*)

lemma *All-mapping-filter*: *All-mapping m Q ⟹ All-mapping (filter P m) Q*
by (*auto simp*: *All-mapping-filter-iff intro*: *All-mapping-mono*)

lemma *All-mapping-map-values*: *All-mapping (map-values f m) P ⟷ All-mapping m (λk v. P k (f k v))*
by (*auto simp*: *All-mapping-def lookup-map-values split*: *option.splits*)

lemma *All-mapping-tabulate*: *(∀x∈set xs. P x (f x)) ⟹ All-mapping (Mapping.tabulate xs f) P*
unfolding *All-mapping-def*
apply (*intro allI*)
apply *transfer*
apply (*auto split*: *option.split dest*!: *map-of-SomeD*)
done

lemma *All-mapping-alist*:
(⋀k v. (k, v) ∈ set xs ⟹ P k v) ⟹ All-mapping (Mapping.of-alist xs) P
by (*auto simp*: *All-mapping-def lookup-of-alist dest*!: *map-of-SomeD split*: *option.splits*)

lemma *combine-empty* [*simp*]: *combine f Mapping.empty y = y combine f y Map-*

ping.empty = y
by (*transfer*; *force*)+

lemma (**in** *abel-semigroup*) *comm-monoid-set-combine*: *comm-monoid-set* (*combine f*) *Mapping.empty*
by *standard* (*transfer fixing*: *f*, *simp add*: *combine-options-ac*[*of f*] *ac-simps*)+

locale *combine-mapping-abel-semigroup* = *abel-semigroup*
begin

sublocale *combine*: *comm-monoid-set combine f Mapping.empty*
by (*rule comm-monoid-set-combine*)

lemma *fold-combine-code*:
combine.F g (*set xs*) = *foldr* (λ*x*. *combine f* (*g x*)) (*remdups xs*) *Mapping.empty*
proof −
have *combine.F g* (*set xs*) = *foldr* (λ*x*. *combine f* (*g x*)) *xs Mapping.empty*
if *distinct xs* **for** *xs*
using *that* **by** (*induction xs*) *simp-all*
from *this*[*of remdups xs*] **show** *?thesis* **by** *simp*
qed

lemma *keys-fold-combine*: *finite A* ⟹ *Mapping.keys* (*combine.F g A*) = (⋃*x*∈*A*. *Mapping.keys* (*g x*))
by (*induct A rule*: *finite-induct*) *simp-all*

end

59.5.1 *entries*, *ordered-entries*, and *fold*

context *linorder*
begin

sublocale *folding-Map-graph*: *folding-insort-key* (≤) (<) *Map.graph m fst* **for** *m*
by *unfold-locales* (*fact inj-on-fst-graph*)

end

lemma *sorted-fst-list-of-set-insort-Map-graph*[*simp*]:
assumes *finite* (*dom m*) *fst x* ∉ *dom m*
shows *sorted-key-list-of-set fst* (*insert x* (*Map.graph m*))
= *insort-key fst x* (*sorted-key-list-of-set fst* (*Map.graph m*))
proof(*cases x*)
case (*Pair k v*)
with ‹*fst x* ∉ *dom m*› **have** *Map.graph m* ⊆ *Map.graph* (*m*(*k* ↦ *v*))
by(*auto simp*: *graph-def*)
moreover from *Pair* ‹*fst x* ∉ *dom m*› **have** (*k*, *v*) ∉ *Map.graph m*
using *graph-domD* **by** *fastforce*
ultimately show *?thesis*

using *Pair assms folding-Map-graph.sorted-key-list-of-set-insert*[**where** *?m*=*m*(*k* ↦ *v*)]
by *auto*
qed

lemma *sorted-fst-list-of-set-insort-insert-Map-graph*[*simp*]:
assumes *finite* (*dom m*) *fst x* ∉ *dom m*
shows *sorted-key-list-of-set fst* (*insert x* (*Map.graph m*))
= *insort-insert-key fst x* (*sorted-key-list-of-set fst* (*Map.graph m*))
proof(*cases x*)
case (*Pair k v*)
with ‹*fst x* ∉ *dom m*› **have** *Map.graph m* ⊆ *Map.graph* (*m*(*k* ↦ *v*))
by(*auto simp*: *graph-def*)
with *assms Pair* **show** *?thesis*
unfolding *sorted-fst-list-of-set-insort-Map-graph*[*OF assms*] *insort-insert-key-def*
using *folding-Map-graph.set-sorted-key-list-of-set in-graphD* **by** (*fastforce split*: *if-splits*)
qed

lemma *linorder-finite-Map-induct*[*consumes 1*, *case-names empty update*]:
fixes *m* :: *'a*::*linorder* ⇀ *'b*
assumes *finite* (*dom m*)
assumes *P Map.empty*
assumes ⋀*k v m*. ⟦ *finite* (*dom m*); *k* ∉ *dom m*; (⋀*k'*. *k'* ∈ *dom m* ⟹ *k'* ≤ *k*); *P m* ⟧
⟹ *P* (*m*(*k* ↦ *v*))
shows *P m*
proof −
let *?key-list* = λ*m*. *sorted-list-of-set* (*dom m*)
from *assms*(*1*,*2*) **show** *?thesis*
proof(*induction length* (*?key-list m*) *arbitrary*: *m*)
case *0*
then have *sorted-list-of-set* (*dom m*) = []
by *auto*
with ‹*finite* (*dom m*)› **have** *m* = *Map.empty*
by *auto*
with ‹*P Map.empty*› **show** *?case* **by** *simp*
next
case (*Suc n*)
then obtain *x xs* **where** *x-xs*: *sorted-list-of-set* (*dom m*) = *xs* @ [*x*]
by (*metis append-butlast-last-id length-greater-0-conv zero-less-Suc*)
have *sorted-list-of-set* (*dom* (*m*(*x* := *None*))) = *xs*
proof −
have *distinct* (*xs* @ [*x*])
by (*metis sorted-list-of-set.distinct-sorted-key-list-of-set x-xs*)
then have *remove1 x* (*xs* @ [*x*]) = *xs*
by (*simp add*: *remove1-append*)
with ‹*finite* (*dom m*)› *x-xs* **show** *?thesis*
by (*simp add*: *sorted-list-of-set-remove*)

```
    qed
    moreover have k ≤ x if k ∈ dom (m(x := None)) for k
    proof −
      from x-xs have sorted (xs @ [x])
        by (metis sorted-list-of-set.sorted-sorted-key-list-of-set)
      moreover from ‹k ∈ dom (m(x := None))› have k ∈ set xs
        using ‹finite (dom m)› ‹sorted-list-of-set (dom (m(x := None))) = xs›
        by auto
      ultimately show k ≤ x
        by (simp add: sorted-append)
    qed
    moreover from ‹finite (dom m)› have finite (dom (m(x := None))) x ∉ dom
(m(x := None))
      by simp-all
    moreover have P (m(x := None))
      using Suc ‹sorted-list-of-set (dom (m(x := None))) = xs› x-xs by auto
    ultimately show ?case
     using assms(3)[where ?m=m(x := None)] by (metis fun-upd-triv fun-upd-upd
not-Some-eq)
  qed
qed

lemma delete-insort-fst[simp]: AList.delete k (insort-key fst (k, v) xs) = AL-
ist.delete k xs
  by (induction xs) simp-all

lemma insort-fst-delete: ⟦ fst x ≠ k2; sorted (List.map fst xs) ⟧
  ⟹ insort-key fst x (AList.delete k2 xs) = AList.delete k2 (insort-key fst x xs)
  by (induction xs) (fastforce simp add: insort-is-Cons order-trans)+

lemma sorted-fst-list-of-set-Map-graph-fun-upd-None[simp]:
  sorted-key-list-of-set fst (Map.graph (m(k := None)))
   = AList.delete k (sorted-key-list-of-set fst (Map.graph m))
proof(cases finite (Map.graph m))
  assume finite (Map.graph m)
  from this[unfolded finite-graph-iff-finite-dom] show ?thesis
  proof(induction rule: finite-Map-induct)
    let ?list-of=sorted-key-list-of-set fst
    case (update k2 v2 m)
    note [simp] = ‹k2 ∉ dom m› ‹finite (dom m)›

    have right-eq: AList.delete k (?list-of (Map.graph (m(k2 ↦ v2))))
      = AList.delete k (insort-key fst (k2, v2) (?list-of (Map.graph m)))
      by simp

    show ?case
    proof(cases k = k2)
      case True
      then have ?list-of (Map.graph ((m(k2 ↦ v2))(k := None)))
```

```
      = AList.delete k (insort-key fst (k2, v2) (?list-of (Map.graph m)))
      using fst-graph-eq-dom update.IH by auto
    then show ?thesis
      using right-eq by metis
  next
    case False
    then have AList.delete k (insort-key fst (k2, v2) (?list-of (Map.graph m)))
      = insort-key fst (k2, v2) (?list-of (Map.graph (m(k := None))))
      by (auto simp add: insort-fst-delete update.IH
                folding-Map-graph.sorted-sorted-key-list-of-set[OF subset-refl])
    also have ... = ?list-of (insert (k2, v2) (Map.graph (m(k := None))))
      by auto
    also from False ‹k2 ∉ dom m› have ... = ?list-of (Map.graph ((m(k2 ↦
v2))(k := None)))
      by (metis graph-map-upd domIff fun-upd-triv fun-upd-twist)
    finally show ?thesis using right-eq by metis
  qed
 qed simp
qed simp

lemma entries-empty[simp]: entries empty = {}
 by transfer (fact graph-empty)

lemma entries-lookup: entries m = Map.graph (lookup m)
 by transfer rule

lemma in-entriesI: lookup m k = Some v ⟹ (k, v) ∈ entries m
 by transfer (fact in-graphI)

lemma in-entriesD: (k, v) ∈ entries m ⟹ lookup m k = Some v
 by transfer (fact in-graphD)

lemma fst-image-entries-eq-keys[simp]: fst ‘ Mapping.entries m = Mapping.keys
m
 by transfer (fact fst-graph-eq-dom)

lemma finite-entries-iff-finite-keys[simp]:
 finite (entries m) = finite (keys m)
 by transfer (fact finite-graph-iff-finite-dom)

lemma entries-update:
 entries (update k v m) = insert (k, v) (entries (delete k m))
 by transfer (fact graph-map-upd)

lemma entries-delete:
 entries (delete k m) = {e ∈ entries m. fst e ≠ k}
 by transfer (fact graph-fun-upd-None)

lemma entries-of-alist[simp]:
```

distinct (*List.map fst xs*) ⟹ *entries* (*of-alist xs*) = *set xs*
by *transfer* (*fact graph-map-of-if-distinct-dom*)

lemma *entries-keysD*:
x ∈ *entries m* ⟹ *fst x* ∈ *keys m*
by *transfer* (*fact graph-domD*)

lemma *set-ordered-entries*[*simp*]:
finite (*keys m*) ⟹ *set* (*ordered-entries m*) = *entries m*
unfolding *ordered-entries-def*
by *transfer* (*auto simp*: *folding-Map-graph.set-sorted-key-list-of-set*[*OF subset-refl*])

lemma *distinct-ordered-entries*[*simp*]: *distinct* (*List.map fst* (*ordered-entries m*))
unfolding *ordered-entries-def*
by *transfer* (*simp add*: *folding-Map-graph.distinct-sorted-key-list-of-set*[*OF subset-refl*])

lemma *sorted-ordered-entries*[*simp*]: *sorted* (*List.map fst* (*ordered-entries m*))
unfolding *ordered-entries-def*
by *transfer* (*auto intro*: *folding-Map-graph.sorted-sorted-key-list-of-set*)

lemma *ordered-entries-infinite*[*simp*]:
¬ *finite* (*Mapping.keys m*) ⟹ *ordered-entries m* = []
by (*simp add*: *ordered-entries-def*)

lemma *ordered-entries-empty*[*simp*]: *ordered-entries empty* = []
by (*simp add*: *ordered-entries-def*)

lemma *ordered-entries-update*[*simp*]:
assumes *finite* (*keys m*)
shows *ordered-entries* (*update k v m*)
= *insort-insert-key fst* (*k*, *v*) (*AList.delete k* (*ordered-entries m*))
proof −
let *?list-of*=*sorted-key-list-of-set fst* **and** *?insort*=*insort-insert-key fst*

have ∗: *?list-of* (*insert* (*k*, *v*) (*Map.graph* (*m*(*k* := *None*))))
= *?insort* (*k*, *v*) (*AList.delete k* (*?list-of* (*Map.graph m*))) **if** *finite* (*dom m*) **for** *m*
proof −
from ‹*finite* (*dom m*)› **have** *?list-of* (*insert* (*k*, *v*) (*Map.graph* (*m*(*k* := *None*))))
= *?insort* (*k*, *v*) (*?list-of* (*Map.graph* (*m*(*k* := *None*))))
by (*intro sorted-fst-list-of-set-insort-insert-Map-graph*) (*simp-all add*: *subset-insertI*)
then show *?thesis* **by** *simp*
qed
from *assms* **show** *?thesis*
unfolding *ordered-entries-def*
apply (*transfer fixing*: *k v*) **using** ∗ **by** *auto*
qed

lemma *ordered-entries-delete*[*simp*]:
ordered-entries (*delete k m*) = *AList.delete k* (*ordered-entries m*)
unfolding *ordered-entries-def* **by** *transfer auto*

lemma *map-fst-ordered-entries*[*simp*]:
List.map fst (*ordered-entries m*) = *ordered-keys m*
proof(*cases finite* (*Mapping.keys m*))
case *True*
then have *set* (*List.map fst* (*Mapping.ordered-entries m*)) = *set* (*Mapping.ordered-keys m*)
unfolding *ordered-entries-def ordered-keys-def*
by (*transfer*) (*simp add*: *folding-Map-graph.set-sorted-key-list-of-set*[*OF subset-refl*] *fst-graph-eq-dom*)
with *True* **show** *List.map fst* (*Mapping.ordered-entries m*) = *Mapping.ordered-keys m*
by (*metis distinct-ordered-entries ordered-keys-def sorted-list-of-set.idem-if-sorted-distinct*
sorted-list-of-set.set-sorted-key-list-of-set sorted-ordered-entries)
next
case *False*
then show *?thesis*
unfolding *ordered-entries-def ordered-keys-def* **by** *simp*
qed

lemma *fold-empty*[*simp*]: *fold f empty a* = *a*
unfolding *fold-def* **by** *simp*

lemma *insort-key-is-snoc-if-sorted-and-distinct*:
assumes *sorted* (*List.map f xs*) *f y* ∉ *f* ' *set xs* ∀ *x* ∈ *set xs*. *f x* ≤ *f y*
shows *insort-key f y xs* = *xs* @ [*y*]
using *assms* **by** (*induction xs*) (*auto dest*!: *insort-is-Cons*)

lemma *fold-update*:
assumes *finite* (*keys m*)
assumes *k* ∉ *keys m* ⋀*k'*. *k'* ∈ *keys m* ⟹ *k'* ≤ *k*
shows *fold f* (*update k v m*) *a* = *f k v* (*fold f m a*)
proof −
from *assms* **have** *k-notin-entries*: *k* ∉ *fst* ' *set* (*ordered-entries m*)
using *entries-keysD* **by** *fastforce*
with *assms* **have** *ordered-entries* (*update k v m*)
= *insort-insert-key fst* (*k*, *v*) (*ordered-entries m*)
by *simp*
also from *k-notin-entries* **have** ... = *ordered-entries m* @ [(*k*, *v*)]
proof −
from *assms* **have** ∀ *x* ∈ *set* (*ordered-entries m*). *fst x* ≤ *fst* (*k*, *v*)
unfolding *ordered-entries-def*
by *transfer* (*fastforce simp*: *folding-Map-graph.set-sorted-key-list-of-set*[*OF order-refl*]

dest: *graph-domD*)
from *insort-key-is-snoc-if-sorted-and-distinct*[*OF* - - *this*] *k-notin-entries* ‹*finite* (*keys m*)›
show *?thesis*
using *sorted-ordered-keys*
unfolding *insort-insert-key-def* **by** *auto*
qed
finally show *?thesis* **unfolding** *fold-def* **by** *simp*
qed

lemma *linorder-finite-Mapping-induct*[*consumes 1*, *case-names empty update*]:
fixes *m* :: ('*a*::*linorder*, '*b*) *mapping*
assumes *finite* (*keys m*)
assumes *P empty*
assumes ⋀*k v m*.
⟦ *finite* (*keys m*); *k* ∉ *keys m*; (⋀*k'*. *k'* ∈ *keys m* ⟹ *k'* ≤ *k*); *P m* ⟧
⟹ *P* (*update k v m*)
shows *P m*
using *assms* **by** *transfer* (*simp add*: *linorder-finite-Map-induct*)

59.6 Code generator setup

hide-const (**open**) *empty is-empty rep lookup lookup-default filter update delete ordered-keys*
keys size replace default map-entry map-default tabulate bulkload map map-values combine of-alist
entries ordered-entries fold

end

60 Monad notation for arbitrary types

theory *Monad-Syntax*
imports *Adhoc-Overloading*
begin

We provide a convenient do-notation for monadic expressions well-known from Haskell. const>*Let* is printed specially in do-expressions.

consts
bind :: '*a* ⇒ ('*b* ⇒ '*c*) ⇒ '*d* (**infixl** ≫= *54*)

notation (*ASCII*)
bind (**infixl** >>= *54*)

abbreviation (*do-notation*)
bind-do :: '*a* ⇒ ('*b* ⇒ '*c*) ⇒ '*d*
where *bind-do* ≡ *bind*

notation (**output**)
bind-do (**infixl** $\gg\!=$ *54*)

notation (*ASCII* **output**)
bind-do (**infixl** >>= *54*)

nonterminal *do-binds* **and** *do-bind*
syntax
-do-block :: *do-binds* ⇒ *'a* (*do* {//(*2* -)//} [*12*] *62*)
-do-bind :: [*pttrn*, *'a*] ⇒ *do-bind* ((*2*- ←/ -) *13*)
-do-let :: [*pttrn*, *'a*] ⇒ *do-bind* ((*2let* - =/ -) [*1000*, *13*] *13*)
-do-then :: *'a* ⇒ *do-bind* (- [*14*] *13*)
-do-final :: *'a* ⇒ *do-binds* (-)
-do-cons :: [*do-bind*, *do-binds*] ⇒ *do-binds* (-;//- [*13*, *12*] *12*)
-thenM :: [*'a*, *'b*] ⇒ *'c* (**infixl** ≫ *54*)

syntax (*ASCII*)
-do-bind :: [*pttrn*, *'a*] ⇒ *do-bind* ((*2*- <−/ -) *13*)
-thenM :: [*'a*, *'b*] ⇒ *'c* (**infixl** >> *54*)

translations
-do-block (*-do-cons* (*-do-then t*) (*-do-final e*))
⇌ *CONST bind-do t* (λ-. *e*)
-do-block (*-do-cons* (*-do-bind p t*) (*-do-final e*))
⇌ *CONST bind-do t* (λ*p*. *e*)
-do-block (*-do-cons* (*-do-let p t*) *bs*)
⇌ *let p* = *t in -do-block bs*
-do-block (*-do-cons b* (*-do-cons c cs*))
⇌ *-do-block* (*-do-cons b* (*-do-final* (*-do-block* (*-do-cons c cs*))))
-do-cons (*-do-let p t*) (*-do-final s*)
⇌ *-do-final* (*let p* = *t in s*)
-do-block (*-do-final e*) ⇀ *e*
(*m* ≫ *n*) ↽ (*m* ≫= (λ-. *n*))

adhoc-overloading
bind Set.bind Predicate.bind Option.bind List.bind

end

61 Less common functions on lists

theory *More-List*
imports *Main*
begin

definition *strip-while* :: (*'a* ⇒ *bool*) ⇒ *'a list* ⇒ *'a list*
where
strip-while P = *rev* ∘ *dropWhile P* ∘ *rev*

lemma *strip-while-rev* [*simp*]:
strip-while P (*rev xs*) = *rev* (*dropWhile P xs*)
by (*simp add*: *strip-while-def*)

lemma *strip-while-Nil* [*simp*]:
strip-while P [] = []
by (*simp add*: *strip-while-def*)

lemma *strip-while-append* [*simp*]:
¬ *P x* ⟹ *strip-while P* (*xs* @ [*x*]) = *xs* @ [*x*]
by (*simp add*: *strip-while-def*)

lemma *strip-while-append-rec* [*simp*]:
P x ⟹ *strip-while P* (*xs* @ [*x*]) = *strip-while P xs*
by (*simp add*: *strip-while-def*)

lemma *strip-while-Cons* [*simp*]:
¬ *P x* ⟹ *strip-while P* (*x* # *xs*) = *x* # *strip-while P xs*
by (*induct xs rule*: *rev-induct*) (*simp-all add*: *strip-while-def*)

lemma *strip-while-eq-Nil* [*simp*]:
strip-while P xs = [] ⟷ (∀ *x*∈*set xs*. *P x*)
by (*simp add*: *strip-while-def*)

lemma *strip-while-eq-Cons-rec*:
strip-while P (*x* # *xs*) = *x* # *strip-while P xs* ⟷ ¬ (*P x* ∧ (∀ *x*∈*set xs*. *P x*))
by (*induct xs rule*: *rev-induct*) (*simp-all add*: *strip-while-def*)

lemma *split-strip-while-append*:
fixes *xs* :: *'a list*
obtains *ys zs* :: *'a list*
where *strip-while P xs* = *ys* **and** ∀ *x*∈*set zs*. *P x* **and** *xs* = *ys* @ *zs*
proof (*rule that*)
show *strip-while P xs* = *strip-while P xs* **..**
show ∀ *x*∈*set* (*rev* (*takeWhile P* (*rev xs*))). *P x* **by** (*simp add*: *takeWhile-eq-all-conv* [*symmetric*])
have *rev xs* = *rev* (*strip-while P xs* @ *rev* (*takeWhile P* (*rev xs*)))
by (*simp add*: *strip-while-def*)
then show *xs* = *strip-while P xs* @ *rev* (*takeWhile P* (*rev xs*))
by (*simp only*: *rev-is-rev-conv*)
qed

lemma *strip-while-snoc* [*simp*]:
strip-while P (*xs* @ [*x*]) = (*if P x then strip-while P xs else xs* @ [*x*])
by (*simp add*: *strip-while-def*)

lemma *strip-while-map*:
strip-while P (*map f xs*) = *map f* (*strip-while* (*P* ∘ *f*) *xs*)

by (*simp add*: *strip-while-def rev-map dropWhile-map*)

lemma *strip-while-dropWhile-commute*:
strip-while P (*dropWhile Q xs*) = *dropWhile Q* (*strip-while P xs*)
proof (*induct xs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons x xs*)
show *?case*
proof (*cases* ∀ *y*∈*set xs*. *P y*)
case *True*
with *dropWhile-append2* [*of rev xs*] **show** *?thesis*
by (*auto simp add*: *strip-while-def dest*: *set-dropWhileD*)
next
case *False*
then obtain *y* **where** *y* ∈ *set xs* **and** ¬ *P y*
by *blast*
with *Cons dropWhile-append3* [*of P y rev xs*] **show** *?thesis*
by (*simp add*: *strip-while-def*)
qed
qed

lemma *dropWhile-strip-while-commute*:
dropWhile P (*strip-while Q xs*) = *strip-while Q* (*dropWhile P xs*)
by (*simp add*: *strip-while-dropWhile-commute*)

definition *no-leading* :: ('*a* ⇒ *bool*) ⇒ '*a list* ⇒ *bool*
where
no-leading P xs ⟷ (*xs* ≠ [] ⟶ ¬ *P* (*hd xs*))

lemma *no-leading-Nil* [*simp*, *intro*!]:
no-leading P []
by (*simp add*: *no-leading-def*)

lemma *no-leading-Cons* [*simp*, *intro*!]:
no-leading P (*x* # *xs*) ⟷ ¬ *P x*
by (*simp add*: *no-leading-def*)

lemma *no-leading-append* [*simp*]:
no-leading P (*xs* @ *ys*) ⟷ *no-leading P xs* ∧ (*xs* = [] ⟶ *no-leading P ys*)
by (*induct xs*) *simp-all*

lemma *no-leading-dropWhile* [*simp*]:
no-leading P (*dropWhile P xs*)
by (*induct xs*) *simp-all*

lemma *dropWhile-eq-obtain-leading*:
assumes *dropWhile P xs = ys*
obtains *zs* **where** *xs = zs @ ys* **and** $\bigwedge z.\ z \in set\ zs \Longrightarrow P\ z$ **and** *no-leading P ys*
proof −
from *assms* **have** $\exists\, zs.\ xs = zs\ @\ ys \wedge (\forall\, z \in set\ zs.\ P\ z) \wedge$ *no-leading P ys*
proof (*induct xs arbitrary*: *ys*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs ys*)
show *?case* **proof** (*cases P x*)
case *True* **with** *Cons.hyps* [*of ys*] *Cons.prems*
have $\exists\, zs.\ xs = zs\ @\ ys \wedge (\forall\, a{\in}set\ zs.\ P\ a) \wedge$ *no-leading P ys*
by *simp*
then obtain *zs* **where** *xs = zs @ ys* **and** $\bigwedge z.\ z \in set\ zs \Longrightarrow P\ z$
and ∗: *no-leading P ys*
by *blast*
with *True* **have** *x # xs = (x # zs) @ ys* **and** $\bigwedge z.\ z \in set\ (x \#\ zs) \Longrightarrow P\ z$
by *auto*
with ∗ **show** *?thesis*
by *blast* **next**
case *False*
with *Cons* **show** *?thesis* **by** (*cases ys*) *simp-all*
qed
qed
with *that* **show** *thesis*
by *blast*
qed

lemma *dropWhile-idem-iff*:
dropWhile P xs = xs ⟷ *no-leading P xs*
by (*cases xs*) (*auto elim*: *dropWhile-eq-obtain-leading*)

abbreviation *no-trailing* :: (′*a* ⇒ *bool*) ⇒ ′*a list* ⇒ *bool*
where
no-trailing P xs ≡ *no-leading P* (*rev xs*)

lemma *no-trailing-unfold*:
no-trailing P xs ⟷ (*xs* ≠ [] ⟶ ¬ *P* (*last xs*))
by (*induct xs*) *simp-all*

lemma *no-trailing-Nil* [*simp*, *intro*!]:
no-trailing P []
by *simp*

lemma *no-trailing-Cons* [*simp*]:
no-trailing P (*x # xs*) ⟷ *no-trailing P xs* ∧ (*xs* = [] ⟶ ¬ *P x*)
by *simp*

lemma *no-trailing-append*:
no-trailing P (xs @ ys) ⟷ no-trailing P ys ∧ (ys = [] ⟶ no-trailing P xs)
by (*induct xs*) *simp-all*

lemma *no-trailing-append-Cons* [*simp*]:
no-trailing P (xs @ y # ys) ⟷ no-trailing P (y # ys)
by *simp*

lemma *no-trailing-strip-while* [*simp*]:
no-trailing P (strip-while P xs)
by (*induct xs rule*: *rev-induct*) *simp-all*

lemma *strip-while-idem* [*simp*]:
no-trailing P xs ⟹ strip-while P xs = xs
by (*cases xs rule*: *rev-cases*) *simp-all*

lemma *strip-while-eq-obtain-trailing*:
assumes *strip-while P xs = ys*
obtains *zs* **where** *xs = ys @ zs* **and** ⋀*z*. *z ∈ set zs ⟹ P z* **and** *no-trailing P ys*
proof −
from *assms* **have** *rev (rev (dropWhile P (rev xs))) = rev ys*
by (*simp add*: *strip-while-def*)
then have *dropWhile P (rev xs) = rev ys*
by *simp*
then obtain *zs* **where** *A*: *rev xs = zs @ rev ys* **and** *B*: ⋀*z*. *z ∈ set zs ⟹ P z*
and *C*: *no-trailing P ys*
using *dropWhile-eq-obtain-leading* **by** *blast*
from *A* **have** *rev (rev xs) = rev (zs @ rev ys)*
by *simp*
then have *xs = ys @ rev zs*
by *simp*
moreover from *B* **have** ⋀*z*. *z ∈ set (rev zs) ⟹ P z*
by *simp*
ultimately show *thesis* **using** *that C* **by** *blast*
qed

lemma *strip-while-idem-iff*:
strip-while P xs = xs ⟷ no-trailing P xs
proof −
define *ys* **where** *ys = rev xs*
moreover have *strip-while P (rev ys) = rev ys ⟷ no-trailing P (rev ys)*
by (*simp add*: *dropWhile-idem-iff*)
ultimately show *?thesis* **by** *simp*
qed

lemma *no-trailing-map*:
no-trailing P (map f xs) ⟷ no-trailing (P ∘ f) xs

by (*simp add*: *last-map no-trailing-unfold*)

lemma *no-trailing-drop* [*simp*]:
no-trailing P (*drop n xs*) **if** *no-trailing P xs*
proof −
from *that* **have** *no-trailing P* (*take n xs* @ *drop n xs*)
by *simp*
then show *?thesis*
by (*simp only*: *no-trailing-append*)
qed

lemma *no-trailing-upt* [*simp*]:
no-trailing P [*n*..<*m*] ⟷ (*n* < *m* ⟶ ¬ *P* (*m* − *1*))
by (*auto simp add*: *no-trailing-unfold*)

definition *nth-default* :: *'a* ⇒ *'a list* ⇒ *nat* ⇒ *'a*
where
nth-default dflt xs n = (*if n* < *length xs then xs* ! *n else dflt*)

lemma *nth-default-nth*:
n < *length xs* ⟹ *nth-default dflt xs n* = *xs* ! *n*
by (*simp add*: *nth-default-def*)

lemma *nth-default-beyond*:
length xs ≤ *n* ⟹ *nth-default dflt xs n* = *dflt*
by (*simp add*: *nth-default-def*)

lemma *nth-default-Nil* [*simp*]:
nth-default dflt [] *n* = *dflt*
by (*simp add*: *nth-default-def*)

lemma *nth-default-Cons*:
nth-default dflt (*x* # *xs*) *n* = (*case n of 0* ⇒ *x* | *Suc n′* ⇒ *nth-default dflt xs n′*)
by (*simp add*: *nth-default-def split*: *nat.split*)

lemma *nth-default-Cons-0* [*simp*]:
nth-default dflt (*x* # *xs*) *0* = *x*
by (*simp add*: *nth-default-Cons*)

lemma *nth-default-Cons-Suc* [*simp*]:
nth-default dflt (*x* # *xs*) (*Suc n*) = *nth-default dflt xs n*
by (*simp add*: *nth-default-Cons*)

lemma *nth-default-replicate-dflt* [*simp*]:
nth-default dflt (*replicate n dflt*) *m* = *dflt*
by (*simp add*: *nth-default-def*)

lemma *nth-default-append*:

nth-default dflt (xs @ ys) n =
(if n < length xs then nth xs n else nth-default dflt ys (n − length xs))
by (*auto simp add*: *nth-default-def nth-append*)

lemma *nth-default-append-trailing* [*simp*]:
nth-default dflt (xs @ replicate n dflt) = nth-default dflt xs
by (*simp add*: *fun-eq-iff nth-default-append*) (*simp add*: *nth-default-def*)

lemma *nth-default-snoc-default* [*simp*]:
nth-default dflt (xs @ [dflt]) = nth-default dflt xs
by (*auto simp add*: *nth-default-def fun-eq-iff nth-append*)

lemma *nth-default-eq-dflt-iff*:
nth-default dflt xs k = dflt ⟷ (k < length xs ⟶ xs ! k = dflt)
by (*simp add*: *nth-default-def*)

lemma *nth-default-take-eq*:
nth-default dflt (take m xs) n =
(if n < m then nth-default dflt xs n else dflt)
by (*simp add*: *nth-default-def*)

lemma *in-enumerate-iff-nth-default-eq*:
x ≠ dflt ⟹ (n, x) ∈ set (enumerate 0 xs) ⟷ nth-default dflt xs n = x
by (*auto simp add*: *nth-default-def in-set-conv-nth enumerate-eq-zip*)

lemma *last-conv-nth-default*:
assumes *xs ≠ []*
shows *last xs = nth-default dflt xs (length xs − 1)*
using *assms* **by** (*simp add*: *nth-default-def last-conv-nth*)

lemma *nth-default-map-eq*:
f dflt′ = dflt ⟹ nth-default dflt (map f xs) n = f (nth-default dflt′ xs n)
by (*simp add*: *nth-default-def*)

lemma *finite-nth-default-neq-default* [*simp*]:
finite {k. nth-default dflt xs k ≠ dflt}
by (*simp add*: *nth-default-def*)

lemma *sorted-list-of-set-nth-default*:
sorted-list-of-set {k. nth-default dflt xs k ≠ dflt} = map fst (filter (λ(-, x). x ≠ dflt) (enumerate 0 xs))
by (*rule sorted-distinct-set-unique*) (*auto simp add*: *nth-default-def in-set-conv-nth sorted-filter distinct-map-filter enumerate-eq-zip intro*: *rev-image-eqI*)

lemma *map-nth-default*:
map (nth-default x xs) [0..<length xs] = xs
proof −
have ∗: *map (nth-default x xs) [0..<length xs] = map (List.nth xs) [0..<length xs]*

```
    by (rule map-cong) (simp-all add: nth-default-nth)
  show ?thesis by (simp add: ∗ map-nth)
qed

lemma range-nth-default [simp]:
  range (nth-default dflt xs) = insert dflt (set xs)
  by (auto simp add: nth-default-def [abs-def] in-set-conv-nth)

lemma nth-strip-while:
  assumes n < length (strip-while P xs)
  shows strip-while P xs ! n = xs ! n
proof −
  have length (dropWhile P (rev xs)) + length (takeWhile P (rev xs)) = length xs
    by (subst add.commute)
      (simp add: arg-cong [where f=length, OF takeWhile-dropWhile-id, unfolded
length-append])
  then show ?thesis using assms
    by (simp add: strip-while-def rev-nth dropWhile-nth)
qed

lemma length-strip-while-le:
  length (strip-while P xs) ≤ length xs
  unfolding strip-while-def o-def length-rev
  by (subst (2) length-rev[symmetric])
    (simp add: strip-while-def length-dropWhile-le del: length-rev)

lemma nth-default-strip-while-dflt [simp]:
  nth-default dflt (strip-while ((=) dflt) xs) = nth-default dflt xs
  by (induct xs rule: rev-induct) auto

lemma nth-default-eq-iff:
  nth-default dflt xs = nth-default dflt ys
    ⟷ strip-while (HOL.eq dflt) xs = strip-while (HOL.eq dflt) ys (is ?P ⟷
?Q)
proof
  let ?xs = strip-while (HOL.eq dflt) xs and ?ys = strip-while (HOL.eq dflt) ys
  assume ?P
  then have eq: nth-default dflt ?xs = nth-default dflt ?ys
    by simp
  have len: length ?xs = length ?ys
  proof (rule ccontr)
    assume len: length ?xs ≠ length ?ys
    { fix xs ys :: 'a list
      let ?xs = strip-while (HOL.eq dflt) xs and ?ys = strip-while (HOL.eq dflt) ys
      assume eq: nth-default dflt ?xs = nth-default dflt ?ys
      assume len: length ?xs < length ?ys
      then have length ?ys > 0 by arith
      then have ?ys ≠ [] by simp
      with last-conv-nth-default [of ?ys dflt]
```

```
    have last ?ys = nth-default dflt ?ys (length ?ys − 1)
      by auto
    moreover from ‹?ys ≠ []› no-trailing-strip-while [of HOL.eq dflt ys]
      have last ?ys ≠ dflt by (simp add: no-trailing-unfold)
    ultimately have nth-default dflt ?xs (length ?ys − 1) ≠ dflt
      using eq by simp
    moreover from len have length ?ys − 1 ≥ length ?xs by simp
    ultimately have False by (simp only: nth-default-beyond) simp
  }
  from this [of xs ys] this [of ys xs] len eq show False
    by (auto simp only: linorder-class.neq-iff)
 qed
 then show ?Q
 proof (rule nth-equalityI [rule-format])
  fix n
  assume n: n < length ?xs
  with len have n < length ?ys
    by simp
  with n have xs: nth-default dflt ?xs n = ?xs ! n
    and ys: nth-default dflt ?ys n = ?ys ! n
    by (simp-all only: nth-default-nth)
  with eq show ?xs ! n = ?ys ! n
    by simp
 qed
next
 assume ?Q
  then have nth-default dflt (strip-while (HOL.eq dflt) xs) = nth-default dflt
(strip-while (HOL.eq dflt) ys)
  by simp
 then show ?P
  by simp
qed

lemma nth-default-map2:
 ‹nth-default d (map2 f xs ys) n = f (nth-default d1 xs n) (nth-default d2 ys n)›
  if ‹length xs = length ys› and ‹f d1 d2 = d› for bs cs
using that proof (induction xs ys arbitrary: n rule: list-induct2)
 case Nil
 then show ?case
  by simp
next
 case (Cons x xs y ys)
 then show ?case
  by (cases n) simp-all
qed

end
```

theory *Cancellation*
imports *Main*
begin

named-theorems *cancelation-simproc-pre* ‹*These theorems are here to normalise the term. Special handling of constructors should be here. Remark that only the simproc @{term NO-MATCH} is also included.*›

named-theorems *cancelation-simproc-post* ‹*These theorems are here to normalise the term, after the cancelation simproc. Normalisation of ‹iterate-add› back to the normale representation should be put here.*›

named-theorems *cancelation-simproc-eq-elim* ‹*These theorems are here to help deriving contradiction (e.g., ‹Suc - = 0›).*›

definition *iterate-add* :: ‹*nat* ⇒ *'a*::*cancel-comm-monoid-add* ⇒ *'a*› **where**
‹*iterate-add n a* = (((+) *a*) ^^ *n*) *0*›

lemma *iterate-add-simps*[*simp*]:
‹*iterate-add 0 a* = *0*›
‹*iterate-add* (*Suc n*) *a* = *a* + *iterate-add n a*›
unfolding *iterate-add-def* **by** *auto*

lemma *iterate-add-empty*[*simp*]: ‹*iterate-add n 0* = *0*›
unfolding *iterate-add-def* **by** (*induction n*) *auto*

lemma *iterate-add-distrib*[*simp*]: ‹*iterate-add* (*m*+*n*) *a* = *iterate-add m a* + *iterate-add n a*›
by (*induction n*) (*auto simp*: *ac-simps*)

lemma *iterate-add-Numeral1*: ‹*iterate-add n Numeral1* = *of-nat n*›
by (*induction n*) *auto*

lemma *iterate-add-1*: ‹*iterate-add n 1* = *of-nat n*›
using *iterate-add-Numeral1* **by** *auto*

lemma *iterate-add-eq-add-iff1*:
‹*i* ≤ *j* ⟹ (*iterate-add j u* + *m* = *iterate-add i u* + *n*) = (*iterate-add* (*j* − *i*) *u* + *m* = *n*)›
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-eq-add-iff2*:
‹*i* ≤ *j* ⟹ (*iterate-add i u* + *m* = *iterate-add j u* + *n*) = (*m* = *iterate-add* (*j* − *i*) *u* + *n*)›

by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-less-iff1*:
$j \leq (i::nat) \Longrightarrow (iterate\text{-}add\ i\ (u::'a::\{cancel\text{-}comm\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\text{-}imp\text{-}le\})$
$+\ m < iterate\text{-}add\ j\ u\ +\ n) = (iterate\text{-}add\ (i-j)\ u\ +\ m < n)$
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-less-iff2*:
$i \leq (j::nat) \Longrightarrow (iterate\text{-}add\ i\ (u::'a::\{cancel\text{-}comm\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\text{-}imp\text{-}le\})$
$+\ m < iterate\text{-}add\ j\ u\ +\ n) = (m < iterate\text{-}add\ (j\ -\ i)\ u\ +\ n)$
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-less-eq-iff1*:
$j \leq (i::nat) \Longrightarrow (iterate\text{-}add\ i\ (u::'a::\{cancel\text{-}comm\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\text{-}imp\text{-}le\})$
$+\ m \leq iterate\text{-}add\ j\ u\ +\ n) = (iterate\text{-}add\ (i-j)\ u\ +\ m \leq n)$
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-less-eq-iff2*:
$i \leq (j::nat) \Longrightarrow (iterate\text{-}add\ i\ (u::'a::\{cancel\text{-}comm\text{-}monoid\text{-}add, ordered\text{-}ab\text{-}semigroup\text{-}add\text{-}imp\text{-}le\})$
$+\ m \leq iterate\text{-}add\ j\ u\ +\ n) = (m \leq iterate\text{-}add\ (j\ -\ i)\ u\ +\ n)$
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-add-eq1*:
$j \leq (i::nat) \Longrightarrow ((iterate\text{-}add\ i\ u\ +\ m)\ -\ (iterate\text{-}add\ j\ u\ +\ n)) = ((iterate\text{-}add$
$(i-j)\ u\ +\ m)\ -\ n)$
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

lemma *iterate-add-diff-add-eq2*:
$i \leq (j::nat) \Longrightarrow ((iterate\text{-}add\ i\ u\ +\ m)\ -\ (iterate\text{-}add\ j\ u\ +\ n)) = (m\ -$
$(iterate\text{-}add\ (j-i)\ u\ +\ n))$
by (*auto dest*!: *le-Suc-ex add-right-imp-eq simp*: *ab-semigroup-add-class.add-ac*(*1*))

61.1 Simproc Set-Up

ML-file ‹*Cancellation/cancel.ML*›
ML-file ‹*Cancellation/cancel-data.ML*›
ML-file ‹*Cancellation/cancel-simprocs.ML*›

end

62 (Finite) Multisets

theory *Multiset*
imports *Cancellation*
begin

62.1 The type of multisets

typedef $'a\ multiset = ‹\{f :: 'a \Rightarrow nat.\ finite\ \{x.\ f\ x > 0\}\}›$

morphisms *count Abs-multiset*
proof
show ‹(λx. 0::nat) ∈ {f. finite {x. f x > 0}}›
by *simp*
qed

setup-lifting *type-definition-multiset*

lemma *count-Abs-multiset*:
‹*count* (*Abs-multiset f*) = *f*› **if** ‹*finite* {x. *f x* > 0}›
by (*rule Abs-multiset-inverse*) (*simp add*: *that*)

lemma *multiset-eq-iff*: *M* = *N* ⟷ (∀ *a*. *count M a* = *count N a*)
by (*simp only*: *count-inject* [*symmetric*] *fun-eq-iff*)

lemma *multiset-eqI*: (⋀x. *count A x* = *count B x*) ⟹ *A* = *B*
using *multiset-eq-iff* **by** *auto*

Preservation of the representing set *multiset*.

lemma *diff-preserves-multiset*:
‹*finite* {x. 0 < *M x* − *N x*}› **if** ‹*finite* {x. 0 < *M x*}› **for** *M N* :: ‹'*a* ⇒ *nat*›
using *that* **by** (*rule rev-finite-subset*) *auto*

lemma *filter-preserves-multiset*:
‹*finite* {x. 0 < (*if P x then M x else* 0)}› **if** ‹*finite* {x. 0 < *M x*}› **for** *M N* ::
‹'*a* ⇒ *nat*›
using *that* **by** (*rule rev-finite-subset*) *auto*

lemmas *in-multiset* = *diff-preserves-multiset filter-preserves-multiset*

62.2 Representing multisets

Multiset enumeration

instantiation *multiset* :: (*type*) *cancel-comm-monoid-add*
begin

lift-definition *zero-multiset* :: ‹'*a multiset*›
is ‹λa. 0›
by *simp*

abbreviation *empty-mset* :: ‹'*a multiset*› (‹{#}›)
where ‹*empty-mset* ≡ 0›

lift-definition *plus-multiset* :: ‹'*a multiset* ⇒ '*a multiset* ⇒ '*a multiset*›
is ‹λM N a. *M a* + *N a*›
by *simp*

lift-definition *minus-multiset* :: ‹'*a multiset* ⇒ '*a multiset* ⇒ '*a multiset*›
is ‹λM N a. *M a* − *N a*›

by (*rule diff-preserves-multiset*)

instance
by (*standard*; *transfer*) (*simp-all add*: *fun-eq-iff*)

end

context
begin

qualified definition *is-empty* :: *'a multiset* ⇒ *bool* **where**
[*code-abbrev*]: *is-empty A* ⟷ *A* = {#}

end

lemma *add-mset-in-multiset*:
‹*finite* {*x*. *0* < (*if x* = *a then Suc* (*M x*) *else M x*)}›
if ‹*finite* {*x*. *0* < *M x*}›
using *that* **by** (*simp add*: *flip*: *insert-Collect*)

lift-definition *add-mset* :: *'a* ⇒ *'a multiset* ⇒ *'a multiset* **is**
λ*a M b*. *if b* = *a then Suc* (*M b*) *else M b*
by (*rule add-mset-in-multiset*)

syntax
-multiset :: *args* ⇒ *'a multiset* ({#(-)#})
translations
{#*x*, *xs*#} == *CONST add-mset x* {#*xs*#}
{#*x*#} == *CONST add-mset x* {#}

lemma *count-empty* [*simp*]: *count* {#} *a* = *0*
by (*simp add*: *zero-multiset.rep-eq*)

lemma *count-add-mset* [*simp*]:
count (*add-mset b A*) *a* = (*if b* = *a then Suc* (*count A a*) *else count A a*)
by (*simp add*: *add-mset.rep-eq*)

lemma *count-single*: *count* {#*b*#} *a* = (*if b* = *a then 1 else 0*)
by *simp*

lemma
add-mset-not-empty [*simp*]: ‹*add-mset a A* ≠ {#}› **and**
empty-not-add-mset [*simp*]: {#} ≠ *add-mset a A*
by (*auto simp*: *multiset-eq-iff*)

lemma *add-mset-add-mset-same-iff* [*simp*]:
add-mset a A = *add-mset a B* ⟷ *A* = *B*
by (*auto simp*: *multiset-eq-iff*)

lemma *add-mset-commute*:
add-mset x (*add-mset y M*) = *add-mset y* (*add-mset x M*)
by (*auto simp*: *multiset-eq-iff*)

62.3 Basic operations

62.3.1 Conversion to set and membership

definition *set-mset* :: ‹*'a multiset* ⇒ *'a set*›
where ‹*set-mset M* = {*x*. *count M x* > *0*}›

abbreviation *member-mset* :: ‹*'a* ⇒ *'a multiset* ⇒ *bool*›
where ‹*member-mset a M* ≡ *a* ∈ *set-mset M*›

notation
member-mset (‹*'*(∈#*'*)›) **and**
member-mset (‹(-/ ∈# -)› [*50*, *51*] *50*)

notation (*ASCII*)
member-mset (‹*'*(:#*'*)›) **and**
member-mset (‹(-/ :# -)› [*50*, *51*] *50*)

abbreviation *not-member-mset* :: ‹*'a* ⇒ *'a multiset* ⇒ *bool*›
where ‹*not-member-mset a M* ≡ *a* ∉ *set-mset M*›

notation
not-member-mset (‹*'*(∉#*'*)›) **and**
not-member-mset (‹(-/ ∉# -)› [*50*, *51*] *50*)

notation (*ASCII*)
not-member-mset (‹*'*(~:#*'*)›) **and**
not-member-mset (‹(-/ ~:# -)› [*50*, *51*] *50*)

context
begin

qualified abbreviation *Ball* :: *'a multiset* ⇒ (*'a* ⇒ *bool*) ⇒ *bool*
where *Ball M* ≡ *Set.Ball* (*set-mset M*)

qualified abbreviation *Bex* :: *'a multiset* ⇒ (*'a* ⇒ *bool*) ⇒ *bool*
where *Bex M* ≡ *Set.Bex* (*set-mset M*)

end

syntax
-MBall :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*∀ -∈#-./ -) [*0*, *0*, *10*] *10*)
-MBex :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*∃ -∈#-./ -) [*0*, *0*, *10*] *10*)

syntax (*ASCII*)
-MBall :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* ((*3*∀ -:#-./ -) [*0*, *0*, *10*] *10*)

-MBex :: *pttrn* ⇒ *'a set* ⇒ *bool* ⇒ *bool* (*(3∃ -:#-./ -) [0, 0, 10] 10*)

translations
∀ *x*∈#*A*. *P* ⇌ *CONST Multiset.Ball A* (λ*x*. *P*)
∃ *x*∈#*A*. *P* ⇌ *CONST Multiset.Bex A* (λ*x*. *P*)

print-translation ‹
[*Syntax-Trans.preserve-binder-abs2-tr'* ***const-syntax*** ‹*Multiset.Ball*› ***syntax-const*** ‹*-MBall*›,
Syntax-Trans.preserve-binder-abs2-tr' ***const-syntax*** ‹*Multiset.Bex*› ***syntax-const*** ‹*-MBex*›]
› — to avoid eta-contraction of body

lemma *count-eq-zero-iff*:
count M x = *0* ⟷ *x* ∉# *M*
by (*auto simp add*: *set-mset-def*)

lemma *not-in-iff*:
x ∉# *M* ⟷ *count M x* = *0*
by (*auto simp add*: *count-eq-zero-iff*)

lemma *count-greater-zero-iff* [*simp*]:
count M x > *0* ⟷ *x* ∈# *M*
by (*auto simp add*: *set-mset-def*)

lemma *count-inI*:
assumes *count M x* = *0* ⟹ *False*
shows *x* ∈# *M*
proof (*rule ccontr*)
assume *x* ∉# *M*
with *assms* **show** *False* **by** (*simp add*: *not-in-iff*)
qed

lemma *in-countE*:
assumes *x* ∈# *M*
obtains *n* **where** *count M x* = *Suc n*
proof −
from *assms* **have** *count M x* > *0* **by** *simp*
then obtain *n* **where** *count M x* = *Suc n*
using *gr0-conv-Suc* **by** *blast*
with *that* **show** *thesis* **.**
qed

lemma *count-greater-eq-Suc-zero-iff* [*simp*]:
count M x ≥ *Suc 0* ⟷ *x* ∈# *M*
by (*simp add*: *Suc-le-eq*)

lemma *count-greater-eq-one-iff* [*simp*]:
count M x ≥ *1* ⟷ *x* ∈# *M*
by *simp*

lemma *set-mset-empty* [*simp*]:
set-mset {#} = {}
by (*simp add*: *set-mset-def*)

lemma *set-mset-single*:
set-mset {#*b*#} = {*b*}
by (*simp add*: *set-mset-def*)

lemma *set-mset-eq-empty-iff* [*simp*]:
set-mset M = {} ⟷ *M* = {#}
by (*auto simp add*: *multiset-eq-iff count-eq-zero-iff*)

lemma *finite-set-mset* [*iff*]:
finite (*set-mset M*)
using *count* [*of M*] **by** *simp*

lemma *set-mset-add-mset-insert* [*simp*]: ‹*set-mset* (*add-mset a A*) = *insert a* (*set-mset A*)›
by (*auto simp flip*: *count-greater-eq-Suc-zero-iff split*: *if-splits*)

lemma *multiset-nonemptyE* [*elim*]:
assumes *A* ≠ {#}
obtains *x* **where** *x* ∈# *A*
proof −
have ∃*x*. *x* ∈# *A* **by** (*rule ccontr*) (*insert assms*, *auto*)
with *that* **show** *?thesis* **by** *blast*
qed

62.3.2 Union

lemma *count-union* [*simp*]:
count (*M* + *N*) *a* = *count M a* + *count N a*
by (*simp add*: *plus-multiset.rep-eq*)

lemma *set-mset-union* [*simp*]:
set-mset (*M* + *N*) = *set-mset M* ∪ *set-mset N*
by (*simp only*: *set-eq-iff count-greater-zero-iff* [*symmetric*] *count-union*) *simp*

lemma *union-mset-add-mset-left* [*simp*]:
add-mset a A + *B* = *add-mset a* (*A* + *B*)
by (*auto simp*: *multiset-eq-iff*)

lemma *union-mset-add-mset-right* [*simp*]:
A + *add-mset a B* = *add-mset a* (*A* + *B*)
by (*auto simp*: *multiset-eq-iff*)

lemma *add-mset-add-single*: ‹*add-mset a A* = *A* + {#*a*#}›
by (*subst union-mset-add-mset-right*, *subst add.comm-neutral*) *standard*

62.3.3 Difference

instance *multiset* :: (*type*) *comm-monoid-diff*
by *standard* (*transfer*; *simp add*: *fun-eq-iff*)

lemma *count-diff* [*simp*]:
count (*M* − *N*) *a* = *count M a* − *count N a*
by (*simp add*: *minus-multiset.rep-eq*)

lemma *add-mset-diff-bothsides*:
‹*add-mset a M* − *add-mset a A* = *M* − *A*›
by (*auto simp*: *multiset-eq-iff*)

lemma *in-diff-count*:
a ∈# *M* − *N* ⟷ *count N a* < *count M a*
by (*simp add*: *set-mset-def*)

lemma *count-in-diffI*:
assumes ⋀*n*. *count N x* = *n* + *count M x* ⟹ *False*
shows *x* ∈# *M* − *N*
proof (*rule ccontr*)
assume *x* ∉# *M* − *N*
then have *count N x* = (*count N x* − *count M x*) + *count M x*
by (*simp add*: *in-diff-count not-less*)
with *assms* **show** *False* **by** *auto*
qed

lemma *in-diff-countE*:
assumes *x* ∈# *M* − *N*
obtains *n* **where** *count M x* = *Suc n* + *count N x*
proof −
from *assms* **have** *count M x* − *count N x* > *0* **by** (*simp add*: *in-diff-count*)
then have *count M x* > *count N x* **by** *simp*
then obtain *n* **where** *count M x* = *Suc n* + *count N x*
using *less-iff-Suc-add* **by** *auto*
with *that* **show** *thesis* **.**
qed

lemma *in-diffD*:
assumes *a* ∈# *M* − *N*
shows *a* ∈# *M*
proof −
have *0* ≤ *count N a* **by** *simp*
also from *assms* **have** *count N a* < *count M a*
by (*simp add*: *in-diff-count*)
finally show *?thesis* **by** *simp*
qed

lemma *set-mset-diff*:
set-mset (*M* − *N*) = {*a*. *count N a* < *count M a*}

by (*simp add*: *set-mset-def*)

lemma *diff-empty* [*simp*]: $M - \{\#\} = M \wedge \{\#\} - M = \{\#\}$
by *rule* (*fact Groups.diff-zero*, *fact Groups.zero-diff*)

lemma *diff-cancel*: $A - A = \{\#\}$
by (*fact Groups.diff-cancel*)

lemma *diff-union-cancelR*: $M + N - N = (M::'a\ multiset)$
by (*fact add-diff-cancel-right'*)

lemma *diff-union-cancelL*: $N + M - N = (M::'a\ multiset)$
by (*fact add-diff-cancel-left'*)

lemma *diff-right-commute*:
fixes $M\ N\ Q :: 'a\ multiset$
shows $M - N - Q = M - Q - N$
by (*fact diff-right-commute*)

lemma *diff-add*:
fixes $M\ N\ Q :: 'a\ multiset$
shows $M - (N + Q) = M - N - Q$
by (*rule sym*) (*fact diff-diff-add*)

lemma *insert-DiffM* [*simp*]: $x \in\# M \Longrightarrow add\text{-}mset\ x\ (M - \{\#x\#\}) = M$
by (*clarsimp simp*: *multiset-eq-iff*)

lemma *insert-DiffM2*: $x \in\# M \Longrightarrow (M - \{\#x\#\}) + \{\#x\#\} = M$
by *simp*

lemma *diff-union-swap*: $a \neq b \Longrightarrow add\text{-}mset\ b\ (M - \{\#a\#\}) = add\text{-}mset\ b\ M - \{\#a\#\}$
by (*auto simp add*: *multiset-eq-iff*)

lemma *diff-add-mset-swap* [*simp*]: $b \notin\# A \Longrightarrow add\text{-}mset\ b\ M - A = add\text{-}mset\ b\ (M - A)$
by (*auto simp add*: *multiset-eq-iff simp*: *not-in-iff*)

lemma *diff-union-swap2* [*simp*]: $y \in\# M \Longrightarrow add\text{-}mset\ x\ M - \{\#y\#\} = add\text{-}mset\ x\ (M - \{\#y\#\})$
by (*metis add-mset-diff-bothsides diff-union-swap diff-zero insert-DiffM*)

lemma *diff-diff-add-mset* [*simp*]: $(M::'a\ multiset) - N - P = M - (N + P)$
by (*rule diff-diff-add*)

lemma *diff-union-single-conv*:
$a \in\# J \Longrightarrow I + J - \{\#a\#\} = I + (J - \{\#a\#\})$
by (*simp add*: *multiset-eq-iff Suc-le-eq*)

lemma *mset-add* [*elim?*]:
assumes $a \in\# A$
obtains B **where** $A = \mathit{add\text{-}mset}\ a\ B$
proof −
from *assms* **have** $A = \mathit{add\text{-}mset}\ a\ (A - \{\#a\#\})$
by *simp*
with *that* **show** *thesis* **.**
qed

lemma *union-iff*:
$a \in\# A + B \longleftrightarrow a \in\# A \vee a \in\# B$
by *auto*

62.3.4 Min and Max

abbreviation *Min-mset* :: *'a*::*linorder multiset* ⇒ *'a* **where**
Min-mset m ≡ *Min* (*set-mset m*)

abbreviation *Max-mset* :: *'a*::*linorder multiset* ⇒ *'a* **where**
Max-mset m ≡ *Max* (*set-mset m*)

62.3.5 Equality of multisets

lemma *single-eq-single* [*simp*]: $\{\#a\#\} = \{\#b\#\} \longleftrightarrow a = b$
by (*auto simp add*: *multiset-eq-iff*)

lemma *union-eq-empty* [*iff*]: $M + N = \{\#\} \longleftrightarrow M = \{\#\} \wedge N = \{\#\}$
by (*auto simp add*: *multiset-eq-iff*)

lemma *empty-eq-union* [*iff*]: $\{\#\} = M + N \longleftrightarrow M = \{\#\} \wedge N = \{\#\}$
by (*auto simp add*: *multiset-eq-iff*)

lemma *multi-self-add-other-not-self* [*simp*]: $M = \mathit{add\text{-}mset}\ x\ M \longleftrightarrow \mathit{False}$
by (*auto simp add*: *multiset-eq-iff*)

lemma *add-mset-remove-trivial* [*simp*]: ‹*add-mset x M* − {#*x*#} = *M*›
by (*auto simp*: *multiset-eq-iff*)

lemma *diff-single-trivial*: ¬ $x \in\# M \Longrightarrow M - \{\#x\#\} = M$
by (*auto simp add*: *multiset-eq-iff not-in-iff*)

lemma *diff-single-eq-union*: $x \in\# M \Longrightarrow M - \{\#x\#\} = N \longleftrightarrow M = \mathit{add\text{-}mset}\ x\ N$
by *auto*

lemma *union-single-eq-diff*: *add-mset x M* = *N* ⟹ *M* = *N* − {#*x*#}
unfolding *add-mset-add-single*[*of - M*] **by** (*fact add-implies-diff*)

lemma *union-single-eq-member*: *add-mset x M* = *N* ⟹ *x* ∈# *N*
by *auto*

lemma *add-mset-remove-trivial-If*:
add-mset a (*N* − {#*a*#}) = (*if a* ∈# *N then N else add-mset a N*)
by (*simp add*: *diff-single-trivial*)

lemma *add-mset-remove-trivial-eq*: ‹*N* = *add-mset a* (*N* − {#*a*#}) ⟷ *a* ∈# *N*›
by (*auto simp*: *add-mset-remove-trivial-If*)

lemma *union-is-single*:
M + *N* = {#*a*#} ⟷ *M* = {#*a*#} ∧ *N* = {#} ∨ *M* = {#} ∧ *N* = {#*a*#}
(**is** *?lhs* = *?rhs*)
proof
show *?lhs* **if** *?rhs* **using** *that* **by** *auto*
show *?rhs* **if** *?lhs*
by (*metis Multiset.diff-cancel add.commute add-diff-cancel-left' diff-add-zero diff-single-trivial insert-DiffM that*)
qed

lemma *single-is-union*: {#*a*#} = *M* + *N* ⟷ {#*a*#} = *M* ∧ *N* = {#} ∨ *M* = {#} ∧ {#*a*#} = *N*
by (*auto simp add*: *eq-commute* [*of* {#*a*#} *M* + *N*] *union-is-single*)

lemma *add-eq-conv-diff*:
add-mset a M = *add-mset b N* ⟷ *M* = *N* ∧ *a* = *b* ∨ *M* = *add-mset b* (*N* − {#*a*#}) ∧ *N* = *add-mset a* (*M* − {#*b*#})
(**is** *?lhs* ⟷ *?rhs*)

proof
show *?lhs* **if** *?rhs*
using *that*
by (*auto simp add*: *add-mset-commute*[*of a b*])
show *?rhs* **if** *?lhs*
proof (*cases a* = *b*)
case *True* **with** ‹*?lhs*› **show** *?thesis* **by** *simp*
next
case *False*
from ‹*?lhs*› **have** *a* ∈# *add-mset b N* **by** (*rule union-single-eq-member*)
with *False* **have** *a* ∈# *N* **by** *auto*
moreover from ‹*?lhs*› **have** *M* = *add-mset b N* − {#*a*#} **by** (*rule union-single-eq-diff*)
moreover note *False*
ultimately show *?thesis* **by** (*auto simp add*: *diff-right-commute* [*of* - {#*a*#}])
qed
qed

lemma *add-mset-eq-single* [*iff*]: *add-mset b M* = {#*a*#} ⟷ *b* = *a* ∧ *M* = {#}
by (*auto simp*: *add-eq-conv-diff*)

lemma *single-eq-add-mset* [*iff*]: {#*a*#} = *add-mset b M* ⟷ *b* = *a* ∧ *M* = {#}

by (*auto simp*: *add-eq-conv-diff*)

lemma *insert-noteq-member*:
assumes *BC*: *add-mset b B* = *add-mset c C*
and *bnotc*: $b \neq c$
shows $c \in\# B$
proof −
have $c \in\#$ *add-mset c C* **by** *simp*
have *nc*: $\neg\ c \in\#\ \{\#b\#\}$ **using** *bnotc* **by** *simp*
then have $c \in\#$ *add-mset b B* **using** *BC* **by** *simp*
then show $c \in\# B$ **using** *nc* **by** *simp*
qed

lemma *add-eq-conv-ex*:
(*add-mset a M* = *add-mset b N*) =
($M = N \wedge a = b \vee (\exists K.\ M =$ *add-mset b K* $\wedge\ N =$ *add-mset a K*))
by (*auto simp add*: *add-eq-conv-diff*)

lemma *multi-member-split*: $x \in\# M \Longrightarrow \exists A.\ M =$ *add-mset x A*
by (*rule exI* [**where** $x = M - \{\#x\#\}$]) *simp*

lemma *multiset-add-sub-el-shuffle*:
assumes $c \in\# B$
and $b \neq c$
shows *add-mset b* ($B - \{\#c\#\}$) = *add-mset b B* $- \{\#c\#\}$
proof −
from ‹$c \in\# B$› **obtain** *A* **where** *B*: *B* = *add-mset c A*
by (*blast dest*: *multi-member-split*)
have *add-mset b A* = *add-mset c* (*add-mset b A*) $- \{\#c\#\}$ **by** *simp*
then have *add-mset b A* = *add-mset b* (*add-mset c A*) $- \{\#c\#\}$
by (*simp add*: ‹$b \neq c$›)
then show *?thesis* **using** *B* **by** *simp*
qed

lemma *add-mset-eq-singleton-iff*[*iff*]:
add-mset x M = $\{\#y\#\} \longleftrightarrow M = \{\#\} \wedge x = y$
by *auto*

62.3.6 Pointwise ordering induced by count

definition *subseteq-mset* :: *'a multiset* ⇒ *'a multiset* ⇒ *bool* (**infix** $\subseteq\#$ *50*)
where $A \subseteq\# B \longleftrightarrow (\forall a.$ *count A a* $\leq$ *count B a*)

definition *subset-mset* :: *'a multiset* ⇒ *'a multiset* ⇒ *bool* (**infix** $\subset\#$ *50*)
where $A \subset\# B \longleftrightarrow A \subseteq\# B \wedge A \neq B$

abbreviation (*input*) *supseteq-mset* :: *'a multiset* ⇒ *'a multiset* ⇒ *bool* (**infix** $\supseteq\#$ *50*)
where *supseteq-mset A B* ≡ $B \subseteq\# A$

abbreviation (*input*) *supset-mset* :: *'a multiset ⇒ 'a multiset ⇒ bool* (**infix** ⊃# *50*)
where *supset-mset A B ≡ B ⊂# A*

notation (*input*)
subseteq-mset (**infix** ≤# *50*) **and**
supseteq-mset (**infix** ≥# *50*)

notation (*ASCII*)
subseteq-mset (**infix** <=# *50*) **and**
subset-mset (**infix** <# *50*) **and**
supseteq-mset (**infix** >=# *50*) **and**
supset-mset (**infix** ># *50*)

global-interpretation *subset-mset*: *ordering* ‹(⊆#)› ‹(⊂#)›
by *standard* (*auto simp add*: *subset-mset-def subseteq-mset-def multiset-eq-iff intro*: *order.trans order.antisym*)

interpretation *subset-mset*: *ordered-ab-semigroup-add-imp-le* ‹(+)› ‹(−)› ‹(⊆#)› ‹(⊂#)›
by *standard* (*auto simp add*: *subset-mset-def subseteq-mset-def multiset-eq-iff intro*: *order-trans antisym*)
— FIXME: avoid junk stemming from type class interpretation

interpretation *subset-mset*: *ordered-ab-semigroup-monoid-add-imp-le* (+) *0* (−) (⊆#) (⊂#)
by *standard*
— FIXME: avoid junk stemming from type class interpretation

lemma *mset-subset-eqI*:
(⋀*a*. *count A a* ≤ *count B a*) ⟹ *A* ⊆# *B*
by (*simp add*: *subseteq-mset-def*)

lemma *mset-subset-eq-count*:
A ⊆# *B* ⟹ *count A a* ≤ *count B a*
by (*simp add*: *subseteq-mset-def*)

lemma *mset-subset-eq-exists-conv*: (*A*::*'a multiset*) ⊆# *B* ⟷ (∃ *C*. *B* = *A* + *C*)
unfolding *subseteq-mset-def*
apply (*rule iffI*)
apply (*rule exI* [**where** *x* = *B* − *A*])
apply (*auto intro*: *multiset-eq-iff* [*THEN iffD2*])
done

interpretation *subset-mset*: *ordered-cancel-comm-monoid-diff* (+) *0* (⊆#) (⊂#) (−)
by *standard* (*simp*, *fact mset-subset-eq-exists-conv*)
— FIXME: avoid junk stemming from type class interpretation

declare *subset-mset.add-diff-assoc*[*simp*] *subset-mset.add-diff-assoc2*[*simp*]

lemma *mset-subset-eq-mono-add-right-cancel*: $(A::'a\ multiset) + C \subseteq\# B + C \longleftrightarrow A \subseteq\# B$
by (*fact subset-mset.add-le-cancel-right*)

lemma *mset-subset-eq-mono-add-left-cancel*: $C + (A::'a\ multiset) \subseteq\# C + B \longleftrightarrow A \subseteq\# B$
by (*fact subset-mset.add-le-cancel-left*)

lemma *mset-subset-eq-mono-add*: $(A::'a\ multiset) \subseteq\# B \Longrightarrow C \subseteq\# D \Longrightarrow A + C \subseteq\# B + D$
by (*fact subset-mset.add-mono*)

lemma *mset-subset-eq-add-left*: $(A::'a\ multiset) \subseteq\# A + B$
by *simp*

lemma *mset-subset-eq-add-right*: $B \subseteq\# (A::'a\ multiset) + B$
by *simp*

lemma *single-subset-iff* [*simp*]:
$\{\#a\#\} \subseteq\# M \longleftrightarrow a \in\# M$
by (*auto simp add*: *subseteq-mset-def Suc-le-eq*)

lemma *mset-subset-eq-single*: $a \in\# B \Longrightarrow \{\#a\#\} \subseteq\# B$
by *simp*

lemma *mset-subset-eq-add-mset-cancel*: ‹*add-mset a A* $\subseteq\#$ *add-mset a B* $\longleftrightarrow$ *A* $\subseteq\#$ *B*›
unfolding *add-mset-add-single*[*of - A*] *add-mset-add-single*[*of - B*]
by (*rule mset-subset-eq-mono-add-right-cancel*)

lemma *multiset-diff-union-assoc*:
fixes *A B C D* :: *'a multiset*
shows $C \subseteq\# B \Longrightarrow A + B - C = A + (B - C)$
by (*fact subset-mset.diff-add-assoc*)

lemma *mset-subset-eq-multiset-union-diff-commute*:
fixes *A B C D* :: *'a multiset*
shows $B \subseteq\# A \Longrightarrow A - B + C = A + C - B$
by (*fact subset-mset.add-diff-assoc2*)

lemma *diff-subset-eq-self*[*simp*]:
$(M::'a\ multiset) - N \subseteq\# M$
by (*simp add*: *subseteq-mset-def*)

lemma *mset-subset-eqD*:
assumes $A \subseteq\# B$ **and** $x \in\# A$

```
  shows x ∈# B
proof −
  from ‹x ∈# A› have count A x > 0 by simp
  also from ‹A ⊆# B› have count A x ≤ count B x
    by (simp add: subseteq-mset-def)
  finally show ?thesis by simp
qed

lemma mset-subsetD:
  A ⊂# B ⟹ x ∈# A ⟹ x ∈# B
  by (auto intro: mset-subset-eqD [of A])

lemma set-mset-mono:
  A ⊆# B ⟹ set-mset A ⊆ set-mset B
  by (metis mset-subset-eqD subsetI)

lemma mset-subset-eq-insertD:
  add-mset x A ⊆# B ⟹ x ∈# B ∧ A ⊂# B
apply (rule conjI)
 apply (simp add: mset-subset-eqD)
 apply (clarsimp simp: subset-mset-def subseteq-mset-def)
 apply safe
  apply (erule-tac x = a in allE)
  apply (auto split: if-split-asm)
done

lemma mset-subset-insertD:
  add-mset x A ⊂# B ⟹ x ∈# B ∧ A ⊂# B
  by (rule mset-subset-eq-insertD) simp

lemma mset-subset-of-empty[simp]: A ⊂# {#} ⟷ False
  by (simp only: subset-mset.not-less-zero)

lemma empty-subset-add-mset[simp]: {#} ⊂# add-mset x M
  by (auto intro: subset-mset.gr-zeroI)

lemma empty-le: {#} ⊆# A
  by (fact subset-mset.zero-le)

lemma insert-subset-eq-iff:
  add-mset a A ⊆# B ⟷ a ∈# B ∧ A ⊆# B − {#a#}
  using le-diff-conv2 [of Suc 0 count B a count A a]
  apply (auto simp add: subseteq-mset-def not-in-iff Suc-le-eq)
  apply (rule ccontr)
  apply (auto simp add: not-in-iff)
  done

lemma insert-union-subset-iff:
  add-mset a A ⊂# B ⟷ a ∈# B ∧ A ⊂# B − {#a#}
```

by (*auto simp add*: *insert-subset-eq-iff subset-mset-def*)

lemma *subset-eq-diff-conv*:
A − C ⊆# B ⟷ A ⊆# B + C
by (*simp add*: *subseteq-mset-def le-diff-conv*)

lemma *multi-psub-of-add-self* [*simp*]: *A ⊂# add-mset x A*
by (*auto simp*: *subset-mset-def subseteq-mset-def*)

lemma *multi-psub-self*: *A ⊂# A = False*
by *simp*

lemma *mset-subset-add-mset* [*simp*]: *add-mset x N ⊂# add-mset x M ⟷ N ⊂# M*
unfolding *add-mset-add-single*[*of - N*] *add-mset-add-single*[*of - M*]
by (*fact subset-mset.add-less-cancel-right*)

lemma *mset-subset-diff-self*: *c ∈# B ⟹ B − {#c#} ⊂# B*
by (*auto simp*: *subset-mset-def elim*: *mset-add*)

lemma *Diff-eq-empty-iff-mset*: *A − B = {#} ⟷ A ⊆# B*
by (*auto simp*: *multiset-eq-iff subseteq-mset-def*)

lemma *add-mset-subseteq-single-iff*[*iff*]: *add-mset a M ⊆# {#b#} ⟷ M = {#} ∧ a = b*
proof
assume *A*: *add-mset a M ⊆# {#b#}*
then have ‹*a = b*›
by (*auto dest*: *mset-subset-eq-insertD*)
then show *M={#} ∧ a=b*
using *A* **by** (*simp add*: *mset-subset-eq-add-mset-cancel*)
qed *simp*

62.3.7 Intersection and bounded union

definition *inter-mset* :: ‹*'a multiset ⇒ 'a multiset ⇒ 'a multiset*› (**infixl** ‹*∩#*› *70*)
where ‹*A ∩# B = A − (A − B)*›

lemma *count-inter-mset* [*simp*]:
‹*count* (*A ∩# B*) *x = min* (*count A x*) (*count B x*)›
by (*simp add*: *inter-mset-def*)

interpretation *subset-mset*: *semilattice-inf* ‹(*∩#*)› ‹(*⊆#*)› ‹(*⊂#*)›
by *standard* (*simp-all add*: *multiset-eq-iff subseteq-mset-def*)
— FIXME: avoid junk stemming from type class interpretation

definition *union-mset* :: ‹'*a multiset* ⇒ '*a multiset* ⇒ '*a multiset*› (**infixl** ‹∪#› *70*)
where ‹*A* ∪# *B* = *A* + (*B* − *A*)›

lemma *count-union-mset* [*simp*]:
‹*count* (*A* ∪# *B*) *x* = *max* (*count A x*) (*count B x*)›
by (*simp add*: *union-mset-def*)

global-interpretation *subset-mset*: *semilattice-neutr-order* ‹(∪#)› ‹{#}› ‹(⊇#)› ‹(⊃#)›
apply *standard*
apply (*simp-all add*: *multiset-eq-iff subseteq-mset-def subset-mset-def max-def*)
apply (*auto simp add*: *le-antisym dest*: *sym*)
apply (*metis nat-le-linear*)+
done

interpretation *subset-mset*: *semilattice-sup* ‹(∪#)› ‹(⊆#)› ‹(⊂#)›
proof −
have [*simp*]: *m* ≤ *n* ⟹ *q* ≤ *n* ⟹ *m* + (*q* − *m*) ≤ *n* **for** *m n q* :: *nat*
by *arith*
show *class.semilattice-sup* (∪#) (⊆#) (⊂#)
by *standard* (*auto simp add*: *union-mset-def subseteq-mset-def*)
qed — FIXME: avoid junk stemming from type class interpretation

interpretation *subset-mset*: *bounded-lattice-bot* (∩#) (⊆#) (⊂#) (∪#) {#}
by *standard auto*
— FIXME: avoid junk stemming from type class interpretation

62.3.8 Additional intersection facts

lemma *set-mset-inter* [*simp*]:
set-mset (*A* ∩# *B*) = *set-mset A* ∩ *set-mset B*
by (*simp only*: *set-mset-def*) *auto*

lemma *diff-intersect-left-idem* [*simp*]:
M − *M* ∩# *N* = *M* − *N*
by (*simp add*: *multiset-eq-iff min-def*)

lemma *diff-intersect-right-idem* [*simp*]:
M − *N* ∩# *M* = *M* − *N*
by (*simp add*: *multiset-eq-iff min-def*)

lemma *multiset-inter-single*[*simp*]: *a* ≠ *b* ⟹ {#*a*#} ∩# {#*b*#} = {#}
by (*rule multiset-eqI*) *auto*

lemma *multiset-union-diff-commute*:
assumes *B* ∩# *C* = {#}
shows *A* + *B* − *C* = *A* − *C* + *B*

proof (*rule multiset-eqI*)
fix *x*
from *assms* **have** *min* (*count B x*) (*count C x*) = *0*
by (*auto simp add*: *multiset-eq-iff*)
then have *count B x* = *0* ∨ *count C x* = *0*
unfolding *min-def* **by** (*auto split*: *if-splits*)
then show *count* (*A* + *B* − *C*) *x* = *count* (*A* − *C* + *B*) *x*
by *auto*
qed

lemma *disjunct-not-in*:
A ∩# *B* = {#} ⟷ (∀ *a*. *a* ∉# *A* ∨ *a* ∉# *B*) (**is** *?P* ⟷ *?Q*)
proof
assume *?P*
show *?Q*
proof
fix *a*
from ‹*?P*› **have** *min* (*count A a*) (*count B a*) = *0*
by (*simp add*: *multiset-eq-iff*)
then have *count A a* = *0* ∨ *count B a* = *0*
by (*cases count A a* ≤ *count B a*) (*simp-all add*: *min-def*)
then show *a* ∉# *A* ∨ *a* ∉# *B*
by (*simp add*: *not-in-iff*)
qed
next
assume *?Q*
show *?P*
proof (*rule multiset-eqI*)
fix *a*
from ‹*?Q*› **have** *count A a* = *0* ∨ *count B a* = *0*
by (*auto simp add*: *not-in-iff*)
then show *count* (*A* ∩# *B*) *a* = *count* {#} *a*
by *auto*
qed
qed

lemma *inter-mset-empty-distrib-right*: *A* ∩# (*B* + *C*) = {#} ⟷ *A* ∩# *B* = {#} ∧ *A* ∩# *C* = {#}
by (*meson disjunct-not-in union-iff*)

lemma *inter-mset-empty-distrib-left*: (*A* + *B*) ∩# *C* = {#} ⟷ *A* ∩# *C* = {#} ∧ *B* ∩# *C* = {#}
by (*meson disjunct-not-in union-iff*)

lemma *add-mset-inter-add-mset* [*simp*]:
add-mset a A ∩# *add-mset a B* = *add-mset a* (*A* ∩# *B*)
by (*rule multiset-eqI*) *simp*

lemma *add-mset-disjoint* [*simp*]:

add-mset a A $\cap\#$ *B* = {#} $\longleftrightarrow$ *a* $\notin\#$ *B* $\wedge$ *A* $\cap\#$ *B* = {#}
{#} = *add-mset a A* $\cap\#$ *B* $\longleftrightarrow$ *a* $\notin\#$ *B* $\wedge$ {#} = *A* $\cap\#$ *B*
by (*auto simp*: *disjunct-not-in*)

lemma *disjoint-add-mset* [*simp*]:
B $\cap\#$ *add-mset a A* = {#} $\longleftrightarrow$ *a* $\notin\#$ *B* $\wedge$ *B* $\cap\#$ *A* = {#}
{#} = *A* $\cap\#$ *add-mset b B* $\longleftrightarrow$ *b* $\notin\#$ *A* $\wedge$ {#} = *A* $\cap\#$ *B*
by (*auto simp*: *disjunct-not-in*)

lemma *inter-add-left1*: $\neg$ *x* $\in\#$ *N* $\Longrightarrow$ (*add-mset x M*) $\cap\#$ *N* = *M* $\cap\#$ *N*
by (*simp add*: *multiset-eq-iff not-in-iff*)

lemma *inter-add-left2*: *x* $\in\#$ *N* $\Longrightarrow$ (*add-mset x M*) $\cap\#$ *N* = *add-mset x* (*M* $\cap\#$ (*N* − {#*x*#}))
by (*auto simp add*: *multiset-eq-iff elim*: *mset-add*)

lemma *inter-add-right1*: $\neg$ *x* $\in\#$ *N* $\Longrightarrow$ *N* $\cap\#$ (*add-mset x M*) = *N* $\cap\#$ *M*
by (*simp add*: *multiset-eq-iff not-in-iff*)

lemma *inter-add-right2*: *x* $\in\#$ *N* $\Longrightarrow$ *N* $\cap\#$ (*add-mset x M*) = *add-mset x* ((*N* − {#*x*#}) $\cap\#$ *M*)
by (*auto simp add*: *multiset-eq-iff elim*: *mset-add*)

lemma *disjunct-set-mset-diff*:
assumes *M* $\cap\#$ *N* = {#}
shows *set-mset* (*M* − *N*) = *set-mset M*
proof (*rule set-eqI*)
fix *a*
from *assms* **have** *a* $\notin\#$ *M* $\vee$ *a* $\notin\#$ *N*
by (*simp add*: *disjunct-not-in*)
then show *a* $\in\#$ *M* − *N* $\longleftrightarrow$ *a* $\in\#$ *M*
by (*auto dest*: *in-diffD*) (*simp add*: *in-diff-count not-in-iff*)
qed

lemma *at-most-one-mset-mset-diff*:
assumes *a* $\notin\#$ *M* − {#*a*#}
shows *set-mset* (*M* − {#*a*#}) = *set-mset M* − {*a*}
using *assms* **by** (*auto simp add*: *not-in-iff in-diff-count set-eq-iff*)

lemma *more-than-one-mset-mset-diff*:
assumes *a* $\in\#$ *M* − {#*a*#}
shows *set-mset* (*M* − {#*a*#}) = *set-mset M*
proof (*rule set-eqI*)
fix *b*
have *Suc 0* < *count M b* $\Longrightarrow$ *count M b* > *0* **by** *arith*
then show *b* $\in\#$ *M* − {#*a*#} $\longleftrightarrow$ *b* $\in\#$ *M*
using *assms* **by** (*auto simp add*: *in-diff-count*)
qed

lemma *inter-iff*:
$a \in\# A \cap\# B \longleftrightarrow a \in\# A \wedge a \in\# B$
by *simp*

lemma *inter-union-distrib-left*:
$A \cap\# B + C = (A + C) \cap\# (B + C)$
by (*simp add*: *multiset-eq-iff min-add-distrib-left*)

lemma *inter-union-distrib-right*:
$C + A \cap\# B = (C + A) \cap\# (C + B)$
using *inter-union-distrib-left* [*of A B C*] **by** (*simp add*: *ac-simps*)

lemma *inter-subset-eq-union*:
$A \cap\# B \subseteq\# A + B$
by (*auto simp add*: *subseteq-mset-def*)

62.3.9 Additional bounded union facts

lemma *set-mset-sup* [*simp*]:
‹*set-mset* ($A \cup\# B$) = *set-mset A* ∪ *set-mset B*›
by (*simp only*: *set-mset-def*) (*auto simp add*: *less-max-iff-disj*)

lemma *sup-union-left1* [*simp*]: $\neg x \in\# N \Longrightarrow$ (*add-mset x M*) $\cup\# N$ = *add-mset x* ($M \cup\# N$)
by (*simp add*: *multiset-eq-iff not-in-iff*)

lemma *sup-union-left2*: $x \in\# N \Longrightarrow$ (*add-mset x M*) $\cup\# N$ = *add-mset x* ($M \cup\# (N - \{\#x\#\})$)
by (*simp add*: *multiset-eq-iff*)

lemma *sup-union-right1* [*simp*]: $\neg x \in\# N \Longrightarrow N \cup\#$ (*add-mset x M*) = *add-mset x* ($N \cup\# M$)
by (*simp add*: *multiset-eq-iff not-in-iff*)

lemma *sup-union-right2*: $x \in\# N \Longrightarrow N \cup\#$ (*add-mset x M*) = *add-mset x* (($N - \{\#x\#\}$) $\cup\# M$)
by (*simp add*: *multiset-eq-iff*)

lemma *sup-union-distrib-left*:
$A \cup\# B + C = (A + C) \cup\# (B + C)$
by (*simp add*: *multiset-eq-iff max-add-distrib-left*)

lemma *union-sup-distrib-right*:
$C + A \cup\# B = (C + A) \cup\# (C + B)$
using *sup-union-distrib-left* [*of A B C*] **by** (*simp add*: *ac-simps*)

lemma *union-diff-inter-eq-sup*:
$A + B - A \cap\# B = A \cup\# B$
by (*auto simp add*: *multiset-eq-iff*)

lemma *union-diff-sup-eq-inter*:
A + B − A ∪# B = A ∩# B
by (*auto simp add*: *multiset-eq-iff*)

lemma *add-mset-union*:
‹*add-mset a A ∪# add-mset a B = add-mset a* (*A ∪# B*)›
by (*auto simp*: *multiset-eq-iff max-def*)

62.4 Replicate and repeat operations

definition *replicate-mset* :: *nat* ⇒ *'a* ⇒ *'a multiset* **where**
replicate-mset n x = (*add-mset x* ^^ *n*) {#}

lemma *replicate-mset-0*[*simp*]: *replicate-mset 0 x* = {#}
unfolding *replicate-mset-def* **by** *simp*

lemma *replicate-mset-Suc* [*simp*]: *replicate-mset* (*Suc n*) *x* = *add-mset x* (*replicate-mset n x*)
unfolding *replicate-mset-def* **by** (*induct n*) (*auto intro*: *add.commute*)

lemma *count-replicate-mset*[*simp*]: *count* (*replicate-mset n x*) *y* = (*if y* = *x then n else 0*)
unfolding *replicate-mset-def* **by** (*induct n*) *auto*

lift-definition *repeat-mset* :: ‹*nat* ⇒ *'a multiset* ⇒ *'a multiset*›
is ‹λ*n M a*. *n* ∗ *M a*› **by** *simp*

lemma *count-repeat-mset* [*simp*]: *count* (*repeat-mset i A*) *a* = *i* ∗ *count A a*
by *transfer rule*

lemma *repeat-mset-0* [*simp*]:
‹*repeat-mset 0 M* = {#}›
by *transfer simp*

lemma *repeat-mset-Suc* [*simp*]:
‹*repeat-mset* (*Suc n*) *M* = *M* + *repeat-mset n M*›
by *transfer simp*

lemma *repeat-mset-right* [*simp*]: *repeat-mset a* (*repeat-mset b A*) = *repeat-mset* (*a* ∗ *b*) *A*
by (*auto simp*: *multiset-eq-iff left-diff-distrib'*)

lemma *left-diff-repeat-mset-distrib'*: ‹*repeat-mset* (*i* − *j*) *u* = *repeat-mset i u* − *repeat-mset j u*›
by (*auto simp*: *multiset-eq-iff left-diff-distrib'*)

lemma *left-add-mult-distrib-mset*:
repeat-mset i u + (*repeat-mset j u* + *k*) = *repeat-mset* (*i*+*j*) *u* + *k*

by (*auto simp*: *multiset-eq-iff add-mult-distrib*)

lemma *repeat-mset-distrib*:
repeat-mset (*m* + *n*) *A* = *repeat-mset m A* + *repeat-mset n A*
by (*auto simp*: *multiset-eq-iff Nat.add-mult-distrib*)

lemma *repeat-mset-distrib2*[*simp*]:
repeat-mset n (*A* + *B*) = *repeat-mset n A* + *repeat-mset n B*
by (*auto simp*: *multiset-eq-iff add-mult-distrib2*)

lemma *repeat-mset-replicate-mset*[*simp*]:
repeat-mset n {#*a*#} = *replicate-mset n a*
by (*auto simp*: *multiset-eq-iff*)

lemma *repeat-mset-distrib-add-mset*[*simp*]:
repeat-mset n (*add-mset a A*) = *replicate-mset n a* + *repeat-mset n A*
by (*auto simp*: *multiset-eq-iff*)

lemma *repeat-mset-empty*[*simp*]: *repeat-mset n* {#} = {#}
by *transfer simp*

62.4.1 Simprocs

lemma *repeat-mset-iterate-add*: ‹*repeat-mset n M* = *iterate-add n M*›
unfolding *iterate-add-def* **by** (*induction n*) *auto*

lemma *mset-subseteq-add-iff1*:
j ≤ (*i*::*nat*) ⟹ (*repeat-mset i u* + *m* ⊆# *repeat-mset j u* + *n*) = (*repeat-mset* (*i*−*j*) *u* + *m* ⊆# *n*)
by (*auto simp add*: *subseteq-mset-def nat-le-add-iff1*)

lemma *mset-subseteq-add-iff2*:
i ≤ (*j*::*nat*) ⟹ (*repeat-mset i u* + *m* ⊆# *repeat-mset j u* + *n*) = (*m* ⊆# *repeat-mset* (*j*−*i*) *u* + *n*)
by (*auto simp add*: *subseteq-mset-def nat-le-add-iff2*)

lemma *mset-subset-add-iff1*:
j ≤ (*i*::*nat*) ⟹ (*repeat-mset i u* + *m* ⊂# *repeat-mset j u* + *n*) = (*repeat-mset* (*i*−*j*) *u* + *m* ⊂# *n*)
unfolding *subset-mset-def repeat-mset-iterate-add*
by (*simp add*: *iterate-add-eq-add-iff1 mset-subseteq-add-iff1*[*unfolded repeat-mset-iterate-add*])

lemma *mset-subset-add-iff2*:
i ≤ (*j*::*nat*) ⟹ (*repeat-mset i u* + *m* ⊂# *repeat-mset j u* + *n*) = (*m* ⊂# *repeat-mset* (*j*−*i*) *u* + *n*)
unfolding *subset-mset-def repeat-mset-iterate-add*
by (*simp add*: *iterate-add-eq-add-iff2 mset-subseteq-add-iff2*[*unfolded repeat-mset-iterate-add*])

ML-file ‹*multiset-simprocs.ML*›

lemma *add-mset-replicate-mset-safe*[*cancelation-simproc-pre*]: ‹*NO-MATCH* {#} *M* ⟹ *add-mset a M* = {#*a*#} + *M*›
by *simp*

declare *repeat-mset-iterate-add*[*cancelation-simproc-pre*]

declare *iterate-add-distrib*[*cancelation-simproc-pre*]
declare *repeat-mset-iterate-add*[*symmetric*, *cancelation-simproc-post*]

declare *add-mset-not-empty*[*cancelation-simproc-eq-elim*]
empty-not-add-mset[*cancelation-simproc-eq-elim*]
subset-mset.le-zero-eq[*cancelation-simproc-eq-elim*]
empty-not-add-mset[*cancelation-simproc-eq-elim*]
add-mset-not-empty[*cancelation-simproc-eq-elim*]
subset-mset.le-zero-eq[*cancelation-simproc-eq-elim*]
le-zero-eq[*cancelation-simproc-eq-elim*]

simproc-setup *mseteq-cancel*
((*l*::'*a multiset*) + *m* = *n* | (*l*::'*a multiset*) = *m* + *n* |
add-mset a m = *n* | *m* = *add-mset a n* |
replicate-mset p a = *n* | *m* = *replicate-mset p a* |
repeat-mset p m = *n* | *m* = *repeat-mset p m*) =
‹*fn phi* => *Cancel-Simprocs.eq-cancel*›

simproc-setup *msetsubset-cancel*
((*l*::'*a multiset*) + *m* ⊂# *n* | (*l*::'*a multiset*) ⊂# *m* + *n* |
add-mset a m ⊂# *n* | *m* ⊂# *add-mset a n* |
replicate-mset p r ⊂# *n* | *m* ⊂# *replicate-mset p r* |
repeat-mset p m ⊂# *n* | *m* ⊂# *repeat-mset p m*) =
‹*fn phi* => *Multiset-Simprocs.subset-cancel-msets*›

simproc-setup *msetsubset-eq-cancel*
((*l*::'*a multiset*) + *m* ⊆# *n* | (*l*::'*a multiset*) ⊆# *m* + *n* |
add-mset a m ⊆# *n* | *m* ⊆# *add-mset a n* |
replicate-mset p r ⊆# *n* | *m* ⊆# *replicate-mset p r* |
repeat-mset p m ⊆# *n* | *m* ⊆# *repeat-mset p m*) =
‹*fn phi* => *Multiset-Simprocs.subseteq-cancel-msets*›

simproc-setup *msetdiff-cancel*
(((*l*::'*a multiset*) + *m*) − *n* | (*l*::'*a multiset*) − (*m* + *n*) |
add-mset a m − *n* | *m* − *add-mset a n* |
replicate-mset p r − *n* | *m* − *replicate-mset p r* |
repeat-mset p m − *n* | *m* − *repeat-mset p m*) =
‹*fn phi* => *Cancel-Simprocs.diff-cancel*›

62.4.2 Conditionally complete lattice

instantiation *multiset* :: (*type*) *Inf*

begin

lift-definition *Inf-multiset* :: *'a multiset set* ⇒ *'a multiset* **is**
  λA i. if A = {} then 0 else Inf ((λf. f i) ‘ A)
proof −
  fix *A* :: *('a ⇒ nat) set*
  assume ∗: *⋀f. f ∈ A ⟹ finite {x. 0 < f x}*
  show ‹*finite {i. 0 < (if A = {} then 0 else INF f∈A. f i)}*›
  proof (*cases A = {}*)
    case *False*
    then obtain *f* **where** *f ∈ A* **by** *blast*
    hence *{i. Inf ((λf. f i) ‘ A) > 0} ⊆ {i. f i > 0}*
      by (*auto intro*: *less-le-trans*[*OF - cInf-lower*])
    moreover from ‹*f ∈ A*› ∗ **have** *finite …* **by** *simp*
    ultimately have *finite {i. Inf ((λf. f i) ‘ A) > 0}* **by** (*rule finite-subset*)
    with *False* **show** *?thesis* **by** *simp*
  qed *simp-all*
qed

instance ..

end

lemma *Inf-multiset-empty*: *Inf {} = {#}*
  by *transfer simp-all*

lemma *count-Inf-multiset-nonempty*: *A ≠ {} ⟹ count (Inf A) x = Inf ((λX. count X x) ‘ A)*
  by *transfer simp-all*

instantiation *multiset* :: (*type*) *Sup*
begin

definition *Sup-multiset* :: *'a multiset set* ⇒ *'a multiset* **where**
  Sup-multiset A = (if A ≠ {} ∧ subset-mset.bdd-above A then
      Abs-multiset (λi. Sup ((λX. count X i) ‘ A)) else {#})

lemma *Sup-multiset-empty*: *Sup {} = {#}*
  by (*simp add*: *Sup-multiset-def*)

lemma *Sup-multiset-unbounded*: *¬ subset-mset.bdd-above A ⟹ Sup A = {#}*
  by (*simp add*: *Sup-multiset-def*)

instance ..

end

lemma *bdd-above-multiset-imp-bdd-above-count*:

assumes *subset-mset.bdd-above* (*A* :: *'a multiset set*)
shows *bdd-above* ((λ*X*. *count X x*) ‘ *A*)
proof −
from *assms* **obtain** *Y* **where** *Y*: ∀ *X*∈*A*. *X* ⊆# *Y*
by (*meson subset-mset.bdd-above.E*)
hence *count X x* ≤ *count Y x* **if** *X* ∈ *A* **for** *X*
using *that* **by** (*auto intro*: *mset-subset-eq-count*)
thus *?thesis* **by** (*intro bdd-aboveI*[*of* - *count Y x*]) *auto*
qed

lemma *bdd-above-multiset-imp-finite-support*:
assumes *A* ≠ {} *subset-mset.bdd-above* (*A* :: *'a multiset set*)
shows *finite* (⋃ *X*∈*A*. {*x*. *count X x* > *0*})
proof −
from *assms* **obtain** *Y* **where** *Y*: ∀ *X*∈*A*. *X* ⊆# *Y*
by (*meson subset-mset.bdd-above.E*)
hence *count X x* ≤ *count Y x* **if** *X* ∈ *A* **for** *X x*
using *that* **by** (*auto intro*: *mset-subset-eq-count*)
hence (⋃ *X*∈*A*. {*x*. *count X x* > *0*}) ⊆ {*x*. *count Y x* > *0*}
by *safe* (*erule less-le-trans*)
moreover have *finite* ... **by** *simp*
ultimately show *?thesis* **by** (*rule finite-subset*)
qed

lemma *Sup-multiset-in-multiset*:
‹*finite* {*i*. *0* < (*SUP M*∈*A*. *count M i*)}›
if ‹*A* ≠ {}› ‹*subset-mset.bdd-above A*›
proof −
have {*i*. *Sup* ((λ*X*. *count X i*) ‘ *A*) > *0*} ⊆ (⋃ *X*∈*A*. {*i*. *0* < *count X i*})
proof *safe*
fix *i* **assume** *pos*: (*SUP X*∈*A*. *count X i*) > *0*
show *i* ∈ (⋃ *X*∈*A*. {*i*. *0* < *count X i*})
proof (*rule ccontr*)
assume *i* ∉ (⋃ *X*∈*A*. {*i*. *0* < *count X i*})
hence ∀ *X*∈*A*. *count X i* ≤ *0* **by** (*auto simp*: *count-eq-zero-iff*)
with *that* **have** (*SUP X*∈*A*. *count X i*) ≤ *0*
by (*intro cSup-least bdd-above-multiset-imp-bdd-above-count*) *auto*
with *pos* **show** *False* **by** *simp*
qed
qed
moreover from *that* **have** *finite* ...
by (*rule bdd-above-multiset-imp-finite-support*)
ultimately show *finite* {*i*. *Sup* ((λ*X*. *count X i*) ‘ *A*) > *0*}
by (*rule finite-subset*)
qed

lemma *count-Sup-multiset-nonempty*:
‹*count* (*Sup A*) *x* = (*SUP X*∈*A*. *count X x*)›
if ‹*A* ≠ {}› ‹*subset-mset.bdd-above A*›

using *that* **by** (*simp add*: *Sup-multiset-def Sup-multiset-in-multiset count-Abs-multiset*)

interpretation *subset-mset*: *conditionally-complete-lattice Inf Sup* (∩#) (⊆#) (⊂#) (∪#)
proof
fix *X* :: *'a multiset* **and** *A*
assume *X* ∈ *A*
show *Inf A* ⊆# *X*
proof (*rule mset-subset-eqI*)
fix *x*
from ‹*X* ∈ *A*› **have** *A* ≠ {} **by** *auto*
hence *count* (*Inf A*) *x* = (*INF X*∈*A*. *count X x*)
by (*simp add*: *count-Inf-multiset-nonempty*)
also from ‹*X* ∈ *A*› **have** ... ≤ *count X x*
by (*intro cInf-lower*) *simp-all*
finally show *count* (*Inf A*) *x* ≤ *count X x* **.**
qed
next
fix *X* :: *'a multiset* **and** *A*
assume *nonempty*: *A* ≠ {} **and** *le*: ⋀*Y*. *Y* ∈ *A* ⟹ *X* ⊆# *Y*
show *X* ⊆# *Inf A*
proof (*rule mset-subset-eqI*)
fix *x*
from *nonempty* **have** *count X x* ≤ (*INF X*∈*A*. *count X x*)
by (*intro cInf-greatest*) (*auto intro*: *mset-subset-eq-count le*)
also from *nonempty* **have** ... = *count* (*Inf A*) *x* **by** (*simp add*: *count-Inf-multiset-nonempty*)
finally show *count X x* ≤ *count* (*Inf A*) *x* **.**
qed
next
fix *X* :: *'a multiset* **and** *A*
assume *X*: *X* ∈ *A* **and** *bdd*: *subset-mset.bdd-above A*
show *X* ⊆# *Sup A*
proof (*rule mset-subset-eqI*)
fix *x*
from *X* **have** *A* ≠ {} **by** *auto*
have *count X x* ≤ (*SUP X*∈*A*. *count X x*)
by (*intro cSUP-upper X bdd-above-multiset-imp-bdd-above-count bdd*)
also from *count-Sup-multiset-nonempty*[*OF* ‹*A* ≠ {}› *bdd*]
have (*SUP X*∈*A*. *count X x*) = *count* (*Sup A*) *x* **by** *simp*
finally show *count X x* ≤ *count* (*Sup A*) *x* **.**
qed
next
fix *X* :: *'a multiset* **and** *A*
assume *nonempty*: *A* ≠ {} **and** *ge*: ⋀*Y*. *Y* ∈ *A* ⟹ *Y* ⊆# *X*
from *ge* **have** *bdd*: *subset-mset.bdd-above A*
by *blast*
show *Sup A* ⊆# *X*
proof (*rule mset-subset-eqI*)
fix *x*

```
    from count-Sup-multiset-nonempty[OF ‹A ≠ {}› bdd]
      have count (Sup A) x = (SUP X∈A. count X x) .
    also from nonempty have ... ≤ count X x
      by (intro cSup-least) (auto intro: mset-subset-eq-count ge)
    finally show count (Sup A) x ≤ count X x .
  qed
qed — FIXME: avoid junk stemming from type class interpretation

lemma set-mset-Inf:
  assumes A ≠ {}
  shows   set-mset (Inf A) = (⋂X∈A. set-mset X)
proof safe
  fix x X assume x ∈# Inf A X ∈ A
  hence nonempty: A ≠ {} by (auto simp: Inf-multiset-empty)
  from ‹x ∈# Inf A› have {#x#} ⊆# Inf A by auto
  also from ‹X ∈ A› have ... ⊆# X by (rule subset-mset.cInf-lower) simp-all
  finally show x ∈# X by simp
next
  fix x assume x: x ∈ (⋂X∈A. set-mset X)
  hence {#x#} ⊆# X if X ∈ A for X using that by auto
  from assms and this have {#x#} ⊆# Inf A by (rule subset-mset.cInf-greatest)
  thus x ∈# Inf A by simp
qed

lemma in-Inf-multiset-iff:
  assumes A ≠ {}
  shows   x ∈# Inf A ⟷ (∀X∈A. x ∈# X)
proof −
  from assms have set-mset (Inf A) = (⋂X∈A. set-mset X) by (rule set-mset-Inf)
  also have x ∈ ... ⟷ (∀X∈A. x ∈# X) by simp
  finally show ?thesis .
qed

lemma in-Inf-multisetD: x ∈# Inf A ⟹ X ∈ A ⟹ x ∈# X
  by (subst (asm) in-Inf-multiset-iff) auto

lemma set-mset-Sup:
  assumes subset-mset.bdd-above A
  shows   set-mset (Sup A) = (⋃X∈A. set-mset X)
proof safe
  fix x assume x ∈# Sup A
  hence nonempty: A ≠ {} by (auto simp: Sup-multiset-empty)
  show x ∈ (⋃X∈A. set-mset X)
  proof (rule ccontr)
    assume x: x ∉ (⋃X∈A. set-mset X)
    have count X x ≤ count (Sup A) x if X ∈ A for X x
      using that by (intro mset-subset-eq-count subset-mset.cSup-upper assms)
    with x have X ⊆# Sup A − {#x#} if X ∈ A for X
      using that by (auto simp: subseteq-mset-def algebra-simps not-in-iff)
```

hence *Sup A ⊆# Sup A − {#x#}* **by** (*intro subset-mset.cSup-least nonempty*)
with ‹*x ∈# Sup A*› **show** *False*
by (*auto simp*: *subseteq-mset-def simp flip*: *count-greater-zero-iff*
dest!: *spec*[*of* - *x*])
qed
next
fix *x X* **assume** *x ∈ set-mset X X ∈ A*
hence *{#x#} ⊆# X* **by** *auto*
also have *X ⊆# Sup A* **by** (*intro subset-mset.cSup-upper* ‹*X ∈ A*› *assms*)
finally show *x ∈ set-mset* (*Sup A*) **by** *simp*
qed

lemma *in-Sup-multiset-iff*:
assumes *subset-mset.bdd-above A*
shows *x ∈# Sup A ⟷* (*∃ X∈A. x ∈# X*)
proof −
from *assms* **have** *set-mset* (*Sup A*) = (*⋃ X∈A. set-mset X*) **by** (*rule set-mset-Sup*)
also have *x ∈ ... ⟷* (*∃ X∈A. x ∈# X*) **by** *simp*
finally show *?thesis* **.**
qed

lemma *in-Sup-multisetD*:
assumes *x ∈# Sup A*
shows *∃ X∈A. x ∈# X*
proof −
have *subset-mset.bdd-above A*
by (*rule ccontr*) (*insert assms*, *simp-all add*: *Sup-multiset-unbounded*)
with *assms* **show** *?thesis* **by** (*simp add*: *in-Sup-multiset-iff*)
qed

interpretation *subset-mset*: *distrib-lattice* (*∩#*) (*⊆#*) (*⊂#*) (*∪#*)
proof
fix *A B C* :: *'a multiset*
show *A ∪# (B ∩# C) = A ∪# B ∩# (A ∪# C)*
by (*intro multiset-eqI*) *simp-all*
qed — FIXME: avoid junk stemming from type class interpretation

62.4.3 Filter (with comprehension syntax)

Multiset comprehension

lift-definition *filter-mset* :: (*'a ⇒ bool*) *⇒ 'a multiset ⇒ 'a multiset*
is *λP M. λx. if P x then M x else 0*
by (*rule filter-preserves-multiset*)

syntax (*ASCII*)
-MCollect :: *pttrn ⇒ 'a multiset ⇒ bool ⇒ 'a multiset* (*(1{#- :# -./ -#})*)
syntax
-MCollect :: *pttrn ⇒ 'a multiset ⇒ bool ⇒ 'a multiset* (*(1{#- ∈# -./ -#})*)
translations

$\{\#x \in\# M.\ P\#\} == CONST\ filter\text{-}mset\ (\lambda x.\ P)\ M$

lemma *count-filter-mset* [*simp*]:
count (*filter-mset P M*) *a* = (*if P a then count M a else 0*)
by (*simp add*: *filter-mset.rep-eq*)

lemma *set-mset-filter* [*simp*]:
set-mset (*filter-mset P M*) = {*a* ∈ *set-mset M*. *P a*}
by (*simp only*: *set-eq-iff count-greater-zero-iff* [*symmetric*] *count-filter-mset*) *simp*

lemma *filter-empty-mset* [*simp*]: *filter-mset P* {#} = {#}
by (*rule multiset-eqI*) *simp*

lemma *filter-single-mset*: *filter-mset P* {#*x*#} = (*if P x then* {#*x*#} *else* {#})
by (*rule multiset-eqI*) *simp*

lemma *filter-union-mset* [*simp*]: *filter-mset P* (*M* + *N*) = *filter-mset P M* + *filter-mset P N*
by (*rule multiset-eqI*) *simp*

lemma *filter-diff-mset* [*simp*]: *filter-mset P* (*M* − *N*) = *filter-mset P M* − *filter-mset P N*
by (*rule multiset-eqI*) *simp*

lemma *filter-inter-mset* [*simp*]: *filter-mset P* (*M* ∩# *N*) = *filter-mset P M* ∩# *filter-mset P N*
by (*rule multiset-eqI*) *simp*

lemma *filter-sup-mset*[*simp*]: *filter-mset P* (*A* ∪# *B*) = *filter-mset P A* ∪# *filter-mset P B*
by (*rule multiset-eqI*) *simp*

lemma *filter-mset-add-mset* [*simp*]:
filter-mset P (*add-mset x A*) =
(*if P x then add-mset x* (*filter-mset P A*) *else filter-mset P A*)
by (*auto simp*: *multiset-eq-iff*)

lemma *multiset-filter-subset*[*simp*]: *filter-mset f M* ⊆# *M*
by (*simp add*: *mset-subset-eqI*)

lemma *multiset-filter-mono*:
assumes *A* ⊆# *B*
shows *filter-mset f A* ⊆# *filter-mset f B*
proof −
from *assms*[*unfolded mset-subset-eq-exists-conv*]
obtain *C* **where** *B*: *B* = *A* + *C* **by** *auto*
show *?thesis* **unfolding** *B* **by** *auto*
qed

lemma *filter-mset-eq-conv*:
filter-mset P M = N ⟷ N ⊆# M ∧ (∀ b∈#N. P b) ∧ (∀ a∈#M − N. ¬ P a)
(**is** *?P ⟷ ?Q*)
proof
assume *?P* **then show** *?Q* **by** *auto* (*simp add*: *multiset-eq-iff in-diff-count*)
next
assume *?Q*
then obtain *Q* **where** *M*: *M = N + Q*
by (*auto simp add*: *mset-subset-eq-exists-conv*)
then have *MN*: *M − N = Q* **by** *simp*
show *?P*
proof (*rule multiset-eqI*)
fix *a*
from ‹*?Q*› *MN* **have** ∗: *¬ P a ⟹ a ∉# N P a ⟹ a ∉# Q*
by *auto*
show *count* (*filter-mset P M*) *a = count N a*
proof (*cases a ∈# M*)
case *True*
with ∗ **show** *?thesis*
by (*simp add*: *not-in-iff M*)
next
case *False* **then have** *count M a = 0*
by (*simp add*: *not-in-iff*)
with *M* **show** *?thesis* **by** *simp*
qed
qed
qed

lemma *filter-filter-mset*: *filter-mset P* (*filter-mset Q M*) = {#*x* ∈# *M*. *Q x ∧ P x*#}
by (*auto simp*: *multiset-eq-iff*)

lemma
filter-mset-True[*simp*]: {#*y* ∈# *M*. *True*#} = *M* **and**
filter-mset-False[*simp*]: {#*y* ∈# *M*. *False*#} = {#}
by (*auto simp*: *multiset-eq-iff*)

62.4.4 Size

definition *wcount* **where** *wcount f M* = (λ*x*. *count M x ∗ Suc* (*f x*))

lemma *wcount-union*: *wcount f* (*M + N*) *a* = *wcount f M a + wcount f N a*
by (*auto simp*: *wcount-def add-mult-distrib*)

lemma *wcount-add-mset*:
wcount f (*add-mset x M*) *a* = (*if x = a then Suc* (*f a*) *else 0*) + *wcount f M a*
unfolding *add-mset-add-single*[*of - M*] *wcount-union* **by** (*auto simp*: *wcount-def*)

definition *size-multiset* :: (′*a* ⇒ *nat*) ⇒ ′*a multiset* ⇒ *nat* **where**

size-multiset f M = sum (wcount f M) (set-mset M)

lemmas *size-multiset-eq = size-multiset-def[unfolded wcount-def]*

instantiation *multiset :: (type) size*
begin

definition *size-multiset* **where**
size-multiset-overloaded-def: size-multiset = Multiset.size-multiset (λ-. 0)
instance ..

end

lemmas *size-multiset-overloaded-eq =*
size-multiset-overloaded-def[THEN fun-cong, unfolded size-multiset-eq, simplified]

lemma *size-multiset-empty [simp]: size-multiset f {#} = 0*
by *(simp add: size-multiset-def)*

lemma *size-empty [simp]: size {#} = 0*
by *(simp add: size-multiset-overloaded-def)*

lemma *size-multiset-single : size-multiset f {#b#} = Suc (f b)*
by *(simp add: size-multiset-eq)*

lemma *size-single: size {#b#} = 1*
by *(simp add: size-multiset-overloaded-def size-multiset-single)*

lemma *sum-wcount-Int:*
finite A ⟹ sum (wcount f N) (A ∩ set-mset N) = sum (wcount f N) A
by *(induct rule: finite-induct)*
(simp-all add: Int-insert-left wcount-def count-eq-zero-iff)

lemma *size-multiset-union [simp]:*
size-multiset f (M + N::'a multiset) = size-multiset f M + size-multiset f N
apply *(simp add: size-multiset-def sum-Un-nat sum.distrib sum-wcount-Int wcount-union)*
apply *(subst Int-commute)*
apply *(simp add: sum-wcount-Int)*
done

lemma *size-multiset-add-mset [simp]:*
size-multiset f (add-mset a M) = Suc (f a) + size-multiset f M
unfolding *add-mset-add-single[of - M] size-multiset-union* **by** *(auto simp: size-multiset-single)*

lemma *size-add-mset [simp]: size (add-mset a A) = Suc (size A)*
by *(simp add: size-multiset-overloaded-def wcount-add-mset)*

lemma *size-union [simp]: size (M + N::'a multiset) = size M + size N*
by *(auto simp add: size-multiset-overloaded-def)*

lemma *size-multiset-eq-0-iff-empty* [*iff*]:
size-multiset f M = 0 $\longleftrightarrow$ *M = {#}*
by (*auto simp add*: *size-multiset-eq count-eq-zero-iff*)

lemma *size-eq-0-iff-empty* [*iff*]: (*size M = 0*) = (*M = {#}*)
by (*auto simp add*: *size-multiset-overloaded-def*)

lemma *nonempty-has-size*: (*S* $\neq$ *{#}*) = (*0 < size S*)
by (*metis gr0I gr-implies-not0 size-empty size-eq-0-iff-empty*)

lemma *size-eq-Suc-imp-elem*: *size M = Suc n* $\Longrightarrow$ $\exists$ *a*. *a* $\in\#$ *M*
apply (*unfold size-multiset-overloaded-eq*)
apply (*drule sum-SucD*)
apply *auto*
done

lemma *size-eq-Suc-imp-eq-union*:
assumes *size M = Suc n*
shows $\exists$ *a N*. *M = add-mset a N*
proof −
from *assms* **obtain** *a* **where** *a* $\in\#$ *M*
by (*erule size-eq-Suc-imp-elem* [*THEN exE*])
then have *M = add-mset a* (*M* − *{#a#}*) **by** *simp*
then show *?thesis* **by** *blast*
qed

lemma *size-mset-mono*:
fixes *A B* :: *'a multiset*
assumes *A* $\subseteq\#$ *B*
shows *size A* $\leq$ *size B*
proof −
from *assms*[*unfolded mset-subset-eq-exists-conv*]
obtain *C* **where** *B*: *B = A + C* **by** *auto*
show *?thesis* **unfolding** *B* **by** (*induct C*) *auto*
qed

lemma *size-filter-mset-lesseq*[*simp*]: *size* (*filter-mset f M*) $\leq$ *size M*
by (*rule size-mset-mono*[*OF multiset-filter-subset*])

lemma *size-Diff-submset*:
M $\subseteq\#$ *M'* $\Longrightarrow$ *size* (*M'* − *M*) = *size M'* − *size*(*M*::*'a multiset*)
by (*metis add-diff-cancel-left' size-union mset-subset-eq-exists-conv*)

62.5 Induction and case splits

theorem *multiset-induct* [*case-names empty add*, *induct type*: *multiset*]:
assumes *empty*: *P {#}*
assumes *add*: $\bigwedge$*x M*. *P M* $\Longrightarrow$ *P* (*add-mset x M*)

```
  shows P M
proof (induct size M arbitrary: M)
  case 0 thus P M by (simp add: empty)
next
  case (Suc k)
  obtain N x where M = add-mset x N
    using ‹Suc k = size M› [symmetric]
    using size-eq-Suc-imp-eq-union by fast
  with Suc add show P M by simp
qed

lemma multiset-induct-min[case-names empty add]:
  fixes M :: 'a::linorder multiset
  assumes
    empty: P {#} and
    add: ⋀x M. P M ⟹ (∀ y ∈# M. y ≥ x) ⟹ P (add-mset x M)
  shows P M
proof (induct size M arbitrary: M)
  case (Suc k)
  note ih = this(1) and Sk-eq-sz-M = this(2)

  let ?y = Min-mset M
  let ?N = M − {#?y#}

  have M: M = add-mset ?y ?N
    by (metis Min-in Sk-eq-sz-M finite-set-mset insert-DiffM lessI not-less-zero
      set-mset-eq-empty-iff size-empty)
  show ?case
    by (subst M, rule add, rule ih, metis M Sk-eq-sz-M nat.inject size-add-mset,
      meson Min-le finite-set-mset in-diffD)
qed (simp add: empty)

lemma multiset-induct-max[case-names empty add]:
  fixes M :: 'a::linorder multiset
  assumes
    empty: P {#} and
    add: ⋀x M. P M ⟹ (∀ y ∈# M. y ≤ x) ⟹ P (add-mset x M)
  shows P M
proof (induct size M arbitrary: M)
  case (Suc k)
  note ih = this(1) and Sk-eq-sz-M = this(2)

  let ?y = Max-mset M
  let ?N = M − {#?y#}

  have M: M = add-mset ?y ?N
    by (metis Max-in Sk-eq-sz-M finite-set-mset insert-DiffM lessI not-less-zero
      set-mset-eq-empty-iff size-empty)
  show ?case
```

by (*subst M*, *rule add*, *rule ih*, *metis M Sk-eq-sz-M nat.inject size-add-mset*, *meson Max-ge finite-set-mset in-diffD*)
qed (*simp add*: *empty*)

lemma *multi-nonempty-split*: $M \neq \{\#\} \Longrightarrow \exists\, A\ a.\ M = \mathit{add\text{-}mset}\ a\ A$
by (*induct M*) *auto*

lemma *multiset-cases* [*cases type*]:
obtains (*empty*) $M = \{\#\}$
| (*add*) *x N* **where** $M = \mathit{add\text{-}mset}\ x\ N$
by (*induct M*) *simp-all*

lemma *multi-drop-mem-not-eq*: $c \in\# B \Longrightarrow B - \{\#c\#\} \neq B$
by (*cases* $B = \{\#\}$) (*auto dest*: *multi-member-split*)

lemma *union-filter-mset-complement*[*simp*]:
$\forall x.\ P\ x = (\neg\ Q\ x) \Longrightarrow \mathit{filter\text{-}mset}\ P\ M + \mathit{filter\text{-}mset}\ Q\ M = M$
by (*subst multiset-eq-iff*) *auto*

lemma *multiset-partition*: $M = \{\#x \in\# M.\ P\ x\#\} + \{\#x \in\# M.\ \neg\ P\ x\#\}$
by *simp*

lemma *mset-subset-size*: $A \subset\# B \Longrightarrow \mathit{size}\ A < \mathit{size}\ B$
proof (*induct A arbitrary*: *B*)
case *empty*
then show *?case*
using *nonempty-has-size* **by** *auto*
next
case (*add x A*)
have *add-mset x A* $\subseteq\#$ *B*
by (*meson add.prems subset-mset-def*)
then show *?case*
by (*metis* (*no-types*) *add.prems add.right-neutral add-diff-cancel-left' leD nat-neq-iff size-Diff-submset size-eq-0-iff-empty size-mset-mono subset-mset.le-iff-add subset-mset-def*)
qed

lemma *size-1-singleton-mset*: $\mathit{size}\ M = 1 \Longrightarrow \exists\, a.\ M = \{\#a\#\}$
by (*cases M*) *auto*

62.5.1 Strong induction and subset induction for multisets

Well-foundedness of strict subset relation

lemma *wf-subset-mset-rel*: *wf* $\{(M, N :: 'a\ \mathit{multiset}).\ M \subset\# N\}$
apply (*rule wf-measure* [*THEN wf-subset*, **where** *f1*=*size*])
apply (*clarsimp simp*: *measure-def inv-image-def mset-subset-size*)
done

lemma *full-multiset-induct* [*case-names less*]:

assumes *ih*: $\bigwedge B.\ \forall (A::'a\ multiset).\ A \subset\# B \longrightarrow P\ A \Longrightarrow P\ B$
shows $P\ B$
apply (*rule wf-subset-mset-rel* [*THEN wf-induct*])
apply (*rule ih*, *auto*)
done

lemma *multi-subset-induct* [*consumes 2*, *case-names empty add*]:
  assumes $F \subseteq\# A$
    and *empty*: $P\ \{\#\}$
    and *insert*: $\bigwedge a\ F.\ a \in\# A \Longrightarrow P\ F \Longrightarrow P\ (add\text{-}mset\ a\ F)$
  shows $P\ F$
proof −
  from ‹$F \subseteq\# A$›
  show *?thesis*
  proof (*induct F*)
    show $P\ \{\#\}$ **by** *fact*
  next
    fix $x\ F$
    assume *P*: $F \subseteq\# A \Longrightarrow P\ F$ **and** *i*: $add\text{-}mset\ x\ F \subseteq\# A$
    show $P\ (add\text{-}mset\ x\ F)$
    proof (*rule insert*)
      from *i* **show** $x \in\# A$ **by** (*auto dest*: *mset-subset-eq-insertD*)
      from *i* **have** $F \subseteq\# A$ **by** (*auto dest*: *mset-subset-eq-insertD*)
      with *P* **show** $P\ F$ **.**
    qed
  qed
qed

62.6 The fold combinator

definition $fold\text{-}mset :: ('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ multiset \Rightarrow 'b$
where
  $fold\text{-}mset\ f\ s\ M = Finite\text{-}Set.fold\ (\lambda x.\ f\ x \frown\frown count\ M\ x)\ s\ (set\text{-}mset\ M)$

lemma *fold-mset-empty* [*simp*]: $fold\text{-}mset\ f\ s\ \{\#\} = s$
  by (*simp add*: *fold-mset-def*)

context *comp-fun-commute*
begin

lemma *fold-mset-add-mset* [*simp*]: $fold\text{-}mset\ f\ s\ (add\text{-}mset\ x\ M) = f\ x\ (fold\text{-}mset\ f\ s\ M)$
proof −
  interpret *mset*: *comp-fun-commute* $\lambda y.\ f\ y \frown\frown count\ M\ y$
    by (*fact comp-fun-commute-funpow*)
  interpret *mset-union*: *comp-fun-commute* $\lambda y.\ f\ y \frown\frown count\ (add\text{-}mset\ x\ M)\ y$
    by (*fact comp-fun-commute-funpow*)
  show *?thesis*
  proof (*cases* $x \in set\text{-}mset\ M$)

```
    case False
    then have ∗: count (add-mset x M) x = 1
      by (simp add: not-in-iff)
   from False have Finite-Set.fold (λy. f y ^^ count (add-mset x M) y) s (set-mset
M) =
      Finite-Set.fold (λy. f y ^^ count M y) s (set-mset M)
      by (auto intro!: Finite-Set.fold-cong comp-fun-commute-on-funpow)
    with False ∗ show ?thesis
      by (simp add: fold-mset-def del: count-add-mset)
  next
    case True
    define N where N = set-mset M − {x}
    from N-def True have ∗: set-mset M = insert x N x ∉ N finite N by auto
    then have Finite-Set.fold (λy. f y ^^ count (add-mset x M) y) s N =
      Finite-Set.fold (λy. f y ^^ count M y) s N
      by (auto intro!: Finite-Set.fold-cong comp-fun-commute-on-funpow)
    with ∗ show ?thesis by (simp add: fold-mset-def del: count-add-mset) simp
  qed
qed

corollary fold-mset-single: fold-mset f s {#x#} = f x s
  by simp

lemma fold-mset-fun-left-comm: f x (fold-mset f s M) = fold-mset f (f x s) M
  by (induct M) (simp-all add: fun-left-comm)

lemma fold-mset-union [simp]: fold-mset f s (M + N) = fold-mset f (fold-mset f
s M) N
  by (induct M) (simp-all add: fold-mset-fun-left-comm)

lemma fold-mset-fusion:
  assumes comp-fun-commute g
    and ∗: ⋀x y. h (g x y) = f x (h y)
  shows h (fold-mset g w A) = fold-mset f (h w) A
proof −
  interpret comp-fun-commute g by (fact assms)
  from ∗ show ?thesis by (induct A) auto
qed

end

lemma union-fold-mset-add-mset: A + B = fold-mset add-mset A B
proof −
  interpret comp-fun-commute add-mset
    by standard auto
  show ?thesis
    by (induction B) auto
qed
```

A note on code generation: When defining some function containing a

subterm *fold-mset F*, code generation is not automatic. When interpreting locale *left-commutative* with *F*, the would be code thms for *fold-mset* become thms like *fold-mset F z {#} = z* where *F* is not a pattern but contains defined symbols, i.e. is not a code thm. Hence a separate constant with its own code thms needs to be introduced for *F*. See the image operator below.

62.7 Image

definition *image-mset* :: *('a ⇒ 'b) ⇒ 'a multiset ⇒ 'b multiset* **where**
image-mset f = fold-mset (add-mset ∘ f) {#}

lemma *comp-fun-commute-mset-image*: *comp-fun-commute (add-mset ∘ f)*
by *unfold-locales* (*simp add*: *fun-eq-iff*)

lemma *image-mset-empty* [*simp*]: *image-mset f {#} = {#}*
by (*simp add*: *image-mset-def*)

lemma *image-mset-single*: *image-mset f {#x#} = {#f x#}*
by (*simp add*: *comp-fun-commute.fold-mset-add-mset comp-fun-commute-mset-image image-mset-def*)

lemma *image-mset-union* [*simp*]: *image-mset f (M + N) = image-mset f M + image-mset f N*
proof −
interpret *comp-fun-commute add-mset ∘ f*
by (*fact comp-fun-commute-mset-image*)
show *?thesis* **by** (*induct N*) (*simp-all add*: *image-mset-def*)
qed

corollary *image-mset-add-mset* [*simp*]:
image-mset f (add-mset a M) = add-mset (f a) (image-mset f M)
unfolding *image-mset-union add-mset-add-single*[*of a M*] **by** (*simp add*: *image-mset-single*)

lemma *set-image-mset* [*simp*]: *set-mset (image-mset f M) = image f (set-mset M)*
by (*induct M*) *simp-all*

lemma *size-image-mset* [*simp*]: *size (image-mset f M) = size M*
by (*induct M*) *simp-all*

lemma *image-mset-is-empty-iff* [*simp*]: *image-mset f M = {#} ⟷ M = {#}*
by (*cases M*) *auto*

lemma *image-mset-If*:
image-mset (λx. if P x then f x else g x) A =
image-mset f (filter-mset P A) + image-mset g (filter-mset (λx. ¬P x) A)
by (*induction A*) *auto*

lemma *image-mset-Diff*:

assumes $B \subseteq\# A$
shows *image-mset f* $(A - B) =$ *image-mset f A* $-$ *image-mset f B*
proof $-$
have *image-mset f* $(A - B + B) =$ *image-mset f* $(A - B) +$ *image-mset f B*
by *simp*
also from *assms* **have** $A - B + B = A$
by (*simp add*: *subset-mset.diff-add*)
finally show *?thesis* **by** *simp*
qed

lemma *count-image-mset*:
‹*count* (*image-mset f A*) $x = (\sum y \in f -` \{x\} \cap$ *set-mset A*. *count A y*)›
proof (*induction A*)
case *empty*
then show *?case* **by** *simp*
next
case (*add x A*)
moreover have $*$: (*if* $x = y$ *then Suc n else n*) $= n +$ (*if* $x = y$ *then 1 else 0*)
for *n y*
by *simp*
ultimately show *?case*
by (*auto simp*: *sum.distrib intro*!: *sum.mono-neutral-left*)
qed

lemma *count-image-mset'*:
‹*count* (*image-mset f X*) $y = (\sum x \mid x \in\# X \wedge y = f\ x$. *count X x*)›
by (*auto simp add*: *count-image-mset simp flip*: *singleton-conv2 simp add*: *Collect-conj-eq ac-simps*)

lemma *image-mset-subseteq-mono*: $A \subseteq\# B \Longrightarrow$ *image-mset f A* $\subseteq\#$ *image-mset f B*
by (*metis image-mset-union subset-mset.le-iff-add*)

lemma *image-mset-subset-mono*: $M \subset\# N \Longrightarrow$ *image-mset f M* $\subset\#$ *image-mset f N*
by (*metis* (*no-types*) *Diff-eq-empty-iff-mset image-mset-Diff image-mset-is-empty-iff image-mset-subseteq-mono subset-mset.less-le-not-le*)

syntax (*ASCII*)
-comprehension-mset :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'b multiset* $\Rightarrow$ *'a multiset* (({#-/. - :# -#}))
syntax
-comprehension-mset :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'b multiset* $\Rightarrow$ *'a multiset* (({#-/. - $\in\#$ -#}))
translations
{#*e*. *x* $\in\#$ *M*#} $\rightleftharpoons$ *CONST image-mset* (λx. *e*) *M*

syntax (*ASCII*)
-comprehension-mset' :: *'a* $\Rightarrow$ *'b* $\Rightarrow$ *'b multiset* $\Rightarrow$ *bool* $\Rightarrow$ *'a multiset* (({#-/ | - :# -./ -#}))
syntax

-comprehension-mset' :: *'a* ⇒ *'b* ⇒ *'b multiset* ⇒ *bool* ⇒ *'a multiset* (({#-/ | -∈# -./ -#}))
translations
{#*e* | *x*∈#*M*. *P*#} ⇀ {#*e*. *x* ∈# {# *x*∈#*M*. *P*#}#}

This allows to write not just filters like {#*x* ∈# *M*. *x* < *c*#} but also images like {#*x* + *x*. *x* ∈# *M*#} and {#*x*+*x*|*x*∈#*M*. *x*<*c*#}, where the latter is currently displayed as {#*x* + *x*. *x* ∈# {#*x* ∈# *M*. *x* < *c*#}#}.

lemma *in-image-mset*: *y* ∈# {#*f x*. *x* ∈# *M*#} ⟷ *y* ∈ *f* ' *set-mset M*
by *simp*

functor *image-mset*: *image-mset*
proof −
fix *f g* **show** *image-mset f* ∘ *image-mset g* = *image-mset* (*f* ∘ *g*)
proof
fix *A*
show (*image-mset f* ∘ *image-mset g*) *A* = *image-mset* (*f* ∘ *g*) *A*
by (*induct A*) *simp-all*
qed
show *image-mset id* = *id*
proof
fix *A*
show *image-mset id A* = *id A*
by (*induct A*) *simp-all*
qed
qed

declare
image-mset.id [*simp*]
image-mset.identity [*simp*]

lemma *image-mset-id*[*simp*]: *image-mset id x* = *x*
unfolding *id-def* **by** *auto*

lemma *image-mset-cong*: (⋀*x*. *x* ∈# *M* ⟹ *f x* = *g x*) ⟹ {#*f x*. *x* ∈# *M*#} = {#*g x*. *x* ∈# *M*#}
by (*induct M*) *auto*

lemma *image-mset-cong-pair*:
(∀ *x y*. (*x*, *y*) ∈# *M* ⟶ *f x y* = *g x y*) ⟹ {#*f x y*. (*x*, *y*) ∈# *M*#} = {#*g x y*. (*x*, *y*) ∈# *M*#}
by (*metis image-mset-cong split-cong*)

lemma *image-mset-const-eq*:
{#*c*. *a* ∈# *M*#} = *replicate-mset* (*size M*) *c*
by (*induct M*) *simp-all*

62.8 Further conversions

primrec *mset* :: *'a list* ⇒ *'a multiset* **where**
mset [] = {#} |
mset (*a* # *x*) = *add-mset a* (*mset x*)

lemma *in-multiset-in-set*:
x ∈# *mset xs* ⟷ *x* ∈ *set xs*
by (*induct xs*) *simp-all*

lemma *count-mset*:
count (*mset xs*) *x* = *length* (*filter* (λ*y*. *x* = *y*) *xs*)
by (*induct xs*) *simp-all*

lemma *mset-zero-iff*[*simp*]: (*mset x* = {#}) = (*x* = [])
by (*induct x*) *auto*

lemma *mset-zero-iff-right*[*simp*]: ({#} = *mset x*) = (*x* = [])
by (*induct x*) *auto*

lemma *count-mset-gt-0*: *x* ∈ *set xs* ⟹ *count* (*mset xs*) *x* > *0*
by (*induction xs*) *auto*

lemma *count-mset-0-iff* [*simp*]: *count* (*mset xs*) *x* = *0* ⟷ *x* ∉ *set xs*
by (*induction xs*) *auto*

lemma *mset-single-iff*[*iff*]: *mset xs* = {#*x*#} ⟷ *xs* = [*x*]
by (*cases xs*) *auto*

lemma *mset-single-iff-right*[*iff*]: {#*x*#} = *mset xs* ⟷ *xs* = [*x*]
by (*cases xs*) *auto*

lemma *set-mset-mset*[*simp*]: *set-mset* (*mset xs*) = *set xs*
by (*induct xs*) *auto*

lemma *set-mset-comp-mset* [*simp*]: *set-mset* ∘ *mset* = *set*
by (*simp add*: *fun-eq-iff*)

lemma *size-mset* [*simp*]: *size* (*mset xs*) = *length xs*
by (*induct xs*) *simp-all*

lemma *mset-append* [*simp*]: *mset* (*xs* @ *ys*) = *mset xs* + *mset ys*
by (*induct xs arbitrary*: *ys*) *auto*

lemma *mset-filter*[*simp*]: *mset* (*filter P xs*) = {#*x* ∈# *mset xs*. *P x* #}
by (*induct xs*) *simp-all*

lemma *mset-rev* [*simp*]:
mset (*rev xs*) = *mset xs*
by (*induct xs*) *simp-all*

lemma *surj-mset*: *surj mset*
apply (*unfold surj-def*)
apply (*rule allI*)
apply (*rule-tac M = y* **in** *multiset-induct*)
apply *auto*
apply (*rule-tac x = x # xa* **in** *exI*)
apply *auto*
done

lemma *distinct-count-atmost-1*:
distinct x = (∀ a. count (mset x) a = (if a ∈ set x then 1 else 0))
proof (*induct x*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*) **show** *?case* (**is** *?lhs ⟷ ?rhs*)
proof
assume *?lhs* **then show** *?rhs* **using** *Cons* **by** *simp*
next
assume *?rhs* **then have** *x ∉ set xs*
by (*simp split*: *if-splits*)
moreover from ‹*?rhs*› **have** (*∀ a. count (mset xs) a =*
(if a ∈ set xs then 1 else 0))
by (*auto split*: *if-splits simp add*: *count-eq-zero-iff*)
ultimately show *?lhs* **using** *Cons* **by** *simp*
qed
qed

lemma *mset-eq-setD*:
assumes *mset xs = mset ys*
shows *set xs = set ys*
proof −
from *assms* **have** *set-mset (mset xs) = set-mset (mset ys)*
by *simp*
then show *?thesis* **by** *simp*
qed

lemma *set-eq-iff-mset-eq-distinct*:
‹*distinct x ⟹ distinct y ⟹ set x = set y ⟷ mset x = mset y*›
by (*auto simp*: *multiset-eq-iff distinct-count-atmost-1*)

lemma *set-eq-iff-mset-remdups-eq*:
‹*set x = set y ⟷ mset (remdups x) = mset (remdups y)*›
apply (*rule iffI*)
apply (*simp add*: *set-eq-iff-mset-eq-distinct*[*THEN iffD1*])
apply (*drule distinct-remdups* [*THEN distinct-remdups*
[*THEN set-eq-iff-mset-eq-distinct* [*THEN iffD2*]]])
apply *simp*
done

lemma *mset-eq-imp-distinct-iff*:
  ‹*distinct xs* ⟷ *distinct ys*› **if** ‹*mset xs* = *mset ys*›
  using *that* **by** (*auto simp add*: *distinct-count-atmost-1 dest*: *mset-eq-setD*)

lemma *nth-mem-mset*: *i* < *length ls* ⟹ (*ls* ! *i*) ∈# *mset ls*
proof (*induct ls arbitrary*: *i*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case *Cons*
  then show *?case* **by** (*cases i*) *auto*
qed

lemma *mset-remove1*[*simp*]: *mset* (*remove1 a xs*) = *mset xs* − {#*a*#}
  by (*induct xs*) (*auto simp add*: *multiset-eq-iff*)

lemma *mset-eq-length*:
  assumes *mset xs* = *mset ys*
  shows *length xs* = *length ys*
  using *assms* **by** (*metis size-mset*)

lemma *mset-eq-length-filter*:
  assumes *mset xs* = *mset ys*
  shows *length* (*filter* (λ*x*. *z* = *x*) *xs*) = *length* (*filter* (λ*y*. *z* = *y*) *ys*)
  using *assms* **by** (*metis count-mset*)

lemma *fold-multiset-equiv*:
  ‹*List.fold f xs* = *List.fold f ys*›
    if *f*: ‹⋀*x y*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *f x* ∘ *f y* = *f y* ∘ *f x*›
    and ‹*mset xs* = *mset ys*›
using *f* ‹*mset xs* = *mset ys*› [*symmetric*] **proof** (*induction xs arbitrary*: *ys*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons x xs*)
  then have ∗: ‹*set ys* = *set* (*x* # *xs*)›
    by (*blast dest*: *mset-eq-setD*)
  have ‹⋀*x y*. *x* ∈ *set ys* ⟹ *y* ∈ *set ys* ⟹ *f x* ∘ *f y* = *f y* ∘ *f x*›
    by (*rule Cons.prems*(*1*)) (*simp-all add*: ∗)
  moreover from ∗ **have** ‹*x* ∈ *set ys*›
    by *simp*
  ultimately have ‹*List.fold f ys* = *List.fold f* (*remove1 x ys*) ∘ *f x*›
    by (*fact fold-remove1-split*)
  moreover from *Cons.prems* **have** ‹*List.fold f xs* = *List.fold f* (*remove1 x ys*)›
    by (*auto intro*: *Cons.IH*)
  ultimately show *?case*
    by *simp*
qed

lemma *fold-permuted-eq*:
  ‹*List.fold* (⊙) *xs z* = *List.fold* (⊙) *ys z*›
    if ‹*mset xs* = *mset ys*›
    and ‹*P z*› **and** *P*: ‹⋀*x z*. *x* ∈ *set xs* ⟹ *P z* ⟹ *P* (*x* ⊙ *z*)›
    and *f*: ‹⋀*x y z*. *x* ∈ *set xs* ⟹ *y* ∈ *set xs* ⟹ *P z* ⟹ *x* ⊙ (*y* ⊙ *z*) = *y* ⊙ (*x* ⊙ *z*)›
  for *f* (**infixl** ‹⊙› *70*)
using ‹*P z*› *P f* ‹*mset xs* = *mset ys*› [*symmetric*] **proof** (*induction xs arbitrary*: *ys z*)
  case *Nil*
  then show *?case* **by** *simp*
next
  case (*Cons x xs*)
  then have ∗: ‹*set ys* = *set* (*x* # *xs*)›
    by (*blast dest*: *mset-eq-setD*)
  have ‹*P z*›
    by (*fact Cons.prems*(*1*))
  moreover have ‹⋀*x z*. *x* ∈ *set ys* ⟹ *P z* ⟹ *P* (*x* ⊙ *z*)›
    by (*rule Cons.prems*(*2*)) (*simp-all add*: ∗)
  moreover have ‹⋀*x y z*. *x* ∈ *set ys* ⟹ *y* ∈ *set ys* ⟹ *P z* ⟹ *x* ⊙ (*y* ⊙ *z*) = *y* ⊙ (*x* ⊙ *z*)›
    by (*rule Cons.prems*(*3*)) (*simp-all add*: ∗)
  moreover from ∗ **have** ‹*x* ∈ *set ys*›
    by *simp*
  ultimately have ‹*fold* (⊙) *ys z* = *fold* (⊙) (*remove1 x ys*) (*x* ⊙ *z*)›
    by (*induction ys arbitrary*: *z*) *auto*
  moreover from *Cons.prems* **have** ‹*fold* (⊙) *xs* (*x* ⊙ *z*) = *fold* (⊙) (*remove1 x ys*) (*x* ⊙ *z*)›
    by (*auto intro*: *Cons.IH*)
  ultimately show *?case*
    by *simp*
qed

lemma *mset-shuffles*: *zs* ∈ *shuffles xs ys* ⟹ *mset zs* = *mset xs* + *mset ys*
  by (*induction xs ys arbitrary*: *zs rule*: *shuffles.induct*) *auto*

lemma *mset-insort* [*simp*]: *mset* (*insort x xs*) = *add-mset x* (*mset xs*)
  by (*induct xs*) *simp-all*

lemma *mset-map*[*simp*]: *mset* (*map f xs*) = *image-mset f* (*mset xs*)
  by (*induct xs*) *simp-all*

global-interpretation *mset-set*: *folding add-mset* {#}
  defines *mset-set* = *folding-on.F add-mset* {#}
  by *standard* (*simp add*: *fun-eq-iff*)

lemma *sum-multiset-singleton* [*simp*]: *sum* (λ*n*. {#*n*#}) *A* = *mset-set A*
  by (*induction A rule*: *infinite-finite-induct*) *auto*

lemma *count-mset-set* [*simp*]:
finite A $\Longrightarrow$ *x* $\in$ *A* $\Longrightarrow$ *count* (*mset-set A*) *x* = *1* (**is** *PROP ?P*)
$\neg$ *finite A* $\Longrightarrow$ *count* (*mset-set A*) *x* = *0* (**is** *PROP ?Q*)
x $\notin$ *A* $\Longrightarrow$ *count* (*mset-set A*) *x* = *0* (**is** *PROP ?R*)
proof −
have ∗: *count* (*mset-set A*) *x* = *0* **if** *x* $\notin$ *A* **for** *A*
proof (*cases finite A*)
case *False* **then show** *?thesis* **by** *simp*
next
case *True* **from** *True* ‹*x* $\notin$ *A*› **show** *?thesis* **by** (*induct A*) *auto*
qed
then show *PROP ?P PROP ?Q PROP ?R*
by (*auto elim*!: *Set.set-insert*)
qed — TODO: maybe define *mset-set* also in terms of *Abs-multiset*

lemma *elem-mset-set*[*simp*, *intro*]: *finite A* $\Longrightarrow$ *x* $\in\#$ *mset-set A* $\longleftrightarrow$ *x* $\in$ *A*
by (*induct A rule*: *finite-induct*) *simp-all*

lemma *mset-set-Union*:
finite A $\Longrightarrow$ *finite B* $\Longrightarrow$ *A* $\cap$ *B* = {} $\Longrightarrow$ *mset-set* (*A* $\cup$ *B*) = *mset-set A* + *mset-set B*
by (*induction A rule*: *finite-induct*) *auto*

lemma *filter-mset-mset-set* [*simp*]:
finite A $\Longrightarrow$ *filter-mset P* (*mset-set A*) = *mset-set* {*x*∈*A*. *P x*}
proof (*induction A rule*: *finite-induct*)
case (*insert x A*)
from *insert.hyps* **have** *filter-mset P* (*mset-set* (*insert x A*)) =
filter-mset P (*mset-set A*) + *mset-set* (*if P x then* {*x*} *else* {})
by *simp*
also have *filter-mset P* (*mset-set A*) = *mset-set* {*x*∈*A*. *P x*}
by (*rule insert.IH*)
also from *insert.hyps*
have ... + *mset-set* (*if P x then* {*x*} *else* {}) =
mset-set ({*x* ∈ *A*. *P x*} ∪ (*if P x then* {*x*} *else* {})) (**is** - = *mset-set ?A*)
by (*intro mset-set-Union* [*symmetric*]) *simp-all*
also from *insert.hyps* **have** *?A* = {*y*∈*insert x A*. *P y*} **by** *auto*
finally show *?case* **.**
qed *simp-all*

lemma *mset-set-Diff*:
assumes *finite A B* $\subseteq$ *A*
shows *mset-set* (*A* − *B*) = *mset-set A* − *mset-set B*
proof −
from *assms* **have** *mset-set* ((*A* − *B*) ∪ *B*) = *mset-set* (*A* − *B*) + *mset-set B*
by (*intro mset-set-Union*) (*auto dest*: *finite-subset*)
also from *assms* **have** *A* − *B* ∪ *B* = *A* **by** *blast*
finally show *?thesis* **by** *simp*

qed

lemma *mset-set-set*: *distinct xs* $\Longrightarrow$ *mset-set* (*set xs*) = *mset xs*
by (*induction xs*) *simp-all*

lemma *count-mset-set'*: *count* (*mset-set A*) *x* = (*if finite A* $\wedge$ *x* $\in$ *A then 1 else 0*)
by *auto*

lemma *subset-imp-msubset-mset-set*:
assumes *A* $\subseteq$ *B finite B*
shows *mset-set A* $\subseteq\#$ *mset-set B*
proof (*rule mset-subset-eqI*)
fix *x* :: *'a*
from *assms* **have** *finite A* **by** (*rule finite-subset*)
with *assms* **show** *count* (*mset-set A*) *x* $\leq$ *count* (*mset-set B*) *x*
by (*cases x* $\in$ *A*; *cases x* $\in$ *B*) *auto*
qed

lemma *mset-set-set-mset-msubset*: *mset-set* (*set-mset A*) $\subseteq\#$ *A*
proof (*rule mset-subset-eqI*)
fix *x* **show** *count* (*mset-set* (*set-mset A*)) *x* $\leq$ *count A x*
by (*cases x* $\in\#$ *A*) *simp-all*
qed

lemma *mset-set-upto-eq-mset-upto*:
‹*mset-set* {..<*n*} = *mset* [*0*..<*n*]›
by (*induction n*) (*auto simp*: *ac-simps lessThan-Suc*)

context *linorder*
begin

definition *sorted-list-of-multiset* :: *'a multiset* $\Rightarrow$ *'a list*
where
sorted-list-of-multiset M = *fold-mset insort* [] *M*

lemma *sorted-list-of-multiset-empty* [*simp*]:
sorted-list-of-multiset {#} = []
by (*simp add*: *sorted-list-of-multiset-def*)

lemma *sorted-list-of-multiset-singleton* [*simp*]:
sorted-list-of-multiset {#*x*#} = [*x*]
proof −
interpret *comp-fun-commute insort* **by** (*fact comp-fun-commute-insort*)
show *?thesis* **by** (*simp add*: *sorted-list-of-multiset-def*)
qed

lemma *sorted-list-of-multiset-insert* [*simp*]:
sorted-list-of-multiset (*add-mset x M*) = *List.insort x* (*sorted-list-of-multiset M*)

proof −
interpret *comp-fun-commute insort* **by** (*fact comp-fun-commute-insort*)
show *?thesis* **by** (*simp add*: *sorted-list-of-multiset-def*)
qed

end

lemma *mset-sorted-list-of-multiset*[*simp*]: *mset* (*sorted-list-of-multiset M*) = *M*
by (*induct M*) *simp-all*

lemma *sorted-list-of-multiset-mset*[*simp*]: *sorted-list-of-multiset* (*mset xs*) = *sort xs*
by (*induct xs*) *simp-all*

lemma *finite-set-mset-mset-set*[*simp*]: *finite A* ⟹ *set-mset* (*mset-set A*) = *A*
by *auto*

lemma *mset-set-empty-iff*: *mset-set A* = {#} ⟷ *A* = {} ∨ *infinite A*
using *finite-set-mset-mset-set* **by** *fastforce*

lemma *infinite-set-mset-mset-set*: ¬ *finite A* ⟹ *set-mset* (*mset-set A*) = {}
by *simp*

lemma *set-sorted-list-of-multiset* [*simp*]:
set (*sorted-list-of-multiset M*) = *set-mset M*
by (*induct M*) (*simp-all add*: *set-insort-key*)

lemma *sorted-list-of-mset-set* [*simp*]:
sorted-list-of-multiset (*mset-set A*) = *sorted-list-of-set A*
by (*cases finite A*) (*induct A rule*: *finite-induct*, *simp-all*)

lemma *mset-upt* [*simp*]: *mset* [*m*..<*n*] = *mset-set* {*m*..<*n*}
by (*induction n*) (*simp-all add*: *atLeastLessThanSuc*)

lemma *image-mset-map-of*:
distinct (*map fst xs*) ⟹ {#*the* (*map-of xs i*). *i* ∈# *mset* (*map fst xs*)#} = *mset* (*map snd xs*)
proof (*induction xs*)
case (*Cons x xs*)
have {#*the* (*map-of* (*x* # *xs*) *i*). *i* ∈# *mset* (*map fst* (*x* # *xs*))#} =
add-mset (*snd x*) {#*the* (*if i* = *fst x then Some* (*snd x*) *else map-of xs i*).
i ∈# *mset* (*map fst xs*)#} (**is** - = *add-mset* - *?A*) **by** *simp*
also from *Cons.prems* **have** *?A* = {#*the* (*map-of xs i*). *i* :# *mset* (*map fst xs*)#}
by (*cases x*, *intro image-mset-cong*) (*auto simp*: *in-multiset-in-set*)
also from *Cons.prems* **have** ... = *mset* (*map snd xs*) **by** (*intro Cons.IH*) *simp-all*
finally show *?case* **by** *simp*
qed *simp-all*

lemma *msubset-mset-set-iff*[*simp*]:
assumes *finite A finite B*
shows *mset-set A* $\subseteq\#$ *mset-set B* $\longleftrightarrow$ $A \subseteq B$
using *assms set-mset-mono subset-imp-msubset-mset-set* **by** *fastforce*

lemma *mset-set-eq-iff*[*simp*]:
assumes *finite A finite B*
shows *mset-set A* = *mset-set B* $\longleftrightarrow$ $A = B$
using *assms* **by** (*fastforce dest*: *finite-set-mset-mset-set*)

lemma *image-mset-mset-set*:
assumes *inj-on f A*
shows *image-mset f* (*mset-set A*) = *mset-set* (*f* ‘ *A*)
proof *cases*
assume *finite A*
from *this* ‹*inj-on f A*› **show** *?thesis*
by (*induct A*) *auto*
next
assume *infinite A*
from *this* ‹*inj-on f A*› **have** *infinite* (*f* ‘ *A*)
using *finite-imageD* **by** *blast*
from ‹*infinite A*› ‹*infinite* (*f* ‘ *A*)› **show** *?thesis* **by** *simp*
qed

62.9 More properties of the replicate and repeat operations

lemma *in-replicate-mset*[*simp*]: $x \in\#$ *replicate-mset n y* $\longleftrightarrow$ $n > 0 \wedge x = y$
unfolding *replicate-mset-def* **by** (*induct n*) *auto*

lemma *set-mset-replicate-mset-subset*[*simp*]: *set-mset* (*replicate-mset n x*) = (*if n* = *0 then* {} *else* {*x*})
by (*auto split*: *if-splits*)

lemma *size-replicate-mset*[*simp*]: *size* (*replicate-mset n M*) = *n*
by (*induct n*, *simp-all*)

lemma *count-le-replicate-mset-subset-eq*: $n \leq$ *count M x* $\longleftrightarrow$ *replicate-mset n x* $\subseteq\#$ *M*
by (*auto simp add*: *mset-subset-eqI*) (*metis count-replicate-mset subseteq-mset-def*)

lemma *filter-eq-replicate-mset*: {#*y* $\in\#$ *D*. *y* = *x*#} = *replicate-mset* (*count D x*) *x*
by (*induct D*) *simp-all*

lemma *replicate-count-mset-eq-filter-eq*: *replicate* (*count* (*mset xs*) *k*) *k* = *filter* (*HOL.eq k*) *xs*
by (*induct xs*) *auto*

lemma *replicate-mset-eq-empty-iff* [*simp*]: *replicate-mset n a* = {#} ⟷ *n* = *0*
by (*induct n*) *simp-all*

lemma *replicate-mset-eq-iff*:
replicate-mset m a = *replicate-mset n b* ⟷ *m* = *0* ∧ *n* = *0* ∨ *m* = *n* ∧ *a* = *b*
by (*auto simp add*: *multiset-eq-iff*)

lemma *repeat-mset-cancel1*: *repeat-mset a A* = *repeat-mset a B* ⟷ *A* = *B* ∨ *a* = *0*
by (*auto simp*: *multiset-eq-iff*)

lemma *repeat-mset-cancel2*: *repeat-mset a A* = *repeat-mset b A* ⟷ *a* = *b* ∨ *A* = {#}
by (*auto simp*: *multiset-eq-iff*)

lemma *repeat-mset-eq-empty-iff*: *repeat-mset n A* = {#} ⟷ *n* = *0* ∨ *A* = {#}
by (*cases n*) *auto*

lemma *image-replicate-mset* [*simp*]:
image-mset f (*replicate-mset n a*) = *replicate-mset n* (*f a*)
by (*induct n*) *simp-all*

lemma *replicate-mset-msubseteq-iff*:
replicate-mset m a ⊆# *replicate-mset n b* ⟷ *m* = *0* ∨ *a* = *b* ∧ *m* ≤ *n*
by (*cases m*)
(*auto simp*: *insert-subset-eq-iff simp flip*: *count-le-replicate-mset-subset-eq*)

lemma *msubseteq-replicate-msetE*:
assumes *A* ⊆# *replicate-mset n a*
obtains *m* **where** *m* ≤ *n* **and** *A* = *replicate-mset m a*
proof (*cases n* = *0*)
case *True*
with *assms that* **show** *thesis*
by *simp*
next
case *False*
from *assms* **have** *set-mset A* ⊆ *set-mset* (*replicate-mset n a*)
by (*rule set-mset-mono*)
with *False* **have** *set-mset A* ⊆ {*a*}
by *simp*
then have ∃ *m*. *A* = *replicate-mset m a*
proof (*induction A*)
case *empty*
then show *?case*
by *simp*
next
case (*add b A*)
then obtain *m* **where** *A* = *replicate-mset m a*
by *auto*

with *add.prems* **show** *?case*
by (*auto intro*: *exI* [*of* - *Suc m*])
qed
then obtain *m* **where** *A*: *A* = *replicate-mset m a* **..**
with *assms* **have** *m* ≤ *n*
by (*auto simp add*: *replicate-mset-msubseteq-iff*)
then show *thesis* **using** *A* **..**
qed

62.10 Big operators

locale *comm-monoid-mset* = *comm-monoid*
begin

interpretation *comp-fun-commute f*
by *standard* (*simp add*: *fun-eq-iff left-commute*)

interpretation *comp?*: *comp-fun-commute f* ∘ *g*
by (*fact comp-comp-fun-commute*)

context
begin

definition *F* :: *'a multiset* ⇒ *'a*
where *eq-fold*: *F M* = *fold-mset f* **1** *M*

lemma *empty* [*simp*]: *F* {#} = **1**
by (*simp add*: *eq-fold*)

lemma *singleton* [*simp*]: *F* {#*x*#} = *x*
proof −
interpret *comp-fun-commute*
by *standard* (*simp add*: *fun-eq-iff left-commute*)
show *?thesis* **by** (*simp add*: *eq-fold*)
qed

lemma *union* [*simp*]: *F* (*M* + *N*) = *F M* ∗ *F N*
proof −
interpret *comp-fun-commute f*
by *standard* (*simp add*: *fun-eq-iff left-commute*)
show *?thesis*
by (*induct N*) (*simp-all add*: *left-commute eq-fold*)
qed

lemma *add-mset* [*simp*]: *F* (*add-mset x N*) = *x* ∗ *F N*
unfolding *add-mset-add-single*[*of x N*] *union* **by** (*simp add*: *ac-simps*)

lemma *insert* [*simp*]:
shows *F* (*image-mset g* (*add-mset x A*)) = *g x* ∗ *F* (*image-mset g A*)

by (*simp add*: *eq-fold*)

lemma *remove*:
assumes $x \in\# A$
shows $F\ A = x * F\ (A - \{\#x\#\})$
using *multi-member-split*[*OF assms*] **by** *auto*

lemma *neutral*:
$\forall x \in\# A.\ x = \mathbf{1} \Longrightarrow F\ A = \mathbf{1}$
by (*induct A*) *simp-all*

lemma *neutral-const* [*simp*]:
$F\ (\mathit{image\text{-}mset}\ (\lambda\text{-}.\ \mathbf{1})\ A) = \mathbf{1}$
by (*simp add*: *neutral*)

private lemma *F-image-mset-product*:
$F\ \{\#g\ x\ j * F\ \{\#g\ i\ j.\ i \in\# A\#\}.\ j \in\# B\#\} =$
$F\ (\mathit{image\text{-}mset}\ (g\ x)\ B) * F\ \{\#F\ \{\#g\ i\ j.\ i \in\# A\#\}.\ j \in\# B\#\}$
by (*induction B*) (*simp-all add*: *left-commute semigroup.assoc semigroup-axioms*)

lemma *swap*:
$F\ (\mathit{image\text{-}mset}\ (\lambda i.\ F\ (\mathit{image\text{-}mset}\ (g\ i)\ B))\ A) =$
$F\ (\mathit{image\text{-}mset}\ (\lambda j.\ F\ (\mathit{image\text{-}mset}\ (\lambda i.\ g\ i\ j)\ A))\ B)$
apply (*induction A*, *simp*)
apply (*induction B*, *auto simp add*: *F-image-mset-product ac-simps*)
done

lemma *distrib*: $F\ (\mathit{image\text{-}mset}\ (\lambda x.\ g\ x * h\ x)\ A) = F\ (\mathit{image\text{-}mset}\ g\ A) * F\ (\mathit{image\text{-}mset}\ h\ A)$
by (*induction A*) (*auto simp*: *ac-simps*)

lemma *union-disjoint*:
$A \cap\# B = \{\#\} \Longrightarrow F\ (\mathit{image\text{-}mset}\ g\ (A \cup\# B)) = F\ (\mathit{image\text{-}mset}\ g\ A) * F\ (\mathit{image\text{-}mset}\ g\ B)$
by (*induction A*) (*auto simp*: *ac-simps*)

end
end

lemma *comp-fun-commute-plus-mset*[*simp*]: *comp-fun-commute* $((+) :: 'a\ \mathit{multiset} \Rightarrow - \Rightarrow -)$
by *standard* (*simp add*: *add-ac comp-def*)

declare *comp-fun-commute.fold-mset-add-mset*[*OF comp-fun-commute-plus-mset*, *simp*]

lemma *in-mset-fold-plus-iff*[*iff*]: $x \in\#$ *fold-mset* $(+)\ M\ NN \longleftrightarrow x \in\# M \vee (\exists N.\ N \in\# NN \wedge x \in\# N)$
by (*induct NN*) *auto*

context *comm-monoid-add*
begin

sublocale *sum-mset*: *comm-monoid-mset plus 0*
  defines *sum-mset* = *sum-mset.F* **..**

lemma *sum-unfold-sum-mset*:
  sum f A = *sum-mset* (*image-mset f* (*mset-set A*))
  by (*cases finite A*) (*induct A rule*: *finite-induct*, *simp-all*)

end

notation *sum-mset* ($\sum_{\#}$)

syntax (*ASCII*)
  -sum-mset-image :: *pttrn* ⇒ *'b set* ⇒ *'a* ⇒ *'a*::*comm-monoid-add* ((*3SUM -:#-.* *-*) [*0*, *51*, *10*] *10*)
syntax
  -sum-mset-image :: *pttrn* ⇒ *'b set* ⇒ *'a* ⇒ *'a*::*comm-monoid-add* ((*3*$\sum$ *-∈#-.* *-*) [*0*, *51*, *10*] *10*)
translations
  $\sum i \in\# A.\ b \rightleftharpoons$ *CONST sum-mset* (*CONST image-mset* (*λi. b*) *A*)

context *comm-monoid-add*
begin

lemma *sum-mset-sum-list*:
  sum-mset (*mset xs*) = *sum-list xs*
  by (*induction xs*) *auto*

end

context *canonically-ordered-monoid-add*
begin

lemma *sum-mset-0-iff* [*simp*]:
  sum-mset M = *0* ⟷ (∀ *x* ∈ *set-mset M. x* = *0*)
  by (*induction M*) *auto*

end

context *ordered-comm-monoid-add*
begin

lemma *sum-mset-mono*:
  sum-mset (*image-mset f K*) ≤ *sum-mset* (*image-mset g K*)
  if ⋀*i*. *i* ∈# *K* ⟹ *f i* ≤ *g i*
  using *that* **by** (*induction K*) (*simp-all add*: *add-mono*)

end

context *cancel-comm-monoid-add*
begin

lemma *sum-mset-diff*:
sum-mset $(M - N) =$ *sum-mset* $M -$ *sum-mset* N **if** $N \subseteq\# M$ **for** $M\ N ::$ *'a multiset*
using *that* **by** (*auto simp add*: *subset-mset.le-iff-add*)

end

context *semiring-0*
begin

lemma *sum-mset-distrib-left*:
$c * (\sum x \in\# M.\ f\ x) = (\sum x \in\# M.\ c * f(x))$
by (*induction M*) (*simp-all add*: *algebra-simps*)

lemma *sum-mset-distrib-right*:
$(\sum x \in\# M.\ f\ x) * c = (\sum x \in\# M.\ f\ x * c)$
by (*induction M*) (*simp-all add*: *algebra-simps*)

end

lemma *sum-mset-product*:
fixes $f ::$ *'a*::{*comm-monoid-add*,*times*} $\Rightarrow$ *'b*::*semiring-0*
shows $(\sum i \in\# A.\ f\ i) * (\sum i \in\# B.\ g\ i) = (\sum i{\in\#}A.\ \sum j{\in\#}B.\ f\ i * g\ j)$
by (*subst sum-mset.swap*) (*simp add*: *sum-mset-distrib-left sum-mset-distrib-right*)

context *semiring-1*
begin

lemma *sum-mset-replicate-mset* [*simp*]:
sum-mset (*replicate-mset n a*) = *of-nat n* $*$ *a*
by (*induction n*) (*simp-all add*: *algebra-simps*)

lemma *sum-mset-delta*:
sum-mset (*image-mset* ($\lambda x.$ *if x* = *y then c else 0*) *A*) = *c* $*$ *of-nat* (*count A y*)
by (*induction A*) (*simp-all add*: *algebra-simps*)

lemma *sum-mset-delta'*:
sum-mset (*image-mset* ($\lambda x.$ *if y* = *x then c else 0*) *A*) = *c* $*$ *of-nat* (*count A y*)
by (*induction A*) (*simp-all add*: *algebra-simps*)

end

lemma *of-nat-sum-mset* [*simp*]:

of-nat (*sum-mset A*) = *sum-mset* (*image-mset of-nat A*)
by (*induction A*) *auto*

lemma *size-eq-sum-mset*:
size M = (∑ *a*∈#*M*. *1*)
using *image-mset-const-eq* [*of 1*::*nat M*] **by** *simp*

lemma *size-mset-set* [*simp*]:
size (*mset-set A*) = *card A*
by (*simp only*: *size-eq-sum-mset card-eq-sum sum-unfold-sum-mset*)

lemma *sum-mset-constant* [*simp*]:
fixes *y* :: *'b*::*semiring-1*
shows ‹(∑ *x*∈#*A*. *y*) = *of-nat* (*size A*) * *y*›
by (*induction A*) (*auto simp*: *algebra-simps*)

lemma *set-mset-Union-mset*[*simp*]: *set-mset* (∑# *MM*) = (⋃ *M* ∈ *set-mset MM*. *set-mset M*)
by (*induct MM*) *auto*

lemma *in-Union-mset-iff*[*iff*]: *x* ∈# ∑# *MM* ⟷ (∃ *M*. *M* ∈# *MM* ∧ *x* ∈# *M*)
by (*induct MM*) *auto*

lemma *count-sum*:
count (*sum f A*) *x* = *sum* (λ*a*. *count* (*f a*) *x*) *A*
by (*induct A rule*: *infinite-finite-induct*) *simp-all*

lemma *sum-eq-empty-iff*:
assumes *finite A*
shows *sum f A* = {#} ⟷ (∀ *a*∈*A*. *f a* = {#})
using *assms* **by** *induct simp-all*

lemma *Union-mset-empty-conv*[*simp*]: ∑# *M* = {#} ⟷ (∀ *i*∈#*M*. *i* = {#})
by (*induction M*) *auto*

lemma *Union-image-single-mset*[*simp*]: ∑# (*image-mset* (λ*x*. {#*x*#}) *m*) = *m*
by(*induction m*) *auto*

context *comm-monoid-mult*
begin

sublocale *prod-mset*: *comm-monoid-mset times 1*
defines *prod-mset* = *prod-mset.F* **..**

lemma *prod-mset-empty*:
prod-mset {#} = *1*
by (*fact prod-mset.empty*)

lemma *prod-mset-singleton*:
prod-mset {#*x*#} = *x*
by (*fact prod-mset.singleton*)

lemma *prod-mset-Un*:
prod-mset (*A* + *B*) = *prod-mset A* ∗ *prod-mset B*
by (*fact prod-mset.union*)

lemma *prod-mset-prod-list*:
prod-mset (*mset xs*) = *prod-list xs*
by (*induct xs*) *auto*

lemma *prod-mset-replicate-mset* [*simp*]:
prod-mset (*replicate-mset n a*) = *a* ^ *n*
by (*induct n*) *simp-all*

lemma *prod-unfold-prod-mset*:
prod f A = *prod-mset* (*image-mset f* (*mset-set A*))
by (*cases finite A*) (*induct A rule*: *finite-induct*, *simp-all*)

lemma *prod-mset-multiplicity*:
prod-mset M = *prod* (λ*x*. *x* ^ *count M x*) (*set-mset M*)
by (*simp add*: *fold-mset-def prod.eq-fold prod-mset.eq-fold funpow-times-power comp-def*)

lemma *prod-mset-delta*: *prod-mset* (*image-mset* (λ*x*. *if x* = *y then c else 1*) *A*) = *c* ^ *count A y*
by (*induction A*) *simp-all*

lemma *prod-mset-delta'*: *prod-mset* (*image-mset* (λ*x*. *if y* = *x then c else 1*) *A*) = *c* ^ *count A y*
by (*induction A*) *simp-all*

lemma *prod-mset-subset-imp-dvd*:
assumes *A* ⊆# *B*
shows *prod-mset A dvd prod-mset B*
proof −
from *assms* **have** *B* = (*B* − *A*) + *A* **by** (*simp add*: *subset-mset.diff-add*)
also have *prod-mset* ... = *prod-mset* (*B* − *A*) ∗ *prod-mset A* **by** *simp*
also have *prod-mset A dvd* ... **by** *simp*
finally show *?thesis* **.**
qed

lemma *dvd-prod-mset*:
assumes *x* ∈# *A*
shows *x dvd prod-mset A*
using *assms prod-mset-subset-imp-dvd* [*of* {#*x*#} *A*] **by** *simp*

end

notation *prod-mset* ($\prod_{\#}$)

syntax (*ASCII*)
-prod-mset-image :: *pttrn* ⇒ *'b set* ⇒ *'a* ⇒ *'a*::*comm-monoid-mult* ((*3PROD -:#-. -*) [*0*, *51*, *10*] *10*)
syntax
-prod-mset-image :: *pttrn* ⇒ *'b set* ⇒ *'a* ⇒ *'a*::*comm-monoid-mult* ((*3*$\prod$ *-∈#-. -*) [*0*, *51*, *10*] *10*)
translations
$\prod i \in\# A.\ b \rightleftharpoons$ *CONST prod-mset* (*CONST image-mset* (λ*i. b*) *A*)

lemma *prod-mset-constant* [*simp*]: ($\prod$ *-∈#A. c*) = *c* ^ *size A*
by (*simp add*: *image-mset-const-eq*)

lemma (**in** *semidom*) *prod-mset-zero-iff* [*iff*]:
prod-mset A = *0* ⟷ *0* ∈# *A*
by (*induct A*) *auto*

lemma (**in** *semidom-divide*) *prod-mset-diff*:
assumes *B* ⊆# *A* **and** *0* ∉# *B*
shows *prod-mset* (*A* − *B*) = *prod-mset A div prod-mset B*
proof −
from *assms* **obtain** *C* **where** *A* = *B* + *C*
by (*metis subset-mset.add-diff-inverse*)
with *assms* **show** *?thesis* **by** *simp*
qed

lemma (**in** *semidom-divide*) *prod-mset-minus*:
assumes *a* ∈# *A* **and** *a* ≠ *0*
shows *prod-mset* (*A* − {#*a*#}) = *prod-mset A div a*
using *assms prod-mset-diff* [*of* {#*a*#} *A*] **by** *auto*

lemma (**in** *normalization-semidom*) *normalize-prod-mset-normalize*:
normalize (*prod-mset* (*image-mset normalize A*)) = *normalize* (*prod-mset A*)
proof (*induction A*)
case (*add x A*)
have *normalize* (*prod-mset* (*image-mset normalize* (*add-mset x A*))) =
normalize (*x* ∗ *normalize* (*prod-mset* (*image-mset normalize A*)))
by *simp*
also note *add.IH*
finally show *?case* **by** *simp*
qed *auto*

lemma (**in** *algebraic-semidom*) *is-unit-prod-mset-iff*:
is-unit (*prod-mset A*) ⟷ (∀ *x* ∈# *A*. *is-unit x*)
by (*induct A*) (*auto simp*: *is-unit-mult-iff*)

lemma (**in** *normalization-semidom-multiplicative*) *normalize-prod-mset*:

normalize (*prod-mset A*) = *prod-mset* (*image-mset normalize A*)
by (*induct A*) (*simp-all add*: *normalize-mult*)

lemma (**in** *normalization-semidom-multiplicative*) *normalized-prod-msetI*:
assumes $\bigwedge a.\ a \in\# A \Longrightarrow$ *normalize a* = *a*
shows *normalize* (*prod-mset A*) = *prod-mset A*
proof −
from *assms* **have** *image-mset normalize A* = *A*
by (*induct A*) *simp-all*
then show *?thesis* **by** (*simp add*: *normalize-prod-mset*)
qed

62.11 Multiset as order-ignorant lists

context *linorder*
begin

lemma *mset-insort* [*simp*]:
mset (*insort-key k x xs*) = *add-mset x* (*mset xs*)
by (*induct xs*) *simp-all*

lemma *mset-sort* [*simp*]:
mset (*sort-key k xs*) = *mset xs*
by (*induct xs*) *simp-all*

This lemma shows which properties suffice to show that a function f with $f\ xs = ys$ behaves like sort.

lemma *properties-for-sort-key*:
assumes *mset ys* = *mset xs*
and $\bigwedge k.\ k \in$ *set ys* $\Longrightarrow$ *filter* ($\lambda x.\ f\ k = f\ x$) *ys* = *filter* ($\lambda x.\ f\ k = f\ x$) *xs*
and *sorted* (*map f ys*)
shows *sort-key f xs* = *ys*
using *assms*
proof (*induct xs arbitrary*: *ys*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*)
from *Cons.prems*(*2*) **have**
$\forall k \in$ *set ys*. *filter* ($\lambda x.\ f\ k = f\ x$) (*remove1 x ys*) = *filter* ($\lambda x.\ f\ k = f\ x$) *xs*
by (*simp add*: *filter-remove1*)
with *Cons.prems* **have** *sort-key f xs* = *remove1 x ys*
by (*auto intro*!: *Cons.hyps simp add*: *sorted-map-remove1*)
moreover from *Cons.prems* **have** *x* ∈# *mset ys*
by *auto*
then have *x* ∈ *set ys*
by *simp*
ultimately show *?case* **using** *Cons.prems* **by** (*simp add*: *insort-key-remove1*)
qed

lemma *properties-for-sort*:
  assumes *multiset*: *mset ys* = *mset xs*
    and *sorted ys*
  shows *sort xs* = *ys*
proof (*rule properties-for-sort-key*)
  from *multiset* **show** *mset ys* = *mset xs* **.**
  from ‹*sorted ys*› **show** *sorted* (*map* (λ*x*. *x*) *ys*) **by** *simp*
  from *multiset* **have** *length* (*filter* (λ*y*. *k* = *y*) *ys*) = *length* (*filter* (λ*x*. *k* = *x*) *xs*) **for** *k*
    by (*rule mset-eq-length-filter*)
  then have *replicate* (*length* (*filter* (λ*y*. *k* = *y*) *ys*)) *k* =
    replicate (*length* (*filter* (λ*x*. *k* = *x*) *xs*)) *k* **for** *k*
    by *simp*
  then show *k* ∈ *set ys* ⟹ *filter* (λ*y*. *k* = *y*) *ys* = *filter* (λ*x*. *k* = *x*) *xs* **for** *k*
    by (*simp add*: *replicate-length-filter*)
qed

lemma *sort-key-inj-key-eq*:
  assumes *mset-equal*: *mset xs* = *mset ys*
    and *inj-on f* (*set xs*)
    and *sorted* (*map f ys*)
  shows *sort-key f xs* = *ys*
proof (*rule properties-for-sort-key*)
  from *mset-equal*
  show *mset ys* = *mset xs* **by** *simp*
  from ‹*sorted* (*map f ys*)›
  show *sorted* (*map f ys*) **.**
  show [*x*←*ys* . *f k* = *f x*] = [*x*←*xs* . *f k* = *f x*] **if** *k* ∈ *set ys* **for** *k*
  proof −
    from *mset-equal*
    have *set-equal*: *set xs* = *set ys* **by** (*rule mset-eq-setD*)
    with *that* **have** *insert k* (*set ys*) = *set ys* **by** *auto*
    with ‹*inj-on f* (*set xs*)› **have** *inj*: *inj-on f* (*insert k* (*set ys*))
      by (*simp add*: *set-equal*)
    from *inj* **have** [*x*←*ys* . *f k* = *f x*] = *filter* (*HOL.eq k*) *ys*
      by (*auto intro*!: *inj-on-filter-key-eq*)
    also have ... = *replicate* (*count* (*mset ys*) *k*) *k*
      by (*simp add*: *replicate-count-mset-eq-filter-eq*)
    also have ... = *replicate* (*count* (*mset xs*) *k*) *k*
      using *mset-equal* **by** *simp*
    also have ... = *filter* (*HOL.eq k*) *xs*
      by (*simp add*: *replicate-count-mset-eq-filter-eq*)
    also have ... = [*x*←*xs* . *f k* = *f x*]
     using *inj* **by** (*auto intro*!: *inj-on-filter-key-eq* [*symmetric*] *simp add*: *set-equal*)
    finally show *?thesis* **.**
  qed
qed

lemma *sort-key-eq-sort-key*:

```
  assumes mset xs = mset ys
    and inj-on f (set xs)
  shows sort-key f xs = sort-key f ys
  by (rule sort-key-inj-key-eq) (simp-all add: assms)

lemma sort-key-by-quicksort:
  sort-key f xs = sort-key f [x←xs. f x < f (xs ! (length xs div 2))]
    @ [x←xs. f x = f (xs ! (length xs div 2))]
    @ sort-key f [x←xs. f x > f (xs ! (length xs div 2))] (is sort-key f ?lhs = ?rhs)
proof (rule properties-for-sort-key)
  show mset ?rhs = mset ?lhs
    by (rule multiset-eqI) auto
  show sorted (map f ?rhs)
    by (auto simp add: sorted-append intro: sorted-map-same)
next
  fix l
  assume l ∈ set ?rhs
  let ?pivot = f (xs ! (length xs div 2))
  have *: ⋀x. f l = f x ⟷ f x = f l by auto
  have [x ← sort-key f xs . f x = f l] = [x ← xs. f x = f l]
   unfolding filter-sort by (rule properties-for-sort-key) (auto intro: sorted-map-same)
  with * have **: [x ← sort-key f xs . f l = f x] = [x ← xs. f l = f x] by simp
  have ⋀x P. P (f x) ?pivot ∧ f l = f x ⟷ P (f l) ?pivot ∧ f l = f x by auto
  then have ⋀P. [x ← sort-key f xs . P (f x) ?pivot ∧ f l = f x] =
    [x ← sort-key f xs. P (f l) ?pivot ∧ f l = f x] by simp
  note *** = this [of (<)] this [of (>)] this [of (=)]
  show [x ← ?rhs. f l = f x] = [x ← ?lhs. f l = f x]
  proof (cases f l ?pivot rule: linorder-cases)
    case less
    then have f l ≠ ?pivot and ¬ f l > ?pivot by auto
    with less show ?thesis
      by (simp add: filter-sort [symmetric] ** ***)
  next
    case equal then show ?thesis
      by (simp add: * less-le)
  next
    case greater
    then have f l ≠ ?pivot and ¬ f l < ?pivot by auto
    with greater show ?thesis
      by (simp add: filter-sort [symmetric] ** ***)
  qed
qed

lemma sort-by-quicksort:
  sort xs = sort [x←xs. x < xs ! (length xs div 2)]
    @ [x←xs. x = xs ! (length xs div 2)]
    @ sort [x←xs. x > xs ! (length xs div 2)] (is sort ?lhs = ?rhs)
  using sort-key-by-quicksort [of λx. x, symmetric] by simp
```

A stable parameterized quicksort

definition *part* :: ('*b* ⇒ '*a*) ⇒ '*a* ⇒ '*b list* ⇒ '*b list* × '*b list* × '*b list* **where**
part f pivot xs = ([*x* ← *xs*. *f x* < *pivot*], [*x* ← *xs*. *f x* = *pivot*], [*x* ← *xs*. *pivot* < *f x*])

lemma *part-code* [*code*]:
part f pivot [] = ([], [], [])
part f pivot (*x* # *xs*) = (*let* (*lts*, *eqs*, *gts*) = *part f pivot xs*; *x'* = *f x in*
if x' < *pivot then* (*x* # *lts*, *eqs*, *gts*)
else if x' > *pivot then* (*lts*, *eqs*, *x* # *gts*)
else (*lts*, *x* # *eqs*, *gts*))
by (*auto simp add*: *part-def Let-def split-def*)

lemma *sort-key-by-quicksort-code* [*code*]:
sort-key f xs =
(*case xs of*
[] ⇒ []
| [*x*] ⇒ *xs*
| [*x*, *y*] ⇒ (*if f x* ≤ *f y then xs else* [*y*, *x*])
| - ⇒
let (*lts*, *eqs*, *gts*) = *part f* (*f* (*xs* ! (*length xs div 2*))) *xs*
in sort-key f lts @ *eqs* @ *sort-key f gts*)
proof (*cases xs*)
case *Nil* **then show** *?thesis* **by** *simp*
next
case (*Cons* - *ys*) **note** *hyps* = *Cons* **show** *?thesis*
proof (*cases ys*)
case *Nil* **with** *hyps* **show** *?thesis* **by** *simp*
next
case (*Cons* - *zs*) **note** *hyps* = *hyps Cons* **show** *?thesis*
proof (*cases zs*)
case *Nil* **with** *hyps* **show** *?thesis* **by** *auto*
next
case *Cons*
from *sort-key-by-quicksort* [*of f xs*]
have *sort-key f xs* = (*let* (*lts*, *eqs*, *gts*) = *part f* (*f* (*xs* ! (*length xs div 2*))) *xs*
in sort-key f lts @ *eqs* @ *sort-key f gts*)
by (*simp only*: *split-def Let-def part-def fst-conv snd-conv*)
with *hyps Cons* **show** *?thesis* **by** (*simp only*: *list.cases*)
qed
qed
qed

end

hide-const (**open**) *part*

lemma *mset-remdups-subset-eq*: *mset* (*remdups xs*) ⊆# *mset xs*
by (*induct xs*) (*auto intro*: *subset-mset.order-trans*)

lemma *mset-update*:
i < length ls ⟹ mset (ls[i := v]) = add-mset v (mset ls − {#ls ! i#})
proof (*induct ls arbitrary*: *i*)
case *Nil* **then show** *?case* **by** *simp*
next
case (*Cons x xs*)
show *?case*
proof (*cases i*)
case *0* **then show** *?thesis* **by** *simp*
next
case (*Suc i′*)
with *Cons* **show** *?thesis*
by (*cases ‹x = xs ! i′›*) *auto*
qed
qed

lemma *mset-swap*:
i < length ls ⟹ j < length ls ⟹
mset (ls[j := ls ! i, i := ls ! j]) = mset ls
by (*cases i = j*) (*simp-all add*: *mset-update nth-mem-mset*)

lemma *mset-eq-finite*:
‹finite {ys. mset ys = mset xs}›
proof −
have *‹{ys. mset ys = mset xs} ⊆ {ys. set ys ⊆ set xs ∧ length ys ≤ length xs}›*
by (*auto simp add*: *dest*: *mset-eq-setD mset-eq-length*)
moreover have *‹finite {ys. set ys ⊆ set xs ∧ length ys ≤ length xs}›*
using *finite-lists-length-le* **by** *blast*
ultimately show *?thesis*
by (*rule finite-subset*)
qed

62.12 The multiset order

62.12.1 Well-foundedness

definition *mult1* :: (*′a* × *′a*) *set* ⇒ (*′a multiset* × *′a multiset*) *set* **where**
mult1 r = {(N, M). ∃ a M0 K. M = add-mset a M0 ∧ N = M0 + K ∧
(∀ b. b ∈# K ⟶ (b, a) ∈ r)}

definition *mult* :: (*′a* × *′a*) *set* ⇒ (*′a multiset* × *′a multiset*) *set* **where**
mult r = (mult1 r)$^+$

lemma *mult1I*:
assumes *M = add-mset a M0* **and** *N = M0 + K* **and** *⋀b. b ∈# K ⟹ (b, a) ∈ r*
shows (*N*, *M*) ∈ *mult1 r*
using *assms* **unfolding** *mult1-def* **by** *blast*

lemma *mult1E*:

assumes $(N, M) \in$ *mult1 r*
obtains *a M0 K* **where** $M =$ *add-mset a M0* $N = M0 + K \bigwedge b.\ b \in\# K \Longrightarrow (b, a) \in r$
using *assms* **unfolding** *mult1-def* **by** *blast*

lemma *mono-mult1*:
assumes $r \subseteq r'$ **shows** *mult1 r* $\subseteq$ *mult1 r'*
unfolding *mult1-def* **using** *assms* **by** *blast*

lemma *mono-mult*:
assumes $r \subseteq r'$ **shows** *mult r* $\subseteq$ *mult r'*
unfolding *mult-def* **using** *mono-mult1*[*OF assms*] *trancl-mono* **by** *blast*

lemma *not-less-empty* [*iff*]: $(M, \{\#\}) \notin$ *mult1 r*
by (*simp add*: *mult1-def*)

lemma *less-add*:
assumes *mult1*: $(N,$ *add-mset a M0*$) \in$ *mult1 r*
shows
$(\exists M.\ (M,$ *M0*$) \in$ *mult1 r* $\wedge$ $N =$ *add-mset a M*$) \vee$
$(\exists K.\ (\forall b.\ b \in\# K \longrightarrow (b, a) \in r) \wedge N = M0 + K)$
proof −
let *?r* $= \lambda K\ a.\ \forall b.\ b \in\# K \longrightarrow (b, a) \in r$
let *?R* $= \lambda N\ M.\ \exists a\ M0\ K.\ M =$ *add-mset a M0* $\wedge\ N = M0 + K \wedge$ *?r K a*
obtain *a' M0' K* **where** *M0*: *add-mset a M0* $=$ *add-mset a' M0'*
and *N*: $N = M0' + K$
and *r*: *?r K a'*
using *mult1* **unfolding** *mult1-def* **by** *auto*
show *?thesis* (**is** *?case1* $\vee$ *?case2*)
proof −
from *M0* **consider** $M0 = M0'\ a = a'$
| K' **where** *M0* $=$ *add-mset a' K'* $M0' =$ *add-mset a K'*
by *atomize-elim* (*simp only*: *add-eq-conv-ex*)
then show *?thesis*
proof *cases*
case *1*
with *N r* **have** *?r K a* $\wedge$ $N = M0 + K$ **by** *simp*
then have *?case2* **..**
then show *?thesis* **..**
next
case *2*
from *N 2*(*2*) **have** *n*: $N =$ *add-mset a* $(K' + K)$ **by** *simp*
with *r 2*(*1*) **have** *?R* $(K' + K)$ *M0* **by** *blast*
with *n* **have** *?case1* **by** (*simp add*: *mult1-def*)
then show *?thesis* **..**
qed
qed
qed

```
lemma all-accessible:
  assumes wf r
  shows ∀ M. M ∈ Wellfounded.acc (mult1 r)
proof
  let ?R = mult1 r
  let ?W = Wellfounded.acc ?R
  {
    fix M M0 a
    assume M0: M0 ∈ ?W
      and wf-hyp: ⋀b. (b, a) ∈ r ⟹ (∀ M ∈ ?W. add-mset b M ∈ ?W)
      and acc-hyp: ∀ M. (M, M0) ∈ ?R ⟶ add-mset a M ∈ ?W
    have add-mset a M0 ∈ ?W
    proof (rule accI [of add-mset a M0])
      fix N
      assume (N, add-mset a M0) ∈ ?R
      then consider M where (M, M0) ∈ ?R N = add-mset a M
        | K where ∀ b. b ∈# K ⟶ (b, a) ∈ r N = M0 + K
        by atomize-elim (rule less-add)
      then show N ∈ ?W
      proof cases
        case 1
        from acc-hyp have (M, M0) ∈ ?R ⟶ add-mset a M ∈ ?W ..
        from this and ‹(M, M0) ∈ ?R› have add-mset a M ∈ ?W ..
        then show N ∈ ?W by (simp only: ‹N = add-mset a M›)
      next
        case 2
        from this(1) have M0 + K ∈ ?W
        proof (induct K)
          case empty
          from M0 show M0 + {#} ∈ ?W by simp
        next
          case (add x K)
          from add.prems have (x, a) ∈ r by simp
          with wf-hyp have ∀ M ∈ ?W. add-mset x M ∈ ?W by blast
          moreover from add have M0 + K ∈ ?W by simp
          ultimately have add-mset x (M0 + K) ∈ ?W ..
          then show M0 + (add-mset x K) ∈ ?W by simp
        qed
        then show N ∈ ?W by (simp only: 2(2))
      qed
    qed
  } note tedious-reasoning = this

  show M ∈ ?W for M
  proof (induct M)
    show {#} ∈ ?W
    proof (rule accI)
      fix b assume (b, {#}) ∈ ?R
      with not-less-empty show b ∈ ?W by contradiction
```

```
    qed

    fix M a assume M ∈ ?W
    from ‹wf r› have ∀ M ∈ ?W. add-mset a M ∈ ?W
    proof induct
      fix a
      assume r: ⋀b. (b, a) ∈ r ⟹ (∀ M ∈ ?W. add-mset b M ∈ ?W)
      show ∀ M ∈ ?W. add-mset a M ∈ ?W
      proof
        fix M assume M ∈ ?W
        then show add-mset a M ∈ ?W
          by (rule acc-induct) (rule tedious-reasoning [OF - r])
      qed
    qed
    from this and ‹M ∈ ?W› show add-mset a M ∈ ?W ..
  qed
qed

theorem wf-mult1: wf r ⟹ wf (mult1 r)
by (rule acc-wfI) (rule all-accessible)

theorem wf-mult: wf r ⟹ wf (mult r)
unfolding mult-def by (rule wf-trancl) (rule wf-mult1)
```

62.12.2 Closure-free presentation

One direction.

```
lemma mult-implies-one-step:
  assumes
    trans: trans r and
    MN: (M, N) ∈ mult r
  shows ∃ I J K. N = I + J ∧ M = I + K ∧ J ≠ {#} ∧ (∀ k ∈ set-mset K. ∃ j
∈ set-mset J. (k, j) ∈ r)
  using MN unfolding mult-def mult1-def
proof (induction rule: converse-trancl-induct)
  case (base y)
  then show ?case by force
next
  case (step y z) note yz = this(1) and zN = this(2) and N-decomp = this(3)
  obtain I J K where
    N: N = I + J z = I + K J ≠ {#} ∀ k∈#K. ∃ j∈#J. (k, j) ∈ r
    using N-decomp by blast
  obtain a M0 K' where
    z: z = add-mset a M0 and y: y = M0 + K' and K: ∀ b. b ∈# K' ⟶ (b, a)
∈ r
    using yz by blast
  show ?case
  proof (cases a ∈# K)
    case True
```

```
    moreover have ∃j∈#J. (k, j) ∈ r if k ∈# K' for k
      using K N trans True by (meson that transE)
    ultimately show ?thesis
      by (rule-tac x = I in exI, rule-tac x = J in exI, rule-tac x = (K − {#a#})
+ K' in exI)
        (use z y N in ‹auto simp del: subset-mset.add-diff-assoc2 dest: in-diffD›)
  next
    case False
    then have a ∈# I by (metis N(2) union-iff union-single-eq-member z)
    moreover have M0 = I + K − {#a#}
      using N(2) z by force
    ultimately show ?thesis
      by (rule-tac x = I − {#a#} in exI, rule-tac x = add-mset a J in exI,
          rule-tac x = K + K' in exI)
        (use z y N False K in ‹auto simp: add.assoc›)
  qed
qed

lemma one-step-implies-mult:
  assumes
    J ≠ {#} and
    ∀k ∈ set-mset K. ∃j ∈ set-mset J. (k, j) ∈ r
  shows (I + K, I + J) ∈ mult r
  using assms
proof (induction size J arbitrary: I J K)
  case 0
  then show ?case by auto
next
  case (Suc n) note IH = this(1) and size-J = this(2)[THEN sym]
  obtain J' a where J: J = add-mset a J'
    using size-J by (blast dest: size-eq-Suc-imp-eq-union)
  show ?case
  proof (cases J' = {#})
    case True
    then show ?thesis
      using J Suc by (fastforce simp add: mult-def mult1-def)
  next
    case [simp]: False
    have K: K = {#x ∈# K. (x, a) ∈ r#} + {#x ∈# K. (x, a) ∉ r#}
      by simp
    have (I + K, (I + {# x ∈# K. (x, a) ∈ r #}) + J') ∈ mult r
      using IH[of J' {# x ∈# K. (x, a) ∉ r#} I + {# x ∈# K. (x, a) ∈ r#}]
        J Suc.prems K size-J by (auto simp: ac-simps)
    moreover have (I + {#x ∈# K. (x, a) ∈ r#} + J', I + J) ∈ mult r
      by (fastforce simp: J mult1-def mult-def)
    ultimately show ?thesis
      unfolding mult-def by simp
  qed
qed
```

lemma *subset-implies-mult*:
assumes *sub*: $A \subset\# B$
shows $(A, B) \in mult\ r$
proof −
have *ApBmA*: $A + (B - A) = B$
using *sub* **by** *simp*
have *BmA*: $B - A \neq \{\#\}$
using *sub* **by** (*simp add*: *Diff-eq-empty-iff-mset subset-mset.less-le-not-le*)
thus *?thesis*
by (*rule one-step-implies-mult*[*of* $B - A$ $\{\#\}$ - *A*, *unfolded ApBmA*, *simplified*])
qed

62.13 The multiset extension is cancellative for multiset union

lemma *mult-cancel*:
assumes *trans s* **and** *irrefl s*
shows $(X + Z, Y + Z) \in mult\ s \longleftrightarrow (X, Y) \in mult\ s$ (**is** *?L* $\longleftrightarrow$ *?R*)
proof
assume *?L* **thus** *?R*
proof (*induct Z*)
case (*add z Z*)
obtain $X'\ Y'\ Z'$ **where** $*$: *add-mset z X* $+ Z = Z' +$ X' *add-mset z Y* $+ Z = Z' +$ Y' $Y' \neq \{\#\}$
$\forall x \in$ *set-mset* X'. $\exists y \in$ *set-mset* Y'. $(x, y) \in s$
using *mult-implies-one-step*[*OF* ‹*trans s*› *add*(*2*)] **by** *auto*
consider *Z2* **where** $Z' =$ *add-mset z Z2* | *X2 Y2* **where** $X' =$ *add-mset z X2* $Y' =$ *add-mset z Y2*
using $*$(*1*,*2*) **by** (*metis add-mset-remove-trivial-If insert-iff set-mset-add-mset-insert union-iff*)
thus *?case*
proof (*cases*)
case *1* **thus** *?thesis* **using** $*$ *one-step-implies-mult*[*of* Y' X' *s Z2*]
by (*auto simp*: *add.commute*[*of* - $\{\#$-$\#\}$] *add.assoc intro*: *add*(*1*))
next
case *2* **then obtain** *y* **where** $y \in$ *set-mset Y2* $(z, y) \in s$ **using** $*$(*4*) ‹*irrefl s*›
by (*auto simp*: *irrefl-def*)
moreover from *this transD*[*OF* ‹*trans s*› - *this*(*2*)]
have $x' \in$ *set-mset X2* $\Longrightarrow \exists y \in$ *set-mset Y2*. $(x', y) \in s$ **for** x'
using *2* $*$(*4*)[*rule-format*, *of* x'] **by** *auto*
ultimately show *?thesis* **using** $*$ *one-step-implies-mult*[*of Y2 X2 s Z'*] *2*
by (*force simp*: *add.commute*[*of* $\{\#$-$\#\}$] *add.assoc*[*symmetric*] *intro*: *add*(*1*))
qed
qed *auto*
next
assume *?R* **then obtain** *I J K*
where $Y = I + J$ $X = I + K$ $J \neq \{\#\}$ $\forall k \in$ *set-mset K*. $\exists j \in$ *set-mset J*. $(k, j) \in s$

using *mult-implies-one-step*[*OF* ‹*trans s*›] **by** *blast*
thus *?L* **using** *one-step-implies-mult*[*of J K s I + Z*] **by** (*auto simp*: *ac-simps*)
qed

lemmas *mult-cancel-add-mset* =
mult-cancel[*of* - - {#-#}, *unfolded union-mset-add-mset-right add.comm-neutral*]

lemma *mult-cancel-max*:
assumes *trans s* **and** *irrefl s*
shows $(X, Y) \in mult\ s \longleftrightarrow (X - X \cap\# Y, Y - X \cap\# Y) \in mult\ s$ (**is** *?L* $\longleftrightarrow$ *?R*)
proof −
have $X - X \cap\# Y + X \cap\# Y = X\ Y - X \cap\# Y + X \cap\# Y = Y$ **by** (*auto simp flip*: *count-inject*)
thus *?thesis* **using** *mult-cancel*[*OF assms*, *of* $X - X \cap\# Y\ X \cap\# Y\ Y - X \cap\# Y$] **by** *auto*
qed

62.14 Quasi-executable version of the multiset extension

Predicate variants of *mult* and the reflexive closure of *mult*, which are executable whenever the given predicate P is. Together with the standard code equations for $(\cap\#)$ and $(-)$ this should yield quadratic (with respect to calls to P) implementations of *multp* and *multeqp*.

definition *multp* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a multiset* ⇒ '*a multiset* ⇒ *bool* **where**
multp P N M =
(*let Z* = *M* ∩# *N*; *X* = *M* − *Z in*
X ≠ {#} ∧ (*let Y* = *N* − *Z in* (∀ *y* ∈ *set-mset Y*. ∃ *x* ∈ *set-mset X*. *P y x*)))

definition *multeqp* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a multiset* ⇒ '*a multiset* ⇒ *bool* **where**
multeqp P N M =
(*let Z* = *M* ∩# *N*; *X* = *M* − *Z*; *Y* = *N* − *Z in*
(∀ *y* ∈ *set-mset Y*. ∃ *x* ∈ *set-mset X*. *P y x*))

lemma *multp-iff*:
assumes *irrefl R* **and** *trans R* **and** [*simp*]: ⋀*x y*. *P x y* ⟷ (*x*, *y*) ∈ *R*
shows *multp P N M* ⟷ (*N*, *M*) ∈ *mult R* (**is** *?L* ⟷ *?R*)
proof −
have ∗: *M* ∩# *N* + (*N* − *M* ∩# *N*) = *N M* ∩# *N* + (*M* − *M* ∩# *N*) = *M*
(*M* − *M* ∩# *N*) ∩# (*N* − *M* ∩# *N*) = {#} **by** (*auto simp flip*: *count-inject*)
show *?thesis*
proof
assume *?L* **thus** *?R*
using *one-step-implies-mult*[*of M* − *M* ∩# *N N* − *M* ∩# *N R M* ∩# *N*] ∗
by (*auto simp*: *multp-def Let-def*)
next
{ fix *I J K* :: '*a multiset* **assume** (*I* + *J*) ∩# (*I* + *K*) = {#}
then have *I* = {#} **by** (*metis inter-union-distrib-right union-eq-empty*)
} note [*dest*!] = *this*

```
    assume ?R thus ?L
      using mult-implies-one-step[OF assms(2), of N − M ∩# N M − M ∩# N]
        mult-cancel-max[OF assms(2,1), of N M] ∗ by (auto simp: multp-def)
  qed
qed

lemma multeqp-iff:
  assumes irefl R and trans R and ⋀x y. P x y ⟷ (x, y) ∈ R
  shows multeqp P N M ⟷ (N, M) ∈ (mult R)=
proof −
  { assume N ≠ M M − M ∩# N = {#}
    then obtain y where count N y ≠ count M y by (auto simp flip: count-inject)
    then have ∃ y. count M y < count N y using ‹M − M ∩# N = {#}›
      by (auto simp flip: count-inject dest!: le-neq-implies-less fun-cong[of - - y])
  }
  then have multeqp P N M ⟷ multp P N M ∨ N = M
    by (auto simp: multeqp-def multp-def Let-def in-diff-count)
  thus ?thesis using multp-iff[OF assms] by simp
qed
```

62.14.1 Partial-order properties

```
lemma (in preorder) mult1-lessE:
  assumes (N, M) ∈ mult1 {(a, b). a < b}
  obtains a M0 K where M = add-mset a M0 N = M0 + K
    a ∉# K ⋀b. b ∈# K ⟹ b < a
proof −
  from assms obtain a M0 K where M = add-mset a M0 N = M0 + K and
    ∗: b ∈# K ⟹ b < a for b by (blast elim: mult1E)
  moreover from ∗ [of a] have a ∉# K by auto
  ultimately show thesis by (auto intro: that)
qed

instantiation multiset :: (preorder) order
begin

definition less-multiset :: 'a multiset ⇒ 'a multiset ⇒ bool
  where M' < M ⟷ (M', M) ∈ mult {(x', x). x' < x}

definition less-eq-multiset :: 'a multiset ⇒ 'a multiset ⇒ bool
  where less-eq-multiset M' M ⟷ M' < M ∨ M' = M

instance
proof −
  have irefl: ¬ M < M for M :: 'a multiset
  proof
    assume M < M
   then have MM: (M, M) ∈ mult {(x, y). x < y} by (simp add: less-multiset-def)
    have trans {(x'::'a, x). x' < x}
```

by (*metis* (*mono-tags*, *lifting*) *case-prodD case-prodI less-trans mem-Collect-eq transI*)
moreover note *MM*
ultimately have $\exists I\ J\ K.\ M = I + J \wedge M = I + K \wedge J \neq \{\#\} \wedge (\forall k{\in}\textit{set-mset}\ K.\ \exists j{\in}\textit{set-mset}\ J.\ (k, j) \in \{(x, y).\ x < y\})$
by (*rule mult-implies-one-step*)
then obtain *I J K* **where** $M = I + J$ **and** $M = I + K$ **and** $J \neq \{\#\}$ **and** $(\forall k{\in}\textit{set-mset}\ K.\ \exists j{\in}\textit{set-mset}\ J.\ (k, j) \in \{(x, y).\ x < y\})$ **by** *blast*
then have $*$: $K \neq \{\#\}$ **and** $**$: $\forall k{\in}\textit{set-mset}\ K.\ \exists j{\in}\textit{set-mset}\ K.\ k < j$ **by** *auto*
have *finite* (*set-mset K*) **by** *simp*
moreover note $**$
ultimately have *set-mset K* = {}
by (*induct rule*: *finite-induct*) (*auto intro*: *order-less-trans*)
with $*$ **show** *False* **by** *simp*
qed
have *trans*: $K < M \Longrightarrow M < N \Longrightarrow K < N$ **for** *K M N* :: *'a multiset*
unfolding *less-multiset-def mult-def* **by** (*blast intro*: *trancl-trans*)
show *OFCLASS*(*'a multiset*, *order-class*)
by *standard* (*auto simp add*: *less-eq-multiset-def irrefl dest*: *trans*)
qed

end

lemma *mset-le-irrefl* [*elim*!]:
fixes *M* :: *'a*::*preorder multiset*
shows $M < M \Longrightarrow R$
by *simp*

62.14.2 Monotonicity of multiset union

lemma *mult1-union*: $(B, D) \in \textit{mult1}\ r \Longrightarrow (C + B, C + D) \in \textit{mult1}\ r$
by (*force simp*: *mult1-def*)

lemma *union-le-mono2*: $B < D \Longrightarrow C + B < C + (D{::}'a{::}\textit{preorder multiset})$
apply (*unfold less-multiset-def mult-def*)
apply (*erule trancl-induct*)
apply (*blast intro*: *mult1-union*)
apply (*blast intro*: *mult1-union trancl-trans*)
done

lemma *union-le-mono1*: $B < D \Longrightarrow B + C < D + (C{::}'a{::}\textit{preorder multiset})$
apply (*subst add.commute* [*of B C*])
apply (*subst add.commute* [*of D C*])
apply (*erule union-le-mono2*)
done

lemma *union-less-mono*:

fixes *A B C D* :: *'a*::*preorder multiset*
shows $A < C \Longrightarrow B < D \Longrightarrow A + B < C + D$
by (*blast intro*!: *union-le-mono1 union-le-mono2 less-trans*)

instantiation *multiset* :: (*preorder*) *ordered-ab-semigroup-add*
begin
instance
by *standard* (*auto simp add*: *less-eq-multiset-def intro*: *union-le-mono2*)
end

62.14.3 Termination proofs with multiset orders

lemma *multi-member-skip*: $x \in\# XS \Longrightarrow x \in\# \{\# y \#\} + XS$
and *multi-member-this*: $x \in\# \{\# x \#\} + XS$
and *multi-member-last*: $x \in\# \{\# x \#\}$
by *auto*

definition *ms-strict* = *mult pair-less*
definition *ms-weak* = *ms-strict* ∪ *Id*

lemma *ms-reduction-pair*: *reduction-pair* (*ms-strict*, *ms-weak*)
unfolding *reduction-pair-def ms-strict-def ms-weak-def pair-less-def*
by (*auto intro*: *wf-mult1 wf-trancl simp*: *mult-def*)

lemma *smsI*:
(*set-mset A*, *set-mset B*) ∈ *max-strict* ⟹ (*Z* + *A*, *Z* + *B*) ∈ *ms-strict*
unfolding *ms-strict-def*
by (*rule one-step-implies-mult*) (*auto simp add*: *max-strict-def pair-less-def elim*!:*max-ext.cases*)

lemma *wmsI*:
(*set-mset A*, *set-mset B*) ∈ *max-strict* ∨ *A* = {#} ∧ *B* = {#}
⟹ (*Z* + *A*, *Z* + *B*) ∈ *ms-weak*
unfolding *ms-weak-def ms-strict-def*
by (*auto simp add*: *pair-less-def max-strict-def elim*!:*max-ext.cases intro*: *one-step-implies-mult*)

inductive *pw-leq*
where
pw-leq-empty: *pw-leq* {#} {#}
| *pw-leq-step*: ⟦(*x*,*y*) ∈ *pair-leq*; *pw-leq X Y* ⟧ ⟹ *pw-leq* ({#*x*#} + *X*) ({#*y*#}
+ *Y*)

lemma *pw-leq-lstep*:
(*x*, *y*) ∈ *pair-leq* ⟹ *pw-leq* {#*x*#} {#*y*#}
by (*drule pw-leq-step*) (*rule pw-leq-empty*, *simp*)

lemma *pw-leq-split*:
assumes *pw-leq X Y*
shows ∃ *A B Z*. *X* = *A* + *Z* ∧ *Y* = *B* + *Z* ∧ ((*set-mset A*, *set-mset B*) ∈
max-strict ∨ (*B* = {#} ∧ *A* = {#}))

```
  using assms
proof induct
  case pw-leq-empty thus ?case by auto
next
  case (pw-leq-step x y X Y)
  then obtain A B Z where
    [simp]: X = A + Z Y = B + Z
     and 1[simp]: (set-mset A, set-mset B) ∈ max-strict ∨ (B = {#} ∧ A = {#})
    by auto
  from pw-leq-step consider x = y | (x, y) ∈ pair-less
    unfolding pair-leq-def by auto
  thus ?case
  proof cases
    case [simp]: 1
    have {#x#} + X = A + ({#y#}+Z) ∧ {#y#} + Y = B + ({#y#}+Z) ∧
      ((set-mset A, set-mset B) ∈ max-strict ∨ (B = {#} ∧ A = {#}))
      by auto
    thus ?thesis by blast
  next
    case 2
    let ?A' = {#x#} + A and ?B' = {#y#} + B
    have {#x#} + X = ?A' + Z
      {#y#} + Y = ?B' + Z
      by auto
    moreover have
      (set-mset ?A', set-mset ?B') ∈ max-strict
      using 1 2 unfolding max-strict-def
      by (auto elim!: max-ext.cases)
    ultimately show ?thesis by blast
  qed
qed

lemma
  assumes pwleq: pw-leq Z Z'
  shows ms-strictI: (set-mset A, set-mset B) ∈ max-strict ⟹ (Z + A, Z' + B)
∈ ms-strict
    and ms-weakI1: (set-mset A, set-mset B) ∈ max-strict ⟹ (Z + A, Z' + B)
∈ ms-weak
    and ms-weakI2: (Z + {#}, Z' + {#}) ∈ ms-weak
proof −
  from pw-leq-split[OF pwleq]
  obtain A' B' Z''
    where [simp]: Z = A' + Z'' Z' = B' + Z''
    and mx-or-empty: (set-mset A', set-mset B') ∈ max-strict ∨ (A' = {#} ∧ B'
= {#})
    by blast
  {
    assume max: (set-mset A, set-mset B) ∈ max-strict
    from mx-or-empty
```

```
    have (Z'' + (A + A'), Z'' + (B + B')) ∈ ms-strict
    proof
      assume max': (set-mset A', set-mset B') ∈ max-strict
      with max have (set-mset (A + A'), set-mset (B + B')) ∈ max-strict
        by (auto simp: max-strict-def intro: max-ext-additive)
      thus ?thesis by (rule smsI)
    next
      assume [simp]: A' = {#} ∧ B' = {#}
      show ?thesis by (rule smsI) (auto intro: max)
    qed
    thus (Z + A, Z' + B) ∈ ms-strict by (simp add: ac-simps)
    thus (Z + A, Z' + B) ∈ ms-weak by (simp add: ms-weak-def)
  }
  from mx-or-empty
  have (Z'' + A', Z'' + B') ∈ ms-weak by (rule wmsI)
  thus (Z + {#}, Z' + {#}) ∈ ms-weak by (simp add: ac-simps)
qed

lemma empty-neutral: {#} + x = x x + {#} = x
and nonempty-plus: {# x #} + rs ≠ {#}
and nonempty-single: {# x #} ≠ {#}
by auto

setup ‹
  let
    fun msetT T = Type ‹multiset T›;

    fun mk-mset T [] = instantiate ‹'a = T in term ‹{#}››
      | mk-mset T [x] = instantiate ‹'a = T and x in term ‹{#x#}››
      | mk-mset T (x :: xs) = Const ‹plus ‹msetT T› for ‹mk-mset T [x]› ‹mk-mset
T xs››

    fun mset-member-tac ctxt m i =
      if m <= 0 then
        resolve-tac ctxt @{thms multi-member-this} i ORELSE
        resolve-tac ctxt @{thms multi-member-last} i
      else
        resolve-tac ctxt @{thms multi-member-skip} i THEN mset-member-tac ctxt
(m − 1) i

    fun mset-nonempty-tac ctxt =
      resolve-tac ctxt @{thms nonempty-plus} ORELSE'
      resolve-tac ctxt @{thms nonempty-single}

    fun regroup-munion-conv ctxt =
     Function-Lib.regroup-conv ctxt const-abbrev ‹empty-mset› const-name ‹plus›
      (map (fn t => t RS eq-reflection) (@{thms ac-simps} @ @{thms empty-neutral}))

    fun unfold-pwleq-tac ctxt i =
```

```
    (resolve-tac ctxt @{thms pw-leq-step} i THEN (fn st => unfold-pwleq-tac ctxt
(i + 1) st))
      ORELSE (resolve-tac ctxt @{thms pw-leq-lstep} i)
      ORELSE (resolve-tac ctxt @{thms pw-leq-empty} i)

  val set-mset-simps = [@{thm set-mset-empty}, @{thm set-mset-single}, @{thm
set-mset-union},
                        @{thm Un-insert-left}, @{thm Un-empty-left}]
 in
  ScnpReconstruct.multiset-setup (ScnpReconstruct.Multiset
  {
   msetT=msetT, mk-mset=mk-mset, mset-regroup-conv=regroup-munion-conv,
   mset-member-tac=mset-member-tac, mset-nonempty-tac=mset-nonempty-tac,
   mset-pwleq-tac=unfold-pwleq-tac, set-of-simps=set-mset-simps,
    smsI'= @{thm ms-strictI}, wmsI2''= @{thm ms-weakI2}, wmsI1= @{thm
ms-weakI1},
   reduction-pair = @{thm ms-reduction-pair}
  })
 end
›
```

62.15 Legacy theorem bindings

lemmas *multi-count-eq* = *multiset-eq-iff* [*symmetric*]

lemma *union-commute*: $M + N = N + (M::'a\ multiset)$
by (*fact add.commute*)

lemma *union-assoc*: $(M + N) + K = M + (N + (K::'a\ multiset))$
by (*fact add.assoc*)

lemma *union-lcomm*: $M + (N + K) = N + (M + (K::'a\ multiset))$
by (*fact add.left-commute*)

lemmas *union-ac* = *union-assoc union-commute union-lcomm add-mset-commute*

lemma *union-right-cancel*: $M + K = N + K \longleftrightarrow M = (N::'a\ multiset)$
by (*fact add-right-cancel*)

lemma *union-left-cancel*: $K + M = K + N \longleftrightarrow M = (N::'a\ multiset)$
by (*fact add-left-cancel*)

lemma *multi-union-self-other-eq*: $(A::'a\ multiset) + X = A + Y \Longrightarrow X = Y$
by (*fact add-left-imp-eq*)

lemma *mset-subset-trans*: $(M::'a\ multiset) \subset\# K \Longrightarrow K \subset\# N \Longrightarrow M \subset\# N$
by (*fact subset-mset.less-trans*)

lemma *multiset-inter-commute*: $A \cap\# B = B \cap\# A$

by (*fact subset-mset.inf.commute*)

lemma *multiset-inter-assoc*: $A \cap\# (B \cap\# C) = A \cap\# B \cap\# C$
by (*fact subset-mset.inf.assoc* [*symmetric*])

lemma *multiset-inter-left-commute*: $A \cap\# (B \cap\# C) = B \cap\# (A \cap\# C)$
by (*fact subset-mset.inf.left-commute*)

lemmas *multiset-inter-ac* =
multiset-inter-commute
multiset-inter-assoc
multiset-inter-left-commute

lemma *mset-le-not-refl*: $\neg M < (M::'a::preorder\ multiset)$
by (*fact less-irrefl*)

lemma *mset-le-trans*: $K < M \Longrightarrow M < N \Longrightarrow K < (N::'a::preorder\ multiset)$
by (*fact less-trans*)

lemma *mset-le-not-sym*: $M < N \Longrightarrow \neg N < (M::'a::preorder\ multiset)$
by (*fact less-not-sym*)

lemma *mset-le-asym*: $M < N \Longrightarrow (\neg P \Longrightarrow N < (M::'a::preorder\ multiset)) \Longrightarrow P$
by (*fact less-asym*)

declaration ‹
let
fun multiset-postproc - maybe-name all-values (*T as Type* (-, [*elem-T*])) (*Const* - $ *t'*) =
let
val (*maybe-opt*, *ps*) =
Nitpick-Model.dest-plain-fun t'
||> (~~)
||> *map* (*apsnd* (*snd o HOLogic.dest-number*))
fun elems-for t =
(*case AList.lookup* (=) *ps t of*
SOME n => *replicate n t*
| *NONE* => [*Const* (*maybe-name*, *elem-T* --> *elem-T*) $ *t*])
in
(*case maps elems-for* (*all-values elem-T*) @
(*if maybe-opt then* [*Const* (*Nitpick-Model.unrep-mixfix* (), *elem-T*)] *else* []) *of*
[] => ***Const*** ‹*Groups.zero T*›
| *ts* => *foldl1* (*fn* (*s*, *t*) => ***Const*** ‹*add-mset elem-T for s t*›) *ts*)
end
| *multiset-postproc - - - - t* = *t*
in Nitpick-Model.register-term-postprocessor ***typ*** ‹*'a multiset*› *multiset-postproc*
end

›

62.16 Naive implementation using lists

code-datatype *mset*

lemma [*code*]: {#} = *mset* []
by *simp*

lemma [*code*]: *add-mset x* (*mset xs*) = *mset* (*x* # *xs*)
by *simp*

lemma [*code*]: *Multiset.is-empty* (*mset xs*) ⟷ *List.null xs*
by (*simp add*: *Multiset.is-empty-def List.null-def*)

lemma *union-code* [*code*]: *mset xs* + *mset ys* = *mset* (*xs* @ *ys*)
by *simp*

lemma [*code*]: *image-mset f* (*mset xs*) = *mset* (*map f xs*)
by *simp*

lemma [*code*]: *filter-mset f* (*mset xs*) = *mset* (*filter f xs*)
by *simp*

lemma [*code*]: *mset xs* − *mset ys* = *mset* (*fold remove1 ys xs*)
by (*rule sym*, *induct ys arbitrary*: *xs*) (*simp-all add*: *diff-add diff-right-commute diff-diff-add*)

lemma [*code*]:
mset xs ∩# *mset ys* =
mset (*snd* (*fold* (λ*x* (*ys*, *zs*).
if x ∈ *set ys then* (*remove1 x ys*, *x* # *zs*) *else* (*ys*, *zs*)) *xs* (*ys*, []))))
proof −
have ⋀*zs*. *mset* (*snd* (*fold* (λ*x* (*ys*, *zs*).
if x ∈ *set ys then* (*remove1 x ys*, *x* # *zs*) *else* (*ys*, *zs*)) *xs* (*ys*, *zs*))) =
(*mset xs* ∩# *mset ys*) + *mset zs*
by (*induct xs arbitrary*: *ys*)
(*auto simp add*: *inter-add-right1 inter-add-right2 ac-simps*)
then show *?thesis* **by** *simp*
qed

lemma [*code*]:
mset xs ∪# *mset ys* =
mset (*case-prod append* (*fold* (λ*x* (*ys*, *zs*). (*remove1 x ys*, *x* # *zs*)) *xs* (*ys*, [])))
proof −
have ⋀*zs*. *mset* (*case-prod append* (*fold* (λ*x* (*ys*, *zs*). (*remove1 x ys*, *x* # *zs*)) *xs* (*ys*, *zs*))) =
(*mset xs* ∪# *mset ys*) + *mset zs*
by (*induct xs arbitrary*: *ys*) (*simp-all add*: *multiset-eq-iff*)

```
  then show ?thesis by simp
qed

declare in-multiset-in-set [code-unfold]

lemma [code]: count (mset xs) x = fold (λy. if x = y then Suc else id) xs 0
proof −
  have ⋀n. fold (λy. if x = y then Suc else id) xs n = count (mset xs) x + n
    by (induct xs) simp-all
  then show ?thesis by simp
qed

declare set-mset-mset [code]

declare sorted-list-of-multiset-mset [code]

lemma [code]: — not very efficient, but representation-ignorant!
  mset-set A = mset (sorted-list-of-set A)
  apply (cases finite A)
  apply simp-all
  apply (induct A rule: finite-induct)
  apply simp-all
  done

declare size-mset [code]

fun subset-eq-mset-impl :: 'a list ⇒ 'a list ⇒ bool option where
  subset-eq-mset-impl [] ys = Some (ys ≠ [])
| subset-eq-mset-impl (Cons x xs) ys = (case List.extract ((=) x) ys of
    None ⇒ None
  | Some (ys1,-,ys2) ⇒ subset-eq-mset-impl xs (ys1 @ ys2))

lemma subset-eq-mset-impl: (subset-eq-mset-impl xs ys = None ⟷ ¬ mset xs
⊆# mset ys) ∧
  (subset-eq-mset-impl xs ys = Some True ⟷ mset xs ⊂# mset ys) ∧
  (subset-eq-mset-impl xs ys = Some False ⟶ mset xs = mset ys)
proof (induct xs arbitrary: ys)
  case (Nil ys)
  show ?case by (auto simp: subset-mset.zero-less-iff-neq-zero)
next
  case (Cons x xs ys)
  show ?case
  proof (cases List.extract ((=) x) ys)
    case None
    hence x: x ∉ set ys by (simp add: extract-None-iff)
    {
      assume mset (x # xs) ⊆# mset ys
      from set-mset-mono[OF this] x have False by simp
    } note nle = this
```

```
    moreover
    {
      assume mset (x # xs) ⊂# mset ys
      hence mset (x # xs) ⊆# mset ys by auto
      from nle[OF this] have False .
    }
    ultimately show ?thesis using None by auto
  next
    case (Some res)
    obtain ys1 y ys2 where res: res = (ys1,y,ys2) by (cases res, auto)
    note Some = Some[unfolded res]
    from extract-SomeE[OF Some] have ys = ys1 @ x # ys2 by simp
    hence id: mset ys = add-mset x (mset (ys1 @ ys2))
      by auto
    show ?thesis unfolding subset-eq-mset-impl.simps
      unfolding Some option.simps split
      unfolding id
      using Cons[of ys1 @ ys2]
      unfolding subset-mset-def subseteq-mset-def by auto
  qed
qed

lemma [code]: mset xs ⊆# mset ys ⟷ subset-eq-mset-impl xs ys ≠ None
  using subset-eq-mset-impl[of xs ys] by (cases subset-eq-mset-impl xs ys, auto)

lemma [code]: mset xs ⊂# mset ys ⟷ subset-eq-mset-impl xs ys = Some True
  using subset-eq-mset-impl[of xs ys] by (cases subset-eq-mset-impl xs ys, auto)

instantiation multiset :: (equal) equal
begin

definition
  [code del]: HOL.equal A (B :: 'a multiset) ⟷ A = B
lemma [code]: HOL.equal (mset xs) (mset ys) ⟷ subset-eq-mset-impl xs ys =
Some False
  unfolding equal-multiset-def
  using subset-eq-mset-impl[of xs ys] by (cases subset-eq-mset-impl xs ys, auto)

instance
  by standard (simp add: equal-multiset-def)

end

declare sum-mset-sum-list [code]

lemma [code]: prod-mset (mset xs) = fold times xs 1
proof −
  have ⋀x. fold times xs x = prod-mset (mset xs) * x
    by (induct xs) (simp-all add: ac-simps)
```

then show *?thesis* **by** *simp*
qed

Exercise for the casual reader: add implementations for $(\leq)$ and $(<)$ (multiset order).

Quickcheck generators

context
includes *term-syntax*
begin

definition
msetify :: *'a::typerep list* × (*unit* ⇒ *Code-Evaluation.term*)
⇒ *'a multiset* × (*unit* ⇒ *Code-Evaluation.term*) **where**
[*code-unfold*]: *msetify xs* = *Code-Evaluation.valtermify mset* {·} *xs*

end

instantiation *multiset* :: (*random*) *random*
begin

context
includes *state-combinator-syntax*
begin

definition
Quickcheck-Random.random i = *Quickcheck-Random.random i* ○→ (λ*xs*. *Pair* (*msetify xs*))

instance ..

end

end

instantiation *multiset* :: (*full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive-multiset* :: (*'a multiset* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
full-exhaustive-multiset f i = *Quickcheck-Exhaustive.full-exhaustive* (λ*xs*. *f* (*msetify xs*)) *i*

instance ..

end

hide-const (**open**) *msetify*

62.17 BNF setup

definition *rel-mset* **where**
rel-mset R X Y ⟷ (∃*xs ys*. *mset xs* = *X* ∧ *mset ys* = *Y* ∧ *list-all2 R xs ys*)

lemma *mset-zip-take-Cons-drop-twice*:
assumes *length xs* = *length ys j* ≤ *length xs*
shows *mset* (*zip* (*take j xs* @ *x* # *drop j xs*) (*take j ys* @ *y* # *drop j ys*)) =
add-mset (*x*,*y*) (*mset* (*zip xs ys*))
using *assms*
proof (*induct xs ys arbitrary*: *x y j rule*: *list-induct2*)
case *Nil*
thus *?case*
by *simp*
next
case (*Cons x xs y ys*)
thus *?case*
proof (*cases j* = *0*)
case *True*
thus *?thesis*
by *simp*
next
case *False*
then obtain *k* **where** *k*: *j* = *Suc k*
by (*cases j*) *simp*
hence *k* ≤ *length xs*
using *Cons.prems* **by** *auto*
hence *mset* (*zip* (*take k xs* @ *x* # *drop k xs*) (*take k ys* @ *y* # *drop k ys*)) =
add-mset (*x*,*y*) (*mset* (*zip xs ys*))
by (*rule Cons.hyps*(*2*))
thus *?thesis*
unfolding *k* **by** *auto*
qed
qed

lemma *ex-mset-zip-left*:
assumes *length xs* = *length ys mset xs′* = *mset xs*
shows ∃*ys′*. *length ys′* = *length xs′* ∧ *mset* (*zip xs′ ys′*) = *mset* (*zip xs ys*)
using *assms*
proof (*induct xs ys arbitrary*: *xs′ rule*: *list-induct2*)
case *Nil*
thus *?case*
by *auto*
next
case (*Cons x xs y ys xs′*)
obtain *j* **where** *j-len*: *j* < *length xs′* **and** *nth-j*: *xs′* ! *j* = *x*
by (*metis Cons.prems in-set-conv-nth list.set-intros*(*1*) *mset-eq-setD*)

define *xsa* **where** *xsa* = *take j xs′* @ *drop* (*Suc j*) *xs′*
have *mset xs′* = {#*x*#} + *mset xsa*

```
    unfolding xsa-def using j-len nth-j
   by (metis Cons-nth-drop-Suc union-mset-add-mset-right add-mset-remove-trivial
add-diff-cancel-left'
        append-take-drop-id mset.simps(2) mset-append)
  hence ms-x: mset xsa = mset xs
    by (simp add: Cons.prems)
  then obtain ysa where
    len-a: length ysa = length xsa and ms-a: mset (zip xsa ysa) = mset (zip xs ys)
    using Cons.hyps(2) by blast

  define ys' where ys' = take j ysa @ y # drop j ysa
  have xs': xs' = take j xsa @ x # drop j xsa
    using ms-x j-len nth-j Cons.prems xsa-def
   by (metis append-eq-append-conv append-take-drop-id diff-Suc-Suc Cons-nth-drop-Suc
length-Cons
      length-drop size-mset)
  have j-len': j ≤ length xsa
    using j-len xs' xsa-def
   by (metis add-Suc-right append-take-drop-id length-Cons length-append less-eq-Suc-le
not-less)
  have length ys' = length xs'
    unfolding ys'-def using Cons.prems len-a ms-x
   by (metis add-Suc-right append-take-drop-id length-Cons length-append mset-eq-length)
  moreover have mset (zip xs' ys') = mset (zip (x # xs) (y # ys))
    unfolding xs' ys'-def
    by (rule trans[OF mset-zip-take-Cons-drop-twice])
      (auto simp: len-a ms-a j-len')
  ultimately show ?case
    by blast
qed

lemma list-all2-reorder-left-invariance:
  assumes rel: list-all2 R xs ys and ms-x: mset xs' = mset xs
  shows ∃ ys'. list-all2 R xs' ys' ∧ mset ys' = mset ys
proof −
  have len: length xs = length ys
    using rel list-all2-conv-all-nth by auto
  obtain ys' where
    len': length xs' = length ys' and ms-xy: mset (zip xs' ys') = mset (zip xs ys)
    using len ms-x by (metis ex-mset-zip-left)
  have list-all2 R xs' ys'
   using assms(1) len' ms-xy unfolding list-all2-iff by (blast dest: mset-eq-setD)
  moreover have mset ys' = mset ys
    using len len' ms-xy map-snd-zip mset-map by metis
  ultimately show ?thesis
    by blast
qed

lemma ex-mset: ∃ xs. mset xs = X
```

```
  by (induct X) (simp, metis mset.simps(2))

inductive pred-mset :: ('a ⇒ bool) ⇒ 'a multiset ⇒ bool
where
  pred-mset P {#}
| ⟦P a; pred-mset P M⟧ ⟹ pred-mset P (add-mset a M)

lemma pred-mset-iff: — TODO: alias for Multiset.Ball
  ‹pred-mset P M ⟷ Multiset.Ball M P› (is ‹?P ⟷ ?Q›)
proof
  assume ?P
  then show ?Q by induction simp-all
next
  assume ?Q
  then show ?P
    by (induction M) (auto intro: pred-mset.intros)
qed

bnf 'a multiset
  map: image-mset
  sets: set-mset
  bd: natLeq
  wits: {#}
  rel: rel-mset
  pred: pred-mset
proof −
  show image-mset id = id
    by (rule image-mset.id)
  show image-mset (g ∘ f) = image-mset g ∘ image-mset f for f g
   unfolding comp-def by (rule ext) (simp add: comp-def image-mset.compositionality)
  show (⋀z. z ∈ set-mset X ⟹ f z = g z) ⟹ image-mset f X = image-mset g
X for f g X
    by (induct X) simp-all
  show set-mset ∘ image-mset f = (`) f ∘ set-mset for f
    by auto
  show card-order natLeq
    by (rule natLeq-card-order)
  show BNF-Cardinal-Arithmetic.cinfinite natLeq
    by (rule natLeq-cinfinite)
  show ordLeq3 (card-of (set-mset X)) natLeq for X
    by transfer
      (auto intro!: ordLess-imp-ordLeq simp: finite-iff-ordLess-natLeq[symmetric])
  show rel-mset R OO rel-mset S ≤ rel-mset (R OO S) for R S
    unfolding rel-mset-def[abs-def] OO-def
    apply clarify
    subgoal for X Z Y xs ys' ys zs
      apply (drule list-all2-reorder-left-invariance [where xs = ys' and ys = zs
and xs' = ys])
      apply (auto intro: list-all2-trans)
```

```
    done
  done
 show rel-mset R =
  (λx y. ∃z. set-mset z ⊆ {(x, y). R x y} ∧
  image-mset fst z = x ∧ image-mset snd z = y) for R
  unfolding rel-mset-def[abs-def]
  apply (rule ext)+
  apply safe
   apply (rule-tac x = mset (zip xs ys) in exI;
     auto simp: in-set-zip list-all2-iff simp flip: mset-map)
  apply (rename-tac XY)
  apply (cut-tac X = XY in ex-mset)
  apply (erule exE)
  apply (rename-tac xys)
  apply (rule-tac x = map fst xys in exI)
  apply (auto simp: mset-map)
  apply (rule-tac x = map snd xys in exI)
  apply (auto simp: mset-map list-all2I subset-eq zip-map-fst-snd)
  done
 show z ∈ set-mset {#} ⟹ False for z
  by auto
 show pred-mset P = (λx. Ball (set-mset x) P) for P
  by (simp add: fun-eq-iff pred-mset-iff)
qed

inductive rel-mset' :: ‹('a ⇒ 'b ⇒ bool) ⇒ 'a multiset ⇒ 'b multiset ⇒ bool›
where
  Zero[intro]: rel-mset' R {#} {#}
| Plus[intro]: ⟦R a b; rel-mset' R M N⟧ ⟹ rel-mset' R (add-mset a M) (add-mset
b N)

lemma rel-mset-Zero: rel-mset R {#} {#}
unfolding rel-mset-def Grp-def by auto

declare multiset.count[simp]
declare count-Abs-multiset[simp]
declare multiset.count-inverse[simp]

lemma rel-mset-Plus:
 assumes ab: R a b
  and MN: rel-mset R M N
 shows rel-mset R (add-mset a M) (add-mset b N)
proof −
 have ∃ya. add-mset a (image-mset fst y) = image-mset fst ya ∧
  add-mset b (image-mset snd y) = image-mset snd ya ∧
  set-mset ya ⊆ {(x, y). R x y}
  if R a b and set-mset y ⊆ {(x, y). R x y} for y
  using that by (intro exI[of - add-mset (a,b) y]) auto
 thus ?thesis
```

using *assms*
unfolding *multiset.rel-compp-Grp Grp-def* **by** *blast*
qed

lemma *rel-mset'-imp-rel-mset*: *rel-mset' R M N* $\Longrightarrow$ *rel-mset R M N*
by (*induct rule*: *rel-mset'.induct*) (*auto simp*: *rel-mset-Zero rel-mset-Plus*)

lemma *rel-mset-size*: *rel-mset R M N* $\Longrightarrow$ *size M* = *size N*
unfolding *multiset.rel-compp-Grp Grp-def* **by** *auto*

lemma *rel-mset-Zero-iff* [*simp*]:
shows *rel-mset rel* {#} *Y* $\longleftrightarrow$ *Y* = {#} **and** *rel-mset rel X* {#} $\longleftrightarrow$ *X* = {#}
by (*auto simp add*: *rel-mset-Zero dest*: *rel-mset-size*)

lemma *multiset-induct2*[*case-names empty addL addR*]:
assumes *empty*: *P* {#} {#}
and *addL*: $\bigwedge$*a M N*. *P M N* $\Longrightarrow$ *P* (*add-mset a M*) *N*
and *addR*: $\bigwedge$*a M N*. *P M N* $\Longrightarrow$ *P M* (*add-mset a N*)
shows *P M N*
apply(*induct N rule*: *multiset-induct*)
apply(*induct M rule*: *multiset-induct*, *rule empty*, *erule addL*)
apply(*induct M rule*: *multiset-induct*, *erule addR*, *erule addR*)
done

lemma *multiset-induct2-size*[*consumes 1*, *case-names empty add*]:
assumes *c*: *size M* = *size N*
and *empty*: *P* {#} {#}
and *add*: $\bigwedge$*a b M N a b*. *P M N* $\Longrightarrow$ *P* (*add-mset a M*) (*add-mset b N*)
shows *P M N*
using *c*
proof (*induct M arbitrary*: *N rule*: *measure-induct-rule*[*of size*])
case (*less M*)
show *?case*
proof(*cases M* = {#})
case *True* **hence** *N* = {#} **using** *less.prems* **by** *auto*
thus *?thesis* **using** *True empty* **by** *auto*
next
case *False* **then obtain** *M1 a* **where** *M*: *M* = *add-mset a M1* **by** (*metis multi-nonempty-split*)
have *N* ≠ {#} **using** *False less.prems* **by** *auto*
then obtain *N1 b* **where** *N*: *N* = *add-mset b N1* **by** (*metis multi-nonempty-split*)
have *size M1* = *size N1* **using** *less.prems* **unfolding** *M N* **by** *auto*
thus *?thesis* **using** *M N less.hyps add* **by** *auto*
qed
qed

lemma *msed-map-invL*:
assumes *image-mset f* (*add-mset a M*) = *N*
shows ∃ *N1*. *N* = *add-mset* (*f a*) *N1* ∧ *image-mset f M* = *N1*

proof −
have *f a* ∈# *N*
using *assms multiset.set-map*[*of f add-mset a M*] **by** *auto*
then obtain *N1* **where** *N*: *N* = *add-mset* (*f a*) *N1* **using** *multi-member-split*
by *metis*
have *image-mset f M* = *N1* **using** *assms* **unfolding** *N* **by** *simp*
thus *?thesis* **using** *N* **by** *blast*
qed

lemma *msed-map-invR*:
assumes *image-mset f M* = *add-mset b N*
shows ∃ *M1 a*. *M* = *add-mset a M1* ∧ *f a* = *b* ∧ *image-mset f M1* = *N*
proof −
obtain *a* **where** *a*: *a* ∈# *M* **and** *fa*: *f a* = *b*
using *multiset.set-map*[*of f M*] **unfolding** *assms*
by (*metis image-iff union-single-eq-member*)
then obtain *M1* **where** *M*: *M* = *add-mset a M1* **using** *multi-member-split* **by**
metis
have *image-mset f M1* = *N* **using** *assms* **unfolding** *M fa*[*symmetric*] **by** *simp*
thus *?thesis* **using** *M fa* **by** *blast*
qed

lemma *msed-rel-invL*:
assumes *rel-mset R* (*add-mset a M*) *N*
shows ∃ *N1 b*. *N* = *add-mset b N1* ∧ *R a b* ∧ *rel-mset R M N1*
proof −
obtain *K* **where** *KM*: *image-mset fst K* = *add-mset a M*
and *KN*: *image-mset snd K* = *N* **and** *sK*: *set-mset K* ⊆ {(*a*, *b*). *R a b*}
using *assms*
unfolding *multiset.rel-compp-Grp Grp-def* **by** *auto*
obtain *K1 ab* **where** *K*: *K* = *add-mset ab K1* **and** *a*: *fst ab* = *a*
and *K1M*: *image-mset fst K1* = *M* **using** *msed-map-invR*[*OF KM*] **by** *auto*
obtain *N1* **where** *N*: *N* = *add-mset* (*snd ab*) *N1* **and** *K1N1*: *image-mset snd*
K1 = *N1*
using *msed-map-invL*[*OF KN*[*unfolded K*]] **by** *auto*
have *Rab*: *R a* (*snd ab*) **using** *sK a* **unfolding** *K* **by** *auto*
have *rel-mset R M N1* **using** *sK K1M K1N1*
unfolding *K multiset.rel-compp-Grp Grp-def* **by** *auto*
thus *?thesis* **using** *N Rab* **by** *auto*
qed

lemma *msed-rel-invR*:
assumes *rel-mset R M* (*add-mset b N*)
shows ∃ *M1 a*. *M* = *add-mset a M1* ∧ *R a b* ∧ *rel-mset R M1 N*
proof −
obtain *K* **where** *KN*: *image-mset snd K* = *add-mset b N*
and *KM*: *image-mset fst K* = *M* **and** *sK*: *set-mset K* ⊆ {(*a*, *b*). *R a b*}
using *assms*
unfolding *multiset.rel-compp-Grp Grp-def* **by** *auto*

obtain *K1 ab* **where** *K*: *K* = *add-mset ab K1* **and** *b*: *snd ab* = *b*
and *K1N*: *image-mset snd K1* = *N* **using** *msed-map-invR*[*OF KN*] **by** *auto*
obtain *M1* **where** *M*: *M* = *add-mset* (*fst ab*) *M1* **and** *K1M1*: *image-mset fst K1* = *M1*
using *msed-map-invL*[*OF KM*[*unfolded K*]] **by** *auto*
have *Rab*: *R* (*fst ab*) *b* **using** *sK b* **unfolding** *K* **by** *auto*
have *rel-mset R M1 N* **using** *sK K1N K1M1*
unfolding *K multiset.rel-compp-Grp Grp-def* **by** *auto*
thus *?thesis* **using** *M Rab* **by** *auto*
qed

lemma *rel-mset-imp-rel-mset'*:
assumes *rel-mset R M N*
shows *rel-mset' R M N*
using *assms* **proof**(*induct M arbitrary*: *N rule*: *measure-induct-rule*[*of size*])
case (*less M*)
have *c*: *size M* = *size N* **using** *rel-mset-size*[*OF less.prems*] **.**
show *?case*
proof(*cases M* = {#})
case *True* **hence** *N* = {#} **using** *c* **by** *simp*
thus *?thesis* **using** *True rel-mset'.Zero* **by** *auto*
next
case *False* **then obtain** *M1 a* **where** *M*: *M* = *add-mset a M1* **by** (*metis multi-nonempty-split*)
obtain *N1 b* **where** *N*: *N* = *add-mset b N1* **and** *R*: *R a b* **and** *ms*: *rel-mset R M1 N1*
using *msed-rel-invL*[*OF less.prems*[*unfolded M*]] **by** *auto*
have *rel-mset' R M1 N1* **using** *less.hyps*[*of M1 N1*] *ms* **unfolding** *M* **by** *simp*
thus *?thesis* **using** *rel-mset'.Plus*[*of R a b*, *OF R*] **unfolding** *M N* **by** *simp*
qed
qed

lemma *rel-mset-rel-mset'*: *rel-mset R M N* = *rel-mset' R M N*
using *rel-mset-imp-rel-mset' rel-mset'-imp-rel-mset* **by** *auto*

The main end product for *rel-mset*: inductive characterization:

lemmas *rel-mset-induct*[*case-names empty add*, *induct pred*: *rel-mset*] =
rel-mset'.induct[*unfolded rel-mset-rel-mset'*[*symmetric*]]

62.18 Size setup

lemma *size-multiset-o-map*: *size-multiset g* ∘ *image-mset f* = *size-multiset* (*g* ∘ *f*)
apply (*rule ext*)
subgoal for *x* **by** (*induct x*) *auto*
done

setup ‹
BNF-LFP-Size.register-size-global ***type-name*** ‹*multiset*› ***const-name*** ‹*size-multiset*›
@{*thm size-multiset-overloaded-def*}

@{*thms size-multiset-empty size-multiset-single size-multiset-union size-empty size-single size-union*}
@{*thms size-multiset-o-map*}
›

62.19 Lemmas about Size

lemma *size-mset-SucE*: *size A = Suc n ⟹ (⋀a B. A = {#a#} + B ⟹ size B = n ⟹ P) ⟹ P*
by (*cases A*) (*auto simp add*: *ac-simps*)

lemma *size-Suc-Diff1*: *x ∈# M ⟹ Suc (size (M − {#x#})) = size M*
using *arg-cong*[*OF insert-DiffM*, *of - - size*] **by** *simp*

lemma *size-Diff-singleton*: *x ∈# M ⟹ size (M − {#x#}) = size M − 1*
by (*simp flip*: *size-Suc-Diff1*)

lemma *size-Diff-singleton-if*: *size (A − {#x#}) = (if x ∈# A then size A − 1 else size A)*
by (*simp add*: *diff-single-trivial size-Diff-singleton*)

lemma *size-Un-Int*: *size A + size B = size (A ∪# B) + size (A ∩# B)*
by (*metis inter-subset-eq-union size-union subset-mset.diff-add union-diff-inter-eq-sup*)

lemma *size-Un-disjoint*: *A ∩# B = {#} ⟹ size (A ∪# B) = size A + size B*
using *size-Un-Int*[*of A B*] **by** *simp*

lemma *size-Diff-subset-Int*: *size (M − M′) = size M − size (M ∩# M′)*
by (*metis diff-intersect-left-idem size-Diff-submset subset-mset.inf-le1*)

lemma *diff-size-le-size-Diff*: *size (M :: - multiset) − size M′ ≤ size (M − M′)*
by (*simp add*: *diff-le-mono2 size-Diff-subset-Int size-mset-mono*)

lemma *size-Diff1-less*: *x∈# M ⟹ size (M − {#x#}) < size M*
by (*rule Suc-less-SucD*) (*simp add*: *size-Suc-Diff1*)

lemma *size-Diff2-less*: *x∈# M ⟹ y∈# M ⟹ size (M − {#x#} − {#y#}) < size M*
by (*metis less-imp-diff-less size-Diff1-less size-Diff-subset-Int*)

lemma *size-Diff1-le*: *size (M − {#x#}) ≤ size M*
by (*cases x ∈# M*) (*simp-all add*: *size-Diff1-less less-imp-le diff-single-trivial*)

lemma *size-psubset*: *M ⊆# M′ ⟹ size M < size M′ ⟹ M ⊂# M′*
using *less-irrefl subset-mset-def* **by** *blast*

hide-const (**open**) *wcount*

end

63 More Theorems about the Multiset Order

theory *Multiset-Order*
imports *Multiset*
begin

63.1 Alternative Characterizations

context *preorder*
begin

lemma *order-mult*: *class.order*
(λ*M N*. (*M*, *N*) ∈ *mult* {(*x*, *y*). *x* < *y*} ∨ *M* = *N*)
(λ*M N*. (*M*, *N*) ∈ *mult* {(*x*, *y*). *x* < *y*})
(**is** *class.order ?le ?less*)
proof −
have *irrefl*: ⋀*M* :: *'a multiset*. ¬ *?less M M*
proof
fix *M* :: *'a multiset*
have *trans* {(*x'*::*'a*, *x*). *x'* < *x*}
by (*rule transI*) (*blast intro*: *less-trans*)
moreover
assume (*M*, *M*) ∈ *mult* {(*x*, *y*). *x* < *y*}
ultimately have ∃ *I J K*. *M* = *I* + *J* ∧ *M* = *I* + *K*
∧ *J* ≠ {#} ∧ (∀ *k*∈*set-mset K*. ∃ *j*∈*set-mset J*. (*k*, *j*) ∈ {(*x*, *y*). *x* < *y*})
by (*rule mult-implies-one-step*)
then obtain *I J K* **where** *M* = *I* + *J* **and** *M* = *I* + *K*
and *J* ≠ {#} **and** (∀ *k*∈*set-mset K*. ∃ *j*∈*set-mset J*. (*k*, *j*) ∈ {(*x*, *y*). *x* < *y*})
by *blast*
then have *aux1*: *K* ≠ {#} **and** *aux2*: ∀ *k*∈*set-mset K*. ∃ *j*∈*set-mset K*. *k* < *j*
by *auto*
have *finite* (*set-mset K*) **by** *simp*
moreover note *aux2*
ultimately have *set-mset K* = {}
by (*induct rule*: *finite-induct*)
(*simp*, *metis* (*mono-tags*) *insert-absorb insert-iff insert-not-empty less-irrefl less-trans*)
with *aux1* **show** *False* **by** *simp*
qed
have *trans*: ⋀*K M N* :: *'a multiset*. *?less K M* ⟹ *?less M N* ⟹ *?less K N*
unfolding *mult-def* **by** (*blast intro*: *trancl-trans*)
show *class.order ?le ?less*
by *standard* (*auto simp add*: *less-eq-multiset-def irrefl dest*: *trans*)
qed

The Dershowitz–Manna ordering:

definition *less-multiset*$_{DM}$ **where**

less-multiset$_{DM}$ M N $\longleftrightarrow$
($\exists$ *X Y*. *X* $\neq$ {#} $\wedge$ *X* $\subseteq$# *N* $\wedge$ *M* = (*N* − *X*) + *Y* $\wedge$ ($\forall$ *k*. *k* $\in$# *Y* $\longrightarrow$ ($\exists$ *a*. *a* $\in$# *X* $\wedge$ *k* < *a*)))

The Huet–Oppen ordering:

definition *less-multiset$_{HO}$* **where**
less-multiset$_{HO}$ M N $\longleftrightarrow$ *M* $\neq$ *N* $\wedge$ ($\forall$ *y*. *count N y* < *count M y* $\longrightarrow$ ($\exists$ *x*. *y* < *x* $\wedge$ *count M x* < *count N x*))

lemma *mult-imp-less-multiset$_{HO}$*:
(*M*, *N*) $\in$ *mult* {(*x*, *y*). *x* < *y*} $\Longrightarrow$ *less-multiset$_{HO}$ M N*
proof (*unfold mult-def*, *induct rule*: *trancl-induct*)
case (*base P*)
then show *?case*
by (*auto elim*!: *mult1-lessE simp add*: *count-eq-zero-iff less-multiset$_{HO}$-def split*: *if-splits dest*!: *Suc-lessD*)
next
case (*step N P*)
from *step*(*3*) **have** *M* $\neq$ *N* **and**
: $\bigwedge$*y*. *count N y* < *count M y* $\Longrightarrow$ ($\exists$ *x*>*y*. *count M x* < *count N x*)
by (*simp-all add*: *less-multiset$_{HO}$-def*)
from *step*(*2*) **obtain** *M0 a K* **where**
: *P* = *add-mset a M0 N* = *M0* + *K a* $\notin$# *K* $\bigwedge$*b*. *b* $\in$# *K* $\Longrightarrow$ *b* < *a*
by (*blast elim*: *mult1-lessE*)
from ‹*M* $\neq$ *N*› ** *(*1*,*2*,*3*) **have** *M* $\neq$ *P* **by** (*force dest*: *(*4*) *elim*!: *less-asym split*: *if-splits*)
moreover
{ assume *count P a* $\leq$ *count M a*
with ‹*a* $\notin$# *K*› **have** *count N a* < *count M a* **unfolding** *(*1*,*2*)
by (*auto simp add*: *not-in-iff*)
with ** **obtain** *z* **where** *z*: *z* > *a count M z* < *count N z*
by *blast*
with * **have** *count N z* $\leq$ *count P z*
by (*auto elim*: *less-asym intro*: *count-inI*)
with *z* **have** $\exists$ *z* > *a*. *count M z* < *count P z* **by** *auto*
} **note** *count-a* = *this*
{ fix *y*
assume *count-y*: *count P y* < *count M y*
have $\exists$ *x*>*y*. *count M x* < *count P x*
proof (*cases y* = *a*)
case *True*
with *count-y count-a* **show** *?thesis* **by** *auto*
next
case *False*
show *?thesis*
proof (*cases y* $\in$# *K*)
case *True*
with *(*4*) **have** *y* < *a* **by** *simp*
then show *?thesis* **by** (*cases count P a* $\leq$ *count M a*) (*auto dest*: *count-a*

intro: *less-trans*)
next
case *False*
with ‹$y \neq a$› **have** *count P y* = *count N y* **unfolding** ∗(*1*,*2*)
by (*simp add*: *not-in-iff*)
with *count-y* ∗∗ **obtain** *z* **where** *z*: $z > y$ *count M z* < *count N z* **by** *auto*
show *?thesis*
proof (*cases z* ∈# *K*)
case *True*
with ∗(*4*) **have** $z < a$ **by** *simp*
with *z*(*1*) **show** *?thesis*
by (*cases count P a* ≤ *count M a*) (*auto dest*!: *count-a intro*: *less-trans*)
next
case *False*
with ‹$a \notin\# K$› **have** *count N z* ≤ *count P z* **unfolding** ∗
by (*auto simp add*: *not-in-iff*)
with *z* **show** *?thesis* **by** *auto*
qed
qed
qed
}
ultimately show *?case* **unfolding** *less-multiset*$_{HO}$*-def* **by** *blast*
qed

lemma *less-multiset*$_{DM}$*-imp-mult*:
less-multiset$_{DM}$ *M N* ⟹ (*M*, *N*) ∈ *mult* {(*x*, *y*). *x* < *y*}
proof −
assume *less-multiset*$_{DM}$ *M N*
then obtain *X Y* **where**
$X \neq \{\#\}$ **and** $X \subseteq\# N$ **and** $M = N - X + Y$ **and** ∀ *k*. *k* ∈# *Y* ⟶ (∃ *a*. *a* ∈# *X* ∧ *k* < *a*)
unfolding *less-multiset*$_{DM}$*-def* **by** *blast*
then have (*N* − *X* + *Y*, *N* − *X* + *X*) ∈ *mult* {(*x*, *y*). *x* < *y*}
by (*intro one-step-implies-mult*) (*auto simp*: *Bex-def trans-def*)
with ‹$M = N - X + Y$› ‹$X \subseteq\# N$› **show** (*M*, *N*) ∈ *mult* {(*x*, *y*). *x* < *y*}
by (*metis subset-mset.diff-add*)
qed

lemma *less-multiset*$_{HO}$*-imp-less-multiset*$_{DM}$: *less-multiset*$_{HO}$ *M N* ⟹ *less-multiset*$_{DM}$ *M N*
unfolding *less-multiset*$_{DM}$*-def*
proof (*intro iffI exI conjI*)
assume *less-multiset*$_{HO}$ *M N*
then obtain *z* **where** *z*: *count M z* < *count N z*
unfolding *less-multiset*$_{HO}$*-def* **by** (*auto simp*: *multiset-eq-iff nat-neq-iff*)
define *X* **where** *X* = *N* − *M*
define *Y* **where** *Y* = *M* − *N*
from *z* **show** *X* ≠ {#} **unfolding** *X-def* **by** (*auto simp*: *multiset-eq-iff not-less-eq-eq Suc-le-eq*)

from *z* **show** $X \subseteq\# N$ **unfolding** *X-def* **by** *auto*
show $M = (N - X) + Y$ **unfolding** *X-def Y-def multiset-eq-iff count-union count-diff* **by** *force*
show $\forall k.\ k \in\# Y \longrightarrow (\exists a.\ a \in\# X \wedge k < a)$
proof (*intro allI impI*)
fix *k*
assume $k \in\# Y$
then have *count N k* < *count M k* **unfolding** *Y-def*
by (*auto simp add*: *in-diff-count*)
with ‹*less-multiset*$_{HO}$ *M N*› **obtain** *a* **where** $k < a$ **and** *count M a* < *count N a*
unfolding *less-multiset*$_{HO}$*-def* **by** *blast*
then show $\exists a.\ a \in\# X \wedge k < a$ **unfolding** *X-def*
by (*auto simp add*: *in-diff-count*)
qed
qed

lemma *mult-less-multiset*$_{DM}$: $(M, N) \in mult\ \{(x, y).\ x < y\} \longleftrightarrow$ *less-multiset*$_{DM}$ *M N*
by (*metis less-multiset*$_{DM}$*-imp-mult less-multiset*$_{HO}$*-imp-less-multiset*$_{DM}$ *mult-imp-less-multiset*$_{HO}$)

lemma *mult-less-multiset*$_{HO}$: $(M, N) \in mult\ \{(x, y).\ x < y\} \longleftrightarrow$ *less-multiset*$_{HO}$ *M N*
by (*metis less-multiset*$_{DM}$*-imp-mult less-multiset*$_{HO}$*-imp-less-multiset*$_{DM}$ *mult-imp-less-multiset*$_{HO}$)

lemmas *mult*$_{DM}$ = *mult-less-multiset*$_{DM}$[*unfolded less-multiset*$_{DM}$*-def*]
lemmas *mult*$_{HO}$ = *mult-less-multiset*$_{HO}$[*unfolded less-multiset*$_{HO}$*-def*]

end

lemma *less-multiset-less-multiset*$_{HO}$: $M < N \longleftrightarrow$ *less-multiset*$_{HO}$ *M N*
unfolding *less-multiset-def mult*$_{HO}$ *less-multiset*$_{HO}$*-def* **..**

lemmas *less-multiset*$_{DM}$ = *mult*$_{DM}$[*folded less-multiset-def*]
lemmas *less-multiset*$_{HO}$ = *mult*$_{HO}$[*folded less-multiset-def*]

lemma *subset-eq-imp-le-multiset*:
shows $M \subseteq\# N \Longrightarrow M \leq N$
unfolding *less-eq-multiset-def less-multiset*$_{HO}$
by (*simp add*: *less-le-not-le subseteq-mset-def*)

lemma *le-multiset-right-total*: $M <$ *add-mset x M*
unfolding *less-eq-multiset-def less-multiset*$_{HO}$ **by** *simp*

lemma *less-eq-multiset-empty-left*[*simp*]:
shows $\{\#\} \leq M$
by (*simp add*: *subset-eq-imp-le-multiset*)

lemma *ex-gt-imp-less-multiset*: $(\exists\, y.\ y \in\# N \wedge (\forall\, x.\ x \in\# M \longrightarrow x < y)) \Longrightarrow M < N$
unfolding *less-multiset*$_{HO}$
by (*metis count-eq-zero-iff count-greater-zero-iff less-le-not-le*)

lemma *less-eq-multiset-empty-right*[*simp*]: $M \neq \{\#\} \Longrightarrow \neg\ M \leq \{\#\}$
by (*metis less-eq-multiset-empty-left antisym*)

lemma *le-multiset-empty-left*[*simp*]: $M \neq \{\#\} \Longrightarrow \{\#\} < M$
by (*simp add*: *less-multiset*$_{HO}$)

lemma *le-multiset-empty-right*[*simp*]: $\neg\ M < \{\#\}$
using *subset-mset.le-zero-eq less-multiset*$_{DM}$ **by** *blast*

lemma *union-le-diff-plus*: $P \subseteq\# M \Longrightarrow N < P \Longrightarrow M - P + N < M$
by (*drule subset-mset.diff-add*[*symmetric*]) (*metis union-le-mono2*)

instantiation *multiset* :: (*preorder*) *ordered-ab-semigroup-monoid-add-imp-le*
begin

lemma *less-eq-multiset*$_{HO}$:
$M \leq N \longleftrightarrow (\forall\, y.\ count\ N\ y < count\ M\ y \longrightarrow (\exists\, x.\ y < x \wedge count\ M\ x < count\ N\ x))$
by (*auto simp*: *less-eq-multiset-def less-multiset*$_{HO}$)

instance by *standard* (*auto simp*: *less-eq-multiset*$_{HO}$)

lemma
fixes $M\ N$ :: *'a multiset*
shows
less-eq-multiset-plus-left: $N \leq (M + N)$ **and**
less-eq-multiset-plus-right: $M \leq (M + N)$
by *simp-all*

lemma
fixes $M\ N$ :: *'a multiset*
shows
le-multiset-plus-left-nonempty: $M \neq \{\#\} \Longrightarrow N < M + N$ **and**
le-multiset-plus-right-nonempty: $N \neq \{\#\} \Longrightarrow M < M + N$
by *simp-all*

end

lemma *all-lt-Max-imp-lt-mset*: $N \neq \{\#\} \Longrightarrow (\forall\, a \in\# M.\ a < Max\ (set\text{-}mset\ N)) \Longrightarrow M < N$
by (*meson Max-in*[*OF finite-set-mset*] *ex-gt-imp-less-multiset set-mset-eq-empty-iff*)

lemma *lt-imp-ex-count-lt*: $M < N \Longrightarrow \exists\, y.\ count\ M\ y < count\ N\ y$
by (*meson less-eq-multiset*$_{HO}$ *less-le-not-le*)

lemma *subset-imp-less-mset*: $A \subset\# B \Longrightarrow A < B$
by (*simp add*: *order.not-eq-order-implies-strict subset-eq-imp-le-multiset*)

lemma *image-mset-strict-mono*:
assumes
mono-f: $\forall\, x \in set\text{-}mset\ M.\ \forall\, y \in set\text{-}mset\ N.\ x < y \longrightarrow f\ x < f\ y$ **and**
less: $M < N$
shows *image-mset f M* $<$ *image-mset f N*
proof −
obtain $Y\ X$ **where**
y-nemp: $Y \neq \{\#\}$ **and** *y-sub-N*: $Y \subseteq\# N$ **and** *M-eq*: $M = N - Y + X$ **and**
ex-y: $\forall\, x.\ x \in\# X \longrightarrow (\exists\, y.\ y \in\# Y \wedge x < y)$
using *less*[*unfolded less-multiset*$_{DM}$] **by** *blast*

have *x-sub-M*: $X \subseteq\# M$
using *M-eq* **by** *simp*

let *?fY* = *image-mset f Y*
let *?fX* = *image-mset f X*

show *?thesis*
unfolding *less-multiset*$_{DM}$
proof (*intro exI conjI*)
show *image-mset f M* = *image-mset f N* − *?fY* + *?fX*
using *M-eq*[*THEN arg-cong*, *of image-mset f*] *y-sub-N*
by (*metis image-mset-Diff image-mset-union*)
next
obtain *y* **where** *y*: $\forall\, x.\ x \in\# X \longrightarrow y\ x \in\# Y \wedge x < y\ x$
using *ex-y* **by** *moura*

show $\forall\, fx.\ fx \in\# ?fX \longrightarrow (\exists\, fy.\ fy \in\# ?fY \wedge fx < fy)$
proof (*intro allI impI*)
fix *fx*
assume *fx* $\in\#$ *?fX*
then obtain *x* **where** *fx*: *fx* = *f x* **and** *x-in*: $x \in\# X$
by *auto*
hence *y-in*: $y\ x \in\# Y$ **and** *y-gt*: $x < y\ x$
using *y*[*rule-format*, *OF x-in*] **by** *blast*+
hence $f\ (y\ x) \in\# ?fY \wedge f\ x < f\ (y\ x)$
using *mono-f y-sub-N x-sub-M x-in*
by (*metis image-eqI in-image-mset mset-subset-eqD*)
thus $\exists\, fy.\ fy \in\# ?fY \wedge fx < fy$
unfolding *fx* **by** *auto*
qed
qed (*auto simp*: *y-nemp y-sub-N image-mset-subseteq-mono*)

qed

lemma *image-mset-mono*:
assumes
mono-f: $\forall x \in$ *set-mset* $M.\ \forall y \in$ *set-mset* $N.\ x < y \longrightarrow f\ x < f\ y$ **and**
less: $M \leq N$
shows *image-mset* $f\ M \leq$ *image-mset* $f\ N$
by (*metis eq-iff image-mset-strict-mono less less-imp-le mono-f order.not-eq-order-implies-strict*)

lemma *mset-lt-single-right-iff* [*simp*]: $M < \{\#y\#\} \longleftrightarrow (\forall x \in\# M.\ x < y)$ **for** y :: *'a*::*linorder*
proof (*rule iffI*)
assume *M-lt-y*: $M < \{\#y\#\}$
show $\forall x \in\# M.\ x < y$
proof
fix x
assume *x-in*: $x \in\# M$
hence *M*: $M - \{\#x\#\} + \{\#x\#\} = M$
by (*meson insert-DiffM2*)
hence $\neg\ \{\#x\#\} < \{\#y\#\} \Longrightarrow x < y$
using *x-in M-lt-y*
by (*metis diff-single-eq-union le-multiset-empty-left less-add-same-cancel2 mset-le-trans*)
also have $\neg\ \{\#y\#\} < M$
using *M-lt-y mset-le-not-sym* **by** *blast*
ultimately show $x < y$
by (*metis* (*no-types*) *Max-ge all-lt-Max-imp-lt-mset empty-iff finite-set-mset insertE*
less-le-trans linorder-less-linear mset-le-not-sym set-mset-add-mset-insert
set-mset-eq-empty-iff x-in)
qed
next
assume *y-max*: $\forall x \in\# M.\ x < y$
show $M < \{\#y\#\}$
by (*rule all-lt-Max-imp-lt-mset*) (*auto intro*!: *y-max*)
qed

lemma *mset-le-single-right-iff* [*simp*]:
$M \leq \{\#y\#\} \longleftrightarrow M = \{\#y\#\} \vee (\forall x \in\# M.\ x < y)$ **for** y :: *'a*::*linorder*
by (*meson less-eq-multiset-def mset-lt-single-right-iff*)

63.2 Simprocs

lemma *mset-le-add-iff1*:
$j \leq (i$::*nat*$) \Longrightarrow$ (*repeat-mset* $i\ u + m \leq$ *repeat-mset* $j\ u + n$) = (*repeat-mset* $(i-j)\ u + m \leq n$)
proof −
assume $j \leq i$
then have $j + (i - j) = i$

using *le-add-diff-inverse* **by** *blast*
then show *?thesis*
by (*metis* (*no-types*) *add-le-cancel-left left-add-mult-distrib-mset*)
qed

lemma *mset-le-add-iff2*:
$i \leq (j::nat) \Longrightarrow (repeat\text{-}mset\ i\ u + m \leq repeat\text{-}mset\ j\ u + n) = (m \leq repeat\text{-}mset\ (j-i)\ u + n)$
proof −
assume $i \leq j$
then have $i + (j - i) = j$
using *le-add-diff-inverse* **by** *blast*
then show *?thesis*
by (*metis* (*no-types*) *add-le-cancel-left left-add-mult-distrib-mset*)
qed

simproc-setup *msetless-cancel*
$((l::'a::preorder\ multiset) + m < n \mid (l::'a\ multiset) < m + n \mid$
add-mset a m $<$ *n* $\mid$ *m* $<$ *add-mset a n* $\mid$
replicate-mset p a $<$ *n* $\mid$ *m* $<$ *replicate-mset p a* $\mid$
repeat-mset p m $<$ *n* $\mid$ *m* $<$ *repeat-mset p n*) =
‹*fn phi => Cancel-Simprocs.less-cancel*›

simproc-setup *msetle-cancel*
$((l::'a::preorder\ multiset) + m \leq n \mid (l::'a\ multiset) \leq m + n \mid$
add-mset a m $\leq$ *n* $\mid$ *m* $\leq$ *add-mset a n* $\mid$
replicate-mset p a $\leq$ *n* $\mid$ *m* $\leq$ *replicate-mset p a* $\mid$
repeat-mset p m $\leq$ *n* $\mid$ *m* $\leq$ *repeat-mset p n*) =
‹*fn phi => Cancel-Simprocs.less-eq-cancel*›

63.3 Additional facts and instantiations

lemma *ex-gt-count-imp-le-multiset*:
$(\forall y :: 'a :: order.\ y \in\# M + N \longrightarrow y \leq x) \Longrightarrow count\ M\ x < count\ N\ x \Longrightarrow M < N$
unfolding $less\text{-}multiset_{HO}$
by (*metis count-greater-zero-iff le-imp-less-or-eq less-imp-not-less not-gr-zero union-iff*)

lemma *mset-lt-single-iff*[*iff*]: $\{\#x\#\} < \{\#y\#\} \longleftrightarrow x < y$
unfolding $less\text{-}multiset_{HO}$ **by** *simp*

lemma *mset-le-single-iff*[*iff*]: $\{\#x\#\} \leq \{\#y\#\} \longleftrightarrow x \leq y$ **for** *x y* :: *'a*::*order*
unfolding $less\text{-}eq\text{-}multiset_{HO}$ **by** *force*

instance *multiset* :: (*linorder*) *linordered-cancel-ab-semigroup-add*
by *standard* (*metis less-eq-multiset*$_{HO}$ *not-less-iff-gr-or-eq*)

lemma *less-eq-multiset-total*:
fixes *M N* :: *'a* :: *linorder multiset*

shows $\neg\ M \leq N \Longrightarrow N \leq M$
by *simp*

instantiation *multiset* :: (*wellorder*) *wellorder*
begin

lemma *wf-less-multiset*: *wf* {(*M* :: *'a multiset*, *N*). *M* < *N*}
unfolding *less-multiset-def* **by** (*auto intro*: *wf-mult wf*)

instance by *standard* (*metis less-multiset-def wf wf-def wf-mult*)

end

instantiation *multiset* :: (*preorder*) *order-bot*
begin

definition *bot-multiset* :: *'a multiset* **where** *bot-multiset* = {#}

instance by *standard* (*simp add*: *bot-multiset-def*)

end

instance *multiset* :: (*preorder*) *no-top*
proof *standard*
fix *x* :: *'a multiset*
obtain *a* :: *'a* **where** *True* **by** *simp*
have *x* < *x* + (*x* + {#*a*#})
by *simp*
then show ∃ *y*. *x* < *y*
by *blast*
qed

instance *multiset* :: (*preorder*) *ordered-cancel-comm-monoid-add*
by *standard*

instantiation *multiset* :: (*linorder*) *distrib-lattice*
begin

definition *inf-multiset* :: *'a multiset* ⇒ *'a multiset* ⇒ *'a multiset* **where**
inf-multiset A B = (*if A* < *B then A else B*)

definition *sup-multiset* :: *'a multiset* ⇒ *'a multiset* ⇒ *'a multiset* **where**
sup-multiset A B = (*if B* > *A then B else A*)

instance
by *intro-classes* (*auto simp*: *inf-multiset-def sup-multiset-def*)

end

end

64 Non-negative, non-positive integers and reals

theory *Nonpos-Ints*
imports *Complex-Main*
begin

64.1 Non-positive integers

The set of non-positive integers on a ring. (in analogy to the set of non-negative integers $\mathbb{N}$) This is useful e.g. for the Gamma function.

definition *nonpos-Ints* ($\mathbb{Z}_{\leq 0}$) **where** $\mathbb{Z}_{\leq 0} = \{of\text{-}int\ n\ |n.\ n \leq 0\}$

lemma *zero-in-nonpos-Ints* [*simp*,*intro*]: $0 \in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** (*auto intro*!: *exI*[*of - 0*::*int*])

lemma *neg-one-in-nonpos-Ints* [*simp*,*intro*]: $-1 \in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** (*auto intro*!: *exI*[*of - −1*::*int*])

lemma *neg-numeral-in-nonpos-Ints* [*simp*,*intro*]: $-numeral\ n \in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** (*auto intro*!: *exI*[*of - −numeral n*::*int*])

lemma *one-notin-nonpos-Ints* [*simp*]: $(1 :: 'a :: ring\text{-}char\text{-}0) \notin \mathbb{Z}_{\leq 0}$
by (*auto simp*: *nonpos-Ints-def*)

lemma *numeral-notin-nonpos-Ints* [*simp*]: $(numeral\ n :: 'a :: ring\text{-}char\text{-}0) \notin \mathbb{Z}_{\leq 0}$
by (*auto simp*: *nonpos-Ints-def*)

lemma *minus-of-nat-in-nonpos-Ints* [*simp*, *intro*]: $-\ of\text{-}nat\ n \in \mathbb{Z}_{\leq 0}$
proof −
have $-\ of\text{-}nat\ n = of\text{-}int\ (-int\ n)$ **by** *simp*
also have $-int\ n \leq 0$ **by** *simp*
hence $of\text{-}int\ (-int\ n) \in \mathbb{Z}_{\leq 0}$ **unfolding** *nonpos-Ints-def* **by** *blast*
finally show *?thesis* **.**
qed

lemma *of-nat-in-nonpos-Ints-iff*: $(of\text{-}nat\ n :: 'a :: \{ring\text{-}1,ring\text{-}char\text{-}0\}) \in \mathbb{Z}_{\leq 0} \longleftrightarrow n = 0$
proof
assume $(of\text{-}nat\ n :: 'a) \in \mathbb{Z}_{\leq 0}$
then obtain m **where** $of\text{-}nat\ n = (of\text{-}int\ m :: 'a)\ m \leq 0$ **by** (*auto simp*: *nonpos-Ints-def*)
hence $(of\text{-}int\ m :: 'a) = of\text{-}nat\ n$ **by** *simp*
also have $... = of\text{-}int\ (int\ n)$ **by** *simp*
finally have $m = int\ n$ **by** (*subst* (*asm*) *of-int-eq-iff*)
with ‹$m \leq 0$› **show** $n = 0$ **by** *auto*
qed *simp*

lemma *nonpos-Ints-of-int*: $n \leq 0 \Longrightarrow$ *of-int* $n \in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def* **by** *blast*

lemma *nonpos-IntsI*:
$x \in \mathbb{Z} \Longrightarrow x \leq 0 \Longrightarrow (x :: 'a :: \textit{linordered-idom}) \in \mathbb{Z}_{\leq 0}$
unfolding *nonpos-Ints-def Ints-def* **by** *auto*

lemma *nonpos-Ints-subset-Ints*: $\mathbb{Z}_{\leq 0} \subseteq \mathbb{Z}$
unfolding *nonpos-Ints-def Ints-def* **by** *blast*

lemma *nonpos-Ints-nonpos* [*dest*]: $x \in \mathbb{Z}_{\leq 0} \Longrightarrow x \leq (0 :: 'a :: \textit{linordered-idom})$
unfolding *nonpos-Ints-def* **by** *auto*

lemma *nonpos-Ints-Int* [*dest*]: $x \in \mathbb{Z}_{\leq 0} \Longrightarrow x \in \mathbb{Z}$
unfolding *nonpos-Ints-def Ints-def* **by** *blast*

lemma *nonpos-Ints-cases*:
assumes $x \in \mathbb{Z}_{\leq 0}$
obtains n **where** $x =$ *of-int* n $n \leq 0$
using *assms* **unfolding** *nonpos-Ints-def* **by** (*auto elim*!: *Ints-cases*)

lemma *nonpos-Ints-cases'*:
assumes $x \in \mathbb{Z}_{\leq 0}$
obtains n **where** $x = -$*of-nat* n
proof −
from *assms* **obtain** m **where** $x =$ *of-int* m **and** m: $m \leq 0$ **by** (*auto elim*!: *nonpos-Ints-cases*)
hence $x = -$ *of-int* $(-m)$ **by** *auto*
also from m **have** (*of-int* $(-m) :: 'a$) = *of-nat* (*nat* $(-m)$) **by** *simp-all*
finally show *?thesis* **by** (*rule that*)
qed

lemma *of-real-in-nonpos-Ints-iff*: (*of-real* $x :: 'a ::$ *real-algebra-1*) $\in \mathbb{Z}_{\leq 0} \longleftrightarrow x \in \mathbb{Z}_{\leq 0}$
proof
assume *of-real* $x \in (\mathbb{Z}_{\leq 0} :: 'a$ *set*)
then obtain n **where** (*of-real* $x :: 'a$) = *of-int* n $n \leq 0$ **by** (*erule nonpos-Ints-cases*)
note ‹*of-real* $x =$ *of-int* n›
also have *of-int* $n =$ *of-real* (*of-int* n) **by** (*rule of-real-of-int-eq* [*symmetric*])
finally have $x =$ *of-int* n **by** (*subst* (*asm*) *of-real-eq-iff*)
with ‹$n \leq 0$› **show** $x \in \mathbb{Z}_{\leq 0}$ **by** (*simp add*: *nonpos-Ints-of-int*)
qed (*auto elim*!: *nonpos-Ints-cases intro*!: *nonpos-Ints-of-int*)

lemma *nonpos-Ints-altdef*: $\mathbb{Z}_{\leq 0} = \{n \in \mathbb{Z}.\ (n :: 'a :: \textit{linordered-idom}) \leq 0\}$
by (*auto intro*!: *nonpos-IntsI elim*!: *nonpos-Ints-cases*)

lemma *uminus-in-Nats-iff*: $-x \in \mathbb{N} \longleftrightarrow x \in \mathbb{Z}_{\leq 0}$
proof

assume $-x \in \mathbb{N}$
then obtain n **where** $n \geq 0$ $-x = \textit{of-int } n$ **by** (*auto simp*: *Nats-altdef1*)
hence $-n \leq 0$ $x = \textit{of-int } (-n)$ **by** (*simp-all add*: *eq-commute minus-equation-iff*[*of* x])
thus $x \in \mathbb{Z}_{\leq 0}$ **unfolding** *nonpos-Ints-def* **by** *blast*
next
assume $x \in \mathbb{Z}_{\leq 0}$
then obtain n **where** $n \leq 0$ $x = \textit{of-int } n$ **by** (*auto simp*: *nonpos-Ints-def*)
hence $-n \geq 0$ $-x = \textit{of-int } (-n)$ **by** (*simp-all add*: *eq-commute minus-equation-iff*[*of* x])
thus $-x \in \mathbb{N}$ **unfolding** *Nats-altdef1* **by** *blast*
qed

lemma *uminus-in-nonpos-Ints-iff*: $-x \in \mathbb{Z}_{\leq 0} \longleftrightarrow x \in \mathbb{N}$
using *uminus-in-Nats-iff*[*of* $-x$] **by** *simp*

lemma *nonpos-Ints-mult*: $x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x * y \in \mathbb{N}$
using *Nats-mult*[*of* $-x$ $-y$] **by** (*simp add*: *uminus-in-Nats-iff*)

lemma *Nats-mult-nonpos-Ints*: $x \in \mathbb{N} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x * y \in \mathbb{Z}_{\leq 0}$
using *Nats-mult*[*of* x $-y$] **by** (*simp add*: *uminus-in-Nats-iff*)

lemma *nonpos-Ints-mult-Nats*:
$x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{N} \Longrightarrow x * y \in \mathbb{Z}_{\leq 0}$
using *Nats-mult*[*of* $-x$ y] **by** (*simp add*: *uminus-in-Nats-iff*)

lemma *nonpos-Ints-add*:
$x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x + y \in \mathbb{Z}_{\leq 0}$
using *Nats-add*[*of* $-x$ $-y$] *uminus-in-Nats-iff*[*of* $y+x$, *simplified minus-add*]
by (*simp add*: *uminus-in-Nats-iff add.commute*)

lemma *nonpos-Ints-diff-Nats*:
$x \in \mathbb{Z}_{\leq 0} \Longrightarrow y \in \mathbb{N} \Longrightarrow x - y \in \mathbb{Z}_{\leq 0}$
using *Nats-add*[*of* $-x$ y] *uminus-in-Nats-iff*[*of* $x-y$, *simplified minus-add*]
by (*simp add*: *uminus-in-Nats-iff add.commute*)

lemma *Nats-diff-nonpos-Ints*:
$x \in \mathbb{N} \Longrightarrow y \in \mathbb{Z}_{\leq 0} \Longrightarrow x - y \in \mathbb{N}$
using *Nats-add*[*of* x $-y$] **by** (*simp add*: *uminus-in-Nats-iff add.commute*)

lemma *plus-of-nat-eq-0-imp*: $z + \textit{of-nat } n = 0 \Longrightarrow z \in \mathbb{Z}_{\leq 0}$
proof $-$
assume $z + \textit{of-nat } n = 0$
hence *A*: $z = - \textit{of-nat } n$ **by** (*simp add*: *eq-neg-iff-add-eq-0*)
show $z \in \mathbb{Z}_{\leq 0}$ **by** (*subst A*) *simp*
qed

64.2 Non-negative reals

definition *nonneg-Reals* :: *'a::real-algebra-1 set* ($\mathbb{R}_{\geq 0}$)
where $\mathbb{R}_{\geq 0} = \{of\text{-}real\ r \mid r.\ r \geq 0\}$

lemma *nonneg-Reals-of-real-iff* [*simp*]: *of-real* $r \in \mathbb{R}_{\geq 0} \longleftrightarrow r \geq 0$
by (*force simp add*: *nonneg-Reals-def*)

lemma *nonneg-Reals-subset-Reals*: $\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$
unfolding *nonneg-Reals-def Reals-def* **by** *blast*

lemma *nonneg-Reals-Real* [*dest*]: $x \in \mathbb{R}_{\geq 0} \Longrightarrow x \in \mathbb{R}$
unfolding *nonneg-Reals-def Reals-def* **by** *blast*

lemma *nonneg-Reals-of-nat-I* [*simp*]: *of-nat* $n \in \mathbb{R}_{\geq 0}$
by (*metis nonneg-Reals-of-real-iff of-nat-0-le-iff of-real-of-nat-eq*)

lemma *nonneg-Reals-cases*:
assumes $x \in \mathbb{R}_{\geq 0}$
obtains r **where** $x =$ *of-real* r $r \geq 0$
using *assms* **unfolding** *nonneg-Reals-def* **by** (*auto elim*!: *Reals-cases*)

lemma *nonneg-Reals-zero-I* [*simp*]: $0 \in \mathbb{R}_{\geq 0}$
unfolding *nonneg-Reals-def* **by** *auto*

lemma *nonneg-Reals-one-I* [*simp*]: $1 \in \mathbb{R}_{\geq 0}$
by (*metis* (*mono-tags*, *lifting*) *nonneg-Reals-of-nat-I of-nat-1*)

lemma *nonneg-Reals-minus-one-I* [*simp*]: $-1 \notin \mathbb{R}_{\geq 0}$
by (*metis nonneg-Reals-of-real-iff le-minus-one-simps*(*3*) *of-real-1 of-real-def real-vector.scale-minus-left*)

lemma *nonneg-Reals-numeral-I* [*simp*]: *numeral* $w \in \mathbb{R}_{\geq 0}$
by (*metis* (*no-types*) *nonneg-Reals-of-nat-I of-nat-numeral*)

lemma *nonneg-Reals-minus-numeral-I* [*simp*]: $-$ *numeral* $w \notin \mathbb{R}_{\geq 0}$
using *nonneg-Reals-of-real-iff not-zero-le-neg-numeral* **by** *fastforce*

lemma *nonneg-Reals-add-I* [*simp*]: $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a + b \in \mathbb{R}_{\geq 0}$
apply (*simp add*: *nonneg-Reals-def*)
apply *clarify*
apply (*rename-tac r s*)
apply (*rule-tac x*=*r*+*s* **in** *exI*, *auto*)
done

lemma *nonneg-Reals-mult-I* [*simp*]: $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a * b \in \mathbb{R}_{\geq 0}$
unfolding *nonneg-Reals-def* **by** (*auto simp*: *of-real-def*)

lemma *nonneg-Reals-inverse-I* [*simp*]:
fixes a :: *'a::real-div-algebra*
shows $a \in \mathbb{R}_{\geq 0} \Longrightarrow$ *inverse* $a \in \mathbb{R}_{\geq 0}$

by (*simp add*: *nonneg-Reals-def image-iff*) (*metis inverse-nonnegative-iff-nonnegative of-real-inverse*)

lemma *nonneg-Reals-divide-I* [*simp*]:
fixes a :: *'a::real-div-algebra*
shows $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a\ /\ b \in \mathbb{R}_{\geq 0}$
by (*simp add*: *divide-inverse*)

lemma *nonneg-Reals-pow-I* [*simp*]: $a \in \mathbb{R}_{\geq 0} \Longrightarrow a\hat{}n \in \mathbb{R}_{\geq 0}$
by (*induction n*) *auto*

lemma *complex-nonneg-Reals-iff*: $z \in \mathbb{R}_{\geq 0} \longleftrightarrow Re\ z \geq 0 \wedge Im\ z = 0$
by (*auto simp*: *nonneg-Reals-def*) (*metis complex-of-real-def complex-surj*)

lemma *ii-not-nonneg-Reals* [*iff*]: $\mathrm{i} \notin \mathbb{R}_{\geq 0}$
by (*simp add*: *complex-nonneg-Reals-iff*)

64.3 Non-positive reals

definition *nonpos-Reals* :: *'a::real-algebra-1 set* ($\mathbb{R}_{\leq 0}$)
where $\mathbb{R}_{\leq 0} = \{of\text{-}real\ r \mid r.\ r \leq 0\}$

lemma *nonpos-Reals-of-real-iff* [*simp*]: *of-real* $r \in \mathbb{R}_{\leq 0} \longleftrightarrow r \leq 0$
by (*force simp add*: *nonpos-Reals-def*)

lemma *nonpos-Reals-subset-Reals*: $\mathbb{R}_{\leq 0} \subseteq \mathbb{R}$
unfolding *nonpos-Reals-def Reals-def* **by** *blast*

lemma *nonpos-Ints-subset-nonpos-Reals*: $\mathbb{Z}_{\leq 0} \subseteq \mathbb{R}_{\leq 0}$
by (*metis nonpos-Ints-cases nonpos-Ints-nonpos nonpos-Ints-of-int nonpos-Reals-of-real-iff of-real-of-int-eq subsetI*)

lemma *nonpos-Reals-of-nat-iff* [*simp*]: *of-nat* $n \in \mathbb{R}_{\leq 0} \longleftrightarrow n{=}0$
by (*metis nonpos-Reals-of-real-iff of-nat-le-0-iff of-real-of-nat-eq*)

lemma *nonpos-Reals-Real* [*dest*]: $x \in \mathbb{R}_{\leq 0} \Longrightarrow x \in \mathbb{R}$
unfolding *nonpos-Reals-def Reals-def* **by** *blast*

lemma *nonpos-Reals-cases*:
assumes $x \in \mathbb{R}_{\leq 0}$
obtains r **where** $x = of\text{-}real\ r\ r \leq 0$
using *assms* **unfolding** *nonpos-Reals-def* **by** (*auto elim*!: *Reals-cases*)

lemma *uminus-nonneg-Reals-iff* [*simp*]: $-x \in \mathbb{R}_{\geq 0} \longleftrightarrow x \in \mathbb{R}_{\leq 0}$
apply (*auto simp*: *nonpos-Reals-def nonneg-Reals-def*)
apply (*metis nonpos-Reals-of-real-iff minus-minus neg-le-0-iff-le of-real-minus*)
done

lemma *uminus-nonpos-Reals-iff* [*simp*]: $-x \in \mathbb{R}_{\leq 0} \longleftrightarrow x \in \mathbb{R}_{\geq 0}$

by (*metis* (*no-types*) *minus-minus uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-zero-I* [*simp*]: $0 \in \mathbb{R}_{\leq 0}$
unfolding *nonpos-Reals-def* **by** *force*

lemma *nonpos-Reals-one-I* [*simp*]: $1 \notin \mathbb{R}_{\leq 0}$
using *nonneg-Reals-minus-one-I uminus-nonneg-Reals-iff* **by** *blast*

lemma *nonpos-Reals-numeral-I* [*simp*]: *numeral* $w \notin \mathbb{R}_{\leq 0}$
using *nonneg-Reals-minus-numeral-I uminus-nonneg-Reals-iff* **by** *blast*

lemma *nonpos-Reals-add-I* [*simp*]: $[\![a \in \mathbb{R}_{\leq 0};\ b \in \mathbb{R}_{\leq 0}]\!] \Longrightarrow a + b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-add-I add-uminus-conv-diff minus-diff-eq minus-minus uminus-nonpos-Reals-iff*)

lemma *nonpos-Reals-mult-I1*: $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\leq 0}]\!] \Longrightarrow a * b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-mult-I mult-minus-right uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-mult-I2*: $[\![a \in \mathbb{R}_{\leq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a * b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-mult-I mult-minus-left uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-mult-of-nat-iff*:
fixes *a*:: *'a* :: *real-div-algebra* **shows** $a * \mathit{of\text{-}nat}\ n \in \mathbb{R}_{\leq 0} \longleftrightarrow a \in \mathbb{R}_{\leq 0} \vee n{=}0$
apply (*auto intro*: *nonpos-Reals-mult-I2*)
apply (*auto simp*: *nonpos-Reals-def*)
apply (*rule-tac x=r/n* **in** *exI*)
apply (*auto simp*: *field-split-simps*)
done

lemma *nonpos-Reals-inverse-I*:
fixes *a* :: *'a*::*real-div-algebra*
shows $a \in \mathbb{R}_{\leq 0} \Longrightarrow \mathit{inverse}\ a \in \mathbb{R}_{\leq 0}$
using *nonneg-Reals-inverse-I uminus-nonneg-Reals-iff* **by** *fastforce*

lemma *nonpos-Reals-divide-I1*:
fixes *a* :: *'a*::*real-div-algebra*
shows $[\![a \in \mathbb{R}_{\geq 0};\ b \in \mathbb{R}_{\leq 0}]\!] \Longrightarrow a\ /\ b \in \mathbb{R}_{\leq 0}$
by (*simp add*: *nonpos-Reals-inverse-I nonpos-Reals-mult-I1 divide-inverse*)

lemma *nonpos-Reals-divide-I2*:
fixes *a* :: *'a*::*real-div-algebra*
shows $[\![a \in \mathbb{R}_{\leq 0};\ b \in \mathbb{R}_{\geq 0}]\!] \Longrightarrow a\ /\ b \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-divide-I minus-divide-left uminus-nonneg-Reals-iff*)

lemma *nonpos-Reals-divide-of-nat-iff*:
fixes *a*:: *'a* :: *real-div-algebra* **shows** $a\ /\ \mathit{of\text{-}nat}\ n \in \mathbb{R}_{\leq 0} \longleftrightarrow a \in \mathbb{R}_{\leq 0} \vee n{=}0$
apply (*auto intro*: *nonpos-Reals-divide-I2*)
apply (*auto simp*: *nonpos-Reals-def*)
apply (*rule-tac x=r*n* **in** *exI*)

apply (*auto simp*: *field-split-simps mult-le-0-iff*)
done

lemma *nonpos-Reals-inverse-iff* [*simp*]:
fixes a :: $'a$::*real-div-algebra*
shows *inverse* $a \in \mathbb{R}_{\leq 0} \longleftrightarrow a \in \mathbb{R}_{\leq 0}$
using *nonpos-Reals-inverse-I* **by** *fastforce*

lemma *nonpos-Reals-pow-I*: $[\![a \in \mathbb{R}_{\leq 0}; \textit{odd } n]\!] \Longrightarrow a \hat{} n \in \mathbb{R}_{\leq 0}$
by (*metis nonneg-Reals-pow-I power-minus-odd uminus-nonneg-Reals-iff*)

lemma *complex-nonpos-Reals-iff*: $z \in \mathbb{R}_{\leq 0} \longleftrightarrow \textit{Re } z \leq 0 \wedge \textit{Im } z = 0$
using *complex-is-Real-iff* **by** (*force simp add*: *nonpos-Reals-def*)

lemma *ii-not-nonpos-Reals* [*iff*]: $\mathrm{i} \notin \mathbb{R}_{\leq 0}$
by (*simp add*: *complex-nonpos-Reals-iff*)

end

65 Numeral Syntax for Types

theory *Numeral-Type*
imports *Cardinality*
begin

65.1 Numeral Types

typedef *num0* = *UNIV* :: *nat set* **..**
typedef *num1* = *UNIV* :: *unit set* **..**

typedef $'a$ *bit0* = {*0* ..< *2* * *int CARD*($'a$::*finite*)}
proof
show *0* ∈ {*0* ..< *2* * *int CARD*($'a$)}
by *simp*
qed

typedef $'a$ *bit1* = {*0* ..< *1* + *2* * *int CARD*($'a$::*finite*)}
proof
show *0* ∈ {*0* ..< *1* + *2* * *int CARD*($'a$)}
by *simp*
qed

lemma *card-num0* [*simp*]: *CARD* (*num0*) = *0*
unfolding *type-definition.card* [*OF type-definition-num0*]
by *simp*

lemma *infinite-num0*: ¬ *finite* (*UNIV* :: *num0 set*)
using *card-num0*[*unfolded card-eq-0-iff*]
by *simp*

lemma *card-num1* [*simp*]: *CARD*(*num1*) = *1*
unfolding *type-definition.card* [*OF type-definition-num1*]
by (*simp only*: *card-UNIV-unit*)

lemma *card-bit0* [*simp*]: *CARD*('*a bit0*) = *2* ∗ *CARD*('*a*::*finite*)
unfolding *type-definition.card* [*OF type-definition-bit0*]
by *simp*

lemma *card-bit1* [*simp*]: *CARD*('*a bit1*) = *Suc* (*2* ∗ *CARD*('*a*::*finite*))
unfolding *type-definition.card* [*OF type-definition-bit1*]
by *simp*

65.2 *num1*

instance *num1* :: *finite*
proof
show *finite* (*UNIV*::*num1 set*)
unfolding *type-definition.univ* [*OF type-definition-num1*]
using *finite* **by** (*rule finite-imageI*)
qed

instantiation *num1* :: *CARD-1*
begin

instance
proof
show *CARD*(*num1*) = *1* **by** *auto*
qed

end

lemma *num1-eq-iff*: (*x*::*num1*) = (*y*::*num1*) ⟷ *True*
by (*induct x*, *induct y*) *simp*

instantiation *num1* :: {*comm-ring*,*comm-monoid-mult*,*numeral*}
begin

instance
by *standard* (*simp-all add*: *num1-eq-iff*)

end

lemma *num1-eqI*:
fixes *a*::*num1* **shows** *a* = *b*
by(*simp add*: *num1-eq-iff*)

lemma *num1-eq1* [*simp*]:
fixes *a*::*num1* **shows** *a* = *1*

by (*rule num1-eqI*)

lemma *forall-1*[*simp*]: $(\forall i{::}num1.\ P\ i) \longleftrightarrow P\ 1$
by (*metis* (*full-types*) *num1-eq-iff*)

lemma *ex-1*[*simp*]: $(\exists x{::}num1.\ P\ x) \longleftrightarrow P\ 1$
by *auto* (*metis* (*full-types*) *num1-eq-iff*)

instantiation *num1* :: *linorder* **begin**
definition $a < b \longleftrightarrow Rep\text{-}num1\ a < Rep\text{-}num1\ b$
definition $a \leq b \longleftrightarrow Rep\text{-}num1\ a \leq Rep\text{-}num1\ b$
instance
by *intro-classes* (*auto simp*: *less-eq-num1-def less-num1-def intro*: *num1-eqI*)
end

instance *num1* :: *wellorder*
by *intro-classes* (*auto simp*: *less-eq-num1-def less-num1-def*)

instance *bit0* :: (*finite*) *card2*
proof
show *finite* (*UNIV*::'*a bit0 set*)
unfolding *type-definition.univ* [*OF type-definition-bit0*]
by *simp*
show $2 \leq CARD('a\ bit0)$
by *simp*
qed

instance *bit1* :: (*finite*) *card2*
proof
show *finite* (*UNIV*::'*a bit1 set*)
unfolding *type-definition.univ* [*OF type-definition-bit1*]
by *simp*
show $2 \leq CARD('a\ bit1)$
by *simp*
qed

65.3 Locales for for modular arithmetic subtypes

locale *mod-type* =
fixes *n* :: *int*
and *Rep* :: '*a*::{*zero*,*one*,*plus*,*times*,*uminus*,*minus*} ⇒ *int*
and *Abs* :: *int* ⇒ '*a*::{*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
assumes *type*: *type-definition Rep Abs* {*0*..<*n*}
and *size1*: $1 < n$
and *zero-def*: $0 = Abs\ 0$
and *one-def*: $1 = Abs\ 1$
and *add-def*: $x + y = Abs\ ((Rep\ x + Rep\ y)\ mod\ n)$
and *mult-def*: $x * y = Abs\ ((Rep\ x * Rep\ y)\ mod\ n)$

and *diff-def*: $x - y = Abs\ ((Rep\ x - Rep\ y)\ mod\ n)$
and *minus-def*: $- x = Abs\ ((- Rep\ x)\ mod\ n)$
begin

lemma *size0*: $0 < n$
using *size1* **by** *simp*

lemmas *definitions* =
zero-def one-def add-def mult-def minus-def diff-def

lemma *Rep-less-n*: $Rep\ x < n$
by (*rule type-definition.Rep* [*OF type*, *simplified*, *THEN conjunct2*])

lemma *Rep-le-n*: $Rep\ x \leq n$
by (*rule Rep-less-n* [*THEN order-less-imp-le*])

lemma *Rep-inject-sym*: $x = y \longleftrightarrow Rep\ x = Rep\ y$
by (*rule type-definition.Rep-inject* [*OF type*, *symmetric*])

lemma *Rep-inverse*: $Abs\ (Rep\ x) = x$
by (*rule type-definition.Rep-inverse* [*OF type*])

lemma *Abs-inverse*: $m \in \{0..<n\} \Longrightarrow Rep\ (Abs\ m) = m$
by (*rule type-definition.Abs-inverse* [*OF type*])

lemma *Rep-Abs-mod*: $Rep\ (Abs\ (m\ mod\ n)) = m\ mod\ n$
by (*simp add*: *Abs-inverse pos-mod-conj* [*OF size0*])

lemma *Rep-Abs-0*: $Rep\ (Abs\ 0) = 0$
by (*simp add*: *Abs-inverse size0*)

lemma *Rep-0*: $Rep\ 0 = 0$
by (*simp add*: *zero-def Rep-Abs-0*)

lemma *Rep-Abs-1*: $Rep\ (Abs\ 1) = 1$
by (*simp add*: *Abs-inverse size1*)

lemma *Rep-1*: $Rep\ 1 = 1$
by (*simp add*: *one-def Rep-Abs-1*)

lemma *Rep-mod*: $Rep\ x\ mod\ n = Rep\ x$
apply (*rule-tac x*=*x* **in** *type-definition.Abs-cases* [*OF type*])
apply (*simp add*: *type-definition.Abs-inverse* [*OF type*])
done

lemmas *Rep-simps* =
Rep-inject-sym Rep-inverse Rep-Abs-mod Rep-mod Rep-Abs-0 Rep-Abs-1

lemma *comm-ring-1*: $OFCLASS('a, comm\text{-}ring\text{-}1\text{-}class)$

apply (*intro-classes*, *unfold definitions*)
apply (*simp-all add*: *Rep-simps mod-simps field-simps*)
done

end

locale *mod-ring* = *mod-type n Rep Abs*
  for *n* :: *int*
  and *Rep* :: *'a*::{*comm-ring-1*} ⇒ *int*
  and *Abs* :: *int* ⇒ *'a*::{*comm-ring-1*}
begin

lemma *of-nat-eq*: *of-nat k* = *Abs* (*int k mod n*)
apply (*induct k*)
apply (*simp add*: *zero-def*)
apply (*simp add*: *Rep-simps add-def one-def mod-simps ac-simps*)
done

lemma *of-int-eq*: *of-int z* = *Abs* (*z mod n*)
apply (*cases z rule*: *int-diff-cases*)
apply (*simp add*: *Rep-simps of-nat-eq diff-def mod-simps*)
done

lemma *Rep-numeral*:
  Rep (*numeral w*) = *numeral w mod n*
using *of-int-eq* [*of numeral w*]
by (*simp add*: *Rep-inject-sym Rep-Abs-mod*)

lemma *iszero-numeral*:
  iszero (*numeral w*::*'a*) ⟷ *numeral w mod n* = *0*
by (*simp add*: *Rep-inject-sym Rep-numeral Rep-0 iszero-def*)

lemma *cases*:
  assumes *1*: ⋀*z*. ⟦(*x*::*'a*) = *of-int z*; *0* ≤ *z*; *z* < *n*⟧ ⟹ *P*
  shows *P*
apply (*cases x rule*: *type-definition.Abs-cases* [*OF type*])
apply (*rule-tac z*=*y* **in** *1*)
apply (*simp-all add*: *of-int-eq*)
done

lemma *induct*:
  (⋀*z*. ⟦*0* ≤ *z*; *z* < *n*⟧ ⟹ *P* (*of-int z*)) ⟹ *P* (*x*::*'a*)
by (*cases x rule*: *cases*) *simp*

lemma *UNIV-eq*: (*UNIV* :: *'a set*) = *Abs* ' {*0*..<*n*}
  using *type type-definition.univ* **by** *blast*

lemma *CARD-eq*: *CARD*(*'a*) = *nat n*
proof −

have *CARD('a) = card (Abs ' {0..<n})*
by (*simp add: UNIV-eq*)
also have *inj-on Abs {0..<n}*
by (*metis Abs-inverse inj-onI*)
hence *card (Abs ' {0..<n}) = nat n*
using *size1* **by** (*subst card-image*) *auto*
finally show *?thesis* **.**
qed

lemma *CHAR-eq* [*simp*]: *CHAR('a) = CARD('a)*
proof (*rule CHAR-eqI*)
show *of-nat (CARD('a)) = (0 :: 'a)*
by (*simp add: CARD-eq of-nat-eq zero-def*)
next
fix *n* **assume** *of-nat n = (0 :: 'a)*
thus *CARD('a) dvd n*
by (*metis CARD-eq Rep-0 Rep-Abs-mod Rep-le-n mod-0-imp-dvd nat-dvd-iff of-nat-eq*)
qed

end

65.4 Ring class instances

Unfortunately *ring-1* instance is not possible for *num1*, since 0 and 1 are not distinct.

instantiation
bit0 **and** *bit1* :: (*finite*) {*zero*,*one*,*plus*,*times*,*uminus*,*minus*}
begin

definition *Abs-bit0' :: int ⇒ 'a bit0* **where**
Abs-bit0' x = Abs-bit0 (x mod int CARD('a bit0))

definition *Abs-bit1' :: int ⇒ 'a bit1* **where**
Abs-bit1' x = Abs-bit1 (x mod int CARD('a bit1))

definition *0 = Abs-bit0 0*
definition *1 = Abs-bit0 1*
definition *x + y = Abs-bit0' (Rep-bit0 x + Rep-bit0 y)*
definition *x * y = Abs-bit0' (Rep-bit0 x * Rep-bit0 y)*
definition *x − y = Abs-bit0' (Rep-bit0 x − Rep-bit0 y)*
definition *− x = Abs-bit0' (− Rep-bit0 x)*

definition *0 = Abs-bit1 0*
definition *1 = Abs-bit1 1*
definition *x + y = Abs-bit1' (Rep-bit1 x + Rep-bit1 y)*
definition *x * y = Abs-bit1' (Rep-bit1 x * Rep-bit1 y)*
definition *x − y = Abs-bit1' (Rep-bit1 x − Rep-bit1 y)*
definition *− x = Abs-bit1' (− Rep-bit1 x)*

instance ..

end

interpretation *bit0*:
  mod-type int CARD('a::finite bit0)
    Rep-bit0 :: 'a::finite bit0 ⇒ int
    Abs-bit0 :: int ⇒ 'a::finite bit0
apply (*rule mod-type.intro*)
apply (*simp add*: *type-definition-bit0*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit0-def*)
apply (*rule one-bit0-def*)
apply (*rule plus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule times-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule minus-bit0-def* [*unfolded Abs-bit0'-def*])
apply (*rule uminus-bit0-def* [*unfolded Abs-bit0'-def*])
done

interpretation *bit1*:
  mod-type int CARD('a::finite bit1)
    Rep-bit1 :: 'a::finite bit1 ⇒ int
    Abs-bit1 :: int ⇒ 'a::finite bit1
apply (*rule mod-type.intro*)
apply (*simp add*: *type-definition-bit1*)
apply (*rule one-less-int-card*)
apply (*rule zero-bit1-def*)
apply (*rule one-bit1-def*)
apply (*rule plus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule times-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule minus-bit1-def* [*unfolded Abs-bit1'-def*])
apply (*rule uminus-bit1-def* [*unfolded Abs-bit1'-def*])
done

instance *bit0* :: (*finite*) *comm-ring-1*
  by (*rule bit0.comm-ring-1*)

instance *bit1* :: (*finite*) *comm-ring-1*
  by (*rule bit1.comm-ring-1*)

interpretation *bit0*:
  mod-ring int CARD('a::finite bit0)
    Rep-bit0 :: 'a::finite bit0 ⇒ int
    Abs-bit0 :: int ⇒ 'a::finite bit0
  ..

interpretation *bit1*:
  mod-ring int CARD('a::finite bit1)

Rep-bit1 :: *'a*::*finite bit1* ⇒ *int*
Abs-bit1 :: *int* ⇒ *'a*::*finite bit1*

..

Set up cases, induction, and arithmetic

lemmas *bit0-cases* [*case-names of-int*, *cases type*: *bit0*] = *bit0.cases*
lemmas *bit1-cases* [*case-names of-int*, *cases type*: *bit1*] = *bit1.cases*

lemmas *bit0-induct* [*case-names of-int*, *induct type*: *bit0*] = *bit0.induct*
lemmas *bit1-induct* [*case-names of-int*, *induct type*: *bit1*] = *bit1.induct*

lemmas *bit0-iszero-numeral* [*simp*] = *bit0.iszero-numeral*
lemmas *bit1-iszero-numeral* [*simp*] = *bit1.iszero-numeral*

lemmas [*simp*] = *eq-numeral-iff-iszero* [**where** *'a*=*'a bit0*] **for** *dummy* :: *'a*::*finite*
lemmas [*simp*] = *eq-numeral-iff-iszero* [**where** *'a*=*'a bit1*] **for** *dummy* :: *'a*::*finite*

65.5 Order instances

instantiation *bit0* **and** *bit1* :: (*finite*) *linorder* **begin**
definition $a < b \longleftrightarrow$ *Rep-bit0 a* < *Rep-bit0 b*
definition $a \leq b \longleftrightarrow$ *Rep-bit0 a* ≤ *Rep-bit0 b*
definition $a < b \longleftrightarrow$ *Rep-bit1 a* < *Rep-bit1 b*
definition $a \leq b \longleftrightarrow$ *Rep-bit1 a* ≤ *Rep-bit1 b*

instance
by(*intro-classes*)
(*auto simp add*: *less-eq-bit0-def less-bit0-def less-eq-bit1-def less-bit1-def Rep-bit0-inject Rep-bit1-inject*)
end

lemma (**in** *preorder*) *tranclp-less*: (<) $^{++}$ = (<)
by(*auto simp add*: *fun-eq-iff intro*: *less-trans elim*: *tranclp.induct*)

instance *bit0* **and** *bit1* :: (*finite*) *wellorder*
proof −
have *wf* {(*x* :: *'a bit0*, *y*). *x* < *y*}
by(*auto simp add*: *trancl-def tranclp-less intro*!: *finite-acyclic-wf acyclicI*)
thus *OFCLASS*(*'a bit0*, *wellorder-class*)
by(*rule wf-wellorderI*) *intro-classes*
next
have *wf* {(*x* :: *'a bit1*, *y*). *x* < *y*}
by(*auto simp add*: *trancl-def tranclp-less intro*!: *finite-acyclic-wf acyclicI*)
thus *OFCLASS*(*'a bit1*, *wellorder-class*)
by(*rule wf-wellorderI*) *intro-classes*
qed

65.6 Code setup and type classes for code generation

Code setup for *num0* and *num1*

definition *Num0* :: *num0* **where** *Num0* = *Abs-num0 0*
code-datatype *Num0*

instantiation *num0* :: *equal* **begin**
definition *equal-num0* :: *num0* ⇒ *num0* ⇒ *bool*
where *equal-num0* = (=)
instance by *intro-classes* (*simp add*: *equal-num0-def*)
end

lemma *equal-num0-code* [*code*]:
equal-class.equal Num0 Num0 = *True*
by(*rule equal-refl*)

code-datatype *1* :: *num1*

instantiation *num1* :: *equal* **begin**
definition *equal-num1* :: *num1* ⇒ *num1* ⇒ *bool*
where *equal-num1* = (=)
instance by *intro-classes* (*simp add*: *equal-num1-def*)
end

lemma *equal-num1-code* [*code*]:
equal-class.equal (*1* :: *num1*) *1* = *True*
by(*rule equal-refl*)

instantiation *num1* :: *enum* **begin**
definition *enum-class.enum* = [*1* :: *num1*]
definition *enum-class.enum-all P* = *P* (*1* :: *num1*)
definition *enum-class.enum-ex P* = *P* (*1* :: *num1*)
instance
by *intro-classes*
(*auto simp add*: *enum-num1-def enum-all-num1-def enum-ex-num1-def num1-eq-iff Ball-def*)
end

instantiation *num0* **and** *num1* :: *card-UNIV* **begin**
definition *finite-UNIV* = *Phantom*(*num0*) *False*
definition *card-UNIV* = *Phantom*(*num0*) *0*
definition *finite-UNIV* = *Phantom*(*num1*) *True*
definition *card-UNIV* = *Phantom*(*num1*) *1*
instance
by *intro-classes*
(*simp-all add*: *finite-UNIV-num0-def card-UNIV-num0-def infinite-num0 finite-UNIV-num1-def card-UNIV-num1-def*)
end

Code setup for *'a bit0* and *'a bit1*

declare
bit0.Rep-inverse[*code abstype*]

bit0.Rep-0[*code abstract*]
bit0.Rep-1[*code abstract*]

lemma *Abs-bit0'-code* [*code abstract*]:
Rep-bit0 (*Abs-bit0'* *x* :: *'a* :: *finite bit0*) = *x mod int* (*CARD*(*'a bit0*))
by(*auto simp add*: *Abs-bit0'-def intro*!: *Abs-bit0-inverse*)

lemma *inj-on-Abs-bit0*:
inj-on (*Abs-bit0* :: *int* ⇒ *'a bit0*) {*0..<2* ∗ *int CARD*(*'a* :: *finite*)}
by(*auto intro*: *inj-onI simp add*: *Abs-bit0-inject*)

declare
bit1.Rep-inverse[*code abstype*]
bit1.Rep-0[*code abstract*]
bit1.Rep-1[*code abstract*]

lemma *Abs-bit1'-code* [*code abstract*]:
Rep-bit1 (*Abs-bit1'* *x* :: *'a* :: *finite bit1*) = *x mod int* (*CARD*(*'a bit1*))
by(*auto simp add*: *Abs-bit1'-def intro*!: *Abs-bit1-inverse*)

lemma *inj-on-Abs-bit1*:
inj-on (*Abs-bit1* :: *int* ⇒ *'a bit1*) {*0..<1* + *2* ∗ *int CARD*(*'a* :: *finite*)}
by(*auto intro*: *inj-onI simp add*: *Abs-bit1-inject*)

instantiation *bit0* **and** *bit1* :: (*finite*) *equal* **begin**

definition *equal-class.equal x y* ⟷ *Rep-bit0 x* = *Rep-bit0 y*
definition *equal-class.equal x y* ⟷ *Rep-bit1 x* = *Rep-bit1 y*

instance
by *intro-classes* (*simp-all add*: *equal-bit0-def equal-bit1-def Rep-bit0-inject Rep-bit1-inject*)

end

instantiation *bit0* :: (*finite*) *enum* **begin**
definition (*enum-class.enum* :: *'a bit0 list*) = *map* (*Abs-bit0'* ∘ *int*) (*upt 0* (*CARD*(*'a bit0*)))
definition *enum-class.enum-all P* = (∀ *b* :: *'a bit0* ∈ *set enum-class.enum*. *P b*)
definition *enum-class.enum-ex P* = (∃ *b* :: *'a bit0* ∈ *set enum-class.enum*. *P b*)

instance proof
show *distinct* (*enum-class.enum* :: *'a bit0 list*)
by (*simp add*: *enum-bit0-def distinct-map inj-on-def Abs-bit0'-def Abs-bit0-inject*)

let *?Abs* = *Abs-bit0* :: - ⇒ *'a bit0*
interpret *type-definition Rep-bit0 ?Abs* {*0..<2* ∗ *int CARD*(*'a*)}
by (*fact type-definition-bit0*)
have *UNIV* = *?Abs* ' {*0..<2* ∗ *int CARD*(*'a*)}
by (*simp add*: *Abs-image*)

also have ... = *?Abs* ' (*int* ' {*0..<2 * CARD('a)*})
by (*simp add*: *image-int-atLeastLessThan*)
also have ... = (*?Abs* ∘ *int*) ' {*0..<2 * CARD('a)*}
by (*simp add*: *image-image cong*: *image-cong*)
also have ... = *set enum-class.enum*
by (*simp add*: *enum-bit0-def Abs-bit0'-def cong*: *image-cong-simp*)
finally show *univ-eq*: (*UNIV* :: *'a bit0 set*) = *set enum-class.enum* **.**

fix *P* :: *'a bit0* ⇒ *bool*
show *enum-class.enum-all P* = *Ball UNIV P*
and *enum-class.enum-ex P* = *Bex UNIV P*
by(*simp-all add*: *enum-all-bit0-def enum-ex-bit0-def univ-eq*)
qed

end

instantiation *bit1* :: (*finite*) *enum* **begin**
definition (*enum-class.enum* :: *'a bit1 list*) = *map* (*Abs-bit1'* ∘ *int*) (*upt 0* (*CARD('a bit1*)))
definition *enum-class.enum-all P* = (∀ *b* :: *'a bit1* ∈ *set enum-class.enum*. *P b*)
definition *enum-class.enum-ex P* = (∃ *b* :: *'a bit1* ∈ *set enum-class.enum*. *P b*)

instance
proof(*intro-classes*)
show *distinct* (*enum-class.enum* :: *'a bit1 list*)
by(*simp only*: *Abs-bit1'-def zmod-int*[*symmetric*] *enum-bit1-def distinct-map Suc-eq-plus1 card-bit1 o-apply inj-on-def*)
(*clarsimp simp add*: *Abs-bit1-inject*)

let *?Abs* = *Abs-bit1* :: - ⇒ *'a bit1*
interpret *type-definition Rep-bit1 ?Abs* {*0..<1 + 2 * int CARD('a)*}
by (*fact type-definition-bit1*)
have *UNIV* = *?Abs* ' {*0..<1 + 2 * int CARD('a)*}
by (*simp add*: *Abs-image*)
also have ... = *?Abs* ' (*int* ' {*0..<1 + 2 * CARD('a)*})
by (*simp add*: *image-int-atLeastLessThan*)
also have ... = (*?Abs* ∘ *int*) ' {*0..<1 + 2 * CARD('a)*}
by (*simp add*: *image-image cong*: *image-cong*)
finally show *univ-eq*: (*UNIV* :: *'a bit1 set*) = *set enum-class.enum*
by (*simp only*: *enum-bit1-def set-map set-upt*) (*simp add*: *Abs-bit1'-def cong*: *image-cong-simp*)

fix *P* :: *'a bit1* ⇒ *bool*
show *enum-class.enum-all P* = *Ball UNIV P*
and *enum-class.enum-ex P* = *Bex UNIV P*
by(*simp-all add*: *enum-all-bit1-def enum-ex-bit1-def univ-eq*)
qed

end

instantiation *bit0* **and** *bit1* :: (*finite*) *finite-UNIV* **begin**
definition *finite-UNIV* = *Phantom*('*a bit0*) *True*
definition *finite-UNIV* = *Phantom*('*a bit1*) *True*
instance by *intro-classes* (*simp-all add*: *finite-UNIV-bit0-def finite-UNIV-bit1-def*)
end

instantiation *bit0* **and** *bit1* :: ({*finite*,*card-UNIV*}) *card-UNIV* **begin**
definition *card-UNIV* = *Phantom*('*a bit0*) (*2* ∗ *of-phantom* (*card-UNIV* :: '*a card-UNIV*))
definition *card-UNIV* = *Phantom*('*a bit1*) (*1* + *2* ∗ *of-phantom* (*card-UNIV* :: '*a card-UNIV*))
instance by *intro-classes* (*simp-all add*: *card-UNIV-bit0-def card-UNIV-bit1-def card-UNIV*)
end

65.7 Syntax

syntax
-NumeralType :: *num-token* => *type* (-)
-NumeralType0 :: *type* (*0*)
-NumeralType1 :: *type* (*1*)

translations
(*type*) *1* == (*type*) *num1*
(*type*) *0* == (*type*) *num0*

parse-translation ‹
let
fun mk-bintype n =
let
fun mk-bit 0 = *Syntax.const* ***type-syntax***‹*bit0*›
| *mk-bit 1* = *Syntax.const* ***type-syntax***‹*bit1*›;
fun bin-of n =
if n = *1 then Syntax.const* ***type-syntax***‹*num1*›
else if n = *0 then Syntax.const* ***type-syntax***‹*num0*›
else if n = ~*1 then raise TERM* (*negative type numeral*, [])
else
let val (*q*, *r*) = *Integer.div-mod n 2*;
in mk-bit r $ *bin-of q end*;
in bin-of n end;

fun numeral-tr [*Free* (*str*, -)] = *mk-bintype* (*the* (*Int.fromString str*))
| *numeral-tr ts* = *raise TERM* (*numeral-tr*, *ts*);

in [(***syntax-const***‹*-NumeralType*›, *K numeral-tr*)] *end*
›

print-translation ‹

```
  let
    fun int-of [] = 0
      | int-of (b :: bs) = b + 2 * int-of bs;

    fun bin-of (Const (type-syntax‹num0›, -)) = []
      | bin-of (Const (type-syntax‹num1›, -)) = [1]
      | bin-of (Const (type-syntax‹bit0›, -) $ bs) = 0 :: bin-of bs
      | bin-of (Const (type-syntax‹bit1›, -) $ bs) = 1 :: bin-of bs
      | bin-of t = raise TERM (bin-of, [t]);

    fun bit-tr' b [t] =
          let
            val rev-digs = b :: bin-of t handle TERM - => raise Match
            val i = int-of rev-digs;
            val num = string-of-int (abs i);
          in
            Syntax.const syntax-const‹-NumeralType› $ Syntax.free num
          end
      | bit-tr' b - = raise Match;
  in
   [(type-syntax‹bit0›, K (bit-tr' 0)),
    (type-syntax‹bit1›, K (bit-tr' 1))]
  end
›
```

65.8 Examples

lemma *CARD(0) = 0* **by** *simp*
lemma *CARD(17) = 17* **by** *simp*
lemma *CHAR(23) = 23* **by** *simp*
lemma *8 * 11 ^ 3 - 6 = (2::5)* **by** *simp*

end

66 ω-words

theory *Omega-Words-Fun*

imports *Infinite-Set*
begin

Note: This theory is based on Stefan Merz's work.

Automata recognize languages, which are sets of words. For the theory of ω-automata, we are mostly interested in ω-words, but it is sometimes useful to reason about finite words, too. We are modeling finite words as lists; this lets us benefit from the existing library. Other formalizations could be investigated, such as representing words as functions whose domains are initial intervals of the natural numbers.

66.1 Type declaration and elementary operations

We represent ω-words as functions from the natural numbers to the alphabet type. Other possible formalizations include a coinductive definition or a uniform encoding of finite and infinite words, as studied by Müller et al.

type-synonym
$'a\ word = nat \Rightarrow 'a$

We can prefix a finite word to an ω-word, and a way to obtain an ω-word from a finite, non-empty word is by ω-iteration.

definition
$conc :: ['a\ list, 'a\ word] \Rightarrow 'a\ word$ (**infixr** $\langle\frown\rangle$ *65*)
where $w \frown x == \lambda n.\ if\ n < length\ w\ then\ w!n\ else\ x\ (n - length\ w)$

definition
$iter :: 'a\ list \Rightarrow 'a\ word$ ($\langle(\text{-}^{\omega})\rangle$ [*1000*])
where $iter\ w == if\ w = []\ then\ undefined\ else\ (\lambda n.\ w!(n\ mod\ (length\ w)))$

lemma *conc-empty*[*simp*]: $[] \frown w = w$
unfolding *conc-def* **by** *auto*

lemma *conc-fst*[*simp*]: $n < length\ w \Longrightarrow (w \frown x)\ n = w!n$
by (*simp add*: *conc-def*)

lemma *conc-snd*[*simp*]: $\neg(n < length\ w) \Longrightarrow (w \frown x)\ n = x\ (n - length\ w)$
by (*simp add*: *conc-def*)

lemma *iter-nth* [*simp*]: $0 < length\ w \Longrightarrow w^{\omega}\ n = w!(n\ mod\ (length\ w))$
by (*simp add*: *iter-def*)

lemma *conc-conc*[*simp*]: $u \frown v \frown w = (u\ @\ v) \frown w$ (**is** *?lhs = ?rhs*)
proof
fix *n*
have *u*: $n < length\ u \Longrightarrow ?lhs\ n = ?rhs\ n$
by (*simp add*: *conc-def nth-append*)
have *v*: $[\![\ \neg(n < length\ u);\ n < length\ u + length\ v\ ]\!] \Longrightarrow ?lhs\ n = ?rhs\ n$
by (*simp add*: *conc-def nth-append*, *arith*)
have *w*: $\neg(n < length\ u + length\ v) \Longrightarrow ?lhs\ n = ?rhs\ n$
by (*simp add*: *conc-def nth-append*, *arith*)
from *u v w* **show** *?lhs n = ?rhs n* **by** *blast*
qed

lemma *range-conc*[*simp*]: $range\ (w_1 \frown w_2) = set\ w_1 \cup range\ w_2$
proof (*intro equalityI subsetI*)
fix *a*
assume $a \in range\ (w_1 \frown w_2)$
then obtain *i* **where** *1*: $a = (w_1 \frown w_2)\ i$ **by** *auto*
then show $a \in set\ w_1 \cup range\ w_2$
unfolding *1* **by** (*cases* $i < length\ w_1$) *simp-all*

next
fix *a*
assume *a*: $a \in set\ w_1 \cup range\ w_2$
then show $a \in range\ (w_1 \frown w_2)$
proof
assume $a \in set\ w_1$
then obtain *i* **where** *1*: $i < length\ w_1\ a = w_1\ !\ i$
using *in-set-conv-nth* **by** *metis*
show *?thesis*
proof
show $a = (w_1 \frown w_2)\ i$ **using** *1* **by** *auto*
show $i \in UNIV$ **by** *rule*
qed
next
assume $a \in range\ w_2$
then obtain *i* **where** *1*: $a = w_2\ i$ **by** *auto*
show *?thesis*
proof
show $a = (w_1 \frown w_2)\ (length\ w_1 + i)$ **using** *1* **by** *simp*
show $length\ w_1 + i \in UNIV$ **by** *rule*
qed
qed
qed

lemma *iter-unroll*: $0 < length\ w \Longrightarrow w^\omega = w \frown w^\omega$
by (*rule ext*) (*simp add*: *conc-def mod-geq*)

66.2 Subsequence, Prefix, and Suffix

definition *suffix* :: [*nat*, *'a word*] ⇒ *'a word*
where $suffix\ k\ x \equiv \lambda n.\ x\ (k+n)$

definition *subsequence* :: *'a word* ⇒ *nat* ⇒ *nat* ⇒ *'a list* (‹- [- → -]› *900*)
where $subsequence\ w\ i\ j \equiv map\ w\ [i..<j]$

abbreviation *prefix* :: *nat* ⇒ *'a word* ⇒ *'a list*
where $prefix\ n\ w \equiv subsequence\ w\ 0\ n$

lemma *suffix-nth* [*simp*]: $(suffix\ k\ x)\ n = x\ (k+n)$
by (*simp add*: *suffix-def*)

lemma *suffix-0* [*simp*]: $suffix\ 0\ x = x$
by (*simp add*: *suffix-def*)

lemma *suffix-suffix* [*simp*]: $suffix\ m\ (suffix\ k\ x) = suffix\ (k+m)\ x$
by (*rule ext*) (*simp add*: *suffix-def add.assoc*)

lemma *subsequence-append*: $prefix\ (i + j)\ w = prefix\ i\ w\ @\ (w\ [i \rightarrow i + j])$

unfolding *map-append*[*symmetric*] *upt-add-eq-append*[*OF le0*] *subsequence-def* **..**

lemma *subsequence-drop*[*simp*]: *drop i* (*w* [*j* → *k*]) = *w* [*j* + *i* → *k*]
by (*simp add*: *subsequence-def drop-map*)

lemma *subsequence-empty*[*simp*]: *w* [*i* → *j*] = [] ⟷ *j* ≤ *i*
by (*auto simp add*: *subsequence-def*)

lemma *subsequence-length*[*simp*]: *length* (*subsequence w i j*) = *j* − *i*
by (*simp add*: *subsequence-def*)

lemma *subsequence-nth*[*simp*]: *k* < *j* − *i* ⟹ (*w* [*i* → *j*]) ! *k* = *w* (*i* + *k*)
unfolding *subsequence-def*
by *auto*

lemma *subseq-to-zero*[*simp*]: *w*[*i*→*0*] = []
by *simp*

lemma *subseq-to-smaller*[*simp*]: *i*≥*j* ⟹ *w*[*i*→*j*] = []
by *simp*

lemma *subseq-to-Suc*[*simp*]: *i*≤*j* ⟹ *w* [*i* → *Suc j*] = *w* [*i* → *j*] @ [*w j*]
by (*auto simp*: *subsequence-def*)

lemma *subsequence-singleton*[*simp*]: *w* [*i* → *Suc i*] = [*w i*]
by (*auto simp*: *subsequence-def*)

lemma *subsequence-prefix-suffix*: *prefix* (*j* − *i*) (*suffix i w*) = *w* [*i* → *j*]
proof (*cases i* ≤ *j*)
case *True*
have *w* [*i* → *j*] = *map w* (*map* (λ*n*. *n* + *i*) [*0*..<*j* − *i*])
unfolding *map-add-upt subsequence-def*
using *le-add-diff-inverse2*[*OF True*] **by** *force*
also
have ... = *map* (λ*n*. *w* (*n* + *i*)) [*0*..<*j* − *i*]
unfolding *map-map comp-def* **by** *blast*
finally
show *?thesis*
unfolding *subsequence-def suffix-def add.commute*[*of i*] **by** *simp*
next
case *False*
then show *?thesis*
by (*simp add*: *subsequence-def*)
qed

lemma *prefix-suffix*: *x* = *prefix n x* ⌢ (*suffix n x*)
by (*rule ext*) (*simp add*: *subsequence-def conc-def*)

declare *prefix-suffix*[*symmetric*, *simp*]

lemma *word-split*: **obtains** v_1 v_2 **where** $v = v_1 \frown v_2$ *length* $v_1 = k$
proof
  show v = *prefix* k $v \frown$ *suffix* k v
    by (*rule prefix-suffix*)
  show *length* (*prefix* k v) = k
    by *simp*
qed

lemma *set-subsequence*[*simp*]: *set* (w[i→j]) = w`{i..<j}
  unfolding *subsequence-def* **by** *auto*

lemma *subsequence-take*[*simp*]: *take* i (w [j → k]) = w [j → *min* (j + i) k]
  by (*simp add*: *subsequence-def take-map min-def*)

lemma *subsequence-shift*[*simp*]: (*suffix* i w) [j → k] = w [i + j → i + k]
  by (*metis add-diff-cancel-left subsequence-prefix-suffix suffix-suffix*)

lemma *suffix-subseq-join*[*simp*]: $i \leq j \Longrightarrow v$ [i → j] $\frown$ *suffix* j v = *suffix* i v
  by (*metis* (*no-types*, *lifting*) *Nat.add-0-right le-add-diff-inverse prefix-suffix*
    subsequence-shift suffix-suffix)

lemma *prefix-conc-fst*[*simp*]:
  assumes $j \leq$ *length* w
  shows *prefix* j ($w \frown w'$) = *take* j w
proof −
  have $\forall\, i < j$. (*prefix* j ($w \frown w'$)) ! i = (*take* j w) ! i
    using *assms* **by** (*simp add*: *conc-fst subsequence-def*)
  thus *?thesis*
    by (*simp add*: *assms list-eq-iff-nth-eq min.absorb2*)
qed

lemma *prefix-conc-snd*[*simp*]:
  assumes $n \geq$ *length* u
  shows *prefix* n ($u \frown v$) = u @ *prefix* (n − *length* u) v
proof (*intro nth-equalityI*)
  show *length* (*prefix* n ($u \frown v$)) = *length* (u @ *prefix* (n − *length* u) v)
    using *assms* **by** *simp*
  fix i
  assume $i <$ *length* (*prefix* n ($u \frown v$))
  then show *prefix* n ($u \frown v$) ! i = (u @ *prefix* (n − *length* u) v) ! i
    by (*cases* $i <$ *length* u) (*auto simp*: *nth-append*)
qed

lemma *prefix-conc-length*[*simp*]: *prefix* (*length* w) ($w \frown w'$) = w
  by *simp*

lemma *suffix-conc-fst*[*simp*]:
assumes $n \leq$ *length* u
shows *suffix* n $(u \frown v)$ = *drop* n $u \frown v$
proof
show *suffix* n $(u \frown v)$ i = $(drop\ n\ u \frown v)$ i **for** i
using *assms* **by** (*cases* $n + i <$ *length* u) (*auto simp*: *algebra-simps*)
qed

lemma *suffix-conc-snd*[*simp*]:
assumes $n \geq$ *length* u
shows *suffix* n $(u \frown v)$ = *suffix* $(n -$ *length* $u)$ v
proof
show *suffix* n $(u \frown v)$ i = *suffix* $(n -$ *length* $u)$ v i **for** i
using *assms* **by** *simp*
qed

lemma *suffix-conc-length*[*simp*]: *suffix* (*length* w) $(w \frown w') = w'$
unfolding *conc-def* **by** *force*

lemma *concat-eq*[*iff*]:
assumes *length* v_1 = *length* v_2
shows $v_1 \frown u_1 = v_2 \frown u_2 \longleftrightarrow v_1 = v_2 \wedge u_1 = u_2$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
proof
assume *?lhs*
then have *1*: $(v_1 \frown u_1)$ i = $(v_2 \frown u_2)$ i **for** i **by** *auto*
show *?rhs*
proof (*intro conjI ext nth-equalityI*)
show *length* v_1 = *length* v_2 **by** (*rule assms*(*1*))
next
fix i
assume *2*: $i <$ *length* v_1
have *3*: $i <$ *length* v_2 **using** *assms*(*1*) *2* **by** *simp*
show v_1 ! i = v_2 ! i **using** *1*[*of i*] *2 3* **by** *simp*
next
show u_1 i = u_2 i **for** i
using *1*[*of length* v_1 + i] *assms*(*1*) **by** *simp*
qed
next
assume *?rhs*
then show *?lhs* **by** *simp*
qed

lemma *same-concat-eq*[*iff*]: $u \frown v = u \frown w \longleftrightarrow v = w$
by *simp*

lemma *comp-concat*[*simp*]: $f \circ u \frown v =$ *map* f $u \frown (f \circ v)$
proof

fix i
show $(f \circ u \frown v)\ i = (map\ f\ u \frown (f \circ v))\ i$
by $(cases\ i < length\ u)$ *simp-all*
qed

66.3 Prepending

primrec $build :: 'a \Rightarrow 'a\ word \Rightarrow 'a\ word$ (**infixr** ‹##› 65)
where $(a\ \#\#\ w)\ 0 = a \mid (a\ \#\#\ w)\ (Suc\ i) = w\ i$

lemma *build-eq*[*iff*]: $a_1\ \#\#\ w_1 = a_2\ \#\#\ w_2 \longleftrightarrow a_1 = a_2 \wedge w_1 = w_2$
proof
assume *1*: $a_1\ \#\#\ w_1 = a_2\ \#\#\ w_2$
have *2*: $(a_1\ \#\#\ w_1)\ i = (a_2\ \#\#\ w_2)\ i$ **for** i
using *1* **by** *auto*
show $a_1 = a_2 \wedge w_1 = w_2$
proof (*intro conjI ext*)
show $a_1 = a_2$
using *2*[*of 0*] **by** *simp*
show $w_1\ i = w_2\ i$ **for** i
using *2*[*of Suc i*] **by** *simp*
qed
next
assume *1*: $a_1 = a_2 \wedge w_1 = w_2$
show $a_1\ \#\#\ w_1 = a_2\ \#\#\ w_2$ **using** *1* **by** *simp*
qed

lemma *build-cons*[*simp*]: $(a\ \#\ u) \frown v = a\ \#\#\ u \frown v$
proof
fix i
show $((a\ \#\ u) \frown v)\ i = (a\ \#\#\ u \frown v)\ i$
proof (*cases i*)
case *0*
show *?thesis* **unfolding** *0* **by** *simp*
next
case (*Suc j*)
show *?thesis* **unfolding** *Suc* **by** (*cases j < length u*, *simp+*)
qed
qed

lemma *build-append*[*simp*]: $(w\ @\ a\ \#\ u) \frown v = w \frown a\ \#\#\ u \frown v$
unfolding *conc-conc*[*symmetric*] **by** *simp*

lemma *build-first*[*simp*]: $w\ 0\ \#\#\ suffix\ (Suc\ 0)\ w = w$
proof
show $(w\ 0\ \#\#\ suffix\ (Suc\ 0)\ w)\ i = w\ i$ **for** i
by (*cases i*) *simp-all*
qed

lemma *build-split*[*intro*]: $w = w\ 0\ \#\#\ suffix\ 1\ w$
by *simp*

lemma *build-range*[*simp*]: $range\ (a\ \#\#\ w) = insert\ a\ (range\ w)$
proof *safe*
show $(a\ \#\#\ w)\ i \notin range\ w \Longrightarrow (a\ \#\#\ w)\ i = a$ **for** i
by (*cases i*) *auto*
show $a \in range\ (a\ \#\#\ w)$
proof (*rule range-eqI*)
show $a = (a\ \#\#\ w)\ 0$ **by** *simp*
qed
show $w\ i \in range\ (a\ \#\#\ w)$ **for** i
proof (*rule range-eqI*)
show $w\ i = (a\ \#\#\ w)\ (Suc\ i)$ **by** *simp*
qed
qed

lemma *suffix-singleton-suffix*[*simp*]: $w\ i\ \#\#\ suffix\ (Suc\ i)\ w = suffix\ i\ w$
using *suffix-subseq-join*[*of i Suc i w*]
by *simp*

Find the first occurrence of a letter from a given set

lemma *word-first-split-set*:
assumes $A \cap range\ w \neq \{\}$
obtains $u\ a\ v$ **where** $w = u \frown [a] \frown v\ A \cap set\ u = \{\}\ a \in A$
proof −
define i **where** $i = (LEAST\ i.\ w\ i \in A)$
show *?thesis*
proof
show $w = prefix\ i\ w \frown [w\ i] \frown suffix\ (Suc\ i)\ w$
by *simp*
show $A \cap set\ (prefix\ i\ w) = \{\}$
apply *safe*
subgoal premises *prems* **for** a
proof −
from *prems* **obtain** k **where** *3*: $k < i\ w\ k = a$
by *auto*
have *4*: $w\ k \notin A$
using *not-less-Least 3*(*1*) **unfolding** *i-def* **.**
show *?thesis*
using *prems*(*1*) *3*(*2*) *4* **by** *auto*
qed
done
show $w\ i \in A$
using *LeastI assms*(*1*) **unfolding** *i-def* **by** *fast*
qed
qed

66.4 The limit set of an ω-word

The limit set (also called infinity set) of an ω-word is the set of letters that appear infinitely often in the word. This set plays an important role in defining acceptance conditions of ω-automata.

definition *limit* :: *'a word* ⇒ *'a set*
  where *limit x* ≡ {*a* . ∃ ∞*n* . *x n* = *a*}

lemma *limit-iff-frequent*: *a* ∈ *limit x* ⟷ (∃ ∞*n* . *x n* = *a*)
  by (*simp add*: *limit-def*)

The following is a different way to define the limit, using the reverse image, making the laws about reverse image applicable to the limit set. (Might want to change the definition above?)

lemma *limit-vimage*: (*a* ∈ *limit x*) = *infinite* (*x* −‘ {*a*})
  by (*simp add*: *limit-def Inf-many-def vimage-def*)

lemma *two-in-limit-iff*:
  ({*a*, *b*} ⊆ *limit x*) =
    ((∃ *n*. *x n* =*a*) ∧ (∀ *n*. *x n* = *a* ⟶ (∃ *m*>*n*. *x m* = *b*)) ∧ (∀ *m*. *x m* = *b* ⟶ (∃ *n*>*m*. *x n* = *a*)))
  (**is** *?lhs* = (*?r1* ∧ *?r2* ∧ *?r3*))
proof
  assume *lhs*: *?lhs*
  hence *1*: *?r1* **by** (*auto simp*: *limit-def elim*: *INFM-EX*)
  from *lhs* **have** ∀ *n*. ∃ *m*>*n*. *x m* = *b* **by** (*auto simp*: *limit-def INFM-nat*)
  hence *2*: *?r2* **by** *simp*
  from *lhs* **have** ∀ *m*. ∃ *n*>*m*. *x n* = *a* **by** (*auto simp*: *limit-def INFM-nat*)
  hence *3*: *?r3* **by** *simp*
  from *1 2 3* **show** *?r1* ∧ *?r2* ∧ *?r3* **by** *simp*
next
  assume *?r1* ∧ *?r2* ∧ *?r3*
  hence *1*: *?r1* **and** *2*: *?r2* **and** *3*: *?r3* **by** *simp*+
  have *infa*: ∀ *m*. ∃ *n*≥*m*. *x n* = *a*
  proof
    fix *m*
    show ∃ *n*≥*m*. *x n* = *a* (**is** *?A m*)
    proof (*induct m*)
      from *1* **show** *?A 0* **by** *simp*
    next
      fix *m*
      assume *ih*: *?A m*
      then obtain *n* **where** *n*: *n* ≥ *m x n* = *a* **by** *auto*
      with *2* **obtain** *k* **where** *k*: *k*>*n x k* = *b* **by** *auto*
      with *3* **obtain** *l* **where** *l*: *l*>*k x l* = *a* **by** *auto*
      from *n k l* **have** *l* ≥ *Suc m* **by** *auto*
      with *l* **show** *?A* (*Suc m*) **by** *auto*
    qed
  qed

hence *infa'*: $\exists_\infty n.\ x\ n = a$ **by** (*simp add*: *INFM-nat-le*)
have $\forall\, n.\ \exists\, m{>}n.\ x\ m = b$
proof
fix *n*
from *infa* **obtain** *k* **where** *k1*: $k{\geq}n$ **and** *k2*: $x\ k = a$ **by** *auto*
from *2 k2* **obtain** *l* **where** *l1*: $l{>}k$ **and** *l2*: $x\ l = b$ **by** *auto*
from *k1 l1* **have** $l > n$ **by** *auto*
with *l2* **show** $\exists\, m{>}n.\ x\ m = b$ **by** *auto*
qed
hence $\exists_\infty m.\ x\ m = b$ **by** (*simp add*: *INFM-nat*)
with *infa'* **show** *?lhs* **by** (*auto simp*: *limit-def*)
qed

For ω-words over a finite alphabet, the limit set is non-empty. Moreover, from some position onward, any such word contains only letters from its limit set.

lemma *limit-nonempty*:
assumes *fin*: *finite* (*range x*)
shows $\exists\, a.\ a \in \mathit{limit}\ x$
proof −
from *fin* **obtain** *a* **where** $a \in \mathit{range}\ x \wedge \mathit{infinite}\ (x -` \{a\})$
by (*rule inf-img-fin-domE*) *auto*
hence $a \in \mathit{limit}\ x$
by (*auto simp add*: *limit-vimage*)
thus *?thesis* **..**
qed

lemmas *limit-nonemptyE* = *limit-nonempty*[*THEN exE*]

lemma *limit-inter-INF*:
assumes *hyp*: $\mathit{limit}\ w \cap S \neq \{\}$
shows $\exists_\infty\ n.\ w\ n \in S$
proof −
from *hyp* **obtain** *x* **where** $\exists_\infty\ n.\ w\ n = x$ **and** $x \in S$
by (*auto simp add*: *limit-def*)
thus *?thesis*
by (*auto elim*: *INFM-mono*)
qed

The reverse implication is true only if S is finite.

lemma *INF-limit-inter*:
assumes *hyp*: $\exists_\infty\ n.\ w\ n \in\ S$
and *fin*: *finite* ($S \cap \mathit{range}\ w$)
shows $\exists\, a.\ a \in \mathit{limit}\ w \cap S$
proof (*rule ccontr*)
assume *contra*: $\neg(\exists\, a.\ a \in \mathit{limit}\ w \cap S)$
hence $\forall\, a{\in}S.\ \mathit{finite}\ \{n.\ w\ n = a\}$
by (*auto simp add*: *limit-def Inf-many-def*)
with *fin* **have** *finite* (*UN a*:$S \cap \mathit{range}\ w.\ \{n.\ w\ n = a\}$)

```
    by auto
  moreover
  have (UN a:S ∩ range w. {n. w n = a}) = {n. w n ∈ S}
    by auto
  moreover
  note hyp
  ultimately show False
    by (simp add: Inf-many-def)
qed

lemma fin-ex-inf-eq-limit: finite A ⟹ (∃ ∞i. w i ∈ A) ⟷ limit w ∩ A ≠ {}
  by (metis INF-limit-inter equals0D finite-Int limit-inter-INF)

lemma limit-in-range-suffix: limit x ⊆ range (suffix k x)
proof
  fix a
  assume a ∈ limit x
  then obtain l where
    kl: k < l and xl: x l = a
    by (auto simp add: limit-def INFM-nat)
  from kl obtain m where l = k+m
    by (auto simp add:  less-iff-Suc-add)
  with xl show a ∈ range (suffix k x)
    by auto
qed

lemma limit-in-range: limit r ⊆ range r
  using limit-in-range-suffix[of r 0] by simp

lemmas limit-in-range-suffixD = limit-in-range-suffix[THEN subsetD]

lemma limit-subset: limit f ⊆ f ‘ {n..}
  using limit-in-range-suffix[of f n] unfolding suffix-def by auto

theorem limit-is-suffix:
  assumes fin: finite (range x)
  shows ∃ k. limit x = range (suffix k x)
proof −
  have ∃ k. range (suffix k x) ⊆ limit x
  proof −
    — The set of letters that are not in the limit is certainly finite.
    from fin have finite (range x − limit x)
      by simp
    — Moreover, any such letter occurs only finitely often
    moreover
    have ∀ a ∈ range x − limit x. finite (x −‘ {a})
      by (auto simp add: limit-vimage)
    — Thus, there are only finitely many occurrences of such letters.
    ultimately have finite (UN a : range x − limit x. x −‘ {a})
```

by (*blast intro*: *finite-UN-I*)
— Therefore these occurrences are within some initial interval.
then obtain *k* **where** (*UN a* : *range x* − *limit x*. *x* −‘ {*a*}) ⊆ {..<*k*}
by (*blast dest*: *finite-nat-bounded*)
— This is just the bound we are looking for.
hence ∀ *m*. *k* ≤ *m* ⟶ *x m* ∈ *limit x*
by (*auto simp add*: *limit-vimage*)
hence *range* (*suffix k x*) ⊆ *limit x*
by *auto*
thus *?thesis* **..**
qed
then obtain *k* **where** *range* (*suffix k x*) ⊆ *limit x* **..**
with *limit-in-range-suffix*
have *limit x* = *range* (*suffix k x*)
by (*rule subset-antisym*)
thus *?thesis* **..**
qed

lemmas *limit-is-suffixE* = *limit-is-suffix*[*THEN exE*]

The limit set enjoys some simple algebraic laws with respect to concatenation, suffixes, iteration, and renaming.

theorem *limit-conc* [*simp*]: *limit* (*w* ⌢ *x*) = *limit x*
proof (*auto*)
fix *a* **assume** *a*: *a* ∈ *limit* (*w* ⌢ *x*)
have ∀ *m*. ∃ *n*. *m*<*n* ∧ *x n* = *a*
proof
fix *m*
from *a* **obtain** *n* **where** *m* + *length w* < *n* ∧ (*w* ⌢ *x*) *n* = *a*
by (*auto simp add*: *limit-def Inf-many-def infinite-nat-iff-unbounded*)
hence *m* < *n* − *length w* ∧ *x* (*n* − *length w*) = *a*
by (*auto simp add*: *conc-def*)
thus ∃ *n*. *m*<*n* ∧ *x n* = *a* **..**
qed
hence *infinite* {*n* . *x n* = *a*}
by (*simp add*: *infinite-nat-iff-unbounded*)
thus *a* ∈ *limit x*
by (*simp add*: *limit-def Inf-many-def*)
next
fix *a* **assume** *a*: *a* ∈ *limit x*
have ∀ *m*. *length w* < *m* ⟶ (∃ *n*. *m*<*n* ∧ (*w* ⌢ *x*) *n* = *a*)
proof (*clarify*)
fix *m*
assume *m*: *length w* < *m*
with *a* **obtain** *n* **where** *m* − *length w* < *n* ∧ *x n* = *a*
by (*auto simp add*: *limit-def Inf-many-def infinite-nat-iff-unbounded*)
with *m* **have** *m* < *n* + *length w* ∧ (*w* ⌢ *x*) (*n* + *length w*) = *a*
by (*simp add*: *conc-def*, *arith*)
thus ∃ *n*. *m*<*n* ∧ (*w* ⌢ *x*) *n* = *a* **..**

```
  qed
  hence infinite {n . (w ⌢ x) n = a}
    by (simp add: unbounded-k-infinite)
  thus a ∈ limit (w ⌢ x)
    by (simp add: limit-def Inf-many-def)
qed

theorem limit-suffix [simp]: limit (suffix n x) = limit x
proof −
  have x = (prefix n x) ⌢ (suffix n x)
    by (simp add: prefix-suffix)
  hence limit x = limit (prefix n x ⌢ suffix n x)
    by simp
  also have ... = limit (suffix n x)
    by (rule limit-conc)
  finally show ?thesis
    by (rule sym)
qed

theorem limit-iter [simp]:
  assumes nempty: 0 < length w
  shows limit w^ω = set w
proof
  have limit w^ω ⊆ range w^ω
    by (auto simp add: limit-def dest: INFM-EX)
  also from nempty have ... ⊆ set w
    by auto
  finally show limit w^ω ⊆ set w .
next
  {
    fix a assume a: a ∈ set w
    then obtain k where k: k < length w ∧ w!k = a
      by (auto simp add: set-conv-nth)
    — the following bound is terrible, but it simplifies the proof
    from nempty k have ∀ m. w^ω ((Suc m)*(length w) + k) = a
      by (simp add: mod-add-left-eq [symmetric])
    moreover
    — why is the following so hard to prove??
    have ∀ m. m < (Suc m)*(length w) + k
    proof
      fix m
      from nempty have 1 ≤ length w by arith
      hence m*1 ≤ m*length w by simp
      hence m ≤ m*length w by simp
      with nempty have m < length w + (m*length w) + k by arith
      thus m < (Suc m)*(length w) + k by simp
    qed
    moreover note nempty
    ultimately have a ∈ limit w^ω
```

```
    by (auto simp add: limit-iff-frequent INFM-nat)
  }
  then show set w ⊆ limit w^ω by auto
qed

lemma limit-o [simp]:
  assumes a: a ∈ limit w
  shows f a ∈ limit (f ∘ w)
proof −
  from a
  have ∃_∞n. w n = a
    by (simp add: limit-iff-frequent)
  hence ∃_∞n. f (w n) = f a
    by (rule INFM-mono, simp)
  thus f a ∈ limit (f ∘ w)
    by (simp add: limit-iff-frequent)
qed
```

The converse relation is not true in general: $f(a)$ can be in the limit of $f \circ w$ even though a is not in the limit of w. However, *limit* commutes with renaming if the function is injective. More generally, if $f(a)$ is the image of only finitely many elements, some of these must be in the limit of w.

```
lemma limit-o-inv:
  assumes fin: finite (f −‘ {x})
    and x: x ∈ limit (f ∘ w)
  shows ∃ a ∈ (f −‘ {x}). a ∈ limit w
proof (rule ccontr)
  assume contra: ¬ ?thesis
  — hence, every element in the pre-image occurs only finitely often
  then have ∀ a ∈ (f −‘ {x}). finite {n. w n = a}
    by (simp add: limit-def Inf-many-def)
  — so there are only finitely many occurrences of any such element
  with fin have finite (⋃ a ∈ (f −‘ {x}). {n. w n = a})
    by auto
  — these are precisely those positions where x occurs in f ∘ w
  moreover
  have (⋃ a ∈ (f −‘ {x}). {n. w n = a}) = {n. f(w n) = x}
    by auto
  ultimately
  — so x can occur only finitely often in the translated word
  have finite {n. f(w n) = x}
    by simp
  — ... which yields a contradiction
  with x show False
    by (simp add: limit-def Inf-many-def)
qed

theorem limit-inj [simp]:
  assumes inj: inj f
```

```
  shows limit (f ∘ w) = f ‘ (limit w)
proof
  show f ‘ limit w ⊆ limit (f ∘ w)
    by auto
  show limit (f ∘ w) ⊆ f ‘ limit w
  proof
    fix x
    assume x: x ∈ limit (f ∘ w)
    from inj have finite (f −‘ {x})
      by (blast intro: finite-vimageI)
    with x obtain a where a: a ∈ (f −‘ {x}) ∧ a ∈ limit w
      by (blast dest: limit-o-inv)
    thus x ∈ f ‘ (limit w)
      by auto
  qed
qed

lemma limit-inter-empty:
  assumes fin: finite (range w)
  assumes hyp: limit w ∩ S = {}
  shows ∀∞n. w n ∉ S
proof −
  from fin obtain k where k-def: limit w = range (suffix k w)
    using limit-is-suffix by blast
  have w (k + k') ∉ S for k'
    using hyp unfolding k-def suffix-def image-def by blast
  thus ?thesis
    unfolding MOST-nat-le using le-Suc-ex by blast
qed
```

If the limit is the suffix of the sequence's range, we may increase the suffix index arbitrarily

```
lemma limit-range-suffix-incr:
  assumes limit r = range (suffix i r)
  assumes j≥i
  shows limit r = range (suffix j r)
    (is ?lhs = ?rhs)
proof −
  have ?lhs = range (suffix i r)
    using assms by simp
  moreover
  have ... ⊇ ?rhs using ‹j≥i›
    by (metis (mono-tags, lifting) assms(2)
        image-subsetI le-Suc-ex range-eqI suffix-def suffix-suffix)
  moreover
  have ... ⊇ ?lhs by (rule limit-in-range-suffix)
  ultimately
  show ?lhs = ?rhs
    by (metis antisym-conv limit-in-range-suffix)
```

qed

For two finite sequences, we can find a common suffix index such that the limits can be represented as these suffixes' ranges.

lemma *common-range-limit*:
assumes *finite* (*range x*)
and *finite* (*range y*)
obtains *i* **where** *limit x* = *range* (*suffix i x*)
and *limit y* = *range* (*suffix i y*)
proof −
obtain *i j* **where** *1*: *limit x* = *range* (*suffix i x*)
and *2*: *limit y* = *range* (*suffix j y*)
using *assms limit-is-suffix* **by** *metis*
have *limit x* = *range* (*suffix* (*max i j*) *x*)
and *limit y* = *range* (*suffix* (*max i j*) *y*)
using *limit-range-suffix-incr*[*OF 1*] *limit-range-suffix-incr*[*OF 2*]
by *auto*
thus *?thesis*
using *that* **by** *metis*
qed

66.5 Index sequences and piecewise definitions

A word can be defined piecewise: given a sequence of words $w_0, w_1, \ldots$ and a strictly increasing sequence of integers $i_0, i_1, \ldots$ where $i_0 = 0$, a single word is obtained by concatenating subwords of the w_n as given by the integers: the resulting word is

$$(w_0)_{i_0} \ldots (w_0)_{i_1-1} (w_1)_{i_1} \ldots (w_1)_{i_2-1} \ldots$$

We prepare the field by proving some trivial facts about such sequences of indexes.

definition *idx-sequence* :: *nat word* ⇒ *bool*
where *idx-sequence idx* ≡ (*idx 0* = *0*) ∧ (∀ *n*. *idx n* < *idx* (*Suc n*))

lemma *idx-sequence-less*:
assumes *iseq*: *idx-sequence idx*
shows *idx n* < *idx* (*Suc*(*n*+*k*))
proof (*induct k*)
from *iseq* **show** *idx n* < *idx* (*Suc* (*n* + *0*))
by (*simp add*: *idx-sequence-def*)
next
fix *k*
assume *ih*: *idx n* < *idx* (*Suc*(*n*+*k*))
from *iseq* **have** *idx* (*Suc*(*n*+*k*)) < *idx* (*Suc*(*n* + *Suc k*))
by (*simp add*: *idx-sequence-def*)
with *ih* **show** *idx n* < *idx* (*Suc*(*n* + *Suc k*))
by (*rule less-trans*)

qed

lemma *idx-sequence-inj*:
assumes *iseq*: *idx-sequence idx*
and *eq*: *idx m* = *idx n*
shows *m* = *n*
proof (*cases m n rule*: *linorder-cases*)
case *greater*
then obtain *k* **where** *m* = *Suc*(*n*+*k*)
by (*auto simp add*: *less-iff-Suc-add*)
with *iseq* **have** *idx n* < *idx m*
by (*simp add*: *idx-sequence-less*)
with *eq* **show** *?thesis*
by *simp*
next
case *less*
then obtain *k* **where** *n* = *Suc*(*m*+*k*)
by (*auto simp add*: *less-iff-Suc-add*)
with *iseq* **have** *idx m* < *idx n*
by (*simp add*: *idx-sequence-less*)
with *eq* **show** *?thesis*
by *simp*
qed

lemma *idx-sequence-mono*:
assumes *iseq*: *idx-sequence idx*
and *m*: *m* ≤ *n*
shows *idx m* ≤ *idx n*
proof (*cases m*=*n*)
case *True*
thus *?thesis* **by** *simp*
next
case *False*
with *m* **have** *m* < *n* **by** *simp*
then obtain *k* **where** *n* = *Suc*(*m*+*k*)
by (*auto simp add*: *less-iff-Suc-add*)
with *iseq* **have** *idx m* < *idx n*
by (*simp add*: *idx-sequence-less*)
thus *?thesis* **by** *simp*
qed

Given an index sequence, every natural number is contained in the interval defined by two adjacent indexes, and in fact this interval is determined uniquely.

lemma *idx-sequence-idx*:
assumes *idx-sequence idx*
shows *idx k* ∈ {*idx k* ..< *idx* (*Suc k*)}
using *assms* **by** (*auto simp add*: *idx-sequence-def*)

lemma *idx-sequence-interval*:
assumes *iseq*: *idx-sequence idx*
shows $\exists k.\ n \in \{idx\ k\ ..<\ idx\ (Suc\ k)\ \}$
(**is** *?P n* **is** $\exists k.$ *?in n k*)
proof (*induct n*)
from *iseq* **have** *0 = idx 0*
by (*simp add*: *idx-sequence-def*)
moreover
from *iseq* **have** $idx\ 0 \in \{idx\ 0\ ..<\ idx\ (Suc\ 0)\ \}$
by (*rule idx-sequence-idx*)
ultimately
show *?P 0* **by** *auto*
next
fix *n*
assume *?P n*
then obtain *k* **where** *k*: *?in n k* **..**
show *?P* (*Suc n*)
proof (*cases Suc n* $<$ *idx* (*Suc k*))
case *True*
with *k* **have** *?in* (*Suc n*) *k*
by *simp*
thus *?thesis* **..**
next
case *False*
with *k* **have** *Suc n* $=$ *idx* (*Suc k*)
by *auto*
with *iseq* **have** *?in* (*Suc n*) (*Suc k*)
by (*simp add*: *idx-sequence-def*)
thus *?thesis* **..**
qed
qed

lemma *idx-sequence-interval-unique*:
assumes *iseq*: *idx-sequence idx*
and *k*: $n \in \{idx\ k\ ..<\ idx\ (Suc\ k)\}$
and *m*: $n \in \{idx\ m\ ..<\ idx\ (Suc\ m)\}$
shows $k = m$
proof (*cases k m rule*: *linorder-cases*)
case *less*
hence *Suc k* $\leq$ *m* **by** *simp*
with *iseq* **have** *idx* (*Suc k*) $\leq$ *idx m*
by (*rule idx-sequence-mono*)
with *m* **have** *idx* (*Suc k*) $\leq$ *n*
by *auto*
with *k* **have** *False*
by *simp*
thus *?thesis* **..**
next
case *greater*

hence *Suc m ≤ k* **by** *simp*
with *iseq* **have** *idx (Suc m) ≤ idx k*
by (*rule idx-sequence-mono*)
with *k* **have** *idx (Suc m) ≤ n*
by *auto*
with *m* **have** *False*
by *simp*
thus *?thesis* **..**
qed

lemma *idx-sequence-unique-interval*:
assumes *iseq*: *idx-sequence idx*
shows ∃! *k*. *n* ∈ {*idx k ..< idx (Suc k)* }
proof (*rule ex-ex1I*)
from *iseq* **show** ∃ *k*. *n* ∈ {*idx k ..< idx (Suc k)*}
by (*rule idx-sequence-interval*)
next
fix *k y*
assume *n* ∈ {*idx k..<idx (Suc k)*} **and** *n* ∈ {*idx y..<idx (Suc y)*}
with *iseq* **show** *k = y* **by** (*auto elim*: *idx-sequence-interval-unique*)
qed

Now we can define the piecewise construction of a word using an index sequence.

definition *merge* :: *'a word word ⇒ nat word ⇒ 'a word*
where *merge ws idx ≡ λn. let i = THE i. n* ∈ {*idx i ..< idx (Suc i)* } *in ws i n*

lemma *merge*:
assumes *idx*: *idx-sequence idx*
and *n*: *n* ∈ {*idx i ..< idx (Suc i)*}
shows *merge ws idx n = ws i n*
proof −
from *n* **have** (*THE k. n* ∈ {*idx k ..< idx (Suc k)* }) = *i*
by (*rule the-equality*[*OF - sym*[*OF idx-sequence-interval-unique*[*OF idx n*]]])
simp
thus *?thesis*
by (*simp add*: *merge-def Let-def*)
qed

lemma *merge0*:
assumes *idx*: *idx-sequence idx*
shows *merge ws idx 0 = ws 0 0*
proof (*rule merge*[*OF idx*])
from *idx* **have** *idx 0 < idx (Suc 0)*
unfolding *idx-sequence-def* **by** *blast*
with *idx* **show** *0* ∈ {*idx 0 ..< idx (Suc 0)*}
by (*simp add*: *idx-sequence-def*)
qed

lemma *merge-Suc*:
  assumes *idx*: *idx-sequence idx*
    and *n*: $n \in \{idx\ i\ ..<\ idx\ (Suc\ i)\}$
  shows *merge ws idx* (*Suc n*) = (*if Suc n* = *idx* (*Suc i*) *then ws* (*Suc i*) *else ws i*) (*Suc n*)
proof *auto*
  assume *eq*: *Suc n* = *idx* (*Suc i*)
  from *idx* **have** *idx* (*Suc i*) < *idx* (*Suc*(*Suc i*))
    unfolding *idx-sequence-def* **by** *blast*
  with *eq idx* **show** *merge ws idx* (*idx* (*Suc i*)) = *ws* (*Suc i*) (*idx* (*Suc i*))
    by (*simp add*: *merge*)
next
  assume *neq*: *Suc n* ≠ *idx* (*Suc i*)
  with *n* **have** *Suc n* ∈ {*idx i ..< idx* (*Suc i*) }
    by *auto*
  with *idx* **show** *merge ws idx* (*Suc n*) = *ws i* (*Suc n*)
    by (*rule merge*)
qed

end

67 Combinator syntax for generic, open state monads (single-threaded monads)

theory *Open-State-Syntax*
imports *Main*
begin

context
  includes *state-combinator-syntax*
begin

67.1 Motivation

The logic HOL has no notion of constructor classes, so it is not possible to model monads the Haskell way in full genericity in Isabelle/HOL.

However, this theory provides substantial support for a very common class of monads: *state monads* (or *single-threaded monads*, since a state is transformed single-threadedly).

To enter from the Haskell world, https://www.engr.mun.ca/~theo/Misc/haskell_and_monads.htm makes a good motivating start. Here we just sketch briefly how those monads enter the game of Isabelle/HOL.

67.2 State transformations and combinators

We classify functions operating on states into two categories:

transformations with type signature $\sigma \Rightarrow \sigma'$, transforming a state.

"yielding" transformations with type signature $\sigma \Rightarrow \alpha \times \sigma'$, "yielding" a side result while transforming a state.

queries with type signature $\sigma \Rightarrow \alpha$, computing a result dependent on a state.

By convention we write σ for types representing states and α, β, γ, ... for types representing side results. Type changes due to transformations are not excluded in our scenario.

We aim to assert that values of any state type σ are used in a single-threaded way: after application of a transformation on a value of type σ, the former value should not be used again. To achieve this, we use a set of monad combinators:

Given two transformations f and g, they may be directly composed using the $(\circ>)$ combinator, forming a forward composition: $(f \circ> g)\ s = f\ (g\ s)$.

After any yielding transformation, we bind the side result immediately using a lambda abstraction. This is the purpose of the $(\circ\rightarrow)$ combinator: $(f \circ\rightarrow (\lambda x.\ g))\ s = (let\ (x,\ s') = f\ s\ in\ g\ s')$.

For queries, the existing *Let* is appropriate.

Naturally, a computation may yield a side result by pairing it to the state from the left; we introduce the suggestive abbreviation *return* for this purpose.

The most crucial distinction to Haskell is that we do not need to introduce distinguished type constructors for different kinds of state. This has two consequences:

- The monad model does not state anything about the kind of state; the model for the state is completely orthogonal and may be specified completely independently.
- There is no distinguished type constructor encapsulating away the state transformation, i.e. transformations may be applied directly without using any lifting or providing and dropping units ("open monad").
- The type of states may change due to a transformation.

67.3 Monad laws

The common monadic laws hold and may also be used as normalization rules for monadic expressions:

lemmas *monad-simp = Pair-scomp scomp-Pair id-fcomp fcomp-id scomp-scomp scomp-fcomp fcomp-scomp fcomp-assoc*

Evaluation of monadic expressions by force:

lemmas *monad-collapse = monad-simp fcomp-apply scomp-apply split-beta*

end

67.4 Do-syntax

nonterminal *sdo-binds* **and** *sdo-bind*

syntax
-sdo-block :: *sdo-binds* ⇒ *'a* (*exec* {//(*2* -)//} [*12*] *62*)
-sdo-bind :: [*pttrn*, *'a*] ⇒ *sdo-bind* ((- ←/ -) *13*)
-sdo-let :: [*pttrn*, *'a*] ⇒ *sdo-bind* ((*2let* - =/ -) [*1000*, *13*] *13*)
-sdo-then :: *'a* ⇒ *sdo-bind* (- [*14*] *13*)
-sdo-final :: *'a* ⇒ *sdo-binds* (-)
-sdo-cons :: [*sdo-bind*, *sdo-binds*] ⇒ *sdo-binds* (-;//- [*13*, *12*] *12*)

syntax (*ASCII*)
-sdo-bind :: [*pttrn*, *'a*] ⇒ *sdo-bind* ((- <−/ -) *13*)

translations
-sdo-block (*-sdo-cons* (*-sdo-bind p t*) (*-sdo-final e*))
== *CONST scomp t* (*λp. e*)
-sdo-block (*-sdo-cons* (*-sdo-then t*) (*-sdo-final e*))
=> *CONST fcomp t e*
-sdo-final (*-sdo-block* (*-sdo-cons* (*-sdo-then t*) (*-sdo-final e*)))
<= *-sdo-final* (*CONST fcomp t e*)
-sdo-block (*-sdo-cons* (*-sdo-then t*) *e*)
<= *CONST fcomp t* (*-sdo-block e*)
-sdo-block (*-sdo-cons* (*-sdo-let p t*) *bs*)
== *let p = t in -sdo-block bs*
-sdo-block (*-sdo-cons b* (*-sdo-cons c cs*))
== *-sdo-block* (*-sdo-cons b* (*-sdo-final* (*-sdo-block* (*-sdo-cons c cs*))))
-sdo-cons (*-sdo-let p t*) (*-sdo-final s*)
== *-sdo-final* (*let p = t in s*)
-sdo-block (*-sdo-final e*) => *e*

For an example, see `~~/src/HOL/Proofs/Extraction/Higman_Extraction.thy`.

end

68 Canonical order on option type

theory *Option-ord*
imports *Main*
begin

unbundle *lattice-syntax*

instantiation *option* :: (*preorder*) *preorder*
begin

definition *less-eq-option* **where**
$x \leq y \longleftrightarrow$ (*case x of None* $\Rightarrow$ *True* | *Some x* $\Rightarrow$ (*case y of None* $\Rightarrow$ *False* | *Some y* $\Rightarrow$ $x \leq y$))

definition *less-option* **where**
$x < y \longleftrightarrow$ (*case y of None* $\Rightarrow$ *False* | *Some y* $\Rightarrow$ (*case x of None* $\Rightarrow$ *True* | *Some x* $\Rightarrow$ $x < y$))

lemma *less-eq-option-None* [*simp*]: *None* $\leq x$
by (*simp add*: *less-eq-option-def*)

lemma *less-eq-option-None-code* [*code*]: *None* $\leq x \longleftrightarrow$ *True*
by *simp*

lemma *less-eq-option-None-is-None*: $x \leq$ *None* $\Longrightarrow x =$ *None*
by (*cases x*) (*simp-all add*: *less-eq-option-def*)

lemma *less-eq-option-Some-None* [*simp*, *code*]: *Some x* $\leq$ *None* $\longleftrightarrow$ *False*
by (*simp add*: *less-eq-option-def*)

lemma *less-eq-option-Some* [*simp*, *code*]: *Some x* $\leq$ *Some y* $\longleftrightarrow x \leq y$
by (*simp add*: *less-eq-option-def*)

lemma *less-option-None* [*simp*, *code*]: $x <$ *None* $\longleftrightarrow$ *False*
by (*simp add*: *less-option-def*)

lemma *less-option-None-is-Some*: *None* $< x \Longrightarrow \exists z.\ x =$ *Some z*
by (*cases x*) (*simp-all add*: *less-option-def*)

lemma *less-option-None-Some* [*simp*]: *None* $<$ *Some x*
by (*simp add*: *less-option-def*)

lemma *less-option-None-Some-code* [*code*]: *None* $<$ *Some x* $\longleftrightarrow$ *True*
by *simp*

lemma *less-option-Some* [*simp*, *code*]: *Some x* $<$ *Some y* $\longleftrightarrow x < y$
by (*simp add*: *less-option-def*)

instance
by *standard*
(*auto simp add*: *less-eq-option-def less-option-def less-le-not-le*
elim: *order-trans split*: *option.splits*)

end

instance *option* :: (*order*) *order*

by *standard* (*auto simp add*: *less-eq-option-def less-option-def split*: *option.splits*)

instance *option* :: (*linorder*) *linorder*
by *standard* (*auto simp add*: *less-eq-option-def less-option-def split*: *option.splits*)

instantiation *option* :: (*order*) *order-bot*
begin

definition *bot-option* **where** $\bot = None$

instance
by *standard* (*simp add*: *bot-option-def*)

end

instantiation *option* :: (*order-top*) *order-top*
begin

definition *top-option* **where** $\top = Some\ \top$

instance
by *standard* (*simp add*: *top-option-def less-eq-option-def split*: *option.split*)

end

instance *option* :: (*wellorder*) *wellorder*
proof
fix P :: $'a\ option \Rightarrow bool$
fix z :: $'a\ option$
assume H: $\bigwedge x.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x$
have $P\ None$ **by** (*rule H*) *simp*
then have *P-Some* [*case-names Some*]: $P\ z$ **if** $\bigwedge x.\ z = Some\ x \Longrightarrow (P \circ Some)\ x$ **for** z
using ‹$P\ None$› *that* **by** (*cases z*) *simp-all*
show $P\ z$
proof (*cases z rule*: *P-Some*)
case (*Some w*)
show $(P \circ Some)\ w$
proof (*induct rule*: *less-induct*)
case (*less x*)
have P (*Some x*)
proof (*rule H*)
fix y :: $'a\ option$
assume $y < Some\ x$
show $P\ y$
proof (*cases y rule*: *P-Some*)
case (*Some v*)
with ‹$y < Some\ x$› **have** $v < x$ **by** *simp*
with *less* **show** $(P \circ Some)\ v$ **.**

```
      qed
    qed
    then show ?case by simp
  qed
 qed
qed

instantiation option :: (inf) inf
begin

definition inf-option where
  x ⊓ y = (case x of None ⇒ None | Some x ⇒ (case y of None ⇒ None | Some
y ⇒ Some (x ⊓ y)))

lemma inf-None-1 [simp, code]: None ⊓ y = None
  by (simp add: inf-option-def)

lemma inf-None-2 [simp, code]: x ⊓ None = None
  by (cases x) (simp-all add: inf-option-def)

lemma inf-Some [simp, code]: Some x ⊓ Some y = Some (x ⊓ y)
  by (simp add: inf-option-def)

instance ..

end

instantiation option :: (sup) sup
begin

definition sup-option where
  x ⊔ y = (case x of None ⇒ y | Some x' ⇒ (case y of None ⇒ x | Some y ⇒
Some (x' ⊔ y)))

lemma sup-None-1 [simp, code]: None ⊔ y = y
  by (simp add: sup-option-def)

lemma sup-None-2 [simp, code]: x ⊔ None = x
  by (cases x) (simp-all add: sup-option-def)

lemma sup-Some [simp, code]: Some x ⊔ Some y = Some (x ⊔ y)
  by (simp add: sup-option-def)

instance ..

end

instance option :: (semilattice-inf) semilattice-inf
proof
```

fix $x\ y\ z :: 'a\ option$
show $x \sqcap y \leq x$
by (*cases x*, *simp-all*, *cases y*, *simp-all*)
show $x \sqcap y \leq y$
by (*cases x*, *simp-all*, *cases y*, *simp-all*)
show $x \leq y \Longrightarrow x \leq z \Longrightarrow x \leq y \sqcap z$
by (*cases x*, *simp-all*, *cases y*, *simp-all*, *cases z*, *simp-all*)
qed

instance *option* :: (*semilattice-sup*) *semilattice-sup*
proof
fix $x\ y\ z :: 'a\ option$
show $x \leq x \sqcup y$
by (*cases x*, *simp-all*, *cases y*, *simp-all*)
show $y \leq x \sqcup y$
by (*cases x*, *simp-all*, *cases y*, *simp-all*)
fix $x\ y\ z :: 'a\ option$
show $y \leq x \Longrightarrow z \leq x \Longrightarrow y \sqcup z \leq x$
by (*cases y*, *simp-all*, *cases z*, *simp-all*, *cases x*, *simp-all*)
qed

instance *option* :: (*lattice*) *lattice* **..**

instance *option* :: (*lattice*) *bounded-lattice-bot* **..**

instance *option* :: (*bounded-lattice-top*) *bounded-lattice-top* **..**

instance *option* :: (*bounded-lattice-top*) *bounded-lattice* **..**

instance *option* :: (*distrib-lattice*) *distrib-lattice*
proof
fix $x\ y\ z :: 'a\ option$
show $x \sqcup y \sqcap z = (x \sqcup y) \sqcap (x \sqcup z)$
by (*cases x*, *simp-all*, *cases y*, *simp-all*, *cases z*, *simp-all add*: *sup-inf-distrib1 inf-commute*)
qed

instantiation *option* :: (*complete-lattice*) *complete-lattice*
begin

definition *Inf-option* :: $'a\ option\ set \Rightarrow 'a\ option$ **where**
$\bigsqcap A = (\mathit{if}\ None \in A\ \mathit{then}\ None\ \mathit{else}\ Some\ (\bigsqcap Option.these\ A))$

lemma *None-in-Inf* [*simp*]: $None \in A \Longrightarrow \bigsqcap A = None$
by (*simp add*: *Inf-option-def*)

definition *Sup-option* :: $'a\ option\ set \Rightarrow 'a\ option$ **where**
$\bigsqcup A = (\mathit{if}\ A = \{\} \vee A = \{None\}\ \mathit{then}\ None\ \mathit{else}\ Some\ (\bigsqcup Option.these\ A))$

lemma *empty-Sup* [*simp*]: $\bigsqcup \{\} = None$
by (*simp add*: *Sup-option-def*)

lemma *singleton-None-Sup* [*simp*]: $\bigsqcup \{None\} = None$
by (*simp add*: *Sup-option-def*)

instance
proof
fix $x :: 'a\ option$ **and** A
assume $x \in A$
then show $\bigsqcap A \leq x$
by (*cases x*) (*auto simp add*: *Inf-option-def in-these-eq intro*: *Inf-lower*)
next
fix $z :: 'a\ option$ **and** A
assume $*$: $\bigwedge x.\ x \in A \Longrightarrow z \leq x$
show $z \leq \bigsqcap A$
proof (*cases z*)
case *None* **then show** *?thesis* **by** *simp*
next
case (*Some y*)
show *?thesis*
by (*auto simp add*: *Inf-option-def in-these-eq Some intro*!: *Inf-greatest dest*!:
$*$)
qed
next
fix $x :: 'a\ option$ **and** A
assume $x \in A$
then show $x \leq \bigsqcup A$
by (*cases x*) (*auto simp add*: *Sup-option-def in-these-eq intro*: *Sup-upper*)
next
fix $z :: 'a\ option$ **and** A
assume $*$: $\bigwedge x.\ x \in A \Longrightarrow x \leq z$
show $\bigsqcup A \leq z$
proof (*cases z*)
case *None*
with $*$ **have** $\bigwedge x.\ x \in A \Longrightarrow x = None$ **by** (*auto dest*: *less-eq-option-None-is-None*)
then have $A = \{\} \vee A = \{None\}$ **by** *blast*
then show *?thesis* **by** (*simp add*: *Sup-option-def*)
next
case (*Some y*)
from $*$ **have** $\bigwedge w.\ Some\ w \in A \Longrightarrow Some\ w \leq z$ **.**
with *Some* **have** $\bigwedge w.\ w \in Option.these\ A \Longrightarrow w \leq y$
by (*simp add*: *in-these-eq*)
then have $\bigsqcup Option.these\ A \leq y$ **by** (*rule Sup-least*)
with *Some* **show** *?thesis* **by** (*simp add*: *Sup-option-def*)
qed
next
show $\bigsqcup \{\} = (\bot :: 'a\ option)$
by (*auto simp*: *bot-option-def*)

show $\bigsqcap\{\} = (\top::'a\ option)$
by (*auto simp*: *top-option-def Inf-option-def*)
qed

end

lemma *Some-Inf*:
$Some\ (\bigsqcap A) = \bigsqcap(Some\ `\ A)$
by (*auto simp add*: *Inf-option-def*)

lemma *Some-Sup*:
$A \neq \{\} \Longrightarrow Some\ (\bigsqcup A) = \bigsqcup(Some\ `\ A)$
by (*auto simp add*: *Sup-option-def*)

lemma *Some-INF*:
$Some\ (\bigsqcap x{\in}A.\ f\ x) = (\bigsqcap x{\in}A.\ Some\ (f\ x))$
by (*simp add*: *Some-Inf image-comp*)

lemma *Some-SUP*:
$A \neq \{\} \Longrightarrow Some\ (\bigsqcup x{\in}A.\ f\ x) = (\bigsqcup x{\in}A.\ Some\ (f\ x))$
by (*simp add*: *Some-Sup image-comp*)

lemma *option-Inf-Sup*: $\bigsqcap(Sup\ `\ A) \leq \bigsqcup(Inf\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y\ \in\ Y\})$
for $A :: ('a::complete\text{-}distrib\text{-}lattice\ option)\ set\ set$
proof (*cases* $\{\} \in A$)
case *True*
then show *?thesis*
by (*rule INF-lower2*, *simp-all*)
next
case *False*
from *this* **have** *X*: $\{\} \notin A$
by *simp*
then show *?thesis*
proof (*cases* $\{None\} \in A$)
case *True*
then show *?thesis*
by (*rule INF-lower2*, *simp-all*)
next
case *False*

{fix *y*
assume *A*: $y \in A$
have $Sup\ (y - \{None\}) = Sup\ y$
by (*metis* (*no-types*, *lifting*) *Sup-option-def insert-Diff-single these-insert-None these-not-empty-eq*)
from *A* **and** *this* **have** $(\exists z.\ y - \{None\} = z - \{None\} \wedge z \in A) \wedge \bigsqcup y = \bigsqcup(y - \{None\})$
by *auto*
}

from *this* **have** *A*: *Sup* ‘ *A* = (*Sup* ‘ {*y* − {*None*} | *y*. *y*∈*A*})
by (*auto simp add*: *image-def*)

have [*simp*]: ⋀*y*. *y* ∈ *A* ⟹ ∃ *ya*. {*ya*. ∃ *x*. *x* ∈ *y* ∧ (∃ *y*. *x* = *Some y*) ∧ *ya* = *the x*}
= {*y*. ∃ *x*∈*ya* − {*None*}. *y* = *the x*} ∧ *ya* ∈ *A*
by (*rule exI*, *auto*)

have [*simp*]: ⋀*y*. *y* ∈ *A* ⟹
(∃ *ya*. *y* − {*None*} = *ya* − {*None*} ∧ *ya* ∈ *A*) ∧ ⋃{*ya*. ∃ *x*∈*y* − {*None*}. *ya* = *the x*}
= ⋃{*ya*. ∃ *x*. *x* ∈ *y* ∧ (∃ *y*. *x* = *Some y*) ∧ *ya* = *the x*}
apply (*safe*, *blast*)
by (*rule arg-cong* [*of* - - *Sup*], *auto*)
{fix *y*
assume [*simp*]: *y* ∈ *A*
have ∃ *x*. (∃ *y*. *x* = {*ya*. ∃ *x*∈*y* − {*None*}. *ya* = *the x*} ∧ *y* ∈ *A*) ∧ ⋃{*ya*. ∃ *x*. *x* ∈ *y* ∧ (∃ *y*. *x* = *Some y*) ∧ *ya* = *the x*} = ⋃ *x*
and ∃ *x*. (∃ *y*. *x* = *y* − {*None*} ∧ *y* ∈ *A*) ∧ ⋃{*ya*. ∃ *x*∈*y* − {*None*}. *ya* = *the x*} = ⋃{*y*. ∃ *xa*. *xa* ∈ *x* ∧ (∃ *y*. *xa* = *Some y*) ∧ *y* = *the xa*}
apply (*rule exI* [*of* - {*ya*. ∃ *x*. *x* ∈ *y* ∧ (∃ *y*. *x* = *Some y*) ∧ *ya* = *the x*}], *simp*)
by (*rule exI* [*of* - *y* − {*None*}], *simp*)
}
from *this* **have** *C*: (λ*x*. (⋃ *Option.these x*)) ‘ {*y* − {*None*} |*y*. *y* ∈ *A*} = (*Sup* ‘ {*the* ‘ (*y* − {*None*}) |*y*. *y* ∈ *A*})
by (*simp add*: *image-def Option.these-def*, *safe*, *simp-all*)

have *D*: ∀ *f* . ∃ *Y*∈*A*. *f Y* ∉ *Y* ⟹ *False*
by (*drule spec* [*of* - λ *Y* . *SOME x* . *x* ∈ *Y*], *simp add*: *X some-in-eq*)

define *F* **where** *F* = (λ *Y* . *SOME x*::*'a option* . *x* ∈ (*Y* − {*None*}))

have *G*: ⋀ *Y* . *Y* ∈ *A* ⟹ ∃ *x* . *x* ∈ *Y* − {*None*}
by (*metis False X all-not-in-conv insert-Diff-single these-insert-None these-not-empty-eq*)

have *F*: ⋀ *Y* . *Y* ∈ *A* ⟹ *F Y* ∈ (*Y* − {*None*})
by (*metis F-def G empty-iff some-in-eq*)

have *Some* ⊥ ≤ *Inf* (*F* ‘ *A*)
by (*metis* (*no-types*, *lifting*) *Diff-iff F Inf-option-def bot.extremum image-iff less-eq-option-Some singletonI*)

from *this* **have** *Inf* (*F* ‘ *A*) ≠ *None*
by (*cases* ⋂*x*∈*A*. *F x*, *simp-all*)

from *this* **have** *Inf* (*F* ‘ *A*) ≠ *None* ∧ *Inf* (*F* ‘ *A*) ∈ *Inf* ‘ {*f* ‘ *A* |*f*. ∀ *Y*∈*A*. *f Y* ∈ *Y*}
using *F* **by** *auto*

from *this* **have** $\exists\ x\ .\ x \neq None \wedge x \in Inf\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y\ \in\ Y\}$
by *blast*

from *this* **have** *E*: $Inf\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y\ \in\ Y\} = \{None\} \Longrightarrow False$
by *blast*

have [*simp*]: $((\bigsqcup x{\in}\{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y\ \in\ Y\}.\ \bigsqcap x) = None) = False$
by (*metis* (*no-types*, *lifting*) *E Sup-option-def* ‹$\exists x.\ x \neq None \wedge x \in Inf\ `\ \{f\ `\ A\ |f.\ \forall\ Y{\in}A.\ f\ Y\ \in\ Y\}$›
ex-in-conv option.simps(*3*))

have *B*: $Option.these\ ((\lambda x.\ Some\ (\bigsqcup Option.these\ x))\ `\ \{y\ -\ \{None\}\ |y.\ y\ \in\ A\})$
$= ((\lambda x.\ (\bigsqcup\ Option.these\ x))\ `\ \{y\ -\ \{None\}\ |y.\ y\ \in\ A\})$
by (*metis image-image these-image-Some-eq*)
{
fix *f*
assume *A*: $\bigwedge\ Y\ .\ (\exists\ y.\ Y = the\ `\ (y\ -\ \{None\}) \wedge y \in A) \Longrightarrow f\ Y \in Y$

have $\bigwedge xa.\ xa \in A \Longrightarrow f\ \{y.\ \exists\ a{\in}xa\ -\ \{None\}.\ y = the\ a\} = f\ (the\ `\ (xa\ -\ \{None\}))$
by (*simp add*: *image-def*)
from *this* **have** [*simp*]: $\bigwedge xa.\ xa \in A \Longrightarrow \exists\ x{\in}A.\ f\ \{y.\ \exists\ a{\in}xa\ -\ \{None\}.\ y = the\ a\} = f\ (the\ `\ (x\ -\ \{None\}))$
by *blast*
have $\bigwedge xa.\ xa \in A \Longrightarrow f\ (the\ `\ (xa\ -\ \{None\})) = f\ \{y.\ \exists\ a \in xa\ -\ \{None\}.\ y = the\ a\} \wedge xa \in A$
by (*simp add*: *image-def*)
from *this* **have** [*simp*]: $\bigwedge xa.\ xa \in A \Longrightarrow \exists\ x.\ f\ (the\ `\ (xa\ -\ \{None\})) = f\ \{y.\ \exists\ a{\in}x\ -\ \{None\}.\ y = the\ a\} \wedge x \in A$
by *blast*

{
fix *Y*
have $Y \in A \Longrightarrow Some\ (f\ (the\ `\ (Y\ -\ \{None\}))) \in Y$
using *A* [*of the* ‘ $(Y\ -\ \{None\})$] **apply** (*simp add*: *image-def*)
using *option.collapse* **by** *fastforce*
}
from *this* **have** [*simp*]: $\bigwedge\ Y\ .\ Y \in A \Longrightarrow Some\ (f\ (the\ `\ (Y\ -\ \{None\}))) \in Y$
by *blast*
have [*simp*]: $(\bigsqcap x{\in}A.\ Some\ (f\ \{y.\ \exists\ x{\in}x\ -\ \{None\}.\ y = the\ x\})) = \bigsqcap\{Some\ (f\ \{y.\ \exists\ a{\in}x\ -\ \{None\}.\ y = the\ a\})\ |x.\ x \in A\}$
by (*simp add*: *Setcompr-eq-image*)

have [*simp*]: $\exists x.\ (\exists f.\ x = \{y.\ \exists\ x{\in}A.\ y = f\ x\} \wedge (\forall\ Y{\in}A.\ f\ Y\ \in\ Y)) \wedge \bigsqcap\{Some\ (f\ \{y.\ \exists\ a{\in}x\ -\ \{None\}.\ y = the\ a\})\ |x.\ x \in A\} = \bigsqcap x$
apply (*rule exI* [*of* - $\{Some\ (f\ \{y.\ \exists\ a{\in}x\ -\ \{None\}.\ y = the\ a\})\ |\ x\ .\ x{\in}$

$A\}$], *safe*)
by (*rule exI* [*of* - (λ *Y* . *Some* (*f* (*the* ' (*Y* − {*None**}*)))))], *safe*, *simp-all*)

{
fix *xb*
have *xb* ∈ *A* ⟹ (⨅ *x*∈{{*ya*. ∃ *x*∈*y* − {*None*}. *ya* = *the x*} |*y*. *y* ∈ *A*}. *f x*) ≤ *f* {*y*. ∃ *x*∈*xb* − {*None*}. *y* = *the x*}
apply (*rule INF-lower2* [*of* {*y*. ∃ *x*∈*xb* − {*None*}. *y* = *the x*}])
by *blast*+
}
from *this* **have** [*simp*]: (⨅ *x*∈{*the* ' (*y* − {*None*}) |*y*. *y* ∈ *A*}. *f x*) ≤ *the* (⨅ *Y*∈*A*. *Some* (*f* (*the* ' (*Y* − {*None*}))))
apply (*simp add*: *Inf-option-def image-def Option.these-def*)
by (*rule Inf-greatest*, *clarsimp*)
have [*simp*]: *the* (⨅ *Y*∈*A*. *Some* (*f* (*the* ' (*Y* − {*None*})))) ∈ *Option.these* (*Inf* ' {*f* ' *A* |*f*. ∀ *Y*∈*A*. *f Y* ∈ *Y*})
apply (*auto simp add*: *Option.these-def*)
apply (*rule imageI*)
apply *auto*
using ‹⋀*Y*. *Y* ∈ *A* ⟹ *Some* (*f* (*the* ' (*Y* − {*None**}*))) ∈ *Y*› **apply** *blast*
apply (*auto simp add*: *Some-INF* [*symmetric*])
done
have (⨅ *x*∈{*the* ' (*y* − {*None*}) |*y*. *y* ∈ *A*}. *f x*) ≤ ⨆ *Option.these* (*Inf* ' {*f* ' *A* |*f*. ∀ *Y*∈*A*. *f Y* ∈ *Y*})
by (*rule Sup-upper2* [*of the* (*Inf* ((λ *Y* . *Some* (*f* (*the* ' (*Y* − {*None**}*))))) ' *A*))], *simp-all*)
}
from *this* **have** *X*: ⋀ *f* . ∀ *Y*. (∃ *y*. *Y* = *the* ' (*y* − {*None*}) ∧ *y* ∈ *A*) ⟶ *f Y* ∈ *Y* ⟹
(⨅ *x*∈{*the* ' (*y* − {*None*}) |*y*. *y* ∈ *A*}. *f x*) ≤ ⨆ *Option.these* (*Inf* ' {*f* ' *A* |*f*. ∀ *Y*∈*A*. *f Y* ∈ *Y*})
by *blast*

have [*simp*]: ⋀ *x* . *x*∈{*y* − {*None*} |*y*. *y* ∈ *A*} ⟹ *x* ≠ {} ∧ *x* ≠ {*None*}
using *F* **by** *fastforce*

have (*Inf* (*Sup* '*A*)) = (*Inf* (*Sup* ' {*y* − {*None*} | *y*. *y*∈*A*}))
by (*subst A*, *simp*)

also have ... = (⨅ *x*∈{*y* − {*None*} |*y*. *y* ∈ *A*}. *if x* = {} ∨ *x* = {*None*} *then None else Some* (⨆ *Option.these x*))
by (*simp add*: *Sup-option-def*)

also have ... = (⨅ *x*∈{*y* − {*None*} |*y*. *y* ∈ *A*}. *Some* (⨆ *Option.these x*))
using *G* **by** *fastforce*

also have ... = *Some* (⨅ *Option.these* ((λ*x*. *Some* (⨆ *Option.these x*)) ' {*y* − {*None*} |*y*. *y* ∈ *A*}))

by (*simp add*: *Inf-option-def*, *safe*)

also have ... = *Some* (⨅ ((λx. (⨆ *Option.these x*)) ‘ {y − {*None*} |y. y ∈ *A*}))
by (*simp add*: *B*)

also have ... = *Some* (*Inf* (*Sup* ‘ {*the* ‘ (y − {*None*}) |y. y ∈ *A*}))
by (*unfold C*, *simp*)
thm *Inf-Sup*
also have ... = *Some* (⨆x∈{f ‘ {*the* ‘ (y − {*None*}) |y. y ∈ *A*} |f. ∀ *Y*. (∃ y. *Y* = *the* ‘ (y − {*None*}) ∧ y ∈ *A*) ⟶ f *Y* ∈ *Y*}. ⨅ x)
by (*simp add*: *Inf-Sup*)

also have ... ≤ ⨆ (*Inf* ‘ {f ‘ *A* |f. ∀ *Y*∈*A*. f *Y* ∈ *Y*})
proof (*cases* ⨆ (*Inf* ‘ {f ‘ *A* |f. ∀ *Y*∈*A*. f *Y* ∈ *Y*}))
case *None*
then show *?thesis* **by** (*simp add*: *less-eq-option-def*)
next
case (*Some a*)
then show *?thesis*
apply *simp*
apply (*rule Sup-least*, *safe*)
apply (*simp add*: *Sup-option-def*)
apply (*cases* (∀ f. ∃ *Y*∈*A*. f *Y* ∉ *Y*) ∨ *Inf* ‘ {f ‘ *A* |f. ∀ *Y*∈*A*. f *Y* ∈ *Y*} = {*None*}, *simp-all*)
by (*drule X*, *simp*)
qed
finally show *?thesis* **by** *simp*
qed
qed

instance *option* :: (*complete-distrib-lattice*) *complete-distrib-lattice*
by (*standard*, *simp add*: *option-Inf-Sup*)

instance *option* :: (*complete-linorder*) *complete-linorder* **..**

unbundle *no-lattice-syntax*

end

69 Futures and parallel lists for code generated towards Isabelle/ML

theory *Parallel*
imports *Main*
begin

69.1 Futures

datatype *'a future = fork unit ⇒ 'a*

primrec *join* :: *'a future ⇒ 'a* **where**
join (*fork f*) = *f* ()

lemma *future-eqI* [*intro*!]:
assumes *join f* = *join g*
shows *f* = *g*
using *assms* **by** (*cases f*, *cases g*) (*simp add*: *ext*)

code-printing
type-constructor *future* ⇀ (*Eval*) - *future*
| **constant** *fork* ⇀ (*Eval*) *Future.fork*
| **constant** *join* ⇀ (*Eval*) *Future.join*

code-reserved *Eval Future future*

69.2 Parallel lists

definition *map* :: (*'a ⇒ 'b*) ⇒ *'a list ⇒ 'b list* **where**
[*simp*]: *map* = *List.map*

definition *forall* :: (*'a ⇒ bool*) ⇒ *'a list ⇒ bool* **where**
forall = *list-all*

lemma *forall-all* [*simp*]:
forall P xs ⟷ (∀ *x*∈*set xs*. *P x*)
by (*simp add*: *forall-def list-all-iff*)

definition *exists* :: (*'a ⇒ bool*) ⇒ *'a list ⇒ bool* **where**
exists = *list-ex*

lemma *exists-ex* [*simp*]:
exists P xs ⟷ (∃ *x*∈*set xs*. *P x*)
by (*simp add*: *exists-def list-ex-iff*)

code-printing
constant *map* ⇀ (*Eval*) *Par'-List.map*
| **constant** *forall* ⇀ (*Eval*) *Par'-List.forall*
| **constant** *exists* ⇀ (*Eval*) *Par'-List.exists*

code-reserved *Eval Par-List*

hide-const (**open**) *fork join map exists forall*

end

70 Input syntax for pattern aliases (or "as-patterns" in Haskell)

theory *Pattern-Aliases*
imports *Main*
begin

Most functional languages (Haskell, ML, Scala) support aliases in patterns. This allows to refer to a subpattern with a variable name. This theory implements this using a check phase. It works well for function definitions (see usage below). All features are packed into a **bundle**.

The following caveats should be kept in mind:

- The translation expects a term of the form $f\ x\ y = rhs$, where x and y are patterns that may contain aliases. The result of the translation is a nested *Let*-expression on the right hand side. The code generator *does not* print Isabelle pattern aliases to target language pattern aliases.
- The translation does not process nested equalities; only the top-level equality is translated.
- Terms that do not adhere to the above shape may either stay untranslated or produce an error message. The **fun** command will complain if pattern aliases are left untranslated. In particular, there are no checks whether the patterns are wellformed or linear.
- The corresponding uncheck phase attempts to reverse the translation (no guarantee). The additionally introduced variables are bound using a "fake quantifier" that does not appear in the output.
- To obtain reasonable induction principles in function definitions, the bundle also declares a custom congruence rule for *Let* that only affects **fun**. This congruence rule might lead to an explosion in term size (although that is rare)! In some circumstances (using *let* to destructure tuples), the internal construction of functions stumbles over this rule and prints an error. To mitigate this, either
  - activate the bundle locally (**context includes ... begin**) or
  - rewrite the *let*-expression to use *case*: *let* $(a,\ b) = x$ *in* $(b,\ a)$ becomes *case* x *of* $(a,\ b) \Rightarrow (b,\ a)$.
- The bundle also adds the *Let ?s ?f* $\equiv$ *?f ?s* rule to the simpset.

70.1 Definition

consts
  as :: *'a* ⇒ *'a* ⇒ *'a*
  fake-quant :: (*'a* ⇒ *prop*) ⇒ *prop*

lemma *let-cong-unfolding*: *M* = *N* ⟹ *f N* = *g N* ⟹ *Let M f* = *Let N g*
by *simp*

translations *P* <= *CONST fake-quant* (λ*x*. *P*)

ML‹
local

```
fun let-typ a b = a --> (a --> b) --> b
fun as-typ a = a --> a --> a

fun strip-all t =
  case try Logic.dest-all-global t of
    NONE => ([], t)
  | SOME (var, t) => apfst (cons var) (strip-all t)

fun all-Frees t =
  fold-aterms (fn Free (x, t) => insert (op =) (x, t) | - => I) t []

fun subst-once (old, new) t =
  let
    fun go t =
      if t = old then
        (new, true)
      else
        case t of
          u $ v =>
            let
              val (u', substituted) = go u
            in
              if substituted then
                (u' $ v, true)
              else
                case go v of (v', substituted) => (u $ v', substituted)
            end
        | Abs (name, typ, t) =>
            (case go t of (t', substituted) => (Abs (name, typ, t'), substituted))
        | - => (t, false)
  in fst (go t) end

(* adapted from logic.ML *)
fun fake-const T = Const (const-name‹fake-quant›, (T --> propT) --> propT);

fun dependent-fake-name v t =
```

```
  let
    val x = Term.term-name v
    val T = Term.fastype-of v
    val t' = Term.abstract-over (v, t)
  in if Term.is-dependent t' then fake-const T $ Abs (x, T, t') else t end

in

fun check-pattern-syntax t =
  case strip-all t of
    (vars, Const ‹Trueprop› $ (Const (const-name ‹HOL.eq›, -) $ lhs $ rhs)) =>
      let
        fun go (Const (const-name ‹as›, -) $ pat $ var, rhs) =
              let
                val (pat', rhs') = go (pat, rhs)
                 val - = if is-Free var then () else error Right−hand side of =: must
be a free variable
                val rhs'' =
                 Const (const-name ‹Let›, let-typ (fastype-of var) (fastype-of rhs)) $
                    pat' $ lambda var rhs'
              in
                (pat', rhs'')
              end
          | go (t $ u, rhs) =
              let
                val (t', rhs') = go (t, rhs)
                val (u', rhs'') = go (u, rhs')
              in (t' $ u', rhs'') end
          | go (t, rhs) = (t, rhs)

        val (lhs', rhs') = go (lhs, rhs)

        val res = HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs', rhs'))

        val frees = filter (member (op =) vars) (all-Frees res)
      in fold (fn v => Logic.dependent-all-name (, v)) (map Free frees) res end
  | - => t

fun uncheck-pattern-syntax ctxt t =
  case strip-all t of
    (vars, Const ‹Trueprop› $ (Const (const-name ‹HOL.eq›, -) $ lhs $ rhs)) =>
      let
        (* restricted to going down abstractions; ignores eta−contracted rhs *)
        fun go lhs (rhs as Const (const-name ‹Let›, -) $ pat $ Abs (name, typ, t))
ctxt frees =
              if exists-subterm (fn t' => t' = pat) lhs then
                let
                  val ([name'], ctxt') = Variable.variant-fixes [name] ctxt
                  val free = Free (name', typ)
```

```
            val subst = (pat, Const (const-name ‹as›, as-typ typ) $ pat $ free)
            val lhs' = subst-once subst lhs
            val rhs' = subst-bound (free, t)
          in
            go lhs' rhs' ctxt' (Free (name', typ) :: frees)
          end
        else
          (lhs, rhs, ctxt, frees)
    | go lhs rhs ctxt frees = (lhs, rhs, ctxt, frees)

    val (lhs', rhs', -, frees) = go lhs rhs ctxt []

    val res =
      HOLogic.mk-Trueprop (HOLogic.mk-eq (lhs', rhs'))
      |> fold (fn v => Logic.dependent-all-name (, v)) (map Free vars)
      |> fold dependent-fake-name frees
  in
    if null frees then t else res
  end
| - => t

end
›

bundle pattern-aliases begin

  notation as (infixr =: 1)

 declaration ‹K (Syntax-Phases.term-check 98 pattern-syntax (K (map check-pattern-syntax)))›
 declaration ‹K (Syntax-Phases.term-uncheck 98 pattern-syntax (map o uncheck-pattern-syntax))›

  declare let-cong-unfolding [fundef-cong]
  declare Let-def [simp]

end

hide-const as
hide-const fake-quant
```

70.2 Usage

context includes *pattern-aliases* **begin**

Not very useful for plain definitions, but works anyway.

private definition *test-1 x* (*y* =: *z*) = *y* + *z*

lemma *test-1 x y* = *y* + *y*
by (*rule test-1-def*[*unfolded Let-def*])

Very useful for function definitions.

private fun *test-2* **where**
test-2 $(y \# (y' \# ys =: x') =: x) = x \mathbin{@} x' \mathbin{@} x' \mid$
test-2 - = []

lemma *test-2* $(y \# y' \# ys) = (y \# y' \# ys) \mathbin{@} (y' \# ys) \mathbin{@} (y' \# ys)$
by (*rule test-2.simps*[*unfolded Let-def*])

ML‹
let
  val actual =
    @{*thm test-2.simps*(*1*)}
    |> *Thm.prop-of*
    |> *Syntax.string-of-term* ***context***
    |> *YXML.content-of*
  val expected = *test-2* $(?y \# (?y' \# ?ys =: x') =: x) = x \mathbin{@} x' \mathbin{@} x'$
in ***assert*** (*actual* = *expected*) *end*
›

end

end

71 Periodic Functions

theory *Periodic-Fun*
imports *Complex-Main*
begin

A locale for periodic functions. The idea is that one proves $f(x + p) = f(x)$ for some period p and gets derived results like $f(x - p) = f(x)$ and $f(x + 2p) = f(x)$ for free.

g and *gm* are "plus/minus k periods" functions. *g1* and *gn1* are "plus/minus one period" functions. This is useful e.g. if the period is one; the lemmas one gets are then $f\ (x + (1::'b)) = f\ x$ instead of $f\ (x + (1::'b) * (1::'b)) = f\ x$ etc.

locale *periodic-fun* =
  fixes $f :: ('a :: \{ring\text{-}1\}) \Rightarrow 'b$ **and** $g\ gm :: 'a \Rightarrow 'a \Rightarrow 'a$ **and** $g1\ gn1 :: 'a \Rightarrow 'a$
  assumes *plus-1*: $f\ (g1\ x) = f\ x$
  assumes *periodic-arg-plus-0*: $g\ x\ 0 = x$
  assumes *periodic-arg-plus-distrib*: $g\ x\ (of\text{-}int\ (m + n)) = g\ (g\ x\ (of\text{-}int\ n))\ (of\text{-}int\ m)$
  assumes *plus-1-eq*: $g\ x\ 1 = g1\ x$ **and** *minus-1-eq*: $g\ x\ (-1) = gn1\ x$
        and *minus-eq*: $g\ x\ (-y) = gm\ x\ y$
begin

lemma *plus-of-nat*: $f\ (g\ x\ (of\text{-}nat\ n)) = f\ x$
  by (*induction n*) (*insert periodic-arg-plus-distrib*[*of - 1 int n* **for** *n*],
                simp-all add: *plus-1 periodic-arg-plus-0 plus-1-eq*)

lemma *minus-of-nat*: $f\ (gm\ x\ (of\text{-}nat\ n)) = f\ x$
proof −
have $f\ (g\ x\ (-\ of\text{-}nat\ n)) = f\ (g\ (g\ x\ (-\ of\text{-}nat\ n))\ (of\text{-}nat\ n))$
by (*rule plus-of-nat*[*symmetric*])
also have $\ldots = f\ (g\ (g\ x\ (of\text{-}int\ (-\ of\text{-}nat\ n)))\ (of\text{-}int\ (of\text{-}nat\ n)))$ **by** *simp*
also have $\ldots = f\ x$
by (*subst periodic-arg-plus-distrib* [*symmetric*]) (*simp add*: *periodic-arg-plus-0*)
finally show *?thesis* **by** (*simp add*: *minus-eq*)
qed

lemma *plus-of-int*: $f\ (g\ x\ (of\text{-}int\ n)) = f\ x$
by (*induction n*) (*simp-all add*: *plus-of-nat minus-of-nat minus-eq del*: *of-nat-Suc*)

lemma *minus-of-int*: $f\ (gm\ x\ (of\text{-}int\ n)) = f\ x$
using *plus-of-int*[*of x of-int* $(-n)$] **by** (*simp add*: *minus-eq*)

lemma *plus-numeral*: $f\ (g\ x\ (numeral\ n)) = f\ x$
by (*subst of-nat-numeral*[*symmetric*], *subst plus-of-nat*) (*rule refl*)

lemma *minus-numeral*: $f\ (gm\ x\ (numeral\ n)) = f\ x$
by (*subst of-nat-numeral*[*symmetric*], *subst minus-of-nat*) (*rule refl*)

lemma *minus-1*: $f\ (gn1\ x) = f\ x$
using *minus-of-nat*[*of x 1*] **by** (*simp flip*: *minus-1-eq minus-eq*)

lemmas *periodic-simps* = *plus-of-nat minus-of-nat plus-of-int minus-of-int*
plus-numeral minus-numeral plus-1 minus-1

end

Specialised case of the *periodic-fun* locale for periods that are not 1. Gives lemmas $f\ (x - period) = f\ x$ etc.

locale *periodic-fun-simple* =
fixes $f :: ('a :: \{ring\text{-}1\}) \Rightarrow 'b$ **and** $period :: 'a$
assumes *plus-period*: $f\ (x + period) = f\ x$
begin
sublocale *periodic-fun* $f\ \lambda z\ x.\ z + x * period\ \lambda z\ x.\ z - x * period$
$\lambda z.\ z + period\ \lambda z.\ z - period$
by *standard* (*simp-all add*: *ring-distribs plus-period*)
end

Specialised case of the *periodic-fun* locale for period 1. Gives lemmas $f\ (x - (1::'b)) = f\ x$ etc.

locale *periodic-fun-simple'* =
fixes $f :: ('a :: \{ring\text{-}1\}) \Rightarrow 'b$
assumes *plus-period*: $f\ (x + 1) = f\ x$
begin
sublocale *periodic-fun* $f\ \lambda z\ x.\ z + x\ \lambda z\ x.\ z - x\ \lambda z.\ z + 1\ \lambda z.\ z - 1$

by *standard* (*simp-all add*: *ring-distribs plus-period*)

lemma *of-nat*: *f* (*of-nat n*) = *f 0* **using** *plus-of-nat*[*of 0 n*] **by** *simp*
lemma *uminus-of-nat*: *f* (−*of-nat n*) = *f 0* **using** *minus-of-nat*[*of 0 n*] **by** *simp*
lemma *of-int*: *f* (*of-int n*) = *f 0* **using** *plus-of-int*[*of 0 n*] **by** *simp*
lemma *uminus-of-int*: *f* (−*of-int n*) = *f 0* **using** *minus-of-int*[*of 0 n*] **by** *simp*
lemma *of-numeral*: *f* (*numeral n*) = *f 0* **using** *plus-numeral*[*of 0 n*] **by** *simp*
lemma *of-neg-numeral*: *f* (−*numeral n*) = *f 0* **using** *minus-numeral*[*of 0 n*] **by** *simp*
lemma *of-1*: *f 1* = *f 0* **using** *plus-of-nat*[*of 0 1*] **by** *simp*
lemma *of-neg-1*: *f* (−*1*) = *f 0* **using** *minus-of-nat*[*of 0 1*] **by** *simp*

lemmas *periodic-simps'* =
of-nat uminus-of-nat of-int uminus-of-int of-numeral of-neg-numeral of-1 of-neg-1

end

lemma *sin-plus-pi*: *sin* ((*z* :: *'a* :: {*real-normed-field*,*banach*}) + *of-real pi*) = − *sin z*
by (*simp add*: *sin-add*)

lemma *cos-plus-pi*: *cos* ((*z* :: *'a* :: {*real-normed-field*,*banach*}) + *of-real pi*) = − *cos z*
by (*simp add*: *cos-add*)

interpretation *sin*: *periodic-fun-simple sin* 2 ∗ *of-real pi* :: *'a* :: {*real-normed-field*,*banach*}
proof
fix *z* :: *'a*
have *sin* (*z* + *2* ∗ *of-real pi*) = *sin* (*z* + *of-real pi* + *of-real pi*) **by** (*simp add*: *ac-simps*)
also have ... = *sin z* **by** (*simp only*: *sin-plus-pi*) *simp*
finally show *sin* (*z* + *2* ∗ *of-real pi*) = *sin z* **.**
qed

interpretation *cos*: *periodic-fun-simple cos* 2 ∗ *of-real pi* :: *'a* :: {*real-normed-field*,*banach*}
proof
fix *z* :: *'a*
have *cos* (*z* + *2* ∗ *of-real pi*) = *cos* (*z* + *of-real pi* + *of-real pi*) **by** (*simp add*: *ac-simps*)
also have ... = *cos z* **by** (*simp only*: *cos-plus-pi*) *simp*
finally show *cos* (*z* + *2* ∗ *of-real pi*) = *cos z* **.**
qed

interpretation *tan*: *periodic-fun-simple tan* 2 ∗ *of-real pi* :: *'a* :: {*real-normed-field*,*banach*}
by *standard* (*simp only*: *tan-def* [*abs-def*] *sin.plus-1 cos.plus-1*)

interpretation *cot*: *periodic-fun-simple cot* 2 ∗ *of-real pi* :: *'a* :: {*real-normed-field*,*banach*}
by *standard* (*simp only*: *cot-def* [*abs-def*] *sin.plus-1 cos.plus-1*)

end

72 Polynomial mapping: combination of almost everywhere zero functions with an algebraic view

theory *Poly-Mapping*
imports *Groups-Big-Fun Fun-Lexorder More-List*
begin

72.1 Preliminary: auxiliary operations for *almost everywhere zero*

A central notion for polynomials are functions being *almost everywhere zero*. For these we provide some auxiliary definitions and lemmas.

lemma *finite-mult-not-eq-zero-leftI*:
  fixes $f :: 'b \Rightarrow 'a :: mult\text{-}zero$
  assumes *finite* $\{a.\ f\ a \neq 0\}$
  shows *finite* $\{a.\ g\ a * f\ a \neq 0\}$
proof −
  have $\{a.\ g\ a * f\ a \neq 0\} \subseteq \{a.\ f\ a \neq 0\}$ **by** *auto*
  then show *?thesis* **using** *assms* **by** (*rule finite-subset*)
qed

lemma *finite-mult-not-eq-zero-rightI*:
  fixes $f :: 'b \Rightarrow 'a :: mult\text{-}zero$
  assumes *finite* $\{a.\ f\ a \neq 0\}$
  shows *finite* $\{a.\ f\ a * g\ a \neq 0\}$
proof −
  have $\{a.\ f\ a * g\ a \neq 0\} \subseteq \{a.\ f\ a \neq 0\}$ **by** *auto*
  then show *?thesis* **using** *assms* **by** (*rule finite-subset*)
qed

lemma *finite-mult-not-eq-zero-prodI*:
  fixes $f\ g :: 'a \Rightarrow 'b :: semiring\text{-}0$
  assumes *finite* $\{a.\ f\ a \neq 0\}$ (**is** *finite ?F*)
  assumes *finite* $\{b.\ g\ b \neq 0\}$ (**is** *finite ?G*)
  shows *finite* $\{(a, b).\ f\ a * g\ b \neq 0\}$
proof −
  from *assms* **have** *finite* $(?F \times ?G)$
    by *blast*
  then have *finite* $\{(a, b).\ f\ a \neq 0 \wedge g\ b \neq 0\}$
    by *simp*
  then show *?thesis*
    by (*rule rev-finite-subset*) *auto*
qed

lemma *finite-not-eq-zero-sumI*:
  fixes $f\ g :: 'a :: monoid\text{-}add \Rightarrow 'b :: semiring\text{-}0$

assumes *finite* {*a*. *f a* ≠ *0*} (**is** *finite ?F*)
assumes *finite* {*b*. *g b* ≠ *0*} (**is** *finite ?G*)
shows *finite* {*a* + *b* | *a b*. *f a* ≠ *0* ∧ *g b* ≠ *0*} (**is** *finite ?FG*)
proof −
from *assms* **have** *finite* (*?F* × *?G*)
by (*simp add*: *finite-cartesian-product-iff*)
then have *finite* (*case-prod plus* ‘ (*?F* × *?G*))
by (*rule finite-imageI*)
also have *case-prod plus* ‘ (*?F* × *?G*) = *?FG*
by *auto*
finally show *?thesis*
by *simp*
qed

lemma *finite-mult-not-eq-zero-sumI*:
fixes *f g* :: *'a*::*monoid-add* ⇒ *'b*::*semiring-0*
assumes *finite* {*a*. *f a* ≠ *0*}
assumes *finite* {*b*. *g b* ≠ *0*}
shows *finite* {*a* + *b* | *a b*. *f a* ∗ *g b* ≠ *0*}
proof −
from *assms*
have *finite* {*a* + *b* | *a b*. *f a* ≠ *0* ∧ *g b* ≠ *0*}
by (*rule finite-not-eq-zero-sumI*)
then show *?thesis*
by (*rule rev-finite-subset*) (*auto dest*: *mult-not-zero*)
qed

lemma *finite-Sum-any-not-eq-zero-weakenI*:
assumes *finite* {*a*. ∃ *b*. *f a b* ≠ *0*}
shows *finite* {*a*. *Sum-any* (*f a*) ≠ *0*}
proof −
have {*a*. *Sum-any* (*f a*) ≠ *0*} ⊆ {*a*. ∃ *b*. *f a b* ≠ *0*}
by (*auto elim*: *Sum-any.not-neutral-obtains-not-neutral*)
then show *?thesis* **using** *assms* **by** (*rule finite-subset*)
qed

context *zero*
begin

definition *when* :: *'a* ⇒ *bool* ⇒ *'a* (**infixl** *when 20*)
where
(*a when P*) = (*if P then a else 0*)

Case distinctions always complicate matters, particularly when nested. The (*when*) operation allows to minimise these if *0*::*'a* is the false-case value and makes proof obligations much more readable.

lemma *when* [*simp*]:
P ⟹ (*a when P*) = *a*
¬ *P* ⟹ (*a when P*) = *0*

by (*simp-all add*: *when-def*)

lemma *when-simps* [*simp*]:
(*a when True*) = *a*
(*a when False*) = *0*
by *simp-all*

lemma *when-cong*:
assumes $P \longleftrightarrow Q$
and $Q \Longrightarrow a = b$
shows (*a when P*) = (*b when Q*)
using *assms* **by** (*simp add*: *when-def*)

lemma *zero-when* [*simp*]:
(*0 when P*) = *0*
by (*simp add*: *when-def*)

lemma *when-when*:
(*a when P when Q*) = (*a when P* ∧ *Q*)
by (*cases Q*) *simp-all*

lemma *when-commute*:
(*a when Q when P*) = (*a when P when Q*)
by (*simp add*: *when-when conj-commute*)

lemma *when-neq-zero* [*simp*]:
(*a when P*) ≠ *0* ⟷ *P* ∧ *a* ≠ *0*
by (*cases P*) *simp-all*

end

context *monoid-add*
begin

lemma *when-add-distrib*:
(*a* + *b when P*) = (*a when P*) + (*b when P*)
by (*simp add*: *when-def*)

end

context *semiring-1*
begin

lemma *zero-power-eq*:
0 ^ *n* = (*1 when n* = *0*)
by (*simp add*: *power-0-left*)

end

context *comm-monoid-add*
begin

lemma *Sum-any-when-equal* [*simp*]:
$(\sum a.\ (f\ a\ when\ a = b)) = f\ b$
by (*simp add*: *when-def*)

lemma *Sum-any-when-equal'* [*simp*]:
$(\sum a.\ (f\ a\ when\ b = a)) = f\ b$
by (*simp add*: *when-def*)

lemma *Sum-any-when-independent*:
$(\sum a.\ g\ a\ when\ P) = ((\sum a.\ g\ a)\ when\ P)$
by (*cases P*) *simp-all*

lemma *Sum-any-when-dependent-prod-right*:
$(\sum (a,\ b).\ g\ a\ when\ b = h\ a) = (\sum a.\ g\ a)$
proof −
have *inj-on* $(\lambda a.\ (a,\ h\ a))\ \{a.\ g\ a \neq 0\}$
by (*rule inj-onI*) *auto*
then show *?thesis* **unfolding** *Sum-any.expand-set*
by (*rule sum.reindex-cong*) *auto*
qed

lemma *Sum-any-when-dependent-prod-left*:
$(\sum (a,\ b).\ g\ b\ when\ a = h\ b) = (\sum b.\ g\ b)$
proof −
have $(\sum (a,\ b).\ g\ b\ when\ a = h\ b) = (\sum (b,\ a).\ g\ b\ when\ a = h\ b)$
by (*rule Sum-any.reindex-cong* [*of prod.swap*]) (*simp-all add*: *fun-eq-iff*)
then show *?thesis* **by** (*simp add*: *Sum-any-when-dependent-prod-right*)
qed

end

context *cancel-comm-monoid-add*
begin

lemma *when-diff-distrib*:
$(a - b\ when\ P) = (a\ when\ P) - (b\ when\ P)$
by (*simp add*: *when-def*)

end

context *group-add*
begin

lemma *when-uminus-distrib*:
$(-\ a\ when\ P) = -\ (a\ when\ P)$
by (*simp add*: *when-def*)

end

context *mult-zero*
begin

lemma *mult-when*:
a ∗ (*b when P*) = (*a* ∗ *b when P*)
by (*cases P*) *simp-all*

lemma *when-mult*:
(*a when P*) ∗ *b* = (*a* ∗ *b when P*)
by (*cases P*) *simp-all*

end

72.2 Type definition

The following type is of central importance:

typedef (**overloaded**) ('*a*, '*b*) *poly-mapping* ((- ⇒$_0$ /-) [*1*, *0*] *0*) =
{*f* :: '*a* ⇒ '*b*::*zero*. *finite* {*x*. *f x* ≠ *0*}}
morphisms *lookup Abs-poly-mapping*
proof −
have (λ-::'*a*. (*0* :: '*b*)) ∈ *?poly-mapping* **by** *simp*
then show *?thesis* **by** (*blast intro*!: *exI*)
qed

declare *lookup-inverse* [*simp*]
declare *lookup-inject* [*simp*]

lemma *lookup-Abs-poly-mapping* [*simp*]:
finite {*x*. *f x* ≠ *0*} ⟹ *lookup* (*Abs-poly-mapping f*) = *f*
using *Abs-poly-mapping-inverse* [*of f*] **by** *simp*

lemma *finite-lookup* [*simp*]:
finite {*k*. *lookup f k* ≠ *0*}
using *poly-mapping.lookup* [*of f*] **by** *simp*

lemma *finite-lookup-nat* [*simp*]:
fixes *f* :: '*a* ⇒$_0$ *nat*
shows *finite* {*k*. *0* < *lookup f k*}
using *poly-mapping.lookup* [*of f*] **by** *simp*

lemma *poly-mapping-eqI*:
assumes ⋀*k*. *lookup f k* = *lookup g k*
shows *f* = *g*
using *assms* **unfolding** *poly-mapping.lookup-inject* [*symmetric*]
by *blast*

lemma *poly-mapping-eq-iff*: $a = b \longleftrightarrow lookup\ a = lookup\ b$
by *auto*

We model the universe of functions being *almost everywhere zero* by means of a separate type $'a \Rightarrow_0 'b$. For convenience we provide a suggestive infix syntax which is a variant of the usual function space syntax. Conversion between both types happens through the morphisms

$lookup::('a \Rightarrow_0 'b) \Rightarrow 'a \Rightarrow 'b$

$Abs\text{-}poly\text{-}mapping::('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow_0 'b$

satisfying

$Abs\text{-}poly\text{-}mapping\ (lookup\ ?x) = ?x$

$finite\ \{x.\ ?f\ x \neq (0::?'b)\} \Longrightarrow lookup\ (Abs\text{-}poly\text{-}mapping\ ?f) = ?f$

Luckily, we have rarely to deal with those low-level morphisms explicitly but rely on Isabelle's *lifting* package with its method *transfer* and its specification tool *lift-definition*.

setup-lifting *type-definition-poly-mapping*
code-datatype *Abs-poly-mapping*— FIXME? workaround for preventing *code-abstype* setup

$'a \Rightarrow_0 'b$ serves distinctive purposes:

1. A clever nesting as $(nat \Rightarrow_0 nat) \Rightarrow_0 'a$ later in theory *MPoly* gives a suitable representation type for polynomials *almost for free*: Interpreting $nat \Rightarrow_0 nat$ as a mapping from variable identifiers to exponents yields monomials, and the whole type maps monomials to coefficients. Lets call this the *ultimate interpretation*.

2. A further more specialised type isomorphic to $nat \Rightarrow_0 'a$ is apt to direct implementation using code generation [**?**].

Note that despite the names *mapping* and *lookup* suggest something implementation-near, it is best to keep $'a \Rightarrow_0 'b$ as an abstract *algebraic* type providing operations like *addition*, *multiplication* without any notion of key-order, data structures etc. This implementations-specific notions are easily introduced later for particular implementations but do not provide any gain for specifying logical properties of polynomials.

72.3 Additive structure

The additive structure covers the usual operations *0*, + and (unary and binary) −. Recalling the ultimate interpretation, it is obvious that these have just lift the corresponding operations on values to mappings.

Isabelle has a rich hierarchy of algebraic type classes, and in such situations of pointwise lifting a typical pattern is to have instantiations for a considerable number of type classes.

The operations themselves are specified using *lift-definition*, where the proofs of the *almost everywhere zero* property can be significantly involved.

The *lookup* operation is supposed to be usable explicitly (unless in other situations where the morphisms between types are somehow internal to the *lifting* package). Hence it is good style to provide explicit rewrite rules how *lookup* acts on operations immediately.

instantiation *poly-mapping* :: (*type*, *zero*) *zero*
begin

lift-definition *zero-poly-mapping* :: $'a \Rightarrow_0 'b$
is $\lambda k.\ 0$
by *simp*

instance ..

end

lemma *Abs-poly-mapping* [*simp*]: *Abs-poly-mapping* $(\lambda k.\ 0) = 0$
by (*simp add*: *zero-poly-mapping.abs-eq*)

lemma *lookup-zero* [*simp*]: *lookup 0 k* = *0*
by *transfer rule*

instantiation *poly-mapping* :: (*type*, *monoid-add*) *monoid-add*
begin

lift-definition *plus-poly-mapping* ::
$('a \Rightarrow_0 'b) \Rightarrow ('a \Rightarrow_0 'b) \Rightarrow 'a \Rightarrow_0 'b$
is $\lambda f1\ f2\ k.\ f1\ k + f2\ k$
proof −
fix *f1 f2* :: $'a \Rightarrow 'b$
assume *finite* $\{k.\ f1\ k \neq 0\}$
and *finite* $\{k.\ f2\ k \neq 0\}$
then have *finite* $(\{k.\ f1\ k \neq 0\} \cup \{k.\ f2\ k \neq 0\})$ **by** *auto*
moreover have $\{x.\ f1\ x + f2\ x \neq 0\} \subseteq \{k.\ f1\ k \neq 0\} \cup \{k.\ f2\ k \neq 0\}$
by *auto*
ultimately show *finite* $\{x.\ f1\ x + f2\ x \neq 0\}$
by (*blast intro*: *finite-subset*)
qed

instance
by *intro-classes* (*transfer*, *simp add*: *fun-eq-iff ac-simps*)+

end

lemma *lookup-add*:
lookup (*f* + *g*) *k* = *lookup f k* + *lookup g k*
by *transfer rule*

instance *poly-mapping* :: (*type*, *comm-monoid-add*) *comm-monoid-add*
by *intro-classes* (*transfer*, *simp add*: *fun-eq-iff ac-simps*)+

lemma *lookup-sum*: *lookup* (*sum pp X*) *i* = *sum* (λ*x*. *lookup* (*pp x*) *i*) *X*
by (*induction rule*: *infinite-finite-induct*) (*auto simp*: *lookup-add*)

instantiation *poly-mapping* :: (*type*, *cancel-comm-monoid-add*) *cancel-comm-monoid-add*
begin

lift-definition *minus-poly-mapping* :: ('*a* ⇒$_0$ '*b*) ⇒ ('*a* ⇒$_0$ '*b*) ⇒ '*a* ⇒$_0$ '*b*
is λ*f1 f2 k*. *f1 k* − *f2 k*
proof −
fix *f1 f2* :: '*a* ⇒ '*b*
assume *finite* {*k*. *f1 k* ≠ *0*}
and *finite* {*k*. *f2 k* ≠ *0*}
then have *finite* ({*k*. *f1 k* ≠ *0*} ∪ {*k*. *f2 k* ≠ *0*}) **by** *auto*
moreover have {*x*. *f1 x* − *f2 x* ≠ *0*} ⊆ {*k*. *f1 k* ≠ *0*} ∪ {*k*. *f2 k* ≠ *0*}
by *auto*
ultimately show *finite* {*x*. *f1 x* − *f2 x* ≠ *0*} **by** (*blast intro*: *finite-subset*)
qed

instance
by *intro-classes* (*transfer*, *simp add*: *fun-eq-iff diff-diff-add*)+

end

instantiation *poly-mapping* :: (*type*, *ab-group-add*) *ab-group-add*
begin

lift-definition *uminus-poly-mapping* :: ('*a* ⇒$_0$ '*b*) ⇒ '*a* ⇒$_0$ '*b*
is *uminus*
by *simp*

instance
by *intro-classes* (*transfer*, *simp add*: *fun-eq-iff ac-simps*)+

end

lemma *lookup-uminus* [*simp*]:
lookup (− *f*) *k* = − *lookup f k*
by *transfer simp*

lemma *lookup-minus*:
lookup (*f* − *g*) *k* = *lookup f k* − *lookup g k*
by *transfer rule*

72.4 Multiplicative structure

instantiation *poly-mapping* :: (*zero*, *zero-neq-one*) *zero-neq-one*
begin

lift-definition *one-poly-mapping* :: *'a* ⇒$_0$ *'b*
is λ*k*. *1 when k* = *0*
by *simp*

instance
by *intro-classes* (*transfer*, *simp add*: *fun-eq-iff*)

end

lemma *lookup-one*:
lookup 1 k = (*1 when k* = *0*)
by *transfer rule*

lemma *lookup-one-zero* [*simp*]:
lookup 1 0 = *1*
by *transfer simp*

definition *prod-fun* :: (*'a* ⇒ *'b*) ⇒ (*'a* ⇒ *'b*) ⇒ *'a*::*monoid-add* ⇒ *'b*::*semiring-0*
where
prod-fun f1 f2 k = (∑ *l*. *f1 l* ∗ (∑ *q*. (*f2 q when k* = *l* + *q*)))

lemma *prod-fun-unfold-prod*:
fixes *f g* :: *'a* :: *monoid-add* ⇒ *'b*::*semiring-0*
assumes *fin-f*: *finite* {*a*. *f a* ≠ *0*}
assumes *fin-g*: *finite* {*b*. *g b* ≠ *0*}
shows *prod-fun f g k* = (∑ (*a*, *b*). *f a* ∗ *g b when k* = *a* + *b*)
proof −
let *?C* = {*a*. *f a* ≠ *0*} × {*b*. *g b* ≠ *0*}
from *fin-f fin-g* **have** *finite ?C* **by** *blast*
moreover have {*a*. ∃ *b*. (*f a* ∗ *g b when k* = *a* + *b*) ≠ *0*} ×
{*b*. ∃ *a*. (*f a* ∗ *g b when k* = *a* + *b*) ≠ *0*} ⊆ {*a*. *f a* ≠ *0*} × {*b*. *g b* ≠ *0*}
by *auto*
ultimately show *?thesis* **using** *fin-g*
by (*auto simp add*: *prod-fun-def*
Sum-any.cartesian-product [*of* {*a*. *f a* ≠ *0*} × {*b*. *g b* ≠ *0*}] *Sum-any-right-distrib*

mult-when)
qed

lemma *finite-prod-fun*:
fixes *f1 f2* :: *'a* :: *monoid-add* ⇒ *'b* :: *semiring-0*
assumes *fin1*: *finite* {*l*. *f1 l* ≠ *0*}
and *fin2*: *finite* {*q*. *f2 q* ≠ *0*}
shows *finite* {*k*. *prod-fun f1 f2 k* ≠ *0*}
proof −
have ∗: *finite* {*k*. (∃ *l*. *f1 l* ≠ *0* ∧ (∃ *q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*))}
using *assms* **by** *simp*
{ fix *k l*
have {*q*. (*f2 q when k* = *l* + *q*) ≠ *0*} ⊆ {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*} **by** *auto*
with *fin2* **have** *sum f2* {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*} = (∑ *q*. (*f2 q when k* = *l* + *q*))
by (*simp add*: *Sum-any.expand-superset* [*of* {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*}]) **}**
note *aux* = *this*
have {*k*. (∑ *l*. *f1 l* ∗ *sum f2* {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*}) ≠ *0*}
⊆ {*k*. (∃ *l*. *f1 l* ∗ *sum f2* {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*} ≠ *0*)}
by (*auto elim*!: *Sum-any.not-neutral-obtains-not-neutral*)
also have ... ⊆ {*k*. (∃ *l*. *f1 l* ≠ *0* ∧ *sum f2* {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*} ≠ *0*)}
by (*auto dest*: *mult-not-zero*)
also have ... ⊆ {*k*. (∃ *l*. *f1 l* ≠ *0* ∧ (∃ *q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*))}
by (*auto elim*!: *sum.not-neutral-contains-not-neutral*)
finally have *finite* {*k*. (∑ *l*. *f1 l* ∗ *sum f2* {*q*. *f2 q* ≠ *0* ∧ *k* = *l* + *q*}) ≠ *0*}
using ∗ **by** (*rule finite-subset*)
with *aux* **have** *finite* {*k*. (∑ *l*. *f1 l* ∗ (∑ *q*. (*f2 q when k* = *l* + *q*))) ≠ *0*}
by *simp*
with *fin2* **show** *?thesis*
by (*simp add*: *prod-fun-def*)
qed

instantiation *poly-mapping* :: (*monoid-add*, *semiring-0*) *semiring-0*
begin

lift-definition *times-poly-mapping* :: (*'a* ⇒$_0$ *'b*) ⇒ (*'a* ⇒$_0$ *'b*) ⇒ *'a* ⇒$_0$ *'b*
is *prod-fun*
by(*rule finite-prod-fun*)

instance
proof
fix *a b c* :: *'a* ⇒$_0$ *'b*
show *a* ∗ *b* ∗ *c* = *a* ∗ (*b* ∗ *c*)
proof *transfer*
fix *f g h* :: *'a* ⇒ *'b*
assume *fin-f*: *finite* {*a*. *f a* ≠ *0*} (**is** *finite ?F*)
assume *fin-g*: *finite* {*b*. *g b* ≠ *0*} (**is** *finite ?G*)
assume *fin-h*: *finite* {*c*. *h c* ≠ *0*} (**is** *finite ?H*)
from *fin-f fin-g* **have** *fin-fg*: *finite* {(*a*, *b*). *f a* ∗ *g b* ≠ *0*} (**is** *finite ?FG*)

by (*rule finite-mult-not-eq-zero-prodI*)
from *fin-g fin-h* **have** *fin-gh*: *finite* {(*b*, *c*). *g b* ∗ *h c* ≠ *0*} (**is** *finite ?GH*)
by (*rule finite-mult-not-eq-zero-prodI*)
from *fin-f fin-g* **have** *fin-fg'*: *finite* {*a* + *b* | *a b*. *f a* ∗ *g b* ≠ *0*} (**is** *finite ?FG'*)
by (*rule finite-mult-not-eq-zero-sumI*)
then have *fin-fg''*: *finite* {*d*. (∑ (*a*, *b*). *f a* ∗ *g b when d* = *a* + *b*) ≠ *0*}
by (*auto intro*: *finite-Sum-any-not-eq-zero-weakenI*)
from *fin-g fin-h* **have** *fin-gh'*: *finite* {*b* + *c* | *b c*. *g b* ∗ *h c* ≠ *0*} (**is** *finite ?GH'*)
by (*rule finite-mult-not-eq-zero-sumI*)
then have *fin-gh''*: *finite* {*d*. (∑ (*b*, *c*). *g b* ∗ *h c when d* = *b* + *c*) ≠ *0*}
by (*auto intro*: *finite-Sum-any-not-eq-zero-weakenI*)
show *prod-fun* (*prod-fun f g*) *h* = *prod-fun f* (*prod-fun g h*) (**is** *?lhs* = *?rhs*)
proof
fix *k*
from *fin-f fin-g fin-h fin-fg''*
have *?lhs k* = (∑ (*ab*, *c*). (∑ (*a*, *b*). *f a* ∗ *g b when ab* = *a* + *b*) ∗ *h c when k* = *ab* + *c*)
by (*simp add*: *prod-fun-unfold-prod*)
also have ... = (∑ (*ab*, *c*). (∑ (*a*, *b*). *f a* ∗ *g b* ∗ *h c when k* = *ab* + *c when ab* = *a* + *b*))
apply (*subst Sum-any-left-distrib*)
using *fin-fg* **apply** (*simp add*: *split-def*)
apply (*subst Sum-any-when-independent* [*symmetric*])
apply (*simp add*: *when-when when-mult mult-when split-def conj-commute*)
done
also have ... = (∑ (*ab*, *c*, *a*, *b*). *f a* ∗ *g b* ∗ *h c when k* = *ab* + *c when ab* = *a* + *b*)
apply (*subst Sum-any.cartesian-product2* [*of* (*?FG'* × *?H*) × *?FG*])
apply (*auto simp add*: *finite-cartesian-product-iff fin-fg fin-fg' fin-h dest*: *mult-not-zero*)
done
also have ... = (∑ (*ab*, *c*, *a*, *b*). *f a* ∗ *g b* ∗ *h c when k* = *a* + *b* + *c when ab* = *a* + *b*)
by (*rule Sum-any.cong*) (*simp add*: *split-def when-def*)
also have ... = (∑ (*ab*, *cab*). (*case cab of* (*c*, *a*, *b*) ⇒ *f a* ∗ *g b* ∗ *h c when k* = *a* + *b* + *c*)
when ab = (*case cab of* (*c*, *a*, *b*) ⇒ *a* + *b*))
by (*simp add*: *split-def*)
also have ... = (∑ (*c*, *a*, *b*). *f a* ∗ *g b* ∗ *h c when k* = *a* + *b* + *c*)
by (*simp add*: *Sum-any-when-dependent-prod-left*)
also have ... = (∑ (*bc*, *cab*). (*case cab of* (*c*, *a*, *b*) ⇒ *f a* ∗ *g b* ∗ *h c when k* = *a* + *b* + *c*)
when bc = (*case cab of* (*c*, *a*, *b*) ⇒ *b* + *c*))
by (*simp add*: *Sum-any-when-dependent-prod-left*)
also have ... = (∑ (*bc*, *c*, *a*, *b*). *f a* ∗ *g b* ∗ *h c when k* = *a* + *b* + *c when bc* = *b* + *c*)
by (*simp add*: *split-def*)
also have ... = (∑ (*bc*, *c*, *a*, *b*). *f a* ∗ *g b* ∗ *h c when bc* = *b* + *c when k* = *a* + *bc*)

```
      by (rule Sum-any.cong) (simp add: split-def when-def ac-simps)
    also have ... = (∑(a, bc, b, c). f a * g b * h c when bc = b + c when k =
a + bc)
    proof −
      have bij (λ(a, d, b, c). (d, c, a, b))
        by (auto intro!: bijI injI surjI [of - λ(d, c, a, b). (a, d, b, c)] simp add:
split-def)
      then show ?thesis
        by (rule Sum-any.reindex-cong) auto
    qed
    also have ... = (∑(a, bc). (∑(b, c). f a * g b * h c when bc = b + c when
k = a + bc))
      apply (subst Sum-any.cartesian-product2 [of (?F × ?GH') × ?GH])
     apply (auto simp add: finite-cartesian-product-iff fin-f fin-gh fin-gh' ac-simps
dest: mult-not-zero)
      done
    also have ... = (∑(a, bc). f a * (∑(b, c). g b * h c when bc = b + c) when
k = a + bc)
      apply (subst Sum-any-right-distrib)
      using fin-gh apply (simp add: split-def)
      apply (subst Sum-any-when-independent [symmetric])
      apply (simp add: when-when when-mult mult-when split-def ac-simps)
      done
    also from fin-f fin-g fin-h fin-gh''
    have ... = ?rhs k
      by (simp add: prod-fun-unfold-prod)
    finally show ?lhs k = ?rhs k .
  qed
 qed
 show (a + b) * c = a * c + b * c
 proof transfer
  fix f g h :: 'a ⇒ 'b
  assume fin-f: finite {k. f k ≠ 0}
  assume fin-g: finite {k. g k ≠ 0}
  assume fin-h: finite {k. h k ≠ 0}
  show prod-fun (λk. f k + g k) h = (λk. prod-fun f h k + prod-fun g h k)
   apply (rule ext)
   apply (auto simp add: prod-fun-def algebra-simps)
   apply (subst Sum-any.distrib)
   using fin-f fin-g apply (auto intro: finite-mult-not-eq-zero-rightI)
   done
 qed
 show a * (b + c) = a * b + a * c
 proof transfer
  fix f g h :: 'a ⇒ 'b
  assume fin-f: finite {k. f k ≠ 0}
  assume fin-g: finite {k. g k ≠ 0}
  assume fin-h: finite {k. h k ≠ 0}
  show prod-fun f (λk. g k + h k) = (λk. prod-fun f g k + prod-fun f h k)
```

```
    apply (rule ext)
   apply (auto simp add: prod-fun-def Sum-any.distrib algebra-simps when-add-distrib)
    apply (subst Sum-any.distrib)
    apply (simp-all add: algebra-simps)
    apply (auto intro: fin-g fin-h)
    apply (subst Sum-any.distrib)
    apply (simp-all add: algebra-simps)
    using fin-f apply (rule finite-mult-not-eq-zero-rightI)
    using fin-f apply (rule finite-mult-not-eq-zero-rightI)
    done
 qed
 show 0 * a = 0
  by transfer (simp add: prod-fun-def [abs-def])
 show a * 0 = 0
  by transfer (simp add: prod-fun-def [abs-def])
qed

end

lemma lookup-mult:
 lookup (f * g) k = (∑ l. lookup f l * (∑ q. lookup g q when k = l + q))
 by transfer (simp add: prod-fun-def)

instance poly-mapping :: (comm-monoid-add, comm-semiring-0) comm-semiring-0
proof
 fix a b c :: 'a ⇒₀ 'b
 show a * b = b * a
 proof transfer
  fix f g :: 'a ⇒ 'b
  assume fin-f: finite {a. f a ≠ 0}
  assume fin-g: finite {b. g b ≠ 0}
  show prod-fun f g = prod-fun g f
  proof
   fix k
   have fin1: ⋀l. finite {a. (f a when k = l + a) ≠ 0}
    using fin-f by auto
   have fin2: ⋀l. finite {b. (g b when k = l + b) ≠ 0}
    using fin-g by auto
   from fin-f fin-g have finite {(a, b). f a ≠ 0 ∧ g b ≠ 0} (is finite ?AB)
    by simp
   show prod-fun f g k = prod-fun g f k
    apply (simp add: prod-fun-def)
    apply (subst Sum-any-right-distrib)
    apply (rule fin2)
    apply (subst Sum-any-right-distrib)
     apply (rule fin1)
    apply (subst Sum-any.swap [of ?AB])
    apply (fact ‹finite ?AB›)
    apply (auto simp add: mult-when ac-simps)
```

```
      done
    qed
  qed
  show (a + b) * c = a * c + b * c
  proof transfer
    fix f g h :: 'a ⇒ 'b
    assume fin-f: finite {k. f k ≠ 0}
    assume fin-g: finite {k. g k ≠ 0}
    assume fin-h: finite {k. h k ≠ 0}
    show prod-fun (λk. f k + g k) h = (λk. prod-fun f h k + prod-fun g h k)
      apply (auto simp add: prod-fun-def fun-eq-iff algebra-simps)
      apply (subst Sum-any.distrib)
      using fin-f apply (rule finite-mult-not-eq-zero-rightI)
      using fin-g apply (rule finite-mult-not-eq-zero-rightI)
      apply simp-all
      done
  qed
qed

instance poly-mapping :: (monoid-add, semiring-0-cancel) semiring-0-cancel
  ..

instance poly-mapping :: (comm-monoid-add, comm-semiring-0-cancel) comm-semiring-0-cancel
  ..

instance poly-mapping :: (monoid-add, semiring-1) semiring-1
proof
  fix a :: 'a ⇒₀ 'b
  show 1 * a = a
    by transfer (simp add: prod-fun-def [abs-def] when-mult)
  show a * 1 = a
    apply transfer
   apply (simp add: prod-fun-def [abs-def] Sum-any-right-distrib Sum-any-left-distrib
mult-when)
    apply (subst when-commute)
    apply simp
    done
qed

instance poly-mapping :: (comm-monoid-add, comm-semiring-1) comm-semiring-1
proof
  fix a :: 'a ⇒₀ 'b
  show 1 * a = a
    by transfer (simp add: prod-fun-def [abs-def])
qed

instance poly-mapping :: (monoid-add, semiring-1-cancel) semiring-1-cancel
  ..
```

instance *poly-mapping* :: (*monoid-add*, *ring*) *ring*
  ..

instance *poly-mapping* :: (*comm-monoid-add*, *comm-ring*) *comm-ring*
  ..

instance *poly-mapping* :: (*monoid-add*, *ring-1*) *ring-1*
  ..

instance *poly-mapping* :: (*comm-monoid-add*, *comm-ring-1*) *comm-ring-1*
  ..

72.5 Single-point mappings

lift-definition *single* :: $'a \Rightarrow 'b \Rightarrow 'a \Rightarrow_0 'b$::*zero*
  is $\lambda k\ v\ k'.\ (v\ when\ k = k')$
  by *simp*

lemma *inj-single* [*iff*]:
  inj (*single k*)
proof (*rule injI*, *transfer*)
  fix $k :: 'b$ **and** $a\ b :: 'a$::*zero*
  assume $(\lambda k'.\ a\ when\ k = k') = (\lambda k'.\ b\ when\ k = k')$
  then have $(\lambda k'.\ a\ when\ k = k')\ k = (\lambda k'.\ b\ when\ k = k')\ k$
    by (*rule arg-cong*)
  then show $a = b$ **by** *simp*
qed

lemma *lookup-single*:
  lookup (*single k v*) $k' = (v\ when\ k = k')$
  by *transfer rule*

lemma *lookup-single-eq* [*simp*]:
  lookup (*single k v*) *k* = *v*
  by *transfer simp*

lemma *lookup-single-not-eq*:
  $k \neq k' \Longrightarrow$ *lookup* (*single k v*) $k' = 0$
  by *transfer simp*

lemma *single-zero* [*simp*]:
  single k 0 = *0*
  by *transfer simp*

lemma *single-one* [*simp*]:
  single 0 1 = *1*
  by *transfer simp*

lemma *single-add*:

single k (a + b) = single k a + single k b
by *transfer* (*simp add*: *fun-eq-iff when-add-distrib*)

lemma *single-uminus*:
single k (− a) = − single k a
by *transfer* (*simp add*: *fun-eq-iff when-uminus-distrib*)

lemma *single-diff*:
single k (a − b) = single k a − single k b
by *transfer* (*simp add*: *fun-eq-iff when-diff-distrib*)

lemma *single-numeral* [*simp*]:
single 0 (numeral n) = numeral n
by (*induct n*) (*simp-all only*: *numeral.simps numeral-add single-zero single-one single-add*)

lemma *lookup-numeral*:
lookup (numeral n) k = (numeral n when k = 0)
proof −
have *lookup (numeral n) k = lookup (single 0 (numeral n)) k*
by *simp*
then show *?thesis* **unfolding** *lookup-single* **by** *simp*
qed

lemma *single-of-nat* [*simp*]:
single 0 (of-nat n) = of-nat n
by (*induct n*) (*simp-all add*: *single-add*)

lemma *lookup-of-nat*:
lookup (of-nat n) k = (of-nat n when k = 0)
proof −
have *lookup (of-nat n) k = lookup (single 0 (of-nat n)) k*
by *simp*
then show *?thesis* **unfolding** *lookup-single* **by** *simp*
qed

lemma *of-nat-single*:
of-nat = single 0 ∘ of-nat
by (*simp add*: *fun-eq-iff*)

lemma *mult-single*:
single k a ∗ single l b = single (k + l) (a ∗ b)
proof *transfer*
fix *k l* :: *'a* **and** *a b* :: *'b*
show *prod-fun* (*λk'. a when k = k'*) (*λk'. b when l = k'*) = (*λk'. a ∗ b when k + l = k'*)
proof
fix *k'*
have *prod-fun* (*λk'. a when k = k'*) (*λk'. b when l = k'*) *k'* = (∑ *n. a ∗ b when*

l = n when k' = k + n)
by (*simp add*: *prod-fun-def Sum-any-right-distrib mult-when when-mult*)
also have ... = (∑ *n*. *a* * *b when k' = k + n when l = n*)
by (*simp add*: *when-when conj-commute*)
also have ... = (*a* * *b when k' = k + l*)
by *simp*
also have ... = (*a* * *b when k + l = k'*)
by (*simp add*: *when-def*)
finally show *prod-fun* (λ*k'*. *a when k = k'*) (λ*k'*. *b when l = k'*) *k'* =
(λ*k'*. *a* * *b when k + l = k'*) *k'* **.**
qed
qed

instance *poly-mapping* :: (*monoid-add*, *semiring-char-0*) *semiring-char-0*
by *intro-classes* (*auto intro*: *inj-compose inj-of-nat simp add*: *of-nat-single*)

instance *poly-mapping* :: (*monoid-add*, *ring-char-0*) *ring-char-0*
..

lemma *single-of-int* [*simp*]:
single 0 (*of-int k*) = *of-int k*
by (*cases k*) (*simp-all add*: *single-diff single-uminus*)

lemma *lookup-of-int*:
lookup (*of-int l*) *k* = (*of-int l when k = 0*)
proof −
have *lookup* (*of-int l*) *k* = *lookup* (*single 0* (*of-int l*)) *k*
by *simp*
then show *?thesis* **unfolding** *lookup-single* **by** *simp*
qed

72.6 Integral domains

instance *poly-mapping* :: ({*ordered-cancel-comm-monoid-add*, *linorder*}, *semiring-no-zero-divisors*)
semiring-no-zero-divisors

The *linorder* constraint is a pragmatic device for the proof — maybe it can be dropped

proof
fix *f g* :: *'a* ⇒$_0$ *'b*
assume *f* ≠ *0* **and** *g* ≠ *0*
then show *f* * *g* ≠ *0*
proof *transfer*
fix *f g* :: *'a* ⇒ *'b*
define *F* **where** *F* = {*a*. *f a* ≠ *0*}
moreover define *G* **where** *G* = {*a*. *g a* ≠ *0*}
ultimately have [*simp*]:
⋀*a*. *f a* ≠ *0* ⟷ *a* ∈ *F*
⋀*b*. *g b* ≠ *0* ⟷ *b* ∈ *G*

by *simp-all*
assume *finite* $\{a.\ f\ a \neq 0\}$
then have [*simp*]: *finite F*
by *simp*
assume *finite* $\{a.\ g\ a \neq 0\}$
then have [*simp*]: *finite G*
by *simp*
assume $f \neq (\lambda a.\ 0)$
then obtain *a* **where** $f\ a \neq 0$
by (*auto simp add*: *fun-eq-iff*)
assume $g \neq (\lambda b.\ 0)$
then obtain *b* **where** $g\ b \neq 0$
by (*auto simp add*: *fun-eq-iff*)
from ‹$f\ a \neq 0$› **and** ‹$g\ b \neq 0$› **have** $F \neq \{\}$ **and** $G \neq \{\}$
by *auto*
note *Max-F* = ‹*finite F*› ‹$F \neq \{\}$›
note *Max-G* = ‹*finite G*› ‹$G \neq \{\}$›
from *Max-F* **and** *Max-G* **have** [*simp*]:
$Max\ F \in F$
$Max\ G \in G$
by *auto*
from *Max-F Max-G* **have** [*dest*!]:
$\bigwedge a.\ a \in F \Longrightarrow a \leq Max\ F$
$\bigwedge b.\ b \in G \Longrightarrow b \leq Max\ G$
by *auto*
define *q* **where** $q = Max\ F + Max\ G$
have $(\sum (a,\ b).\ f\ a * g\ b\ when\ q = a + b) =$
$(\sum (a,\ b).\ f\ a * g\ b\ when\ q = a + b\ when\ a \in F \wedge b \in G)$
by (*rule Sum-any.cong*) (*auto simp add*: *split-def when-def q-def intro*: *ccontr*)
also have ... =
$(\sum (a,\ b).\ f\ a * g\ b\ when\ (Max\ F,\ Max\ G) = (a,\ b))$
proof (*rule Sum-any.cong*)
fix $ab :: 'a \times 'a$
obtain *a b* **where** [*simp*]: $ab = (a,\ b)$
by (*cases ab*) *simp-all*
have [*dest*!]:
$a \leq Max\ F \Longrightarrow Max\ F \neq a \Longrightarrow a < Max\ F$
$b \leq Max\ G \Longrightarrow Max\ G \neq b \Longrightarrow b < Max\ G$
by *auto*
show (*case ab of* $(a,\ b) \Rightarrow f\ a * g\ b\ when\ q = a + b\ when\ a \in F \wedge b \in G$) =
(*case ab of* $(a,\ b) \Rightarrow f\ a * g\ b\ when\ (Max\ F,\ Max\ G) = (a,\ b)$)
by (*auto simp add*: *split-def when-def q-def dest*: *add-strict-mono* [*of a Max F b Max G*])
qed
also have ... = $(\sum ab.$ (*case ab of* $(a,\ b) \Rightarrow f\ a * g\ b$) *when*
$(Max\ F,\ Max\ G) = ab$)
unfolding *split-def when-def* **by** *auto*
also have ... $\neq 0$
by *simp*

```
    finally have prod-fun f g q ≠ 0
      by (simp add: prod-fun-unfold-prod)
    then show prod-fun f g ≠ (λk. 0)
      by (auto simp add: fun-eq-iff)
  qed
qed

instance poly-mapping :: ({ordered-cancel-comm-monoid-add, linorder}, ring-no-zero-divisors)
ring-no-zero-divisors
  ..

instance poly-mapping :: ({ordered-cancel-comm-monoid-add, linorder}, ring-1-no-zero-divisors)
ring-1-no-zero-divisors
  ..

instance poly-mapping :: ({ordered-cancel-comm-monoid-add, linorder}, idom) idom
  ..
```

72.7 Mapping order

```
instantiation poly-mapping :: (linorder, {zero, linorder}) linorder
begin

lift-definition less-poly-mapping :: ('a ⇒₀ 'b) ⇒ ('a ⇒₀ 'b) ⇒ bool
  is less-fun
  .

lift-definition less-eq-poly-mapping :: ('a ⇒₀ 'b) ⇒ ('a ⇒₀ 'b) ⇒ bool
  is λf g. less-fun f g ∨ f = g
  .

instance proof (rule class.Orderings.linorder.of-class.intro)
  show class.linorder (less-eq :: (- ⇒₀ -) ⇒ -) less
  proof (rule linorder-strictI, rule order-strictI)
    fix f g h :: 'a ⇒₀ 'b
    show f ≤ g ⟷ f < g ∨ f = g
      by transfer (rule refl)
    show ¬ f < f
      by transfer (rule less-fun-irrefl)
    show f < g ∨ f = g ∨ g < f
    proof transfer
      fix f g :: 'a ⇒ 'b
      assume finite {k. f k ≠ 0} and finite {k. g k ≠ 0}
      then have finite ({k. f k ≠ 0} ∪ {k. g k ≠ 0})
        by simp
      moreover have {k. f k ≠ g k} ⊆ {k. f k ≠ 0} ∪ {k. g k ≠ 0}
        by auto
      ultimately have finite {k. f k ≠ g k}
        by (rule rev-finite-subset)
```

```
    then show less-fun f g ∨ f = g ∨ less-fun g f
      by (rule less-fun-trichotomy)
  qed
  assume f < g then show ¬ g < f
    by transfer (rule less-fun-asym)
  note ‹f < g› moreover assume g < h
    ultimately show f < h
    by transfer (rule less-fun-trans)
 qed
qed

end

instance poly-mapping :: (linorder, {ordered-comm-monoid-add, ordered-ab-semigroup-add-imp-le,
linorder}) ordered-ab-semigroup-add
proof (intro-classes, transfer)
 fix f g h :: 'a ⇒ 'b
 assume ∗: less-fun f g ∨ f = g
 { assume less-fun f g
  then obtain k where f k < g k (⋀k'. k' < k ⟹ f k' = g k')
    by (blast elim!: less-funE)
  then have h k + f k < h k + g k (⋀k'. k' < k ⟹ h k' + f k' = h k' + g k')
    by simp-all
  then have less-fun (λk. h k + f k) (λk. h k + g k)
    by (blast intro: less-funI)
 }
 with ∗ show less-fun (λk. h k + f k) (λk. h k + g k) ∨ (λk. h k + f k) = (λk.
h k + g k)
  by (auto simp add: fun-eq-iff)
qed

instance poly-mapping :: (linorder, {ordered-comm-monoid-add, ordered-ab-semigroup-add-imp-le,
cancel-comm-monoid-add, linorder}) linordered-cancel-ab-semigroup-add
 ..

instance poly-mapping :: (linorder, {ordered-comm-monoid-add, ordered-ab-semigroup-add-imp-le,
cancel-comm-monoid-add, linorder}) ordered-comm-monoid-add
 ..

instance poly-mapping :: (linorder, {ordered-comm-monoid-add, ordered-ab-semigroup-add-imp-le,
cancel-comm-monoid-add, linorder}) ordered-cancel-comm-monoid-add
 ..

instance poly-mapping :: (linorder, linordered-ab-group-add) linordered-ab-group-add
 ..
```

For pragmatism we leave out the final elements in the hierarchy: *linordered-ring*, *linordered-ring-strict*, *linordered-idom*; remember that the order instance is a mere technical device, not a deeper algebraic property.

72.8 Fundamental mapping notions

lift-definition *keys* :: $('a \Rightarrow_0 'b{::}zero) \Rightarrow 'a\ set$
is $\lambda f.\ \{k.\ f\ k \neq 0\}$ **.**

lift-definition *range* :: $('a \Rightarrow_0 'b{::}zero) \Rightarrow 'b\ set$
is $\lambda f :: 'a \Rightarrow 'b.\ Set.range\ f - \{0\}$ **.**

lemma *finite-keys* [*simp*]:
finite (*keys f*)
by *transfer*

lemma *not-in-keys-iff-lookup-eq-zero*:
$k \notin keys\ f \longleftrightarrow lookup\ f\ k = 0$
by *transfer simp*

lemma *lookup-not-eq-zero-eq-in-keys*:
$lookup\ f\ k \neq 0 \longleftrightarrow k \in keys\ f$
by *transfer simp*

lemma *lookup-eq-zero-in-keys-contradict* [*dest*]:
$lookup\ f\ k = 0 \Longrightarrow \neg\ k \in keys\ f$
by (*simp add*: *not-in-keys-iff-lookup-eq-zero*)

lemma *finite-range* [*simp*]: *finite* (*Poly-Mapping.range p*)
proof *transfer*
fix $f :: 'b \Rightarrow 'a$
assume $*$: *finite* $\{x.\ f\ x \neq 0\}$
have $Set.range\ f - \{0\} \subseteq f\ `\ \{x.\ f\ x \neq 0\}$
by *auto*
thus *finite* (*Set.range f* $-$ $\{0\}$)
by(*rule finite-subset*)(*rule finite-imageI*[*OF* $*$])
qed

lemma *in-keys-lookup-in-range* [*simp*]:
$k \in keys\ f \Longrightarrow lookup\ f\ k \in range\ f$
by *transfer simp*

lemma *in-keys-iff*: $x \in (keys\ s) = (lookup\ s\ x \neq 0)$
by (*transfer*, *simp*)

lemma *keys-zero* [*simp*]:
$keys\ 0 = \{\}$
by *transfer simp*

lemma *range-zero* [*simp*]:
$range\ 0 = \{\}$
by *transfer auto*

lemma *keys-add*:

keys (*f* + *g*) ⊆ *keys f* ∪ *keys g*
by *transfer auto*

lemma *keys-one* [*simp*]:
keys 1 = {*0*}
by *transfer simp*

lemma *range-one* [*simp*]:
range 1 = {*1*}
by *transfer* (*auto simp add*: *when-def*)

lemma *keys-single* [*simp*]:
keys (*single k v*) = (*if v* = *0 then* {} *else* {*k*})
by *transfer simp*

lemma *range-single* [*simp*]:
range (*single k v*) = (*if v* = *0 then* {} *else* {*v*})
by *transfer* (*auto simp add*: *when-def*)

lemma *keys-mult*:
keys (*f* ∗ *g*) ⊆ {*a* + *b* | *a b*. *a* ∈ *keys f* ∧ *b* ∈ *keys g*}
apply *transfer*
apply (*auto simp add*: *prod-fun-def dest*!: *mult-not-zero elim*!: *Sum-any.not-neutral-obtains-not-neutral*)
apply *blast*
done

lemma *setsum-keys-plus-distrib*:
assumes *hom-0*: ⋀*k*. *f k 0* = *0*
and *hom-plus*: ⋀*k*. *k* ∈ *Poly-Mapping.keys p* ∪ *Poly-Mapping.keys q* ⟹ *f k* (*Poly-Mapping.lookup p k* + *Poly-Mapping.lookup q k*) = *f k* (*Poly-Mapping.lookup p k*) + *f k* (*Poly-Mapping.lookup q k*)
shows
(∑ *k*∈*Poly-Mapping.keys* (*p* + *q*). *f k* (*Poly-Mapping.lookup* (*p* + *q*) *k*)) =
(∑ *k*∈*Poly-Mapping.keys p*. *f k* (*Poly-Mapping.lookup p k*)) +
(∑ *k*∈*Poly-Mapping.keys q*. *f k* (*Poly-Mapping.lookup q k*))
(**is** *?lhs* = *?p* + *?q*)
proof −
let *?A* = *Poly-Mapping.keys p* ∪ *Poly-Mapping.keys q*
have *?lhs* = (∑ *k*∈*?A*. *f k* (*Poly-Mapping.lookup p k* + *Poly-Mapping.lookup q k*))
apply(*rule sum.mono-neutral-cong-left*)
apply(*simp-all add*: *Poly-Mapping.keys-add*)
apply(*transfer fixing*: *f*)
apply(*auto simp add*: *hom-0*)[*1*]
apply(*transfer fixing*: *f*)
apply(*auto simp add*: *hom-0*)[*1*]
done
also have ... = (∑ *k*∈*?A*. *f k* (*Poly-Mapping.lookup p k*) + *f k* (*Poly-Mapping.lookup q k*))

by(*rule sum.cong*)(*simp-all add*: *hom-plus*)
also have ... = ($\sum$ *k*∈*?A*. *f k* (*Poly-Mapping.lookup p k*)) + ($\sum$ *k*∈*?A*. *f k* (*Poly-Mapping.lookup q k*))
(**is** - = *?p'* + *?q'*)
by(*simp add*: *sum.distrib*)
also have *?p'* = *?p*
by (*simp add*: *hom-0 in-keys-iff sum.mono-neutral-cong-right*)
also have *?q'* = *?q*
by (*simp add*: *hom-0 in-keys-iff sum.mono-neutral-cong-right*)
finally show *?thesis* **.**
qed

72.9 Degree

definition *degree* :: (*nat* ⇒$_0$ *'a*::*zero*) ⇒ *nat*
where
degree f = *Max* (*insert 0* (*Suc* ` *keys f*))

lemma *degree-zero* [*simp*]:
degree 0 = *0*
unfolding *degree-def* **by** *transfer simp*

lemma *degree-one* [*simp*]:
degree 1 = *1*
unfolding *degree-def* **by** *transfer simp*

lemma *degree-single-zero* [*simp*]:
degree (*single k 0*) = *0*
unfolding *degree-def* **by** *transfer simp*

lemma *degree-single-not-zero* [*simp*]:
v ≠ *0* ⟹ *degree* (*single k v*) = *Suc k*
unfolding *degree-def* **by** *transfer simp*

lemma *degree-zero-iff* [*simp*]:
degree f = *0* ⟷ *f* = *0*
unfolding *degree-def* **proof** *transfer*
fix *f* :: *nat* ⇒ *'a*
assume *finite* {*n*. *f n* ≠ *0*}
then have *fin*: *finite* (*insert 0* (*Suc* ` {*n*. *f n* ≠ *0*})) **by** *auto*
show *Max* (*insert 0* (*Suc* ` {*n*. *f n* ≠ *0*})) = *0* ⟷ *f* = (λ*n*. *0*) (**is** *?P* ⟷ *?Q*)
proof
assume *?P*
have {*n*. *f n* ≠ *0*} = {}
proof (*rule ccontr*)
assume {*n*. *f n* ≠ *0*} ≠ {}
then obtain *n* **where** *n* ∈ {*n*. *f n* ≠ *0*} **by** *blast*
then have {*n*. *f n* ≠ *0*} = *insert n* {*n*. *f n* ≠ *0*} **by** *auto*
then have *Suc* ` {*n*. *f n* ≠ *0*} = *insert* (*Suc n*) (*Suc* ` {*n*. *f n* ≠ *0*}) **by** *auto*

```
      with ‹?P› have Max (insert 0 (insert (Suc n) (Suc ` {n. f n ≠ 0}))) = 0
by simp
     then have Max (insert (Suc n) (insert 0 (Suc ` {n. f n ≠ 0}))) = 0
       by (simp add: insert-commute)
     with fin have max (Suc n) (Max (insert 0 (Suc ` {n. f n ≠ 0}))) = 0
       by simp
     then show False by simp
   qed
   then show ?Q by (simp add: fun-eq-iff)
  next
   assume ?Q then show ?P by simp
  qed
qed

lemma degree-greater-zero-in-keys:
  assumes 0 < degree f
  shows degree f − 1 ∈ keys f
proof −
  from assms have keys f ≠ {}
    by (auto simp add: degree-def)
  then show ?thesis unfolding degree-def
    by (simp add: mono-Max-commute [symmetric] mono-Suc)
qed

lemma in-keys-less-degree:
  n ∈ keys f ⟹ n < degree f
unfolding degree-def by transfer (auto simp add: Max-gr-iff)

lemma beyond-degree-lookup-zero:
  degree f ≤ n ⟹ lookup f n = 0
  unfolding degree-def by transfer auto

lemma degree-add:
  degree (f + g) ≤ max (degree f) (Poly-Mapping.degree g)
unfolding degree-def proof transfer
  fix f g :: nat ⇒ 'a
  assume f: finite {x. f x ≠ 0}
  assume g: finite {x. g x ≠ 0}
  let ?f = Max (insert 0 (Suc ` {k. f k ≠ 0}))
  let ?g = Max (insert 0 (Suc ` {k. g k ≠ 0}))
  have Max (insert 0 (Suc ` {k. f k + g k ≠ 0})) ≤ Max (insert 0 (Suc ` ({k. f k
≠ 0} ∪ {k. g k ≠ 0})))
    by (rule Max.subset-imp) (insert f g, auto)
  also have ... = max ?f ?g
    using f g by (simp-all add: image-Un Max-Un [symmetric])
  finally show Max (insert 0 (Suc ` {k. f k + g k ≠ 0}))
    ≤ max (Max (insert 0 (Suc ` {k. f k ≠ 0}))) (Max (insert 0 (Suc ` {k. g k ≠
0})))
    .
```

qed

lemma *sorted-list-of-set-keys*:
sorted-list-of-set (*keys f*) = *filter* (λ*k*. *k* ∈ *keys f*) [*0*..<*degree f*] (**is** - = *?r*)
proof −
have *keys f* = *set ?r*
by (*auto dest*: *in-keys-less-degree*)
moreover have *sorted-list-of-set* (*set ?r*) = *?r*
unfolding *sorted-list-of-set-sort-remdups*
by (*simp add*: *remdups-filter filter-sort* [*symmetric*])
ultimately show *?thesis* **by** *simp*
qed

72.10 Inductive structure

lift-definition *update* :: *'a* ⇒ *'b* ⇒ (*'a* ⇒$_0$ *'b*::*zero*) ⇒ *'a* ⇒$_0$ *'b*
is λ*k v f*. *f*(*k* := *v*)
proof −
fix *f* :: *'a* ⇒ *'b* **and** *k' v*
assume *finite* {*k*. *f k* ≠ *0*}
then have *finite* (*insert k'* {*k*. *f k* ≠ *0*})
by *simp*
then show *finite* {*k*. (*f*(*k'* := *v*)) *k* ≠ *0*}
by (*rule rev-finite-subset*) *auto*
qed

lemma *update-induct* [*case-names const update*]:
assumes *const'*: *P 0*
assumes *update'*: ⋀*f a b*. *a* ∉ *keys f* ⟹ *b* ≠ *0* ⟹ *P f* ⟹ *P* (*update a b f*)
shows *P f*
proof −
obtain *g* **where** *f* = *Abs-poly-mapping g* **and** *finite* {*a*. *g a* ≠ *0*}
by (*cases f*) *simp-all*
define *Q* **where** *Q g* = *P* (*Abs-poly-mapping g*) **for** *g*
from ‹*finite* {*a*. *g a* ≠ *0*}› **have** *Q g*
proof (*induct g rule*: *finite-update-induct*)
case *const* **with** *const' Q-def* **show** *?case*
by *simp*
next
case (*update a b g*)
from ‹*finite* {*a*. *g a* ≠ *0*}› ‹*g a* = *0*› **have** *a* ∉ *keys* (*Abs-poly-mapping g*)
by (*simp add*: *Abs-poly-mapping-inverse keys.rep-eq*)
moreover note ‹*b* ≠ *0*›
moreover from ‹*Q g*› **have** *P* (*Abs-poly-mapping g*)
by (*simp add*: *Q-def*)
ultimately have *P* (*update a b* (*Abs-poly-mapping g*))
by (*rule update'*)
also from ‹*finite* {*a*. *g a* ≠ *0*}›
have *update a b* (*Abs-poly-mapping g*) = *Abs-poly-mapping* (*g*(*a* := *b*))

```
    by (simp add: update.abs-eq eq-onp-same-args)
  finally show ?case
    by (simp add: Q-def fun-upd-def)
 qed
 then show ?thesis by (simp add: Q-def ‹f = Abs-poly-mapping g›)
qed

lemma lookup-update:
 lookup (update k v f) k' = (if k = k' then v else lookup f k')
 by transfer simp

lemma keys-update:
 keys (update k v f) = (if v = 0 then keys f - {k} else insert k (keys f))
 by transfer auto
```

72.11 Quasi-functorial structure

```
lift-definition map :: ('b::zero ⇒ 'c::zero)
 ⇒ ('a ⇒₀ 'b) ⇒ ('a ⇒₀ 'c::zero)
 is λg f k. g (f k) when f k ≠ 0
 by simp

context
 fixes f :: 'b ⇒ 'a
 assumes inj-f: inj f
begin

lift-definition map-key :: ('a ⇒₀ 'c::zero) ⇒ 'b ⇒₀ 'c
 is λp. p ∘ f
proof -
 fix g :: 'c ⇒ 'd and p :: 'a ⇒ 'c
 assume finite {x. p x ≠ 0}
 hence finite (f ` {y. p (f y) ≠ 0})
   by(rule finite-subset[rotated]) auto
 thus finite {x. (p ∘ f) x ≠ 0} unfolding o-def
   by(rule finite-imageD)(rule subset-inj-on[OF inj-f], simp)
qed

end

lemma map-key-compose:
 assumes [transfer-rule]: inj f inj g
 shows map-key f (map-key g p) = map-key (g ∘ f) p
proof -
 from assms have [transfer-rule]: inj (g ∘ f)
   by(simp add: inj-compose)
 show ?thesis by transfer(simp add: o-assoc)
qed
```

lemma *map-key-id*:
map-key (λ*x*. *x*) *p* = *p*
proof −
have [*transfer-rule*]: *inj* (λ*x*. *x*) **by** *simp*
show *?thesis* **by** *transfer*(*simp add*: *o-def*)
qed

context
fixes *f* :: *'a* ⇒ *'b*
assumes *inj-f* [*transfer-rule*]: *inj f*
begin

lemma *map-key-map*:
map-key f (*map g p*) = *map g* (*map-key f p*)
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *map-key-plus*:
map-key f (*p* + *q*) = *map-key f p* + *map-key f q*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *keys-map-key*:
keys (*map-key f p*) = *f* −‘ *keys p*
by *transfer auto*

lemma *map-key-zero* [*simp*]:
map-key f 0 = *0*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *map-key-single* [*simp*]:
map-key f (*single* (*f k*) *v*) = *single k v*
by *transfer* (*simp add*: *fun-eq-iff inj-onD* [*OF inj-f*] *when-def*)

end

lemma *mult-map-scale-conv-mult*: *map* ((∗) *s*) *p* = *single 0 s* ∗ *p*
proof(*transfer fixing*: *s*)
fix *p* :: *'a* ⇒ *'b*
assume ∗: *finite* {*x*. *p x* ≠ *0*}
{ fix *x*
have *prod-fun* (λ*k'*. *s when 0* = *k'*) *p x* =
(∑ *l* :: *'a*. *if l* = *0 then s* ∗ (∑ *q*. *p q when x* = *q*) *else 0*)
by(*auto simp add*: *prod-fun-def when-def intro*: *Sum-any.cong simp del*: *Sum-any.delta*)
also have ... = (λ*k*. *s* ∗ *p k when p k* ≠ *0*) *x* **by**(*simp add*: *when-def*)
also note *calculation* **}**
then show (λ*k*. *s* ∗ *p k when p k* ≠ *0*) = *prod-fun* (λ*k'*. *s when 0* = *k'*) *p*
by(*simp add*: *fun-eq-iff*)
qed

lemma *map-single* [*simp*]:
  $(c = 0 \Longrightarrow f\ 0 = 0) \Longrightarrow map\ f\ (single\ x\ c) = single\ x\ (f\ c)$
by *transfer*(*auto simp add*: *fun-eq-iff when-def*)

lemma *map-eq-zero-iff*: $map\ f\ p = 0 \longleftrightarrow (\forall k \in keys\ p.\ f\ (lookup\ p\ k) = 0)$
by *transfer*(*auto simp add*: *fun-eq-iff when-def*)

72.12 Canonical dense representation of $nat \Rightarrow_0 'a$

abbreviation *no-trailing-zeros* :: $'a$ :: *zero list* ⇒ *bool*
where
  no-trailing-zeros ≡ *no-trailing* ((=) *0*)

lift-definition *nth* :: $'a$ *list* ⇒ (*nat* ⇒$_0$ $'a$::*zero*)
  is *nth-default 0*
  by (*fact finite-nth-default-neq-default*)

The opposite direction is directly specified on (later) type *nat-mapping*.

lemma *nth-Nil* [*simp*]:
  nth [] = *0*
  by *transfer* (*simp add*: *fun-eq-iff*)

lemma *nth-singleton* [*simp*]:
  nth [*v*] = *single 0 v*
proof (*transfer*, *rule ext*)
  fix *n* :: *nat* **and** *v* :: $'a$
  show *nth-default 0* [*v*] *n* = (*v when 0* = *n*)
    by (*auto simp add*: *nth-default-def nth-append*)
qed

lemma *nth-replicate* [*simp*]:
  nth (*replicate n 0* @ [*v*]) = *single n v*
proof (*transfer*, *rule ext*)
  fix *m n* :: *nat* **and** *v* :: $'a$
  show *nth-default 0* (*replicate n 0* @ [*v*]) *m* = (*v when n* = *m*)
    by (*auto simp add*: *nth-default-def nth-append*)
qed

lemma *nth-strip-while* [*simp*]:
  nth (*strip-while* ((=) *0*) *xs*) = *nth xs*
  by *transfer* (*fact nth-default-strip-while-dflt*)

lemma *nth-strip-while′* [*simp*]:
  nth (*strip-while* (λ*k*. *k* = *0*) *xs*) = *nth xs*
  by (*subst eq-commute*) (*fact nth-strip-while*)

lemma *nth-eq-iff*:
  nth xs = *nth ys* ⟷ *strip-while* (*HOL.eq 0*) *xs* = *strip-while* (*HOL.eq 0*) *ys*
  by *transfer* (*simp add*: *nth-default-eq-iff*)

lemma *lookup-nth* [*simp*]:
lookup (*nth xs*) = *nth-default 0 xs*
by (*fact nth.rep-eq*)

lemma *keys-nth* [*simp*]:
keys (*nth xs*) = *fst* ‘ {(*n*, *v*) ∈ *set* (*enumerate 0 xs*). *v* ≠ *0*}
proof *transfer*
fix *xs* :: *'a list*
{ fix *n*
assume *nth-default 0 xs n* ≠ *0*
then have *n* < *length xs* **and** *xs* ! *n* ≠ *0*
by (*auto simp add*: *nth-default-def split*: *if-splits*)
then have (*n*, *xs* ! *n*) ∈ {(*n*, *v*). (*n*, *v*) ∈ *set* (*enumerate 0 xs*) ∧ *v* ≠ *0*} (**is** *?x* ∈ *?A*)
by (*auto simp add*: *in-set-conv-nth enumerate-eq-zip*)
then have *fst ?x* ∈ *fst* ‘ *?A*
by *blast*
then have *n* ∈ *fst* ‘ {(*n*, *v*). (*n*, *v*) ∈ *set* (*enumerate 0 xs*) ∧ *v* ≠ *0*}
by *simp*
}
then show {*k*. *nth-default 0 xs k* ≠ *0*} = *fst* ‘ {(*n*, *v*). (*n*, *v*) ∈ *set* (*enumerate 0 xs*) ∧ *v* ≠ *0*}
by (*auto simp add*: *in-enumerate-iff-nth-default-eq*)
qed

lemma *range-nth* [*simp*]:
range (*nth xs*) = *set xs* − {*0*}
by *transfer simp*

lemma *degree-nth*:
no-trailing-zeros xs ⟹ *degree* (*nth xs*) = *length xs*
unfolding *degree-def* **proof** *transfer*
fix *xs* :: *'a list*
assume ∗: *no-trailing-zeros xs*
let *?A* = {*n*. *nth-default 0 xs n* ≠ *0*}
let *?f* = *nth-default 0 xs*
let *?bound* = *Max* (*insert 0* (*Suc* ‘ {*n*. *?f n* ≠ *0*}))
show *?bound* = *length xs*
proof (*cases xs* = [])
case *False*
with ∗ **obtain** *n* **where** *n*: *n* < *length xs xs* ! *n* ≠ *0*
by (*fastforce simp add*: *no-trailing-unfold last-conv-nth neq-Nil-conv*)
then have *?bound* = *Max* (*Suc* ‘ {*k*. (*k* < *length xs* ⟶ *xs* ! *k* ≠ (*0*::*'a*)) ∧ *k* < *length xs*})
by (*subst Max-insert*) (*auto simp add*: *nth-default-def*)
also let *?A* = {*k*. *k* < *length xs* ∧ *xs* ! *k* ≠ *0*}
have {*k*. (*k* < *length xs* ⟶ *xs* ! *k* ≠ (*0*::*'a*)) ∧ *k* < *length xs*} = *?A* **by** *auto*
also have *Max* (*Suc* ‘ *?A*) = *Suc* (*Max ?A*) **using** *n*
by (*subst mono-Max-commute* [**where** *f* = *Suc*, *symmetric*]) (*auto simp add*:

```
mono-Suc)
   also {
     have Max ?A ∈ ?A using n Max-in [of ?A] by fastforce
     hence Suc (Max ?A) ≤ length xs by simp
     moreover from ∗ False have length xs − 1 ∈ ?A
       by(auto simp add: no-trailing-unfold last-conv-nth)
     hence length xs − 1 ≤ Max ?A using Max-ge[of ?A length xs − 1] by auto
     hence length xs ≤ Suc (Max ?A) by simp
     ultimately have Suc (Max ?A) = length xs by simp }
   finally show ?thesis .
 qed simp
qed

lemma nth-trailing-zeros [simp]:
 nth (xs @ replicate n 0) = nth xs
 by transfer simp

lemma nth-idem:
 nth (List.map (lookup f) [0..<degree f]) = f
 unfolding degree-def by transfer
   (auto simp add: nth-default-def fun-eq-iff not-less)

lemma nth-idem-bound:
 assumes degree f ≤ n
 shows nth (List.map (lookup f) [0..<n]) = f
proof −
 from assms obtain m where n = degree f + m
   by (blast dest: le-Suc-ex)
 then have [0..<n] = [0..<degree f] @ [degree f..<degree f + m]
   by (simp add: upt-add-eq-append [of 0])
 moreover have List.map (lookup f) [degree f..<degree f + m] = replicate m 0
   by (rule replicate-eqI) (auto simp add: beyond-degree-lookup-zero)
 ultimately show ?thesis by (simp add: nth-idem)
qed
```

72.13 Canonical sparse representation of $'a \Rightarrow_0 'b$

```
lift-definition the-value :: ('a × 'b) list ⇒ 'a ⇒₀ 'b::zero
 is λxs k. case map-of xs k of None ⇒ 0 | Some v ⇒ v
proof −
 fix xs :: ('a × 'b) list
 have fin: finite {k. ∃ v. map-of xs k = Some v}
   using finite-dom-map-of [of xs] unfolding dom-def by auto
 then show finite {k. (case map-of xs k of None ⇒ 0 | Some v ⇒ v) ≠ 0}
   using fin by (simp split: option.split)
qed

definition items :: ('a::linorder ⇒₀ 'b::zero) ⇒ ('a × 'b) list
where
```

items f = List.map (λk. (k, lookup f k)) (sorted-list-of-set (keys f))

For the canonical sparse representation we provide both directions of morphisms since the specification of ordered association lists in theory *OAL-ist* will support arbitrary linear orders *linorder* as keys, not just natural numbers *nat*.

lemma *the-value-items* [*simp*]:
the-value (items f) = f
unfolding *items-def*
by *transfer* (*simp add*: *fun-eq-iff map-of-map-restrict restrict-map-def*)

lemma *lookup-the-value*:
lookup (the-value xs) k = (case map-of xs k of None ⇒ 0 | Some v ⇒ v)
by *transfer rule*

lemma *items-the-value*:
assumes *sorted* (*List.map fst xs*) **and** *distinct* (*List.map fst xs*) **and** *0 ∉ snd ' set xs*
shows *items* (*the-value xs*) = *xs*
proof −
from *assms* **have** *sorted-list-of-set* (*set* (*List.map fst xs*)) = *List.map fst xs*
unfolding *sorted-list-of-set-sort-remdups* **by** (*simp add*: *distinct-remdups-id sorted-sort-id*)
moreover from *assms* **have** *keys* (*the-value xs*) = *fst ' set xs*
by *transfer* (*auto simp add*: *image-def split*: *option.split dest*: *set-map-of-compr*)
ultimately show *?thesis*
unfolding *items-def* **using** *assms*
by (*auto simp add*: *lookup-the-value intro*: *map-idI*)
qed

lemma *the-value-Nil* [*simp*]:
the-value [] = 0
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *the-value-Cons* [*simp*]:
the-value (x # xs) = update (fst x) (snd x) (the-value xs)
by *transfer* (*simp add*: *fun-eq-iff*)

lemma *items-zero* [*simp*]:
items 0 = []
unfolding *items-def* **by** *simp*

lemma *items-one* [*simp*]:
items 1 = [(0, 1)]
unfolding *items-def* **by** *transfer simp*

lemma *items-single* [*simp*]:
items (single k v) = (if v = 0 then [] else [(k, v)])
unfolding *items-def* **by** *simp*

lemma *in-set-items-iff* [*simp*]:
  $(k, v) \in set\ (items\ f) \longleftrightarrow k \in keys\ f \wedge lookup\ f\ k = v$
  unfolding *items-def* **by** *transfer auto*

72.14 Size estimation

context
  fixes $f :: 'a \Rightarrow nat$
  and $g :: 'b :: zero \Rightarrow nat$
begin

definition *poly-mapping-size* :: $('a \Rightarrow_0 'b) \Rightarrow nat$
where
  $poly\text{-}mapping\text{-}size\ m = g\ 0 + (\sum k \in keys\ m.\ Suc\ (f\ k + g\ (lookup\ m\ k)))$

lemma *poly-mapping-size-0* [*simp*]:
  $poly\text{-}mapping\text{-}size\ 0 = g\ 0$
  by (*simp add*: *poly-mapping-size-def*)

lemma *poly-mapping-size-single* [*simp*]:
  $poly\text{-}mapping\text{-}size\ (single\ k\ v) = (if\ v = 0\ then\ g\ 0\ else\ g\ 0 + f\ k + g\ v + 1)$
  unfolding *poly-mapping-size-def* **by** *transfer simp*

lemma *keys-less-poly-mapping-size*:
  $k \in keys\ m \Longrightarrow f\ k + g\ (lookup\ m\ k) < poly\text{-}mapping\text{-}size\ m$
unfolding *poly-mapping-size-def*
proof *transfer*
  fix $k :: 'a$ **and** $m :: 'a \Rightarrow 'b$ **and** $f :: 'a \Rightarrow nat$ **and** g
  let *?keys* $= \{k.\ m\ k \neq 0\}$
  assume $*$: *finite ?keys* $k \in$ *?keys*
  then have $f\ k + g\ (m\ k) = (\sum k' \in$ *?keys*. $f\ k' + g\ (m\ k')$ *when* $k' = k)$
    by (*simp add*: *sum.delta when-def*)
  also have $\ldots < (\sum k' \in$ *?keys*. *Suc* $(f\ k' + g\ (m\ k')))$ **using** $*$
    by (*intro sum-strict-mono*) (*auto simp add*: *when-def*)
  also have $\ldots \leq g\ 0 + \ldots$ **by** *simp*
  finally have $f\ k + g\ (m\ k) < \ldots$ **.**
  then show $f\ k + g\ (m\ k) < g\ 0 + (\sum k \mid m\ k \neq 0.\ Suc\ (f\ k + g\ (m\ k)))$
    by *simp*
qed

lemma *lookup-le-poly-mapping-size*:
  $g\ (lookup\ m\ k) \leq poly\text{-}mapping\text{-}size\ m$
proof (*cases* $k \in keys\ m$)
  case *True*
  with *keys-less-poly-mapping-size* [*of k m*]
  show *?thesis* **by** *simp*
next
  case *False*

then show *?thesis*
by (*simp add*: *Poly-Mapping.poly-mapping-size-def in-keys-iff*)
qed

lemma *poly-mapping-size-estimation*:
$k \in keys\ m \Longrightarrow y \leq f\ k + g\ (lookup\ m\ k) \Longrightarrow y < poly\text{-}mapping\text{-}size\ m$
using *keys-less-poly-mapping-size* **by** (*auto intro*: *le-less-trans*)

lemma *poly-mapping-size-estimation2*:
assumes $v \in range\ m$ **and** $y \leq g\ v$
shows $y < poly\text{-}mapping\text{-}size\ m$
proof −
from *assms* **obtain** k **where** $*$: $lookup\ m\ k = v\ v \neq 0$
by *transfer blast*
from $*$ **have** $k \in keys\ m$
by (*simp add*: *in-keys-iff*)
then show *?thesis*
proof (*rule poly-mapping-size-estimation*)
from *assms* $*$ **have** $y \leq g\ (lookup\ m\ k)$
by *simp*
then show $y \leq f\ k + g\ (lookup\ m\ k)$
by *simp*
qed
qed

end

lemma *poly-mapping-size-one* [*simp*]:
$poly\text{-}mapping\text{-}size\ f\ g\ 1 = g\ 0 + f\ 0 + g\ 1 + 1$
unfolding *poly-mapping-size-def* **by** *transfer simp*

lemma *poly-mapping-size-cong* [*fundef-cong*]:
$m = m' \Longrightarrow g\ 0 = g'\ 0 \Longrightarrow (\bigwedge k.\ k \in keys\ m' \Longrightarrow f\ k = f'\ k)$
$\Longrightarrow (\bigwedge v.\ v \in range\ m' \Longrightarrow g\ v = g'\ v)$
$\Longrightarrow poly\text{-}mapping\text{-}size\ f\ g\ m = poly\text{-}mapping\text{-}size\ f'\ g'\ m'$
by (*auto simp add*: *poly-mapping-size-def intro*!: *sum.cong*)

instantiation *poly-mapping* :: (*type*, *zero*) *size*
begin

definition $size = poly\text{-}mapping\text{-}size\ (\lambda\text{-}.\ 0)\ (\lambda\text{-}.\ 0)$

instance ..

end

72.15 Further mapping operations and properties

It is like in algebra: there are many definitions, some are also used

lift-definition *mapp* ::
(*'a* ⇒ *'b* :: *zero* ⇒ *'c* :: *zero*) ⇒ (*'a* ⇒$_0$ *'b*) ⇒ (*'a* ⇒$_0$ *'c*)
is λ*f p k*. (*if k* ∈ *keys p then f k* (*lookup p k*) *else 0*)
by *simp*

lemma *mapp-cong* [*fundef-cong*]:
⟦ *m* = *m'*; ⋀*k*. *k* ∈ *keys m'* ⟹ *f k* (*lookup m' k*) = *f' k* (*lookup m' k*) ⟧
⟹ *mapp f m* = *mapp f' m'*
by *transfer* (*auto simp add*: *fun-eq-iff*)

lemma *lookup-mapp*:
lookup (*mapp f p*) *k* = (*f k* (*lookup p k*) *when k* ∈ *keys p*)
by (*simp add*: *mapp.rep-eq*)

lemma *keys-mapp-subset*: *keys* (*mapp f p*) ⊆ *keys p*
by (*meson in-keys-iff mapp.rep-eq subsetI*)

72.16 Free Abelian Groups Over a Type

abbreviation *frag-of* :: *'a* ⇒ *'a* ⇒$_0$ *int*
where *frag-of c* ≡ *Poly-Mapping.single c* (*1*::*int*)

lemma *lookup-frag-of* [*simp*]:
Poly-Mapping.lookup(*frag-of c*) = (λ*x*. *if x* = *c then 1 else 0*)
by (*force simp add*: *lookup-single-not-eq*)

lemma *frag-of-nonzero* [*simp*]: *frag-of a* ≠ *0*
proof −
let *?f* = λ*x*. *if x* = *a then 1 else* (*0*::*int*)
have *?f* ≠ (λ*x*. *0*::*int*)
by (*auto simp*: *fun-eq-iff*)
then have *Poly-Mapping.lookup* (*Abs-poly-mapping ?f*) ≠ *Poly-Mapping.lookup* (*Abs-poly-mapping* (λ*x*. *0*))
by *fastforce*
then show *?thesis*
by (*metis lookup-single-eq lookup-zero*)
qed

definition *frag-cmul* :: *int* ⇒ (*'a* ⇒$_0$ *int*) ⇒ (*'a* ⇒$_0$ *int*)
where *frag-cmul c a* = *Abs-poly-mapping* (λ*x*. *c* ∗ *Poly-Mapping.lookup a x*)

lemma *frag-cmul-zero* [*simp*]: *frag-cmul 0 x* = *0*
by (*simp add*: *frag-cmul-def*)

lemma *frag-cmul-zero2* [*simp*]: *frag-cmul c 0* = *0*
by (*simp add*: *frag-cmul-def*)

lemma *frag-cmul-one* [*simp*]: *frag-cmul 1 x* = *x*
by (*auto simp*: *frag-cmul-def Poly-Mapping.poly-mapping.lookup-inverse*)

lemma *frag-cmul-minus-one* [*simp*]: *frag-cmul* (−*1*) *x* = −*x*
by (*simp add*: *frag-cmul-def uminus-poly-mapping-def poly-mapping-eqI*)

lemma *frag-cmul-cmul* [*simp*]: *frag-cmul c* (*frag-cmul d x*) = *frag-cmul* (*c*∗*d*) *x*
by (*simp add*: *frag-cmul-def mult-ac*)

lemma *lookup-frag-cmul* [*simp*]: *poly-mapping.lookup* (*frag-cmul c x*) *i* = *c* ∗ *poly-mapping.lookup x i*
by (*simp add*: *frag-cmul-def*)

lemma *minus-frag-cmul* [*simp*]: − *frag-cmul k x* = *frag-cmul* (−*k*) *x*
by (*simp add*: *poly-mapping-eqI*)

lemma *keys-frag-of*: *Poly-Mapping.keys*(*frag-of a*) = {*a*}
by *simp*

lemma *finite-cmul-nonzero*: *finite* {*x*. *c* ∗ *Poly-Mapping.lookup a x* ≠ (*0*::*int*)}
by *simp*

lemma *keys-cmul*: *Poly-Mapping.keys*(*frag-cmul c a*) ⊆ *Poly-Mapping.keys a*
using *finite-cmul-nonzero* [*of c a*]
by (*metis lookup-frag-cmul mult-zero-right not-in-keys-iff-lookup-eq-zero subsetI*)

lemma *keys-cmul-iff* [*iff*]: *i* ∈ *Poly-Mapping.keys* (*frag-cmul c x*) ⟷ *i* ∈ *Poly-Mapping.keys x* ∧ *c* ≠ *0*
apply (*auto simp*:)
apply (*meson subsetD keys-cmul*)
by (*metis in-keys-iff lookup-frag-cmul mult-eq-0-iff*)

lemma *keys-minus* [*simp*]: *Poly-Mapping.keys*(−*a*) = *Poly-Mapping.keys a*
by (*metis* (*no-types*, *opaque-lifting*) *in-keys-iff lookup-uminus neg-equal-0-iff-equal subsetI subset-antisym*)

lemma *keys-diff*:
Poly-Mapping.keys(*a* − *b*) ⊆ *Poly-Mapping.keys a* ∪ *Poly-Mapping.keys b*
by (*auto simp add*: *in-keys-iff lookup-minus*)

lemma *keys-eq-empty* [*simp*]: *Poly-Mapping.keys c* = {} ⟷ *c* = *0*
by (*metis in-keys-iff keys-zero lookup-zero poly-mapping-eqI*)

lemma *frag-cmul-eq-0-iff* [*simp*]: *frag-cmul k c* = *0* ⟷ *k*=*0* ∨ *c*=*0*
by *auto* (*metis subsetI subset-antisym keys-cmul-iff keys-eq-empty*)

lemma *frag-of-eq*: *frag-of x* = *frag-of y* ⟷ *x* = *y*
by (*metis lookup-single-eq lookup-single-not-eq zero-neq-one*)

lemma *frag-cmul-distrib*: *frag-cmul* (*c*+*d*) *a* = *frag-cmul c a* + *frag-cmul d a*

by (*simp add*: *frag-cmul-def plus-poly-mapping-def int-distrib*)

lemma *frag-cmul-distrib2*: *frag-cmul c* (*a*+*b*) = *frag-cmul c a* + *frag-cmul c b*
proof −
have *finite* {*x*. *poly-mapping.lookup a x* + *poly-mapping.lookup b x* ≠ *0*}
using *keys-add* [*of a b*]
by (*metis* (*no-types*, *lifting*) *finite-keys finite-subset keys.rep-eq lookup-add mem-Collect-eq subsetI*)
then show *?thesis*
by (*simp add*: *frag-cmul-def plus-poly-mapping-def int-distrib*)
qed

lemma *frag-cmul-diff-distrib*: *frag-cmul* (*a* − *b*) *c* = *frag-cmul a c* − *frag-cmul b c*
by (*auto simp*: *left-diff-distrib lookup-minus poly-mapping-eqI*)

lemma *frag-cmul-sum*:
frag-cmul a (*sum b I*) = (∑ *i*∈*I*. *frag-cmul a* (*b i*))
proof (*induction rule*: *infinite-finite-induct*)
case (*insert i I*)
then show *?case*
by (*auto simp*: *algebra-simps frag-cmul-distrib2*)
qed *auto*

lemma *keys-sum*: *Poly-Mapping.keys*(*sum b I*) ⊆ (⋃ *i* ∈*I*. *Poly-Mapping.keys*(*b i*))
proof (*induction I rule*: *infinite-finite-induct*)
case (*insert i I*)
then show *?case*
using *keys-add* [*of b i sum b I*] **by** *auto*
qed *auto*

definition *frag-extend* :: ('*b* ⇒ '*a* ⇒$_0$ *int*) ⇒ ('*b* ⇒$_0$ *int*) ⇒ '*a* ⇒$_0$ *int*
where *frag-extend b x* ≡ (∑ *i* ∈ *Poly-Mapping.keys x*. *frag-cmul* (*Poly-Mapping.lookup x i*) (*b i*))

lemma *frag-extend-0* [*simp*]: *frag-extend b 0* = *0*
by (*simp add*: *frag-extend-def*)

lemma *frag-extend-of* [*simp*]: *frag-extend f* (*frag-of a*) = *f a*
by (*simp add*: *frag-extend-def*)

lemma *frag-extend-cmul*:
frag-extend f (*frag-cmul c x*) = *frag-cmul c* (*frag-extend f x*)
by (*auto simp*: *frag-extend-def frag-cmul-sum intro*: *sum.mono-neutral-cong-left*)

lemma *frag-extend-minus*:
frag-extend f (− *x*) = − (*frag-extend f x*)
using *frag-extend-cmul* [*of f* −*1*] **by** *simp*

lemma *frag-extend-add*:
frag-extend f (a+b) = (frag-extend f a) + (frag-extend f b)
proof −
have ∗: (∑ *i*∈*Poly-Mapping.keys a. frag-cmul* (*poly-mapping.lookup a i*) (*f i*))
= (∑ *i*∈*Poly-Mapping.keys a* ∪ *Poly-Mapping.keys b. frag-cmul* (*poly-mapping.lookup a i*) (*f i*))
(∑ *i*∈*Poly-Mapping.keys b. frag-cmul* (*poly-mapping.lookup b i*) (*f i*))
= (∑ *i*∈*Poly-Mapping.keys a* ∪ *Poly-Mapping.keys b. frag-cmul* (*poly-mapping.lookup b i*) (*f i*))
by (*auto simp*: *in-keys-iff intro*: *sum.mono-neutral-cong-left*)
have *frag-extend f (a+b)* = (∑ *i*∈*Poly-Mapping.keys* (*a* + *b*).
frag-cmul (*poly-mapping.lookup a i*) (*f i*) + *frag-cmul* (*poly-mapping.lookup b i*) (*f i*))
by (*auto simp*: *frag-extend-def Poly-Mapping.lookup-add frag-cmul-distrib*)
also have ... = (∑ *i* ∈ *Poly-Mapping.keys a* ∪ *Poly-Mapping.keys b. frag-cmul* (*poly-mapping.lookup a i*) (*f i*)
+ *frag-cmul* (*poly-mapping.lookup b i*) (*f i*))
apply (*rule sum.mono-neutral-cong-left*)
using *keys-add* [*of a b*]
apply (*auto simp add*: *in-keys-iff plus-poly-mapping.rep-eq frag-cmul-distrib* [*symmetric*])
done
also have ... = (*frag-extend f a*) + (*frag-extend f b*)
by (*auto simp*: ∗ *sum.distrib frag-extend-def*)
finally show *?thesis* **.**
qed

lemma *frag-extend-diff*:
frag-extend f (a−b) = (frag-extend f a) − (frag-extend f b)
by (*metis* (*no-types*, *opaque-lifting*) *add-uminus-conv-diff frag-extend-add frag-extend-minus*)

lemma *frag-extend-sum*:
finite I ⟹ *frag-extend f* (∑ *i*∈*I*. *g i*) = *sum* (*frag-extend f o g*) *I*
by (*induction I rule*: *finite-induct*) (*simp-all add*: *frag-extend-add*)

lemma *frag-extend-eq*:
(⋀*f*. *f* ∈ *Poly-Mapping.keys c* ⟹ *g f* = *h f*) ⟹ *frag-extend g c* = *frag-extend h c*
by (*simp add*: *frag-extend-def*)

lemma *frag-extend-eq-0*:
(⋀*x*. *x* ∈ *Poly-Mapping.keys c* ⟹ *f x* = *0*) ⟹ *frag-extend f c* = *0*
by (*simp add*: *frag-extend-def*)

lemma *keys-frag-extend*: *Poly-Mapping.keys*(*frag-extend f c*) ⊆ (⋃ *x* ∈ *Poly-Mapping.keys c. Poly-Mapping.keys*(*f x*))
unfolding *frag-extend-def*
using *keys-sum* **by** *fastforce*

lemma *frag-expansion*: $a = \textit{frag-extend frag-of } a$
proof −
have ∗: *finite I*
$\Longrightarrow$ *Poly-Mapping.lookup* $(\sum i{\in}I.$ *frag-cmul* (*Poly-Mapping.lookup a i*) (*frag-of i*)) *j* =
(*if j* ∈ *I then Poly-Mapping.lookup a j else 0*) **for** *I j*
by (*induction I rule*: *finite-induct*) (*auto simp*: *lookup-single lookup-add*)
show *?thesis*
unfolding *frag-extend-def*
by (*rule poly-mapping-eqI*) (*fastforce simp add*: *in-keys-iff* ∗)
qed

lemma *frag-closure-minus-cmul*:
assumes *P 0* **and** *P*: $\bigwedge x\ y.\ [\![P\ x;\ P\ y]\!] \Longrightarrow P(x - y)\ P\ c$
shows *P*(*frag-cmul k c*)
proof −
have *P* (*frag-cmul* (*int n*) *c*) **for** *n*
apply (*induction n*)
apply (*simp-all add*: *assms frag-cmul-distrib*)
by (*metis add.left-neutral add-diff-cancel-right′ add-uminus-conv-diff P*)
then show *?thesis*
by (*metis* (*no-types*, *opaque-lifting*) *add-diff-eq assms*(*2*) *diff-add-cancel frag-cmul-distrib int-diff-cases*)
qed

lemma *frag-induction* [*consumes 1*, *case-names zero one diff*]:
assumes *supp*: *Poly-Mapping.keys c* ⊆ *S*
and *0*: *P 0* **and** *sing*: $\bigwedge x.\ x \in S \Longrightarrow P(\textit{frag-of } x)$
and *diff*: $\bigwedge a\ b.\ [\![P\ a;\ P\ b]\!] \Longrightarrow P(a - b)$
shows *P c*
proof −
have *P* $(\sum i{\in}I.$ *frag-cmul* (*poly-mapping.lookup c i*) (*frag-of i*))
if *I* ⊆ *Poly-Mapping.keys c* **for** *I*
using *finite-subset* [*OF that finite-keys* [*of c*]] *that supp*
proof (*induction I arbitrary*: *c rule*: *finite-induct*)
case *empty*
then show *?case*
by (*auto simp*: *0*)
next
case (*insert i I c*)
have *ab*: $a{+}b = a - (0 - b)$ **for** *a b* :: $'a \Rightarrow_0 int$
by *simp*
have *Pfrag*: *P* (*frag-cmul* (*poly-mapping.lookup c i*) (*frag-of i*))
by (*metis 0 diff frag-closure-minus-cmul insert.prems insert-subset sing subset-iff*)
show *?case*
apply (*simp add*: *insert.hyps*)
apply (*subst ab*)

```
    using insert apply (blast intro: assms Pfrag)
    done
  qed
  then show ?thesis
    by (subst frag-expansion) (auto simp add: frag-extend-def)
qed

lemma frag-extend-compose:
  frag-extend f (frag-extend (frag-of o g) c) = frag-extend (f o g) c
  using subset-UNIV
  by (induction c rule: frag-induction) (auto simp: frag-extend-diff)

lemma frag-split:
  fixes c :: 'a ⇒₀ int
  assumes Poly-Mapping.keys c ⊆ S ∪ T
  obtains d e where Poly-Mapping.keys d ⊆ S Poly-Mapping.keys e ⊆ T d + e
= c
proof
  let ?d = frag-extend (λf. if f ∈ S then frag-of f else 0) c
  let ?e = frag-extend (λf. if f ∈ S then 0 else frag-of f) c
  show Poly-Mapping.keys ?d ⊆ S Poly-Mapping.keys ?e ⊆ T
   using assms by (auto intro!: order-trans [OF keys-frag-extend] split: if-split-asm)
  show ?d + ?e = c
    using assms
  proof (induction c rule: frag-induction)
    case (diff a b)
    then show ?case
     by (metis (no-types, lifting) frag-extend-diff add-diff-eq diff-add-eq diff-add-eq-diff-diff-swap)
  qed auto
qed

hide-const (open) lookup single update keys range map map-key degree nth the-value
items foldr mapp

end
```

73 Exponentiation by Squaring

```
theory Power-By-Squaring
  imports Main
begin

context
  fixes f :: 'a ⇒ 'a ⇒ 'a
begin

function efficient-funpow :: 'a ⇒ 'a ⇒ nat ⇒ 'a where
  efficient-funpow y x 0 = y
| efficient-funpow y x (Suc 0) = f x y
```

| $n \neq 0 \Longrightarrow even\ n \Longrightarrow efficient\text{-}funpow\ y\ x\ n = efficient\text{-}funpow\ y\ (f\ x\ x)\ (n\ div\ 2)$
| $n \neq 1 \Longrightarrow odd\ n \Longrightarrow efficient\text{-}funpow\ y\ x\ n = efficient\text{-}funpow\ (f\ x\ y)\ (f\ x\ x)\ (n\ div\ 2)$
by *force+*
termination by (*relation measure* (*snd* ∘ *snd*)) (*auto elim*: *oddE*)

lemma *efficient-funpow-code* [*code*]:
efficient-funpow y x n =
(*if n* = *0 then y*
else if n = *1 then f x y*
else if even n then efficient-funpow y (*f x x*) (*n div 2*)
else efficient-funpow (*f x y*) (*f x x*) (*n div 2*))
by (*induction y x n rule*: *efficient-funpow.induct*) *auto*

end

lemma *efficient-funpow-correct*:
assumes *f-assoc*: ⋀*x z*. *f x* (*f x z*) = *f* (*f x x*) *z*
shows *efficient-funpow f y x n* = (*f x* ^^ *n*) *y*
proof −
have [*simp*]: *f* ^^ *2* = (λ*x*. *f* (*f x*)) **for** *f* :: *'a* ⇒ *'a*
by (*simp add*: *eval-nat-numeral o-def*)
show *?thesis*
by (*induction y x n rule*: *efficient-funpow.induct*[*of* - *f*])
(*auto elim*!: *evenE oddE simp*: *funpow-mult* [*symmetric*] *funpow-Suc-right f-assoc*
simp del: *funpow.simps*(*2*))
qed

context *monoid-mult*
begin

lemma *power-by-squaring*: *efficient-funpow* (∗) (*1* :: *'a*) = (^)
proof (*intro ext*)
fix *x* :: *'a* **and** *n*
have *efficient-funpow* (∗) *1 x n* = ((∗) *x* ^^ *n*) *1*
by (*subst efficient-funpow-correct*) (*simp-all add*: *mult.assoc*)
also have ... = *x* ^ *n*
by (*induction n*) *simp-all*
finally show *efficient-funpow* (∗) *1 x n* = *x* ^ *n* **.**
qed

end

end

74 Preorders with explicit equivalence relation

theory *Preorder*
imports *Main*
begin

class *preorder-equiv* = *preorder*
begin

definition *equiv* :: $'a \Rightarrow 'a \Rightarrow bool$
  where *equiv x y* $\longleftrightarrow x \leq y \wedge y \leq x$

notation
  equiv ($'(\approx')$) **and**
  equiv ((-/ $\approx$ -) [*51*, *51*] *50*)

lemma *equivD1*: $x \leq y$ **if** $x \approx y$
  using *that* **by** (*simp add*: *equiv-def*)

lemma *equivD2*: $y \leq x$ **if** $x \approx y$
  using *that* **by** (*simp add*: *equiv-def*)

lemma *equiv-refl* [*iff*]: $x \approx x$
  by (*simp add*: *equiv-def*)

lemma *equiv-sym*: $x \approx y \longleftrightarrow y \approx x$
  by (*auto simp add*: *equiv-def*)

lemma *equiv-trans*: $x \approx y \Longrightarrow y \approx z \Longrightarrow x \approx z$
  by (*auto simp*: *equiv-def intro*: *order-trans*)

lemma *equiv-antisym*: $x \leq y \Longrightarrow y \leq x \Longrightarrow x \approx y$
  by (*simp only*: *equiv-def*)

lemma *less-le*: $x < y \longleftrightarrow x \leq y \wedge \neg\, x \approx y$
  by (*auto simp add*: *equiv-def less-le-not-le*)

lemma *le-less*: $x \leq y \longleftrightarrow x < y \vee x \approx y$
  by (*auto simp add*: *equiv-def less-le*)

lemma *le-imp-less-or-equiv*: $x \leq y \Longrightarrow x < y \vee x \approx y$
  by (*simp add*: *less-le*)

lemma *less-imp-not-equiv*: $x < y \Longrightarrow \neg\, x \approx y$
  by (*simp add*: *less-le*)

lemma *not-equiv-le-trans*: $\neg\, a \approx b \Longrightarrow a \leq b \Longrightarrow a < b$
  by (*simp add*: *less-le*)

lemma *le-not-equiv-trans*: $a \leq b \Longrightarrow \neg\ a \approx b \Longrightarrow a < b$
by (*rule not-equiv-le-trans*)

lemma *antisym-conv*: $y \leq x \Longrightarrow x \leq y \longleftrightarrow x \approx y$
by (*simp add*: *equiv-def*)

end

ML-file ‹~~/*src/Provers/preorder.ML*›

ML ‹
structure Quasi = Quasi-Tac(
struct

val le-trans = @{thm order-trans};
val le-refl = @{thm order-refl};
val eqD1 = @{thm equivD1};
val eqD2 = @{thm equivD2};
val less-reflE = @{thm less-irrefl};
val less-imp-le = @{thm less-imp-le};
val le-neq-trans = @{thm le-not-equiv-trans};
val neq-le-trans = @{thm not-equiv-le-trans};
val less-imp-neq = @{thm less-imp-not-equiv};

fun decomp-quasi thy (Const (@{const-name less-eq}, -) $ t1 $ t2) = SOME (t1, <=, t2)
| decomp-quasi thy (Const (@{const-name less}, -) $ t1 $ t2) = SOME (t1, <, t2)
| decomp-quasi thy (Const (@{const-name equiv}, -) $ t1 $ t2) = SOME (t1, =, t2)
| decomp-quasi thy (Const (@{const-name Not}, -) $ (Const (@{const-name equiv}, -) $ t1 $ t2)) = SOME (t1, ~=, t2)
| decomp-quasi thy - = NONE;

fun decomp-trans thy t = case decomp-quasi thy t of
x as SOME (t1, <=, t2) => x
| - => NONE;

end
);
›

end

75 Additive group operations on product types

theory *Product-Plus*
imports *Main*
begin

75.1 Operations

instantiation *prod* :: (*zero*, *zero*) *zero*
begin

definition *zero-prod-def*: $0 = (0, 0)$

instance ..
end

instantiation *prod* :: (*plus*, *plus*) *plus*
begin

definition *plus-prod-def*:
$x + y = (fst\ x + fst\ y, snd\ x + snd\ y)$

instance ..
end

instantiation *prod* :: (*minus*, *minus*) *minus*
begin

definition *minus-prod-def*:
$x - y = (fst\ x - fst\ y, snd\ x - snd\ y)$

instance ..
end

instantiation *prod* :: (*uminus*, *uminus*) *uminus*
begin

definition *uminus-prod-def*:
$- x = (- fst\ x, - snd\ x)$

instance ..
end

lemma *fst-zero* [*simp*]: $fst\ 0 = 0$
unfolding *zero-prod-def* **by** *simp*

lemma *snd-zero* [*simp*]: $snd\ 0 = 0$
unfolding *zero-prod-def* **by** *simp*

lemma *fst-add* [*simp*]: $fst\ (x + y) = fst\ x + fst\ y$
unfolding *plus-prod-def* **by** *simp*

lemma *snd-add* [*simp*]: $snd\ (x + y) = snd\ x + snd\ y$
unfolding *plus-prod-def* **by** *simp*

lemma *fst-diff* [*simp*]: $fst\ (x - y) = fst\ x - fst\ y$

unfolding *minus-prod-def* **by** *simp*

lemma *snd-diff* [*simp*]: *snd* $(x - y) =$ *snd* $x -$ *snd* y
unfolding *minus-prod-def* **by** *simp*

lemma *fst-uminus* [*simp*]: *fst* $(- x) = -$ *fst* x
unfolding *uminus-prod-def* **by** *simp*

lemma *snd-uminus* [*simp*]: *snd* $(- x) = -$ *snd* x
unfolding *uminus-prod-def* **by** *simp*

lemma *add-Pair* [*simp*]: $(a, b) + (c, d) = (a + c, b + d)$
unfolding *plus-prod-def* **by** *simp*

lemma *diff-Pair* [*simp*]: $(a, b) - (c, d) = (a - c, b - d)$
unfolding *minus-prod-def* **by** *simp*

lemma *uminus-Pair* [*simp*, *code*]: $- (a, b) = (- a, - b)$
unfolding *uminus-prod-def* **by** *simp*

75.2 Class instances

instance *prod* :: (*semigroup-add*, *semigroup-add*) *semigroup-add*
by *standard* (*simp add*: *prod-eq-iff add.assoc*)

instance *prod* :: (*ab-semigroup-add*, *ab-semigroup-add*) *ab-semigroup-add*
by *standard* (*simp add*: *prod-eq-iff add.commute*)

instance *prod* :: (*monoid-add*, *monoid-add*) *monoid-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

instance *prod* :: (*comm-monoid-add*, *comm-monoid-add*) *comm-monoid-add*
by *standard* (*simp add*: *prod-eq-iff*)

instance *prod* :: (*cancel-semigroup-add*, *cancel-semigroup-add*) *cancel-semigroup-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

instance *prod* :: (*cancel-ab-semigroup-add*, *cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
by *standard* (*simp-all add*: *prod-eq-iff diff-diff-eq*)

instance *prod* :: (*cancel-comm-monoid-add*, *cancel-comm-monoid-add*) *cancel-comm-monoid-add*
..

instance *prod* :: (*group-add*, *group-add*) *group-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

instance *prod* :: (*ab-group-add*, *ab-group-add*) *ab-group-add*
by *standard* (*simp-all add*: *prod-eq-iff*)

lemma *fst-sum*: *fst* $(\sum x{\in}A.\ f\ x) = (\sum x{\in}A.\ fst\ (f\ x))$
proof (*cases finite A*)
  case *True*
  then show *?thesis* **by** *induct simp-all*
next
  case *False*
  then show *?thesis* **by** *simp*
qed

lemma *snd-sum*: *snd* $(\sum x{\in}A.\ f\ x) = (\sum x{\in}A.\ snd\ (f\ x))$
proof (*cases finite A*)
  case *True*
  then show *?thesis* **by** *induct simp-all*
next
  case *False*
  then show *?thesis* **by** *simp*
qed

lemma *sum-prod*: $(\sum x{\in}A.\ (f\ x,\ g\ x)) = (\sum x{\in}A.\ f\ x,\ \sum x{\in}A.\ g\ x)$
proof (*cases finite A*)
  case *True*
  then show *?thesis* **by** *induct* (*simp-all add*: *zero-prod-def*)
next
  case *False*
  then show *?thesis* **by** (*simp add*: *zero-prod-def*)
qed

end

76 Roots of real quadratics

theory *Quadratic-Discriminant*
imports *Complex-Main*
begin

definition *discrim* :: *real* ⇒ *real* ⇒ *real* ⇒ *real*
  where $discrim\ a\ b\ c \equiv b^2 - 4 * a * c$

lemma *complete-square*:
  $a \neq 0 \Longrightarrow a * x^2 + b * x + c = 0 \longleftrightarrow (2 * a * x + b)^2 = discrim\ a\ b\ c$
by (*simp add*: *discrim-def*) *algebra*

lemma *discriminant-negative*:
  fixes *a b c x* :: *real*
  assumes $a \neq 0$
    and $discrim\ a\ b\ c < 0$
  shows $a * x^2 + b * x + c \neq 0$
proof −
  have $(2 * a * x + b)^2 \geq 0$

by *simp*
with ‹*discrim a b c* < *0*› **have** $(2 * a * x + b)^2 \neq$ *discrim a b c*
by *arith*
with *complete-square* **and** ‹$a \neq 0$› **show** $a * x^2 + b * x + c \neq 0$
by *simp*
qed

lemma *plus-or-minus-sqrt*:
fixes *x y* :: *real*
assumes $y \geq 0$
shows $x^2 = y \longleftrightarrow x =$ *sqrt y* $\vee x = -$ *sqrt y*
proof
assume $x^2 = y$
then have *sqrt* $(x^2) =$ *sqrt y*
by *simp*
then have *sqrt y* $= |x|$
by *simp*
then show $x =$ *sqrt y* $\vee x = -$ *sqrt y*
by *auto*
next
assume $x =$ *sqrt y* $\vee x = -$ *sqrt y*
then have $x^2 =$ (*sqrt y*)$^2 \vee x^2 = (-$ *sqrt y*$)^2$
by *auto*
with ‹$y \geq 0$› **show** $x^2 = y$
by *simp*
qed

lemma *divide-non-zero*:
fixes *x y z* :: *real*
assumes $x \neq 0$
shows $x * y = z \longleftrightarrow y = z$ / x
proof
show $y = z$ / x **if** $x * y = z$
using ‹$x \neq 0$› *that* **by** (*simp add*: *field-simps*)
show $x * y = z$ **if** $y = z$ / x
using ‹$x \neq 0$› *that* **by** *simp*
qed

lemma *discriminant-nonneg*:
fixes *a b c x* :: *real*
assumes $a \neq 0$
and *discrim a b c* ≥ 0
shows $a * x^2 + b * x + c = 0 \longleftrightarrow$
$x = (-b +$ *sqrt* (*discrim a b c*)) / $(2 * a) \vee$
$x = (-b -$ *sqrt* (*discrim a b c*)) / $(2 * a)$
proof −
from *complete-square* **and** *plus-or-minus-sqrt* **and** *assms*
have $a * x^2 + b * x + c = 0 \longleftrightarrow$
$(2 * a) * x + b =$ *sqrt* (*discrim a b c*) $\vee$

$(2 * a) * x + b = - sqrt\ (discrim\ a\ b\ c)$
by *simp*
also have $\ldots \longleftrightarrow (2 * a) * x = (-b + sqrt\ (discrim\ a\ b\ c)) \vee$
$(2 * a) * x = (-b - sqrt\ (discrim\ a\ b\ c))$
by *auto*
also from ‹$a \neq 0$› **and** *divide-non-zero* [*of* $2 * a\ x$]
have $\ldots \longleftrightarrow x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a)$
by *simp*
finally show $a * x^2 + b * x + c = 0 \longleftrightarrow$
$x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a)$ **.**
qed

lemma *discriminant-zero*:
fixes $a\ b\ c\ x :: real$
assumes $a \neq 0$
and $discrim\ a\ b\ c = 0$
shows $a * x^2 + b * x + c = 0 \longleftrightarrow x = -b / (2 * a)$
by (*simp add*: *discriminant-nonneg assms*)

theorem *discriminant-iff*:
fixes $a\ b\ c\ x :: real$
assumes $a \neq 0$
shows $a * x^2 + b * x + c = 0 \longleftrightarrow$
$discrim\ a\ b\ c \geq 0 \wedge$
$(x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a))$
proof
assume $a * x^2 + b * x + c = 0$
with *discriminant-negative* **and** ‹$a \neq 0$› **have** $\neg(discrim\ a\ b\ c < 0)$
by *auto*
then have $discrim\ a\ b\ c \geq 0$
by *simp*
with *discriminant-nonneg* **and** ‹$a * x^2 + b * x + c = 0$› **and** ‹$a \neq 0$›
have $x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a)$
by *simp*
with ‹$discrim\ a\ b\ c \geq 0$›
show $discrim\ a\ b\ c \geq 0 \wedge$
$(x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a))$ **..**
next
assume $discrim\ a\ b\ c \geq 0 \wedge$
$(x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a))$
then have $discrim\ a\ b\ c \geq 0$ **and**
$x = (-b + sqrt\ (discrim\ a\ b\ c)) / (2 * a) \vee$
$x = (-b - sqrt\ (discrim\ a\ b\ c)) / (2 * a)$

```
    by simp-all
  with discriminant-nonneg and ‹a ≠ 0› show a * x^2 + b * x + c = 0
    by simp
qed

lemma discriminant-nonneg-ex:
  fixes a b c :: real
  assumes a ≠ 0
    and discrim a b c ≥ 0
  shows ∃ x. a * x^2 + b * x + c = 0
  by (auto simp: discriminant-nonneg assms)

lemma discriminant-pos-ex:
  fixes a b c :: real
  assumes a ≠ 0
    and discrim a b c > 0
  shows ∃x y. x ≠ y ∧ a * x^2 + b * x + c = 0 ∧ a * y^2 + b * y + c = 0
proof −
  let ?x = (−b + sqrt (discrim a b c)) / (2 * a)
  let ?y = (−b − sqrt (discrim a b c)) / (2 * a)
  from ‹discrim a b c > 0› have sqrt (discrim a b c) ≠ 0
    by simp
  then have sqrt (discrim a b c) ≠ − sqrt (discrim a b c)
    by arith
  with ‹a ≠ 0› have ?x ≠ ?y
    by simp
  moreover from assms have a * ?x^2 + b * ?x + c = 0 and a * ?y^2 + b * ?y
+ c = 0
    using discriminant-nonneg [of a b c ?x]
      and discriminant-nonneg [of a b c ?y]
    by simp-all
  ultimately show ?thesis
    by blast
qed

lemma discriminant-pos-distinct:
  fixes a b c x :: real
  assumes a ≠ 0
    and discrim a b c > 0
  shows ∃ y. x ≠ y ∧ a * y^2 + b * y + c = 0
proof −
  from discriminant-pos-ex and ‹a ≠ 0› and ‹discrim a b c > 0›
  obtain w and z where w ≠ z
    and a * w^2 + b * w + c = 0 and a * z^2 + b * z + c = 0
    by blast
  show ∃y. x ≠ y ∧ a * y^2 + b * y + c = 0
  proof (cases x = w)
    case True
    with ‹w ≠ z› have x ≠ z
```

```
    by simp
  with ‹a * z^2 + b * z + c = 0› show ?thesis
    by auto
 next
  case False
  with ‹a * w^2 + b * w + c = 0› show ?thesis
    by auto
 qed
qed

lemma Rats-solution-QE:
 assumes a ∈ ℚ b ∈ ℚ a ≠ 0
 and a*x^2 + b*x + c = 0
 and sqrt (discrim a b c) ∈ ℚ
 shows x ∈ ℚ
using assms(1,2,5) discriminant-iff[THEN iffD1, OF assms(3,4)] by auto

lemma Rats-solution-QE-converse:
 assumes a ∈ ℚ b ∈ ℚ
 and a*x^2 + b*x + c = 0
 and x ∈ ℚ
 shows sqrt (discrim a b c) ∈ ℚ
proof −
 from assms(3) have discrim a b c = (2*a*x+b)^2 unfolding discrim-def by
algebra
 hence sqrt (discrim a b c) = |2*a*x+b| by (simp)
 thus ?thesis using ‹a ∈ ℚ› ‹b ∈ ℚ› ‹x ∈ ℚ› by (simp)
qed

end
```

77 Pretty syntax for Quotient operations

```
theory Quotient-Syntax
imports Main
begin

notation
 rel-conj (infixr OOO 75) and
 map-fun (infixr −−−> 55) and
 rel-fun (infixr ===> 55)

end
```

78 Quotient infrastructure for the set type

```
theory Quotient-Set
imports Quotient-Syntax
```

begin

78.1 Contravariant set map (vimage) and set relator, rules for the Quotient package

definition *rel-vset R xs ys* ≡ ∀ *x y*. *R x y* ⟶ *x* ∈ *xs* ⟷ *y* ∈ *ys*

lemma *rel-vset-eq* [*id-simps*]:
rel-vset (=) = (=)
by (*subst fun-eq-iff*, *subst fun-eq-iff*) (*simp add*: *set-eq-iff rel-vset-def*)

lemma *rel-vset-equivp*:
assumes *e*: *equivp R*
shows *rel-vset R xs ys* ⟷ *xs* = *ys* ∧ (∀ *x y*. *x* ∈ *xs* ⟶ *R x y* ⟶ *y* ∈ *xs*)
unfolding *rel-vset-def*
using *equivp-reflp*[*OF e*]
by *auto* (*metis*, *metis equivp-symp*[*OF e*])

lemma *set-quotient* [*quot-thm*]:
assumes *Quotient3 R Abs Rep*
shows *Quotient3* (*rel-vset R*) (*vimage Rep*) (*vimage Abs*)
proof (*rule Quotient3I*)
from *assms* **have** ⋀*x*. *Abs* (*Rep x*) = *x* **by** (*rule Quotient3-abs-rep*)
then show ⋀*xs*. *Rep* −‘ (*Abs* −‘ *xs*) = *xs*
unfolding *vimage-def* **by** *auto*
next
show ⋀*xs*. *rel-vset R* (*Abs* −‘ *xs*) (*Abs* −‘ *xs*)
unfolding *rel-vset-def vimage-def*
by *auto* (*metis Quotient3-rel-abs*[*OF assms*])+
next
fix *r s*
show *rel-vset R r s* = (*rel-vset R r r* ∧ *rel-vset R s s* ∧ *Rep* −‘ *r* = *Rep* −‘ *s*)
unfolding *rel-vset-def vimage-def set-eq-iff*
by *auto* (*metis rep-abs-rsp*[*OF assms*] *assms*[*simplified Quotient3-def*])+
qed

declare [[*mapQ3 set* = (*rel-vset*, *set-quotient*)]]

lemma *empty-set-rsp*[*quot-respect*]:
rel-vset R {} {}
unfolding *rel-vset-def* **by** *simp*

lemma *collect-rsp*[*quot-respect*]:
assumes *Quotient3 R Abs Rep*
shows ((*R* ===> (=)) ===> *rel-vset R*) *Collect Collect*
by (*intro rel-funI*) (*simp add*: *rel-fun-def rel-vset-def*)

lemma *collect-prs*[*quot-preserve*]:
assumes *Quotient3 R Abs Rep*

shows ((*Abs* −−−> *id*) −−−> (− ‘) *Rep*) *Collect* = *Collect*
unfolding *fun-eq-iff*
by (*simp add*: *Quotient3-abs-rep*[*OF assms*])

lemma *union-rsp*[*quot-respect*]:
assumes *Quotient3 R Abs Rep*
shows (*rel-vset R* ===> *rel-vset R* ===> *rel-vset R*) (∪) (∪)
by (*intro rel-funI*) (*simp add*: *rel-vset-def*)

lemma *union-prs*[*quot-preserve*]:
assumes *Quotient3 R Abs Rep*
shows ((− ‘) *Abs* −−−> (− ‘) *Abs* −−−> (− ‘) *Rep*) (∪) = (∪)
unfolding *fun-eq-iff*
by (*simp add*: *Quotient3-abs-rep*[*OF set-quotient*[*OF assms*]])

lemma *diff-rsp*[*quot-respect*]:
assumes *Quotient3 R Abs Rep*
shows (*rel-vset R* ===> *rel-vset R* ===> *rel-vset R*) (−) (−)
by (*intro rel-funI*) (*simp add*: *rel-vset-def*)

lemma *diff-prs*[*quot-preserve*]:
assumes *Quotient3 R Abs Rep*
shows ((− ‘) *Abs* −−−> (− ‘) *Abs* −−−> (− ‘) *Rep*) (−) = (−)
unfolding *fun-eq-iff*
by (*simp add*: *Quotient3-abs-rep*[*OF set-quotient*[*OF assms*]] *vimage-Diff*)

lemma *inter-rsp*[*quot-respect*]:
assumes *Quotient3 R Abs Rep*
shows (*rel-vset R* ===> *rel-vset R* ===> *rel-vset R*) (∩) (∩)
by (*intro rel-funI*) (*auto simp add*: *rel-vset-def*)

lemma *inter-prs*[*quot-preserve*]:
assumes *Quotient3 R Abs Rep*
shows ((− ‘) *Abs* −−−> (− ‘) *Abs* −−−> (− ‘) *Rep*) (∩) = (∩)
unfolding *fun-eq-iff*
by (*simp add*: *Quotient3-abs-rep*[*OF set-quotient*[*OF assms*]])

lemma *mem-prs*[*quot-preserve*]:
assumes *Quotient3 R Abs Rep*
shows (*Rep* −−−> (− ‘) *Abs* −−−> *id*) (∈) = (∈)
by (*simp add*: *fun-eq-iff Quotient3-abs-rep*[*OF assms*])

lemma *mem-rsp*[*quot-respect*]:
shows (*R* ===> *rel-vset R* ===> (=)) (∈) (∈)
by (*intro rel-funI*) (*simp add*: *rel-vset-def*)

end

79 Quotient infrastructure for the product type

theory *Quotient-Product*
imports *Quotient-Syntax*
begin

79.1 Rules for the Quotient package

lemma *map-prod-id* [*id-simps*]:
shows *map-prod id id = id*
by (*simp add*: *fun-eq-iff*)

lemma *rel-prod-eq* [*id-simps*]:
shows *rel-prod* (=) (=) = (=)
by (*simp add*: *fun-eq-iff*)

lemma *prod-equivp* [*quot-equiv*]:
assumes *equivp R1*
assumes *equivp R2*
shows *equivp* (*rel-prod R1 R2*)
using *assms* **by** (*auto intro*!: *equivpI reflpI sympI transpI elim*!: *equivpE elim*: *reflpE sympE transpE*)

lemma *prod-quotient* [*quot-thm*]:
assumes *Quotient3 R1 Abs1 Rep1*
assumes *Quotient3 R2 Abs2 Rep2*
shows *Quotient3* (*rel-prod R1 R2*) (*map-prod Abs1 Abs2*) (*map-prod Rep1 Rep2*)
apply (*rule Quotient3I*)
apply (*simp add*: *map-prod.compositionality comp-def map-prod.identity Quotient3-abs-rep* [*OF assms*(*1*)] *Quotient3-abs-rep* [*OF assms*(*2*)])
apply (*simp add*: *split-paired-all Quotient3-rel-rep* [*OF assms*(*1*)] *Quotient3-rel-rep* [*OF assms*(*2*)])
using *Quotient3-rel* [*OF assms*(*1*)] *Quotient3-rel* [*OF assms*(*2*)]
apply (*auto simp add*: *split-paired-all*)
done

declare [[*mapQ3 prod* = (*rel-prod*, *prod-quotient*)]]

lemma *Pair-rsp* [*quot-respect*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*R1* ===> *R2* ===> *rel-prod R1 R2*) *Pair Pair*
by (*rule Pair-transfer*)

lemma *Pair-prs* [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep1* ---> *Rep2* ---> (*map-prod Abs1 Abs2*)) *Pair* = *Pair*
apply(*simp add*: *fun-eq-iff*)
apply(*simp add*: *Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*])

done

lemma *fst-rsp* [*quot-respect*]:
assumes *Quotient3 R1 Abs1 Rep1*
assumes *Quotient3 R2 Abs2 Rep2*
shows (*rel-prod R1 R2 ===> R1*) *fst fst*
by *auto*

lemma *fst-prs* [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*map-prod Rep1 Rep2 ---> Abs1*) *fst* = *fst*
by (*simp add*: *fun-eq-iff Quotient3-abs-rep*[*OF q1*])

lemma *snd-rsp* [*quot-respect*]:
assumes *Quotient3 R1 Abs1 Rep1*
assumes *Quotient3 R2 Abs2 Rep2*
shows (*rel-prod R1 R2 ===> R2*) *snd snd*
by *auto*

lemma *snd-prs* [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*map-prod Rep1 Rep2 ---> Abs2*) *snd* = *snd*
by (*simp add*: *fun-eq-iff Quotient3-abs-rep*[*OF q2*])

lemma *case-prod-rsp* [*quot-respect*]:
shows ((*R1 ===> R2 ===>* (=)) *===>* (*rel-prod R1 R2*) *===>* (=)) *case-prod case-prod*
by (*rule case-prod-transfer*)

lemma *split-prs* [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows (((*Abs1 ---> Abs2 ---> id*) *---> map-prod Rep1 Rep2 ---> id*) *case-prod*) = *case-prod*
by (*simp add*: *fun-eq-iff Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*])

lemma [*quot-respect*]:
shows ((*R2 ===> R2 ===>* (=)) *===>* (*R1 ===> R1 ===>* (=)) *===>*
rel-prod R2 R1 ===> rel-prod R2 R1 ===> (=)) *rel-prod rel-prod*
by (*rule prod.rel-transfer*)

lemma [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 abs1 rep1*
and *q2*: *Quotient3 R2 abs2 rep2*
shows ((*abs1 ---> abs1 ---> id*) *---> (abs2 ---> abs2 ---> id*) *--->*
map-prod rep1 rep2 ---> map-prod rep1 rep2 ---> id) *rel-prod* = *rel-prod*

by (*simp add*: *fun-eq-iff Quotient3-abs-rep*[*OF q1*] *Quotient3-abs-rep*[*OF q2*])

lemma [*quot-preserve*]:
shows(*rel-prod* ((*rep1* −−−> *rep1* −−−> *id*) *R1*) ((*rep2* −−−> *rep2* −−−> *id*) *R2*)
(*l1*, *l2*) (*r1*, *r2*)) = (*R1* (*rep1 l1*) (*rep1 r1*) ∧ *R2* (*rep2 l2*) (*rep2 r2*))
by *simp*

declare *prod.inject*[*quot-preserve*]

end

80 Quotient infrastructure for the option type

theory *Quotient-Option*
imports *Quotient-Syntax*
begin

80.1 Rules for the Quotient package

lemma *rel-option-map1*:
rel-option R (*map-option f x*) *y* ⟷ *rel-option* (λ*x*. *R* (*f x*)) *x y*
by (*simp add*: *rel-option-iff split*: *option.split*)

lemma *rel-option-map2*:
rel-option R x (*map-option f y*) ⟷ *rel-option* (λ*x y*. *R x* (*f y*)) *x y*
by (*simp add*: *rel-option-iff split*: *option.split*)

declare
map-option.id [*id-simps*]
option.rel-eq [*id-simps*]

lemma *reflp-rel-option*:
reflp R ⟹ *reflp* (*rel-option R*)
unfolding *reflp-def split-option-all* **by** *simp*

lemma *option-symp*:
symp R ⟹ *symp* (*rel-option R*)
unfolding *symp-def split-option-all*
by (*simp only*: *option.rel-inject option.rel-distinct*) *fast*

lemma *option-transp*:
transp R ⟹ *transp* (*rel-option R*)
unfolding *transp-def split-option-all*
by (*simp only*: *option.rel-inject option.rel-distinct*) *fast*

lemma *option-equivp* [*quot-equiv*]:
equivp R ⟹ *equivp* (*rel-option R*)
by (*blast intro*: *equivpI reflp-rel-option option-symp option-transp elim*: *equivpE*)

lemma *option-quotient* [*quot-thm*]:
assumes *Quotient3 R Abs Rep*
shows *Quotient3* (*rel-option R*) (*map-option Abs*) (*map-option Rep*)
apply (*rule Quotient3I*)
apply (*simp-all add*: *option.map-comp comp-def option.map-id*[*unfolded id-def*] *option.rel-eq rel-option-map1 rel-option-map2 Quotient3-abs-rep* [*OF assms*] *Quotient3-rel-rep* [*OF assms*])
using *Quotient3-rel* [*OF assms*]
apply (*simp add*: *rel-option-iff split*: *option.split*)
done

declare [[*mapQ3 option* = (*rel-option*, *option-quotient*)]]

lemma *option-None-rsp* [*quot-respect*]:
assumes *q*: *Quotient3 R Abs Rep*
shows *rel-option R None None*
by (*rule option.ctr-transfer*(*1*))

lemma *option-Some-rsp* [*quot-respect*]:
assumes *q*: *Quotient3 R Abs Rep*
shows (*R* ===> *rel-option R*) *Some Some*
by (*rule option.ctr-transfer*(*2*))

lemma *option-None-prs* [*quot-preserve*]:
assumes *q*: *Quotient3 R Abs Rep*
shows *map-option Abs None* = *None*
by (*rule Option.option.map*(*1*))

lemma *option-Some-prs* [*quot-preserve*]:
assumes *q*: *Quotient3 R Abs Rep*
shows (*Rep* ---> *map-option Abs*) *Some* = *Some*
apply(*simp add*: *fun-eq-iff*)
apply(*simp add*: *Quotient3-abs-rep*[*OF q*])
done

end

81 Quotient infrastructure for the list type

theory *Quotient-List*
imports *Quotient-Set Quotient-Product Quotient-Option*
begin

81.1 Rules for the Quotient package

lemma *map-id* [*id-simps*]:
map id = *id*
by (*fact List.map.id*)

lemma *list-all2-eq* [*id-simps*]:
list-all2 (=) = (=)
proof (*rule ext*)+
fix *xs ys*
show *list-all2* (=) *xs ys* ⟷ *xs* = *ys*
by (*induct xs ys rule*: *list-induct2'*) *simp-all*
qed

lemma *reflp-list-all2*:
assumes *reflp R*
shows *reflp* (*list-all2 R*)
proof (*rule reflpI*)
from *assms* **have** ∗: ⋀*xs*. *R xs xs* **by** (*rule reflpE*)
fix *xs*
show *list-all2 R xs xs*
by (*induct xs*) (*simp-all add*: ∗)
qed

lemma *list-symp*:
assumes *symp R*
shows *symp* (*list-all2 R*)
proof (*rule sympI*)
from *assms* **have** ∗: ⋀*xs ys*. *R xs ys* ⟹ *R ys xs* **by** (*rule sympE*)
fix *xs ys*
assume *list-all2 R xs ys*
then show *list-all2 R ys xs*
by (*induct xs ys rule*: *list-induct2'*) (*simp-all add*: ∗)
qed

lemma *list-transp*:
assumes *transp R*
shows *transp* (*list-all2 R*)
proof (*rule transpI*)
from *assms* **have** ∗: ⋀*xs ys zs*. *R xs ys* ⟹ *R ys zs* ⟹ *R xs zs* **by** (*rule transpE*)
fix *xs ys zs*
assume *list-all2 R xs ys* **and** *list-all2 R ys zs*
then show *list-all2 R xs zs*
by (*induct arbitrary*: *zs*) (*auto simp*: *list-all2-Cons1 intro*: ∗)
qed

lemma *list-equivp* [*quot-equiv*]:
equivp R ⟹ *equivp* (*list-all2 R*)
by (*blast intro*: *equivpI reflp-list-all2 list-symp list-transp elim*: *equivpE*)

lemma *list-quotient3* [*quot-thm*]:
assumes *Quotient3 R Abs Rep*
shows *Quotient3* (*list-all2 R*) (*map Abs*) (*map Rep*)
proof (*rule Quotient3I*)

from *assms* **have** $\bigwedge x.\ Abs\ (Rep\ x) = x$ **by** (*rule Quotient3-abs-rep*)
then show $\bigwedge xs.\ map\ Abs\ (map\ Rep\ xs) = xs$ **by** (*simp add*: *comp-def*)
next
from *assms* **have** $\bigwedge x\ y.\ R\ (Rep\ x)\ (Rep\ y) \longleftrightarrow x = y$ **by** (*rule Quotient3-rel-rep*)
then show $\bigwedge xs.$ *list-all2 R* (*map Rep xs*) (*map Rep xs*)
by (*simp add*: *list-all2-map1 list-all2-map2 list-all2-eq*)
next
fix *xs ys*
from *assms* **have** $\bigwedge x\ y.\ R\ x\ x \wedge R\ y\ y \wedge Abs\ x = Abs\ y \longleftrightarrow R\ x\ y$ **by** (*rule Quotient3-rel*)
then show *list-all2 R xs ys* $\longleftrightarrow$ *list-all2 R xs xs* $\wedge$ *list-all2 R ys ys* $\wedge$ *map Abs xs* = *map Abs ys*
by (*induct xs ys rule*: *list-induct2'*) *auto*
qed

declare [[*mapQ3 list* = (*list-all2*, *list-quotient3*)]]

lemma *cons-prs* [*quot-preserve*]:
assumes *q*: *Quotient3 R Abs Rep*
shows (*Rep* −−−> (*map Rep*) −−−> (*map Abs*)) (#) = (#)
by (*auto simp add*: *fun-eq-iff comp-def Quotient3-abs-rep* [*OF q*])

lemma *cons-rsp* [*quot-respect*]:
assumes *q*: *Quotient3 R Abs Rep*
shows (*R* ===> *list-all2 R* ===> *list-all2 R*) (#) (#)
by *auto*

lemma *nil-prs* [*quot-preserve*]:
assumes *q*: *Quotient3 R Abs Rep*
shows *map Abs* [] = []
by *simp*

lemma *nil-rsp* [*quot-respect*]:
assumes *q*: *Quotient3 R Abs Rep*
shows *list-all2 R* [] []
by *simp*

lemma *map-prs-aux*:
assumes *a*: *Quotient3 R1 abs1 rep1*
and *b*: *Quotient3 R2 abs2 rep2*
shows (*map abs2*) (*map* ((*abs1* −−−> *rep2*) *f*) (*map rep1 l*)) = *map f l*
by (*induct l*)
(*simp-all add*: *Quotient3-abs-rep*[*OF a*] *Quotient3-abs-rep*[*OF b*])

lemma *map-prs* [*quot-preserve*]:
assumes *a*: *Quotient3 R1 abs1 rep1*
and *b*: *Quotient3 R2 abs2 rep2*
shows ((*abs1* −−−> *rep2*) −−−> (*map rep1*) −−−> (*map abs2*)) *map* = *map*
and ((*abs1* −−−> *id*) −−−> *map rep1* −−−> *id*) *map* = *map*

by (*simp-all only*: *fun-eq-iff map-prs-aux*[*OF a b*] *comp-def*)
(*simp-all add*: *Quotient3-abs-rep*[*OF a*] *Quotient3-abs-rep*[*OF b*])

lemma *map-rsp* [*quot-respect*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows ((*R1* ===> *R2*) ===> (*list-all2 R1*) ===> *list-all2 R2*) *map map*
and ((*R1* ===> (=)) ===> (*list-all2 R1*) ===> (=)) *map map*
unfolding *list-all2-eq* [*symmetric*] **by** (*rule list.map-transfer*)+

lemma *foldr-prs-aux*:
assumes *a*: *Quotient3 R1 abs1 rep1*
and *b*: *Quotient3 R2 abs2 rep2*
shows *abs2* (*foldr* ((*abs1* ---> *abs2* ---> *rep2*) *f*) (*map rep1 l*) (*rep2 e*))
= *foldr f l e*
by (*induct l*) (*simp-all add*: *Quotient3-abs-rep*[*OF a*] *Quotient3-abs-rep*[*OF b*])

lemma *foldr-prs* [*quot-preserve*]:
assumes *a*: *Quotient3 R1 abs1 rep1*
and *b*: *Quotient3 R2 abs2 rep2*
shows ((*abs1* ---> *abs2* ---> *rep2*) ---> (*map rep1*) ---> *rep2* --->
abs2) *foldr* = *foldr*
apply (*simp add*: *fun-eq-iff*)
by (*simp only*: *fun-eq-iff foldr-prs-aux*[*OF a b*])
(*simp*)

lemma *foldl-prs-aux*:
assumes *a*: *Quotient3 R1 abs1 rep1*
and *b*: *Quotient3 R2 abs2 rep2*
shows *abs1* (*foldl* ((*abs1* ---> *abs2* ---> *rep1*) *f*) (*rep1 e*) (*map rep2 l*)) =
foldl f e l
by (*induct l arbitrary*:*e*) (*simp-all add*: *Quotient3-abs-rep*[*OF a*] *Quotient3-abs-rep*[*OF b*])

lemma *foldl-prs* [*quot-preserve*]:
assumes *a*: *Quotient3 R1 abs1 rep1*
and *b*: *Quotient3 R2 abs2 rep2*
shows ((*abs1* ---> *abs2* ---> *rep1*) ---> *rep1* ---> (*map rep2*) --->
abs1) *foldl* = *foldl*
by (*simp add*: *fun-eq-iff foldl-prs-aux* [*OF a b*])

lemma *foldl-rsp*[*quot-respect*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows ((*R1* ===> *R2* ===> *R1*) ===> *R1* ===> *list-all2 R2* ===> *R1*)
foldl foldl
by (*rule foldl-transfer*)

lemma *foldr-rsp*[*quot-respect*]:

assumes *q1*: *Quotient3 R1 Abs1 Rep1*
and *q2*: *Quotient3 R2 Abs2 Rep2*
shows *((R1 ===> R2 ===> R2) ===> list-all2 R1 ===> R2 ===> R2) foldr foldr*
by (*rule foldr-transfer*)

lemma *list-all2-rsp*:
assumes *r*: $\forall x\ y.\ R\ x\ y \longrightarrow (\forall a\ b.\ R\ a\ b \longrightarrow S\ x\ a = T\ y\ b)$
and *l1*: *list-all2 R x y*
and *l2*: *list-all2 R a b*
shows *list-all2 S x a = list-all2 T y b*
using *l1 l2*
by (*induct arbitrary*: *a b rule*: *list-all2-induct*,
auto simp: *list-all2-Cons1 list-all2-Cons2 r*)

lemma [*quot-respect*]:
((R ===> R ===> (=)) ===> list-all2 R ===> list-all2 R ===> (=)) list-all2 list-all2
by (*rule list.rel-transfer*)

lemma [*quot-preserve*]:
assumes *a*: *Quotient3 R abs1 rep1*
shows *((abs1 ---> abs1 ---> id) ---> map rep1 ---> map rep1 ---> id) list-all2 = list-all2*
apply (*simp add*: *fun-eq-iff*)
apply *clarify*
apply (*induct-tac xa xb rule*: *list-induct2'*)
apply (*simp-all add*: *Quotient3-abs-rep*[*OF a*])
done

lemma [*quot-preserve*]:
assumes *a*: *Quotient3 R abs1 rep1*
shows *(list-all2 ((rep1 ---> rep1 ---> id) R) l m) = (l = m)*
by (*induct l m rule*: *list-induct2'*) (*simp-all add*: *Quotient3-rel-rep*[*OF a*])

lemma *list-all2-find-element*:
assumes *a*: $x \in set\ a$
and *b*: *list-all2 R a b*
shows $\exists y.\ (y \in set\ b \wedge R\ x\ y)$
using *b a* **by** *induct auto*

lemma *list-all2-refl*:
assumes *a*: $\bigwedge x\ y.\ R\ x\ y = (R\ x = R\ y)$
shows *list-all2 R x x*
by (*induct x*) (*auto simp add*: *a*)

end

82 Quotient infrastructure for the sum type

theory *Quotient-Sum*
imports *Quotient-Syntax*
begin

82.1 Rules for the Quotient package

lemma *rel-sum-map1*:
rel-sum R1 R2 (map-sum f1 f2 x) y ⟷ rel-sum (λx. R1 (f1 x)) (λx. R2 (f2 x)) x y
by (*rule sum.rel-map*(*1*))

lemma *rel-sum-map2*:
rel-sum R1 R2 x (map-sum f1 f2 y) ⟷ rel-sum (λx y. R1 x (f1 y)) (λx y. R2 x (f2 y)) x y
by (*rule sum.rel-map*(*2*))

lemma *map-sum-id* [*id-simps*]:
map-sum id id = id
by (*simp add*: *id-def map-sum.identity fun-eq-iff*)

lemma *rel-sum-eq* [*id-simps*]:
rel-sum (=) (=) = (=)
by (*rule sum.rel-eq*)

lemma *reflp-rel-sum*:
reflp R1 ⟹ reflp R2 ⟹ reflp (rel-sum R1 R2)
unfolding *reflp-def split-sum-all rel-sum-simps* **by** *fast*

lemma *sum-symp*:
symp R1 ⟹ symp R2 ⟹ symp (rel-sum R1 R2)
unfolding *symp-def split-sum-all rel-sum-simps* **by** *fast*

lemma *sum-transp*:
transp R1 ⟹ transp R2 ⟹ transp (rel-sum R1 R2)
unfolding *transp-def split-sum-all rel-sum-simps* **by** *fast*

lemma *sum-equivp* [*quot-equiv*]:
equivp R1 ⟹ equivp R2 ⟹ equivp (rel-sum R1 R2)
by (*blast intro*: *equivpI reflp-rel-sum sum-symp sum-transp elim*: *equivpE*)

lemma *sum-quotient* [*quot-thm*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows *Quotient3 (rel-sum R1 R2) (map-sum Abs1 Abs2) (map-sum Rep1 Rep2)*
apply (*rule Quotient3I*)
apply (*simp-all add*: *map-sum.compositionality comp-def map-sum.identity rel-sum-eq rel-sum-map1 rel-sum-map2*
Quotient3-abs-rep [*OF q1*] *Quotient3-rel-rep* [*OF q1*] *Quotient3-abs-rep* [*OF q2*]

Quotient3-rel-rep [*OF q2*])
using *Quotient3-rel* [*OF q1*] *Quotient3-rel* [*OF q2*]
apply (*fastforce elim*!: *rel-sum.cases simp add*: *comp-def split*: *sum.split*)
done

declare [[*mapQ3 sum* = (*rel-sum*, *sum-quotient*)]]

lemma *sum-Inl-rsp* [*quot-respect*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*R1* ===> *rel-sum R1 R2*) *Inl Inl*
by *auto*

lemma *sum-Inr-rsp* [*quot-respect*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*R2* ===> *rel-sum R1 R2*) *Inr Inr*
by *auto*

lemma *sum-Inl-prs* [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep1* ---> *map-sum Abs1 Abs2*) *Inl* = *Inl*
apply(*simp add*: *fun-eq-iff*)
apply(*simp add*: *Quotient3-abs-rep*[*OF q1*])
done

lemma *sum-Inr-prs* [*quot-preserve*]:
assumes *q1*: *Quotient3 R1 Abs1 Rep1*
assumes *q2*: *Quotient3 R2 Abs2 Rep2*
shows (*Rep2* ---> *map-sum Abs1 Abs2*) *Inr* = *Inr*
apply(*simp add*: *fun-eq-iff*)
apply(*simp add*: *Quotient3-abs-rep*[*OF q2*])
done

end

83 Quotient types

theory *Quotient-Type*
imports *Main*
begin

We introduce the notion of quotient types over equivalence relations via type classes.

83.1 Equivalence relations and quotient types

Type class *equiv* models equivalence relations $\sim :: 'a \Rightarrow 'a \Rightarrow bool$.

class *eqv* =
  fixes *eqv* :: *'a* ⇒ *'a* ⇒ *bool* (**infixl** ∼ *50*)

class *equiv* = *eqv* +
  assumes *equiv-refl* [*intro*]: *x* ∼ *x*
    and *equiv-trans* [*trans*]: *x* ∼ *y* ⟹ *y* ∼ *z* ⟹ *x* ∼ *z*
    and *equiv-sym* [*sym*]: *x* ∼ *y* ⟹ *y* ∼ *x*
begin

lemma *equiv-not-sym* [*sym*]: ¬ *x* ∼ *y* ⟹ ¬ *y* ∼ *x*
proof −
  assume ¬ *x* ∼ *y*
  then show ¬ *y* ∼ *x* **by** (*rule contrapos-nn*) (*rule equiv-sym*)
qed

lemma *not-equiv-trans1* [*trans*]: ¬ *x* ∼ *y* ⟹ *y* ∼ *z* ⟹ ¬ *x* ∼ *z*
proof −
  assume ¬ *x* ∼ *y* **and** *y* ∼ *z*
  show ¬ *x* ∼ *z*
  proof
    assume *x* ∼ *z*
    also from ‹*y* ∼ *z*› **have** *z* ∼ *y* **..**
    finally have *x* ∼ *y* **.**
    with ‹¬ *x* ∼ *y*› **show** *False* **by** *contradiction*
  qed
qed

lemma *not-equiv-trans2* [*trans*]: *x* ∼ *y* ⟹ ¬ *y* ∼ *z* ⟹ ¬ *x* ∼ *z*
proof −
  assume ¬ *y* ∼ *z*
  then have ¬ *z* ∼ *y* **..**
  also
  assume *x* ∼ *y*
  then have *y* ∼ *x* **..**
  finally have ¬ *z* ∼ *x* **.**
  then show ¬ *x* ∼ *z* **..**
qed

end

The quotient type *'a quot* consists of all *equivalence classes* over elements of the base type *'a*.

definition (**in** *eqv*) *quot* = {{*x*. *a* ∼ *x*} | *a*. *True*}

typedef (**overloaded**) *'a quot* = *quot* :: *'a*::*eqv set set*
  unfolding *quot-def* **by** *blast*

lemma *quotI* [*intro*]: {*x*. *a* ∼ *x*} ∈ *quot*
  unfolding *quot-def* **by** *blast*

lemma *quotE* [*elim*]:
assumes $R \in quot$
obtains a **where** $R = \{x.\ a \sim x\}$
using *assms* **unfolding** *quot-def* **by** *blast*

Abstracted equivalence classes are the canonical representation of elements of a quotient type.

definition *class* :: $'a$::*equiv* $\Rightarrow$ $'a$ *quot* ($\lfloor$-$\rfloor$)
where $\lfloor a \rfloor = Abs\text{-}quot\ \{x.\ a \sim x\}$

theorem *quot-exhaust*: $\exists\, a.\ A = \lfloor a \rfloor$
proof (*cases A*)
fix R
assume R: $A = Abs\text{-}quot\ R$
assume $R \in quot$
then have $\exists\, a.\ R = \{x.\ a \sim x\}$ **by** *blast*
with R **have** $\exists\, a.\ A = Abs\text{-}quot\ \{x.\ a \sim x\}$ **by** *blast*
then show *?thesis* **unfolding** *class-def* **.**
qed

lemma *quot-cases* [*cases type*: *quot*]:
obtains a **where** $A = \lfloor a \rfloor$
using *quot-exhaust* **by** *blast*

83.2 Equality on quotients

Equality of canonical quotient elements coincides with the original relation.

theorem *quot-equality* [*iff?*]: $\lfloor a \rfloor = \lfloor b \rfloor \longleftrightarrow a \sim b$
proof
assume *eq*: $\lfloor a \rfloor = \lfloor b \rfloor$
show $a \sim b$
proof −
from *eq* **have** $\{x.\ a \sim x\} = \{x.\ b \sim x\}$
by (*simp only*: *class-def Abs-quot-inject quotI*)
moreover have $a \sim a$ **..**
ultimately have $a \in \{x.\ b \sim x\}$ **by** *blast*
then have $b \sim a$ **by** *blast*
then show *?thesis* **..**
qed
next
assume *ab*: $a \sim b$
show $\lfloor a \rfloor = \lfloor b \rfloor$
proof −
have $\{x.\ a \sim x\} = \{x.\ b \sim x\}$
proof (*rule Collect-cong*)
fix x **show** $(a \sim x) = (b \sim x)$
proof
from *ab* **have** $b \sim a$ **..**

```
      also assume a ∼ x
      finally show b ∼ x .
    next
      note ab
      also assume b ∼ x
      finally show a ∼ x .
    qed
  qed
  then show ?thesis by (simp only: class-def)
 qed
qed
```

83.3 Picking representing elements

```
definition pick :: 'a::equiv quot ⇒ 'a
 where pick A = (SOME a. A = ⌊a⌋)

theorem pick-equiv [intro]: pick ⌊a⌋ ∼ a
proof (unfold pick-def)
 show (SOME x. ⌊a⌋ = ⌊x⌋) ∼ a
 proof (rule someI2)
  show ⌊a⌋ = ⌊a⌋ ..
  fix x assume ⌊a⌋ = ⌊x⌋
  then have a ∼ x ..
  then show x ∼ a ..
 qed
qed

theorem pick-inverse [intro]: ⌊pick A⌋ = A
proof (cases A)
 fix a assume a: A = ⌊a⌋
 then have pick A ∼ a by (simp only: pick-equiv)
 then have ⌊pick A⌋ = ⌊a⌋ ..
 with a show ?thesis by simp
qed
```

The following rules support canonical function definitions on quotient types (with up to two arguments). Note that the stripped-down version without additional conditions is sufficient most of the time.

```
theorem quot-cond-function:
 assumes eq: ⋀X Y. P X Y ⟹ f X Y ≡ g (pick X) (pick Y)
  and cong: ⋀x x' y y'. ⌊x⌋ = ⌊x'⌋ ⟹ ⌊y⌋ = ⌊y'⌋
    ⟹ P ⌊x⌋ ⌊y⌋ ⟹ P ⌊x'⌋ ⌊y'⌋ ⟹ g x y = g x' y'
  and P: P ⌊a⌋ ⌊b⌋
 shows f ⌊a⌋ ⌊b⌋ = g a b
proof −
 from eq and P have f ⌊a⌋ ⌊b⌋ = g (pick ⌊a⌋) (pick ⌊b⌋) by (simp only:)
 also have ... = g a b
 proof (rule cong)
```

show $\lfloor pick\ \lfloor a \rfloor \rfloor = \lfloor a \rfloor$ **..**
moreover
show $\lfloor pick\ \lfloor b \rfloor \rfloor = \lfloor b \rfloor$ **..**
moreover
show $P\ \lfloor a \rfloor\ \lfloor b \rfloor$ **by** (*rule P*)
ultimately show $P\ \lfloor pick\ \lfloor a \rfloor \rfloor\ \lfloor pick\ \lfloor b \rfloor \rfloor$ **by** (*simp only:*)
qed
finally show *?thesis* **.**
qed

theorem *quot-function*:
assumes $\bigwedge X\ Y.\ f\ X\ Y \equiv g\ (pick\ X)\ (pick\ Y)$
and $\bigwedge x\ x'\ y\ y'.\ \lfloor x \rfloor = \lfloor x' \rfloor \Longrightarrow \lfloor y \rfloor = \lfloor y' \rfloor \Longrightarrow g\ x\ y = g\ x'\ y'$
shows $f\ \lfloor a \rfloor\ \lfloor b \rfloor = g\ a\ b$
using *assms* **and** *TrueI*
by (*rule quot-cond-function*)

theorem *quot-function'*:
$(\bigwedge X\ Y.\ f\ X\ Y \equiv g\ (pick\ X)\ (pick\ Y)) \Longrightarrow$
$(\bigwedge x\ x'\ y\ y'.\ x \sim x' \Longrightarrow y \sim y' \Longrightarrow g\ x\ y = g\ x'\ y') \Longrightarrow$
$f\ \lfloor a \rfloor\ \lfloor b \rfloor = g\ a\ b$
by (*rule quot-function*) (*simp-all only*: *quot-equality*)

end

84 Ramsey's Theorem

theory *Ramsey*
imports *Infinite-Set FuncSet*
begin

84.1 Preliminary definitions

abbreviation *strict-sorted* :: *'a*::*linorder list* ⇒ *bool* **where**
strict-sorted ≡ *sorted-wrt* (<)

84.1.1 The n-element subsets of a set A

definition *nsets* :: [*'a set*, *nat*] ⇒ *'a set set* (([-]^-) [*0*,*999*] *999*)
where *nsets A n* ≡ {*N*. *N* ⊆ *A* ∧ *finite N* ∧ *card N* = *n*}

lemma *nsets-mono*: *A* ⊆ *B* ⟹ *nsets A n* ⊆ *nsets B n*
by (*auto simp*: *nsets-def*)

lemma *nsets-Pi-contra*: *A'* ⊆ *A* ⟹ *Pi* ([*A*]^*n*) *B* ⊆ *Pi* ([*A'*]^*n*) *B*
by (*auto simp*: *nsets-def*)

lemma *nsets-2-eq*: *nsets A 2* = (⋃*x*∈*A*. ⋃*y*∈*A* − {*x*}. {{*x*, *y*}})
by (*auto simp*: *nsets-def card-2-iff*)

lemma *nsets-doubleton-2-eq* [*simp*]: $[\{x,\ y\}]^2 = (\mathit{if}\ x{=}y\ \mathit{then}\ \{\}\ \mathit{else}\ \{\{x,\ y\}\})$
by (*auto simp*: *nsets-2-eq*)

lemma *doubleton-in-nsets-2* [*simp*]: $\{x,y\} \in [A]^2 \longleftrightarrow x \in A \wedge y \in A \wedge x \neq y$
by (*auto simp*: *nsets-2-eq Set.doubleton-eq-iff*)

lemma *nsets-3-eq*: *nsets A 3* = $(\bigcup x{\in}A.\ \bigcup y{\in}A - \{x\}.\ \bigcup z{\in}A - \{x,y\}.\ \{\{x,y,z\}\})$
by (*simp add*: *eval-nat-numeral nsets-def card-Suc-eq*) *blast*

lemma *nsets-4-eq*: $[A]^4 = (\bigcup u{\in}A.\ \bigcup x{\in}A - \{u\}.\ \bigcup y{\in}A - \{u,x\}.\ \bigcup z{\in}A - \{u,x,y\}.\ \{\{u,x,y,z\}\})$
(**is** - = *?rhs*)
proof
show $[A]^4 \subseteq$ *?rhs*
by (*clarsimp simp add*: *nsets-def eval-nat-numeral card-Suc-eq*) *blast*
show *?rhs* $\subseteq [A]^4$
apply (*clarsimp simp add*: *nsets-def eval-nat-numeral card-Suc-eq*)
by (*metis insert-iff singletonD*)
qed

lemma *nsets-disjoint-2*:
$X \cap Y = \{\} \Longrightarrow [X \cup Y]^2 = [X]^2 \cup [Y]^2 \cup (\bigcup x{\in}X.\ \bigcup y{\in}Y.\ \{\{x,y\}\})$
by (*fastforce simp*: *nsets-2-eq Set.doubleton-eq-iff*)

lemma *ordered-nsets-2-eq*:
fixes *A* :: *'a*::*linorder set*
shows *nsets A 2* = $\{\{x,y\} \mid x\ y.\ x \in A \wedge y \in A \wedge x{<}y\}$
(**is** - = *?rhs*)
proof
show *nsets A 2* $\subseteq$ *?rhs*
unfolding *numeral-nat*
apply (*clarsimp simp add*: *nsets-def card-Suc-eq Set.doubleton-eq-iff not-less*)
by (*metis antisym*)
show *?rhs* $\subseteq$ *nsets A 2*
unfolding *numeral-nat* **by** (*auto simp*: *nsets-def card-Suc-eq*)
qed

lemma *ordered-nsets-3-eq*:
fixes *A* :: *'a*::*linorder set*
shows *nsets A 3* = $\{\{x,y,z\} \mid x\ y\ z.\ x \in A \wedge y \in A \wedge z \in A \wedge x{<}y \wedge y{<}z\}$
(**is** - = *?rhs*)
proof
show *nsets A 3* $\subseteq$ *?rhs*
apply (*clarsimp simp add*: *nsets-def card-Suc-eq eval-nat-numeral*)
by (*metis insert-commute linorder-cases*)
show *?rhs* $\subseteq$ *nsets A 3*
apply (*clarsimp simp add*: *nsets-def card-Suc-eq eval-nat-numeral*)
by (*metis empty-iff insert-iff not-less-iff-gr-or-eq*)

qed

lemma *ordered-nsets-4-eq*:
fixes *A* :: *'a*::*linorder set*
shows $[A]^4 = \{U.\ \exists u\ x\ y\ z.\ U = \{u,x,y,z\} \wedge u \in A \wedge x \in A \wedge y \in A \wedge z \in A \wedge u < x \wedge x < y \wedge y < z\}$
(**is** - = *Collect ?RHS*)
proof −
{ fix *U*
assume $U \in [A]^4$
then obtain *l* **where** *strict-sorted l List.set l* = *U length l* = *4 U* ⊆ *A*
by (*simp add*: *nsets-def*) (*metis finite-set-strict-sorted*)
then have *?RHS U*
unfolding *numeral-nat length-Suc-conv* **by** *auto blast* **}**
moreover
have *Collect ?RHS* ⊆ $[A]^4$
apply (*clarsimp simp add*: *nsets-def eval-nat-numeral*)
apply (*subst card-insert-disjoint*, *auto*)+
done
ultimately show *?thesis*
by *auto*
qed

lemma *ordered-nsets-5-eq*:
fixes *A* :: *'a*::*linorder set*
shows $[A]^5 = \{U.\ \exists u\ v\ x\ y\ z.\ U = \{u,v,x,y,z\} \wedge u \in A \wedge v \in A \wedge x \in A \wedge y \in A \wedge z \in A \wedge u < v \wedge v < x \wedge x < y \wedge y < z\}$
(**is** - = *Collect ?RHS*)
proof −
{ fix *U*
assume $U \in [A]^5$
then obtain *l* **where** *strict-sorted l List.set l* = *U length l* = *5 U* ⊆ *A*
apply (*simp add*: *nsets-def*)
by (*metis finite-set-strict-sorted*)
then have *?RHS U*
unfolding *numeral-nat length-Suc-conv* **by** *auto blast* **}**
moreover
have *Collect ?RHS* ⊆ $[A]^5$
apply (*clarsimp simp add*: *nsets-def eval-nat-numeral*)
apply (*subst card-insert-disjoint*, *auto*)+
done
ultimately show *?thesis*
by *auto*
qed

lemma *binomial-eq-nsets*: *n choose k* = *card* (*nsets* {*0*..<*n*} *k*)
apply (*simp add*: *binomial-def nsets-def*)
by (*meson subset-eq-atLeast0-lessThan-finite*)

lemma *nsets-eq-empty-iff*: *nsets A r* = {} ⟷ *finite A* ∧ *card A* < *r*
unfolding *nsets-def*
proof (*intro iffI conjI*)
assume *that*: {*N*. *N* ⊆ *A* ∧ *finite N* ∧ *card N* = *r*} = {}
show *finite A*
using *infinite-arbitrarily-large that* **by** *auto*
then have ¬ *r* ≤ *card A*
using *that* **by** (*simp add*: *set-eq-iff*) (*metis obtain-subset-with-card-n*)
then show *card A* < *r*
using *not-less* **by** *blast*
next
show {*N*. *N* ⊆ *A* ∧ *finite N* ∧ *card N* = *r*} = {}
if *finite A* ∧ *card A* < *r*
using *that card-mono leD* **by** *auto*
qed

lemma *nsets-eq-empty*: ⟦*finite A*; *card A* < *r*⟧ ⟹ *nsets A r* = {}
by (*simp add*: *nsets-eq-empty-iff*)

lemma *nsets-empty-iff*: *nsets* {} *r* = (*if r=0 then* {{}} *else* {})
by (*auto simp*: *nsets-def*)

lemma *nsets-singleton-iff*: *nsets* {*a*} *r* = (*if r=0 then* {{}} *else if r=1 then* {{*a*}} *else* {})
by (*auto simp*: *nsets-def card-gt-0-iff subset-singleton-iff*)

lemma *nsets-self* [*simp*]: *nsets* {..<*m*} *m* = {{..<*m*}}
unfolding *nsets-def*
apply *auto*
by (*metis add.left-neutral lessThan-atLeast0 lessThan-iff subset-card-intvl-is-intvl*)

lemma *nsets-zero* [*simp*]: *nsets A 0* = {{}}
by (*auto simp*: *nsets-def*)

lemma *nsets-one*: *nsets A* (*Suc 0*) = (λ*x*. {*x*}) ‘ *A*
using *card-eq-SucD* **by** (*force simp*: *nsets-def*)

lemma *inj-on-nsets*:
assumes *inj-on f A*
shows *inj-on* (λ*X*. *f* ‘ *X*) ([*A*]^*n*)
using *assms* **unfolding** *nsets-def*
by (*metis* (*no-types*, *lifting*) *inj-on-inverseI inv-into-image-cancel mem-Collect-eq*)

lemma *bij-betw-nsets*:
assumes *bij-betw f A B*
shows *bij-betw* (λ*X*. *f* ‘ *X*) ([*A*]^*n*) ([*B*]^*n*)
proof −
have (‘) *f* ‘ [*A*]^*n* = [*f* ‘ *A*]^*n*
using *assms*

apply (*auto simp*: *nsets-def bij-betw-def image-iff card-image inj-on-subset*)
by (*metis card-image inj-on-finite order-refl subset-image-inj*)
with *assms* **show** *?thesis*
by (*auto simp*: *bij-betw-def inj-on-nsets*)
qed

lemma *nset-image-obtains*:
assumes $X \in [f`A]^k$ *inj-on f A*
obtains Y **where** $Y \in [A]^k$ $X = f \mathbin{`} Y$
using *assms*
apply (*clarsimp simp add*: *nsets-def subset-image-iff*)
by (*metis card-image finite-imageD inj-on-subset*)

lemma *nsets-image-funcset*:
assumes $g \in S \to T$ **and** *inj-on g S*
shows $(\lambda X.\ g \mathbin{`} X) \in [S]^k \to [T]^k$
using *assms*
by (*fastforce simp add*: *nsets-def card-image inj-on-subset subset-iff simp flip*: *image-subset-iff-funcset*)

lemma *nsets-compose-image-funcset*:
assumes *f*: $f \in [T]^k \to D$ **and** $g \in S \to T$ **and** *inj-on g S*
shows $f \circ (\lambda X.\ g \mathbin{`} X) \in [S]^k \to D$
proof −
have $(\lambda X.\ g \mathbin{`} X) \in [S]^k \to [T]^k$
using *assms* **by** (*simp add*: *nsets-image-funcset*)
then show *?thesis*
using *f* **by** *fastforce*
qed

84.1.2 Partition predicates

definition *partn* :: *'a set* ⇒ *nat* ⇒ *nat* ⇒ *'b set* ⇒ *bool*
where *partn* $\beta\ \alpha\ \gamma\ \delta \equiv \forall f \in nsets\ \beta\ \gamma \to \delta.\ \exists H \in nsets\ \beta\ \alpha.\ \exists \xi \in \delta.\ f \mathbin{`} (nsets\ H\ \gamma) \subseteq \{\xi\}$

definition *partn-lst* :: *'a set* ⇒ *nat list* ⇒ *nat* ⇒ *bool*
where *partn-lst* $\beta\ \alpha\ \gamma \equiv \forall f \in nsets\ \beta\ \gamma \to \{..<length\ \alpha\}.$
$\exists i < length\ \alpha.\ \exists H \in nsets\ \beta\ (\alpha!i).\ f \mathbin{`} (nsets\ H\ \gamma) \subseteq \{i\}$

lemma *partn-lst-greater-resource*:
fixes *M*::*nat*
assumes *M*: *partn-lst* $\{..<M\}\ \alpha\ \gamma$ **and** $M \le N$
shows *partn-lst* $\{..<N\}\ \alpha\ \gamma$
proof (*clarsimp simp*: *partn-lst-def*)
fix *f*
assume $f \in nsets\ \{..<N\}\ \gamma \to \{..<length\ \alpha\}$
then have $f \in nsets\ \{..<M\}\ \gamma \to \{..<length\ \alpha\}$
by (*meson Pi-anti-mono* ‹$M \le N$› *lessThan-subset-iff nsets-mono subsetD*)

then obtain *i H* **where** *i*: $i < length\ \alpha$ **and** *H*: $H \in nsets\ \{..<M\}\ (\alpha\ !\ i)$ **and** *subi*: $f\ `\ nsets\ H\ \gamma \subseteq \{i\}$
using *M partn-lst-def* **by** *blast*
have $H \in nsets\ \{..<N\}\ (\alpha\ !\ i)$
using ‹$M \leq N$› *H* **by** (*auto simp*: *nsets-def subset-iff*)
then show $\exists i{<}length\ \alpha.\ \exists H{\in}nsets\ \{..<N\}\ (\alpha\ !\ i).\ f\ `\ nsets\ H\ \gamma \subseteq \{i\}$
using *i subi* **by** *blast*
qed

lemma *partn-lst-fewer-colours*:
assumes *major*: *partn-lst* $\beta\ (n\#\alpha)\ \gamma$ **and** $n \geq \gamma$
shows *partn-lst* $\beta\ \alpha\ \gamma$
proof (*clarsimp simp*: *partn-lst-def*)
fix *f* :: *'a set* ⇒ *nat*
assume *f*: $f \in [\beta]^\gamma \to \{..<length\ \alpha\}$
then obtain *i H* **where** *i*: $i < Suc\ (length\ \alpha)$
and *H*: $H \in [\beta]^{((n\ \#\ \alpha)\ !\ i)}$
and *hom*: $\forall x{\in}[H]^\gamma.\ Suc\ (f\ x) = i$
using ‹$n \geq \gamma$› *major* [*unfolded partn-lst-def*, *rule-format*, *of Suc o f*]
by (*fastforce simp*: *image-subset-iff nsets-eq-empty-iff*)
show $\exists i{<}length\ \alpha.\ \exists H{\in}nsets\ \beta\ (\alpha\ !\ i).\ f\ `\ [H]^\gamma \subseteq \{i\}$
proof (*cases i*)
case *0*
then have $[H]^\gamma = \{\}$
using *hom* **by** *blast*
then show *?thesis*
using *0 H* ‹$n \geq \gamma$›
by (*simp add*: *nsets-eq-empty-iff*) (*simp add*: *nsets-def*)
next
case (*Suc i'*)
then show *?thesis*
using *i H hom* **by** *auto*
qed
qed

lemma *partn-lst-eq-partn*: *partn-lst* {..<*n*} [*m*,*m*] *2* = *partn* {..<*n*} *m 2* {..<*2*::*nat*}
apply (*simp add*: *partn-lst-def partn-def numeral-2-eq-2*)
by (*metis less-2-cases numeral-2-eq-2 lessThan-iff nth-Cons-0 nth-Cons-Suc*)

84.2 Finite versions of Ramsey's theorem

To distinguish the finite and infinite ones, lower and upper case names are used.

84.2.1 Trivial cases

Vacuous, since we are dealing with 0-sets!

lemma *ramsey0*: ∃ *N*::*nat*. *partn-lst* {..<*N*} [*q1*,*q2*] *0*

by (*force simp*: *partn-lst-def ex-in-conv less-Suc-eq nsets-eq-empty-iff*)

Just the pigeon hole principle, since we are dealing with 1-sets

lemma *ramsey1*: $\exists N$::*nat*. *partn-lst* $\{..<N\}$ $[q0,q1]$ *1*
proof −
have $\exists i<Suc\ (Suc\ 0).\ \exists H\in nsets\ \{..<Suc\ (q0\ +\ q1)\}\ ([q0,\ q1]\ !\ i).\ f\ `\ nsets\ H\ (Suc\ 0) \subseteq \{i\}$
if $f \in nsets\ \{..<Suc\ (q0\ +\ q1)\}\ (Suc\ 0) \to \{..<Suc\ (Suc\ 0)\}$ **for** f
proof −
define A **where** $A \equiv \lambda i.\ \{q.\ q \le q0+q1 \wedge f\ \{q\} = i\}$
have $A\ 0 \cup A\ 1 = \{..q0\ +\ q1\}$
using *that* **by** (*auto simp*: *A-def PiE-iff nsets-one lessThan-Suc-atMost le-Suc-eq*)
moreover have $A\ 0 \cap A\ 1 = \{\}$
by (*auto simp*: *A-def*)
ultimately have $q0\ +\ q1 \le card\ (A\ 0)\ +\ card\ (A\ 1)$
by (*metis card-Un-le card-atMost eq-imp-le le-SucI le-trans*)
then consider $card\ (A\ 0) \ge q0 \mid card\ (A\ 1) \ge q1$
by *linarith*
then obtain i **where** $i < Suc\ (Suc\ 0)\ card\ (A\ i) \ge [q0,\ q1]\ !\ i$
by (*metis One-nat-def lessI nth-Cons-0 nth-Cons-Suc zero-less-Suc*)
then obtain B **where** $B \subseteq A\ i\ card\ B = [q0,\ q1]\ !\ i\ finite\ B$
by (*meson obtain-subset-with-card-n*)
then have $B \in nsets\ \{..<Suc\ (q0\ +\ q1)\}\ ([q0,\ q1]\ !\ i) \wedge f\ `\ nsets\ B\ (Suc\ 0) \subseteq \{i\}$
by (*auto simp*: *A-def nsets-def card-1-singleton-iff*)
then show *?thesis*
using ‹$i < Suc\ (Suc\ 0)$› **by** *auto*
qed
then show *?thesis*
by (*clarsimp simp*: *partn-lst-def*) *blast*
qed

84.2.2 Ramsey's theorem with two colours and arbitrary exponents (hypergraph version)

proposition *ramsey2-full*: $\exists N$::*nat*. *partn-lst* $\{..<N\}$ $[q1,q2]$ *r*
proof (*induction r arbitrary*: *q1 q2*)
case *0*
then show *?case*
by (*simp add*: *ramsey0*)
next
case (*Suc r*)
note *outer* = *this*
show *?case*
proof (*cases r* = *0*)
case *True*
then show *?thesis*
using *ramsey1* **by** *auto*

```
  next
    case False
    then have r > 0
      by simp
    show ?thesis
      using Suc.prems
    proof (induct k ≡ q1 + q2 arbitrary: q1 q2)
      case 0
      show ?case
      proof
        show partn-lst {..<1::nat} [q1, q2] (Suc r)
          using nsets-empty-iff subset-insert 0
            by (fastforce simp: partn-lst-def funcset-to-empty-iff nsets-eq-empty im-
age-subset-iff)
      qed
    next
      case (Suc k)
      consider q1 = 0 ∨ q2 = 0 | q1 ≠ 0 q2 ≠ 0 by auto
      then show ?case
      proof cases
        case 1
        then have partn-lst {..< Suc 0} [q1, q2] (Suc r)
          unfolding partn-lst-def using ‹r > 0›
          by (fastforce simp add: nsets-empty-iff nsets-singleton-iff lessThan-Suc)
        then show ?thesis by blast
      next
        case 2
        with Suc have k = (q1 − 1) + q2 k = q1 + (q2 − 1) by auto
        then obtain p1 p2::nat where p1: partn-lst {..<p1} [q1−1,q2] (Suc r)
and p2: partn-lst {..<p2} [q1,q2−1] (Suc r)
          using Suc.hyps by blast
        then obtain p::nat where p: partn-lst {..<p} [p1,p2] r
          using outer Suc.prems by auto
        show ?thesis
        proof (intro exI conjI)
          have ∃i<Suc (Suc 0). ∃H∈nsets {..p} ([q1,q2] ! i). f ‘ nsets H (Suc r)
⊆ {i}
            if f: f ∈ nsets {..p} (Suc r) → {..<Suc (Suc 0)} for f
          proof −
            define g where g ≡ λR. f (insert p R)
            have f (insert p i) ∈ {..<Suc (Suc 0)} if i ∈ nsets {..<p} r for i
              using that card-insert-if by (fastforce simp: nsets-def intro!: Pi-mem
[OF f])
            then have g: g ∈ nsets {..<p} r → {..<Suc (Suc 0)}
              by (force simp: g-def PiE-iff)
            then obtain i U where i: i < Suc (Suc 0) and gi: g ‘ nsets U r ⊆ {i}
              and U: U ∈ nsets {..<p} ([p1, p2] ! i)
              using p by (auto simp: partn-lst-def)
            then have Usub: U ⊆ {..<p}
```

```
      by (auto simp: nsets-def)
    consider (izero) i = 0 | (ione) i = Suc 0
      using i by linarith
    then show ?thesis
    proof cases
      case izero
      then have U ∈ nsets {..<p} p1
        using U by simp
      then obtain u where u: bij-betw u {..<p1} U
       using ex-bij-betw-nat-finite lessThan-atLeast0 by (fastforce simp add:
nsets-def)
      have u-nsets: u ` X ∈ nsets {..p} n if X ∈ nsets {..<p1} n for X n
      proof −
        have inj-on u X
             using u that bij-betw-imp-inj-on inj-on-subset by (force simp:
nsets-def)
        then show ?thesis
          using Usub u that bij-betwE
          by (fastforce simp add: nsets-def card-image)
      qed
      define h where h ≡ λR. f (u ` R)
      have h ∈ nsets {..<p1} (Suc r) → {..<Suc (Suc 0)}
        unfolding h-def using f u-nsets by auto
      then obtain j V where j: j <Suc (Suc 0) and hj: h ` nsets V (Suc
r) ⊆ {j}
        and V: V ∈ nsets {..<p1} ([q1 − Suc 0, q2] ! j)
        using p1 by (auto simp: partn-lst-def)
      then have Vsub: V ⊆ {..<p1}
        by (auto simp: nsets-def)
     have invinv-eq: u ` inv-into {..<p1} u ` X = X if X ⊆ u ` {..<p1} for
X
        by (simp add: image-inv-into-cancel that)
      let ?W = insert p (u ` V)
      consider (jzero) j = 0 | (jone) j = Suc 0
        using j by linarith
      then show ?thesis
      proof cases
        case jzero
        then have V ∈ nsets {..<p1} (q1 − Suc 0)
          using V by simp
        then have u ` V ∈ nsets {..<p} (q1 − Suc 0)
          using u-nsets [of - q1 − Suc 0] nsets-mono [OF Vsub] Usub u
          unfolding bij-betw-def nsets-def
          by (fastforce elim!: subsetD)
        then have inq1: ?W ∈ nsets {..p} q1
          unfolding nsets-def using ‹q1 ≠ 0› card-insert-if by fastforce
        have invu-nsets: inv-into {..<p1} u ` X ∈ nsets V r
          if X ∈ nsets (u ` V) r for X r
        proof −
```

```
      have X ⊆ u ‘ V ∧ finite X ∧ card X = r
        using nsets-def that by auto
      then have [simp]: card (inv-into {..<p1} u ‘ X) = card X
    by (meson Vsub bij-betw-def bij-betw-inv-into card-image image-mono
inj-on-subset u)
      show ?thesis
        using that u Vsub by (fastforce simp: nsets-def bij-betw-def)
    qed
    have f X = i if X: X ∈ nsets ?W (Suc r) for X
    proof (cases p ∈ X)
      case True
      then have Xp: X − {p} ∈ nsets (u ‘ V) r
        using X by (auto simp: nsets-def)
      moreover have u ‘ V ⊆ U
        using Vsub bij-betwE u by blast
      ultimately have X − {p} ∈ nsets U r
        by (meson in-mono nsets-mono)
      then have g (X − {p}) = i
        using gi by blast
      have f X = i
        using gi True ‹X − {p} ∈ nsets U r› insert-Diff
        by (fastforce simp add: g-def image-subset-iff)
      then show ?thesis
        by (simp add: ‹f X = i› ‹g (X − {p}) = i›)
    next
      case False
      then have Xim: X ∈ nsets (u ‘ V) (Suc r)
        using X by (auto simp: nsets-def subset-insert)
      then have u ‘ inv-into {..<p1} u ‘ X = X
        using Vsub bij-betw-imp-inj-on u
        by (fastforce simp: nsets-def image-mono invinv-eq subset-trans)
      then show ?thesis
        using izero jzero hj Xim invu-nsets unfolding h-def
        by (fastforce simp add: image-subset-iff)
    qed
    moreover have insert p (u ‘ V) ∈ nsets {..p} q1
      by (simp add: izero inq1)
    ultimately show ?thesis
      by (metis izero image-subsetI insertI1 nth-Cons-0 zero-less-Suc)
  next
    case jone
    then have u ‘ V ∈ nsets {..p} q2
      using V u-nsets by auto
    moreover have f ‘ nsets (u ‘ V) (Suc r) ⊆ {j}
      using hj
     by (force simp add: h-def image-subset-iff nsets-def subset-image-inj
card-image dest: finite-imageD)
    ultimately show ?thesis
      using jone not-less-eq by fastforce
```

```
      qed
    next
      case ione
      then have U ∈ nsets {..<p} p2
        using U by simp
      then obtain u where u: bij-betw u {..<p2} U
       using ex-bij-betw-nat-finite lessThan-atLeast0 by (fastforce simp add:
nsets-def)
      have u-nsets: u ` X ∈ nsets {..p} n if X ∈ nsets {..<p2} n for X n
      proof −
        have inj-on u X
             using u that bij-betw-imp-inj-on inj-on-subset by (force simp:
nsets-def)
        then show ?thesis
          using Usub u that bij-betwE
          by (fastforce simp add: nsets-def card-image)
      qed
      define h where h ≡ λR. f (u ` R)
      have h ∈ nsets {..<p2} (Suc r) → {..<Suc (Suc 0)}
        unfolding h-def using f u-nsets by auto
       then obtain j V where j: j <Suc (Suc 0) and hj: h ` nsets V (Suc
r) ⊆ {j}
        and V: V ∈ nsets {..<p2} ([q1, q2 − Suc 0] ! j)
        using p2 by (auto simp: partn-lst-def)
      then have Vsub: V ⊆ {..<p2}
        by (auto simp: nsets-def)
      have invinv-eq: u ` inv-into {..<p2} u ` X = X if X ⊆ u ` {..<p2} for
X
        by (simp add: image-inv-into-cancel that)
      let ?W = insert p (u ` V)
      consider (jzero) j = 0 | (jone) j = Suc 0
        using j by linarith
      then show ?thesis
      proof cases
        case jone
        then have V ∈ nsets {..<p2} (q2 − Suc 0)
          using V by simp
        then have u ` V ∈ nsets {..<p} (q2 − Suc 0)
          using u-nsets [of - q2 − Suc 0] nsets-mono [OF Vsub] Usub u
          unfolding bij-betw-def nsets-def
          by (fastforce elim!: subsetD)
        then have inq1: ?W ∈ nsets {..p} q2
          unfolding nsets-def using ‹q2 ≠ 0› card-insert-if by fastforce
        have invu-nsets: inv-into {..<p2} u ` X ∈ nsets V r
          if X ∈ nsets (u ` V) r for X r
        proof −
          have X ⊆ u ` V ∧ finite X ∧ card X = r
            using nsets-def that by auto
          then have [simp]: card (inv-into {..<p2} u ` X) = card X
```

```
            by (meson Vsub bij-betw-def bij-betw-inv-into card-image image-mono
inj-on-subset u)
             show ?thesis
               using that u Vsub by (fastforce simp: nsets-def bij-betw-def)
            qed
            have f X = i if X: X ∈ nsets ?W (Suc r) for X
            proof (cases p ∈ X)
             case True
             then have Xp: X − {p} ∈ nsets (u ‘ V) r
               using X by (auto simp: nsets-def)
             moreover have u ‘ V ⊆ U
               using Vsub bij-betwE u by blast
             ultimately have X − {p} ∈ nsets U r
               by (meson in-mono nsets-mono)
             then have g (X − {p}) = i
               using gi by blast
             have f X = i
               using gi True ‹X − {p} ∈ nsets U r› insert-Diff
               by (fastforce simp add: g-def image-subset-iff)
             then show ?thesis
               by (simp add: ‹f X = i› ‹g (X − {p}) = i›)
            next
             case False
             then have Xim: X ∈ nsets (u ‘ V) (Suc r)
               using X by (auto simp: nsets-def subset-insert)
             then have u ‘ inv-into {..<p2} u ‘ X = X
               using Vsub bij-betw-imp-inj-on u
               by (fastforce simp: nsets-def image-mono invinv-eq subset-trans)
             then show ?thesis
               using ione jone hj Xim invu-nsets unfolding h-def
               by (fastforce simp add: image-subset-iff)
            qed
            moreover have insert p (u ‘ V) ∈ nsets {..p} q2
             by (simp add: ione inq1)
            ultimately show ?thesis
            by (metis ione image-subsetI insertI1 lessI nth-Cons-0 nth-Cons-Suc)
           next
            case jzero
            then have u ‘ V ∈ nsets {..p} q1
              using V u-nsets by auto
            moreover have f ‘ nsets (u ‘ V) (Suc r) ⊆ {j}
              using hj
              apply (clarsimp simp add: h-def image-subset-iff nsets-def)
              by (metis Zero-not-Suc card-eq-0-iff card-image subset-image-inj)
            ultimately show ?thesis
              using jzero not-less-eq by fastforce
           qed
          qed
         qed
```

```
        then show partn-lst {..<Suc p} [q1,q2] (Suc r)
          using lessThan-Suc lessThan-Suc-atMost by (auto simp: partn-lst-def
insert-commute)
      qed
    qed
  qed
 qed
qed
```

84.2.3 Full Ramsey's theorem with multiple colours and arbitrary exponents

```
theorem ramsey-full: ∃N::nat. partn-lst {..<N} qs r
proof (induction k ≡ length qs arbitrary: qs)
 case 0
 then show ?case
  by (rule-tac x= r in exI) (simp add: partn-lst-def)
next
 case (Suc k)
 note IH = this
 show ?case
 proof (cases k)
  case 0
  with Suc obtain q where qs = [q]
   by (metis length-0-conv length-Suc-conv)
  then show ?thesis
   by (rule-tac x=q in exI) (auto simp: partn-lst-def funcset-to-empty-iff)
 next
  case (Suc k')
  then obtain q1 q2 l where qs: qs = q1#q2#l
   by (metis Suc.hyps(2) length-Suc-conv)
  then obtain q::nat where q: partn-lst {..<q} [q1,q2] r
   using ramsey2-full by blast
  then obtain p::nat where p: partn-lst {..<p} (q#l) r
   using IH ‹qs = q1 # q2 # l› by fastforce
  have keq: Suc (length l) = k
   using IH qs by auto
  show ?thesis
  proof (intro exI conjI)
   show partn-lst {..<p} qs r
   proof (auto simp: partn-lst-def)
    fix f
    assume f: f ∈ nsets {..<p} r → {..<length qs}
    define g where g ≡ λX. if f X < Suc (Suc 0) then 0 else f X − Suc 0
    have g ∈ nsets {..<p} r → {..<k}
     unfolding g-def using f Suc IH
     by (auto simp: Pi-def not-less)
    then obtain i U where i: i < k and gi: g ‘ nsets U r ⊆ {i}
          and U: U ∈ nsets {..<p} ((q#l) ! i)
```

```
      using p keq by (auto simp: partn-lst-def)
    show ∃i<length qs. ∃H∈nsets {..<p} (qs ! i). f ` nsets H r ⊆ {i}
    proof (cases i = 0)
      case True
      then have U ∈ nsets {..<p} q and f01: f ` nsets U r ⊆ {0, Suc 0}
        using U gi unfolding g-def by (auto simp: image-subset-iff)
      then obtain u where u: bij-betw u {..<q} U
        using ex-bij-betw-nat-finite lessThan-atLeast0 by (fastforce simp add:
nsets-def)
      then have Usub: U ⊆ {..<p}
        by (smt (verit) U mem-Collect-eq nsets-def)
      have u-nsets: u ` X ∈ nsets {..<p} n if X ∈ nsets {..<q} n for X n
      proof −
        have inj-on u X
          using u that bij-betw-imp-inj-on inj-on-subset
          by (force simp: nsets-def)
        then show ?thesis
          using Usub u that bij-betwE
          by (fastforce simp add: nsets-def card-image)
      qed
      define h where h ≡ λX. f (u ` X)
      have f (u ` X) < Suc (Suc 0) if X ∈ nsets {..<q} r for X
      proof −
        have u ` X ∈ nsets U r
          using u u-nsets that by (auto simp: nsets-def bij-betwE subset-eq)
        then show ?thesis
          using f01 by auto
      qed
      then have h ∈ nsets {..<q} r → {..<Suc (Suc 0)}
        unfolding h-def by blast
      then obtain j V where j: j < Suc (Suc 0) and hj: h ` nsets V r ⊆ {j}
        and V: V ∈ nsets {..<q} ([q1,q2] ! j)
        using q by (auto simp: partn-lst-def)
      show ?thesis
      proof (intro exI conjI bexI)
        show j < length qs
          using Suc Suc.hyps(2) j by linarith
        have nsets (u ` V) r ⊆ (λx. (u ` x)) ` nsets V r
          apply (clarsimp simp add: nsets-def image-iff)
          by (metis card-eq-0-iff card-image image-is-empty subset-image-inj)
        then have f ` nsets (u ` V) r ⊆ h ` nsets V r
          by (auto simp: h-def)
        then show f ` nsets (u ` V) r ⊆ {j}
          using hj by auto
        show (u ` V) ∈ nsets {..<p} (qs ! j)
          using V j less-2-cases numeral-2-eq-2 qs u-nsets by fastforce
      qed
    next
      case False
```

```
        show ?thesis
        proof (intro exI conjI bexI)
          show Suc i < length qs
            using Suc.hyps(2) i by auto
          show f ‘ nsets U r ⊆ {Suc i}
            using i gi False
            apply (auto simp: g-def image-subset-iff)
                by (metis Suc-lessD Suc-pred g-def gi image-subset-iff not-less-eq
singleton-iff)
          show U ∈ nsets {..<p} (qs ! (Suc i))
            using False U qs by auto
        qed
      qed
    qed
  qed
qed
```

84.2.4 Simple graph version

This is the most basic version in terms of cliques and independent sets, i.e. the version for graphs and 2 colours.

```
definition clique V E ⟷ (∀ v∈V. ∀ w∈V. v ≠ w ⟶ {v, w} ∈ E)
definition indep V E ⟷ (∀ v∈V. ∀ w∈V. v ≠ w ⟶ {v, w} ∉ E)

lemma ramsey2:
  ∃ r≥1. ∀ (V::'a set) (E::'a set set). finite V ∧ card V ≥ r ⟶
    (∃ R ⊆ V. card R = m ∧ clique R E ∨ card R = n ∧ indep R E)
proof −
  obtain N where N ≥ Suc 0 and N: partn-lst {..<N} [m,n] 2
    using ramsey2-full nat-le-linear partn-lst-greater-resource by blast
  have ∃ R⊆V. card R = m ∧ clique R E ∨ card R = n ∧ indep R E
    if finite V N ≤ card V for V :: 'a set and E :: 'a set set
  proof −
    from that
    obtain v where u: inj-on v {..<N} v ‘ {..<N} ⊆ V
      by (metis card-le-inj card-lessThan finite-lessThan)
    define f where f ≡ λe. if v ‘ e ∈ E then 0 else Suc 0
    have f: f ∈ nsets {..<N} 2 → {..<Suc (Suc 0)}
      by (simp add: f-def)
    then obtain i U where i: i < 2 and gi: f ‘ nsets U 2 ⊆ {i}
      and U: U ∈ nsets {..<N} ([m,n] ! i)
      using N numeral-2-eq-2 by (auto simp: partn-lst-def)
    show ?thesis
    proof (intro exI conjI)
      show v ‘ U ⊆ V
        using U u by (auto simp: image-subset-iff nsets-def)
      show card (v ‘ U) = m ∧ clique (v ‘ U) E ∨ card (v ‘ U) = n ∧ indep (v ‘
U) E
```

```
      using i unfolding numeral-2-eq-2
        using gi U u
      apply (simp add: image-subset-iff nsets-2-eq clique-def indep-def less-Suc-eq)
      apply (auto simp: f-def nsets-def card-image inj-on-subset split: if-split-asm)
        done
    qed
  qed
  then show ?thesis
    using ‹Suc 0 ≤ N› by auto
qed
```

84.3 Preliminaries

84.3.1 "Axiom" of Dependent Choice

```
primrec choice :: ('a ⇒ bool) ⇒ ('a × 'a) set ⇒ nat ⇒ 'a
  where — An integer-indexed chain of choices
    choice-0: choice P r 0 = (SOME x. P x)
  | choice-Suc: choice P r (Suc n) = (SOME y. P y ∧ (choice P r n, y) ∈ r)

lemma choice-n:
  assumes P0: P x0
    and Pstep: ⋀x. P x ⟹ ∃ y. P y ∧ (x, y) ∈ r
  shows P (choice P r n)
proof (induct n)
  case 0
  show ?case by (force intro: someI P0)
next
  case Suc
  then show ?case by (auto intro: someI2-ex [OF Pstep])
qed

lemma dependent-choice:
  assumes trans: trans r
    and P0: P x0
    and Pstep: ⋀x. P x ⟹ ∃ y. P y ∧ (x, y) ∈ r
  obtains f :: nat ⇒ 'a where ⋀n. P (f n) and ⋀n m. n < m ⟹ (f n, f m) ∈ r
proof
  fix n
  show P (choice P r n)
    by (blast intro: choice-n [OF P0 Pstep])
next
  fix n m :: nat
  assume n < m
  from Pstep [OF choice-n [OF P0 Pstep]] have (choice P r k, choice P r (Suc
k)) ∈ r for k
    by (auto intro: someI2-ex)
  then show (choice P r n, choice P r m) ∈ r
    by (auto intro: less-Suc-induct [OF ‹n < m›] transD [OF trans])
qed
```

84.3.2 Partition functions

definition *part-fn* :: *nat* ⇒ *nat* ⇒ *'a set* ⇒ (*'a set* ⇒ *nat*) ⇒ *bool*
— the function f partitions the r-subsets of the typically infinite set Y into s distinct categories.
where *part-fn r s Y f* ⟷ (*f* ∈ *nsets Y r* → {..<*s*})

For induction, we decrease the value of r in partitions.

lemma *part-fn-Suc-imp-part-fn*:
⟦*infinite Y*; *part-fn* (*Suc r*) *s Y f*; *y* ∈ *Y*⟧ ⟹ *part-fn r s* (*Y* − {*y*}) (λ*u*. *f* (*insert y u*))
by (*simp add*: *part-fn-def nsets-def Pi-def subset-Diff-insert*)

lemma *part-fn-subset*: *part-fn r s YY f* ⟹ *Y* ⊆ *YY* ⟹ *part-fn r s Y f*
unfolding *part-fn-def nsets-def* **by** *blast*

84.4 Ramsey's Theorem: Infinitary Version

lemma *Ramsey-induction*:
fixes *s r* :: *nat*
and *YY* :: *'a set*
and *f* :: *'a set* ⇒ *nat*
assumes *infinite YY part-fn r s YY f*
shows ∃ *Y' t'*. *Y'* ⊆ *YY* ∧ *infinite Y'* ∧ *t'* < *s* ∧ (∀ *X*. *X* ⊆ *Y'* ∧ *finite X* ∧ *card X* = *r* ⟶ *f X* = *t'*)
using *assms*
proof (*induct r arbitrary*: *YY f*)
case *0*
then show *?case*
by (*auto simp add*: *part-fn-def card-eq-0-iff cong*: *conj-cong*)
next
case (*Suc r*)
show *?case*
proof −
from *Suc.prems infinite-imp-nonempty* **obtain** *yy* **where** *yy*: *yy* ∈ *YY*
by *blast*
let *?ramr* = {((*y*, *Y*, *t*), (*y'*, *Y'*, *t'*)). *y'* ∈ *Y* ∧ *Y'* ⊆ *Y*}
let *?propr* = λ(*y*, *Y*, *t*).
y ∈ *YY* ∧ *y* ∉ *Y* ∧ *Y* ⊆ *YY* ∧ *infinite Y* ∧ *t* < *s*
∧ (∀ *X*. *X*⊆*Y* ∧ *finite X* ∧ *card X* = *r* ⟶ (*f* ∘ *insert y*) *X* = *t*)
from *Suc.prems* **have** *infYY'*: *infinite* (*YY* − {*yy*}) **by** *auto*
from *Suc.prems* **have** *partf'*: *part-fn r s* (*YY* − {*yy*}) (*f* ∘ *insert yy*)
by (*simp add*: *o-def part-fn-Suc-imp-part-fn yy*)
have *transr*: *trans ?ramr* **by** (*force simp add*: *trans-def*)
from *Suc.hyps* [*OF infYY' partf'*]
obtain *Y0* **and** *t0* **where** *Y0* ⊆ *YY* − {*yy*} *infinite Y0 t0* < *s*
X ⊆ *Y0* ∧ *finite X* ∧ *card X* = *r* ⟶ (*f* ∘ *insert yy*) *X* = *t0* **for** *X*
by *blast*
with *yy* **have** *propr0*: *?propr*(*yy*, *Y0*, *t0*) **by** *blast*
have *proprstep*: ∃ *y*. *?propr y* ∧ (*x*, *y*) ∈ *?ramr* **if** *x*: *?propr x* **for** *x*

```
    proof (cases x)
      case (fields yx Yx tx)
      with x obtain yx' where yx': yx' ∈ Yx
        by (blast dest: infinite-imp-nonempty)
      from fields x have infYx': infinite (Yx − {yx'}) by auto
      with fields x yx' Suc.prems have partfx': part-fn r s (Yx − {yx'}) (f ∘ insert
yx')
       by (simp add: o-def part-fn-Suc-imp-part-fn part-fn-subset [where YY=YY
and Y=Yx])
      from Suc.hyps [OF infYx' partfx'] obtain Y' and t'
        where Y': Y' ⊆ Yx − {yx'} infinite Y' t' < s
          X ⊆ Y' ∧ finite X ∧ card X = r ⟶ (f ∘ insert yx') X = t' for X
        by blast
      from fields x Y' yx' have ?propr (yx', Y', t') ∧ (x, (yx', Y', t')) ∈ ?ramr
        by blast
      then show ?thesis ..
    qed
    from dependent-choice [OF transr propr0 proprstep]
    obtain g where pg: ?propr (g n) and rg: n < m ⟹ (g n, g m) ∈ ?ramr for
n m :: nat
      by blast
    let ?gy = fst ∘ g
    let ?gt = snd ∘ snd ∘ g
    have rangeg: ∃ k. range ?gt ⊆ {..<k}
    proof (intro exI subsetI)
      fix x
      assume x ∈ range ?gt
      then obtain n where x = ?gt n ..
      with pg [of n] show x ∈ {..<s} by (cases g n) auto
    qed
    from rangeg have finite (range ?gt)
      by (simp add: finite-nat-iff-bounded)
    then obtain s' and n' where s': s' = ?gt n' and infeqs': infinite {n. ?gt n =
s'}
      by (rule inf-img-fin-domE) (auto simp add: vimage-def intro: infinite-UNIV-nat)
    with pg [of n'] have less': s'<s by (cases g n') auto
    have inj-gy: inj ?gy
    proof (rule linorder-injI)
      fix m m' :: nat
      assume m < m'
      from rg [OF this] pg [of m] show ?gy m ≠ ?gy m'
        by (cases g m, cases g m') auto
    qed
    show ?thesis
    proof (intro exI conjI)
      from pg show ?gy ` {n. ?gt n = s'} ⊆ YY
        by (auto simp add: Let-def split-beta)
      from infeqs' show infinite (?gy ` {n. ?gt n = s'})
        by (blast intro: inj-gy [THEN subset-inj-on] dest: finite-imageD)
```

```
      show s' < s by (rule less')
      show ∀X. X ⊆ ?gy ` {n. ?gt n = s'} ∧ finite X ∧ card X = Suc r ⟶ f X
= s'
      proof −
        have f X = s'
          if X: X ⊆ ?gy ` {n. ?gt n = s'}
          and cardX: finite X card X = Suc r
          for X
        proof −
          from X obtain AA where AA: AA ⊆ {n. ?gt n = s'} and Xeq: X =
?gy`AA
            by (auto simp add: subset-image-iff)
          with cardX have AA ≠ {} by auto
         then have AAleast: (LEAST x. x ∈ AA) ∈ AA by (auto intro: LeastI-ex)
          show ?thesis
          proof (cases g (LEAST x. x ∈ AA))
            case (fields ya Ya ta)
            with AAleast Xeq have ya: ya ∈ X by (force intro!: rev-image-eqI)
           then have f X = f (insert ya (X − {ya})) by (simp add: insert-absorb)
            also have ... = ta
            proof −
              have ∗: X − {ya} ⊆ Ya
              proof
                fix x assume x: x ∈ X − {ya}
                then obtain a' where xeq: x = ?gy a' and a': a' ∈ AA
                  by (auto simp add: Xeq)
                with fields x have a' ≠ (LEAST x. x ∈ AA) by auto
               with Least-le [of λx. x ∈ AA, OF a'] have (LEAST x. x ∈ AA) < a'
                  by arith
                from xeq fields rg [OF this] show x ∈ Ya by auto
              qed
              have card (X − {ya}) = r
                by (simp add: cardX ya)
              with pg [of LEAST x. x ∈ AA] fields cardX ∗ show ?thesis
                by (auto simp del: insert-Diff-single)
            qed
            also from AA AAleast fields have ... = s' by auto
            finally show ?thesis .
          qed
        qed
        then show ?thesis by blast
      qed
    qed
  qed
qed


theorem Ramsey:
  fixes s r :: nat
```

and $Z :: 'a\ set$
and $f :: 'a\ set \Rightarrow nat$
shows
$[\![infinite\ Z;$
$\forall X.\ X \subseteq Z \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X < s]\!]$
$\Longrightarrow \exists Y\ t.\ Y \subseteq Z \wedge infinite\ Y \wedge t < s$
$\wedge\ (\forall X.\ X \subseteq Y \wedge finite\ X \wedge card\ X = r \longrightarrow f\ X = t)$
by (*blast intro*: *Ramsey-induction* [*unfolded part-fn-def nsets-def*])

corollary *Ramsey2*:
fixes $s :: nat$
and $Z :: 'a\ set$
and $f :: 'a\ set \Rightarrow nat$
assumes *infZ*: *infinite Z*
and *part*: $\forall x{\in}Z.\ \forall y{\in}Z.\ x \neq y \longrightarrow f\ \{x, y\} < s$
shows $\exists Y\ t.\ Y \subseteq Z \wedge infinite\ Y \wedge t < s \wedge (\forall x{\in}Y.\ \forall y{\in}Y.\ x{\neq}y \longrightarrow f\ \{x, y\} = t)$
proof −
from *part* **have** *part2*: $\forall X.\ X \subseteq Z \wedge finite\ X \wedge card\ X = 2 \longrightarrow f\ X < s$
by (*fastforce simp add*: *eval-nat-numeral card-Suc-eq*)
obtain $Y\ t$ **where** $*$:
$Y \subseteq Z\ infinite\ Y\ t < s\ (\forall X.\ X \subseteq Y \wedge finite\ X \wedge card\ X = 2 \longrightarrow f\ X = t)$
by (*insert Ramsey* [*OF infZ part2*]) *auto*
then have $\forall x{\in}Y.\ \forall y{\in}Y.\ x \neq y \longrightarrow f\ \{x, y\} = t$ **by** *auto*
with $*$ **show** *?thesis* **by** *iprover*
qed

corollary *Ramsey-nsets*:
fixes $f :: 'a\ set \Rightarrow nat$
assumes *infinite Z f ' nsets Z r* $\subseteq \{..{<}s\}$
obtains $Y\ t$ **where** $Y \subseteq Z\ infinite\ Y\ t < s\ f\ `\ nsets\ Y\ r \subseteq \{t\}$
using *Ramsey* [*of Z r f s*] *assms* **by** (*auto simp*: *nsets-def image-subset-iff*)

84.5 Disjunctive Well-Foundedness

An application of Ramsey's theorem to program termination. See [3].

definition *disj-wf* :: $('a \times 'a)\ set \Rightarrow bool$
where *disj-wf r* $\longleftrightarrow$ $(\exists T.\ \exists n{::}nat.\ (\forall i{<}n.\ wf\ (T\ i)) \wedge r = (\bigcup i{<}n.\ T\ i))$

definition *transition-idx* :: $(nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow ('a \times 'a)\ set) \Rightarrow nat\ set \Rightarrow nat$
where *transition-idx s T A* = $(LEAST\ k.\ \exists i\ j.\ A = \{i, j\} \wedge i < j \wedge (s\ j, s\ i) \in T\ k)$

lemma *transition-idx-less*:
assumes $i < j\ (s\ j, s\ i) \in T\ k\ k < n$
shows *transition-idx s T* $\{i, j\} < n$
proof −
from *assms*(*1*,*2*) **have** *transition-idx s T* $\{i, j\} \leq k$

by (*simp add*: *transition-idx-def*, *blast intro*: *Least-le*)
with *assms*(*3*) **show** *?thesis* **by** *simp*
qed

lemma *transition-idx-in*:
assumes $i < j$ $(s\ j,\ s\ i) \in T\ k$
shows $(s\ j,\ s\ i) \in T\ (\mathit{transition\text{-}idx}\ s\ T\ \{i,\ j\})$
using *assms*
by (*simp add*: *transition-idx-def doubleton-eq-iff conj-disj-distribR cong*: *conj-cong*) (*erule LeastI*)

To be equal to the union of some well-founded relations is equivalent to being the subset of such a union.

lemma *disj-wf*: *disj-wf* $r \longleftrightarrow (\exists\, T.\ \exists\, n{::}\mathit{nat}.\ (\forall\, i{<}n.\ \mathit{wf}(T\ i)) \wedge r \subseteq (\bigcup i{<}n.\ T\ i))$
proof −
have ∗: $\bigwedge T\ n.\ [\![\forall\, i{<}n.\ \mathit{wf}\ (T\ i);\ r \subseteq \bigcup\ (T\ `\ \{..{<}n\})]\!]$
$\Longrightarrow (\forall\, i{<}n.\ \mathit{wf}\ (T\ i \cap r)) \wedge r = (\bigcup i{<}n.\ T\ i \cap r)$
by (*force simp add*: *wf-Int1*)
show *?thesis*
unfolding *disj-wf-def* **by** *auto* (*metis* ∗)
qed

theorem *trans-disj-wf-implies-wf*:
assumes *trans r*
and *disj-wf r*
shows *wf r*
proof (*simp only*: *wf-iff-no-infinite-down-chain*, *rule notI*)
assume $\exists\, s.\ \forall\, i.\ (s\ (\mathit{Suc}\ i),\ s\ i) \in r$
then obtain *s* **where** *sSuc*: $\forall\, i.\ (s\ (\mathit{Suc}\ i),\ s\ i) \in r$ **..**
from ‹*disj-wf r*› **obtain** *T* **and** *n* :: *nat* **where** *wfT*: $\forall\, k{<}n.\ \mathit{wf}(T\ k)$ **and** *r*: $r = (\bigcup k{<}n.\ T\ k)$
by (*auto simp add*: *disj-wf-def*)
have *s-in-T*: $\exists\, k.\ (s\ j,\ s\ i) \in T\ k \wedge k{<}n$ **if** $i < j$ **for** *i j*
proof −
from ‹$i < j$› **have** $(s\ j,\ s\ i) \in r$
proof (*induct rule*: *less-Suc-induct*)
case *1*
then show *?case* **by** (*simp add*: *sSuc*)
next
case *2*
with ‹*trans r*› **show** *?case*
unfolding *trans-def* **by** *blast*
qed
then show *?thesis* **by** (*auto simp add*: *r*)
qed
have $i < j \Longrightarrow \mathit{transition\text{-}idx}\ s\ T\ \{i,\ j\} < n$ **for** *i j*
using *s-in-T transition-idx-less* **by** *blast*
then have *trless*: $i \neq j \Longrightarrow \mathit{transition\text{-}idx}\ s\ T\ \{i,\ j\} < n$ **for** *i j*

by (*metis doubleton-eq-iff less-linear*)
have $\exists K\ k.\ K \subseteq UNIV \wedge infinite\ K \wedge k < n \wedge$
$(\forall i{\in}K.\ \forall j{\in}K.\ i \neq j \longrightarrow transition\text{-}idx\ s\ T\ \{i, j\} = k)$
by (*rule Ramsey2*) (*auto intro*: *trless infinite-UNIV-nat*)
then obtain K **and** k **where** *infK*: *infinite* K **and** $k < n$
and *allk*: $\forall i{\in}K.\ \forall j{\in}K.\ i \neq j \longrightarrow transition\text{-}idx\ s\ T\ \{i, j\} = k$
by *auto*
have (*s* (*enumerate K* (*Suc m*)), *s* (*enumerate K m*)) ∈ *T k* **for** *m* :: *nat*
proof −
let *?j* = *enumerate K* (*Suc m*)
let *?i* = *enumerate K m*
have *ij*: *?i* < *?j* **by** (*simp add*: *enumerate-step infK*)
have *?j* ∈ *K ?i* ∈ *K* **by** (*simp-all add*: *enumerate-in-set infK*)
with *ij* **have** *k*: *k* = *transition-idx s T* {*?i*, *?j*} **by** (*simp add*: *allk*)
from *s-in-T* [*OF ij*] **obtain** *k′* **where** (*s ?j*, *s ?i*) ∈ *T k′ k′*<*n* **by** *blast*
then show (*s ?j*, *s ?i*) ∈ *T k* **by** (*simp add*: *k transition-idx-in ij*)
qed
then have ¬ *wf* (*T k*)
unfolding *wf-iff-no-infinite-down-chain* **by** *iprover*
with *wfT* ‹*k* < *n*› **show** *False* **by** *blast*
qed

end

85 Generic reflection and reification

theory *Reflection*
imports *Main*
begin

ML-file ‹~~/*src*/*HOL*/*Tools*/*reflection.ML*›

method-setup *reify* = ‹
Attrib.thms −−
Scan.option (*Scan.lift* (*Args.*$$$ () |−− *Args.term* −−| *Scan.lift* (*Args.*$$$)))
>>
(*fn* (*user-eqs*, *to*) => *fn ctxt* => *SIMPLE-METHOD′* (*Reflection.default-reify-tac ctxt user-eqs to*))
› *partial automatic reification*

method-setup *reflection* = ‹
let
fun keyword k = *Scan.lift* (*Args.*$$$ *k* −− *Args.colon*) >> *K* ();
val onlyN = *only*;
val rulesN = *rules*;
val any-keyword = *keyword onlyN* || *keyword rulesN*;
val thms = *Scan.repeats* (*Scan.unless any-keyword Attrib.multi-thm*);
val terms = *thms* >> *map* (*Thm.term-of o Drule.dest-term*);
in

thms −− *Scan.optional* (*keyword rulesN* |−− *thms*) [] −−
Scan.option (*keyword onlyN* |−− *Args.term*) >>
(*fn* ((*user-eqs*, *user-thms*), *to*) => *fn ctxt* =>
SIMPLE-METHOD' (*Reflection.default-reflection-tac ctxt user-thms user-eqs to*))
end
› *partial automatic reflection*

end

theory *Rewrite*
imports *Main*
begin

consts *rewrite-HOLE* :: *'a*::{} (⌑)

lemma *eta-expand*:
fixes *f* :: *'a*::{} ⇒ *'b*::{}
shows *f* ≡ λ*x*. *f x* **.**

lemma *imp-cong-eq*:
(*PROP A* ⟹ (*PROP B* ⟹ *PROP C*) ≡ (*PROP B'* ⟹ *PROP C'*)) ≡
((*PROP B* ⟹ *PROP A* ⟹ *PROP C*) ≡ (*PROP B'* ⟹ *PROP A* ⟹ *PROP C'*))
apply (*intro Pure.equal-intr-rule*)
apply (*drule* (*1*) *cut-rl*; *drule Pure.equal-elim-rule1 Pure.equal-elim-rule2*; *assumption*)+
apply (*drule Pure.equal-elim-rule1 Pure.equal-elim-rule2*; *assumption*)+
done

ML-file ‹*cconv.ML*›
ML-file ‹*rewrite.ML*›

end

86 Assigning lengths to types by type classes

theory *Type-Length*
imports *Numeral-Type*
begin

The aim of this is to allow any type as index type, but to provide a default instantiation for numeral types. This independence requires some duplication with the definitions in `Numeral_Type.thy`.

class *len0* =
fixes *len-of* :: *'a itself* ⇒ *nat*

syntax *-type-length* :: *type* ⇒ *nat* (‹(*1LENGTH*/(*1'*(-')))›)

translations *LENGTH('a)* ⇀
CONST len-of (CONST Pure.type :: 'a itself)

print-translation ‹
let
*fun len-of-itself-tr' ctxt [Const (**const-syntax**‹Pure.type›, Type (-, [T]))] =*
*Syntax.const **syntax-const**‹-type-length› $ Syntax-Phases.term-of-typ ctxt T*
*in [(**const-syntax**‹len-of›, len-of-itself-tr')] end*
›

Some theorems are only true on words with length greater 0.

class *len = len0 +*
assumes *len-gt-0 [iff]: 0 < LENGTH('a)*
begin

lemma *len-not-eq-0 [simp]:*
LENGTH('a) ≠ 0
by *simp*

end

instantiation *num0* **and** *num1 :: len0*
begin

definition *len-num0: len-of (- :: num0 itself) = 0*
definition *len-num1: len-of (- :: num1 itself) = 1*

instance ..

end

instantiation *bit0* **and** *bit1 :: (len0) len0*
begin

definition *len-bit0: len-of (- :: 'a::len0 bit0 itself) = 2 * LENGTH('a)*
definition *len-bit1: len-of (- :: 'a::len0 bit1 itself) = 2 * LENGTH('a) + 1*

instance ..

end

lemmas *len-of-numeral-defs [simp] = len-num0 len-num1 len-bit0 len-bit1*

instance *num1 :: len*
by *standard simp*
instance *bit0 :: (len) len*
by *standard simp*
instance *bit1 :: (len0) len*

by *standard simp*

instantiation *Enum.finite-1* :: *len*
begin

definition
len-of-finite-1 (*x* :: *Enum.finite-1 itself*) ≡ (*1* :: *nat*)

instance
by *standard* (*auto simp*: *len-of-finite-1-def*)

end

instantiation *Enum.finite-2* :: *len*
begin

definition
len-of-finite-2 (*x* :: *Enum.finite-2 itself*) ≡ (*2* :: *nat*)

instance
by *standard* (*auto simp*: *len-of-finite-2-def*)

end

instantiation *Enum.finite-3* :: *len*
begin

definition
len-of-finite-3 (*x* :: *Enum.finite-3 itself*) ≡ (*4* :: *nat*)

instance
by *standard* (*auto simp*: *len-of-finite-3-def*)

end

lemma *length-not-greater-eq-2-iff* [*simp*]:
‹¬ *2* ≤ *LENGTH*('*a*::*len*) ⟷ *LENGTH*('*a*) = *1*›
by (*auto simp add*: *not-le dest*: *less-2-cases*)

context *linordered-idom*
begin

lemma *two-less-eq-exp-length* [*simp*]:
‹*2* ≤ *2* ^ *LENGTH*('*b*::*len*)›
using *mult-left-mono* [*of 1* ‹*2* ^ (*LENGTH*('*b*::*len*) − *1*)› *2*]
by (*cases* ‹*LENGTH*('*b*::*len*)›) *simp-all*

end

lemma *less-eq-decr-length-iff* [*simp*]:
‹$n \leq LENGTH('a::len) - Suc\ 0 \longleftrightarrow n < LENGTH('a)$›
by (*cases* ‹$LENGTH('a)$›) (*simp-all add*: *less-Suc-eq le-less*)

lemma *decr-length-less-iff* [*simp*]:
‹$LENGTH('a::len) - Suc\ 0 < n \longleftrightarrow LENGTH('a) \leq n$›
by (*cases* ‹$LENGTH('a)$›) *auto*

end

87 Saturated arithmetic

theory *Saturated*
imports *Numeral-Type Type-Length*
begin

87.1 The type of saturated naturals

typedef (**overloaded**) ($'a::len$) *sat* = {.. $LENGTH('a)$}
morphisms *nat-of Abs-sat*
by *auto*

lemma *sat-eqI*:
nat-of m = *nat-of* $n \Longrightarrow m = n$
by (*simp add*: *nat-of-inject*)

lemma *sat-eq-iff*:
$m = n \longleftrightarrow$ *nat-of* m = *nat-of* n
by (*simp add*: *nat-of-inject*)

lemma *Abs-sat-nat-of* [*code abstype*]:
Abs-sat (*nat-of* n) = n
by (*fact nat-of-inverse*)

definition *Abs-sat'* :: *nat* ⇒ $'a::len$ *sat* **where**
Abs-sat' n = *Abs-sat* (*min* ($LENGTH('a)$) n)

lemma *nat-of-Abs-sat'* [*simp*]:
nat-of (*Abs-sat'* n :: ($'a::len$) *sat*) = *min* ($LENGTH('a)$) n
unfolding *Abs-sat'-def* **by** (*rule Abs-sat-inverse*) *simp*

lemma *nat-of-le-len-of* [*simp*]:
nat-of (n :: ($'a::len$) *sat*) ≤ $LENGTH('a)$
using *nat-of* [**where** $x = n$] **by** *simp*

lemma *min-len-of-nat-of* [*simp*]:
min ($LENGTH('a)$) (*nat-of* (n::($'a::len$) *sat*)) = *nat-of* n
by (*rule min.absorb2* [*OF nat-of-le-len-of*])

lemma *min-nat-of-len-of* [*simp*]:
min (*nat-of* (*n*::(*'a*::*len*) *sat*)) (*LENGTH*(*'a*)) = *nat-of n*
by (*subst min.commute*) *simp*

lemma *Abs-sat'-nat-of* [*simp*]:
Abs-sat' (*nat-of n*) = *n*
by (*simp add*: *Abs-sat'-def nat-of-inverse*)

instantiation *sat* :: (*len*) *linorder*
begin

definition
less-eq-sat-def: *x* ≤ *y* ⟷ *nat-of x* ≤ *nat-of y*

definition
less-sat-def: *x* < *y* ⟷ *nat-of x* < *nat-of y*

instance
by *standard*
(*auto simp add*: *less-eq-sat-def less-sat-def not-le sat-eq-iff min.coboundedI1 mult.commute*)

end

instantiation *sat* :: (*len*) {*minus*, *comm-semiring-1*}
begin

definition
0 = *Abs-sat'* *0*

definition
1 = *Abs-sat'* *1*

lemma *nat-of-zero-sat* [*simp*, *code abstract*]:
nat-of 0 = *0*
by (*simp add*: *zero-sat-def*)

lemma *nat-of-one-sat* [*simp*, *code abstract*]:
nat-of 1 = *min 1* (*LENGTH*(*'a*))
by (*simp add*: *one-sat-def*)

definition
x + *y* = *Abs-sat'* (*nat-of x* + *nat-of y*)

lemma *nat-of-plus-sat* [*simp*, *code abstract*]:
nat-of (*x* + *y*) = *min* (*nat-of x* + *nat-of y*) (*LENGTH*(*'a*))
by (*simp add*: *plus-sat-def*)

definition

x − y = Abs-sat′ (nat-of x − nat-of y)

lemma *nat-of-minus-sat* [*simp*, *code abstract*]:
nat-of (x − y) = nat-of x − nat-of y
proof −
from *nat-of-le-len-of* [*of x*] **have** *nat-of x − nat-of y ≤ LENGTH('a)* **by** *arith*
then show *?thesis* **by** (*simp add*: *minus-sat-def*)
qed

definition
x ∗ y = Abs-sat′ (nat-of x ∗ nat-of y)

lemma *nat-of-times-sat* [*simp*, *code abstract*]:
nat-of (x ∗ y) = min (nat-of x ∗ nat-of y) (LENGTH('a))
by (*simp add*: *times-sat-def*)

instance
proof
fix *a b c* :: *'a*::*len sat*
show *a ∗ b ∗ c = a ∗ (b ∗ c)*
proof(*cases a = 0*)
case *True* **thus** *?thesis* **by** (*simp add*: *sat-eq-iff*)
next
case *False* **show** *?thesis*
proof(*cases c = 0*)
case *True* **thus** *?thesis* **by** (*simp add*: *sat-eq-iff*)
next
case *False* **with** ‹*a ≠ 0*› **show** *?thesis*
by (*simp add*: *sat-eq-iff nat-mult-min-left nat-mult-min-right mult.assoc min.assoc min.absorb2*)
qed
qed
show *1 ∗ a = a*
by (*simp add*: *sat-eq-iff min-def not-le not-less*)
show *(a + b) ∗ c = a ∗ c + b ∗ c*
proof(*cases c = 0*)
case *True* **thus** *?thesis* **by** (*simp add*: *sat-eq-iff*)
next
case *False* **thus** *?thesis*
by (*simp add*: *sat-eq-iff nat-mult-min-left add-mult-distrib min-add-distrib-left min-add-distrib-right min.assoc min.absorb2*)
qed
qed (*simp-all add*: *sat-eq-iff mult.commute*)

end

instantiation *sat* :: (*len*) *ordered-comm-semiring*
begin

instance
by *standard*
(*auto simp add*: *less-eq-sat-def less-sat-def not-le sat-eq-iff min.coboundedI1 mult.commute*)

end

lemma *Abs-sat'-eq-of-nat*: *Abs-sat' n* = *of-nat n*
by (*rule sat-eqI*, *induct n*, *simp-all*)

abbreviation *Sat* :: *nat* ⇒ *'a*::*len sat* **where**
Sat ≡ *of-nat*

lemma *nat-of-Sat* [*simp*]:
nat-of (*Sat n* :: (*'a*::*len*) *sat*) = *min* (*LENGTH*(*'a*)) *n*
by (*rule nat-of-Abs-sat'* [*unfolded Abs-sat'-eq-of-nat*])

lemma [*code-abbrev*]:
of-nat (*numeral k*) = (*numeral k* :: *'a*::*len sat*)
by *simp*

context
begin

qualified definition *sat-of-nat* :: *nat* ⇒ (*'a*::*len*) *sat*
where [*code-abbrev*]: *sat-of-nat* = *of-nat*

lemma [*code abstract*]:
nat-of (*sat-of-nat n* :: (*'a*::*len*) *sat*) = *min* (*LENGTH*(*'a*)) *n*
by (*simp add*: *sat-of-nat-def*)

end

instance *sat* :: (*len*) *finite*
proof
show *finite* (*UNIV*::*'a sat set*)
unfolding *type-definition.univ* [*OF type-definition-sat*]
using *finite* **by** *simp*
qed

instantiation *sat* :: (*len*) *equal*
begin

definition *HOL.equal A B* ⟷ *nat-of A* = *nat-of B*

instance
by *standard* (*simp add*: *equal-sat-def nat-of-inject*)

end

instantiation *sat* :: (*len*) {*bounded-lattice*, *distrib-lattice*}
begin

definition (*inf* :: *'a sat* ⇒ *'a sat* ⇒ *'a sat*) = *min*
definition (*sup* :: *'a sat* ⇒ *'a sat* ⇒ *'a sat*) = *max*
definition *bot* = (*0* :: *'a sat*)
definition *top* = *Sat* (*LENGTH*(*'a*))

instance
  by *standard*
    (*simp-all add*: *inf-sat-def sup-sat-def bot-sat-def top-sat-def max-min-distrib2*,
      simp-all add: *less-eq-sat-def*)

end

instantiation *sat* :: (*len*) {*Inf*, *Sup*}
begin

global-interpretation *Inf-sat*: *semilattice-neutr-set min* ‹*top* :: *'a sat*›
  defines *Inf-sat* = *Inf-sat.F*
  by *standard* (*simp add*: *min-def*)

global-interpretation *Sup-sat*: *semilattice-neutr-set max* ‹*bot* :: *'a sat*›
  defines *Sup-sat* = *Sup-sat.F*
  by *standard* (*simp add*: *max-def bot.extremum-unique*)

instance ..

end

instance *sat* :: (*len*) *complete-lattice*
proof
  fix *x* :: *'a sat*
  fix *A* :: *'a sat set*
  note *finite*
  moreover assume *x* ∈ *A*
  ultimately show *Inf A* ≤ *x*
    by (*induct A*) (*auto intro*: *min.coboundedI2*)
next
  fix *z* :: *'a sat*
  fix *A* :: *'a sat set*
  note *finite*
  moreover assume *z*: ⋀*x*. *x* ∈ *A* ⟹ *z* ≤ *x*
  ultimately show *z* ≤ *Inf A* **by** (*induct A*) *simp-all*
next
  fix *x* :: *'a sat*
  fix *A* :: *'a sat set*
  note *finite*

moreover assume $x \in A$
ultimately show $x \leq Sup\ A$
by (*induct A*) (*auto intro*: *max.coboundedI2*)
next
fix $z :: 'a\ sat$
fix $A :: 'a\ sat\ set$
note *finite*
moreover assume z: $\bigwedge x.\ x \in A \Longrightarrow x \leq z$
ultimately show $Sup\ A \leq z$ **by** (*induct A*) *auto*
next
show $Inf\ \{\} = (top::'a\ sat)$
by (*auto simp*: *top-sat-def*)
show $Sup\ \{\} = (bot::'a\ sat)$
by (*auto simp*: *bot-sat-def*)
qed

end

88 Set Idioms

theory *Set-Idioms*
imports *Countable-Set*

begin

88.1 Idioms for being a suitable union/intersection of something

definition *union-of* :: $('a\ set\ set \Rightarrow bool) \Rightarrow ('a\ set \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
(**infixr** *union'-of 60*)
where $P\ union\text{-}of\ Q \equiv \lambda S.\ \exists \mathcal{U}.\ P\ \mathcal{U} \wedge \mathcal{U} \subseteq Collect\ Q \wedge \bigcup \mathcal{U} = S$

definition *intersection-of* :: $('a\ set\ set \Rightarrow bool) \Rightarrow ('a\ set \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow bool$
(**infixr** *intersection'-of 60*)
where $P\ intersection\text{-}of\ Q \equiv \lambda S.\ \exists \mathcal{U}.\ P\ \mathcal{U} \wedge \mathcal{U} \subseteq Collect\ Q \wedge \bigcap \mathcal{U} = S$

definition *arbitrary*:: $'a\ set\ set \Rightarrow bool$ **where** $arbitrary\ \mathcal{U} \equiv True$

lemma *union-of-inc*: $[\![P\ \{S\};\ Q\ S]\!] \Longrightarrow (P\ union\text{-}of\ Q)\ S$
by (*auto simp*: *union-of-def*)

lemma *intersection-of-inc*:
$[\![P\ \{S\};\ Q\ S]\!] \Longrightarrow (P\ intersection\text{-}of\ Q)\ S$
by (*auto simp*: *intersection-of-def*)

lemma *union-of-mono*:
$[\![(P\ union\text{-}of\ Q)\ S;\ \bigwedge x.\ Q\ x \Longrightarrow Q'\ x]\!] \Longrightarrow (P\ union\text{-}of\ Q')\ S$
by (*auto simp*: *union-of-def*)

lemma *intersection-of-mono*:
$[\![(P \textit{ intersection-of } Q)\ S;\ \bigwedge x.\ Q\ x \Longrightarrow Q'\ x]\!] \Longrightarrow (P \textit{ intersection-of } Q')\ S$
by (*auto simp*: *intersection-of-def*)

lemma *all-union-of*:
$(\forall S.\ (P \textit{ union-of } Q)\ S \longrightarrow R\ S) \longleftrightarrow (\forall T.\ P\ T \wedge T \subseteq \textit{Collect } Q \longrightarrow R(\bigcup T))$
by (*auto simp*: *union-of-def*)

lemma *all-intersection-of*:
$(\forall S.\ (P \textit{ intersection-of } Q)\ S \longrightarrow R\ S) \longleftrightarrow (\forall T.\ P\ T \wedge T \subseteq \textit{Collect } Q \longrightarrow R(\bigcap T))$
by (*auto simp*: *intersection-of-def*)

lemma *intersection-ofE*:
$[\![(P \textit{ intersection-of } Q)\ S;\ \bigwedge T.\ [\![P\ T;\ T \subseteq \textit{Collect } Q]\!] \Longrightarrow R(\bigcap T)]\!] \Longrightarrow R\ S$
by (*auto simp*: *intersection-of-def*)

lemma *union-of-empty*:
$P\ \{\} \Longrightarrow (P \textit{ union-of } Q)\ \{\}$
by (*auto simp*: *union-of-def*)

lemma *intersection-of-empty*:
$P\ \{\} \Longrightarrow (P \textit{ intersection-of } Q)\ \textit{UNIV}$
by (*auto simp*: *intersection-of-def*)

The arbitrary and finite cases

lemma *arbitrary-union-of-alt*:
$(\textit{arbitrary union-of } Q)\ S \longleftrightarrow (\forall x \in S.\ \exists U.\ Q\ U \wedge x \in U \wedge U \subseteq S)$
(**is** *?lhs = ?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*force simp*: *union-of-def arbitrary-def*)
next
assume *?rhs*
then have $\{U.\ Q\ U \wedge U \subseteq S\} \subseteq \textit{Collect } Q \bigcup \{U.\ Q\ U \wedge U \subseteq S\} = S$
by *auto*
then show *?lhs*
unfolding *union-of-def arbitrary-def* **by** *blast*
qed

lemma *arbitrary-union-of-empty* [*simp*]: (*arbitrary union-of P*) {}
by (*force simp*: *union-of-def arbitrary-def*)

lemma *arbitrary-intersection-of-empty* [*simp*]:
(*arbitrary intersection-of P*) *UNIV*
by (*force simp*: *intersection-of-def arbitrary-def*)

lemma *arbitrary-union-of-inc*:

P S $\Longrightarrow$ (*arbitrary union-of P*) *S*
by (*force simp*: *union-of-inc arbitrary-def*)

lemma *arbitrary-intersection-of-inc*:
P S $\Longrightarrow$ (*arbitrary intersection-of P*) *S*
by (*force simp*: *intersection-of-inc arbitrary-def*)

lemma *arbitrary-union-of-complement*:
(*arbitrary union-of P*) *S* $\longleftrightarrow$ (*arbitrary intersection-of* ($\lambda S.$ *P*(− *S*))) (− *S*)
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then obtain $\mathcal{U}$ **where** $\mathcal{U} \subseteq$ *Collect P S* = $\bigcup \mathcal{U}$
by (*auto simp*: *union-of-def arbitrary-def*)
then show *?rhs*
unfolding *intersection-of-def arbitrary-def*
by (*rule-tac x*=*uminus* ' $\mathcal{U}$ **in** *exI*) *auto*
next
assume *?rhs*
then obtain $\mathcal{U}$ **where** $\mathcal{U} \subseteq \{S.\ P\ (-\ S)\}$ $\bigcap \mathcal{U}$ = − *S*
by (*auto simp*: *union-of-def intersection-of-def arbitrary-def*)
then show *?lhs*
unfolding *union-of-def arbitrary-def*
by (*rule-tac x*=*uminus* ' $\mathcal{U}$ **in** *exI*) *auto*
qed

lemma *arbitrary-intersection-of-complement*:
(*arbitrary intersection-of P*) *S* $\longleftrightarrow$ (*arbitrary union-of* ($\lambda S.$ *P*(− *S*))) (− *S*)
by (*simp add*: *arbitrary-union-of-complement*)

lemma *arbitrary-union-of-idempot* [*simp*]:
arbitrary union-of arbitrary union-of P = *arbitrary union-of P*
proof −
have *1*: $\exists \mathcal{U}' \subseteq$ *Collect P*. $\bigcup \mathcal{U}' = \bigcup \mathcal{U}$ **if** $\mathcal{U} \subseteq \{S.\ \exists \mathcal{V} \subseteq$ *Collect P*. $\bigcup \mathcal{V} = S\}$ **for** $\mathcal{U}$
proof −
let *?*$\mathcal{W}$ = $\{V.\ \exists \mathcal{V}.\ \mathcal{V} \subseteq$ *Collect P* $\wedge$ $V \in \mathcal{V}$ $\wedge$ $(\exists S \in \mathcal{U}.\ \bigcup \mathcal{V} = S)\}$
have $*$: $\bigwedge x\ U.\ [\![x \in U;\ U \in \mathcal{U}]\!] \Longrightarrow x \in \bigcup$ *?*$\mathcal{W}$
using *that*
apply *simp*
apply (*drule subsetD*, *assumption*, *auto*)
done
show *?thesis*
apply (*rule-tac x*=$\{V.\ \exists \mathcal{V}.\ \mathcal{V} \subseteq$ *Collect P* $\wedge$ $V \in \mathcal{V}$ $\wedge$ $(\exists S \in \mathcal{U}.\ \bigcup \mathcal{V} = S)\}$ **in** *exI*)
using *that* **by** (*blast intro*: $*$)
qed
have *2*: $\exists \mathcal{U}' \subseteq \{S.\ \exists \mathcal{U} \subseteq$ *Collect P*. $\bigcup \mathcal{U} = S\}$. $\bigcup \mathcal{U}' = \bigcup \mathcal{U}$ **if** $\mathcal{U} \subseteq$ *Collect P* **for** $\mathcal{U}$

by (*metis* (*mono-tags*, *lifting*) *union-of-def arbitrary-union-of-inc that*)
show *?thesis*
unfolding *union-of-def arbitrary-def* **by** (*force simp*: *1 2*)
qed

lemma *arbitrary-intersection-of-idempot*:
arbitrary intersection-of arbitrary intersection-of P = *arbitrary intersection-of P*
(**is** *?lhs* = *?rhs*)
proof −
have − *?lhs* = − *?rhs*
unfolding *arbitrary-intersection-of-complement* **by** *simp*
then show *?thesis*
by *simp*
qed

lemma *arbitrary-union-of-Union*:
(⋀*S*. *S* ∈ *𝒰* ⟹ (*arbitrary union-of P*) *S*) ⟹ (*arbitrary union-of P*) (⋃*𝒰*)
by (*metis union-of-def arbitrary-def arbitrary-union-of-idempot mem-Collect-eq subsetI*)

lemma *arbitrary-union-of-Un*:
⟦(*arbitrary union-of P*) *S*; (*arbitrary union-of P*) *T*⟧
⟹ (*arbitrary union-of P*) (*S* ∪ *T*)
using *arbitrary-union-of-Union* [*of* {*S*, *T*}] **by** *auto*

lemma *arbitrary-intersection-of-Inter*:
(⋀*S*. *S* ∈ *𝒰* ⟹ (*arbitrary intersection-of P*) *S*) ⟹ (*arbitrary intersection-of P*) (⋂*𝒰*)
by (*metis intersection-of-def arbitrary-def arbitrary-intersection-of-idempot mem-Collect-eq subsetI*)

lemma *arbitrary-intersection-of-Int*:
⟦(*arbitrary intersection-of P*) *S*; (*arbitrary intersection-of P*) *T*⟧
⟹ (*arbitrary intersection-of P*) (*S* ∩ *T*)
using *arbitrary-intersection-of-Inter* [*of* {*S*, *T*}] **by** *auto*

lemma *arbitrary-union-of-Int-eq*:
(∀ *S T*. (*arbitrary union-of P*) *S* ∧ (*arbitrary union-of P*) *T*
⟶ (*arbitrary union-of P*) (*S* ∩ *T*))
⟷ (∀ *S T*. *P S* ∧ *P T* ⟶ (*arbitrary union-of P*) (*S* ∩ *T*)) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *arbitrary-union-of-inc*)
next
assume *R*: *?rhs*
show *?lhs*
proof *clarify*
fix *S* :: *'a set* **and** *T* :: *'a set*

```
    assume (arbitrary union-of P) S and (arbitrary union-of P) T
    then obtain 𝒰 𝒱 where *: 𝒰 ⊆ Collect P ⋃𝒰 = S 𝒱 ⊆ Collect P ⋃𝒱 = T
      by (auto simp: union-of-def)
    then have (arbitrary union-of P) (⋃ C∈𝒰. ⋃ D∈𝒱. C ∩ D)
     using R by (blast intro: arbitrary-union-of-Union)
   then show (arbitrary union-of P) (S ∩ T)
     by (simp add: Int-UN-distrib2 *)
 qed
qed

lemma arbitrary-intersection-of-Un-eq:
   (∀ S T. (arbitrary intersection-of P) S ∧ (arbitrary intersection-of P) T
              ⟶ (arbitrary intersection-of P) (S ∪ T)) ⟷
       (∀ S T. P S ∧ P T ⟶ (arbitrary intersection-of P) (S ∪ T))
  apply (simp add: arbitrary-intersection-of-complement)
  using arbitrary-union-of-Int-eq [of λS. P (− S)]
  by (metis (no-types, lifting) arbitrary-def double-compl union-of-inc)

lemma finite-union-of-empty [simp]: (finite union-of P) {}
  by (simp add: union-of-empty)

lemma finite-intersection-of-empty [simp]: (finite intersection-of P) UNIV
  by (simp add: intersection-of-empty)

lemma finite-union-of-inc:
   P S ⟹ (finite union-of P) S
  by (simp add: union-of-inc)

lemma finite-intersection-of-inc:
   P S ⟹ (finite intersection-of P) S
  by (simp add: intersection-of-inc)

lemma finite-union-of-complement:
  (finite union-of P) S ⟷ (finite intersection-of (λS. P(− S))) (− S)
  unfolding union-of-def intersection-of-def
  apply safe
   apply (rule-tac x=uminus ` 𝒰 in exI, fastforce)+
  done

lemma finite-intersection-of-complement:
   (finite intersection-of P) S ⟷ (finite union-of (λS. P(− S))) (− S)
  by (simp add: finite-union-of-complement)

lemma finite-union-of-idempot [simp]:
  finite union-of finite union-of P = finite union-of P
proof −
  have (finite union-of P) S if S: (finite union-of finite union-of P) S for S
  proof −
   obtain 𝒰 where finite 𝒰 S = ⋃𝒰 and 𝒰: ∀ U∈𝒰. ∃𝒰. finite 𝒰 ∧ (𝒰 ⊆ Collect
```

$P) \wedge \bigcup \mathcal{U} = U$
using S **unfolding** *union-of-def* **by** (*auto simp*: *subset-eq*)
then obtain f **where** $\forall\, U{\in}\mathcal{U}.$ *finite* $(f\ U) \wedge (f\ U \subseteq$ *Collect* $P) \wedge \bigcup (f\ U) = U$
by *metis*
then show *?thesis*
unfolding *union-of-def* ‹$S = \bigcup \mathcal{U}$›
by (*rule-tac* x = *snd* ' *Sigma* $\mathcal{U}\ f$ **in** *exI*) (*fastforce simp*: ‹*finite* $\mathcal{U}$›)
qed
moreover
have (*finite union-of finite union-of* P) S **if** (*finite union-of* P) S **for** S
by (*simp add*: *finite-union-of-inc that*)
ultimately show *?thesis*
by *force*
qed

lemma *finite-intersection-of-idempot* [*simp*]:
finite intersection-of finite intersection-of P = *finite intersection-of* P
by (*force simp*: *finite-intersection-of-complement*)

lemma *finite-union-of-Union*:
⟦*finite* $\mathcal{U}$; $\bigwedge S.\ S \in \mathcal{U} \Longrightarrow$ (*finite union-of* P) S⟧ $\Longrightarrow$ (*finite union-of* P) $(\bigcup \mathcal{U})$
using *finite-union-of-idempot* [*of* P]
by (*metis mem-Collect-eq subsetI union-of-def*)

lemma *finite-union-of-Un*:
⟦(*finite union-of* P) S; (*finite union-of* P) T⟧ $\Longrightarrow$ (*finite union-of* P) $(S \cup T)$
by (*auto simp*: *union-of-def*)

lemma *finite-intersection-of-Inter*:
⟦*finite* $\mathcal{U}$; $\bigwedge S.\ S \in \mathcal{U} \Longrightarrow$ (*finite intersection-of* P) S⟧ $\Longrightarrow$ (*finite intersection-of* P) $(\bigcap \mathcal{U})$
using *finite-intersection-of-idempot* [*of* P]
by (*metis intersection-of-def mem-Collect-eq subsetI*)

lemma *finite-intersection-of-Int*:
⟦(*finite intersection-of* P) S; (*finite intersection-of* P) T⟧
$\Longrightarrow$ (*finite intersection-of* P) $(S \cap T)$
by (*auto simp*: *intersection-of-def*)

lemma *finite-union-of-Int-eq*:
$(\forall\, S\ T.$ (*finite union-of* P) $S \wedge$ (*finite union-of* P) $T \longrightarrow$ (*finite union-of* P) $(S \cap T))$
$\longleftrightarrow (\forall\, S\ T.\ P\ S \wedge P\ T \longrightarrow$ (*finite union-of* P) $(S \cap T))$
(**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
then show *?rhs*
by (*simp add*: *finite-union-of-inc*)
next

assume *R*: *?rhs*
show *?lhs*
proof *clarify*
fix S :: *'a set* **and** T :: *'a set*
assume (*finite union-of P*) S **and** (*finite union-of P*) T
then obtain $\mathcal{U}$ $\mathcal{V}$ **where** $*$: $\mathcal{U} \subseteq$ *Collect P* $\bigcup\mathcal{U} = S$ *finite* $\mathcal{U}$ $\mathcal{V} \subseteq$ *Collect P* $\bigcup\mathcal{V} = T$ *finite* $\mathcal{V}$
by (*auto simp*: *union-of-def*)
then have (*finite union-of P*) $(\bigcup C{\in}\mathcal{U}.\ \bigcup D{\in}\mathcal{V}.\ C \cap D)$
using *R*
by (*blast intro*: *finite-union-of-Union*)
then show (*finite union-of P*) $(S \cap T)$
by (*simp add*: *Int-UN-distrib2* $*$)
qed
qed

lemma *finite-intersection-of-Un-eq*:
$(\forall S\ T.$ (*finite intersection-of P*) $S \wedge$
(*finite intersection-of P*) T
$\longrightarrow$ (*finite intersection-of P*) $(S \cup T)) \longleftrightarrow$
$(\forall S\ T.\ P\ S \wedge P\ T \longrightarrow$ (*finite intersection-of P*) $(S \cup T))$
apply (*simp add*: *finite-intersection-of-complement*)
using *finite-union-of-Int-eq* [*of* $\lambda S.\ P\ (-\ S)$]
by (*metis* (*no-types*, *lifting*) *double-compl*)

abbreviation *finite'* :: *'a set* $\Rightarrow$ *bool*
where *finite' A* $\equiv$ *finite A* $\wedge$ $A \neq \{\}$

lemma *finite'-intersection-of-Int*:
$[\![$(*finite' intersection-of P*) S; (*finite' intersection-of P*) $T]\!]$
$\Longrightarrow$ (*finite' intersection-of P*) $(S \cap T)$
by (*auto simp*: *intersection-of-def*)

lemma *finite'-intersection-of-inc*:
P S $\Longrightarrow$ (*finite' intersection-of P*) S
by (*simp add*: *intersection-of-inc*)

88.2 The "Relative to" operator

A somewhat cheap but handy way of getting localized forms of various topological concepts (open, closed, borel, fsigma, gdelta etc.)

definition *relative-to* :: [*'a set* $\Rightarrow$ *bool*, *'a set*, *'a set*] $\Rightarrow$ *bool* (**infixl** *relative'-to* *55*)
where *P relative-to S* $\equiv \lambda T.\ \exists U.\ P\ U \wedge S \cap U = T$

lemma *relative-to-UNIV* [*simp*]: (*P relative-to UNIV*) $S \longleftrightarrow P\ S$
by (*simp add*: *relative-to-def*)

lemma *relative-to-imp-subset*:
(*P relative-to S*) *T* ⟹ *T* ⊆ *S*
by (*auto simp*: *relative-to-def*)

lemma *all-relative-to*: (∀ *S*. (*P relative-to U*) *S* ⟶ *Q S*) ⟷ (∀ *S*. *P S* ⟶ *Q*(*U* ∩ *S*))
by (*auto simp*: *relative-to-def*)

lemma *relative-toE*: ⟦(*P relative-to U*) *S*; ⋀*S*. *P S* ⟹ *Q*(*U* ∩ *S*)⟧ ⟹ *Q S*
by (*auto simp*: *relative-to-def*)

lemma *relative-to-inc*:
P S ⟹ (*P relative-to U*) (*U* ∩ *S*)
by (*auto simp*: *relative-to-def*)

lemma *relative-to-relative-to* [*simp*]:
P relative-to S relative-to T = *P relative-to* (*S* ∩ *T*)
unfolding *relative-to-def*
by *auto*

lemma *relative-to-compl*:
S ⊆ *U* ⟹ ((*P relative-to U*) (*U* − *S*) ⟷ ((λ*c*. *P*(− *c*)) *relative-to U*) *S*)
unfolding *relative-to-def*
by (*metis Diff-Diff-Int Diff-eq double-compl inf*.*absorb-iff2*)

lemma *relative-to-subset*:
S ⊆ *T* ∧ *P S* ⟹ (*P relative-to T*) *S*
unfolding *relative-to-def* **by** *auto*

lemma *relative-to-subset-trans*:
(*P relative-to U*) *S* ∧ *S* ⊆ *T* ∧ *T* ⊆ *U* ⟹ (*P relative-to T*) *S*
unfolding *relative-to-def* **by** *auto*

lemma *relative-to-mono*:
⟦(*P relative-to U*) *S*; ⋀*S*. *P S* ⟹ *Q S*⟧ ⟹ (*Q relative-to U*) *S*
unfolding *relative-to-def* **by** *auto*

lemma *relative-to-subset-inc*: ⟦*S* ⊆ *U*; *P S*⟧ ⟹ (*P relative-to U*) *S*
unfolding *relative-to-def* **by** *auto*

lemma *relative-to-Int*:
⟦(*P relative-to S*) *C*; (*P relative-to S*) *D*; ⋀*X Y*. ⟦*P X*; *P Y*⟧ ⟹ *P*(*X* ∩ *Y*)⟧
⟹ (*P relative-to S*) (*C* ∩ *D*)
unfolding *relative-to-def* **by** *auto*

lemma *relative-to-Un*:
⟦(*P relative-to S*) *C*; (*P relative-to S*) *D*; ⋀*X Y*. ⟦*P X*; *P Y*⟧ ⟹ *P*(*X* ∪ *Y*)⟧
⟹ (*P relative-to S*) (*C* ∪ *D*)
unfolding *relative-to-def* **by** *auto*

lemma *arbitrary-union-of-relative-to*:
(((*arbitrary union-of* *P*) *relative-to* *U*) = (*arbitrary union-of* (*P relative-to* *U*)))
(**is** *?lhs* = *?rhs*)
proof −
have *?rhs S* **if** *L*: *?lhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $S = U \cap \bigcup \mathcal{U}\ \mathcal{U} \subseteq$ *Collect P*
using *L* **unfolding** *relative-to-def union-of-def* **by** *auto*
then show *?thesis*
unfolding *relative-to-def union-of-def arbitrary-def*
by (*rule-tac x*=($\lambda X.\ U \cap X$) ‘ $\mathcal{U}$ **in** *exI*) *auto*
qed
moreover have *?lhs S* **if** *R*: *?rhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $S = \bigcup \mathcal{U} \ \forall\, T{\in}\mathcal{U}.\ \exists\, V.\ P\ V \wedge U \cap V = T$
using *R* **unfolding** *relative-to-def union-of-def* **by** *auto*
then obtain *f* **where** *f*: $\bigwedge T.\ T \in \mathcal{U} \Longrightarrow P\ (f\ T)\ \bigwedge T.\ T \in \mathcal{U} \Longrightarrow U \cap (f\ T) = T$
by *metis*
then have $\exists \mathcal{U}' {\subseteq} \textit{Collect } P.\ \bigcup \mathcal{U}' = \bigcup\ (f\ ‘\ \mathcal{U})$
by (*metis image-subset-iff mem-Collect-eq*)
moreover have *eq*: $U \cap \bigcup\ (f\ ‘\ \mathcal{U}) = \bigcup \mathcal{U}$
using *f* **by** *auto*
ultimately show *?thesis*
unfolding *relative-to-def union-of-def arbitrary-def* ‹$S = \bigcup \mathcal{U}$›
by *metis*
qed
ultimately show *?thesis*
by *blast*
qed

lemma *finite-union-of-relative-to*:
(((*finite union-of* *P*) *relative-to* *U*) = (*finite union-of* (*P relative-to* *U*))) (**is** *?lhs* = *?rhs*)
proof −
have *?rhs S* **if** *L*: *?lhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $S = U \cap \bigcup \mathcal{U}\ \mathcal{U} \subseteq$ *Collect P finite* $\mathcal{U}$
using *L* **unfolding** *relative-to-def union-of-def* **by** *auto*
then show *?thesis*
unfolding *relative-to-def union-of-def*
by (*rule-tac x*=($\lambda X.\ U \cap X$) ‘ $\mathcal{U}$ **in** *exI*) *auto*
qed
moreover have *?lhs S* **if** *R*: *?rhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $S = \bigcup \mathcal{U} \ \forall\, T{\in}\mathcal{U}.\ \exists\, V.\ P\ V \wedge U \cap V = T$ *finite* $\mathcal{U}$
using *R* **unfolding** *relative-to-def union-of-def* **by** *auto*
then obtain *f* **where** *f*: $\bigwedge T.\ T \in \mathcal{U} \Longrightarrow P\ (f\ T)\ \bigwedge T.\ T \in \mathcal{U} \Longrightarrow U \cap (f\ T)$

```
= T
      by metis
    then have ∃U'⊆Collect P. ⋃U' = ⋃ (f ` U)
      by (metis image-subset-iff mem-Collect-eq)
    moreover have eq: U ∩ ⋃ (f ` U) = ⋃U
      using f by auto
    ultimately show ?thesis
      using ‹finite U› f
      unfolding relative-to-def union-of-def ‹S = ⋃U›
    by (rule-tac x=⋃ (f ` U) in exI) (metis finite-imageI image-subsetI mem-Collect-eq)
  qed
  ultimately show ?thesis
    by blast
qed

lemma countable-union-of-relative-to:
  ((countable union-of P) relative-to U) = (countable union-of (P relative-to U))
(is ?lhs = ?rhs)
proof −
  have ?rhs S if L: ?lhs S for S
  proof −
    obtain U where S = U ∩ ⋃U U ⊆ Collect P countable U
      using L unfolding relative-to-def union-of-def by auto
    then show ?thesis
      unfolding relative-to-def union-of-def
      by (rule-tac x=(λX. U ∩ X) ` U in exI) auto
  qed
  moreover have ?lhs S if R: ?rhs S for S
  proof −
    obtain U where S = ⋃U ∀ T∈U. ∃ V. P V ∧ U ∩ V = T countable U
      using R unfolding relative-to-def union-of-def by auto
    then obtain f where f: ⋀T. T ∈ U ⟹ P (f T) ⋀T. T ∈ U ⟹ U ∩ (f T)
= T
      by metis
    then have ∃U'⊆Collect P. ⋃U' = ⋃ (f ` U)
      by (metis image-subset-iff mem-Collect-eq)
    moreover have eq: U ∩ ⋃ (f ` U) = ⋃U
      using f by auto
    ultimately show ?thesis
      using ‹countable U› f
      unfolding relative-to-def union-of-def ‹S = ⋃U›
        by (rule-tac x=⋃ (f ` U) in exI) (metis countable-image image-subsetI
mem-Collect-eq)
  qed
  ultimately show ?thesis
    by blast
qed
```

lemma *arbitrary-intersection-of-relative-to*:
(((*arbitrary intersection-of* *P*) *relative-to* *U*) = ((*arbitrary intersection-of* (*P relative-to* *U*)) *relative-to* *U*) (**is** *?lhs* = *?rhs*)
proof −
have *?rhs S* **if** *L*: *?lhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $\mathcal{U}$: $S = U \cap \bigcap \mathcal{U}$ $\mathcal{U} \subseteq$ *Collect P*
using *L* **unfolding** *relative-to-def intersection-of-def* **by** *auto*
show *?thesis*
unfolding *relative-to-def intersection-of-def arbitrary-def*
proof (*intro exI conjI*)
show $U \cap (\bigcap X{\in}\mathcal{U}.\ U \cap X) = S$ (∩) U ‘ $\mathcal{U} \subseteq \{T.\ \exists Ua.\ P\ Ua \wedge U \cap Ua = T\}$
using $\mathcal{U}$ **by** *blast+*
qed *auto*
qed
moreover have *?lhs S* **if** *R*: *?rhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $S = U \cap \bigcap \mathcal{U}$ $\forall T{\in}\mathcal{U}.\ \exists V.\ P\ V \wedge U \cap V = T$
using *R* **unfolding** *relative-to-def intersection-of-def* **by** *auto*
then obtain *f* **where** *f*: $\bigwedge T.\ T \in \mathcal{U} \Longrightarrow P\ (f\ T)$ $\bigwedge T.\ T \in \mathcal{U} \Longrightarrow U \cap (f\ T) = T$
by *metis*
then have *f* ‘ $\mathcal{U} \subseteq$ *Collect P*
by *auto*
moreover have *eq*: $U \cap \bigcap (f\ ‘\ \mathcal{U}) = U \cap \bigcap \mathcal{U}$
using *f* **by** *auto*
ultimately show *?thesis*
unfolding *relative-to-def intersection-of-def arbitrary-def* ‹$S = U \cap \bigcap \mathcal{U}$›
by *auto*
qed
ultimately show *?thesis*
by *blast*
qed

lemma *finite-intersection-of-relative-to*:
(((*finite intersection-of* *P*) *relative-to* *U*) = ((*finite intersection-of* (*P relative-to* *U*)) *relative-to* *U*) (**is** *?lhs* = *?rhs*)
proof −
have *?rhs S* **if** *L*: *?lhs S* **for** *S*
proof −
obtain $\mathcal{U}$ **where** $\mathcal{U}$: $S = U \cap \bigcap \mathcal{U}$ $\mathcal{U} \subseteq$ *Collect P finite* $\mathcal{U}$
using *L* **unfolding** *relative-to-def intersection-of-def* **by** *auto*
show *?thesis*
unfolding *relative-to-def intersection-of-def*
proof (*intro exI conjI*)
show $U \cap (\bigcap X{\in}\mathcal{U}.\ U \cap X) = S$ (∩) U ‘ $\mathcal{U} \subseteq \{T.\ \exists Ua.\ P\ Ua \wedge U \cap Ua = T\}$
using $\mathcal{U}$ **by** *blast+*

```
    show finite ((∩) U ` 𝒰)
      by (simp add: ‹finite 𝒰›)
  qed auto
 qed
 moreover have ?lhs S if R: ?rhs S for S
 proof −
  obtain 𝒰 where S = U ∩ ⋂𝒰 ∀ T∈𝒰. ∃ V. P V ∧ U ∩ V = T finite 𝒰
    using R unfolding relative-to-def intersection-of-def by auto
  then obtain f where f: ⋀T. T ∈ 𝒰 ⟹ P (f T) ⋀T. T ∈ 𝒰 ⟹ U ∩ (f T)
= T
    by metis
  then have f ` 𝒰 ⊆ Collect P
    by auto
  moreover have eq: U ∩ ⋂ (f ` 𝒰) = U ∩ ⋂ 𝒰
    using f by auto
  ultimately show ?thesis
    unfolding relative-to-def intersection-of-def ‹S = U ∩ ⋂𝒰›
    using ‹finite 𝒰›
    by auto
 qed
 ultimately show ?thesis
  by blast
qed

lemma countable-intersection-of-relative-to:
  ((countable intersection-of P) relative-to U) = ((countable intersection-of (P
relative-to U)) relative-to U) (is ?lhs = ?rhs)
proof −
 have ?rhs S if L: ?lhs S for S
 proof −
  obtain 𝒰 where 𝒰: S = U ∩ ⋂𝒰 𝒰 ⊆ Collect P countable 𝒰
    using L unfolding relative-to-def intersection-of-def by auto
  show ?thesis
    unfolding relative-to-def intersection-of-def
  proof (intro exI conjI)
    show U ∩ (⋂X∈𝒰. U ∩ X) = S (∩) U ` 𝒰 ⊆ {T. ∃ Ua. P Ua ∧ U ∩ Ua =
T}
      using 𝒰 by blast+
    show countable ((∩) U ` 𝒰)
      by (simp add: ‹countable 𝒰›)
  qed auto
 qed
 moreover have ?lhs S if R: ?rhs S for S
 proof −
  obtain 𝒰 where S = U ∩ ⋂𝒰 ∀ T∈𝒰. ∃ V. P V ∧ U ∩ V = T countable 𝒰
    using R unfolding relative-to-def intersection-of-def by auto
  then obtain f where f: ⋀T. T ∈ 𝒰 ⟹ P (f T) ⋀T. T ∈ 𝒰 ⟹ U ∩ (f T)
= T
    by metis
```

```
    then have f ‘ 𝒰 ⊆ Collect P
      by auto
    moreover have eq: U ∩ ⋂ (f ‘ 𝒰) = U ∩ ⋂ 𝒰
      using f by auto
    ultimately show ?thesis
      unfolding relative-to-def intersection-of-def ‹S = U ∩ ⋂𝒰›
      using ‹countable 𝒰› countable-image
      by auto
  qed
  ultimately show ?thesis
    by blast
qed

end
```

88.3 Signed division: negative results rounded towards zero rather than minus infinity.

```
theory Signed-Division
  imports Main
begin

class signed-division =
  fixes signed-divide :: ‹'a ⇒ 'a ⇒ 'a› (infixl sdiv 70)
  and signed-modulo :: ‹'a ⇒ 'a ⇒ 'a› (infixl smod 70)

instantiation int :: signed-division
begin

definition signed-divide-int :: ‹int ⇒ int ⇒ int›
  where ‹k sdiv l = sgn k * sgn l * (|k| div |l|)› for k l :: int

definition signed-modulo-int :: ‹int ⇒ int ⇒ int›
  where ‹k smod l = k − (k sdiv l) * l› for k l :: int

instance ..

end

lemma int-sdiv-simps [simp]:
    (a :: int) sdiv 1 = a
    (a :: int) sdiv 0 = 0
    (a :: int) sdiv −1 = −a
  apply (auto simp: signed-divide-int-def sgn-if)
  done

lemma sgn-div-eq-sgn-mult:
    a div b ≠ 0 ⟹ sgn ((a :: int) div b) = sgn (a * b)
  apply (clarsimp simp: sgn-if zero-le-mult-iff neg-imp-zdiv-nonneg-iff not-less)
```

apply (*metis less-le mult-le-0-iff neg-imp-zdiv-neg-iff not-less pos-imp-zdiv-neg-iff zdiv-eq-0-iff*)
done

lemma *sgn-sdiv-eq-sgn-mult*:
*a sdiv b ≠ 0 ⟹ sgn ((a :: int) sdiv b) = sgn (a * b)*
by (*auto simp*: *signed-divide-int-def sgn-div-eq-sgn-mult sgn-mult*)

lemma *int-sdiv-same-is-1* [*simp*]:
a ≠ 0 ⟹ ((a :: int) sdiv b = a) = (b = 1)
apply (*rule iffI*)
apply (*clarsimp simp*: *signed-divide-int-def*)
apply (*subgoal-tac b > 0*)
apply (*case-tac a > 0*)
apply (*clarsimp simp*: *sgn-if*)
apply (*simp-all add*: *not-less algebra-split-simps sgn-if split*: *if-splits*)
using *int-div-less-self* [*of a b*] **apply** *linarith*
apply (*metis add.commute add.inverse-inverse group-cancel.rule0 int-div-less-self linorder-neqE-linordered-idom neg-0-le-iff-le not-less verit-comp-simplify1*(*1*) *zless-imp-add1-zle*)
apply (*metis div-minus-right neg-imp-zdiv-neg-iff neg-le-0-iff-le not-less order.not-eq-order-implies-strict*)
apply (*metis abs-le-zero-iff abs-of-nonneg neg-imp-zdiv-nonneg-iff order.not-eq-order-implies-strict*)
done

lemma *int-sdiv-negated-is-minus1* [*simp*]:
a ≠ 0 ⟹ ((a :: int) sdiv b = − a) = (b = −1)
apply (*clarsimp simp*: *signed-divide-int-def*)
apply (*rule iffI*)
apply (*subgoal-tac b < 0*)
apply (*case-tac a > 0*)
apply (*clarsimp simp*: *sgn-if algebra-split-simps not-less*)
apply (*case-tac sgn (a * b) = −1*)
apply (*simp-all add*: *not-less algebra-split-simps sgn-if split*: *if-splits*)
apply (*metis add.inverse-inverse int-div-less-self int-one-le-iff-zero-less less-le neg-0-less-iff-less*)
apply (*metis add.inverse-inverse div-minus-right int-div-less-self int-one-le-iff-zero-less less-le neg-0-less-iff-less*)
apply (*metis less-le neg-less-0-iff-less not-less pos-imp-zdiv-neg-iff*)
apply (*metis div-minus-right dual-order.eq-iff neg-imp-zdiv-nonneg-iff neg-less-0-iff-less*)
done

lemma *sdiv-int-range*:
(a :: int) sdiv b ∈ { − (abs a) .. (abs a) }
apply (*unfold signed-divide-int-def*)
apply (*subgoal-tac (abs a) div (abs b) ≤ (abs a)*)
apply (*auto simp add*: *sgn-if not-less*)
apply (*metis le-less le-less-trans neg-equal-0-iff-equal neg-less-iff-less not-le pos-imp-zdiv-neg-iff*)
apply (*metis add.inverse-neutral div-int-pos-iff le-less neg-le-iff-le order-trans*)
apply (*metis div-minus-right le-less-trans neg-imp-zdiv-neg-iff neg-less-0-iff-less*

not-le)
using *div-int-pos-iff* **apply** *fastforce*
apply (*auto simp add*: *abs-if not-less*)
apply (*metis add.inverse-inverse add-0-left div-by-1 div-minus-right less-le neg-0-le-iff-le not-le not-one-le-zero zdiv-mono2 zless-imp-add1-zle*)
apply (*metis div-by-1 neg-0-less-iff-less pos-imp-zdiv-pos-iff zdiv-mono2 zero-less-one*)
apply (*metis add.inverse-neutral div-by-0 div-by-1 int-div-less-self int-one-le-iff-zero-less less-le less-minus-iff order-refl*)
apply (*metis div-by-1 divide-int-def int-div-less-self less-le linorder-neqE-linordered-idom order-refl unique-euclidean-semiring-numeral-class.div-less*)
done

lemma *sdiv-int-div-0* [*simp*]:
(*x* :: *int*) *sdiv* 0 = 0
by (*clarsimp simp*: *signed-divide-int-def*)

lemma *sdiv-int-0-div* [*simp*]:
0 *sdiv* (*x* :: *int*) = 0
by (*clarsimp simp*: *signed-divide-int-def*)

lemma *smod-int-alt-def*:
(*a*::*int*) *smod b* = *sgn* (*a*) ∗ (*abs a mod abs b*)
apply (*clarsimp simp*: *signed-modulo-int-def signed-divide-int-def*)
apply (*clarsimp simp*: *minus-div-mult-eq-mod* [*symmetric*] *abs-sgn sgn-mult sgn-if algebra-split-simps*)
done

lemma *smod-int-range*:
b ≠ 0 ⟹ (*a*::*int*) *smod b* ∈ { − *abs b* + 1 .. *abs b* − 1 }
apply (*case-tac* *b* > 0)
apply (*insert pos-mod-conj* [**where** *a*=*a* **and** *b*=*b*])[*1*]
apply (*insert pos-mod-conj* [**where** *a*=−*a* **and** *b*=*b*])[*1*]
apply (*auto simp*: *smod-int-alt-def algebra-simps sgn-if abs-if not-less add1-zle-eq* [*simplified add.commute*])[*1*]
apply (*metis add-nonneg-nonneg int-one-le-iff-zero-less le-less less-add-same-cancel2 not-le pos-mod-conj*)
apply (*metis* (*full-types*) *add.inverse-inverse eucl-rel-int eucl-rel-int-iff le-less-trans neg-0-le-iff-le*)
apply (*insert neg-mod-conj* [**where** *a*=*a* **and** *b*=*b*])[*1*]
apply (*insert neg-mod-conj* [**where** *a*=−*a* **and** *b*=*b*])[*1*]
apply (*clarsimp simp*: *smod-int-alt-def algebra-simps sgn-if abs-if not-less add1-zle-eq* [*simplified add.commute*])
apply (*metis neg-0-less-iff-less neg-mod-conj not-le not-less-iff-gr-or-eq order-trans pos-mod-conj*)
done

lemma *smod-int-compares*:
⟦ 0 ≤ *a*; 0 < *b* ⟧ ⟹ (*a* :: *int*) *smod b* < *b*
⟦ 0 ≤ *a*; 0 < *b* ⟧ ⟹ 0 ≤ (*a* :: *int*) *smod b*

$[\![\ a \leq 0;\ 0 < b\ ]\!] \Longrightarrow -b < (a :: int)\ smod\ b$
$[\![\ a \leq 0;\ 0 < b\ ]\!] \Longrightarrow (a :: int)\ smod\ b \leq 0$
$[\![\ 0 \leq a;\ b < 0\ ]\!] \Longrightarrow (a :: int)\ smod\ b < -\ b$
$[\![\ 0 \leq a;\ b < 0\ ]\!] \Longrightarrow 0 \leq (a :: int)\ smod\ b$
$[\![\ a \leq 0;\ b < 0\ ]\!] \Longrightarrow (a :: int)\ smod\ b \leq 0$
$[\![\ a \leq 0;\ b < 0\ ]\!] \Longrightarrow b \leq (a :: int)\ smod\ b$
apply (*insert smod-int-range* [**where** *a*=*a* **and** *b*=*b*])
apply (*auto simp*: *add1-zle-eq smod-int-alt-def sgn-if*)
done

lemma *smod-int-mod-0* [*simp*]:
$x\ smod\ (0 :: int) = x$
by (*clarsimp simp*: *signed-modulo-int-def*)

lemma *smod-int-0-mod* [*simp*]:
$0\ smod\ (x :: int) = 0$
by (*clarsimp simp*: *smod-int-alt-def*)

lemma *smod-mod-positive*:
$[\![\ 0 \leq (a :: int);\ 0 \leq b\ ]\!] \Longrightarrow a\ smod\ b = a\ mod\ b$
by (*clarsimp simp*: *smod-int-alt-def zsgn-def*)

lemma *minus-sdiv-eq* [*simp*]:
‹$-\ k\ sdiv\ l = -\ (k\ sdiv\ l)$› **for** $k\ l :: int$
by (*simp add*: *signed-divide-int-def*)

lemma *sdiv-minus-eq* [*simp*]:
‹$k\ sdiv\ -\ l = -\ (k\ sdiv\ l)$› **for** $k\ l :: int$
by (*simp add*: *signed-divide-int-def*)

lemma *sdiv-int-numeral-numeral* [*simp*]:
‹$numeral\ m\ sdiv\ numeral\ n = numeral\ m\ div\ (numeral\ n :: int)$›
by (*simp add*: *signed-divide-int-def*)

lemma *minus-smod-eq* [*simp*]:
‹$-\ k\ smod\ l = -\ (k\ smod\ l)$› **for** $k\ l :: int$
by (*simp add*: *smod-int-alt-def*)

lemma *smod-minus-eq* [*simp*]:
‹$k\ smod\ -\ l = k\ smod\ l$› **for** $k\ l :: int$
by (*simp add*: *smod-int-alt-def*)

lemma *smod-int-numeral-numeral* [*simp*]:
‹$numeral\ m\ smod\ numeral\ n = numeral\ m\ mod\ (numeral\ n :: int)$›
by (*simp add*: *smod-int-alt-def*)

end

89 State monad

theory *State-Monad*
imports *Monad-Syntax*
begin

datatype ('*s*, '*a*) *state* = *State* (*run-state*: '*s* ⇒ ('*a* × '*s*))

lemma *set-state-iff*: *x* ∈ *set-state m* ⟷ (∃ *s s'*. *run-state m s* = (*x*, *s'*))
by (*cases m*) (*simp add*: *prod-set-defs eq-fst-iff*)

lemma *pred-stateI*[*intro*]:
  assumes ⋀*a s s'*. *run-state m s* = (*a*, *s'*) ⟹ *P a*
  shows *pred-state P m*
proof (*subst state.pred-set*, *rule*)
  fix *x*
  assume *x* ∈ *set-state m*
  then obtain *s s'* **where** *run-state m s* = (*x*, *s'*)
    by (*auto simp*: *set-state-iff*)
  with *assms* **show** *P x* **.**
qed

lemma *pred-stateD*[*dest*]:
  assumes *pred-state P m run-state m s* = (*a*, *s'*)
  shows *P a*
proof (*rule state.exhaust*[*of m*])
  fix *f*
  assume *m* = *State f*
  with *assms* **have** *pred-fun* (λ-. *True*) (*pred-prod P top*) *f*
    by (*metis state.pred-inject*)
  moreover have *f s* = (*a*, *s'*)
    using *assms* **unfolding** ‹*m* = -› **by** *auto*
  ultimately show *P a*
    unfolding *pred-prod-beta pred-fun-def*
    by (*metis fst-conv*)
qed

lemma *pred-state-run-state*: *pred-state P m* ⟹ *P* (*fst* (*run-state m s*))
by (*meson pred-stateD prod.exhaust-sel*)

definition *state-io-rel* :: ('*s* ⇒ '*s* ⇒ *bool*) ⇒ ('*s*, '*a*) *state* ⇒ *bool* **where**
state-io-rel P m = (∀ *s*. *P s* (*snd* (*run-state m s*)))

lemma *state-io-relI*[*intro*]:
  assumes ⋀*a s s'*. *run-state m s* = (*a*, *s'*) ⟹ *P s s'*
  shows *state-io-rel P m*
using *assms* **unfolding** *state-io-rel-def*
by (*metis prod.collapse*)

lemma *state-io-relD*[*dest*]:
assumes *state-io-rel P m run-state m s* = (*a*, *s'*)
shows *P s s'*
using *assms* **unfolding** *state-io-rel-def*
by (*metis snd-conv*)

lemma *state-io-rel-mono*[*mono*]: *P* ≤ *Q* ⟹ *state-io-rel P* ≤ *state-io-rel Q*
by *blast*

lemma *state-ext*:
assumes ⋀*s*. *run-state m s* = *run-state n s*
shows *m* = *n*
using *assms*
by (*cases m*; *cases n*) *auto*

context begin

qualified definition *return* :: *'a* ⇒ (*'s*, *'a*) *state* **where**
return a = *State* (*Pair a*)

lemma *run-state-return*[*simp*]: *run-state* (*return x*) *s* = (*x*, *s*)
unfolding *return-def*
by *simp*

qualified definition *ap* :: (*'s*, *'a* ⇒ *'b*) *state* ⇒ (*'s*, *'a*) *state* ⇒ (*'s*, *'b*) *state* **where**
ap f x = *State* (λ*s*. *case run-state f s of* (*g*, *s'*) ⇒ *case run-state x s' of* (*y*, *s''*) ⇒ (*g y*, *s''*))

lemma *run-state-ap*[*simp*]:
run-state (*ap f x*) *s* = (*case run-state f s of* (*g*, *s'*) ⇒ *case run-state x s' of* (*y*, *s''*) ⇒ (*g y*, *s''*))
unfolding *ap-def* **by** *auto*

qualified definition *bind* :: (*'s*, *'a*) *state* ⇒ (*'a* ⇒ (*'s*, *'b*) *state*) ⇒ (*'s*, *'b*) *state* **where**
bind x f = *State* (λ*s*. *case run-state x s of* (*a*, *s'*) ⇒ *run-state* (*f a*) *s'*)

lemma *run-state-bind*[*simp*]:
run-state (*bind x f*) *s* = (*case run-state x s of* (*a*, *s'*) ⇒ *run-state* (*f a*) *s'*)
unfolding *bind-def* **by** *auto*

adhoc-overloading *Monad-Syntax.bind bind*

lemma *bind-left-identity*[*simp*]: *bind* (*return a*) *f* = *f a*
unfolding *return-def bind-def* **by** *simp*

lemma *bind-right-identity*[*simp*]: *bind m return* = *m*
unfolding *return-def bind-def* **by** *simp*

lemma *bind-assoc*[*simp*]: *bind* (*bind m f*) *g* = *bind m* (λ*x*. *bind* (*f x*) *g*)
unfolding *bind-def* **by** (*auto split*: *prod.splits*)

lemma *bind-predI*[*intro*]:
assumes *pred-state* (λ*x*. *pred-state P* (*f x*)) *m*
shows *pred-state P* (*bind m f*)
apply (*rule pred-stateI*)
unfolding *bind-def*
using *assms* **by** (*auto split*: *prod.splits*)

qualified definition *get* :: ('*s*, '*s*) *state* **where**
get = *State* (λ*s*. (*s*, *s*))

lemma *run-state-get*[*simp*]: *run-state get s* = (*s*, *s*)
unfolding *get-def* **by** *simp*

qualified definition *set* :: '*s* ⇒ ('*s*, *unit*) *state* **where**
set s' = *State* (λ-. ((), *s'*))

lemma *run-state-set*[*simp*]: *run-state* (*set s'*) *s* = ((), *s'*)
unfolding *set-def* **by** *simp*

lemma *get-set*[*simp*]: *bind get set* = *return* ()
unfolding *bind-def get-def set-def return-def*
by *simp*

lemma *set-set*[*simp*]: *bind* (*set s*) (λ-. *set s'*) = *set s'*
unfolding *bind-def set-def*
by *simp*

lemma *get-bind-set*[*simp*]: *bind get* (λ*s*. *bind* (*set s*) (*f s*)) = *bind get* (λ*s*. *f s* ())
unfolding *bind-def get-def set-def*
by *simp*

lemma *get-const*[*simp*]: *bind get* (λ-. *m*) = *m*
unfolding *get-def bind-def*
by *simp*

fun *traverse-list* :: ('*a* ⇒ ('*b*, '*c*) *state*) ⇒ '*a list* ⇒ ('*b*, '*c list*) *state* **where**
traverse-list - [] = *return* [] |
traverse-list f (*x* # *xs*) = *do* {
x ← *f x*;
xs ← *traverse-list f xs*;
return (*x* # *xs*)
}

lemma *traverse-list-app*[*simp*]: *traverse-list f* (*xs* @ *ys*) = *do* {
xs ← *traverse-list f xs*;

ys ← *traverse-list f ys*;
return (*xs* @ *ys*)
}
by (*induction xs*) *auto*

lemma *traverse-comp*[*simp*]: *traverse-list* (*g* ∘ *f*) *xs* = *traverse-list g* (*map f xs*)
by (*induction xs*) *auto*

abbreviation *mono-state* :: ('*s*::*preorder*, '*a*) *state* ⇒ *bool* **where**
mono-state ≡ *state-io-rel* (≤)

abbreviation *strict-mono-state* :: ('*s*::*preorder*, '*a*) *state* ⇒ *bool* **where**
strict-mono-state ≡ *state-io-rel* (<)

corollary *strict-mono-implies-mono*: *strict-mono-state m* ⟹ *mono-state m*
unfolding *state-io-rel-def*
by (*simp add*: *less-imp-le*)

lemma *return-mono*[*simp*, *intro*]: *mono-state* (*return x*)
unfolding *return-def* **by** *auto*

lemma *get-mono*[*simp*, *intro*]: *mono-state get*
unfolding *get-def* **by** *auto*

lemma *put-mono*:
assumes ⋀*x*. *s'* ≥ *x*
shows *mono-state* (*set s'*)
using *assms* **unfolding** *set-def*
by *auto*

lemma *map-mono*[*intro*]: *mono-state m* ⟹ *mono-state* (*map-state f m*)
by (*auto intro*!: *state-io-relI split*: *prod.splits simp*: *map-prod-def state.map-sel*)

lemma *map-strict-mono*[*intro*]: *strict-mono-state m* ⟹ *strict-mono-state* (*map-state f m*)
by (*auto intro*!: *state-io-relI split*: *prod.splits simp*: *map-prod-def state.map-sel*)

lemma *bind-mono-strong*:
assumes *mono-state m*
assumes ⋀*x s s'*. *run-state m s* = (*x*, *s'*) ⟹ *mono-state* (*f x*)
shows *mono-state* (*bind m f*)
unfolding *bind-def*
apply (*rule state-io-relI*)
using *assms* **by** (*auto split*: *prod.splits dest*!: *state-io-relD intro*: *order-trans*)

lemma *bind-strict-mono-strong1*:
assumes *mono-state m*
assumes ⋀*x s s'*. *run-state m s* = (*x*, *s'*) ⟹ *strict-mono-state* (*f x*)
shows *strict-mono-state* (*bind m f*)

unfolding *bind-def*
apply (*rule state-io-relI*)
using *assms* **by** (*auto split*: *prod.splits dest*!: *state-io-relD intro*: *le-less-trans*)

lemma *bind-strict-mono-strong2*:
assumes *strict-mono-state m*
assumes $\bigwedge x\ s\ s'$. *run-state m s* $= (x, s') \Longrightarrow$ *mono-state* (*f x*)
shows *strict-mono-state* (*bind m f*)
unfolding *bind-def*
apply (*rule state-io-relI*)
using *assms* **by** (*auto split*: *prod.splits dest*!: *state-io-relD intro*: *less-le-trans*)

corollary *bind-strict-mono-strong*:
assumes *strict-mono-state m*
assumes $\bigwedge x\ s\ s'$. *run-state m s* $= (x, s') \Longrightarrow$ *strict-mono-state* (*f x*)
shows *strict-mono-state* (*bind m f*)
using *assms* **by** (*auto intro*: *bind-strict-mono-strong1 strict-mono-implies-mono*)

qualified definition *update* :: ('*s* ⇒ '*s*) ⇒ ('*s*, *unit*) *state* **where**
update f = *bind get* (*set* ∘ *f*)

lemma *update-id*[*simp*]: *update* (λ*x*. *x*) = *return* ()
unfolding *update-def return-def get-def set-def bind-def*
by *auto*

lemma *update-comp*[*simp*]: *bind* (*update f*) (λ-. *update g*) = *update* (*g* ∘ *f*)
unfolding *update-def return-def get-def set-def bind-def*
by *auto*

lemma *set-update*[*simp*]: *bind* (*set s*) (λ-. *update f*) = *set* (*f s*)
unfolding *set-def update-def bind-def get-def set-def*
by *simp*

lemma *set-bind-update*[*simp*]: *bind* (*set s*) (λ-. *bind* (*update f*) *g*) = *bind* (*set* (*f s*)) *g*
unfolding *set-def update-def bind-def get-def set-def*
by *simp*

lemma *update-mono*:
assumes $\bigwedge x$. $x \leq f\ x$
shows *mono-state* (*update f*)
using *assms* **unfolding** *update-def get-def set-def bind-def*
by (*auto intro*!: *state-io-relI*)

lemma *update-strict-mono*:
assumes $\bigwedge x$. $x < f\ x$
shows *strict-mono-state* (*update f*)
using *assms* **unfolding** *update-def get-def set-def bind-def*
by (*auto intro*!: *state-io-relI*)

end

end

theory *Comparator*
imports *Main*
begin

90 Comparators on linear quasi-orders

90.1 Basic properties

datatype *comp* = *Less* | *Equiv* | *Greater*

locale *comparator* =
fixes *cmp* :: $'a \Rightarrow 'a \Rightarrow$ *comp*
assumes *refl* [*simp*]: $\bigwedge a.$ *cmp a a* = *Equiv*
and *trans-equiv*: $\bigwedge a\ b\ c.$ *cmp a b* = *Equiv* $\Longrightarrow$ *cmp b c* = *Equiv* $\Longrightarrow$ *cmp a c* = *Equiv*
assumes *trans-less*: *cmp a b* = *Less* $\Longrightarrow$ *cmp b c* = *Less* $\Longrightarrow$ *cmp a c* = *Less*
and *greater-iff-sym-less*: $\bigwedge b\ a.$ *cmp b a* = *Greater* $\longleftrightarrow$ *cmp a b* = *Less*
begin

Dual properties

lemma *trans-greater*:
cmp a c = *Greater* **if** *cmp a b* = *Greater cmp b c* = *Greater*
using *that greater-iff-sym-less trans-less* **by** *blast*

lemma *less-iff-sym-greater*:
cmp b a = *Less* $\longleftrightarrow$ *cmp a b* = *Greater*
by (*simp add*: *greater-iff-sym-less*)

The equivalence part

lemma *sym*:
cmp b a = *Equiv* $\longleftrightarrow$ *cmp a b* = *Equiv*
by (*metis* (*full-types*) *comp.exhaust greater-iff-sym-less*)

lemma *reflp*:
reflp ($\lambda a\ b.$ *cmp a b* = *Equiv*)
by (*rule reflpI*) *simp*

lemma *symp*:
symp ($\lambda a\ b.$ *cmp a b* = *Equiv*)
by (*rule sympI*) (*simp add*: *sym*)

lemma *transp*:
transp ($\lambda a\ b.$ *cmp a b* = *Equiv*)

by (*rule transpI*) (*fact trans-equiv*)

lemma *equivp*:
equivp (λ*a b*. *cmp a b* = *Equiv*)
using *reflp symp transp* **by** (*rule equivpI*)

The strict part

lemma *irreflp-less*:
irreflp (λ*a b*. *cmp a b* = *Less*)
by (*rule irreflpI*) *simp*

lemma *irreflp-greater*:
irreflp (λ*a b*. *cmp a b* = *Greater*)
by (*rule irreflpI*) *simp*

lemma *asym-less*:
cmp b a ≠ *Less* **if** *cmp a b* = *Less*
using *that greater-iff-sym-less* **by** *force*

lemma *asym-greater*:
cmp b a ≠ *Greater* **if** *cmp a b* = *Greater*
using *that greater-iff-sym-less* **by** *force*

lemma *asymp-less*:
asymp (λ*a b*. *cmp a b* = *Less*)
using *irreflp-less* **by** (*auto intro*: *asympI dest*: *asym-less*)

lemma *asymp-greater*:
asymp (λ*a b*. *cmp a b* = *Greater*)
using *irreflp-greater* **by** (*auto intro*!: *asympI dest*: *asym-greater*)

lemma *trans-equiv-less*:
cmp a c = *Less* **if** *cmp a b* = *Equiv* **and** *cmp b c* = *Less*
using *that*
by (*metis* (*full-types*) *comp.exhaust greater-iff-sym-less trans-equiv trans-less*)

lemma *trans-less-equiv*:
cmp a c = *Less* **if** *cmp a b* = *Less* **and** *cmp b c* = *Equiv*
using *that*
by (*metis* (*full-types*) *comp.exhaust greater-iff-sym-less trans-equiv trans-less*)

lemma *trans-equiv-greater*:
cmp a c = *Greater* **if** *cmp a b* = *Equiv* **and** *cmp b c* = *Greater*
using *that* **by** (*simp add*: *sym* [*of a b*] *greater-iff-sym-less trans-less-equiv*)

lemma *trans-greater-equiv*:
cmp a c = *Greater* **if** *cmp a b* = *Greater* **and** *cmp b c* = *Equiv*
using *that* **by** (*simp add*: *sym* [*of b c*] *greater-iff-sym-less trans-equiv-less*)

lemma *transp-less*:
transp (*λa b. cmp a b = Less*)
by (*rule transpI*) (*fact trans-less*)

lemma *transp-greater*:
transp (*λa b. cmp a b = Greater*)
by (*rule transpI*) (*fact trans-greater*)

The reflexive part

lemma *reflp-not-less*:
reflp (*λa b. cmp a b ≠ Less*)
by (*rule reflpI*) *simp*

lemma *reflp-not-greater*:
reflp (*λa b. cmp a b ≠ Greater*)
by (*rule reflpI*) *simp*

lemma *quasisym-not-less*:
cmp a b = Equiv **if** *cmp a b ≠ Less* **and** *cmp b a ≠ Less*
using *that comp.exhaust greater-iff-sym-less* **by** *auto*

lemma *quasisym-not-greater*:
cmp a b = Equiv **if** *cmp a b ≠ Greater* **and** *cmp b a ≠ Greater*
using *that comp.exhaust greater-iff-sym-less* **by** *auto*

lemma *trans-not-less*:
cmp a c ≠ Less **if** *cmp a b ≠ Less cmp b c ≠ Less*
using *that* **by** (*metis comp.exhaust greater-iff-sym-less trans-equiv trans-less*)

lemma *trans-not-greater*:
cmp a c ≠ Greater **if** *cmp a b ≠ Greater cmp b c ≠ Greater*
using *that greater-iff-sym-less trans-not-less* **by** *blast*

lemma *transp-not-less*:
transp (*λa b. cmp a b ≠ Less*)
by (*rule transpI*) (*fact trans-not-less*)

lemma *transp-not-greater*:
transp (*λa b. cmp a b ≠ Greater*)
by (*rule transpI*) (*fact trans-not-greater*)

Substitution under equivalences

lemma *equiv-subst-left*:
cmp z y = comp ⟷ cmp x y = comp **if** *cmp z x = Equiv* **for** *comp*
proof −
from *that* **have** *cmp x z = Equiv*
by (*simp add*: *sym*)
with *that* **show** *?thesis*
by (*cases comp*) (*auto intro*: *trans-equiv trans-equiv-less trans-equiv-greater*)

qed

lemma *equiv-subst-right*:
cmp x z = comp $\longleftrightarrow$ *cmp x y = comp* **if** *cmp z y = Equiv* **for** *comp*
proof −
from *that* **have** *cmp y z = Equiv*
by (*simp add*: *sym*)
with *that* **show** *?thesis*
by (*cases comp*) (*auto intro*: *trans-equiv trans-less-equiv trans-greater-equiv*)
qed

end

typedef *'a comparator* = {*cmp* :: *'a* ⇒ *'a* ⇒ *comp*. *comparator cmp*}
morphisms *compare Abs-comparator*
proof −
have *comparator* (λ- -. *Equiv*)
by *standard simp-all*
then show *?thesis*
by *auto*
qed

setup-lifting *type-definition-comparator*

global-interpretation *compare*: *comparator compare cmp*
using *compare* [*of cmp*] **by** *simp*

lift-definition *flat* :: *'a comparator*
is λ- -. *Equiv* **by** *standard simp-all*

instantiation *comparator* :: (*linorder*) *default*
begin

lift-definition *default-comparator* :: *'a comparator*
is λ*x y*. *if x < y then Less else if x > y then Greater else Equiv*
by *standard* (*auto split*: *if-splits*)

instance ..

end

A rudimentary quickcheck setup

instantiation *comparator* :: (*enum*) *equal*
begin

lift-definition *equal-comparator* :: *'a comparator* ⇒ *'a comparator* ⇒ *bool*
is λ*f g*. ∀ *x* ∈ *set Enum.enum*. *f x = g x* .

instance

by (*standard*; *transfer*) (*auto simp add*: *enum-UNIV*)

end

lemma [*code*]:
HOL.equal cmp1 cmp2 ⟷ *Enum.enum-all* (λ*x*. *compare cmp1 x* = *compare cmp2 x*)
by *transfer* (*simp add*: *enum-UNIV*)

lemma [*code nbe*]:
HOL.equal (*cmp* :: 'a::*enum comparator*) *cmp* ⟷ *True*
by (*fact equal-refl*)

instantiation *comparator* :: ({*linorder*, *typerep*}) *full-exhaustive*
begin

definition *full-exhaustive-comparator* ::
('*a comparator* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*)
⇒ *natural* ⇒ (*bool* × *term list*) *option*
where *full-exhaustive-comparator f s* =
Quickcheck-Exhaustive.orelse
(*f* (*flat*, (λ*u*. *Code-Evaluation.Const* (*STR* ''*Comparator.flat*'') *TYPEREP*('*a comparator*))))
(*f* (*default*, (λ*u*. *Code-Evaluation.Const* (*STR* ''*HOL.default-class.default*'') *TYPEREP*('*a comparator*))))

instance ..

end

90.2 Fundamental comparator combinators

lift-definition *reversed* :: '*a comparator* ⇒ '*a comparator*
is λ*cmp a b*. *cmp b a*
proof −
fix *cmp* :: '*a* ⇒ '*a* ⇒ *comp*
assume *comparator cmp*
then interpret *comparator cmp* .
show *comparator* (λ*a b*. *cmp b a*)
by *standard* (*auto intro*: *trans-equiv trans-less simp*: *greater-iff-sym-less*)
qed

lift-definition *key* :: ('*b* ⇒ '*a*) ⇒ '*a comparator* ⇒ '*b comparator*
is λ*f cmp a b*. *cmp* (*f a*) (*f b*)
proof −
fix *cmp* :: '*a* ⇒ '*a* ⇒ *comp* **and** *f* :: '*b* ⇒ '*a*
assume *comparator cmp*
then interpret *comparator cmp* .
show *comparator* (λ*a b*. *cmp* (*f a*) (*f b*))

by *standard* (*auto intro*: *trans-equiv trans-less simp*: *greater-iff-sym-less*)
qed

90.3 Direct implementations for linear orders on selected types

definition *comparator-bool* :: *bool comparator*
where [*simp*, *code-abbrev*]: *comparator-bool* = *default*

lemma *compare-comparator-bool* [*code abstract*]:
compare comparator-bool = (λ*p q*.
if p then if q then Equiv else Greater
else if q then Less else Equiv)
by (*auto simp add*: *fun-eq-iff*) (*transfer*; *simp*)+

definition *raw-comparator-nat* :: *nat* ⇒ *nat* ⇒ *comp*
where [*simp*]: *raw-comparator-nat* = *compare default*

lemma *default-comparator-nat* [*simp*, *code*]:
raw-comparator-nat (*0*::*nat*) *0* = *Equiv*
raw-comparator-nat (*Suc m*) *0* = *Greater*
raw-comparator-nat 0 (*Suc n*) = *Less*
raw-comparator-nat (*Suc m*) (*Suc n*) = *raw-comparator-nat m n*
by (*transfer*; *simp*)+

definition *comparator-nat* :: *nat comparator*
where [*simp*, *code-abbrev*]: *comparator-nat* = *default*

lemma *compare-comparator-nat* [*code abstract*]:
compare comparator-nat = *raw-comparator-nat*
by *simp*

definition *comparator-linordered-group* :: *'a*::*linordered-ab-group-add comparator*
where [*simp*, *code-abbrev*]: *comparator-linordered-group* = *default*

lemma *comparator-linordered-group* [*code abstract*]:
compare comparator-linordered-group = (λ*a b*.
let c = *a* − *b in if c* < *0 then Less*
else if c = *0 then Equiv else Greater*)
proof (*rule ext*)+
fix *a b* :: *'a*
show *compare comparator-linordered-group a b* =
(*let c* = *a* − *b in if c* < *0 then Less*
else if c = *0 then Equiv else Greater*)
by (*simp add*: *Let-def not-less*) (*transfer*; *auto*)
qed

end

theory *Sorting-Algorithms*
imports *Main Multiset Comparator*
begin

91 Stably sorted lists

abbreviation (*input*) *stable-segment* :: *'a comparator* ⇒ *'a* ⇒ *'a list* ⇒ *'a list*
where *stable-segment cmp x* ≡ *filter* (λ*y*. *compare cmp x y* = *Equiv*)

fun *sorted* :: *'a comparator* ⇒ *'a list* ⇒ *bool*
where *sorted-Nil*: *sorted cmp* [] ⟷ *True*
| *sorted-single*: *sorted cmp* [*x*] ⟷ *True*
| *sorted-rec*: *sorted cmp* (*y* # *x* # *xs*) ⟷ *compare cmp y x* ≠ *Greater* ∧ *sorted cmp* (*x* # *xs*)

lemma *sorted-ConsI*:
sorted cmp (*x* # *xs*) **if** *sorted cmp xs*
and ⋀*y ys*. *xs* = *y* # *ys* ⟹ *compare cmp x y* ≠ *Greater*
using *that* **by** (*cases xs*) *simp-all*

lemma *sorted-Cons-imp-sorted*:
sorted cmp xs **if** *sorted cmp* (*x* # *xs*)
using *that* **by** (*cases xs*) *simp-all*

lemma *sorted-Cons-imp-not-less*:
compare cmp y x ≠ *Greater* **if** *sorted cmp* (*y* # *xs*)
and *x* ∈ *set xs*
using *that* **by** (*induction xs arbitrary*: *y*) (*auto dest*: *compare.trans-not-greater*)

lemma *sorted-induct* [*consumes 1*, *case-names Nil Cons*, *induct pred*: *sorted*]:
P xs **if** *sorted cmp xs* **and** *P* []
and ∗: ⋀*x xs*. *sorted cmp xs* ⟹ *P xs*
⟹ (⋀*y*. *y* ∈ *set xs* ⟹ *compare cmp x y* ≠ *Greater*) ⟹ *P* (*x* # *xs*)
using ‹*sorted cmp xs*› **proof** (*induction xs*)
case *Nil*
show *?case*
by (*rule* ‹*P* []›)
next
case (*Cons x xs*)
from ‹*sorted cmp* (*x* # *xs*)› **have** *sorted cmp xs*
by (*cases xs*) *simp-all*
moreover have *P xs* **using** ‹*sorted cmp xs*›
by (*rule Cons.IH*)
moreover have *compare cmp x y* ≠ *Greater* **if** *y* ∈ *set xs* **for** *y*
using *that* ‹*sorted cmp* (*x* # *xs*)› **proof** (*induction xs*)
case *Nil*
then show *?case*
by *simp*

```
  next
    case (Cons z zs)
    then show ?case
    proof (cases zs)
      case Nil
      with Cons.prems show ?thesis
        by simp
    next
      case (Cons w ws)
        with Cons.prems have compare cmp z w ≠ Greater compare cmp x z ≠
Greater
        by auto
      then have compare cmp x w ≠ Greater
        by (auto dest: compare.trans-not-greater)
      with Cons show ?thesis
        using Cons.prems Cons.IH by auto
    qed
  qed
  ultimately show ?case
    by (rule ∗)
qed

lemma sorted-induct-remove1 [consumes 1, case-names Nil minimum]:
  P xs if sorted cmp xs and P []
    and ∗: ⋀x xs. sorted cmp xs ⟹ P (remove1 x xs)
     ⟹ x ∈ set xs ⟹ hd (stable-segment cmp x xs) = x ⟹ (⋀y. y ∈ set xs ⟹
compare cmp x y ≠ Greater)
    ⟹ P xs
using ‹sorted cmp xs› proof (induction xs)
  case Nil
  show ?case
    by (rule ‹P []›)
next
  case (Cons x xs)
  then have sorted cmp (x # xs)
    by (simp add: sorted-ConsI)
  moreover note Cons.IH
  moreover have ⋀y. compare cmp x y = Greater ⟹ y ∈ set xs ⟹ False
    using Cons.hyps by simp
  ultimately show ?case
    by (auto intro!: ∗ [of x # xs x]) blast
qed

lemma sorted-remove1:
  sorted cmp (remove1 x xs) if sorted cmp xs
proof (cases x ∈ set xs)
  case False
  with that show ?thesis
    by (simp add: remove1-idem)
```

```
next
  case True
  with that show ?thesis proof (induction xs)
    case Nil
    then show ?case
      by simp
  next
    case (Cons y ys)
    show ?case proof (cases x = y)
      case True
      with Cons.hyps show ?thesis
        by simp
    next
      case False
      then have sorted cmp (remove1 x ys)
        using Cons.IH Cons.prems by auto
      then have sorted cmp (y # remove1 x ys)
      proof (rule sorted-ConsI)
        fix z zs
        assume remove1 x ys = z # zs
        with ‹x ≠ y› have z ∈ set ys
          using notin-set-remove1 [of z ys x] by auto
        then show compare cmp y z ≠ Greater
          by (rule Cons.hyps(2))
      qed
      with False show ?thesis
        by simp
    qed
  qed
qed

lemma sorted-stable-segment:
  sorted cmp (stable-segment cmp x xs)
proof (induction xs)
  case Nil
  show ?case
    by simp
next
  case (Cons y ys)
  then show ?case
    by (auto intro!: sorted-ConsI simp add: filter-eq-Cons-iff compare.sym)
     (auto dest: compare.trans-equiv simp add: compare.sym compare.greater-iff-sym-less)

qed

primrec insort :: 'a comparator ⇒ 'a ⇒ 'a list ⇒ 'a list
  where insort cmp y [] = [y]
  | insort cmp y (x # xs) = (if compare cmp y x ≠ Greater
      then y # x # xs
```

else x # insort cmp y xs)

lemma *mset-insort* [*simp*]:
mset (*insort cmp x xs*) = *add-mset x* (*mset xs*)
by (*induction xs*) *simp-all*

lemma *length-insort* [*simp*]:
length (*insort cmp x xs*) = *Suc* (*length xs*)
by (*induction xs*) *simp-all*

lemma *sorted-insort*:
sorted cmp (*insort cmp x xs*) **if** *sorted cmp xs*
using *that* **proof** (*induction xs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons y ys*)
then show *?case* **by** (*cases ys*)
(*auto*, *simp-all add*: *compare.greater-iff-sym-less*)
qed

lemma *stable-insort-equiv*:
stable-segment cmp y (*insort cmp x xs*) = *x # stable-segment cmp y xs*
if *compare cmp y x* = *Equiv*
proof (*induction xs*)
case *Nil*
from *that* **show** *?case*
by *simp*
next
case (*Cons z xs*)
moreover from *that* **have** *compare cmp y z* = *Equiv* ⟹ *compare cmp z x* = *Equiv*
by (*auto intro*: *compare.trans-equiv simp add*: *compare.sym*)
ultimately show *?case*
using *that* **by** (*auto simp add*: *compare.greater-iff-sym-less*)
qed

lemma *stable-insort-not-equiv*:
stable-segment cmp y (*insort cmp x xs*) = *stable-segment cmp y xs*
if *compare cmp y x* ≠ *Equiv*
using *that* **by** (*induction xs*) *simp-all*

lemma *remove1-insort-same-eq* [*simp*]:
remove1 x (*insort cmp x xs*) = *xs*
by (*induction xs*) *simp-all*

lemma *insort-eq-ConsI*:
insort cmp x xs = *x # xs*

```
    if sorted cmp xs ⋀y. y ∈ set xs ⟹ compare cmp x y ≠ Greater
  using that by (induction xs) (simp-all add: compare.greater-iff-sym-less)

lemma remove1-insort-not-same-eq [simp]:
  remove1 y (insort cmp x xs) = insort cmp x (remove1 y xs)
    if sorted cmp xs x ≠ y
using that proof (induction xs)
  case Nil
  then show ?case
    by simp
next
  case (Cons z zs)
  show ?case
  proof (cases compare cmp x z = Greater)
    case True
    with Cons show ?thesis
      by simp
  next
    case False
    then have compare cmp x y ≠ Greater if y ∈ set zs for y
      using that Cons.hyps
      by (auto dest: compare.trans-not-greater)
    with Cons show ?thesis
      by (simp add: insort-eq-ConsI)
  qed
qed

lemma insort-remove1-same-eq:
  insort cmp x (remove1 x xs) = xs
    if sorted cmp xs and x ∈ set xs and hd (stable-segment cmp x xs) = x
using that proof (induction xs)
  case Nil
  then show ?case
    by simp
next
  case (Cons y ys)
  then have compare cmp x y ≠ Less
    by (auto simp add: compare.greater-iff-sym-less)
  then consider compare cmp x y = Greater | compare cmp x y = Equiv
    by (cases compare cmp x y) auto
  then show ?case proof cases
    case 1
    with Cons.prems Cons.IH show ?thesis
      by auto
  next
    case 2
    with Cons.prems have x = y
      by simp
    with Cons.hyps show ?thesis
```

```
    by (simp add: insort-eq-ConsI)
  qed
qed

lemma sorted-append-iff:
  sorted cmp (xs @ ys) ⟷ sorted cmp xs ∧ sorted cmp ys
    ∧ (∀x ∈ set xs. ∀y ∈ set ys. compare cmp x y ≠ Greater) (is ?P ⟷ ?R ∧
?S ∧ ?Q)
proof
  assume ?P
  have ?R
    using ‹?P› by (induction xs)
      (auto simp add: sorted-Cons-imp-not-less,
        auto simp add: sorted-Cons-imp-sorted intro: sorted-ConsI)
  moreover have ?S
    using ‹?P› by (induction xs) (auto dest: sorted-Cons-imp-sorted)
  moreover have ?Q
    using ‹?P› by (induction xs) (auto simp add: sorted-Cons-imp-not-less,
      simp add: sorted-Cons-imp-sorted)
  ultimately show ?R ∧ ?S ∧ ?Q
    by simp
next
  assume ?R ∧ ?S ∧ ?Q
  then have ?R ?S ?Q
    by simp-all
  then show ?P
    by (induction xs)
      (auto simp add: append-eq-Cons-conv intro!: sorted-ConsI)
qed

definition sort :: 'a comparator ⇒ 'a list ⇒ 'a list
  where sort cmp xs = foldr (insort cmp) xs []

lemma sort-simps [simp]:
  sort cmp [] = []
  sort cmp (x # xs) = insort cmp x (sort cmp xs)
  by (simp-all add: sort-def)

lemma mset-sort [simp]:
  mset (sort cmp xs) = mset xs
  by (induction xs) simp-all

lemma length-sort [simp]:
  length (sort cmp xs) = length xs
  by (induction xs) simp-all

lemma sorted-sort [simp]:
  sorted cmp (sort cmp xs)
  by (induction xs) (simp-all add: sorted-insort)
```

lemma *stable-sort*:
stable-segment cmp x (*sort cmp xs*) = *stable-segment cmp x xs*
by (*induction xs*) (*simp-all add*: *stable-insort-equiv stable-insort-not-equiv*)

lemma *sort-remove1-eq* [*simp*]:
sort cmp (*remove1 x xs*) = *remove1 x* (*sort cmp xs*)
by (*induction xs*) *simp-all*

lemma *set-insort* [*simp*]:
set (*insort cmp x xs*) = *insert x* (*set xs*)
by (*induction xs*) *auto*

lemma *set-sort* [*simp*]:
set (*sort cmp xs*) = *set xs*
by (*induction xs*) *auto*

lemma *sort-eqI*:
sort cmp ys = *xs*
if *permutation*: *mset ys* = *mset xs*
and *sorted*: *sorted cmp xs*
and *stable*: ⋀*y*. *y* ∈ *set ys* ⟹
stable-segment cmp y ys = *stable-segment cmp y xs*
proof −
have *stable′*: *stable-segment cmp y ys* =
stable-segment cmp y xs **for** *y*
proof (*cases* ∃ *x*∈*set ys*. *compare cmp y x* = *Equiv*)
case *True*
then obtain *z* **where** *z* ∈ *set ys* **and** *compare cmp y z* = *Equiv*
by *auto*
then have *compare cmp y x* = *Equiv* ⟷ *compare cmp z x* = *Equiv* **for** *x*
by (*meson compare.sym compare.trans-equiv*)
moreover have *stable-segment cmp z ys* =
stable-segment cmp z xs
using ‹*z* ∈ *set ys*› **by** (*rule stable*)
ultimately show *?thesis*
by *simp*
next
case *False*
moreover from *permutation* **have** *set ys* = *set xs*
by (*rule mset-eq-setD*)
ultimately show *?thesis*
by *simp*
qed
show *?thesis*
using *sorted permutation stable′* **proof** (*induction xs arbitrary*: *ys rule*: *sorted-induct-remove1*)
case *Nil*
then show *?case*
by *simp*

```
  next
    case (minimum x xs)
    from ‹mset ys = mset xs› have ys: set ys = set xs
      by (rule mset-eq-setD)
    then have compare cmp x y ≠ Greater if y ∈ set ys for y
      using that minimum.hyps by simp
    from minimum.prems have stable: stable-segment cmp x ys = stable-segment
cmp x xs
      by simp
    have sort cmp (remove1 x ys) = remove1 x xs
      by (rule minimum.IH) (simp-all add: minimum.prems filter-remove1)
    then have remove1 x (sort cmp ys) = remove1 x xs
      by simp
    then have insort cmp x (remove1 x (sort cmp ys)) =
      insort cmp x (remove1 x xs)
      by simp
    also from minimum.hyps ys stable have insort cmp x (remove1 x (sort cmp
ys)) = sort cmp ys
      by (simp add: stable-sort insort-remove1-same-eq)
    also from minimum.hyps have insort cmp x (remove1 x xs) = xs
      by (simp add: insort-remove1-same-eq)
    finally show ?case .
  qed
qed

lemma filter-insort:
  filter P (insort cmp x xs) = insort cmp x (filter P xs)
    if sorted cmp xs and P x
  using that by (induction xs)
    (auto simp add: compare.trans-not-greater insort-eq-ConsI)

lemma filter-insort-triv:
  filter P (insort cmp x xs) = filter P xs
    if ¬ P x
  using that by (induction xs) simp-all

lemma filter-sort:
  filter P (sort cmp xs) = sort cmp (filter P xs)
  by (induction xs) (auto simp add: filter-insort filter-insort-triv)
```

92 Alternative sorting algorithms

92.1 Quicksort

```
definition quicksort :: 'a comparator ⇒ 'a list ⇒ 'a list
  where quicksort-is-sort [simp]: quicksort = sort

lemma sort-by-quicksort:
  sort = quicksort
```

by *simp*

lemma *sort-by-quicksort-rec*:
sort cmp xs = sort cmp [x←xs. compare cmp x (xs ! (length xs div 2)) = Less]
@ stable-segment cmp (xs ! (length xs div 2)) xs
@ sort cmp [x←xs. compare cmp x (xs ! (length xs div 2)) = Greater] (**is** *- = ?rhs*)
proof (*rule sort-eqI*)
show *mset xs = mset ?rhs*
by (*rule multiset-eqI*) (*auto simp add*: *compare.sym intro*: *comp.exhaust*)
next
show *sorted cmp ?rhs*
by (*auto simp add*: *sorted-append-iff sorted-stable-segment compare.equiv-subst-right dest*: *compare.trans-greater*)
next
let *?pivot = xs ! (length xs div 2)*
fix *l*
have *compare cmp x ?pivot = comp ∧ compare cmp l x = Equiv*
⟷ compare cmp l ?pivot = comp ∧ compare cmp l x = Equiv **for** *x comp*
proof −
have *compare cmp x ?pivot = comp ⟷ compare cmp l ?pivot = comp*
if *compare cmp l x = Equiv*
using *that* **by** (*simp add*: *compare.equiv-subst-left compare.sym*)
then show *?thesis* **by** *blast*
qed
then show *stable-segment cmp l xs = stable-segment cmp l ?rhs*
by (*simp add*: *stable-sort compare.sym* [*of - ?pivot*])
(*cases compare cmp l ?pivot*, *simp-all*)
qed

context
begin

qualified definition *partition* :: *'a comparator ⇒ 'a ⇒ 'a list ⇒ 'a list × 'a list × 'a list*
where *partition cmp pivot xs =*
([*x ← xs. compare cmp x pivot = Less*], *stable-segment cmp pivot xs*, [*x ← xs. compare cmp x pivot = Greater*])

qualified lemma *partition-code* [*code*]:
partition cmp pivot [] = ([], [], [])
partition cmp pivot (x # xs) =
(*let* (*lts*, *eqs*, *gts*) = *partition cmp pivot xs*
in case compare cmp x pivot of
Less ⇒ (*x # lts*, *eqs*, *gts*)
| Equiv ⇒ (*lts*, *x # eqs*, *gts*)
| Greater ⇒ (*lts*, *eqs*, *x # gts*))
using *comp.exhaust* **by** (*auto simp add*: *partition-def Let-def compare.sym* [*of - pivot*])

lemma *quicksort-code* [*code*]:
  quicksort cmp xs =
    (*case xs of*
      [] ⇒ []
    | [*x*] ⇒ *xs*
    | [*x*, *y*] ⇒ (*if compare cmp x y* ≠ *Greater then xs else* [*y*, *x*])
    | - ⇒
        let (*lts*, *eqs*, *gts*) = *partition cmp* (*xs* ! (*length xs div 2*)) *xs*
        in quicksort cmp lts @ *eqs* @ *quicksort cmp gts*)
proof (*cases length xs* ≥ *3*)
  case *False*
  then have *length xs* ∈ {*0*, *1*, *2*}
    by (*auto simp add*: *not-le le-less less-antisym*)
  then consider *xs* = [] | *x* **where** *xs* = [*x*] | *x y* **where** *xs* = [*x*, *y*]
    by (*auto simp add*: *length-Suc-conv numeral-2-eq-2*)
  then show *?thesis*
    by *cases simp-all*
next
  case *True*
  then obtain *x y z zs* **where** *xs* = *x* # *y* # *z* # *zs*
   by (*metis le-0-eq length-0-conv length-Cons list.exhaust not-less-eq-eq numeral-3-eq-3*)
  moreover have *quicksort cmp xs* =
    (*let* (*lts*, *eqs*, *gts*) = *partition cmp* (*xs* ! (*length xs div 2*)) *xs*
    in quicksort cmp lts @ *eqs* @ *quicksort cmp gts*)
    using *sort-by-quicksort-rec* [*of cmp xs*] **by** (*simp add*: *partition-def*)
  ultimately show *?thesis*
    by *simp*
qed

end

92.2 Mergesort

definition *mergesort* :: *'a comparator* ⇒ *'a list* ⇒ *'a list*
  where *mergesort-is-sort* [*simp*]: *mergesort* = *sort*

lemma *sort-by-mergesort*:
  sort = *mergesort*
  by *simp*

context
  fixes *cmp* :: *'a comparator*
begin

qualified function *merge* :: *'a list* ⇒ *'a list* ⇒ *'a list*
  where *merge* [] *ys* = *ys*
  | *merge xs* [] = *xs*
  | *merge* (*x* # *xs*) (*y* # *ys*) = (*if compare cmp x y* = *Greater*

```
    then y # merge (x # xs) ys else x # merge xs (y # ys))
  by pat-completeness auto

qualified termination by lexicographic-order

lemma mset-merge:
  mset (merge xs ys) = mset xs + mset ys
  by (induction xs ys rule: merge.induct) simp-all

lemma merge-eq-Cons-imp:
  xs ≠ [] ∧ z = hd xs ∨ ys ≠ [] ∧ z = hd ys
    if merge xs ys = z # zs
  using that by (induction xs ys rule: merge.induct) (auto split: if-splits)

lemma filter-merge:
  filter P (merge xs ys) = merge (filter P xs) (filter P ys)
    if sorted cmp xs and sorted cmp ys
using that proof (induction xs ys rule: merge.induct)
  case (1 ys)
  then show ?case
    by simp
next
  case (2 xs)
  then show ?case
    by simp
next
  case (3 x xs y ys)
  show ?case
  proof (cases compare cmp x y = Greater)
    case True
    with 3 have hyp: filter P (merge (x # xs) ys) =
      merge (filter P (x # xs)) (filter P ys)
      by (simp add: sorted-Cons-imp-sorted)
    show ?thesis
    proof (cases ¬ P x ∧ P y)
      case False
      with ‹compare cmp x y = Greater› show ?thesis
        by (auto simp add: hyp)
    next
      case True
      from ‹compare cmp x y = Greater› 3.prems
      have ∗: compare cmp z y = Greater if z ∈ set (filter P xs) for z
      using that by (auto dest: compare.trans-not-greater sorted-Cons-imp-not-less)
      from ‹compare cmp x y = Greater› show ?thesis
        by (cases filter P xs) (simp-all add: hyp ∗)
    qed
  next
    case False
    with 3 have hyp: filter P (merge xs (y # ys)) =
```

```
      merge (filter P xs) (filter P (y # ys))
      by (simp add: sorted-Cons-imp-sorted)
    show ?thesis
    proof (cases P x ∧ ¬ P y)
      case False
      with ‹compare cmp x y ≠ Greater› show ?thesis
        by (auto simp add: hyp)
    next
      case True
      from ‹compare cmp x y ≠ Greater› 3.prems
      have ∗: compare cmp x z ≠ Greater if z ∈ set (filter P ys) for z
      using that by (auto dest: compare.trans-not-greater sorted-Cons-imp-not-less)
      from ‹compare cmp x y ≠ Greater› show ?thesis
        by (cases filter P ys) (simp-all add: hyp ∗)
    qed
  qed
qed

lemma sorted-merge:
  sorted cmp (merge xs ys) if sorted cmp xs and sorted cmp ys
using that proof (induction xs ys rule: merge.induct)
  case (1 ys)
  then show ?case
    by simp
next
  case (2 xs)
  then show ?case
    by simp
next
  case (3 x xs y ys)
  show ?case
  proof (cases compare cmp x y = Greater)
    case True
    with 3 have sorted cmp (merge (x # xs) ys)
      by (simp add: sorted-Cons-imp-sorted)
    then have sorted cmp (y # merge (x # xs) ys)
    proof (rule sorted-ConsI)
      fix z zs
      assume merge (x # xs) ys = z # zs
      with 3(4) True show compare cmp y z ≠ Greater
        by (clarsimp simp add: sorted-Cons-imp-sorted dest!: merge-eq-Cons-imp)
          (auto simp add: compare.asym-greater sorted-Cons-imp-not-less)
    qed
    with True show ?thesis
      by simp
  next
    case False
    with 3 have sorted cmp (merge xs (y # ys))
      by (simp add: sorted-Cons-imp-sorted)
```

```
    then have sorted cmp (x # merge xs (y # ys))
    proof (rule sorted-ConsI)
      fix z zs
      assume merge xs (y # ys) = z # zs
      with 3(3) False show compare cmp x z ≠ Greater
        by (clarsimp simp add: sorted-Cons-imp-sorted dest!: merge-eq-Cons-imp)
          (auto simp add: compare.asym-greater sorted-Cons-imp-not-less)
    qed
    with False show ?thesis
      by simp
  qed
qed

lemma merge-eq-appendI:
  merge xs ys = xs @ ys
    if ⋀x y. x ∈ set xs ⟹ y ∈ set ys ⟹ compare cmp x y ≠ Greater
  using that by (induction xs ys rule: merge.induct) simp-all

lemma merge-stable-segments:
  merge (stable-segment cmp l xs) (stable-segment cmp l ys) =
     stable-segment cmp l xs @ stable-segment cmp l ys
  by (rule merge-eq-appendI) (auto dest: compare.trans-equiv-greater)

lemma sort-by-mergesort-rec:
  sort cmp xs =
    merge (sort cmp (take (length xs div 2) xs))
      (sort cmp (drop (length xs div 2) xs)) (is - = ?rhs)
proof (rule sort-eqI)
  have mset (take (length xs div 2) xs) + mset (drop (length xs div 2) xs) =
    mset (take (length xs div 2) xs @ drop (length xs div 2) xs)
    by (simp only: mset-append)
  then show mset xs = mset ?rhs
    by (simp add: mset-merge)
next
  show sorted cmp ?rhs
    by (simp add: sorted-merge)
next
  fix l
  have stable-segment cmp l (take (length xs div 2) xs) @ stable-segment cmp l
(drop (length xs div 2) xs)
    = stable-segment cmp l xs
    by (simp only: filter-append [symmetric] append-take-drop-id)
  have merge (stable-segment cmp l (take (length xs div 2) xs))
    (stable-segment cmp l (drop (length xs div 2) xs)) =
    stable-segment cmp l (take (length xs div 2) xs) @ stable-segment cmp l (drop
(length xs div 2) xs)
    by (rule merge-eq-appendI) (auto simp add: compare.trans-equiv-greater)
  also have ... = stable-segment cmp l xs
    by (simp only: filter-append [symmetric] append-take-drop-id)
```

```
  finally show stable-segment cmp l xs = stable-segment cmp l ?rhs
    by (simp add: stable-sort filter-merge)
qed

lemma mergesort-code [code]:
  mergesort cmp xs =
    (case xs of
      [] ⇒ []
    | [x] ⇒ xs
    | [x, y] ⇒ (if compare cmp x y ≠ Greater then xs else [y, x])
    | - ⇒
        let
          half = length xs div 2;
          ys = take half xs;
          zs = drop half xs
        in merge (mergesort cmp ys) (mergesort cmp zs))
proof (cases length xs ≥ 3)
  case False
  then have length xs ∈ {0, 1, 2}
    by (auto simp add: not-le le-less less-antisym)
  then consider xs = [] | x where xs = [x] | x y where xs = [x, y]
    by (auto simp add: length-Suc-conv numeral-2-eq-2)
  then show ?thesis
    by cases simp-all
next
  case True
  then obtain x y z zs where xs = x # y # z # zs
   by (metis le-0-eq length-0-conv length-Cons list.exhaust not-less-eq-eq numeral-3-eq-3)
  moreover have mergesort cmp xs =
    (let
       half = length xs div 2;
       ys = take half xs;
       zs = drop half xs
     in merge (mergesort cmp ys) (mergesort cmp zs))
    using sort-by-mergesort-rec [of xs] by (simp add: Let-def)
  ultimately show ?thesis
    by simp
qed

end

end
```

93 A decision procedure for universal multivariate real arithmetic with addition, multiplication and ordering using semidefinite programming

theory *Sum-of-Squares*
imports *Complex-Main*
begin

ML-file ‹*Sum-of-Squares/positivstellensatz.ML*›
ML-file ‹*Sum-of-Squares/positivstellensatz-tools.ML*›
ML-file ‹*Sum-of-Squares/sum-of-squares.ML*›
ML-file ‹*Sum-of-Squares/sos-wrapper.ML*›

end

94 A table-based implementation of the reflexive transitive closure

theory *Transitive-Closure-Table*
imports *Main*
begin

inductive *rtrancl-path* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a \Rightarrow 'a\ list \Rightarrow 'a \Rightarrow bool$
  for $r :: 'a \Rightarrow 'a \Rightarrow bool$
where
  base: *rtrancl-path r x* [] *x*
| *step*: $r\ x\ y \Longrightarrow$ *rtrancl-path r y ys z* $\Longrightarrow$ *rtrancl-path r x* $(y \# ys)\ z$

lemma *rtranclp-eq-rtrancl-path*: $r^{**}\ x\ y \longleftrightarrow (\exists xs.$ *rtrancl-path r x xs y*$)$
proof
  show $\exists xs.$ *rtrancl-path r x xs y* **if** $r^{**}\ x\ y$
    using *that*
  proof (*induct rule*: *converse-rtranclp-induct*)
    case *base*
    have *rtrancl-path r y* [] *y* **by** (*rule rtrancl-path.base*)
    then show *?case* **..**
  next
    case (*step x z*)
    from ‹$\exists xs.$ *rtrancl-path r z xs y*›
    obtain *xs* **where** *rtrancl-path r z xs y* **..**
    with ‹*r x z*› **have** *rtrancl-path r x* $(z \# xs)\ y$
      by (*rule rtrancl-path.step*)
    then show *?case* **..**
  qed
  show $r^{**}\ x\ y$ **if** $\exists xs.$ *rtrancl-path r x xs y*
  proof −

```
    from that obtain xs where rtrancl-path r x xs y ..
    then show ?thesis
    proof induct
      case (base x)
      show ?case
        by (rule rtranclp.rtrancl-refl)
    next
      case (step x y ys z)
      from ‹r x y› ‹r** y z› show ?case
        by (rule converse-rtranclp-into-rtranclp)
    qed
  qed
qed

lemma rtrancl-path-trans:
  assumes xy: rtrancl-path r x xs y
    and yz: rtrancl-path r y ys z
  shows rtrancl-path r x (xs @ ys) z using xy yz
proof (induct arbitrary: z)
  case (base x)
  then show ?case by simp
next
  case (step x y xs)
  then have rtrancl-path r y (xs @ ys) z
    by simp
  with ‹r x y› have rtrancl-path r x (y # (xs @ ys)) z
    by (rule rtrancl-path.step)
  then show ?case by simp
qed

lemma rtrancl-path-appendE:
  assumes xz: rtrancl-path r x (xs @ y # ys) z
  obtains rtrancl-path r x (xs @ [y]) y and rtrancl-path r y ys z
  using xz
proof (induct xs arbitrary: x)
  case Nil
  then have rtrancl-path r x (y # ys) z by simp
  then obtain xy: r x y and yz: rtrancl-path r y ys z
    by cases auto
  from xy have rtrancl-path r x [y] y
    by (rule rtrancl-path.step [OF - rtrancl-path.base])
  then have rtrancl-path r x ([] @ [y]) y by simp
  then show thesis using yz by (rule Nil)
next
  case (Cons a as)
  then have rtrancl-path r x (a # (as @ y # ys)) z by simp
  then obtain xa: r x a and az: rtrancl-path r a (as @ y # ys) z
    by cases auto
  show thesis
```

```
  proof (rule Cons(1) [OF - az])
    assume rtrancl-path r y ys z
    assume rtrancl-path r a (as @ [y]) y
    with xa have rtrancl-path r x (a # (as @ [y])) y
      by (rule rtrancl-path.step)
    then have rtrancl-path r x ((a # as) @ [y]) y
      by simp
    then show thesis using ‹rtrancl-path r y ys z›
      by (rule Cons(2))
  qed
qed

lemma rtrancl-path-distinct:
  assumes xy: rtrancl-path r x xs y
  obtains xs' where rtrancl-path r x xs' y and distinct (x # xs') and set xs' ⊆
set xs
  using xy
proof (induct xs rule: measure-induct-rule [of length])
  case (less xs)
  show ?case
  proof (cases distinct (x # xs))
    case True
    with ‹rtrancl-path r x xs y› show ?thesis by (rule less) simp
  next
    case False
    then have ∃ as bs cs a. x # xs = as @ [a] @ bs @ [a] @ cs
      by (rule not-distinct-decomp)
    then obtain as bs cs a where xxs: x # xs = as @ [a] @ bs @ [a] @ cs
      by iprover
    show ?thesis
    proof (cases as)
      case Nil
      with xxs have x: x = a and xs: xs = bs @ a # cs
        by auto
      from x xs ‹rtrancl-path r x xs y› have cs: rtrancl-path r x cs y set cs ⊆ set xs
        by (auto elim: rtrancl-path-appendE)
      from xs have length cs < length xs by simp
      then show ?thesis
        by (rule less(1))(blast intro: cs less(2) order-trans del: subsetI)+
    next
      case (Cons d ds)
      with xxs have xs: xs = ds @ a # (bs @ [a] @ cs)
        by auto
      with ‹rtrancl-path r x xs y› obtain xa: rtrancl-path r x (ds @ [a]) a
        and ay: rtrancl-path r a (bs @ a # cs) y
        by (auto elim: rtrancl-path-appendE)
      from ay have rtrancl-path r a cs y by (auto elim: rtrancl-path-appendE)
      with xa have xy: rtrancl-path r x ((ds @ [a]) @ cs) y
        by (rule rtrancl-path-trans)
```

```
    from xs have set: set ((ds @ [a]) @ cs) ⊆ set xs by auto
    from xs have length ((ds @ [a]) @ cs) < length xs by simp
    then show ?thesis
      by (rule less(1))(blast intro: xy less(2) set[THEN subsetD])+
  qed
 qed
qed

inductive rtrancl-tab :: ('a ⇒ 'a ⇒ bool) ⇒ 'a list ⇒ 'a ⇒ 'a ⇒ bool
 for r :: 'a ⇒ 'a ⇒ bool
where
 base: rtrancl-tab r xs x x
| step: x ∉ set xs ⟹ r x y ⟹ rtrancl-tab r (x # xs) y z ⟹ rtrancl-tab r xs x z

lemma rtrancl-path-imp-rtrancl-tab:
 assumes path: rtrancl-path r x xs y
   and x: distinct (x # xs)
   and ys: ({x} ∪ set xs) ∩ set ys = {}
 shows rtrancl-tab r ys x y
 using path x ys
proof (induct arbitrary: ys)
 case base
 show ?case
   by (rule rtrancl-tab.base)
next
 case (step x y zs z)
 then have x ∉ set ys
   by auto
 from step have distinct (y # zs)
   by simp
 moreover from step have ({y} ∪ set zs) ∩ set (x # ys) = {}
   by auto
 ultimately have rtrancl-tab r (x # ys) y z
   by (rule step)
 with ‹x ∉ set ys› ‹r x y› show ?case
   by (rule rtrancl-tab.step)
qed

lemma rtrancl-tab-imp-rtrancl-path:
 assumes tab: rtrancl-tab r ys x y
 obtains xs where rtrancl-path r x xs y
 using tab
proof induct
 case base
 from rtrancl-path.base show ?case
   by (rule base)
next
 case step
 show ?case
```

by (*iprover intro*: *step rtrancl-path.step*)
qed

lemma *rtranclp-eq-rtrancl-tab-nil*: $r^{**}\ x\ y \longleftrightarrow$ *rtrancl-tab r* [] *x y*
proof
show *rtrancl-tab r* [] *x y* **if** $r^{**}\ x\ y$
proof −
from *that* **obtain** *xs* **where** *rtrancl-path r x xs y*
by (*auto simp add*: *rtranclp-eq-rtrancl-path*)
then obtain *xs′* **where** *xs′*: *rtrancl-path r x xs′ y* **and** *distinct*: *distinct* (*x* # *xs′*)
by (*rule rtrancl-path-distinct*)
have $(\{x\} \cup set\ xs') \cap set\ [] = \{\}$
by *simp*
with *xs′ distinct* **show** *?thesis*
by (*rule rtrancl-path-imp-rtrancl-tab*)
qed
show $r^{**}\ x\ y$ **if** *rtrancl-tab r* [] *x y*
proof −
from *that* **obtain** *xs* **where** *rtrancl-path r x xs y*
by (*rule rtrancl-tab-imp-rtrancl-path*)
then show *?thesis*
by (*auto simp add*: *rtranclp-eq-rtrancl-path*)
qed
qed

declare *rtranclp-rtrancl-eq* [*code del*]
declare *rtranclp-eq-rtrancl-tab-nil* [*THEN iffD2*, *code-pred-intro*]

code-pred *rtranclp*
using *rtranclp-eq-rtrancl-tab-nil* [*THEN iffD1*] **by** *fastforce*

lemma *rtrancl-path-Range*: ⟦ *rtrancl-path R x xs y*; *z* ∈ *set xs* ⟧ ⟹ *Rangep R z*
by(*induction rule*: *rtrancl-path.induct*) *auto*

lemma *rtrancl-path-Range-end*: ⟦ *rtrancl-path R x xs y*; *xs* ≠ [] ⟧ ⟹ *Rangep R y*
by(*induction rule*: *rtrancl-path.induct*)(*auto elim*: *rtrancl-path.cases*)

lemma *rtrancl-path-nth*:
⟦ *rtrancl-path R x xs y*; *i* < *length xs* ⟧ ⟹ *R* ((*x* # *xs*) ! *i*) (*xs* ! *i*)
proof(*induction arbitrary*: *i rule*: *rtrancl-path.induct*)
case *step* **thus** *?case* **by**(*cases i*) *simp-all*
qed *simp*

lemma *rtrancl-path-last*: ⟦ *rtrancl-path R x xs y*; *xs* ≠ [] ⟧ ⟹ *last xs* = *y*
by(*induction rule*: *rtrancl-path.induct*)(*auto elim*: *rtrancl-path.cases*)

lemma *rtrancl-path-mono*:
⟦ *rtrancl-path R x p y*; ⋀*x y*. *R x y* ⟹ *S x y* ⟧ ⟹ *rtrancl-path S x p y*

by(*induction rule*: *rtrancl-path.induct*)(*auto intro*: *rtrancl-path.intros*)

end

95 Binary Tree

theory *Tree*
imports *Main*
begin

datatype *'a tree* =
  Leaf (⟨⟩) |
  Node 'a tree (*value*: *'a*) *'a tree* ((*1*⟨-,/ -,/ -⟩))
datatype-compat *tree*

primrec *left* :: *'a tree* ⇒ *'a tree* **where**
left (*Node l v r*) = *l* |
left Leaf = *Leaf*

primrec *right* :: *'a tree* ⇒ *'a tree* **where**
right (*Node l v r*) = *r* |
right Leaf = *Leaf*

Counting the number of leaves rather than nodes:

fun *size1* :: *'a tree* ⇒ *nat* **where**
size1 ⟨⟩ = *1* |
size1 ⟨*l*, *x*, *r*⟩ = *size1 l* + *size1 r*

fun *subtrees* :: *'a tree* ⇒ *'a tree set* **where**
subtrees ⟨⟩ = {⟨⟩} |
subtrees (⟨*l*, *a*, *r*⟩) = {⟨*l*, *a*, *r*⟩} ∪ *subtrees l* ∪ *subtrees r*

fun *mirror* :: *'a tree* ⇒ *'a tree* **where**
mirror ⟨⟩ = *Leaf* |
mirror ⟨*l*,*x*,*r*⟩ = ⟨*mirror r*, *x*, *mirror l*⟩

class *height* = **fixes** *height* :: *'a* ⇒ *nat*

instantiation *tree* :: (*type*)*height*
begin

fun *height-tree* :: *'a tree* => *nat* **where**
height Leaf = *0* |
height (*Node l a r*) = *max* (*height l*) (*height r*) + *1*

instance ..

end

fun *min-height* :: *'a tree* ⇒ *nat* **where**
min-height Leaf = *0* |
min-height (*Node l - r*) = *min* (*min-height l*) (*min-height r*) + *1*

fun *complete* :: *'a tree* ⇒ *bool* **where**
complete Leaf = *True* |
complete (*Node l x r*) = (*height l* = *height r* ∧ *complete l* ∧ *complete r*)

Almost complete:

definition *acomplete* :: *'a tree* ⇒ *bool* **where**
acomplete t = (*height t* − *min-height t* ≤ *1*)

Weight balanced:

fun *wbalanced* :: *'a tree* ⇒ *bool* **where**
wbalanced Leaf = *True* |
wbalanced (*Node l x r*) = (*abs*(*int*(*size l*) − *int*(*size r*)) ≤ *1* ∧ *wbalanced l* ∧ *wbalanced r*)

Internal path length:

fun *ipl* :: *'a tree* ⇒ *nat* **where**
ipl Leaf = *0* |
ipl (*Node l - r*) = *ipl l* + *size l* + *ipl r* + *size r*

fun *preorder* :: *'a tree* ⇒ *'a list* **where**
preorder ⟨⟩ = [] |
preorder ⟨*l*, *x*, *r*⟩ = *x* # *preorder l* @ *preorder r*

fun *inorder* :: *'a tree* ⇒ *'a list* **where**
inorder ⟨⟩ = [] |
inorder ⟨*l*, *x*, *r*⟩ = *inorder l* @ [*x*] @ *inorder r*

A linear version avoiding append:

fun *inorder2* :: *'a tree* ⇒ *'a list* ⇒ *'a list* **where**
inorder2 ⟨⟩ *xs* = *xs* |
inorder2 ⟨*l*, *x*, *r*⟩ *xs* = *inorder2 l* (*x* # *inorder2 r xs*)

fun *postorder* :: *'a tree* ⇒ *'a list* **where**
postorder ⟨⟩ = [] |
postorder ⟨*l*, *x*, *r*⟩ = *postorder l* @ *postorder r* @ [*x*]

Binary Search Tree:

fun *bst-wrt* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a tree* ⇒ *bool* **where**
bst-wrt P ⟨⟩ ⟷ *True* |
bst-wrt P ⟨*l*, *a*, *r*⟩ ⟷
(∀ *x*∈*set-tree l*. *P x a*) ∧ (∀ *x*∈*set-tree r*. *P a x*) ∧ *bst-wrt P l* ∧ *bst-wrt P r*

abbreviation *bst* :: (*'a*::*linorder*) *tree* ⇒ *bool* **where**
bst ≡ *bst-wrt* (<)

fun (**in** *linorder*) *heap* :: *'a tree* ⇒ *bool* **where**
heap Leaf = *True* |
heap (*Node l m r*) =
  ((∀ *x* ∈ *set-tree l* ∪ *set-tree r*. *m* ≤ *x*) ∧ *heap l* ∧ *heap r*)

95.1 *map-tree*

lemma *eq-map-tree-Leaf*[*simp*]: *map-tree f t* = *Leaf* ⟷ *t* = *Leaf*
by (*rule tree.map-disc-iff*)

lemma *eq-Leaf-map-tree*[*simp*]: *Leaf* = *map-tree f t* ⟷ *t* = *Leaf*
by (*cases t*) *auto*

95.2 *size*

lemma *size1-size*: *size1 t* = *size t* + *1*
by (*induction t*) *simp-all*

lemma *size1-ge0*[*simp*]: *0* < *size1 t*
by (*simp add*: *size1-size*)

lemma *eq-size-0*[*simp*]: *size t* = *0* ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *eq-0-size*[*simp*]: *0* = *size t* ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *neq-Leaf-iff*: (*t* ≠ ⟨⟩) = (∃ *l a r*. *t* = ⟨*l*, *a*, *r*⟩)
by (*cases t*) *auto*

lemma *size-map-tree*[*simp*]: *size* (*map-tree f t*) = *size t*
by (*induction t*) *auto*

lemma *size1-map-tree*[*simp*]: *size1* (*map-tree f t*) = *size1 t*
by (*simp add*: *size1-size*)

95.3 *set-tree*

lemma *eq-set-tree-empty*[*simp*]: *set-tree t* = {} ⟷ *t* = *Leaf*
by (*cases t*) *auto*

lemma *eq-empty-set-tree*[*simp*]: {} = *set-tree t* ⟷ *t* = *Leaf*
by (*cases t*) *auto*

lemma *finite-set-tree*[*simp*]: *finite*(*set-tree t*)
by(*induction t*) *auto*

95.4 *subtrees*

lemma *neq-subtrees-empty*[*simp*]: *subtrees t* ≠ {}

by (*cases t*)(*auto*)

lemma *neq-empty-subtrees*[*simp*]: {} ≠ *subtrees t*
by (*cases t*)(*auto*)

lemma *size-subtrees*: *s* ∈ *subtrees t* ⟹ *size s* ≤ *size t*
by(*induction t*)(*auto*)

lemma *set-treeE*: *a* ∈ *set-tree t* ⟹ ∃ *l r*. ⟨*l*, *a*, *r*⟩ ∈ *subtrees t*
by (*induction t*)(*auto*)

lemma *Node-notin-subtrees-if*[*simp*]: *a* ∉ *set-tree t* ⟹ *Node l a r* ∉ *subtrees t*
by (*induction t*) *auto*

lemma *in-set-tree-if*: ⟨*l*, *a*, *r*⟩ ∈ *subtrees t* ⟹ *a* ∈ *set-tree t*
by (*metis Node-notin-subtrees-if*)

95.5 *height* and *min-height*

lemma *eq-height-0*[*simp*]: *height t* = *0* ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *eq-0-height*[*simp*]: *0* = *height t* ⟷ *t* = *Leaf*
by(*cases t*) *auto*

lemma *height-map-tree*[*simp*]: *height* (*map-tree f t*) = *height t*
by (*induction t*) *auto*

lemma *height-le-size-tree*: *height t* ≤ *size* (*t*::'*a tree*)
by (*induction t*) *auto*

lemma *size1-height*: *size1 t* ≤ *2* ^ *height* (*t*::'*a tree*)
proof(*induction t*)
 case (*Node l a r*)
 show *?case*
 proof (*cases height l* ≤ *height r*)
  case *True*
  have *size1*(*Node l a r*) = *size1 l* + *size1 r* **by** *simp*
  also have ... ≤ *2* ^ *height l* + *2* ^ *height r* **using** *Node.IH* **by** *arith*
  also have ... ≤ *2* ^ *height r* + *2* ^ *height r* **using** *True* **by** *simp*
  also have ... = *2* ^ *height* (*Node l a r*)
   using *True* **by** (*auto simp*: *max-def mult-2*)
  finally show *?thesis* **.**
 next
  case *False*
  have *size1*(*Node l a r*) = *size1 l* + *size1 r* **by** *simp*
  also have ... ≤ *2* ^ *height l* + *2* ^ *height r* **using** *Node.IH* **by** *arith*
  also have ... ≤ *2* ^ *height l* + *2* ^ *height l* **using** *False* **by** *simp*
  finally show *?thesis* **using** *False* **by** (*auto simp*: *max-def mult-2*)

qed
qed *simp*

corollary *size-height*: *size t* ≤ *2* ^ *height* (*t*::*'a tree*) − *1*
using *size1-height*[*of t*, *unfolded size1-size*] **by**(*arith*)

lemma *height-subtrees*: *s* ∈ *subtrees t* ⟹ *height s* ≤ *height t*
by (*induction t*) *auto*

lemma *min-height-le-height*: *min-height t* ≤ *height t*
by(*induction t*) *auto*

lemma *min-height-map-tree*[*simp*]: *min-height* (*map-tree f t*) = *min-height t*
by (*induction t*) *auto*

lemma *min-height-size1*: *2* ^ *min-height t* ≤ *size1 t*
proof(*induction t*)
case (*Node l a r*)
have (*2*::*nat*) ^ *min-height* (*Node l a r*) ≤ *2* ^ *min-height l* + *2* ^ *min-height r*
by (*simp add*: *min-def*)
also have ... ≤ *size1*(*Node l a r*) **using** *Node.IH* **by** *simp*
finally show *?case* **.**
qed *simp*

95.6 *complete*

lemma *complete-iff-height*: *complete t* ⟷ (*min-height t* = *height t*)
apply(*induction t*)
apply *simp*
apply (*simp add*: *min-def max-def*)
by (*metis le-antisym le-trans min-height-le-height*)

lemma *size1-if-complete*: *complete t* ⟹ *size1 t* = *2* ^ *height t*
by (*induction t*) *auto*

lemma *size-if-complete*: *complete t* ⟹ *size t* = *2* ^ *height t* − *1*
using *size1-if-complete*[*simplified size1-size*] **by** *fastforce*

lemma *size1-height-if-incomplete*:
¬ *complete t* ⟹ *size1 t* < *2* ^ *height t*
proof(*induction t*)
case *Leaf* **thus** *?case* **by** *simp*
next
case (*Node l x r*)
have *1*: *?case* **if** *h*: *height l* < *height r*
using *h size1-height*[*of l*] *size1-height*[*of r*] *power-strict-increasing*[*OF h*, *of* *2*::*nat*]
by(*auto simp*: *max-def simp del*: *power-strict-increasing-iff*)

have *2*: *?case* **if** *h*: *height l* > *height r*
using *h size1-height*[*of l*] *size1-height*[*of r*] *power-strict-increasing*[*OF h*, *of 2*::*nat*]
by(*auto simp*: *max-def simp del*: *power-strict-increasing-iff*)
have *3*: *?case* **if** *h*: *height l* = *height r* **and** *c*: ¬ *complete l*
using *h size1-height*[*of r*] *Node.IH*(*1*)[*OF c*] **by**(*simp*)
have *4*: *?case* **if** *h*: *height l* = *height r* **and** *c*: ¬ *complete r*
using *h size1-height*[*of l*] *Node.IH*(*2*)[*OF c*] **by**(*simp*)
from *1 2 3 4 Node.prems* **show** *?case* **apply** (*simp add*: *max-def*) **by** *linarith*
qed

lemma *complete-iff-min-height*: *complete t* ⟷ (*height t* = *min-height t*)
by(*auto simp add*: *complete-iff-height*)

lemma *min-height-size1-if-incomplete*:
¬ *complete t* ⟹ *2* ^ *min-height t* < *size1 t*
proof(*induction t*)
case *Leaf* **thus** *?case* **by** *simp*
next
case (*Node l x r*)
have *1*: *?case* **if** *h*: *min-height l* < *min-height r*
using *h min-height-size1*[*of l*] *min-height-size1*[*of r*] *power-strict-increasing*[*OF h*, *of 2*::*nat*]
by(*auto simp*: *max-def simp del*: *power-strict-increasing-iff*)
have *2*: *?case* **if** *h*: *min-height l* > *min-height r*
using *h min-height-size1*[*of l*] *min-height-size1*[*of r*] *power-strict-increasing*[*OF h*, *of 2*::*nat*]
by(*auto simp*: *max-def simp del*: *power-strict-increasing-iff*)
have *3*: *?case* **if** *h*: *min-height l* = *min-height r* **and** *c*: ¬ *complete l*
using *h min-height-size1*[*of r*] *Node.IH*(*1*)[*OF c*] **by**(*simp add*: *complete-iff-min-height*)
have *4*: *?case* **if** *h*: *min-height l* = *min-height r* **and** *c*: ¬ *complete r*
using *h min-height-size1*[*of l*] *Node.IH*(*2*)[*OF c*] **by**(*simp add*: *complete-iff-min-height*)
from *1 2 3 4 Node.prems* **show** *?case*
by (*fastforce simp*: *complete-iff-min-height*[*THEN iffD1*])
qed

lemma *complete-if-size1-height*: *size1 t* = *2* ^ *height t* ⟹ *complete t*
using *size1-height-if-incomplete* **by** *fastforce*

lemma *complete-if-size1-min-height*: *size1 t* = *2* ^ *min-height t* ⟹ *complete t*
using *min-height-size1-if-incomplete* **by** *fastforce*

lemma *complete-iff-size1*: *complete t* ⟷ *size1 t* = *2* ^ *height t*
using *complete-if-size1-height size1-if-complete* **by** *blast*

95.7 *acomplete*

lemma *acomplete-subtreeL*: *acomplete* (*Node l x r*) ⟹ *acomplete l*
by(*simp add*: *acomplete-def*)

lemma *acomplete-subtreeR*: *acomplete* (*Node l x r*) ⟹ *acomplete r*
by(*simp add*: *acomplete-def*)

lemma *acomplete-subtrees*: ⟦ *acomplete t*; *s* ∈ *subtrees t* ⟧ ⟹ *acomplete s*
using [[*simp-depth-limit=1*]]
by(*induction t arbitrary*: *s*)
(*auto simp add*: *acomplete-subtreeL acomplete-subtreeR*)

Balanced trees have optimal height:

lemma *acomplete-optimal*:
fixes *t* :: *'a tree* **and** *t'* :: *'b tree*
assumes *acomplete t size t* ≤ *size t'* **shows** *height t* ≤ *height t'*
proof (*cases complete t*)
case *True*
have (*2*::*nat*) ^ *height t* ≤ *2* ^ *height t'*
proof −
have *2* ^ *height t* = *size1 t*
using *True* **by** (*simp add*: *size1-if-complete*)
also have ... ≤ *size1 t'* **using** *assms*(*2*) **by**(*simp add*: *size1-size*)
also have ... ≤ *2* ^ *height t'* **by** (*rule size1-height*)
finally show *?thesis* **.**
qed
thus *?thesis* **by** (*simp*)
next
case *False*
have (*2*::*nat*) ^ *min-height t* < *2* ^ *height t'*
proof −
have (*2*::*nat*) ^ *min-height t* < *size1 t*
by(*rule min-height-size1-if-incomplete*[*OF False*])
also have ... ≤ *size1 t'* **using** *assms*(*2*) **by** (*simp add*: *size1-size*)
also have ... ≤ *2* ^ *height t'* **by**(*rule size1-height*)
finally have (*2*::*nat*) ^ *min-height t* < (*2*::*nat*) ^ *height t'* **.**
thus *?thesis* **.**
qed
hence ∗: *min-height t* < *height t'* **by** *simp*
have *min-height t* + *1* = *height t*
using *min-height-le-height*[*of t*] *assms*(*1*) *False*
by (*simp add*: *complete-iff-height acomplete-def*)
with ∗ **show** *?thesis* **by** *arith*
qed

95.8 *wbalanced*

lemma *wbalanced-subtrees*: ⟦ *wbalanced t*; *s* ∈ *subtrees t* ⟧ ⟹ *wbalanced s*
using [[*simp-depth-limit=1*]] **by**(*induction t arbitrary*: *s*) *auto*

95.9 *ipl*

The internal path length of a tree:

lemma *ipl-if-complete-int*:
  complete t $\Longrightarrow$ *int*(*ipl t*) = (*int*(*height t*) − *2*) ∗ *2*^(*height t*) + *2*
apply(*induction t*)
 apply *simp*
apply *simp*
apply (*simp add*: *algebra-simps size-if-complete of-nat-diff*)
done

95.10 List of entries

lemma *eq-inorder-Nil*[*simp*]: *inorder t* = [] $\longleftrightarrow$ *t* = *Leaf*
by (*cases t*) *auto*

lemma *eq-Nil-inorder*[*simp*]: [] = *inorder t* $\longleftrightarrow$ *t* = *Leaf*
by (*cases t*) *auto*

lemma *set-inorder*[*simp*]: *set* (*inorder t*) = *set-tree t*
by (*induction t*) *auto*

lemma *set-preorder*[*simp*]: *set* (*preorder t*) = *set-tree t*
by (*induction t*) *auto*

lemma *set-postorder*[*simp*]: *set* (*postorder t*) = *set-tree t*
by (*induction t*) *auto*

lemma *length-preorder*[*simp*]: *length* (*preorder t*) = *size t*
by (*induction t*) *auto*

lemma *length-inorder*[*simp*]: *length* (*inorder t*) = *size t*
by (*induction t*) *auto*

lemma *length-postorder*[*simp*]: *length* (*postorder t*) = *size t*
by (*induction t*) *auto*

lemma *preorder-map*: *preorder* (*map-tree f t*) = *map f* (*preorder t*)
by (*induction t*) *auto*

lemma *inorder-map*: *inorder* (*map-tree f t*) = *map f* (*inorder t*)
by (*induction t*) *auto*

lemma *postorder-map*: *postorder* (*map-tree f t*) = *map f* (*postorder t*)
by (*induction t*) *auto*

lemma *inorder2-inorder*: *inorder2 t xs* = *inorder t* @ *xs*
by (*induction t arbitrary*: *xs*) *auto*

95.11 Binary Search Tree

lemma *bst-wrt-mono*: (⋀*x y*. *P x y* $\Longrightarrow$ *Q x y*) $\Longrightarrow$ *bst-wrt P t* $\Longrightarrow$ *bst-wrt Q t*
by (*induction t*) (*auto*)

lemma *bst-wrt-le-if-bst*: *bst t* $\Longrightarrow$ *bst-wrt* $(\leq)$ *t*
using *bst-wrt-mono less-imp-le* **by** *blast*

lemma *bst-wrt-le-iff-sorted*: *bst-wrt* $(\leq)$ *t* $\longleftrightarrow$ *sorted* (*inorder t*)
apply (*induction t*)
apply(*simp*)
by (*fastforce simp*: *sorted-append intro*: *less-imp-le less-trans*)

lemma *bst-iff-sorted-wrt-less*: *bst t* $\longleftrightarrow$ *sorted-wrt* $(<)$ (*inorder t*)
apply (*induction t*)
apply *simp*
apply (*fastforce simp*: *sorted-wrt-append*)
done

95.12 *heap*

95.13 *mirror*

lemma *mirror-Leaf*[*simp*]: *mirror t* $= \langle\rangle \longleftrightarrow t = \langle\rangle$
by (*induction t*) *simp-all*

lemma *Leaf-mirror*[*simp*]: $\langle\rangle$ = *mirror t* $\longleftrightarrow t = \langle\rangle$
using *mirror-Leaf* **by** *fastforce*

lemma *size-mirror*[*simp*]: *size*(*mirror t*) = *size t*
by (*induction t*) *simp-all*

lemma *size1-mirror*[*simp*]: *size1*(*mirror t*) = *size1 t*
by (*simp add*: *size1-size*)

lemma *height-mirror*[*simp*]: *height*(*mirror t*) = *height t*
by (*induction t*) *simp-all*

lemma *min-height-mirror* [*simp*]: *min-height* (*mirror t*) = *min-height t*
by (*induction t*) *simp-all*

lemma *ipl-mirror* [*simp*]: *ipl* (*mirror t*) = *ipl t*
by (*induction t*) *simp-all*

lemma *inorder-mirror*: *inorder*(*mirror t*) = *rev*(*inorder t*)
by (*induction t*) *simp-all*

lemma *map-mirror*: *map-tree f* (*mirror t*) = *mirror* (*map-tree f t*)
by (*induction t*) *simp-all*

lemma *mirror-mirror*[*simp*]: *mirror*(*mirror t*) = *t*
by (*induction t*) *simp-all*

end

96 Multiset of Elements of Binary Tree

theory *Tree-Multiset*
imports *Multiset Tree*
begin

Kept separate from theory *HOL−Library.Tree* to avoid importing all of theory *HOL−Library.Multiset* into *HOL−Library.Tree*. Should be merged if *HOL−Library.Multiset* ever becomes part of *Main*.

fun *mset-tree* :: *'a tree* ⇒ *'a multiset* **where**
mset-tree Leaf = {#} |
mset-tree (*Node l a r*) = {#*a*#} + *mset-tree l* + *mset-tree r*

fun *subtrees-mset* :: *'a tree* ⇒ *'a tree multiset* **where**
subtrees-mset Leaf = {#*Leaf*#} |
subtrees-mset (*Node l x r*) = *add-mset* (*Node l x r*) (*subtrees-mset l* + *subtrees-mset r*)

lemma *mset-tree-empty-iff*[*simp*]: *mset-tree t* = {#} ⟷ *t* = *Leaf*
by (*cases t*) *auto*

lemma *set-mset-tree*[*simp*]: *set-mset* (*mset-tree t*) = *set-tree t*
by(*induction t*) *auto*

lemma *size-mset-tree*[*simp*]: *size*(*mset-tree t*) = *size t*
by(*induction t*) *auto*

lemma *mset-map-tree*: *mset-tree* (*map-tree f t*) = *image-mset f* (*mset-tree t*)
by (*induction t*) *auto*

lemma *mset-iff-set-tree*: *x* ∈# *mset-tree t* ⟷ *x* ∈ *set-tree t*
by(*induction t arbitrary*: *x*) *auto*

lemma *mset-preorder*[*simp*]: *mset* (*preorder t*) = *mset-tree t*
by (*induction t*) (*auto simp*: *ac-simps*)

lemma *mset-inorder*[*simp*]: *mset* (*inorder t*) = *mset-tree t*
by (*induction t*) (*auto simp*: *ac-simps*)

lemma *map-mirror*: *mset-tree* (*mirror t*) = *mset-tree t*
by (*induction t*) (*simp-all add*: *ac-simps*)

lemma *in-subtrees-mset-iff*[*simp*]: *s* ∈# *subtrees-mset t* ⟷ *s* ∈ *subtrees t*
by(*induction t*) *auto*

end

theory *Tree-Real*
imports
Complex-Main
Tree
begin

This theory is separate from *HOL−Library.Tree* because the former is discrete and builds on *Main* whereas this theory builds on *Complex-Main*.

lemma *size1-height-log*: *log 2 (size1 t) ≤ height t*
by (*simp add*: *log2-of-power-le size1-height*)

lemma *min-height-size1-log*: *min-height t ≤ log 2 (size1 t)*
by (*simp add*: *le-log2-of-power min-height-size1*)

lemma *size1-log-if-complete*: *complete t ⟹ height t = log 2 (size1 t)*
by (*simp add*: *size1-if-complete*)

lemma *min-height-size1-log-if-incomplete*:
¬ complete t ⟹ min-height t < log 2 (size1 t)
by (*simp add*: *less-log2-of-power min-height-size1-if-incomplete*)

lemma *min-height-acomplete*: **assumes** *acomplete t*
shows *min-height t = nat*(*floor*(*log 2 (size1 t)*))
proof *cases*
assume ∗: *complete t*
hence *size1 t = 2 ^ min-height t*
by (*simp add*: *complete-iff-height size1-if-complete*)
from *log2-of-power-eq*[*OF this*] **show** *?thesis* **by** *linarith*
next
assume ∗: *¬ complete t*
hence *height t = min-height t + 1*
using *assms min-height-le-height*[*of t*]
by(*auto simp*: *acomplete-def complete-iff-height*)
hence *size1 t < 2 ^ (min-height t + 1)* **by** (*metis* ∗ *size1-height-if-incomplete*)
from *floor-log-nat-eq-if*[*OF min-height-size1 this*] **show** *?thesis* **by** *simp*
qed

lemma *height-acomplete*: **assumes** *acomplete t*
shows *height t = nat*(*ceiling*(*log 2 (size1 t)*))
proof *cases*
assume ∗: *complete t*
hence *size1 t = 2 ^ height t* **by** (*simp add*: *size1-if-complete*)
from *log2-of-power-eq*[*OF this*] **show** *?thesis* **by** *linarith*
next
assume ∗: *¬ complete t*
hence ∗∗: *height t = min-height t + 1*
using *assms min-height-le-height*[*of t*]
by(*auto simp add*: *acomplete-def complete-iff-height*)

```
  hence size1 t ≤ 2 ^ (min-height t + 1) by (metis size1-height)
  from log2-of-power-le[OF this size1-ge0] min-height-size1-log-if-incomplete[OF *]
**
  show ?thesis by linarith
qed

lemma acomplete-Node-if-wbal1:
assumes acomplete l acomplete r size l = size r + 1
shows acomplete ⟨l, x, r⟩
proof −
  from assms(3) have [simp]: size1 l = size1 r + 1 by(simp add: size1-size)
  have nat ⌈log 2 (1 + size1 r)⌉ ≥ nat ⌈log 2 (size1 r)⌉
    by(rule nat-mono[OF ceiling-mono]) simp
  hence 1: height(Node l x r) = nat ⌈log 2 (1 + size1 r)⌉ + 1
    using height-acomplete[OF assms(1)] height-acomplete[OF assms(2)]
    by (simp del: nat-ceiling-le-eq add: max-def)
  have nat ⌊log 2 (1 + size1 r)⌋ ≥ nat ⌊log 2 (size1 r)⌋
    by(rule nat-mono[OF floor-mono]) simp
  hence 2: min-height(Node l x r) = nat ⌊log 2 (size1 r)⌋ + 1
    using min-height-acomplete[OF assms(1)] min-height-acomplete[OF assms(2)]
    by (simp)
  have size1 r ≥ 1 by(simp add: size1-size)
  then obtain i where i: 2 ^ i ≤ size1 r size1 r < 2 ^ (i + 1)
    using ex-power-ivl1[of 2 size1 r] by auto
  hence i1: 2 ^ i < size1 r + 1 size1 r + 1 ≤ 2 ^ (i + 1) by auto
  from 1 2 floor-log-nat-eq-if[OF i] ceiling-log-nat-eq-if[OF i1]
  show ?thesis by(simp add:acomplete-def)
qed

lemma acomplete-sym: acomplete ⟨l, x, r⟩ ⟹ acomplete ⟨r, y, l⟩
by(auto simp: acomplete-def)

lemma acomplete-Node-if-wbal2:
assumes acomplete l acomplete r abs(int(size l) − int(size r)) ≤ 1
shows acomplete ⟨l, x, r⟩
proof −
  have size l = size r ∨ (size l = size r + 1 ∨ size r = size l + 1) (is ?A ∨ ?B)
    using assms(3) by linarith
  thus ?thesis
  proof
    assume ?A
    thus ?thesis using assms(1,2)
      apply(simp add: acomplete-def min-def max-def)
      by (metis assms(1,2) acomplete-optimal le-antisym le-less)
  next
    assume ?B
    thus ?thesis
      by (meson assms(1,2) acomplete-sym acomplete-Node-if-wbal1)
  qed
```

qed

lemma *acomplete-if-wbalanced*: *wbalanced t* $\Longrightarrow$ *acomplete t*
proof(*induction t*)
  case *Leaf* **show** *?case* **by** (*simp add*: *acomplete-def*)
next
  case (*Node l x r*)
  thus *?case* **by**(*simp add*: *acomplete-Node-if-wbal2*)
qed

end

97 Unordered pairs

theory *Uprod* **imports** *Main* **begin**

typedef ('*a*, '*b*) *commute* = {*f* :: '*a* $\Rightarrow$ '*a* $\Rightarrow$ '*b*. $\forall$ *x y*. *f x y* = *f y x*}
  morphisms *apply-commute Abs-commute*
  by *auto*

setup-lifting *type-definition-commute*

lemma *apply-commute-commute*: *apply-commute f x y* = *apply-commute f y x*
by(*transfer*) *simp*

context includes *lifting-syntax* **begin**

lift-definition *rel-commute* :: ('*a* $\Rightarrow$ '*b* $\Rightarrow$ *bool*) $\Rightarrow$ ('*c* $\Rightarrow$ '*d* $\Rightarrow$ *bool*) $\Rightarrow$ ('*a*, '*c*) *commute* $\Rightarrow$ ('*b*, '*d*) *commute* $\Rightarrow$ *bool*
is $\lambda A\ B.\ A$ ===> A ===> B **.**

end

definition *eq-upair* :: ('*a* $\times$ '*a*) $\Rightarrow$ ('*a* $\times$ '*a*) $\Rightarrow$ *bool*
where *eq-upair* = (λ(*a*, *b*) (*c*, *d*). *a* = *c* $\wedge$ *b* = *d* $\vee$ *a* = *d* $\wedge$ *b* = *c*)

lemma *eq-upair-simps* [*simp*]:
  eq-upair (*a*, *b*) (*c*, *d*) $\longleftrightarrow$ *a* = *c* $\wedge$ *b* = *d* $\vee$ *a* = *d* $\wedge$ *b* = *c*
by(*simp add*: *eq-upair-def*)

lemma *equivp-eq-upair*: *equivp eq-upair*
by(*auto simp add*: *equivp-def fun-eq-iff*)

quotient-type '*a uprod* = '*a* $\times$ '*a* / *eq-upair* **by**(*rule equivp-eq-upair*)

lift-definition *Upair* :: '*a* $\Rightarrow$ '*a* $\Rightarrow$ '*a uprod* **is** *Pair* **parametric** *Pair-transfer*[*of A A* **for** *A*] **.**

lemma *uprod-exhaust* [*case-names Upair*, *cases type*: *uprod*]:

obtains *a b* **where** $x = Upair\ a\ b$
by *transfer fastforce*

lemma *Upair-inject* [*simp*]: $Upair\ a\ b = Upair\ c\ d \longleftrightarrow a = c \wedge b = d \vee a = d \wedge b = c$
by *transfer auto*

code-datatype *Upair*

lift-definition *case-uprod* :: $('a, 'b)\ commute \Rightarrow 'a\ uprod \Rightarrow 'b$ **is** *case-prod*
parametric *case-prod-transfer*[*of A A* **for** *A*] **by** *auto*

lemma *case-uprod-simps* [*simp*, *code*]: $case\text{-}uprod\ f\ (Upair\ x\ y) = apply\text{-}commute\ f\ x\ y$
by *transfer auto*

lemma *uprod-split*: $P\ (case\text{-}uprod\ f\ x) \longleftrightarrow (\forall a\ b.\ x = Upair\ a\ b \longrightarrow P\ (apply\text{-}commute\ f\ a\ b))$
by *transfer auto*

lemma *uprod-split-asm*: $P\ (case\text{-}uprod\ f\ x) \longleftrightarrow \neg\ (\exists a\ b.\ x = Upair\ a\ b \wedge \neg\ P\ (apply\text{-}commute\ f\ a\ b))$
by *transfer auto*

lift-definition *not-equal* :: $('a, bool)\ commute$ **is** $(\neq)$ **by** *auto*

lemma *apply-not-equal* [*simp*]: $apply\text{-}commute\ not\text{-}equal\ x\ y \longleftrightarrow x \neq y$
by *transfer simp*

definition *proper-uprod* :: $'a\ uprod \Rightarrow bool$
where $proper\text{-}uprod = case\text{-}uprod\ not\text{-}equal$

lemma *proper-uprod-simps* [*simp*, *code*]: $proper\text{-}uprod\ (Upair\ x\ y) \longleftrightarrow x \neq y$
by(*simp add*: *proper-uprod-def*)

context includes *lifting-syntax* **begin**

private lemma *set-uprod-parametric'*:
$(rel\text{-}prod\ A\ A ===> rel\text{-}set\ A)\ (\lambda(a, b).\ \{a, b\})\ (\lambda(a, b).\ \{a, b\})$
by *transfer-prover*

lift-definition *set-uprod* :: $'a\ uprod \Rightarrow 'a\ set$ **is** $\lambda(a, b).\ \{a, b\}$
parametric *set-uprod-parametric'* **by** *auto*

lemma *set-uprod-simps* [*simp*, *code*]: $set\text{-}uprod\ (Upair\ x\ y) = \{x, y\}$
by *transfer simp*

lemma *finite-set-uprod* [*simp*]: *finite* $(set\text{-}uprod\ x)$
by(*cases x*) *simp*

private lemma *map-uprod-parametric'*:
((A ===> B) ===> *rel-prod* A A ===> *rel-prod* B B) (λf. *map-prod* f f) (λf. *map-prod* f f)
by *transfer-prover*

lift-definition *map-uprod* :: ($'a \Rightarrow 'b$) $\Rightarrow$ $'a$ *uprod* $\Rightarrow$ $'b$ *uprod* **is** λf. *map-prod* f f
parametric *map-uprod-parametric'* **by** *auto*

lemma *map-uprod-simps* [*simp*, *code*]: *map-uprod* f (*Upair* x y) = *Upair* (f x) (f y)
by *transfer simp*

private lemma *rel-uprod-transfer'*:
((A ===> B ===> (=)) ===> *rel-prod* A A ===> *rel-prod* B B ===> (=))
(λR (a, b) (c, d). R a c $\wedge$ R b d $\vee$ R a d $\wedge$ R b c) (λR (a, b) (c, d). R a c $\wedge$ R b d $\vee$ R a d $\wedge$ R b c)
by *transfer-prover*

lift-definition *rel-uprod* :: ($'a \Rightarrow 'b \Rightarrow$ *bool*) $\Rightarrow$ $'a$ *uprod* $\Rightarrow$ $'b$ *uprod* $\Rightarrow$ *bool*
is λR (a, b) (c, d). R a c $\wedge$ R b d $\vee$ R a d $\wedge$ R b c **parametric** *rel-uprod-transfer'*
by *auto*

lemma *rel-uprod-simps* [*simp*, *code*]:
rel-uprod R (*Upair* a b) (*Upair* c d) $\longleftrightarrow$ R a c $\wedge$ R b d $\vee$ R a d $\wedge$ R b c
by *transfer auto*

lemma *Upair-parametric* [*transfer-rule*]: (A ===> A ===> *rel-uprod* A) *Upair* *Upair*
unfolding *rel-fun-def* **by** *transfer auto*

lemma *case-uprod-parametric* [*transfer-rule*]:
(*rel-commute* A B ===> *rel-uprod* A ===> B) *case-uprod* *case-uprod*
unfolding *rel-fun-def* **by** *transfer*(*force dest*: *rel-funD*)

end

bnf *uprod*: $'a$ *uprod*
map: *map-uprod*
sets: *set-uprod*
bd: *natLeq*
rel: *rel-uprod*
proof −
show *map-uprod* *id* = *id* **unfolding** *fun-eq-iff* **by** *transfer auto*
show *map-uprod* (g $\circ$ f) = *map-uprod* g $\circ$ *map-uprod* f **for** f :: $'a \Rightarrow 'b$ **and** g :: $'b \Rightarrow 'c$
unfolding *fun-eq-iff* **by** *transfer auto*
show *map-uprod* f x = *map-uprod* g x **if** $\bigwedge z.\ z \in$ *set-uprod* $x \Longrightarrow f\ z = g\ z$

for $f :: 'a \Rightarrow 'b$ **and** $g\ x$ **using** *that* **by** *transfer auto*
show *set-uprod* $\circ$ *map-uprod* $f = (`)\ f \circ$ *set-uprod* **for** $f :: 'a \Rightarrow 'b$ **by** *transfer auto*
show *card-order natLeq* **by**(*rule natLeq-card-order*)
show *BNF-Cardinal-Arithmetic.cinfinite natLeq* **by**(*rule natLeq-cinfinite*)
show *ordLeq3* (*card-of* (*set-uprod x*)) *natLeq* **for** $x :: 'a$ *uprod*
by (*auto simp flip*: *finite-iff-ordLess-natLeq intro*: *ordLess-imp-ordLeq*)
show *rel-uprod R OO rel-uprod S* $\leq$ *rel-uprod* (*R OO S*)
for $R :: 'a \Rightarrow 'b \Rightarrow$ *bool* **and** $S :: 'b \Rightarrow 'c \Rightarrow$ *bool* **by**(*rule predicate2I*)(*transfer*; *auto*)
show *rel-uprod R* = $(\lambda x\ y.\ \exists z.$ *set-uprod* $z \subseteq \{(x, y).\ R\ x\ y\} \wedge$ *map-uprod fst* $z = x \wedge$ *map-uprod snd* $z = y)$
for $R :: 'a \Rightarrow 'b \Rightarrow$ *bool* **by** *transfer*(*auto simp add*: *fun-eq-iff*)
qed

lemma *pred-uprod-code* [*simp*, *code*]: *pred-uprod P* (*Upair x y*) $\longleftrightarrow$ *P x* $\wedge$ *P y*
by(*simp add*: *pred-uprod-def*)

instantiation *uprod* :: (*equal*) *equal* **begin**

definition *equal-uprod* :: $'a$ *uprod* $\Rightarrow$ $'a$ *uprod* $\Rightarrow$ *bool*
where *equal-uprod* = (=)

lemma *equal-uprod-code* [*code*]:
HOL.equal (*Upair x y*) (*Upair z u*) $\longleftrightarrow$ $x = z \wedge y = u \vee x = u \wedge y = z$
unfolding *equal-uprod-def* **by** *simp*

instance by *standard*(*simp add*: *equal-uprod-def*)
end

quickcheck-generator *uprod constructors*: *Upair*

lemma *UNIV-uprod*: *UNIV* = $(\lambda x.$ *Upair x x*) ` *UNIV* $\cup$ $(\lambda(x, y).$ *Upair x y*) ` *Sigma UNIV* $(\lambda x.$ *UNIV* $-$ $\{x\})$
apply(*rule set-eqI*)
subgoal for *x* **by**(*cases x*) *auto*
done

context begin
private lift-definition *upair-inv* :: $'a$ *uprod* $\Rightarrow$ $'a$
is $\lambda(x, y).$ *if x* = *y then x else undefined* **by** *auto*

lemma *finite-UNIV-prod* [*simp*]:
finite (*UNIV* :: $'a$ *uprod set*) $\longleftrightarrow$ *finite* (*UNIV* :: $'a$ *set*) (**is** *?lhs* = *?rhs*)
proof
assume *?lhs*
hence *finite* (*range* $(\lambda x :: 'a.$ *Upair x x*)) **by**(*rule finite-subset*[*rotated*]) *simp*
hence *finite* (*upair-inv* ` *range* $(\lambda x :: 'a.$ *Upair x x*)) **by**(*rule finite-imageI*)
also have *upair-inv* (*Upair x x*) = *x* **for** $x :: 'a$ **by** *transfer simp*

then have *upair-inv ‘ range* (λ*x* :: *'a*. *Upair x x*) = *UNIV* **by**(*auto simp add*: *image-image*)
finally show *?rhs* **.**
qed(*simp add*: *UNIV-uprod*)

end

lemma *card-UNIV-uprod*:
card (*UNIV* :: *'a uprod set*) = *card* (*UNIV* :: *'a set*) ∗ (*card* (*UNIV* :: *'a set*) + *1*) *div 2*
(**is** *?UPROD* = *?A* ∗ - *div* -)
proof(*cases finite* (*UNIV* :: *'a set*))
case *True*
from *True* **obtain** *f* :: *nat* ⇒ *'a* **where** *bij*: *bij-betw f* {*0*..<*?A*} *UNIV*
by (*blast dest*: *ex-bij-betw-nat-finite*)
hence [*simp*]: *f ‘* {*0*..<*?A*} = *UNIV* **by**(*rule bij-betw-imp-surj-on*)
have *UNIV* = (λ(*x*, *y*). *Upair* (*f x*) (*f y*)) ‘ (*SIGMA x*:{*0*..<*?A*}. {..*x*})
apply(*rule set-eqI*)
subgoal for *x*
apply(*cases x*)
apply(*clarsimp*)
subgoal for *a b*
apply(*cases inv-into* {*0*..<*?A*} *f a* ≤ *inv-into* {*0*..<*?A*} *f b*)
subgoal by(*rule rev-image-eqI*[**where** *x*=(*inv-into* {*0*..<*?A*} *f* -, *inv-into* {*0*..<*?A*} *f* -)])
(*auto simp add*: *inv-into-into*[**where** *A*={*0*..<*?A*} **and** *f*=*f*, *simplified*] *intro*: *f-inv-into-f*[**where** *f*=*f*, *symmetric*])
subgoal
apply(*simp only*: *not-le*)
apply(*drule less-imp-le*)
apply(*rule rev-image-eqI*[**where** *x*=(*inv-into* {*0*..<*?A*} *f* -, *inv-into* {*0*..<*?A*} *f* -)])
apply(*auto simp add*: *inv-into-into*[**where** *A*={*0*..<*?A*} **and** *f*=*f*, *simplified*] *intro*: *f-inv-into-f*[**where** *f*=*f*, *symmetric*])
done
done
done
done
hence *?UPROD* = *card* ... **by** *simp*
also have ... = *card* (*SIGMA x*:{*0*..<*?A*}. {..*x*})
apply(*rule card-image*)
using *bij*[*THEN bij-betw-imp-inj-on*]
by(*simp add*: *inj-on-def Ball-def*)(*metis leD le-eq-less-or-eq le-less-trans*)
also have ... = *sum Suc* {*0*..<*?A*}
by (*subst card-SigmaI*) *simp-all*
also have ... = *sum of-nat* {*Suc 0*..*?A*}
using *sum.atLeastLessThan-reindex* [*symmetric*, *of Suc 0 ?A id*]
by (*simp del*: *sum.op-ivl-Suc add*: *atLeastLessThanSuc-atLeastAtMost*)
also have ... = *?A* ∗ (*?A* + *1*) *div 2*

using *gauss-sum-from-Suc-0* [*of ?A*, **where** *?'a = nat*] **by** *simp*
finally show *?thesis* **.**
qed *simp*

end

98 A type of finite bit strings

theory *Word*
imports
HOL−Library.Type-Length
begin

98.1 Preliminaries

lemma *signed-take-bit-decr-length-iff*:
‹*signed-take-bit* (*LENGTH*('*a*::*len*) − *Suc 0*) *k* = *signed-take-bit* (*LENGTH*('*a*) − *Suc 0*) *l*
⟷ *take-bit LENGTH*('*a*) *k* = *take-bit LENGTH*('*a*) *l*›
by (*cases* ‹*LENGTH*('*a*)›)
(*simp-all add*: *signed-take-bit-eq-iff-take-bit-eq*)

98.2 Fundamentals

98.2.1 Type definition

quotient-type (**overloaded**) '*a word* = *int* / ‹λ*k l*. *take-bit LENGTH*('*a*) *k* = *take-bit LENGTH*('*a*::*len*) *l*›
morphisms *rep Word* **by** (*auto intro*!: *equivpI reflpI sympI transpI*)

hide-const (**open**) *rep* — only for foundational purpose
hide-const (**open**) *Word* — only for code generation

98.2.2 Basic arithmetic

instantiation *word* :: (*len*) *comm-ring-1*
begin

lift-definition *zero-word* :: ‹'*a word*›
is *0* **.**

lift-definition *one-word* :: ‹'*a word*›
is *1* **.**

lift-definition *plus-word* :: ‹'*a word* ⇒ '*a word* ⇒ '*a word*›
is ‹(+)›
by (*auto simp add*: *take-bit-eq-mod intro*: *mod-add-cong*)

lift-definition *minus-word* :: ‹'*a word* ⇒ '*a word* ⇒ '*a word*›
is ‹(−)›

by (*auto simp add*: *take-bit-eq-mod intro*: *mod-diff-cong*)

lift-definition *uminus-word* :: ‹*'a word* ⇒ *'a word*›
is *uminus*
by (*auto simp add*: *take-bit-eq-mod intro*: *mod-minus-cong*)

lift-definition *times-word* :: ‹*'a word* ⇒ *'a word* ⇒ *'a word*›
is ‹(∗)›
by (*auto simp add*: *take-bit-eq-mod intro*: *mod-mult-cong*)

instance
by (*standard*; *transfer*) (*simp-all add*: *algebra-simps*)

end

context
includes *lifting-syntax*
notes
power-transfer [*transfer-rule*]
transfer-rule-of-bool [*transfer-rule*]
transfer-rule-numeral [*transfer-rule*]
transfer-rule-of-nat [*transfer-rule*]
transfer-rule-of-int [*transfer-rule*]
begin

lemma *power-transfer-word* [*transfer-rule*]:
‹(*pcr-word* ===> (=) ===> *pcr-word*) (^) (^)›
by *transfer-prover*

lemma [*transfer-rule*]:
‹((=) ===> *pcr-word*) *of-bool of-bool*›
by *transfer-prover*

lemma [*transfer-rule*]:
‹((=) ===> *pcr-word*) *numeral numeral*›
by *transfer-prover*

lemma [*transfer-rule*]:
‹((=) ===> *pcr-word*) *int of-nat*›
by *transfer-prover*

lemma [*transfer-rule*]:
‹((=) ===> *pcr-word*) (λ*k*. *k*) *of-int*›
proof −
have ‹((=) ===> *pcr-word*) *of-int of-int*›
by *transfer-prover*
then show *?thesis* **by** (*simp add*: *id-def*)
qed

lemma [*transfer-rule*]:
‹(*pcr-word* ===> (⟷)) *even* ((*dvd*) *2* :: *'a*::*len word* ⇒ *bool*)›
proof −
have *even-word-unfold*: *even k* ⟷ (∃ *l*. *take-bit LENGTH*('*a*) *k* = *take-bit LENGTH*('*a*) (*2* ∗ *l*)) (**is** *?P* ⟷ *?Q*)
for *k* :: *int*
proof
assume *?P*
then show *?Q*
by *auto*
next
assume *?Q*
then obtain *l* **where** *take-bit LENGTH*('*a*) *k* = *take-bit LENGTH*('*a*) (*2* ∗ *l*) **..**
then have *even* (*take-bit LENGTH*('*a*) *k*)
by *simp*
then show *?P*
by *simp*
qed
show *?thesis* **by** (*simp only*: *even-word-unfold* [*abs-def*] *dvd-def* [**where** *?'a* = *'a word*, *abs-def*])
transfer-prover
qed

end

lemma *exp-eq-zero-iff* [*simp*]:
‹*2* ^ *n* = (*0* :: *'a*::*len word*) ⟷ *n* ≥ *LENGTH*('*a*)›
by *transfer auto*

lemma *word-exp-length-eq-0* [*simp*]:
‹(*2* :: *'a*::*len word*) ^ *LENGTH*('*a*) = *0*›
by *simp*

98.2.3 Basic tool setup

ML-file ‹*Tools/word-lib.ML*›

98.2.4 Basic code generation setup

context
begin

qualified lift-definition *the-int* :: ‹*'a*::*len word* ⇒ *int*›
is ‹*take-bit LENGTH*('*a*)› **.**

end

lemma [*code abstype*]:
‹*Word.Word* (*Word.the-int w*) = *w*›

by *transfer simp*

lemma *Word-eq-word-of-int* [*code-post*, *simp*]:
‹*Word.Word* = *of-int*›
by (*rule*; *transfer*) *simp*

quickcheck-generator *word*
constructors:
‹*0* :: *'a*::*len word*›,
‹*numeral* :: *num* ⇒ *'a*::*len word*›

instantiation *word* :: (*len*) *equal*
begin

lift-definition *equal-word* :: ‹*'a word* ⇒ *'a word* ⇒ *bool*›
is ‹λ*k l*. *take-bit LENGTH*(*'a*) *k* = *take-bit LENGTH*(*'a*) *l*›
by *simp*

instance
by (*standard*; *transfer*) *rule*

end

lemma [*code*]:
‹*HOL.equal v w* ⟷ *HOL.equal* (*Word.the-int v*) (*Word.the-int w*)›
by *transfer* (*simp add*: *equal*)

lemma [*code*]:
‹*Word.the-int 0* = *0*›
by *transfer simp*

lemma [*code*]:
‹*Word.the-int 1* = *1*›
by *transfer simp*

lemma [*code*]:
‹*Word.the-int* (*v* + *w*) = *take-bit LENGTH*(*'a*) (*Word.the-int v* + *Word.the-int w*)›
for *v w* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *take-bit-add*)

lemma [*code*]:
‹*Word.the-int* (− *w*) = (*let k* = *Word.the-int w in if w* = *0 then 0 else 2* ^ *LENGTH*(*'a*) − *k*)›
for *w* :: ‹*'a*::*len word*›
by *transfer* (*auto simp add*: *take-bit-eq-mod zmod-zminus1-eq-if*)

lemma [*code*]:
‹*Word.the-int* (*v* − *w*) = *take-bit LENGTH*(*'a*) (*Word.the-int v* − *Word.the-int*

w)›
for *v w* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *take-bit-diff*)

lemma [*code*]:
‹*Word.the-int* (*v* ∗ *w*) = *take-bit LENGTH*('*a*) (*Word.the-int v* ∗ *Word.the-int w*)›
for *v w* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *take-bit-mult*)

98.2.5 Basic conversions

abbreviation *word-of-nat* :: ‹*nat* ⇒ *'a*::*len word*›
where ‹*word-of-nat* ≡ *of-nat*›

abbreviation *word-of-int* :: ‹*int* ⇒ *'a*::*len word*›
where ‹*word-of-int* ≡ *of-int*›

lemma *word-of-nat-eq-iff*:
‹*word-of-nat m* = (*word-of-nat n* :: *'a*::*len word*) ⟷ *take-bit LENGTH*('*a*) *m* = *take-bit LENGTH*('*a*) *n*›
by *transfer* (*simp add*: *take-bit-of-nat*)

lemma *word-of-int-eq-iff*:
‹*word-of-int k* = (*word-of-int l* :: *'a*::*len word*) ⟷ *take-bit LENGTH*('*a*) *k* = *take-bit LENGTH*('*a*) *l*›
by *transfer rule*

lemma *word-of-nat-eq-0-iff*:
‹*word-of-nat n* = (*0* :: *'a*::*len word*) ⟷ *2* ^ *LENGTH*('*a*) *dvd n*›
using *word-of-nat-eq-iff* [**where** *?'a* = *'a*, *of n 0*] **by** (*simp add*: *take-bit-eq-0-iff*)

lemma *word-of-int-eq-0-iff*:
‹*word-of-int k* = (*0* :: *'a*::*len word*) ⟷ *2* ^ *LENGTH*('*a*) *dvd k*›
using *word-of-int-eq-iff* [**where** *?'a* = *'a*, *of k 0*] **by** (*simp add*: *take-bit-eq-0-iff*)

context *semiring-1*
begin

lift-definition *unsigned* :: ‹*'b*::*len word* ⇒ *'a*›
is ‹*of-nat* ∘ *nat* ∘ *take-bit LENGTH*('*b*)›
by *simp*

lemma *unsigned-0* [*simp*]:
‹*unsigned 0* = *0*›
by *transfer simp*

lemma *unsigned-1* [*simp*]:
‹*unsigned 1* = *1*›

by *transfer simp*

lemma *unsigned-numeral* [*simp*]:
‹*unsigned* (*numeral n* :: *'b*::*len word*) = *of-nat* (*take-bit LENGTH*('*b*) (*numeral n*))›
by *transfer* (*simp add*: *nat-take-bit-eq*)

lemma *unsigned-neg-numeral* [*simp*]:
‹*unsigned* (− *numeral n* :: *'b*::*len word*) = *of-nat* (*nat* (*take-bit LENGTH*('*b*) (− *numeral n*)))›
by *transfer simp*

end

context *semiring-1*
begin

lemma *unsigned-of-nat*:
‹*unsigned* (*word-of-nat n* :: *'b*::*len word*) = *of-nat* (*take-bit LENGTH*('*b*) *n*)›
by *transfer* (*simp add*: *nat-eq-iff take-bit-of-nat*)

lemma *unsigned-of-int*:
‹*unsigned* (*word-of-int k* :: *'b*::*len word*) = *of-nat* (*nat* (*take-bit LENGTH*('*b*) *k*))›
by *transfer simp*

end

context *semiring-char-0*
begin

lemma *unsigned-word-eqI*:
‹*v* = *w*› **if** ‹*unsigned v* = *unsigned w*›
using *that* **by** *transfer* (*simp add*: *eq-nat-nat-iff*)

lemma *word-eq-iff-unsigned*:
‹*v* = *w* ⟷ *unsigned v* = *unsigned w*›
by (*auto intro*: *unsigned-word-eqI*)

lemma *inj-unsigned* [*simp*]:
‹*inj unsigned*›
by (*rule injI*) (*simp add*: *unsigned-word-eqI*)

lemma *unsigned-eq-0-iff*:
‹*unsigned w* = *0* ⟷ *w* = *0*›
using *word-eq-iff-unsigned* [*of w 0*] **by** *simp*

end

context *ring-1*

begin

lift-definition *signed* :: ‹*'b*::*len word* ⇒ *'a*›
is ‹*of-int* ∘ *signed-take-bit* (*LENGTH*(*'b*) − *Suc 0*)›
by (*simp flip*: *signed-take-bit-decr-length-iff*)

lemma *signed-0* [*simp*]:
‹*signed 0* = *0*›
by *transfer simp*

lemma *signed-1* [*simp*]:
‹*signed* (*1* :: *'b*::*len word*) = (*if LENGTH*(*'b*) = *1 then* − *1 else 1*)›
by (*transfer fixing*: *uminus*; *cases* ‹*LENGTH*(*'b*)›) (*auto dest*: *gr0-implies-Suc*)

lemma *signed-minus-1* [*simp*]:
‹*signed* (− *1* :: *'b*::*len word*) = − *1*›
by (*transfer fixing*: *uminus*) *simp*

lemma *signed-numeral* [*simp*]:
‹*signed* (*numeral n* :: *'b*::*len word*) = *of-int* (*signed-take-bit* (*LENGTH*(*'b*) − *1*) (*numeral n*))›
by *transfer simp*

lemma *signed-neg-numeral* [*simp*]:
‹*signed* (− *numeral n* :: *'b*::*len word*) = *of-int* (*signed-take-bit* (*LENGTH*(*'b*) − *1*) (− *numeral n*))›
by *transfer simp*

lemma *signed-of-nat*:
‹*signed* (*word-of-nat n* :: *'b*::*len word*) = *of-int* (*signed-take-bit* (*LENGTH*(*'b*) − *Suc 0*) (*int n*))›
by *transfer simp*

lemma *signed-of-int*:
‹*signed* (*word-of-int n* :: *'b*::*len word*) = *of-int* (*signed-take-bit* (*LENGTH*(*'b*) − *Suc 0*) *n*)›
by *transfer simp*

end

context *ring-char-0*
begin

lemma *signed-word-eqI*:
‹*v* = *w*› **if** ‹*signed v* = *signed w*›
using *that* **by** *transfer* (*simp flip*: *signed-take-bit-decr-length-iff*)

lemma *word-eq-iff-signed*:
‹*v* = *w* ⟷ *signed v* = *signed w*›

by (*auto intro*: *signed-word-eqI*)

lemma *inj-signed* [*simp*]:
‹*inj signed*›
by (*rule injI*) (*simp add*: *signed-word-eqI*)

lemma *signed-eq-0-iff*:
‹*signed w* = *0* ⟷ *w* = *0*›
using *word-eq-iff-signed* [*of w 0*] **by** *simp*

end

abbreviation *unat* :: ‹'*a*::*len word* ⇒ *nat*›
where ‹*unat* ≡ *unsigned*›

abbreviation *uint* :: ‹'*a*::*len word* ⇒ *int*›
where ‹*uint* ≡ *unsigned*›

abbreviation *sint* :: ‹'*a*::*len word* ⇒ *int*›
where ‹*sint* ≡ *signed*›

abbreviation *ucast* :: ‹'*a*::*len word* ⇒ '*b*::*len word*›
where ‹*ucast* ≡ *unsigned*›

abbreviation *scast* :: ‹'*a*::*len word* ⇒ '*b*::*len word*›
where ‹*scast* ≡ *signed*›

context
includes *lifting-syntax*
begin

lemma [*transfer-rule*]:
‹(*pcr-word* ===> (=)) (*nat* ∘ *take-bit LENGTH*('*a*)) (*unat* :: '*a*::*len word* ⇒ *nat*)›
using *unsigned.transfer* [**where** *?'a* = *nat*] **by** *simp*

lemma [*transfer-rule*]:
‹(*pcr-word* ===> (=)) (*take-bit LENGTH*('*a*)) (*uint* :: '*a*::*len word* ⇒ *int*)›
using *unsigned.transfer* [**where** *?'a* = *int*] **by** (*simp add*: *comp-def*)

lemma [*transfer-rule*]:
‹(*pcr-word* ===> (=)) (*signed-take-bit* (*LENGTH*('*a*) − *Suc 0*)) (*sint* :: '*a*::*len word* ⇒ *int*)›
using *signed.transfer* [**where** *?'a* = *int*] **by** *simp*

lemma [*transfer-rule*]:
‹(*pcr-word* ===> *pcr-word*) (*take-bit LENGTH*('*a*)) (*ucast* :: '*a*::*len word* ⇒ '*b*::*len word*)›
proof (*rule rel-funI*)

fix *k* :: *int* **and** *w* :: ‹*'a word*›
assume ‹*pcr-word k w*›
then have ‹*w = word-of-int k*›
by (*simp add*: *pcr-word-def cr-word-def relcompp-apply*)
moreover have ‹*pcr-word* (*take-bit LENGTH('a) k*) (*ucast* (*word-of-int k* :: *'a word*))›
by *transfer* (*simp add*: *pcr-word-def cr-word-def relcompp-apply*)
ultimately show ‹*pcr-word* (*take-bit LENGTH('a) k*) (*ucast w*)›
by *simp*
qed

lemma [*transfer-rule*]:
‹(*pcr-word* ===> *pcr-word*) (*signed-take-bit* (*LENGTH('a) − Suc 0*)) (*scast* :: *'a*::*len word* ⇒ *'b*::*len word*)›
proof (*rule rel-funI*)
fix *k* :: *int* **and** *w* :: ‹*'a word*›
assume ‹*pcr-word k w*›
then have ‹*w = word-of-int k*›
by (*simp add*: *pcr-word-def cr-word-def relcompp-apply*)
moreover have ‹*pcr-word* (*signed-take-bit* (*LENGTH('a) − Suc 0*) *k*) (*scast* (*word-of-int k* :: *'a word*))›
by *transfer* (*simp add*: *pcr-word-def cr-word-def relcompp-apply*)
ultimately show ‹*pcr-word* (*signed-take-bit* (*LENGTH('a) − Suc 0*) *k*) (*scast w*)›
by *simp*
qed

end

lemma *of-nat-unat* [*simp*]:
‹*of-nat* (*unat w*) = *unsigned w*›
by *transfer simp*

lemma *of-int-uint* [*simp*]:
‹*of-int* (*uint w*) = *unsigned w*›
by *transfer simp*

lemma *of-int-sint* [*simp*]:
‹*of-int* (*sint a*) = *signed a*›
by *transfer* (*simp-all add*: *take-bit-signed-take-bit*)

lemma *nat-uint-eq* [*simp*]:
‹*nat* (*uint w*) = *unat w*›
by *transfer simp*

lemma *sgn-uint-eq* [*simp*]:
‹*sgn* (*uint w*) = *of-bool* (*w ≠ 0*)›
by *transfer* (*simp add*: *less-le*)

Aliasses only for code generation

context
begin

qualified lift-definition *of-int* :: ‹*int* ⇒ *'a*::*len word*›
is ‹*take-bit LENGTH*(*'a*)› **.**

qualified lift-definition *of-nat* :: ‹*nat* ⇒ *'a*::*len word*›
is ‹*int* ∘ *take-bit LENGTH*(*'a*)› **.**

qualified lift-definition *the-nat* :: ‹*'a*::*len word* ⇒ *nat*›
is ‹*nat* ∘ *take-bit LENGTH*(*'a*)› **by** *simp*

qualified lift-definition *the-signed-int* :: ‹*'a*::*len word* ⇒ *int*›
is ‹*signed-take-bit* (*LENGTH*(*'a*) − *Suc 0*)› **by** (*simp add*: *signed-take-bit-decr-length-iff*)

qualified lift-definition *cast* :: ‹*'a*::*len word* ⇒ *'b*::*len word*›
is ‹*take-bit LENGTH*(*'a*)› **by** *simp*

qualified lift-definition *signed-cast* :: ‹*'a*::*len word* ⇒ *'b*::*len word*›
is ‹*signed-take-bit* (*LENGTH*(*'a*) − *Suc 0*)› **by** (*metis signed-take-bit-decr-length-iff*)

end

lemma [*code-abbrev*, *simp*]:
‹*Word.the-int* = *uint*›
by *transfer rule*

lemma [*code*]:
‹*Word.the-int* (*Word.of-int k* :: *'a*::*len word*) = *take-bit LENGTH*(*'a*) *k*›
by *transfer simp*

lemma [*code-abbrev*, *simp*]:
‹*Word.of-int* = *word-of-int*›
by (*rule*; *transfer*) *simp*

lemma [*code*]:
‹*Word.the-int* (*Word.of-nat n* :: *'a*::*len word*) = *take-bit LENGTH*(*'a*) (*int n*)›
by *transfer* (*simp add*: *take-bit-of-nat*)

lemma [*code-abbrev*, *simp*]:
‹*Word.of-nat* = *word-of-nat*›
by (*rule*; *transfer*) (*simp add*: *take-bit-of-nat*)

lemma [*code*]:
‹*Word.the-nat w* = *nat* (*Word.the-int w*)›
by *transfer simp*

lemma [*code-abbrev*, *simp*]:
‹*Word.the-nat* = *unat*›

by (*rule*; *transfer*) *simp*

lemma [*code*]:
‹*Word.the-signed-int w* = *signed-take-bit* (*LENGTH*('*a*) − *Suc 0*) (*Word.the-int w*)›
for *w* :: ‹'*a*::*len word*›
by *transfer* (*simp add*: *signed-take-bit-take-bit*)

lemma [*code-abbrev*, *simp*]:
‹*Word.the-signed-int* = *sint*›
by (*rule*; *transfer*) *simp*

lemma [*code*]:
‹*Word.the-int* (*Word.cast w* :: '*b*::*len word*) = *take-bit LENGTH*('*b*) (*Word.the-int w*)›
for *w* :: ‹'*a*::*len word*›
by *transfer simp*

lemma [*code-abbrev*, *simp*]:
‹*Word.cast* = *ucast*›
by (*rule*; *transfer*) *simp*

lemma [*code*]:
‹*Word.the-int* (*Word.signed-cast w* :: '*b*::*len word*) = *take-bit LENGTH*('*b*) (*Word.the-signed-int w*)›
for *w* :: ‹'*a*::*len word*›
by *transfer simp*

lemma [*code-abbrev*, *simp*]:
‹*Word.signed-cast* = *scast*›
by (*rule*; *transfer*) *simp*

lemma [*code*]:
‹*unsigned w* = *of-nat* (*nat* (*Word.the-int w*))›
by *transfer simp*

lemma [*code*]:
‹*signed w* = *of-int* (*Word.the-signed-int w*)›
by *transfer simp*

98.2.6 Basic ordering

instantiation *word* :: (*len*) *linorder*
begin

lift-definition *less-eq-word* :: '*a word* ⇒ '*a word* ⇒ *bool*
is λ*a b*. *take-bit LENGTH*('*a*) *a* ≤ *take-bit LENGTH*('*a*) *b*
by *simp*

lift-definition *less-word* :: *'a word* ⇒ *'a word* ⇒ *bool*
is λ*a b*. *take-bit LENGTH('a) a* < *take-bit LENGTH('a) b*
by *simp*

instance
by (*standard*; *transfer*) *auto*

end

interpretation *word-order*: *ordering-top* ‹(≤)› ‹(<)› ‹− *1* :: *'a*::*len word*›
by (*standard*; *transfer*) (*simp add*: *take-bit-eq-mod zmod-minus1*)

interpretation *word-coorder*: *ordering-top* ‹(≥)› ‹(>)› ‹*0* :: *'a*::*len word*›
by (*standard*; *transfer*) *simp*

lemma *word-of-nat-less-eq-iff*:
‹*word-of-nat m* ≤ (*word-of-nat n* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) m* ≤ *take-bit LENGTH('a) n*›
by *transfer* (*simp add*: *take-bit-of-nat*)

lemma *word-of-int-less-eq-iff*:
‹*word-of-int k* ≤ (*word-of-int l* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) k* ≤ *take-bit LENGTH('a) l*›
by *transfer rule*

lemma *word-of-nat-less-iff*:
‹*word-of-nat m* < (*word-of-nat n* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) m* < *take-bit LENGTH('a) n*›
by *transfer* (*simp add*: *take-bit-of-nat*)

lemma *word-of-int-less-iff*:
‹*word-of-int k* < (*word-of-int l* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) k* < *take-bit LENGTH('a) l*›
by *transfer rule*

lemma *word-le-def* [*code*]:
a ≤ *b* ⟷ *uint a* ≤ *uint b*
by *transfer rule*

lemma *word-less-def* [*code*]:
a < *b* ⟷ *uint a* < *uint b*
by *transfer rule*

lemma *word-greater-zero-iff*:
‹*a* > *0* ⟷ *a* ≠ *0*› **for** *a* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *less-le*)

lemma *of-nat-word-less-eq-iff*:
‹*of-nat m* ≤ (*of-nat n* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) m* ≤ *take-bit*

LENGTH('a) n›
by *transfer* (*simp add*: *take-bit-of-nat*)

lemma *of-nat-word-less-iff*:
‹*of-nat m* < (*of-nat n* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) m* < *take-bit LENGTH('a) n*›
by *transfer* (*simp add*: *take-bit-of-nat*)

lemma *of-int-word-less-eq-iff*:
‹*of-int k* ≤ (*of-int l* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) k* ≤ *take-bit LENGTH('a) l*›
by *transfer rule*

lemma *of-int-word-less-iff*:
‹*of-int k* < (*of-int l* :: *'a*::*len word*) ⟷ *take-bit LENGTH('a) k* < *take-bit LENGTH('a) l*›
by *transfer rule*

98.3 Enumeration

lemma *inj-on-word-of-nat*:
‹*inj-on* (*word-of-nat* :: *nat* ⇒ *'a*::*len word*) {*0*..<*2* ^ *LENGTH('a)*}›
by (*rule inj-onI*; *transfer*) (*simp-all add*: *take-bit-int-eq-self*)

lemma *UNIV-word-eq-word-of-nat*:
‹(*UNIV* :: *'a*::*len word set*) = *word-of-nat* ` {*0*..<*2* ^ *LENGTH('a)*}› (**is** ‹- = *?A*›)
proof
show ‹*word-of-nat* ` {*0*..<*2* ^ *LENGTH('a)*} ⊆ *UNIV*›
by *simp*
show ‹*UNIV* ⊆ *?A*›
proof
fix *w* :: ‹*'a word*›
show ‹*w* ∈ (*word-of-nat* ` {*0*..<*2* ^ *LENGTH('a)*} :: *'a word set*)›
by (*rule image-eqI* [*of* - - ‹*unat w*›]; *transfer*) *simp-all*
qed
qed

instantiation *word* :: (*len*) *enum*
begin

definition *enum-word* :: ‹*'a word list*›
where ‹*enum-word* = *map word-of-nat* [*0*..<*2* ^ *LENGTH('a)*]›

definition *enum-all-word* :: ‹(*'a word* ⇒ *bool*) ⇒ *bool*›
where ‹*enum-all-word* = *Ball UNIV*›

definition *enum-ex-word* :: ‹(*'a word* ⇒ *bool*) ⇒ *bool*›
where ‹*enum-ex-word* = *Bex UNIV*›

lemma [*code*]:
‹*Enum.enum-all P* ⟷ *Ball UNIV P*›
‹*Enum.enum-ex P* ⟷ *Bex UNIV P*› **for** *P* :: ‹*'a word* ⇒ *bool*›
by (*simp-all add*: *enum-all-word-def enum-ex-word-def*)

instance
by *standard* (*simp-all add*: *UNIV-word-eq-word-of-nat inj-on-word-of-nat enum-word-def enum-all-word-def enum-ex-word-def distinct-map*)

end

98.4 Bit-wise operations

instantiation *word* :: (*len*) *semiring-modulo*
begin

lift-definition *divide-word* :: ‹*'a word* ⇒ *'a word* ⇒ *'a word*›
is ‹λ*a b*. *take-bit LENGTH*('*a*) *a div take-bit LENGTH*('*a*) *b*›
by *simp*

lift-definition *modulo-word* :: ‹*'a word* ⇒ *'a word* ⇒ *'a word*›
is ‹λ*a b*. *take-bit LENGTH*('*a*) *a mod take-bit LENGTH*('*a*) *b*›
by *simp*

instance proof
show *a div b* ∗ *b* + *a mod b* = *a* **for** *a b* :: *'a word*
proof *transfer*
fix *k l* :: *int*
define *r* :: *int* **where** *r* = *2* ^ *LENGTH*('*a*)
then have *r*: *take-bit LENGTH*('*a*) *k* = *k mod r* **for** *k*
by (*simp add*: *take-bit-eq-mod*)
have *k mod r* = ((*k mod r*) *div* (*l mod r*) ∗ (*l mod r*)
+ (*k mod r*) *mod* (*l mod r*)) *mod r*
by (*simp add*: *div-mult-mod-eq*)
also have ... = (((*k mod r*) *div* (*l mod r*) ∗ (*l mod r*)) *mod r*
+ (*k mod r*) *mod* (*l mod r*)) *mod r*
by (*simp add*: *mod-add-left-eq*)
also have ... = (((*k mod r*) *div* (*l mod r*) ∗ *l*) *mod r*
+ (*k mod r*) *mod* (*l mod r*)) *mod r*
by (*simp add*: *mod-mult-right-eq*)
finally have *k mod r* = ((*k mod r*) *div* (*l mod r*) ∗ *l*
+ (*k mod r*) *mod* (*l mod r*)) *mod r*
by (*simp add*: *mod-simps*)
with *r* **show** *take-bit LENGTH*('*a*) (*take-bit LENGTH*('*a*) *k div take-bit LENGTH*('*a*) *l* ∗ *l*
+ *take-bit LENGTH*('*a*) *k mod take-bit LENGTH*('*a*) *l*) = *take-bit LENGTH*('*a*) *k*
by *simp*

```
  qed
qed

end

instance word :: (len) semiring-parity
proof
  show ¬ 2 dvd (1::'a word)
    by transfer simp
  show even-iff-mod-2-eq-0: 2 dvd a ⟷ a mod 2 = 0
    for a :: 'a word
    by transfer (simp-all add: mod-2-eq-odd take-bit-Suc)
  show ¬ 2 dvd a ⟷ a mod 2 = 1
    for a :: 'a word
    by transfer (simp-all add: mod-2-eq-odd take-bit-Suc)
qed

lemma word-bit-induct [case-names zero even odd]:
  ‹P a› if word-zero: ‹P 0›
    and word-even: ‹⋀a. P a ⟹ 0 < a ⟹ a < 2 ^ (LENGTH('a) - Suc 0) ⟹
P (2 * a)›
    and word-odd: ‹⋀a. P a ⟹ a < 2 ^ (LENGTH('a) - Suc 0) ⟹ P (1 + 2
* a)›
  for P and a :: ‹'a::len word›
proof -
  define m :: nat where ‹m = LENGTH('a) - Suc 0›
  then have l: ‹LENGTH('a) = Suc m›
    by simp
  define n :: nat where ‹n = unat a›
  then have ‹n < 2 ^ LENGTH('a)›
    by transfer (simp add: take-bit-eq-mod)
  then have ‹n < 2 * 2 ^ m›
    by (simp add: l)
  then have ‹P (of-nat n)›
  proof (induction n rule: nat-bit-induct)
    case zero
    show ?case
      by simp (rule word-zero)
  next
    case (even n)
    then have ‹n < 2 ^ m›
      by simp
    with even.IH have ‹P (of-nat n)›
      by simp
    moreover from ‹n < 2 ^ m› even.hyps have ‹0 < (of-nat n :: 'a word)›
      by (auto simp add: word-greater-zero-iff l word-of-nat-eq-0-iff)
    moreover from ‹n < 2 ^ m› have ‹(of-nat n :: 'a word) < 2 ^ (LENGTH('a)
- Suc 0)›
      using of-nat-word-less-iff [where ?'a = 'a, of n ‹2 ^ m›]
```

```
      by (simp add: l take-bit-eq-mod)
    ultimately have ‹P (2 * of-nat n)›
      by (rule word-even)
    then show ?case
      by simp
  next
    case (odd n)
    then have ‹Suc n ≤ 2 ^ m›
      by simp
    with odd.IH have ‹P (of-nat n)›
      by simp
   moreover from ‹Suc n ≤ 2 ^ m› have ‹(of-nat n :: 'a word) < 2 ^ (LENGTH('a)
− Suc 0)›
      using of-nat-word-less-iff [where ?'a = 'a, of n ‹2 ^ m›]
      by (simp add: l take-bit-eq-mod)
    ultimately have ‹P (1 + 2 * of-nat n)›
      by (rule word-odd)
    then show ?case
      by simp
  qed
  moreover have ‹of-nat (nat (uint a)) = a›
    by transfer simp
  ultimately show ?thesis
    by (simp add: n-def)
qed

lemma bit-word-half-eq:
  ‹(of-bool b + a * 2) div 2 = a›
    if ‹a < 2 ^ (LENGTH('a) − Suc 0)›
    for a :: ‹'a::len word›
proof (cases ‹2 ≤ LENGTH('a::len)›)
  case False
  have ‹of-bool (odd k) < (1 :: int) ⟷ even k› for k :: int
    by auto
  with False that show ?thesis
    by transfer (simp add: eq-iff)
next
  case True
  obtain n where length: ‹LENGTH('a) = Suc n›
    by (cases ‹LENGTH('a)›) simp-all
  show ?thesis proof (cases b)
    case False
    moreover have ‹a * 2 div 2 = a›
    using that proof transfer
      fix k :: int
      from length have ‹k * 2 mod 2 ^ LENGTH('a) = (k mod 2 ^ n) * 2›
        by simp
      moreover assume ‹take-bit LENGTH('a) k < take-bit LENGTH('a) (2 ^
(LENGTH('a) − Suc 0))›
```

```
      with ‹LENGTH('a) = Suc n›
      have ‹k mod 2 ^ LENGTH('a) = k mod 2 ^ n›
        by (simp add: take-bit-eq-mod divmod-digit-0)
      ultimately have ‹take-bit LENGTH('a) (k * 2) = take-bit LENGTH('a) k
* 2›
        by (simp add: take-bit-eq-mod)
      with True show ‹take-bit LENGTH('a) (take-bit LENGTH('a) (k * 2) div
take-bit LENGTH('a) 2)
        = take-bit LENGTH('a) k›
        by simp
    qed
    ultimately show ?thesis
      by simp
  next
    case True
    moreover have ‹(1 + a * 2) div 2 = a›
    using that proof transfer
      fix k :: int
      from length have ‹(1 + k * 2) mod 2 ^ LENGTH('a) = 1 + (k mod 2 ^ n)
* 2›
        using pos-zmod-mult-2 [of ‹2 ^ n› k] by (simp add: ac-simps)
      moreover assume ‹take-bit LENGTH('a) k < take-bit LENGTH('a) (2 ^
(LENGTH('a) - Suc 0))›
      with ‹LENGTH('a) = Suc n›
      have ‹k mod 2 ^ LENGTH('a) = k mod 2 ^ n›
        by (simp add: take-bit-eq-mod divmod-digit-0)
       ultimately have ‹take-bit LENGTH('a) (1 + k * 2) = 1 + take-bit
LENGTH('a) k * 2›
        by (simp add: take-bit-eq-mod)
      with True show ‹take-bit LENGTH('a) (take-bit LENGTH('a) (1 + k * 2)
div take-bit LENGTH('a) 2)
        = take-bit LENGTH('a) k›
        by (auto simp add: take-bit-Suc)
    qed
    ultimately show ?thesis
      by simp
  qed
qed

lemma even-mult-exp-div-word-iff:
  ‹even (a * 2 ^ m div 2 ^ n) ⟷ ¬ (
    m ≤ n ∧
    n < LENGTH('a) ∧ odd (a div 2 ^ (n - m)))› for a :: ‹'a::len word›
  by transfer
   (auto simp flip: drop-bit-eq-div simp add: even-drop-bit-iff-not-bit bit-take-bit-iff,
     simp-all flip: push-bit-eq-mult add: bit-push-bit-iff-int)

instantiation word :: (len) semiring-bits
begin
```

lift-definition *bit-word* :: ‹*'a word* ⇒ *nat* ⇒ *bool*›
is ‹λ*k n*. *n* < *LENGTH*('*a*) ∧ *bit k n*›
proof
fix *k l* :: *int* **and** *n* :: *nat*
assume ∗: ‹*take-bit LENGTH*('*a*) *k* = *take-bit LENGTH*('*a*) *l*›
show ‹*n* < *LENGTH*('*a*) ∧ *bit k n* ⟷ *n* < *LENGTH*('*a*) ∧ *bit l n*›
proof (*cases* ‹*n* < *LENGTH*('*a*)›)
case *True*
from ∗ **have** ‹*bit* (*take-bit LENGTH*('*a*) *k*) *n* ⟷ *bit* (*take-bit LENGTH*('*a*) *l*) *n*›
by *simp*
then show *?thesis*
by (*simp add*: *bit-take-bit-iff*)
next
case *False*
then show *?thesis*
by *simp*
qed
qed

instance proof
show ‹*P a*› **if** *stable*: ‹⋀*a*. *a div 2* = *a* ⟹ *P a*›
and *rec*: ‹⋀*a b*. *P a* ⟹ (*of-bool b* + *2* ∗ *a*) *div 2* = *a* ⟹ *P* (*of-bool b* + *2* ∗ *a*)›
for *P* **and** *a* :: ‹*'a word*›
proof (*induction a rule*: *word-bit-induct*)
case *zero*
have ‹*0 div 2* = (*0*::*'a word*)›
by *transfer simp*
with *stable* [*of 0*] **show** *?case*
by *simp*
next
case (*even a*)
with *rec* [*of a False*] **show** *?case*
using *bit-word-half-eq* [*of a False*] **by** (*simp add*: *ac-simps*)
next
case (*odd a*)
with *rec* [*of a True*] **show** *?case*
using *bit-word-half-eq* [*of a True*] **by** (*simp add*: *ac-simps*)
qed
show ‹*bit a n* ⟷ *odd* (*a div 2* ^ *n*)› **for** *a* :: ‹*'a word*› **and** *n*
by *transfer* (*simp flip*: *drop-bit-eq-div add*: *drop-bit-take-bit bit-iff-odd-drop-bit*)
show ‹*0 div a* = *0*›
for *a* :: ‹*'a word*›
by *transfer simp*
show ‹*a div 1* = *a*›
for *a* :: ‹*'a word*›
by *transfer simp*

have §: $\bigwedge i\ n.\ (i::int)\ mod\ 2\ \hat{}\ n = 0 \vee 0 < i\ mod\ 2\ \hat{}\ n$
by (*metis le-less take-bit-eq-mod take-bit-nonnegative*)
have *less-power*: $\bigwedge n\ i\ p.\ (i::int)\ mod\ numeral\ p\ \hat{}\ n < numeral\ p\ \hat{}\ n$
by *simp*
show ‹*a mod b div b = 0*›
for *a b* :: ‹*'a word*›
apply *transfer*
apply (*simp add*: *take-bit-eq-mod mod-eq-0-iff-dvd dvd-def*)
by (*metis* (*no-types*, *opaque-lifting*) § *Euclidean-Division.pos-mod-bound Euclidean-Division.pos-mod-sign le-less-trans mult-eq-0-iff take-bit-eq-mod take-bit-nonnegative zdiv-eq-0-iff zmod-le-nonneg-dividend*)
show ‹*(1 + a) div 2 = a div 2*›
if ‹*even a*›
for *a* :: ‹*'a word*›
using *that* **by** *transfer*
(*auto dest*: *le-Suc-ex simp add*: *take-bit-Suc elim*!: *evenE*)
show ‹*(2 :: 'a word) ^ m div 2 ^ n = of-bool ((2 :: 'a word) ^ m ≠ 0 ∧ n ≤ m) * 2 ^ (m − n)*›
for *m n* :: *nat*
by *transfer* (*simp*, *simp add*: *exp-div-exp-eq*)
show *a div 2 ^ m div 2 ^ n = a div 2 ^ (m + n)*
for *a* :: *'a word* **and** *m n* :: *nat*
apply *transfer*
apply (*auto simp add*: *not-less take-bit-drop-bit ac-simps simp flip*: *drop-bit-eq-div*)
apply (*simp add*: *drop-bit-take-bit*)
done
show *a mod 2 ^ m mod 2 ^ n = a mod 2 ^ min m n*
for *a* :: *'a word* **and** *m n* :: *nat*
by *transfer* (*auto simp flip*: *take-bit-eq-mod simp add*: *ac-simps*)
show ‹*a * 2 ^ m mod 2 ^ n = a mod 2 ^ (n − m) * 2 ^ m*›
if ‹*m ≤ n*› **for** *a* :: *'a word* **and** *m n* :: *nat*
using *that* **apply** *transfer*
apply (*auto simp flip*: *take-bit-eq-mod*)
apply (*auto simp flip*: *push-bit-eq-mult simp add*: *push-bit-take-bit split*: *split-min-lin*)
done
show ‹*a div 2 ^ n mod 2 ^ m = a mod (2 ^ (n + m)) div 2 ^ n*›
for *a* :: *'a word* **and** *m n* :: *nat*
by *transfer* (*auto simp add*: *not-less take-bit-drop-bit ac-simps simp flip*: *take-bit-eq-mod drop-bit-eq-div split*: *split-min-lin*)
show ‹*even ((2 ^ m − 1) div (2::'a word) ^ n) ⟷ 2 ^ n = (0::'a word) ∨ m ≤ n*›
for *m n* :: *nat*
by *transfer*
(*simp flip*: *drop-bit-eq-div mask-eq-exp-minus-1 add*: *bit-simps even-drop-bit-iff-not-bit not-less*)
show ‹*even (a * 2 ^ m div 2 ^ n) ⟷ n < m ∨ (2::'a word) ^ n = 0 ∨ m ≤ n ∧ even (a div 2 ^ (n − m))*›
for *a* :: ‹*'a word*› **and** *m n* :: *nat*

```
  proof transfer
    show ‹even (take-bit LENGTH('a) (k * 2 ^ m) div take-bit LENGTH('a) (2 ^
n)) ⟷
      n < m
      ∨ take-bit LENGTH('a) ((2::int) ^ n) = take-bit LENGTH('a) 0
      ∨ (m ≤ n ∧ even (take-bit LENGTH('a) k div take-bit LENGTH('a) (2 ^ (n
- m))))›
    for m n :: nat and k l :: int
      by (auto simp flip: take-bit-eq-mod drop-bit-eq-div push-bit-eq-mult
        simp add: div-push-bit-of-1-eq-drop-bit drop-bit-take-bit drop-bit-push-bit-int
[of n m])
  qed
qed

end

lemma bit-word-eqI:
  ‹a = b› if ‹⋀n. n < LENGTH('a) ⟹ bit a n ⟷ bit b n›
  for a b :: ‹'a::len word›
  using that by transfer (auto simp add: nat-less-le bit-eq-iff bit-take-bit-iff)

lemma bit-imp-le-length:
  ‹n < LENGTH('a)› if ‹bit w n›
    for w :: ‹'a::len word›
  using that by transfer simp

lemma not-bit-length [simp]:
  ‹¬ bit w LENGTH('a)› for w :: ‹'a::len word›
  by transfer simp

lemma finite-bit-word [simp]:
  ‹finite {n. bit w n}›
  for w :: ‹'a::len word›
proof -
  have ‹{n. bit w n} ⊆ {0..LENGTH('a)}›
    by (auto dest: bit-imp-le-length)
  moreover have ‹finite {0..LENGTH('a)}›
    by simp
  ultimately show ?thesis
    by (rule finite-subset)
qed

lemma bit-numeral-word-iff [simp]:
  ‹bit (numeral w :: 'a::len word) n
    ⟷ n < LENGTH('a) ∧ bit (numeral w :: int) n›
  by transfer simp

lemma bit-neg-numeral-word-iff [simp]:
  ‹bit (- numeral w :: 'a::len word) n
```

```
    ⟷ n < LENGTH('a) ∧ bit (- numeral w :: int) n›
  by transfer simp

instantiation word :: (len) ring-bit-operations
begin

lift-definition not-word :: ‹'a word ⇒ 'a word›
  is not
  by (simp add: take-bit-not-iff)

lift-definition and-word :: ‹'a word ⇒ 'a word ⇒ 'a word›
  is ‹and›
  by simp

lift-definition or-word :: ‹'a word ⇒ 'a word ⇒ 'a word›
  is or
  by simp

lift-definition xor-word ::  ‹'a word ⇒ 'a word ⇒ 'a word›
  is xor
  by simp

lift-definition mask-word :: ‹nat ⇒ 'a word›
  is mask
  .

lift-definition set-bit-word :: ‹nat ⇒ 'a word ⇒ 'a word›
  is set-bit
  by (simp add: set-bit-def)

lift-definition unset-bit-word :: ‹nat ⇒ 'a word ⇒ 'a word›
  is unset-bit
  by (simp add: unset-bit-def)

lift-definition flip-bit-word :: ‹nat ⇒ 'a word ⇒ 'a word›
  is flip-bit
  by (simp add: flip-bit-def)

lift-definition push-bit-word :: ‹nat ⇒ 'a word ⇒ 'a word›
  is push-bit
proof -
  show ‹take-bit LENGTH('a) (push-bit n k) = take-bit LENGTH('a) (push-bit n
l)›
    if ‹take-bit LENGTH('a) k = take-bit LENGTH('a) l› for k l :: int and n ::
nat
  proof -
    from that
    have ‹take-bit (LENGTH('a) - n) (take-bit LENGTH('a) k)
      = take-bit (LENGTH('a) - n) (take-bit LENGTH('a) l)›
```

```
    by simp
  moreover have ‹min (LENGTH('a) - n) LENGTH('a) = LENGTH('a) -
n›
    by simp
  ultimately show ?thesis
    by (simp add: take-bit-push-bit)
 qed
qed

lift-definition drop-bit-word :: ‹nat ⇒ 'a word ⇒ 'a word›
 is ‹λn. drop-bit n ∘ take-bit LENGTH('a)›
 by (simp add: take-bit-eq-mod)

lift-definition take-bit-word :: ‹nat ⇒ 'a word ⇒ 'a word›
 is ‹λn. take-bit (min LENGTH('a) n)›
 by (simp add: ac-simps) (simp only: flip: take-bit-take-bit)

instance apply (standard; transfer)
 apply (auto simp add: minus-eq-not-minus-1 mask-eq-exp-minus-1
   bit-simps set-bit-def flip-bit-def take-bit-drop-bit
   simp flip: drop-bit-eq-div take-bit-eq-mod)
  apply (simp-all add: drop-bit-take-bit flip: push-bit-eq-mult)
 done

end

lemma [code]:
 ‹push-bit n w = w * 2 ^ n› for w :: ‹'a::len word›
 by (fact push-bit-eq-mult)

lemma [code]:
 ‹Word.the-int (drop-bit n w) = drop-bit n (Word.the-int w)›
 by transfer (simp add: drop-bit-take-bit min-def le-less less-diff-conv)

lemma [code]:
 ‹Word.the-int (take-bit n w) = (if n < LENGTH('a::len) then take-bit n (Word.the-int
w) else Word.the-int w)›
 for w :: ‹'a::len word›
 by transfer (simp add: not-le not-less ac-simps min-absorb2)

lemma [code-abbrev]:
 ‹push-bit n 1 = (2 :: 'a::len word) ^ n›
 by (fact push-bit-of-1)

context
 includes bit-operations-syntax
begin

lemma [code]:
```

‹*NOT w = Word.of-int (NOT (Word.the-int w))*›
for *w* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *take-bit-not-take-bit*)

lemma [*code*]:
‹*Word.the-int (v AND w) = Word.the-int v AND Word.the-int w*›
by *transfer simp*

lemma [*code*]:
‹*Word.the-int (v OR w) = Word.the-int v OR Word.the-int w*›
by *transfer simp*

lemma [*code*]:
‹*Word.the-int (v XOR w) = Word.the-int v XOR Word.the-int w*›
by *transfer simp*

lemma [*code*]:
‹*Word.the-int (mask n* :: *'a*::*len word) = mask (min LENGTH('a) n)*›
by *transfer simp*

lemma [*code*]:
‹*set-bit n w = w OR push-bit n 1*› **for** *w* :: ‹*'a*::*len word*›
by (*fact set-bit-eq-or*)

lemma [*code*]:
‹*unset-bit n w = w AND NOT (push-bit n 1)*› **for** *w* :: ‹*'a*::*len word*›
by (*fact unset-bit-eq-and-not*)

lemma [*code*]:
‹*flip-bit n w = w XOR push-bit n 1*› **for** *w* :: ‹*'a*::*len word*›
by (*fact flip-bit-eq-xor*)

context
includes *lifting-syntax*
begin

lemma *set-bit-word-transfer* [*transfer-rule*]:
‹((=) ===> *pcr-word* ===> *pcr-word*) *set-bit set-bit*›
by (*unfold set-bit-def*) *transfer-prover*

lemma *unset-bit-word-transfer* [*transfer-rule*]:
‹((=) ===> *pcr-word* ===> *pcr-word*) *unset-bit unset-bit*›
by (*unfold unset-bit-def*) *transfer-prover*

lemma *flip-bit-word-transfer* [*transfer-rule*]:
‹((=) ===> *pcr-word* ===> *pcr-word*) *flip-bit flip-bit*›
by (*unfold flip-bit-def*) *transfer-prover*

lemma *signed-take-bit-word-transfer* [*transfer-rule*]:

‹((=) ===> *pcr-word* ===> *pcr-word*)
(λ*n k*. *signed-take-bit n* (*take-bit LENGTH*('*a*::*len*) *k*))
(*signed-take-bit* :: *nat* ⇒ '*a word* ⇒ '*a word*)›
proof −
let *?K* = ‹λ*n* (*k* :: *int*). *take-bit* (*min LENGTH*('*a*) *n*) *k OR of-bool* (*n* < *LENGTH*('*a*) ∧ *bit k n*) ∗ *NOT* (*mask n*)›
let *?W* = ‹λ*n* (*w* :: '*a word*). *take-bit n w OR of-bool* (*bit w n*) ∗ *NOT* (*mask n*)›
have ‹((=) ===> *pcr-word* ===> *pcr-word*) *?K ?W*›
by *transfer-prover*
also have ‹*?K* = (λ*n k*. *signed-take-bit n* (*take-bit LENGTH*('*a*::*len*) *k*))›
by (*simp add*: *fun-eq-iff signed-take-bit-def bit-take-bit-iff ac-simps*)
also have ‹*?W* = *signed-take-bit*›
by (*simp add*: *fun-eq-iff signed-take-bit-def*)
finally show *?thesis* **.**
qed

end

end

98.5 Conversions including casts

98.5.1 Generic unsigned conversion

context *semiring-bits*
begin

lemma *bit-unsigned-iff* [*bit-simps*]:
‹*bit* (*unsigned w*) *n* ⟷ *possible-bit TYPE*('*a*) *n* ∧ *bit w n*›
for *w* :: ‹'*b*::*len word*›
by (*transfer fixing*: *bit*) (*simp add*: *bit-of-nat-iff bit-nat-iff bit-take-bit-iff*)

end

lemma *possible-bit-word*[*simp*]:
‹*possible-bit TYPE*(('*a* :: *len*) *word*) *m* ⟷ *m* < *LENGTH*('*a*)›
by (*simp add*: *possible-bit-def linorder-not-le*)

context *semiring-bit-operations*
begin

lemma *unsigned-minus-1-eq-mask*:
‹*unsigned* (− *1* :: '*b*::*len word*) = *mask LENGTH*('*b*)›
by (*transfer fixing*: *mask*) (*simp add*: *nat-mask-eq of-nat-mask-eq*)

lemma *unsigned-push-bit-eq*:
‹*unsigned* (*push-bit n w*) = *take-bit LENGTH*('*b*) (*push-bit n* (*unsigned w*))›
for *w* :: ‹'*b*::*len word*›
proof (*rule bit-eqI*)

fix *m*
assume ‹*possible-bit TYPE('a) m*›
show ‹*bit* (*unsigned* (*push-bit n w*)) *m* = *bit* (*take-bit LENGTH('b)* (*push-bit n* (*unsigned w*))) *m*›
proof (*cases* ‹*n* ≤ *m*›)
case *True*
with ‹*possible-bit TYPE('a) m*› **have** ‹*possible-bit TYPE('a)* (*m* − *n*)›
by (*simp add*: *possible-bit-less-imp*)
with *True* **show** *?thesis*
by (*simp add*: *bit-unsigned-iff bit-push-bit-iff Bit-Operations.bit-push-bit-iff bit-take-bit-iff not-le ac-simps*)
next
case *False*
then show *?thesis*
by (*simp add*: *not-le bit-unsigned-iff bit-push-bit-iff Bit-Operations.bit-push-bit-iff bit-take-bit-iff*)
qed
qed

lemma *unsigned-take-bit-eq*:
‹*unsigned* (*take-bit n w*) = *take-bit n* (*unsigned w*)›
for *w* :: ‹*'b*::*len word*›
by (*rule bit-eqI*) (*simp add*: *bit-unsigned-iff bit-take-bit-iff Bit-Operations.bit-take-bit-iff*)

end

context *unique-euclidean-semiring-with-bit-operations*
begin

lemma *unsigned-drop-bit-eq*:
‹*unsigned* (*drop-bit n w*) = *drop-bit n* (*take-bit LENGTH('b)* (*unsigned w*))›
for *w* :: ‹*'b*::*len word*›
by (*rule bit-eqI*) (*auto simp add*: *bit-unsigned-iff bit-take-bit-iff bit-drop-bit-eq Bit-Operations.bit-drop-bit-eq possible-bit-def dest*: *bit-imp-le-length*)

end

lemma *ucast-drop-bit-eq*:
‹*ucast* (*drop-bit n w*) = *drop-bit n* (*ucast w* :: *'b*::*len word*)›
if ‹*LENGTH('a)* ≤ *LENGTH('b)*› **for** *w* :: ‹*'a*::*len word*›
by (*rule bit-word-eqI*) (*use that* **in** ‹*auto simp add*: *bit-unsigned-iff bit-drop-bit-eq dest*: *bit-imp-le-length*›)

context *semiring-bit-operations*
begin

context
includes *bit-operations-syntax*
begin

lemma *unsigned-and-eq*:
‹*unsigned* (*v AND w*) = *unsigned v AND unsigned w*›
for *v w* :: ‹'*b*::*len word*›
by (*simp add*: *bit-eq-iff bit-simps*)

lemma *unsigned-or-eq*:
‹*unsigned* (*v OR w*) = *unsigned v OR unsigned w*›
for *v w* :: ‹'*b*::*len word*›
by (*simp add*: *bit-eq-iff bit-simps*)

lemma *unsigned-xor-eq*:
‹*unsigned* (*v XOR w*) = *unsigned v XOR unsigned w*›
for *v w* :: ‹'*b*::*len word*›
by (*simp add*: *bit-eq-iff bit-simps*)

end

end

context *ring-bit-operations*
begin

context
includes *bit-operations-syntax*
begin

lemma *unsigned-not-eq*:
‹*unsigned* (*NOT w*) = *take-bit LENGTH*('*b*) (*NOT* (*unsigned w*))›
for *w* :: ‹'*b*::*len word*›
by (*simp add*: *bit-eq-iff bit-simps*)

end

end

context *unique-euclidean-semiring-numeral*
begin

lemma *unsigned-greater-eq* [*simp*]:
‹*0* ≤ *unsigned w*› **for** *w* :: ‹'*b*::*len word*›
by (*transfer fixing*: *less-eq*) *simp*

lemma *unsigned-less* [*simp*]:
‹*unsigned w* < *2* ^ *LENGTH*('*b*)› **for** *w* :: ‹'*b*::*len word*›
by (*transfer fixing*: *less*) *simp*

end

context *linordered-semidom*
begin

lemma *word-less-eq-iff-unsigned*:
  $a \leq b \longleftrightarrow$ *unsigned* $a \leq$ *unsigned* b
  by (*transfer fixing*: *less-eq*) (*simp add*: *nat-le-eq-zle*)

lemma *word-less-iff-unsigned*:
  $a < b \longleftrightarrow$ *unsigned* $a <$ *unsigned* b
 by (*transfer fixing*: *less*) (*auto dest*: *preorder-class.le-less-trans* [*OF take-bit-nonnegative*])

end

98.5.2 Generic signed conversion

context *ring-bit-operations*
begin

lemma *bit-signed-iff* [*bit-simps*]:
  ‹*bit* (*signed* w) $n \longleftrightarrow$ *possible-bit* *TYPE*('a) $n \wedge$ *bit* w (*min* (*LENGTH*('b) − *Suc 0*) n)›
  for w :: ‹'b::*len word*›
  by (*transfer fixing*: *bit*)
   (*auto simp add*: *bit-of-int-iff Bit-Operations.bit-signed-take-bit-iff min-def*)

lemma *signed-push-bit-eq*:
  ‹*signed* (*push-bit* n w) = *signed-take-bit* (*LENGTH*('b) − *Suc 0*) (*push-bit* n (*signed* w :: 'a))›
  for w :: ‹'b::*len word*›
  apply (*simp add*: *bit-eq-iff bit-simps possible-bit-less-imp min-less-iff-disj*)
  apply (*cases* n, *simp-all add*: *min-def*)
  done

lemma *signed-take-bit-eq*:
  ‹*signed* (*take-bit* n w) = (*if* $n <$ *LENGTH*('b) *then take-bit* n (*signed* w) *else signed* w)›
  for w :: ‹'b::*len word*›
  by (*transfer fixing*: *take-bit*; *cases* ‹*LENGTH*('b)›)
   (*auto simp add*: *Bit-Operations.signed-take-bit-take-bit Bit-Operations.take-bit-signed-take-bit take-bit-of-int min-def less-Suc-eq*)

context
  includes *bit-operations-syntax*
begin

lemma *signed-not-eq*:
  ‹*signed* (*NOT* w) = *signed-take-bit* *LENGTH*('b) (*NOT* (*signed* w))›
  for w :: ‹'b::*len word*›
  by (*simp add*: *bit-eq-iff bit-simps possible-bit-less-imp min-less-iff-disj*)

(*auto simp*: *min-def*)

lemma *signed-and-eq*:
‹*signed* (*v AND w*) = *signed v AND signed w*›
for *v w* :: ‹'*b*::*len word*›
by (*rule bit-eqI*) (*simp add*: *bit-signed-iff bit-and-iff Bit-Operations.bit-and-iff*)

lemma *signed-or-eq*:
‹*signed* (*v OR w*) = *signed v OR signed w*›
for *v w* :: ‹'*b*::*len word*›
by (*rule bit-eqI*) (*simp add*: *bit-signed-iff bit-or-iff Bit-Operations.bit-or-iff*)

lemma *signed-xor-eq*:
‹*signed* (*v XOR w*) = *signed v XOR signed w*›
for *v w* :: ‹'*b*::*len word*›
by (*rule bit-eqI*) (*simp add*: *bit-signed-iff bit-xor-iff Bit-Operations.bit-xor-iff*)

end

end

98.5.3 More

lemma *sint-greater-eq*:
‹− (2 ^ (*LENGTH*('*a*) − *Suc 0*)) ≤ *sint w*› **for** *w* :: ‹'*a*::*len word*›
proof (*cases* ‹*bit w* (*LENGTH*('*a*) − *Suc 0*)›)
case *True*
then show *?thesis*
by *transfer* (*simp add*: *signed-take-bit-eq-if-negative minus-exp-eq-not-mask or-greater-eq ac-simps*)
next
have ∗: ‹− (2 ^ (*LENGTH*('*a*) − *Suc 0*)) ≤ (*0*::*int*)›
by *simp*
case *False*
then show *?thesis*
by *transfer* (*auto simp add*: *signed-take-bit-eq intro*: *order-trans* ∗)
qed

lemma *sint-less*:
‹*sint w* < 2 ^ (*LENGTH*('*a*) − *Suc 0*)› **for** *w* :: ‹'*a*::*len word*›
by (*cases* ‹*bit w* (*LENGTH*('*a*) − *Suc 0*)›; *transfer*)
(*simp-all add*: *signed-take-bit-eq signed-take-bit-def not-eq-complement mask-eq-exp-minus-1 OR-upper*)

lemma *unat-div-distrib*:
‹*unat* (*v div w*) = *unat v div unat w*›
proof *transfer*
fix *k l*
have ‹*nat* (*take-bit LENGTH*('*a*) *k*) *div nat* (*take-bit LENGTH*('*a*) *l*) ≤ *nat*

(*take-bit LENGTH('a) k*)›
by (*rule div-le-dividend*)
also have ‹*nat* (*take-bit LENGTH('a) k*) < *2* ^ *LENGTH('a)*›
by (*simp add*: *nat-less-iff*)
finally show ‹(*nat* ∘ *take-bit LENGTH('a)*) (*take-bit LENGTH('a) k div take-bit LENGTH('a) l*) =
(*nat* ∘ *take-bit LENGTH('a)*) *k div* (*nat* ∘ *take-bit LENGTH('a)*) *l*›
by (*simp add*: *nat-take-bit-eq div-int-pos-iff nat-div-distrib take-bit-nat-eq-self-iff*)
qed

lemma *unat-mod-distrib*:
‹*unat* (*v mod w*) = *unat v mod unat w*›
proof *transfer*
fix *k l*
have ‹*nat* (*take-bit LENGTH('a) k*) *mod nat* (*take-bit LENGTH('a) l*) ≤ *nat* (*take-bit LENGTH('a) k*)›
by (*rule mod-less-eq-dividend*)
also have ‹*nat* (*take-bit LENGTH('a) k*) < *2* ^ *LENGTH('a)*›
by (*simp add*: *nat-less-iff*)
finally show ‹(*nat* ∘ *take-bit LENGTH('a)*) (*take-bit LENGTH('a) k mod take-bit LENGTH('a) l*) =
(*nat* ∘ *take-bit LENGTH('a)*) *k mod* (*nat* ∘ *take-bit LENGTH('a)*) *l*›
by (*simp add*: *nat-take-bit-eq mod-int-pos-iff less-le nat-mod-distrib take-bit-nat-eq-self-iff*)
qed

lemma *uint-div-distrib*:
‹*uint* (*v div w*) = *uint v div uint w*›
proof −
have ‹*int* (*unat* (*v div w*)) = *int* (*unat v div unat w*)›
by (*simp add*: *unat-div-distrib*)
then show *?thesis*
by (*simp add*: *of-nat-div*)
qed

lemma *unat-drop-bit-eq*:
‹*unat* (*drop-bit n w*) = *drop-bit n* (*unat w*)›
by (*rule bit-eqI*) (*simp add*: *bit-unsigned-iff bit-drop-bit-eq*)

lemma *uint-mod-distrib*:
‹*uint* (*v mod w*) = *uint v mod uint w*›
proof −
have ‹*int* (*unat* (*v mod w*)) = *int* (*unat v mod unat w*)›
by (*simp add*: *unat-mod-distrib*)
then show *?thesis*
by (*simp add*: *of-nat-mod*)
qed

context *semiring-bit-operations*
begin

lemma *unsigned-ucast-eq*:
‹*unsigned* (*ucast w* :: *'c*::*len word*) = *take-bit LENGTH*('*c*) (*unsigned w*)›
for *w* :: ‹*'b*::*len word*›
by (*rule bit-eqI*) (*simp add*: *bit-unsigned-iff Word.bit-unsigned-iff bit-take-bit-iff not-le*)

end

context *ring-bit-operations*
begin

lemma *signed-ucast-eq*:
‹*signed* (*ucast w* :: *'c*::*len word*) = *signed-take-bit* (*LENGTH*('*c*) − *Suc 0*) (*unsigned w*)›
for *w* :: ‹*'b*::*len word*›
by (*simp add*: *bit-eq-iff bit-simps min-less-iff-disj*)

lemma *signed-scast-eq*:
‹*signed* (*scast w* :: *'c*::*len word*) = *signed-take-bit* (*LENGTH*('*c*) − *Suc 0*) (*signed w*)›
for *w* :: ‹*'b*::*len word*›
by (*simp add*: *bit-eq-iff bit-simps min-less-iff-disj*)

end

lemma *uint-nonnegative*: *0* ≤ *uint w*
by (*fact unsigned-greater-eq*)

lemma *uint-bounded*: *uint w* < *2* ^ *LENGTH*('*a*)
for *w* :: *'a*::*len word*
by (*fact unsigned-less*)

lemma *uint-idem*: *uint w mod 2* ^ *LENGTH*('*a*) = *uint w*
for *w* :: *'a*::*len word*
by *transfer* (*simp add*: *take-bit-eq-mod*)

lemma *word-uint-eqI*: *uint a* = *uint b* ⟹ *a* = *b*
by (*fact unsigned-word-eqI*)

lemma *word-uint-eq-iff*: *a* = *b* ⟷ *uint a* = *uint b*
by (*fact word-eq-iff-unsigned*)

lemma *uint-word-of-int-eq*:
‹*uint* (*word-of-int k* :: *'a*::*len word*) = *take-bit LENGTH*('*a*) *k*›
by *transfer rule*

lemma *uint-word-of-int*: *uint* (*word-of-int k* :: *'a*::*len word*) = *k mod 2* ^ *LENGTH*('*a*)
by (*simp add*: *uint-word-of-int-eq take-bit-eq-mod*)

lemma *word-of-int-uint*: *word-of-int* (*uint w*) = *w*
by *transfer simp*

lemma *word-div-def* [*code*]:
a div b = *word-of-int* (*uint a div uint b*)
by *transfer rule*

lemma *word-mod-def* [*code*]:
a mod b = *word-of-int* (*uint a mod uint b*)
by *transfer rule*

lemma *split-word-all*: (⋀*x*::*'a*::*len word*. *PROP P x*) ≡ (⋀*x*. *PROP P* (*word-of-int x*))
proof
fix *x* :: *'a word*
assume ⋀*x*. *PROP P* (*word-of-int x*)
then have *PROP P* (*word-of-int* (*uint x*)) **.**
then show *PROP P x*
by (*simp only*: *word-of-int-uint*)
qed

lemma *sint-uint*:
‹*sint w* = *signed-take-bit* (*LENGTH*(*'a*) − *Suc 0*) (*uint w*)›
for *w* :: ‹*'a*::*len word*›
by (*cases* ‹*LENGTH*(*'a*)›; *transfer*) (*simp-all add*: *signed-take-bit-take-bit*)

lemma *unat-eq-nat-uint*:
‹*unat w* = *nat* (*uint w*)›
by *simp*

lemma *ucast-eq*:
‹*ucast w* = *word-of-int* (*uint w*)›
by *transfer simp*

lemma *scast-eq*:
‹*scast w* = *word-of-int* (*sint w*)›
by *transfer simp*

lemma *uint-0-eq*:
‹*uint 0* = *0*›
by (*fact unsigned-0*)

lemma *uint-1-eq*:
‹*uint 1* = *1*›
by (*fact unsigned-1*)

lemma *word-m1-wi*: − *1* = *word-of-int* (− *1*)
by *simp*

lemma *uint-0-iff*: *uint x = 0 ⟷ x = 0*
by (*auto simp add*: *unsigned-word-eqI*)

lemma *unat-0-iff*: *unat x = 0 ⟷ x = 0*
by (*auto simp add*: *unsigned-word-eqI*)

lemma *unat-0*: *unat 0 = 0*
by (*fact unsigned-0*)

lemma *unat-gt-0*: *0 < unat x ⟷ x ≠ 0*
by (*auto simp*: *unat-0-iff* [*symmetric*])

lemma *ucast-0*: *ucast 0 = 0*
by (*fact unsigned-0*)

lemma *sint-0*: *sint 0 = 0*
by (*fact signed-0*)

lemma *scast-0*: *scast 0 = 0*
by (*fact signed-0*)

lemma *sint-n1*: *sint (− 1) = − 1*
by (*fact signed-minus-1*)

lemma *scast-n1*: *scast (− 1) = − 1*
by (*fact signed-minus-1*)

lemma *uint-1*: *uint (1::'a::len word) = 1*
by (*fact uint-1-eq*)

lemma *unat-1*: *unat (1::'a::len word) = 1*
by (*fact unsigned-1*)

lemma *ucast-1*: *ucast (1::'a::len word) = 1*
by (*fact unsigned-1*)

instantiation *word* :: (*len*) *size*
begin

lift-definition *size-word* :: ‹*'a word ⇒ nat*›
is ‹*λ-. LENGTH('a)*› **..**

instance ..

end

lemma *word-size* [*code*]:
‹*size w = LENGTH('a)*› **for** *w* :: ‹*'a::len word*›

by (*fact size-word.rep-eq*)

lemma *word-size-gt-0* [*iff*]: *0 < size w*
for *w* :: *'a::len word*
by (*simp add*: *word-size*)

lemmas *lens-gt-0 = word-size-gt-0 len-gt-0*

lemma *lens-not-0* [*iff*]:
‹*size w ≠ 0*› **for** *w* :: ‹*'a::len word*›
by *auto*

lift-definition *source-size* :: ‹(*'a::len word ⇒ 'b*) ⇒ *nat*›
is ‹λ-. *LENGTH('a)*› **.**

lift-definition *target-size* :: ‹(*'a ⇒ 'b::len word*) ⇒ *nat*›
is ‹λ-. *LENGTH('b)*› **..**

lift-definition *is-up* :: ‹(*'a::len word ⇒ 'b::len word*) ⇒ *bool*›
is ‹λ-. *LENGTH('a) ≤ LENGTH('b)*› **..**

lift-definition *is-down* :: ‹(*'a::len word ⇒ 'b::len word*) ⇒ *bool*›
is ‹λ-. *LENGTH('a) ≥ LENGTH('b)*› **..**

lemma *is-up-eq*:
‹*is-up f ⟷ source-size f ≤ target-size f*›
for *f* :: ‹*'a::len word ⇒ 'b::len word*›
by (*simp add*: *source-size.rep-eq target-size.rep-eq is-up.rep-eq*)

lemma *is-down-eq*:
‹*is-down f ⟷ target-size f ≤ source-size f*›
for *f* :: ‹*'a::len word ⇒ 'b::len word*›
by (*simp add*: *source-size.rep-eq target-size.rep-eq is-down.rep-eq*)

lift-definition *word-int-case* :: ‹(*int ⇒ 'b*) ⇒ *'a::len word ⇒ 'b*›
is ‹λ*f*. *f ∘ take-bit LENGTH('a)*› **by** *simp*

lemma *word-int-case-eq-uint* [*code*]:
‹*word-int-case f w = f (uint w)*›
by *transfer simp*

translations
case x of XCONST of-int y ⇒ b ⇌ CONST word-int-case (λ*y*. *b*) *x*
case x of (*XCONST of-int* :: *'a*) *y ⇒ b ⇀ CONST word-int-case* (λ*y*. *b*) *x*

98.6 Arithmetic operations

lemma *div-word-self*:
‹*w div w = 1*› **if** ‹*w ≠ 0*› **for** *w* :: ‹*'a::len word*›

```
  using that by transfer simp

lemma mod-word-self [simp]:
  ‹w mod w = 0› for w :: ‹'a::len word›
  apply (cases ‹w = 0›)
  apply auto
  using div-mult-mod-eq [of w w] by (simp add: div-word-self)

lemma div-word-less:
  ‹w div v = 0› if ‹w < v› for w v :: ‹'a::len word›
  using that by transfer simp

lemma mod-word-less:
  ‹w mod v = w› if ‹w < v› for w v :: ‹'a::len word›
  using div-mult-mod-eq [of w v] using that by (simp add: div-word-less)

lemma div-word-one [simp]:
  ‹1 div w = of-bool (w = 1)› for w :: ‹'a::len word›
proof transfer
  fix k :: int
  show ‹take-bit LENGTH('a) (take-bit LENGTH('a) 1 div take-bit LENGTH('a)
k) =
      take-bit LENGTH('a) (of-bool (take-bit LENGTH('a) k = take-bit LENGTH('a)
1))›
  proof (cases ‹take-bit LENGTH('a) k > 1›)
    case False
    with take-bit-nonnegative [of ‹LENGTH('a)› k]
    have ‹take-bit LENGTH('a) k = 0 ∨ take-bit LENGTH('a) k = 1›
      by linarith
    then show ?thesis
      by auto
  next
    case True
    then show ?thesis
      by simp
  qed
qed

lemma mod-word-one [simp]:
  ‹1 mod w = 1 - w * of-bool (w = 1)› for w :: ‹'a::len word›
  using div-mult-mod-eq [of 1 w] by simp

lemma div-word-by-minus-1-eq [simp]:
  ‹w div - 1 = of-bool (w = - 1)› for w :: ‹'a::len word›
 by (auto intro: div-word-less simp add: div-word-self word-order.not-eq-extremum)

lemma mod-word-by-minus-1-eq [simp]:
  ‹w mod - 1 = w * of-bool (w < - 1)› for w :: ‹'a::len word›
  apply (cases ‹w = - 1›)
```

apply (*auto simp add*: *word-order.not-eq-extremum*)
using *div-mult-mod-eq* [*of w ‹− 1›*] **by** *simp*

Legacy theorems:

lemma *word-add-def* [*code*]:
a + b = word-of-int (uint a + uint b)
by *transfer* (*simp add*: *take-bit-add*)

lemma *word-sub-wi* [*code*]:
a − b = word-of-int (uint a − uint b)
by *transfer* (*simp add*: *take-bit-diff*)

lemma *word-mult-def* [*code*]:
*a * b = word-of-int (uint a * uint b)*
by *transfer* (*simp add*: *take-bit-eq-mod mod-simps*)

lemma *word-minus-def* [*code*]:
− a = word-of-int (− uint a)
by *transfer* (*simp add*: *take-bit-minus*)

lemma *word-0-wi*:
0 = word-of-int 0
by *transfer simp*

lemma *word-1-wi*:
1 = word-of-int 1
by *transfer simp*

lift-definition *word-succ* :: *'a*::*len word ⇒ 'a word* **is** *λx. x + 1*
by (*auto simp add*: *take-bit-eq-mod intro*: *mod-add-cong*)

lift-definition *word-pred* :: *'a*::*len word ⇒ 'a word* **is** *λx. x − 1*
by (*auto simp add*: *take-bit-eq-mod intro*: *mod-diff-cong*)

lemma *word-succ-alt* [*code*]:
word-succ a = word-of-int (uint a + 1)
by *transfer* (*simp add*: *take-bit-eq-mod mod-simps*)

lemma *word-pred-alt* [*code*]:
word-pred a = word-of-int (uint a − 1)
by *transfer* (*simp add*: *take-bit-eq-mod mod-simps*)

lemmas *word-arith-wis* =
word-add-def word-sub-wi word-mult-def
word-minus-def word-succ-alt word-pred-alt
word-0-wi word-1-wi

lemma *wi-homs*:
shows *wi-hom-add*: *word-of-int a + word-of-int b = word-of-int (a + b)*

and *wi-hom-sub*: *word-of-int a − word-of-int b = word-of-int (a − b)*
and *wi-hom-mult*: *word-of-int a * word-of-int b = word-of-int (a * b)*
and *wi-hom-neg*: *− word-of-int a = word-of-int (− a)*
and *wi-hom-succ*: *word-succ (word-of-int a) = word-of-int (a + 1)*
and *wi-hom-pred*: *word-pred (word-of-int a) = word-of-int (a − 1)*
by (*transfer*, *simp*)+

lemmas *wi-hom-syms = wi-homs* [*symmetric*]

lemmas *word-of-int-homs = wi-homs word-0-wi word-1-wi*

lemmas *word-of-int-hom-syms = word-of-int-homs* [*symmetric*]

lemma *double-eq-zero-iff*:
‹*2 * a = 0 ⟷ a = 0 ∨ a = 2 ^ (LENGTH('a) − Suc 0)*›
for *a* :: ‹*'a*::*len word*›
proof −
define *n* **where** ‹*n = LENGTH('a) − Suc 0*›
then have *: ‹*LENGTH('a) = Suc n*›
by *simp*
have ‹*a = 0*› **if** ‹*2 * a = 0*› **and** ‹*a ≠ 2 ^ (LENGTH('a) − Suc 0)*›
using *that* **by** *transfer*
(*auto simp add*: *take-bit-eq-0-iff take-bit-eq-mod* *)
moreover have ‹*2 ^ LENGTH('a) = (0 :: 'a word)*›
by *transfer simp*
then have ‹*2 * 2 ^ (LENGTH('a) − Suc 0) = (0 :: 'a word)*›
by (*simp add*: *)
ultimately show *?thesis*
by *auto*
qed

98.7 Ordering

lift-definition *word-sle* :: ‹*'a*::*len word ⇒ 'a word ⇒ bool*›
is ‹*λk l. signed-take-bit (LENGTH('a) − Suc 0) k ≤ signed-take-bit (LENGTH('a) − Suc 0) l*›
by (*simp flip*: *signed-take-bit-decr-length-iff*)

lift-definition *word-sless* :: ‹*'a*::*len word ⇒ 'a word ⇒ bool*›
is ‹*λk l. signed-take-bit (LENGTH('a) − Suc 0) k < signed-take-bit (LENGTH('a) − Suc 0) l*›
by (*simp flip*: *signed-take-bit-decr-length-iff*)

notation
word-sle (*'(≤s')*) **and**
word-sle ((*-/ ≤s -*) [*51*, *51*] *50*) **and**
word-sless (*'(<s')*) **and**
word-sless ((*-/ <s -*) [*51*, *51*] *50*)

notation (*input*)
word-sle ((-/ <=*s* -) [*51*, *51*] *50*)

lemma *word-sle-eq* [*code*]:
‹*a* <=*s* *b* ⟷ *sint a* ≤ *sint b*›
by *transfer simp*

lemma [*code*]:
‹*a* <*s* *b* ⟷ *sint a* < *sint b*›
by *transfer simp*

lemma *signed-ordering*: ‹*ordering word-sle word-sless*›
apply (*standard*; *transfer*)
using *signed-take-bit-decr-length-iff* **by** *force+*

lemma *signed-linorder*: ‹*class.linorder word-sle word-sless*›
by (*standard*; *transfer*) (*auto simp add*: *signed-take-bit-decr-length-iff*)

interpretation *signed*: *linorder word-sle word-sless*
by (*fact signed-linorder*)

lemma *word-sless-eq*:
‹*x* <*s* *y* ⟷ *x* <=*s* *y* ∧ *x* ≠ *y*›
by (*fact signed.less-le*)

lemma *word-less-alt*: *a* < *b* ⟷ *uint a* < *uint b*
by (*fact word-less-def*)

lemma *word-zero-le* [*simp*]: *0* ≤ *y*
for *y* :: *'a*::*len word*
by (*fact word-coorder.extremum*)

lemma *word-m1-ge* [*simp*] : *word-pred 0* ≥ *y*
by *transfer* (*simp add*: *mask-eq-exp-minus-1*)

lemma *word-n1-ge* [*simp*]: *y* ≤ *−1*
for *y* :: *'a*::*len word*
by (*fact word-order.extremum*)

lemmas *word-not-simps* [*simp*] =
word-zero-le [*THEN leD*] *word-m1-ge* [*THEN leD*] *word-n1-ge* [*THEN leD*]

lemma *word-gt-0*: *0* < *y* ⟷ *0* ≠ *y*
for *y* :: *'a*::*len word*
by (*simp add*: *less-le*)

lemmas *word-gt-0-no* [*simp*] = *word-gt-0* [*of numeral y*] **for** *y*

lemma *word-sless-alt*: *a* <*s* *b* ⟷ *sint a* < *sint b*

by *transfer simp*

lemma *word-le-nat-alt*: $a \leq b \longleftrightarrow unat\ a \leq unat\ b$
by *transfer* (*simp add*: *nat-le-eq-zle*)

lemma *word-less-nat-alt*: $a < b \longleftrightarrow unat\ a < unat\ b$
by *transfer* (*auto simp add*: *less-le* [*of 0*])

lemmas *unat-mono* = *word-less-nat-alt* [*THEN iffD1*]

instance *word* :: (*len*) *wellorder*
proof
fix P :: $'a\ word \Rightarrow bool$ **and** a
assume $*$: $(\bigwedge b.\ (\bigwedge a.\ a < b \Longrightarrow P\ a) \Longrightarrow P\ b)$
have *wf* (*measure unat*) **..**
moreover have $\{(a, b :: ('a::len)\ word).\ a < b\} \subseteq measure\ unat$
by (*auto simp add*: *word-less-nat-alt*)
ultimately have *wf* $\{(a, b :: ('a::len)\ word).\ a < b\}$
by (*rule wf-subset*)
then show $P\ a$ **using** $*$
by *induction blast*
qed

lemma *wi-less*:
(*word-of-int n* < (*word-of-int m* :: *'a::len word*)) =
(*n mod 2* ^ *LENGTH*('*a*) < *m mod 2* ^ *LENGTH*('*a*))
by *transfer* (*simp add*: *take-bit-eq-mod*)

lemma *wi-le*:
(*word-of-int n* ≤ (*word-of-int m* :: *'a::len word*)) =
(*n mod 2* ^ *LENGTH*('*a*) ≤ *m mod 2* ^ *LENGTH*('*a*))
by *transfer* (*simp add*: *take-bit-eq-mod*)

98.8 Bit-wise operations

context
includes *bit-operations-syntax*
begin

lemma *uint-take-bit-eq*:
‹*uint* (*take-bit n w*) = *take-bit n* (*uint w*)›
by *transfer* (*simp add*: *ac-simps*)

lemma *take-bit-word-eq-self*:
‹*take-bit n w* = *w*› **if** ‹*LENGTH*('*a*) ≤ *n*› **for** *w* :: ‹*'a::len word*›
using *that* **by** *transfer simp*

lemma *take-bit-length-eq* [*simp*]:
‹*take-bit LENGTH*('*a*) *w* = *w*› **for** *w* :: ‹*'a::len word*›

by (*rule take-bit-word-eq-self*) *simp*

lemma *bit-word-of-int-iff*:
‹*bit* (*word-of-int k* :: *'a*::*len word*) *n* ⟷ *n* < *LENGTH*(*'a*) ∧ *bit k n*›
by *transfer rule*

lemma *bit-uint-iff*:
‹*bit* (*uint w*) *n* ⟷ *n* < *LENGTH*(*'a*) ∧ *bit w n*›
for *w* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *bit-take-bit-iff*)

lemma *bit-sint-iff*:
‹*bit* (*sint w*) *n* ⟷ *n* ≥ *LENGTH*(*'a*) ∧ *bit w* (*LENGTH*(*'a*) − *1*) ∨ *bit w n*›
for *w* :: ‹*'a*::*len word*›
by *transfer* (*auto simp add*: *bit-signed-take-bit-iff min-def le-less not-less*)

lemma *bit-word-ucast-iff*:
‹*bit* (*ucast w* :: *'b*::*len word*) *n* ⟷ *n* < *LENGTH*(*'a*) ∧ *n* < *LENGTH*(*'b*) ∧ *bit w n*›
for *w* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *bit-take-bit-iff ac-simps*)

lemma *bit-word-scast-iff*:
‹*bit* (*scast w* :: *'b*::*len word*) *n* ⟷
n < *LENGTH*(*'b*) ∧ (*bit w n* ∨ *LENGTH*(*'a*) ≤ *n* ∧ *bit w* (*LENGTH*(*'a*) − *Suc 0*))›
for *w* :: ‹*'a*::*len word*›
by *transfer* (*auto simp add*: *bit-signed-take-bit-iff le-less min-def*)

lemma *bit-word-iff-drop-bit-and* [*code*]:
‹*bit a n* ⟷ *drop-bit n a AND 1* = *1*› **for** *a* :: ‹*'a*::*len word*›
by (*simp add*: *bit-iff-odd-drop-bit odd-iff-mod-2-eq-one and-one-eq*)

lemma
word-not-def: *NOT* (*a*::*'a*::*len word*) = *word-of-int* (*NOT* (*uint a*))
and *word-and-def*: (*a*::*'a word*) *AND b* = *word-of-int* (*uint a AND uint b*)
and *word-or-def*: (*a*::*'a word*) *OR b* = *word-of-int* (*uint a OR uint b*)
and *word-xor-def*: (*a*::*'a word*) *XOR b* = *word-of-int* (*uint a XOR uint b*)
by (*transfer*, *simp add*: *take-bit-not-take-bit*)+

definition *even-word* :: ‹*'a*::*len word* ⇒ *bool*›
where [*code-abbrev*]: ‹*even-word* = *even*›

lemma *even-word-iff* [*code*]:
‹*even-word a* ⟷ *a AND 1* = *0*›
by (*simp add*: *and-one-eq even-iff-mod-2-eq-zero even-word-def*)

lemma *map-bit-range-eq-if-take-bit-eq*:
‹*map* (*bit k*) [*0*..<*n*] = *map* (*bit l*) [*0*..<*n*]›

```
  if ‹take-bit n k = take-bit n l› for k l :: int
using that proof (induction n arbitrary: k l)
  case 0
  then show ?case
    by simp
next
  case (Suc n)
  from Suc.prems have ‹take-bit n (k div 2) = take-bit n (l div 2)›
    by (simp add: take-bit-Suc)
  then have ‹map (bit (k div 2)) [0..<n] = map (bit (l div 2)) [0..<n]›
    by (rule Suc.IH)
  moreover have ‹bit (r div 2) = bit r ∘ Suc› for r :: int
    by (simp add: fun-eq-iff bit-Suc)
  moreover from Suc.prems have ‹even k ⟷ even l›
    by (auto simp add: take-bit-Suc elim!: evenE oddE) arith+
  ultimately show ?case
    by (simp only: map-Suc-upt upt-conv-Cons flip: list.map-comp) simp
qed

lemma
   take-bit-word-Bit0-eq [simp]: ‹take-bit (numeral n) (numeral (num.Bit0 m) ::
'a::len word)
    = 2 * take-bit (pred-numeral n) (numeral m)› (is ?P)
  and take-bit-word-Bit1-eq [simp]: ‹take-bit (numeral n) (numeral (num.Bit1 m)
:: 'a::len word)
    = 1 + 2 * take-bit (pred-numeral n) (numeral m)› (is ?Q)
 and take-bit-word-minus-Bit0-eq [simp]: ‹take-bit (numeral n) (- numeral (num.Bit0
m) :: 'a::len word)
    = 2 * take-bit (pred-numeral n) (- numeral m)› (is ?R)
 and take-bit-word-minus-Bit1-eq [simp]: ‹take-bit (numeral n) (- numeral (num.Bit1
m) :: 'a::len word)
    = 1 + 2 * take-bit (pred-numeral n) (- numeral (Num.inc m))› (is ?S)
proof -
  define w :: ‹'a::len word›
    where ‹w = numeral m›
  moreover define q :: nat
    where ‹q = pred-numeral n›
  ultimately have num:
    ‹numeral m = w›
    ‹numeral (num.Bit0 m) = 2 * w›
    ‹numeral (num.Bit1 m) = 1 + 2 * w›
    ‹numeral (Num.inc m) = 1 + w›
    ‹pred-numeral n = q›
    ‹numeral n = Suc q›
   by (simp-all only: w-def q-def numeral-Bit0 [of m] numeral-Bit1 [of m] ac-simps
      numeral-inc numeral-eq-Suc flip: mult-2)
  have even: ‹take-bit (Suc q) (2 * w) = 2 * take-bit q w› for w :: ‹'a::len word›
    by (rule bit-word-eqI)
      (auto simp add: bit-take-bit-iff bit-double-iff)
```

have *odd*: ‹*take-bit* (*Suc q*) (*1 + 2 * w*) = *1 + 2 * take-bit q w*› **for** *w* :: ‹*'a*::*len word*›
by (*rule bit-eqI*)
(*auto simp add*: *bit-take-bit-iff bit-double-iff even-bit-succ-iff*)
show *?P*
using *even* [*of w*] **by** (*simp add*: *num*)
show *?Q*
using *odd* [*of w*] **by** (*simp add*: *num*)
show *?R*
using *even* [*of* ‹*− w*›] **by** (*simp add*: *num*)
show *?S*
using *odd* [*of* ‹*− (1 + w)*›] **by** (*simp add*: *num*)
qed

98.9 More shift operations

lift-definition *signed-drop-bit* :: ‹*nat* ⇒ *'a word* ⇒ *'a*::*len word*›
is ‹*λn. drop-bit n ∘ signed-take-bit* (*LENGTH*('*a*) *− Suc 0*)›
using *signed-take-bit-decr-length-iff*
by (*simp add*: *take-bit-drop-bit*) *force*

lemma *bit-signed-drop-bit-iff* [*bit-simps*]:
‹*bit* (*signed-drop-bit m w*) *n* ⟷ *bit w* (*if LENGTH*('*a*) *− m* ≤ *n* ∧ *n* < *LENGTH*('*a*) *then LENGTH*('*a*) *− 1 else m + n*)›
for *w* :: ‹*'a*::*len word*›
apply *transfer*
apply (*auto simp add*: *bit-drop-bit-eq bit-signed-take-bit-iff not-le min-def*)
apply (*metis add.commute le-antisym less-diff-conv less-eq-decr-length-iff*)
apply (*metis le-antisym less-eq-decr-length-iff*)
done

lemma [*code*]:
‹*Word.the-int* (*signed-drop-bit n w*) = *take-bit LENGTH*('*a*) (*drop-bit n* (*Word.the-signed-int w*))›
for *w* :: ‹*'a*::*len word*›
by *transfer simp*

lemma *signed-drop-bit-of-0* [*simp*]:
‹*signed-drop-bit n 0* = *0*›
by *transfer simp*

lemma *signed-drop-bit-of-minus-1* [*simp*]:
‹*signed-drop-bit n* (*− 1*) = *− 1*›
by *transfer simp*

lemma *signed-drop-bit-signed-drop-bit* [*simp*]:
‹*signed-drop-bit m* (*signed-drop-bit n w*) = *signed-drop-bit* (*m + n*) *w*›
for *w* :: ‹*'a*::*len word*›
proof (*cases* ‹*LENGTH*('*a*)›)

case *0*
then show *?thesis*
using *len-not-eq-0* **by** *blast*
next
case (*Suc n*)
then show *?thesis*
by (*force simp add*: *bit-signed-drop-bit-iff not-le less-diff-conv ac-simps intro*!: *bit-word-eqI*)
qed

lemma *signed-drop-bit-0* [*simp*]:
‹*signed-drop-bit 0 w = w*›
by *transfer* (*simp add*: *take-bit-signed-take-bit*)

lemma *sint-signed-drop-bit-eq*:
‹*sint* (*signed-drop-bit n w*) = *drop-bit n* (*sint w*)›
proof (*cases* ‹*LENGTH*('*a*) = *0* ∨ *n*=*0*›)
case *False*
then show *?thesis*
apply *simp*
apply (*rule bit-eqI*)
by (*auto simp add*: *bit-sint-iff bit-drop-bit-eq bit-signed-drop-bit-iff dest*: *bit-imp-le-length*)
qed *auto*

98.10 Rotation

lift-definition *word-rotr* :: ‹*nat* ⇒ '*a*::*len word* ⇒ '*a*::*len word*›
is ‹λ*n k*. *concat-bit* (*LENGTH*('*a*) − *n mod LENGTH*('*a*))
(*drop-bit* (*n mod LENGTH*('*a*)) (*take-bit LENGTH*('*a*) *k*))
(*take-bit* (*n mod LENGTH*('*a*)) *k*)›
subgoal for *n k l*
by (*simp add*: *concat-bit-def nat-le-iff less-imp-le*
take-bit-tightened [*of* ‹*LENGTH*('*a*)› *k l* ‹*n mod LENGTH*('*a*::*len*)›])
done

lift-definition *word-rotl* :: ‹*nat* ⇒ '*a*::*len word* ⇒ '*a*::*len word*›
is ‹λ*n k*. *concat-bit* (*n mod LENGTH*('*a*))
(*drop-bit* (*LENGTH*('*a*) − *n mod LENGTH*('*a*)) (*take-bit LENGTH*('*a*) *k*))
(*take-bit* (*LENGTH*('*a*) − *n mod LENGTH*('*a*)) *k*)›
subgoal for *n k l*
by (*simp add*: *concat-bit-def nat-le-iff less-imp-le*
take-bit-tightened [*of* ‹*LENGTH*('*a*)› *k l* ‹*LENGTH*('*a*) − *n mod LENGTH*('*a*::*len*)›])
done

lift-definition *word-roti* :: ‹*int* ⇒ '*a*::*len word* ⇒ '*a*::*len word*›
is ‹λ*r k*. *concat-bit* (*LENGTH*('*a*) − *nat* (*r mod int LENGTH*('*a*)))
(*drop-bit* (*nat* (*r mod int LENGTH*('*a*))) (*take-bit LENGTH*('*a*) *k*))
(*take-bit* (*nat* (*r mod int LENGTH*('*a*))) *k*)›
subgoal for *r k l*

by (*simp add*: *concat-bit-def nat-le-iff less-imp-le*
take-bit-tightened [*of* ‹*LENGTH*('*a*)› *k l* ‹*nat* (*r mod int LENGTH*('*a*::*len*))›])
done

lemma *word-rotl-eq-word-rotr* [*code*]:
‹*word-rotl n* = (*word-rotr* (*LENGTH*('*a*) − *n mod LENGTH*('*a*)) :: '*a*::*len word* ⇒ '*a word*)›
by (*rule ext*, *cases* ‹*n mod LENGTH*('*a*) = *0*›; *transfer*) *simp-all*

lemma *word-roti-eq-word-rotr-word-rotl* [*code*]:
‹*word-roti i w* =
(*if i* ≥ *0 then word-rotr* (*nat i*) *w else word-rotl* (*nat* (− *i*)) *w*)›
proof (*cases* ‹*i* ≥ *0*›)
case *True*
moreover define *n* **where** ‹*n* = *nat i*›
ultimately have ‹*i* = *int n*›
by *simp*
moreover have ‹*word-roti* (*int n*) = (*word-rotr n* :: - ⇒ '*a word*)›
by (*rule ext*, *transfer*) (*simp add*: *nat-mod-distrib*)
ultimately show *?thesis*
by *simp*
next
case *False*
moreover define *n* **where** ‹*n* = *nat* (− *i*)›
ultimately have ‹*i* = − *int n*› ‹*n* > *0*›
by *simp-all*
moreover have ‹*word-roti* (− *int n*) = (*word-rotl n* :: - ⇒ '*a word*)›
by (*rule ext*, *transfer*)
(*simp add*: *zmod-zminus1-eq-if flip*: *of-nat-mod of-nat-diff*)
ultimately show *?thesis*
by *simp*
qed

lemma *bit-word-rotr-iff* [*bit-simps*]:
‹*bit* (*word-rotr m w*) *n* ⟷
n < *LENGTH*('*a*) ∧ *bit w* ((*n* + *m*) *mod LENGTH*('*a*))›
for *w* :: ‹'*a*::*len word*›
proof *transfer*
fix *k* :: *int* **and** *m n* :: *nat*
define *q* **where** ‹*q* = *m mod LENGTH*('*a*)›
have ‹*q* < *LENGTH*('*a*)›
by (*simp add*: *q-def*)
then have ‹*q* ≤ *LENGTH*('*a*)›
by *simp*
have ‹*m mod LENGTH*('*a*) = *q*›
by (*simp add*: *q-def*)
moreover have ‹(*n* + *m*) *mod LENGTH*('*a*) = (*n* + *q*) *mod LENGTH*('*a*)›
by (*subst mod-add-right-eq* [*symmetric*]) (*simp add*: ‹*m mod LENGTH*('*a*) = *q*›)

```
  moreover have ‹n < LENGTH('a) ∧
      bit (concat-bit (LENGTH('a) − q) (drop-bit q (take-bit LENGTH('a) k))
(take-bit q k)) n ⟷
    n < LENGTH('a) ∧ bit k ((n + q) mod LENGTH('a))›
    using ‹q < LENGTH('a)›
    by (cases ‹q + n ≥ LENGTH('a)›)
     (auto simp add: bit-concat-bit-iff bit-drop-bit-eq
        bit-take-bit-iff le-mod-geq ac-simps)
  ultimately show ‹n < LENGTH('a) ∧
    bit (concat-bit (LENGTH('a) − m mod LENGTH('a))
      (drop-bit (m mod LENGTH('a)) (take-bit LENGTH('a) k))
      (take-bit (m mod LENGTH('a)) k)) n
    ⟷ n < LENGTH('a) ∧
      (n + m) mod LENGTH('a) < LENGTH('a) ∧
      bit k ((n + m) mod LENGTH('a))›
    by simp
qed

lemma bit-word-rotl-iff [bit-simps]:
  ‹bit (word-rotl m w) n ⟷
    n < LENGTH('a) ∧ bit w ((n + (LENGTH('a) − m mod LENGTH('a))) mod
LENGTH('a))›
  for w :: ‹'a::len word›
  by (simp add: word-rotl-eq-word-rotr bit-word-rotr-iff)

lemma bit-word-roti-iff [bit-simps]:
  ‹bit (word-roti k w) n ⟷
    n < LENGTH('a) ∧ bit w (nat ((int n + k) mod int LENGTH('a)))›
  for w :: ‹'a::len word›
proof transfer
  fix k l :: int and n :: nat
  define m where ‹m = nat (k mod int LENGTH('a))›
  have ‹m < LENGTH('a)›
    by (simp add: nat-less-iff m-def)
  then have ‹m ≤ LENGTH('a)›
    by simp
  have ‹k mod int LENGTH('a) = int m›
    by (simp add: nat-less-iff m-def)
   moreover have ‹(int n + k) mod int LENGTH('a) = int ((n + m) mod
LENGTH('a))›
     by (subst mod-add-right-eq [symmetric]) (simp add: of-nat-mod ‹k mod int
LENGTH('a) = int m›)
  moreover have ‹n < LENGTH('a) ∧
      bit (concat-bit (LENGTH('a) − m) (drop-bit m (take-bit LENGTH('a) l))
(take-bit m l)) n ⟷
    n < LENGTH('a) ∧ bit l ((n + m) mod LENGTH('a))›
    using ‹m < LENGTH('a)›
    by (cases ‹m + n ≥ LENGTH('a)›)
     (auto simp add: bit-concat-bit-iff bit-drop-bit-eq
```

bit-take-bit-iff nat-less-iff not-le not-less ac-simps
le-diff-conv le-mod-geq)
ultimately show ‹*n* < *LENGTH*('*a*)
∧ *bit* (*concat-bit* (*LENGTH*('*a*) − *nat* (*k mod int LENGTH*('*a*)))
(*drop-bit* (*nat* (*k mod int LENGTH*('*a*))) (*take-bit LENGTH*('*a*) *l*))
(*take-bit* (*nat* (*k mod int LENGTH*('*a*))) *l*)) *n* ⟷
n < *LENGTH*('*a*)
∧ *nat* ((*int n* + *k*) *mod int LENGTH*('*a*)) < *LENGTH*('*a*)
∧ *bit l* (*nat* ((*int n* + *k*) *mod int LENGTH*('*a*)))›
by *simp*
qed

lemma *uint-word-rotr-eq*:
‹*uint* (*word-rotr n w*) = *concat-bit* (*LENGTH*('*a*) − *n mod LENGTH*('*a*))
(*drop-bit* (*n mod LENGTH*('*a*)) (*uint w*))
(*uint* (*take-bit* (*n mod LENGTH*('*a*)) *w*))›
for *w* :: ‹'*a*::*len word*›
by *transfer* (*simp add*: *take-bit-concat-bit-eq*)

lemma [*code*]:
‹*Word.the-int* (*word-rotr n w*) = *concat-bit* (*LENGTH*('*a*) − *n mod LENGTH*('*a*))
(*drop-bit* (*n mod LENGTH*('*a*)) (*Word.the-int w*))
(*Word.the-int* (*take-bit* (*n mod LENGTH*('*a*)) *w*))›
for *w* :: ‹'*a*::*len word*›
using *uint-word-rotr-eq* [*of n w*] **by** *simp*

98.11 Split and cat operations

lift-definition *word-cat* :: ‹'*a*::*len word* ⇒ '*b*::*len word* ⇒ '*c*::*len word*›
is ‹λ*k l*. *concat-bit LENGTH*('*b*) *l* (*take-bit LENGTH*('*a*) *k*)›
by (*simp add*: *bit-eq-iff bit-concat-bit-iff bit-take-bit-iff*)

lemma *word-cat-eq*:
‹(*word-cat v w* :: '*c*::*len word*) = *push-bit LENGTH*('*b*) (*ucast v*) + *ucast w*›
for *v* :: ‹'*a*::*len word*› **and** *w* :: ‹'*b*::*len word*›
by *transfer* (*simp add*: *concat-bit-eq ac-simps*)

lemma *word-cat-eq'* [*code*]:
‹*word-cat a b* = *word-of-int* (*concat-bit LENGTH*('*b*) (*uint b*) (*uint a*))›
for *a* :: ‹'*a*::*len word*› **and** *b* :: ‹'*b*::*len word*›
by *transfer* (*simp add*: *concat-bit-take-bit-eq*)

lemma *bit-word-cat-iff* [*bit-simps*]:
‹*bit* (*word-cat v w* :: '*c*::*len word*) *n* ⟷ *n* < *LENGTH*('*c*) ∧ (*if n* < *LENGTH*('*b*)
then bit w n else bit v (*n* − *LENGTH*('*b*)))›
for *v* :: ‹'*a*::*len word*› **and** *w* :: ‹'*b*::*len word*›
by *transfer* (*simp add*: *bit-concat-bit-iff bit-take-bit-iff*)

definition *word-split* :: ‹'*a*::*len word* ⇒ '*b*::*len word* × '*c*::*len word*›

where ‹*word-split w* =
(*ucast* (*drop-bit LENGTH*('*c*) *w*) :: '*b*::*len word*, *ucast w* :: '*c*::*len word*)›

definition *word-rcat* :: ‹'*a*::*len word list* ⇒ '*b*::*len word*›
where ‹*word-rcat* = *word-of-int* ∘ *horner-sum uint* (*2* ^ *LENGTH*('*a*)) ∘ *rev*›

98.12 More on conversions

lemma *int-word-sint*:
‹*sint* (*word-of-int x* :: '*a*::*len word*) = (*x* + *2* ^ (*LENGTH*('*a*) − *1*)) *mod* *2* ^ *LENGTH*('*a*) − *2* ^ (*LENGTH*('*a*) − *1*)›
by *transfer* (*simp flip*: *take-bit-eq-mod add*: *signed-take-bit-eq-take-bit-shift*)

lemma *sint-sbintrunc'*: *sint* (*word-of-int bin* :: '*a word*) = *signed-take-bit* (*LENGTH*('*a*::*len*) − *1*) *bin*
by (*simp add*: *signed-of-int*)

lemma *uint-sint*: *uint w* = *take-bit LENGTH*('*a*) (*sint w*)
for *w* :: '*a*::*len word*
by *transfer* (*simp add*: *take-bit-signed-take-bit*)

lemma *bintr-uint*: *LENGTH*('*a*) ≤ *n* ⟹ *take-bit n* (*uint w*) = *uint w*
for *w* :: '*a*::*len word*
by *transfer* (*simp add*: *min-def*)

lemma *wi-bintr*:
LENGTH('*a*::*len*) ≤ *n* ⟹
word-of-int (*take-bit n w*) = (*word-of-int w* :: '*a word*)
by *transfer simp*

lemma *word-numeral-alt*: *numeral b* = *word-of-int* (*numeral b*)
by (*induct b*, *simp-all only*: *numeral.simps word-of-int-homs*)

declare *word-numeral-alt* [*symmetric*, *code-abbrev*]

lemma *word-neg-numeral-alt*: − *numeral b* = *word-of-int* (− *numeral b*)
by (*simp only*: *word-numeral-alt wi-hom-neg*)

declare *word-neg-numeral-alt* [*symmetric*, *code-abbrev*]

lemma *uint-bintrunc* [*simp*]:
uint (*numeral bin* :: '*a word*) =
take-bit (*LENGTH*('*a*::*len*)) (*numeral bin*)
by *transfer rule*

lemma *uint-bintrunc-neg* [*simp*]:
uint (− *numeral bin* :: '*a word*) = *take-bit* (*LENGTH*('*a*::*len*)) (− *numeral bin*)
by *transfer rule*

lemma *sint-sbintrunc* [*simp*]:
sint (*numeral bin* :: *'a word*) = *signed-take-bit* (*LENGTH*('a::*len*) − *1*) (*numeral bin*)
by *transfer simp*

lemma *sint-sbintrunc-neg* [*simp*]:
sint (− *numeral bin* :: *'a word*) = *signed-take-bit* (*LENGTH*('a::*len*) − *1*) (− *numeral bin*)
by *transfer simp*

lemma *unat-bintrunc* [*simp*]:
unat (*numeral bin* :: *'a*::*len word*) = *nat* (*take-bit* (*LENGTH*('a)) (*numeral bin*))
by *transfer simp*

lemma *unat-bintrunc-neg* [*simp*]:
unat (− *numeral bin* :: *'a*::*len word*) = *nat* (*take-bit* (*LENGTH*('a)) (− *numeral bin*))
by *transfer simp*

lemma *size-0-eq*: *size w* = *0* ⟹ *v* = *w*
for *v w* :: *'a*::*len word*
by *transfer simp*

lemma *uint-ge-0* [*iff*]: *0* ≤ *uint x*
by (*fact unsigned-greater-eq*)

lemma *uint-lt2p* [*iff*]: *uint x* < *2* ^ *LENGTH*('a)
for *x* :: *'a*::*len word*
by (*fact unsigned-less*)

lemma *sint-ge*: − (*2* ^ (*LENGTH*('a) − *1*)) ≤ *sint x*
for *x* :: *'a*::*len word*
using *sint-greater-eq* [*of x*] **by** *simp*

lemma *sint-lt*: *sint x* < *2* ^ (*LENGTH*('a) − *1*)
for *x* :: *'a*::*len word*
using *sint-less* [*of x*] **by** *simp*

lemma *uint-m2p-neg*: *uint x* − *2* ^ *LENGTH*('a) < *0*
for *x* :: *'a*::*len word*
by (*simp only*: *diff-less-0-iff-less uint-lt2p*)

lemma *uint-m2p-not-non-neg*: ¬ *0* ≤ *uint x* − *2* ^ *LENGTH*('a)
for *x* :: *'a*::*len word*
by (*simp only*: *not-le uint-m2p-neg*)

lemma *lt2p-lem*: *LENGTH*('a) ≤ *n* ⟹ *uint w* < *2* ^ *n*
for *w* :: *'a*::*len word*
using *uint-bounded* [*of w*] **by** (*rule less-le-trans*) *simp*

lemma *uint-le-0-iff* [*simp*]: *uint x ≤ 0 ⟷ uint x = 0*
by (*fact uint-ge-0* [*THEN leD*, *THEN antisym-conv1*])

lemma *uint-nat*: *uint w = int* (*unat w*)
by *transfer simp*

lemma *uint-numeral*: *uint* (*numeral b* :: *'a*::*len word*) = *numeral b mod 2* ^ *LENGTH*(*'a*)
by (*simp flip*: *take-bit-eq-mod add*: *of-nat-take-bit*)

lemma *uint-neg-numeral*: *uint* (− *numeral b* :: *'a*::*len word*) = − *numeral b mod 2* ^ *LENGTH*(*'a*)
by (*simp flip*: *take-bit-eq-mod add*: *of-nat-take-bit*)

lemma *unat-numeral*: *unat* (*numeral b* :: *'a*::*len word*) = *numeral b mod 2* ^ *LENGTH*(*'a*)
by *transfer* (*simp add*: *take-bit-eq-mod nat-mod-distrib nat-power-eq*)

lemma *sint-numeral*:
sint (*numeral b* :: *'a*::*len word*) =
(*numeral b* + *2* ^ (*LENGTH*(*'a*) − *1*)) *mod 2* ^ *LENGTH*(*'a*) − *2* ^ (*LENGTH*(*'a*) − *1*)
by (*metis int-word-sint word-numeral-alt*)

lemma *word-of-int-0* [*simp*, *code-post*]: *word-of-int 0 = 0*
by (*fact of-int-0*)

lemma *word-of-int-1* [*simp*, *code-post*]: *word-of-int 1 = 1*
by (*fact of-int-1*)

lemma *word-of-int-neg-1* [*simp*]: *word-of-int* (− *1*) = − *1*
by (*simp add*: *wi-hom-syms*)

lemma *word-of-int-numeral* [*simp*] : (*word-of-int* (*numeral bin*) :: *'a*::*len word*) = *numeral bin*
by (*fact of-int-numeral*)

lemma *word-of-int-neg-numeral* [*simp*]:
(*word-of-int* (− *numeral bin*) :: *'a*::*len word*) = − *numeral bin*
by (*fact of-int-neg-numeral*)

lemma *word-int-case-wi*:
word-int-case f (*word-of-int i* :: *'b word*) = *f* (*i mod 2* ^ *LENGTH*(*'b*::*len*))
by *transfer* (*simp add*: *take-bit-eq-mod*)

lemma *word-int-split*:
P (*word-int-case f x*) =
(∀ *i*. *x* = (*word-of-int i* :: *'b*::*len word*) ∧ *0* ≤ *i* ∧ *i* < *2* ^ *LENGTH*(*'b*) ⟶ *P*

(*f i*))
by *transfer* (*auto simp add*: *take-bit-eq-mod*)

lemma *word-int-split-asm*:
P (*word-int-case f x*) =
(∄*n*. *x* = (*word-of-int n* :: *'b*::*len word*) ∧ *0* ≤ *n* ∧ *n* < *2* ^ *LENGTH*(*'b*::*len*)
∧ ¬ *P* (*f n*))
by *transfer* (*auto simp add*: *take-bit-eq-mod*)

lemma *uint-range-size*: *0* ≤ *uint w* ∧ *uint w* < *2* ^ *size w*
by *transfer simp*

lemma *sint-range-size*: − (*2* ^ (*size w* − *Suc 0*)) ≤ *sint w* ∧ *sint w* < *2* ^ (*size w*
− *Suc 0*)
by (*simp add*: *word-size sint-greater-eq sint-less*)

lemma *sint-above-size*: *2* ^ (*size w* − *1*) ≤ *x* ⟹ *sint w* < *x*
for *w* :: *'a*::*len word*
unfolding *word-size* **by** (*rule less-le-trans* [*OF sint-lt*])

lemma *sint-below-size*: *x* ≤ − (*2* ^ (*size w* − *1*)) ⟹ *x* ≤ *sint w*
for *w* :: *'a*::*len word*
unfolding *word-size* **by** (*rule order-trans* [*OF* - *sint-ge*])

lemma *word-unat-eq-iff*:
‹*v* = *w* ⟷ *unat v* = *unat w*›
for *v w* :: ‹*'a*::*len word*›
by (*fact word-eq-iff-unsigned*)

98.13 Testing bits

lemma *bin-nth-uint-imp*: *bit* (*uint w*) *n* ⟹ *n* < *LENGTH*(*'a*)
for *w* :: *'a*::*len word*
by *transfer* (*simp add*: *bit-take-bit-iff*)

lemma *bin-nth-sint*:
LENGTH(*'a*) ≤ *n* ⟹
bit (*sint w*) *n* = *bit* (*sint w*) (*LENGTH*(*'a*) − *1*)
for *w* :: *'a*::*len word*
by (*transfer fixing*: *n*) (*simp add*: *bit-signed-take-bit-iff le-diff-conv min-def*)

lemma *num-of-bintr'*:
take-bit (*LENGTH*(*'a*::*len*)) (*numeral a* :: *int*) = (*numeral b*) ⟹
numeral a = (*numeral b* :: *'a word*)
proof (*transfer fixing*: *a b*)
assume ‹*take-bit LENGTH*(*'a*) (*numeral a* :: *int*) = *numeral b*›
then have ‹*take-bit LENGTH*(*'a*) (*take-bit LENGTH*(*'a*) (*numeral a* :: *int*)) =
take-bit LENGTH(*'a*) (*numeral b*)›
by *simp*

then show ‹*take-bit LENGTH('a) (numeral a :: int) = take-bit LENGTH('a) (numeral b)*›
by *simp*
qed

lemma *num-of-sbintr'*:
signed-take-bit (LENGTH('a::len) − 1) (numeral a :: int) = (numeral b) ⟹
numeral a = (numeral b :: 'a word)
proof (*transfer fixing*: *a b*)
assume ‹*signed-take-bit (LENGTH('a) − 1) (numeral a :: int) = numeral b*›
then have ‹*take-bit LENGTH('a) (signed-take-bit (LENGTH('a) − 1) (numeral a :: int)) = take-bit LENGTH('a) (numeral b)*›
by *simp*
then show ‹*take-bit LENGTH('a) (numeral a :: int) = take-bit LENGTH('a) (numeral b)*›
by (*simp add*: *take-bit-signed-take-bit*)
qed

lemma *num-abs-bintr*:
(numeral x :: 'a word) =
word-of-int (take-bit (LENGTH('a::len)) (numeral x))
by *transfer simp*

lemma *num-abs-sbintr*:
(numeral x :: 'a word) =
word-of-int (signed-take-bit (LENGTH('a::len) − 1) (numeral x))
by *transfer* (*simp add*: *take-bit-signed-take-bit*)

cast – note, no arg for new length, as it's determined by type of result, thus in *cast w = w*, the type means cast to length of *w*!

lemma *bit-ucast-iff*:
‹*bit (ucast a :: 'a::len word) n ⟷ n < LENGTH('a::len) ∧ bit a n*›
by *transfer* (*simp add*: *bit-take-bit-iff*)

lemma *ucast-id* [*simp*]: *ucast w = w*
by *transfer simp*

lemma *scast-id* [*simp*]: *scast w = w*
by *transfer* (*simp add*: *take-bit-signed-take-bit*)

lemma *ucast-mask-eq*:
‹*ucast (mask n :: 'b word) = mask (min LENGTH('b::len) n)*›
by (*simp add*: *bit-eq-iff*) (*auto simp add*: *bit-mask-iff bit-ucast-iff*)

— literal u(s)cast
lemma *ucast-bintr* [*simp*]:
ucast (numeral w :: 'a::len word) =
word-of-int (take-bit (LENGTH('a)) (numeral w))
by *transfer simp*

lemma *scast-sbintr* [*simp*]:
scast (*numeral w* ::'*a*::*len word*) =
word-of-int (*signed-take-bit* (*LENGTH*('*a*) − *Suc 0*) (*numeral w*))
by *transfer simp*

lemma *source-size*: *source-size* (*c*::'*a*::*len word* ⇒ -) = *LENGTH*('*a*)
by *transfer simp*

lemma *target-size*: *target-size* (*c*::- ⇒ '*b*::*len word*) = *LENGTH*('*b*)
by *transfer simp*

lemma *is-down*: *is-down c* ⟷ *LENGTH*('*b*) ≤ *LENGTH*('*a*)
for *c* :: '*a*::*len word* ⇒ '*b*::*len word*
by *transfer simp*

lemma *is-up*: *is-up c* ⟷ *LENGTH*('*a*) ≤ *LENGTH*('*b*)
for *c* :: '*a*::*len word* ⇒ '*b*::*len word*
by *transfer simp*

lemma *is-up-down*:
‹*is-up c* ⟷ *is-down d*›
for *c* :: ‹'*a*::*len word* ⇒ '*b*::*len word*›
and *d* :: ‹'*b*::*len word* ⇒ '*a*::*len word*›
by *transfer simp*

context
fixes *dummy-types* :: ‹'*a*::*len* × '*b*::*len*›
begin

private abbreviation (*input*) *UCAST* :: ‹'*a*::*len word* ⇒ '*b*::*len word*›
where ‹*UCAST* == *ucast*›

private abbreviation (*input*) *SCAST* :: ‹'*a*::*len word* ⇒ '*b*::*len word*›
where ‹*SCAST* == *scast*›

lemma *down-cast-same*:
‹*UCAST* = *scast*› **if** ‹*is-down UCAST*›
by (*rule ext*, *use that* **in** *transfer*) (*simp add*: *take-bit-signed-take-bit*)

lemma *sint-up-scast*:
‹*sint* (*SCAST w*) = *sint w*› **if** ‹*is-up SCAST*›
using *that* **by** *transfer* (*simp add*: *min-def Suc-leI le-diff-iff*)

lemma *uint-up-ucast*:
‹*uint* (*UCAST w*) = *uint w*› **if** ‹*is-up UCAST*›
using *that* **by** *transfer* (*simp add*: *min-def*)

lemma *ucast-up-ucast*:
‹*ucast* (*UCAST w*) = *ucast w*› **if** ‹*is-up UCAST*›
using *that* **by** *transfer* (*simp add*: *ac-simps*)

lemma *ucast-up-ucast-id*:
‹*ucast* (*UCAST w*) = *w*› **if** ‹*is-up UCAST*›
using *that* **by** (*simp add*: *ucast-up-ucast*)

lemma *scast-up-scast*:
‹*scast* (*SCAST w*) = *scast w*› **if** ‹*is-up SCAST*›
using *that* **by** *transfer* (*simp add*: *ac-simps*)

lemma *scast-up-scast-id*:
‹*scast* (*SCAST w*) = *w*› **if** ‹*is-up SCAST*›
using *that* **by** (*simp add*: *scast-up-scast*)

lemma *isduu*:
‹*is-up UCAST*› **if** ‹*is-down d*›
for *d* :: ‹*'b word* ⇒ *'a word*›
using *that* *is-up-down* [*of UCAST d*] **by** *simp*

lemma *isdus*:
‹*is-up SCAST*› **if** ‹*is-down d*›
for *d* :: ‹*'b word* ⇒ *'a word*›
using *that* *is-up-down* [*of SCAST d*] **by** *simp*

lemmas *ucast-down-ucast-id* = *isduu* [*THEN ucast-up-ucast-id*]
lemmas *scast-down-scast-id* = *isdus* [*THEN scast-up-scast-id*]

lemma *up-ucast-surj*:
‹*surj* (*ucast* :: *'b word* ⇒ *'a word*)› **if** ‹*is-up UCAST*›
by (*rule surjI*) (*use that* **in** ‹*rule ucast-up-ucast-id*›)

lemma *up-scast-surj*:
‹*surj* (*scast* :: *'b word* ⇒ *'a word*)› **if** ‹*is-up SCAST*›
by (*rule surjI*) (*use that* **in** ‹*rule scast-up-scast-id*›)

lemma *down-ucast-inj*:
‹*inj-on UCAST A*› **if** ‹*is-down* (*ucast* :: *'b word* ⇒ *'a word*)›
by (*rule inj-on-inverseI*) (*use that* **in** ‹*rule ucast-down-ucast-id*›)

lemma *down-scast-inj*:
‹*inj-on SCAST A*› **if** ‹*is-down* (*scast* :: *'b word* ⇒ *'a word*)›
by (*rule inj-on-inverseI*) (*use that* **in** ‹*rule scast-down-scast-id*›)

lemma *ucast-down-wi*:
‹*UCAST* (*word-of-int x*) = *word-of-int x*› **if** ‹*is-down UCAST*›
using *that* **by** *transfer simp*

lemma *ucast-down-no*:
  ‹*UCAST* (*numeral bin*) = *numeral bin*› **if** ‹*is-down UCAST*›
  using *that* **by** *transfer simp*

end

lemmas *word-log-defs* = *word-and-def word-or-def word-xor-def word-not-def*

lemma *bit-last-iff*:
  ‹*bit w* (*LENGTH*('*a*) − *Suc 0*) ⟷ *sint w* < *0*› (**is** ‹*?P* ⟷ *?Q*›)
  for *w* :: ‹'*a*::*len word*›
proof −
  have ‹*?P* ⟷ *bit* (*uint w*) (*LENGTH*('*a*) − *Suc 0*)›
    by (*simp add*: *bit-uint-iff*)
  also have ‹... ⟷ *?Q*›
    by (*simp add*: *sint-uint*)
  finally show *?thesis* **.**
qed

lemma *drop-bit-eq-zero-iff-not-bit-last*:
  ‹*drop-bit* (*LENGTH*('*a*) − *Suc 0*) *w* = *0* ⟷ ¬ *bit w* (*LENGTH*('*a*) − *Suc 0*)›
  for *w* :: '*a*::*len word*
proof (*cases* ‹*LENGTH*('*a*)›)
  case (*Suc n*)
  then show *?thesis*
    apply *transfer*
    apply (*simp add*: *take-bit-drop-bit*)
    by (*simp add*: *bit-iff-odd-drop-bit drop-bit-take-bit odd-iff-mod-2-eq-one*)
qed *auto*

lemma *unat-div*:
  ‹*unat* (*x div y*) = *unat x div unat y*›
  by (*fact unat-div-distrib*)

lemma *unat-mod*:
  ‹*unat* (*x mod y*) = *unat x mod unat y*›
  by (*fact unat-mod-distrib*)

98.14 Word Arithmetic

lemmas *less-eq-word-numeral-numeral* [*simp*] =
  word-le-def [*of* ‹*numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
  for *a b*
lemmas *less-word-numeral-numeral* [*simp*] =
  word-less-def [*of* ‹*numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
  for *a b*

lemmas *less-eq-word-minus-numeral-numeral* [*simp*] =
word-le-def [*of* ‹− *numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-word-minus-numeral-numeral* [*simp*] =
word-less-def [*of* ‹− *numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-eq-word-numeral-minus-numeral* [*simp*] =
word-le-def [*of* ‹*numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-word-numeral-minus-numeral* [*simp*] =
word-less-def [*of* ‹*numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-eq-word-minus-numeral-minus-numeral* [*simp*] =
word-le-def [*of* ‹− *numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-word-minus-numeral-minus-numeral* [*simp*] =
word-less-def [*of* ‹− *numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-word-numeral-minus-1* [*simp*] =
word-less-def [*of* ‹*numeral a*› ‹− *1*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *less-word-minus-numeral-minus-1* [*simp*] =
word-less-def [*of* ‹− *numeral a*› ‹− *1*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*

lemmas *sless-eq-word-numeral-numeral* [*simp*] =
word-sle-eq [*of* ‹*numeral a*› ‹*numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-word-numeral-numeral* [*simp*] =
word-sless-alt [*of* ‹*numeral a*› ‹*numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-eq-word-minus-numeral-numeral* [*simp*] =
word-sle-eq [*of* ‹− *numeral a*› ‹*numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-word-minus-numeral-numeral* [*simp*] =
word-sless-alt [*of* ‹− *numeral a*› ‹*numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-eq-word-numeral-minus-numeral* [*simp*] =
word-sle-eq [*of* ‹*numeral a*› ‹− *numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-word-numeral-minus-numeral* [*simp*] =

word-sless-alt [*of* ‹*numeral a*› ‹− *numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-eq-word-minus-numeral-minus-numeral* [*simp*] =
word-sle-eq [*of* ‹− *numeral a*› ‹− *numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*
lemmas *sless-word-minus-numeral-minus-numeral* [*simp*] =
word-sless-alt [*of* ‹− *numeral a*› ‹− *numeral b*›, *simplified sint-sbintrunc sint-sbintrunc-neg*]
for *a b*

lemmas *div-word-numeral-numeral* [*simp*] =
word-div-def [*of* ‹*numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *div-word-minus-numeral-numeral* [*simp*] =
word-div-def [*of* ‹− *numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *div-word-numeral-minus-numeral* [*simp*] =
word-div-def [*of* ‹*numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *div-word-minus-numeral-minus-numeral* [*simp*] =
word-div-def [*of* ‹− *numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *div-word-minus-1-numeral* [*simp*] =
word-div-def [*of* ‹− *1*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *div-word-minus-1-minus-numeral* [*simp*] =
word-div-def [*of* ‹− *1*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*

lemmas *mod-word-numeral-numeral* [*simp*] =
word-mod-def [*of* ‹*numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *mod-word-minus-numeral-numeral* [*simp*] =
word-mod-def [*of* ‹− *numeral a*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *mod-word-numeral-minus-numeral* [*simp*] =
word-mod-def [*of* ‹*numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *mod-word-minus-numeral-minus-numeral* [*simp*] =
word-mod-def [*of* ‹− *numeral a*› ‹− *numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]

for *a b*
lemmas *mod-word-minus-1-numeral* [*simp*] =
word-mod-def [*of* ‹*− 1*› ‹*numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*
lemmas *mod-word-minus-1-minus-numeral* [*simp*] =
word-mod-def [*of* ‹*− 1*› ‹*− numeral b*›, *simplified uint-bintrunc uint-bintrunc-neg unsigned-minus-1-eq-mask mask-eq-exp-minus-1*]
for *a b*

lemma *signed-drop-bit-of-1* [*simp*]:
‹*signed-drop-bit n* (*1* :: *'a*::*len word*) = *of-bool* (*LENGTH*('*a*) = *1* ∨ *n* = *0*)›
apply (*transfer fixing*: *n*)
apply (*cases* ‹*LENGTH*('*a*)›)
apply (*auto simp add*: *take-bit-signed-take-bit*)
apply (*auto simp add*: *take-bit-drop-bit gr0-conv-Suc simp flip*: *take-bit-eq-self-iff-drop-bit-eq-0*)
done

lemma *take-bit-word-beyond-length-eq*:
‹*take-bit n w* = *w*› **if** ‹*LENGTH*('*a*) ≤ *n*› **for** *w* :: ‹*'a*::*len word*›
using *that* **by** *transfer simp*

lemmas *word-div-no* [*simp*] = *word-div-def* [*of numeral a numeral b*] **for** *a b*
lemmas *word-mod-no* [*simp*] = *word-mod-def* [*of numeral a numeral b*] **for** *a b*
lemmas *word-less-no* [*simp*] = *word-less-def* [*of numeral a numeral b*] **for** *a b*
lemmas *word-le-no* [*simp*] = *word-le-def* [*of numeral a numeral b*] **for** *a b*
lemmas *word-sless-no* [*simp*] = *word-sless-eq* [*of numeral a numeral b*] **for** *a b*
lemmas *word-sle-no* [*simp*] = *word-sle-eq* [*of numeral a numeral b*] **for** *a b*

lemma *size-0-same'*: *size w* = *0* ⟹ *w* = *v*
for *v w* :: *'a*::*len word*
by (*unfold word-size*) *simp*

lemmas *size-0-same* = *size-0-same'* [*unfolded word-size*]

lemmas *unat-eq-0* = *unat-0-iff*
lemmas *unat-eq-zero* = *unat-0-iff*

lemma *mask-1*: *mask 1* = *1*
by *simp*

lemma *mask-Suc-0*: *mask* (*Suc 0*) = *1*
by *simp*

lemma *bin-last-bintrunc*: *odd* (*take-bit l n*) ⟷ *l* > *0* ∧ *odd n*
by *simp*

lemma *push-bit-word-beyond* [*simp*]:
‹*push-bit n w* = *0*› **if** ‹*LENGTH*('*a*) ≤ *n*› **for** *w* :: ‹*'a*::*len word*›

using *that* **by** (*transfer fixing*: *n*) (*simp add*: *take-bit-push-bit*)

lemma *drop-bit-word-beyond* [*simp*]:
‹*drop-bit n w* = *0*› **if** ‹*LENGTH*('*a*) ≤ *n*› **for** *w* :: ‹'*a*::*len word*›
using *that* **by** (*transfer fixing*: *n*) (*simp add*: *drop-bit-take-bit*)

lemma *signed-drop-bit-beyond*:
‹*signed-drop-bit n w* = (*if bit w* (*LENGTH*('*a*) − *Suc 0*) *then* − *1 else 0*)›
if ‹*LENGTH*('*a*) ≤ *n*› **for** *w* :: ‹'*a*::*len word*›
by (*rule bit-word-eqI*) (*simp add*: *bit-signed-drop-bit-iff that*)

lemma *take-bit-numeral-minus-numeral-word* [*simp*]:
‹*take-bit* (*numeral m*) (− *numeral n* :: '*a*::*len word*) =
(*case take-bit-num* (*numeral m*) *n of None* ⇒ *0* | *Some q* ⇒ *take-bit* (*numeral m*) (*2* ^ *numeral m* − *numeral q*))› (**is** ‹*?lhs* = *?rhs*›)
proof (*cases* ‹*LENGTH*('*a*) ≤ *numeral m*›)
case *True*
then have ∗: ‹(*take-bit* (*numeral m*) :: '*a word* ⇒ '*a word*) = *id*›
by (*simp add*: *fun-eq-iff take-bit-word-eq-self*)
have ∗∗: ‹*2* ^ *numeral m* = (*0* :: '*a word*)›
using *True* **by** (*simp flip*: *exp-eq-zero-iff*)
show *?thesis*
by (*auto simp only*: ∗ ∗∗ *split*: *option.split*
dest!: *take-bit-num-eq-None-imp* [**where** *?'a* = ‹'*a word*›] *take-bit-num-eq-Some-imp* [**where** *?'a* = ‹'*a word*›])
simp-all
next
case *False*
then show *?thesis*
by (*transfer fixing*: *m n*) *simp*
qed

lemma *of-nat-inverse*:
‹*word-of-nat r* = *a* ⟹ *r* < *2* ^ *LENGTH*('*a*) ⟹ *unat a* = *r*›
for *a* :: ‹'*a*::*len word*›
by (*metis id-apply of-nat-eq-id take-bit-nat-eq-self-iff unsigned-of-nat*)

98.15 Transferring goals from words to ints

lemma *word-ths*:
shows *word-succ-p1*: *word-succ a* = *a* + *1*
and *word-pred-m1*: *word-pred a* = *a* − *1*
and *word-pred-succ*: *word-pred* (*word-succ a*) = *a*
and *word-succ-pred*: *word-succ* (*word-pred a*) = *a*
and *word-mult-succ*: *word-succ a* ∗ *b* = *b* + *a* ∗ *b*
by (*transfer*, *simp add*: *algebra-simps*)+

lemma *uint-cong*: *x* = *y* ⟹ *uint x* = *uint y*
by *simp*

lemma *uint-word-ariths*:
fixes *a b* :: *'a*::*len word*
shows *uint* (*a* + *b*) = (*uint a* + *uint b*) *mod* 2 ^ *LENGTH*(*'a*::*len*)
and *uint* (*a* − *b*) = (*uint a* − *uint b*) *mod* 2 ^ *LENGTH*(*'a*)
and *uint* (*a* ∗ *b*) = *uint a* ∗ *uint b mod* 2 ^ *LENGTH*(*'a*)
and *uint* (− *a*) = − *uint a mod* 2 ^ *LENGTH*(*'a*)
and *uint* (*word-succ a*) = (*uint a* + *1*) *mod* 2 ^ *LENGTH*(*'a*)
and *uint* (*word-pred a*) = (*uint a* − *1*) *mod* 2 ^ *LENGTH*(*'a*)
and *uint* (*0* :: *'a word*) = *0 mod* 2 ^ *LENGTH*(*'a*)
and *uint* (*1* :: *'a word*) = *1 mod* 2 ^ *LENGTH*(*'a*)
by (*simp-all only*: *word-arith-wis uint-word-of-int-eq flip*: *take-bit-eq-mod*)

lemma *uint-word-arith-bintrs*:
fixes *a b* :: *'a*::*len word*
shows *uint* (*a* + *b*) = *take-bit* (*LENGTH*(*'a*)) (*uint a* + *uint b*)
and *uint* (*a* − *b*) = *take-bit* (*LENGTH*(*'a*)) (*uint a* − *uint b*)
and *uint* (*a* ∗ *b*) = *take-bit* (*LENGTH*(*'a*)) (*uint a* ∗ *uint b*)
and *uint* (− *a*) = *take-bit* (*LENGTH*(*'a*)) (− *uint a*)
and *uint* (*word-succ a*) = *take-bit* (*LENGTH*(*'a*)) (*uint a* + *1*)
and *uint* (*word-pred a*) = *take-bit* (*LENGTH*(*'a*)) (*uint a* − *1*)
and *uint* (*0* :: *'a word*) = *take-bit* (*LENGTH*(*'a*)) *0*
and *uint* (*1* :: *'a word*) = *take-bit* (*LENGTH*(*'a*)) *1*
by (*simp-all add*: *uint-word-ariths take-bit-eq-mod*)

lemma *sint-word-ariths*:
fixes *a b* :: *'a*::*len word*
shows *sint* (*a* + *b*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) (*sint a* + *sint b*)
and *sint* (*a* − *b*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) (*sint a* − *sint b*)
and *sint* (*a* ∗ *b*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) (*sint a* ∗ *sint b*)
and *sint* (− *a*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) (− *sint a*)
and *sint* (*word-succ a*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) (*sint a* + *1*)
and *sint* (*word-pred a*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) (*sint a* − *1*)
and *sint* (*0* :: *'a word*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) *0*
and *sint* (*1* :: *'a word*) = *signed-take-bit* (*LENGTH*(*'a*) − *1*) *1*
subgoal
by *transfer* (*simp add*: *signed-take-bit-add*)
subgoal
by *transfer* (*simp add*: *signed-take-bit-diff*)
subgoal
by *transfer* (*simp add*: *signed-take-bit-mult*)
subgoal
by *transfer* (*simp add*: *signed-take-bit-minus*)
apply (*metis of-int-sint scast-id sint-sbintrunc' wi-hom-succ*)
apply (*metis of-int-sint scast-id sint-sbintrunc' wi-hom-pred*)
apply (*simp-all add*: *sint-uint*)
done

lemma *word-pred-0-n1*: *word-pred 0* = *word-of-int* (− *1*)

unfolding *word-pred-m1* **by** *simp*

lemma *succ-pred-no* [*simp*]:
word-succ (*numeral w*) = *numeral w* + *1*
word-pred (*numeral w*) = *numeral w* − *1*
word-succ (− *numeral w*) = − *numeral w* + *1*
word-pred (− *numeral w*) = − *numeral w* − *1*
by (*simp-all add*: *word-succ-p1 word-pred-m1*)

lemma *word-sp-01* [*simp*]:
word-succ (− *1*) = *0* ∧ *word-succ 0* = *1* ∧ *word-pred 0* = − *1* ∧ *word-pred 1* = *0*
by (*simp-all add*: *word-succ-p1 word-pred-m1*)

— alternative approach to lifting arithmetic equalities
lemma *word-of-int-Ex*: ∃ *y*. *x* = *word-of-int y*
by (*rule-tac x*=*uint x* **in** *exI*) *simp*

98.16 Order on fixed-length words

lift-definition *udvd* :: ‹'*a*::*len word* ⇒ '*a*::*len word* ⇒ *bool*› (**infixl** ‹*udvd*› *50*)
is ‹λ*k l*. *take-bit LENGTH*('*a*) *k dvd take-bit LENGTH*('*a*) *l*› **by** *simp*

lemma *udvd-iff-dvd*:
‹*x udvd y* ⟷ *unat x dvd unat y*›
by *transfer* (*simp add*: *nat-dvd-iff*)

lemma *udvd-iff-dvd-int*:
‹*v udvd w* ⟷ *uint v dvd uint w*›
by *transfer rule*

lemma *udvdI* [*intro*]:
‹*v udvd w*› **if** ‹*unat w* = *unat v* ∗ *unat u*›
proof −
from *that* **have** ‹*unat v dvd unat w*› **..**
then show *?thesis*
by (*simp add*: *udvd-iff-dvd*)
qed

lemma *udvdE* [*elim*]:
fixes *v w* :: ‹'*a*::*len word*›
assumes ‹*v udvd w*›
obtains *u* :: ‹'*a word*› **where** ‹*unat w* = *unat v* ∗ *unat u*›
proof (*cases* ‹*v* = *0*›)
case *True*
moreover from *True* ‹*v udvd w*› **have** ‹*w* = *0*›
by *transfer simp*
ultimately show *thesis*
using *that* **by** *simp*

```
next
  case False
  then have ‹unat v > 0›
    by (simp add: unat-gt-0)
  from ‹v udvd w› have ‹unat v dvd unat w›
    by (simp add: udvd-iff-dvd)
  then obtain n where ‹unat w = unat v * n› ..
  moreover have ‹n < 2 ^ LENGTH('a)›
  proof (rule ccontr)
    assume ‹¬ n < 2 ^ LENGTH('a)›
    then have ‹n ≥ 2 ^ LENGTH('a)›
      by (simp add: not-le)
    then have ‹unat v * n ≥ 2 ^ LENGTH('a)›
      using ‹unat v > 0› mult-le-mono [of 1 ‹unat v› ‹2 ^ LENGTH('a)› n]
      by simp
    with ‹unat w = unat v * n›
    have ‹unat w ≥ 2 ^ LENGTH('a)›
      by simp
    with unsigned-less [of w, where ?'a = nat] show False
      by linarith
  qed
  ultimately have ‹unat w = unat v * unat (word-of-nat n :: 'a word)›
    by (auto simp add: take-bit-nat-eq-self-iff unsigned-of-nat intro: sym)
  with that show thesis .
qed

lemma udvd-imp-mod-eq-0:
  ‹w mod v = 0› if ‹v udvd w›
  using that by transfer simp

lemma mod-eq-0-imp-udvd [intro?]:
  ‹v udvd w› if ‹w mod v = 0›
proof −
  from that have ‹unat (w mod v) = unat 0›
    by simp
  then have ‹unat w mod unat v = 0›
    by (simp add: unat-mod-distrib)
  then have ‹unat v dvd unat w› ..
  then show ?thesis
    by (simp add: udvd-iff-dvd)
qed

lemma udvd-imp-dvd:
  ‹v dvd w› if ‹v udvd w› for v w :: ‹'a::len word›
proof −
  from that obtain u :: ‹'a word› where ‹unat w = unat v * unat u› ..
  then have ‹(word-of-nat (unat w) :: 'a word) = word-of-nat (unat v * unat u)›
    by simp
  then have ‹w = v * u›
```

```
    by simp
  then show ‹v dvd w› ..
qed

lemma exp-dvd-iff-exp-udvd:
  ‹2 ^ n dvd w ⟷ 2 ^ n udvd w› for v w :: ‹'a::len word›
proof
  assume ‹2 ^ n udvd w› then show ‹2 ^ n dvd w›
    by (rule udvd-imp-dvd)
next
  assume ‹2 ^ n dvd w›
  then obtain u :: ‹'a word› where ‹w = 2 ^ n * u› ..
  then have ‹w = push-bit n u›
    by (simp add: push-bit-eq-mult)
  then show ‹2 ^ n udvd w›
    by transfer (simp add: take-bit-push-bit dvd-eq-mod-eq-0 flip: take-bit-eq-mod)
qed

lemma udvd-nat-alt:
  ‹a udvd b ⟷ (∃ n. unat b = n * unat a)›
  by (auto simp add: udvd-iff-dvd)

lemma udvd-unfold-int:
  ‹a udvd b ⟷ (∃ n≥0. uint b = n * uint a)›
  unfolding udvd-iff-dvd-int
  by (metis dvd-div-mult-self dvd-triv-right uint-div-distrib uint-ge-0)

lemma unat-minus-one:
  ‹unat (w - 1) = unat w - 1› if ‹w ≠ 0›
proof -
  have 0 ≤ uint w by (fact uint-nonnegative)
  moreover from that have 0 ≠ uint w
    by (simp add: uint-0-iff)
  ultimately have 1 ≤ uint w
    by arith
  from uint-lt2p [of w] have uint w - 1 < 2 ^ LENGTH('a)
    by arith
  with ‹1 ≤ uint w› have (uint w - 1) mod 2 ^ LENGTH('a) = uint w - 1
    by (auto intro: mod-pos-pos-trivial)
  with ‹1 ≤ uint w› have nat ((uint w - 1) mod 2 ^ LENGTH('a)) = nat (uint
w) - 1
    by (auto simp del: nat-uint-eq)
  then show ?thesis
    by (simp only: unat-eq-nat-uint word-arith-wis mod-diff-right-eq)
      (metis of-int-1 uint-word-of-int unsigned-1)
qed

lemma measure-unat: p ≠ 0 ⟹ unat (p - 1) < unat p
  by (simp add: unat-minus-one) (simp add: unat-0-iff [symmetric])
```

lemmas *uint-add-ge0* [*simp*] = *add-nonneg-nonneg* [*OF uint-ge-0 uint-ge-0*]
lemmas *uint-mult-ge0* [*simp*] = *mult-nonneg-nonneg* [*OF uint-ge-0 uint-ge-0*]

lemma *uint-sub-lt2p* [*simp*]: *uint x − uint y < 2 ^ LENGTH('a)*
  for *x :: 'a::len word* **and** *y :: 'b::len word*
  using *uint-ge-0* [*of y*] *uint-lt2p* [*of x*] **by** *arith*

98.17 Conditions for the addition (etc) of two words to overflow

lemma *uint-add-lem*:
  (uint x + uint y < 2 ^ LENGTH('a)) =
    (uint (x + y) = uint x + uint y)
  for *x y :: 'a::len word*
  by (*metis add.right-neutral add-mono-thms-linordered-semiring(1) mod-pos-pos-trivial of-nat-0-le-iff uint-lt2p uint-nat uint-word-ariths(1)*)

lemma *uint-mult-lem*:
  *(uint x * uint y < 2 ^ LENGTH('a)) =*
    *(uint (x * y) = uint x * uint y)*
  for *x y :: 'a::len word*
  by (*metis mod-pos-pos-trivial uint-lt2p uint-mult-ge0 uint-word-ariths(3)*)

lemma *uint-sub-lem*: *uint x ≥ uint y ⟷ uint (x − y) = uint x − uint y*
  by (*metis diff-ge-0-iff-ge of-nat-0-le-iff uint-nat uint-sub-lt2p uint-word-of-int unique-euclidean-semiring-numeral-class.mod-less word-sub-wi*)

lemma *uint-add-le*: *uint (x + y) ≤ uint x + uint y*
  unfolding *uint-word-ariths* **by** (*simp add*: *zmod-le-nonneg-dividend*)

lemma *uint-sub-ge*: *uint (x − y) ≥ uint x − uint y*
  unfolding *uint-word-ariths*
  by (*simp flip*: *take-bit-eq-mod add*: *take-bit-int-greater-eq-self-iff*)

lemma *int-mod-ge*: ‹*a ≤ a mod n*› **if** ‹*a < n*› ‹*0 < n*›
  for *a n :: int*
proof (*cases* ‹*a < 0*›)
  case *True*
  with ‹*0 < n*› **show** *?thesis*
    by (*metis less-trans not-less pos-mod-conj*)

next
  case *False*
  with ‹*a < n*› **show** *?thesis*
    by *simp*
qed

lemma *mod-add-if-z*:

⟦*x* < *z*; *y* < *z*; *0* ≤ *y*; *0* ≤ *x*; *0* ≤ *z*⟧ ⟹
(*x* + *y*) *mod z* = (*if x* + *y* < *z then x* + *y else x* + *y* − *z*)
for *x y z* :: *int*
apply (*simp add*: *not-less*)
by (*metis* (*no-types*) *add-strict-mono diff-ge-0-iff-ge diff-less-eq minus-mod-self2 mod-pos-pos-trivial*)

lemma *uint-plus-if'*:
uint (*a* + *b*) =
(*if uint a* + *uint b* < *2* ^ *LENGTH*('*a*) *then uint a* + *uint b*
else uint a + *uint b* − *2* ^ *LENGTH*('*a*))
for *a b* :: '*a*::*len word*
using *mod-add-if-z* [*of uint a* - *uint b*] **by** (*simp add*: *uint-word-ariths*)

lemma *mod-sub-if-z*:
⟦*x* < *z*; *y* < *z*; *0* ≤ *y*; *0* ≤ *x*; *0* ≤ *z*⟧ ⟹
(*x* − *y*) *mod z* = (*if y* ≤ *x then x* − *y else x* − *y* + *z*)
for *x y z* :: *int*
using *mod-pos-pos-trivial* [*of x* − *y* + *z z*] **by** (*auto simp add*: *not-le*)

lemma *uint-sub-if'*:
uint (*a* − *b*) =
(*if uint b* ≤ *uint a then uint a* − *uint b*
else uint a − *uint b* + *2* ^ *LENGTH*('*a*))
for *a b* :: '*a*::*len word*
using *mod-sub-if-z* [*of uint a* - *uint b*] **by** (*simp add*: *uint-word-ariths*)

lemma *word-of-int-inverse*:
word-of-int r = *a* ⟹ *0* ≤ *r* ⟹ *r* < *2* ^ *LENGTH*('*a*) ⟹ *uint a* = *r*
for *a* :: '*a*::*len word*
by *transfer* (*simp add*: *take-bit-int-eq-self*)

lemma *unat-split*: *P* (*unat x*) ⟷ (∀ *n*. *of-nat n* = *x* ∧ *n* < *2*^*LENGTH*('*a*) ⟶ *P n*)
for *x* :: '*a*::*len word*
by (*auto simp add*: *unsigned-of-nat take-bit-nat-eq-self*)

lemma *unat-split-asm*: *P* (*unat x*) ⟷ (∄ *n*. *of-nat n* = *x* ∧ *n* < *2*^*LENGTH*('*a*) ∧ ¬ *P n*)
for *x* :: '*a*::*len word*
by (*auto simp add*: *unsigned-of-nat take-bit-nat-eq-self*)

lemma *un-ui-le*:
‹*unat a* ≤ *unat b* ⟷ *uint a* ≤ *uint b*›
by *transfer* (*simp add*: *nat-le-iff*)

lemma *unat-plus-if'*:
‹*unat* (*a* + *b*) =
(*if unat a* + *unat b* < *2* ^ *LENGTH*('*a*)

then unat a + unat b
else unat a + unat b − 2 ^ LENGTH('a)› **for** *a b* :: ‹*'a*::*len word*›
apply (*auto simp add*: *not-less le-iff-add*)
apply (*metis* (*mono-tags*, *lifting*) *of-nat-add of-nat-unat take-bit-nat-eq-self-iff unsigned-less unsigned-of-nat unsigned-word-eqI*)
apply (*smt* (*verit*, *ccfv-SIG*) *dbl-simps*(*3*) *dbl-simps*(*5*) *numerals*(*1*) *of-nat-0-le-iff of-nat-add of-nat-eq-iff of-nat-numeral of-nat-power of-nat-unat uint-plus-if' unsigned-1*)
done

lemma *unat-sub-if-size*:
unat (*x* − *y*) =
(*if unat y* ≤ *unat x*
then unat x − *unat y*
else unat x + *2* ^ *size x* − *unat y*)
proof −
{ assume *xy*: ¬ *uint y* ≤ *uint x*
have *nat* (*uint x* − *uint y* + *2* ^ *LENGTH('a)*) = *nat* (*uint x* + *2* ^ *LENGTH('a)* − *uint y*)
by *simp*
also have ... = *nat* (*uint x* + *2* ^ *LENGTH('a)*) − *nat* (*uint y*)
by (*simp add*: *nat-diff-distrib'*)
also have ... = *nat* (*uint x*) + *2* ^ *LENGTH('a)* − *nat* (*uint y*)
by (*metis nat-add-distrib nat-eq-numeral-power-cancel-iff order-less-imp-le unsigned-0 unsigned-greater-eq unsigned-less*)
finally have *nat* (*uint x* − *uint y* + *2* ^ *LENGTH('a)*) = *nat* (*uint x*) + *2* ^ *LENGTH('a)* − *nat* (*uint y*) **.**
}
then show *?thesis*
by (*simp add*: *word-size*) (*metis nat-diff-distrib' uint-sub-if' un-ui-le unat-eq-nat-uint unsigned-greater-eq*)
qed

lemmas *unat-sub-if'* = *unat-sub-if-size* [*unfolded word-size*]

lemma *uint-split*:
P (*uint x*) = (∀ *i*. *word-of-int i* = *x* ∧ *0* ≤ *i* ∧ *i* < *2*^*LENGTH('a)* ⟶ *P i*)
for *x* :: *'a*::*len word*
by *transfer* (*auto simp add*: *take-bit-eq-mod*)

lemma *uint-split-asm*:
P (*uint x*) = (∄ *i*. *word-of-int i* = *x* ∧ *0* ≤ *i* ∧ *i* < *2*^*LENGTH('a)* ∧ ¬ *P i*)
for *x* :: *'a*::*len word*
by (*auto simp add*: *unsigned-of-int take-bit-int-eq-self*)

98.18 Some proof tool support

lemma *power-False-cong*: *False* ⟹ *a* ^ *b* = *c* ^ *d*
by *auto*

lemmas *unat-splits = unat-split unat-split-asm*

lemmas *unat-arith-simps =*
word-le-nat-alt word-less-nat-alt
word-unat-eq-iff
unat-sub-if' unat-plus-if' unat-div unat-mod

lemmas *uint-splits = uint-split uint-split-asm*

lemmas *uint-arith-simps =*
word-le-def word-less-alt
word-uint-eq-iff
uint-sub-if' uint-plus-if'

— *unat-arith-tac*: tactic to reduce word arithmetic to *nat*, try to solve via *arith*
ML ‹
val unat-arith-simpset =
@{context} (∗ TODO: completely explicitly determined simpset ∗)
|> fold Simplifier.add-simp @{thms unat-arith-simps}
|> fold Splitter.add-split @{thms if-split-asm}
|> fold Simplifier.add-cong @{thms power-False-cong}
|> simpset-of

fun unat-arith-tacs ctxt =
let
fun arith-tac' n t =
Arith-Data.arith-tac ctxt n t
handle Cooper.COOPER - => Seq.empty;
in
[clarify-tac ctxt 1,
full-simp-tac (put-simpset unat-arith-simpset ctxt) 1,
ALLGOALS (full-simp-tac
(put-simpset HOL-ss ctxt
|> fold Splitter.add-split @{thms unat-splits}
|> fold Simplifier.add-cong @{thms power-False-cong})),
rewrite-goals-tac ctxt @{thms word-size},
ALLGOALS (fn n => REPEAT (resolve-tac ctxt [allI, impI] n) THEN
REPEAT (eresolve-tac ctxt [conjE] n) THEN
REPEAT (dresolve-tac ctxt @{thms of-nat-inverse} n THEN
assume-tac ctxt n)),
TRYALL arith-tac']
end

fun unat-arith-tac ctxt = SELECT-GOAL (EVERY (unat-arith-tacs ctxt))
›

method-setup *unat-arith =*
‹*Scan.succeed (SIMPLE-METHOD' o unat-arith-tac)*›

solving word arithmetic via natural numbers and arith

— *uint-arith-tac*: reduce to arithmetic on int, try to solve by arith
ML ‹
val uint-arith-simpset =
@{context} (∗ TODO: completely explicitly determined simpset ∗)
|> fold Simplifier.add-simp @{thms uint-arith-simps}
|> fold Splitter.add-split @{thms if-split-asm}
|> fold Simplifier.add-cong @{thms power-False-cong}
|> simpset-of;

fun uint-arith-tacs ctxt =
let
fun arith-tac′ n t =
Arith-Data.arith-tac ctxt n t
handle Cooper.COOPER - => Seq.empty;
in
[clarify-tac ctxt 1,
full-simp-tac (put-simpset uint-arith-simpset ctxt) 1,
ALLGOALS (full-simp-tac
(put-simpset HOL-ss ctxt
|> fold Splitter.add-split @{thms uint-splits}
|> fold Simplifier.add-cong @{thms power-False-cong})),
rewrite-goals-tac ctxt @{thms word-size},
ALLGOALS (fn n => REPEAT (resolve-tac ctxt [allI, impI] n) THEN
REPEAT (eresolve-tac ctxt [conjE] n) THEN
REPEAT (dresolve-tac ctxt @{thms word-of-int-inverse} n
THEN assume-tac ctxt n
THEN assume-tac ctxt n)),
TRYALL arith-tac′]
end

fun uint-arith-tac ctxt = SELECT-GOAL (EVERY (uint-arith-tacs ctxt))
›

method-setup *uint-arith =*
‹Scan.succeed (SIMPLE-METHOD′ o uint-arith-tac)›
solving word arithmetic via integers and arith

98.19 More on overflows and monotonicity

lemma *no-plus-overflow-uint-size*: $x \leq x + y \longleftrightarrow uint\ x + uint\ y < 2\ \hat{}\ size\ x$
for *x y :: 'a::len word*
by (*auto simp add*: *word-size word-le-def uint-add-lem uint-sub-lem*)

lemmas *no-olen-add = no-plus-overflow-uint-size [unfolded word-size]*

lemma *no-ulen-sub*: $x \geq x - y \longleftrightarrow uint\ y \leq uint\ x$
for *x y :: 'a::len word*

by (*auto simp add*: *word-size word-le-def uint-add-lem uint-sub-lem*)

lemma *no-olen-add'*: $x \leq y + x \longleftrightarrow uint\ y + uint\ x < 2 \hat{} LENGTH('a)$
for *x y* :: *'a*::*len word*
by (*simp add*: *ac-simps no-olen-add*)

lemmas *olen-add-eqv* = *trans* [*OF no-olen-add no-olen-add'* [*symmetric*]]

lemmas *uint-plus-simple-iff* = *trans* [*OF no-olen-add uint-add-lem*]
lemmas *uint-plus-simple* = *uint-plus-simple-iff* [*THEN iffD1*]
lemmas *uint-minus-simple-iff* = *trans* [*OF no-ulen-sub uint-sub-lem*]
lemmas *uint-minus-simple-alt* = *uint-sub-lem* [*folded word-le-def*]
lemmas *word-sub-le-iff* = *no-ulen-sub* [*folded word-le-def*]
lemmas *word-sub-le* = *word-sub-le-iff* [*THEN iffD2*]

lemma *word-less-sub1*: $x \neq 0 \Longrightarrow 1 < x \longleftrightarrow 0 < x - 1$
for *x* :: *'a*::*len word*
by *transfer* (*simp add*: *take-bit-decr-eq*)

lemma *word-le-sub1*: $x \neq 0 \Longrightarrow 1 \leq x \longleftrightarrow 0 \leq x - 1$
for *x* :: *'a*::*len word*
by *transfer* (*simp add*: *int-one-le-iff-zero-less less-le*)

lemma *sub-wrap-lt*: $x < x - z \longleftrightarrow x < z$
for *x z* :: *'a*::*len word*
by (*simp add*: *word-less-def uint-sub-lem*)
(*meson linorder-not-le uint-minus-simple-iff uint-sub-lem word-less-iff-unsigned*)

lemma *sub-wrap*: $x \leq x - z \longleftrightarrow z = 0 \vee x < z$
for *x z* :: *'a*::*len word*
by (*simp add*: *le-less sub-wrap-lt ac-simps*)

lemma *plus-minus-not-NULL-ab*: $x \leq ab - c \Longrightarrow c \leq ab \Longrightarrow c \neq 0 \Longrightarrow x + c \neq 0$
for *x ab c* :: *'a*::*len word*
by *uint-arith*

lemma *plus-minus-no-overflow-ab*: $x \leq ab - c \Longrightarrow c \leq ab \Longrightarrow x \leq x + c$
for *x ab c* :: *'a*::*len word*
by *uint-arith*

lemma *le-minus'*: $a + c \leq b \Longrightarrow a \leq a + c \Longrightarrow c \leq b - a$
for *a b c* :: *'a*::*len word*
by *uint-arith*

lemma *le-plus'*: $a \leq b \Longrightarrow c \leq b - a \Longrightarrow a + c \leq b$
for *a b c* :: *'a*::*len word*
by *uint-arith*

lemmas *le-plus* = *le-plus'* [*rotated*]

lemmas *le-minus* = *leD* [*THEN thin-rl*, *THEN le-minus'*]

lemma *word-plus-mono-right*: $y \leq z \Longrightarrow x \leq x + z \Longrightarrow x + y \leq x + z$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-less-minus-cancel*: $y - x < z - x \Longrightarrow x \leq z \Longrightarrow y < z$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-less-minus-mono-left*: $y < z \Longrightarrow x \leq y \Longrightarrow y - x < z - x$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-less-minus-mono*: $a < c \Longrightarrow d < b \Longrightarrow a - b < a \Longrightarrow c - d < c \Longrightarrow a - b < c - d$
for *a b c d* :: *'a*::*len word*
by *uint-arith*

lemma *word-le-minus-cancel*: $y - x \leq z - x \Longrightarrow x \leq z \Longrightarrow y \leq z$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-le-minus-mono-left*: $y \leq z \Longrightarrow x \leq y \Longrightarrow y - x \leq z - x$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-le-minus-mono*:
$a \leq c \Longrightarrow d \leq b \Longrightarrow a - b \leq a \Longrightarrow c - d \leq c \Longrightarrow a - b \leq c - d$
for *a b c d* :: *'a*::*len word*
by *uint-arith*

lemma *plus-le-left-cancel-wrap*: $x + y' < x \Longrightarrow x + y < x \Longrightarrow x + y' < x + y \longleftrightarrow y' < y$
for *x y y'* :: *'a*::*len word*
by *uint-arith*

lemma *plus-le-left-cancel-nowrap*: $x \leq x + y' \Longrightarrow x \leq x + y \Longrightarrow x + y' < x + y \longleftrightarrow y' < y$
for *x y y'* :: *'a*::*len word*
by *uint-arith*

lemma *word-plus-mono-right2*: $a \leq a + b \Longrightarrow c \leq b \Longrightarrow a \leq a + c$
for *a b c* :: *'a*::*len word*
by *uint-arith*

lemma *word-less-add-right*: $x < y - z \Longrightarrow z \leq y \Longrightarrow x + z < y$

for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-less-sub-right*: $x < y + z \Longrightarrow y \leq x \Longrightarrow x - y < z$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-le-plus-either*: $x \leq y \vee x \leq z \Longrightarrow y \leq y + z \Longrightarrow x \leq y + z$
for *x y z* :: *'a*::*len word*
by *uint-arith*

lemma *word-less-nowrapI*: $x < z - k \Longrightarrow k \leq z \Longrightarrow 0 < k \Longrightarrow x < x + k$
for *x z k* :: *'a*::*len word*
by *uint-arith*

lemma *inc-le*: $i < m \Longrightarrow i + 1 \leq m$
for *i m* :: *'a*::*len word*
by *uint-arith*

lemma *inc-i*: $1 \leq i \Longrightarrow i < m \Longrightarrow 1 \leq i + 1 \wedge i + 1 \leq m$
for *i m* :: *'a*::*len word*
by *uint-arith*

lemma *udvd-incr-lem*:
$up < uq \Longrightarrow up = ua + n * uint\ K \Longrightarrow$
$uq = ua + n' * uint\ K \Longrightarrow up + uint\ K \leq uq$
by *auto* (*metis int-distrib*(*1*) *linorder-not-less mult.left-neutral mult-right-mono uint-nonnegative zless-imp-add1-zle*)

lemma *udvd-incr'*:
$p < q \Longrightarrow uint\ p = ua + n * uint\ K \Longrightarrow$
$uint\ q = ua + n' * uint\ K \Longrightarrow p + K \leq q$
unfolding *word-less-alt word-le-def*
by (*metis* (*full-types*) *order-trans udvd-incr-lem uint-add-le*)

lemma *udvd-decr'*:
assumes $p < q$ *uint* $p = ua + n * uint\ K$ *uint* $q = ua + n' * uint\ K$
shows *uint* $q = ua + n' * uint\ K \Longrightarrow p \leq q - K$
proof −
have $\bigwedge w\ wa.\ uint\ (w::'a\ word) \leq uint\ wa + uint\ (w - wa)$
by (*metis* (*no-types*) *add-diff-cancel-left' diff-add-cancel uint-add-le*)
moreover have *uint K* + *uint p* ≤ *uint q*
using *assms* **by** (*metis* (*no-types*) *add-diff-cancel-left' diff-add-cancel udvd-incr-lem word-less-def*)
ultimately show *?thesis*
by (*meson add-le-cancel-left order-trans word-less-eq-iff-unsigned*)
qed

lemmas *udvd-incr-lem0* = *udvd-incr-lem* [**where** *ua=0*, *unfolded add-0-left*]

lemmas *udvd-incr0* = *udvd-incr'* [**where** *ua=0*, *unfolded add-0-left*]
lemmas *udvd-decr0* = *udvd-decr'* [**where** *ua=0*, *unfolded add-0-left*]

lemma *udvd-minus-le'*: $xy < k \Longrightarrow z$ *udvd* $xy \Longrightarrow z$ *udvd* $k \Longrightarrow xy \leq k - z$
unfolding *udvd-unfold-int*
by (*meson udvd-decr0*)

lemma *udvd-incr2-K*:
$p < a + s \Longrightarrow a \leq a + s \Longrightarrow K$ *udvd* $s \Longrightarrow K$ *udvd* $p - a \Longrightarrow a \leq p \Longrightarrow$
$0 < K \Longrightarrow p \leq p + K \wedge p + K \leq a + s$
unfolding *udvd-unfold-int*
apply (*simp add*: *uint-arith-simps split*: *if-split-asm*)
apply (*metis* (*no-types*, *opaque-lifting*) *le-add-diff-inverse le-less-trans udvd-incr-lem*)
using *uint-lt2p* [*of s*] **by** *simp*

98.20 Arithmetic type class instantiations

lemmas *word-le-0-iff* [*simp*] =
word-zero-le [*THEN leD*, *THEN antisym-conv1*]

lemma *word-of-int-nat*: $0 \leq x \Longrightarrow$ *word-of-int* $x =$ *of-nat* (*nat* x)
by *simp*

note that *iszero-def* is only for class *comm-semiring-1-cancel*, which requires word length ≥ 1, ie *'a*::*len word*

lemma *iszero-word-no* [*simp*]:
iszero (*numeral bin* :: *'a*::*len word*) =
iszero (*take-bit LENGTH*(*'a*) (*numeral bin* :: *int*))
by (*metis iszero-def uint-0-iff uint-bintrunc*)

Use *iszero* to simplify equalities between word numerals.

lemmas *word-eq-numeral-iff-iszero* [*simp*] =
eq-numeral-iff-iszero [**where** *'a='a*::*len word*]

98.21 Word and nat

lemma *word-nchotomy*: $\forall w$ :: *'a*::*len word*. $\exists n$. $w =$ *of-nat* $n \wedge n < 2 \hat{} LENGTH('a)$
by (*metis of-nat-unat ucast-id unsigned-less*)

lemma *of-nat-eq*: *of-nat* $n = w \longleftrightarrow (\exists q.\ n =$ *unat* $w + q * 2 \hat{} LENGTH('a))$
for w :: *'a*::*len word*
using *mod-div-mult-eq* [*of n* $2 \hat{} LENGTH('a)$, *symmetric*]
by (*auto simp flip*: *take-bit-eq-mod simp add*: *unsigned-of-nat*)

lemma *of-nat-eq-size*: *of-nat* $n = w \longleftrightarrow (\exists q.\ n =$ *unat* $w + q * 2 \hat{}$ *size* $w)$
unfolding *word-size* **by** (*rule of-nat-eq*)

lemma *of-nat-0*: *of-nat* $m = (0$::*'a*::*len word*$) \longleftrightarrow (\exists q.\ m = q * 2 \hat{} LENGTH('a))$
by (*simp add*: *of-nat-eq*)

lemma *of-nat-2p* [*simp*]: *of-nat* (*2* ^ *LENGTH*('*a*)) = (*0*::'*a*::*len word*)
by (*fact mult-1* [*symmetric*, *THEN iffD2* [*OF of-nat-0 exI*]])

lemma *of-nat-gt-0*: *of-nat k* ≠ *0* ⟹ *0* < *k*
by (*cases k*) *auto*

lemma *of-nat-neq-0*: *0* < *k* ⟹ *k* < *2* ^ *LENGTH*('*a*::*len*) ⟹ *of-nat k* ≠ (*0* :: '*a word*)
by (*auto simp add* : *of-nat-0*)

lemma *Abs-fnat-hom-add*: *of-nat a* + *of-nat b* = *of-nat* (*a* + *b*)
by *simp*

lemma *Abs-fnat-hom-mult*: *of-nat a* ∗ *of-nat b* = (*of-nat* (*a* ∗ *b*) :: '*a*::*len word*)
by (*simp add*: *wi-hom-mult*)

lemma *Abs-fnat-hom-Suc*: *word-succ* (*of-nat a*) = *of-nat* (*Suc a*)
by *transfer* (*simp add*: *ac-simps*)

lemma *Abs-fnat-hom-0*: (*0*::'*a*::*len word*) = *of-nat 0*
by *simp*

lemma *Abs-fnat-hom-1*: (*1*::'*a*::*len word*) = *of-nat* (*Suc 0*)
by *simp*

lemmas *Abs-fnat-homs* =
Abs-fnat-hom-add Abs-fnat-hom-mult Abs-fnat-hom-Suc
Abs-fnat-hom-0 Abs-fnat-hom-1

lemma *word-arith-nat-add*: *a* + *b* = *of-nat* (*unat a* + *unat b*)
by *simp*

lemma *word-arith-nat-mult*: *a* ∗ *b* = *of-nat* (*unat a* ∗ *unat b*)
by *simp*

lemma *word-arith-nat-Suc*: *word-succ a* = *of-nat* (*Suc* (*unat a*))
by (*subst Abs-fnat-hom-Suc* [*symmetric*]) *simp*

lemma *word-arith-nat-div*: *a div b* = *of-nat* (*unat a div unat b*)
by (*metis of-int-of-nat-eq of-nat-unat of-nat-div word-div-def*)

lemma *word-arith-nat-mod*: *a mod b* = *of-nat* (*unat a mod unat b*)
by (*metis of-int-of-nat-eq of-nat-mod of-nat-unat word-mod-def*)

lemmas *word-arith-nat-defs* =
word-arith-nat-add word-arith-nat-mult
word-arith-nat-Suc Abs-fnat-hom-0
Abs-fnat-hom-1 word-arith-nat-div

word-arith-nat-mod

lemma *unat-cong*: $x = y \Longrightarrow unat\ x = unat\ y$
by (*fact arg-cong*)

lemma *unat-of-nat*:
‹*unat* (*word-of-nat x* :: *'a*::*len word*) = *x mod 2* ^ *LENGTH*(*'a*)›
by *transfer* (*simp flip*: *take-bit-eq-mod add*: *nat-take-bit-eq*)

lemmas *unat-word-ariths* = *word-arith-nat-defs*
[*THEN trans* [*OF unat-cong unat-of-nat*]]

lemmas *word-sub-less-iff* = *word-sub-le-iff*
[*unfolded linorder-not-less* [*symmetric*] *Not-eq-iff*]

lemma *unat-add-lem*:
unat x + *unat y* < *2* ^ *LENGTH*(*'a*) ⟷ *unat* (*x* + *y*) = *unat x* + *unat y*
for *x y* :: *'a*::*len word*
by (*metis mod-less unat-word-ariths*(*1*) *unsigned-less*)

lemma *unat-mult-lem*:
unat x * *unat y* < *2* ^ *LENGTH*(*'a*) ⟷ *unat* (*x* * *y*) = *unat x* * *unat y*
for *x y* :: *'a*::*len word*
by (*metis mod-less unat-word-ariths*(*2*) *unsigned-less*)

lemma *le-no-overflow*: $x \leq b \Longrightarrow a \leq a + b \Longrightarrow x \leq a + b$
for *a b x* :: *'a*::*len word*
using *word-le-plus-either* **by** *blast*

lemma *uint-div*:
‹*uint* (*x div y*) = *uint x div uint y*›
by (*fact uint-div-distrib*)

lemma *uint-mod*:
‹*uint* (*x mod y*) = *uint x mod uint y*›
by (*fact uint-mod-distrib*)

lemma *no-plus-overflow-unat-size*: $x \leq x + y \longleftrightarrow unat\ x + unat\ y < 2\ \hat{}\ size\ x$
for *x y* :: *'a*::*len word*
unfolding *word-size* **by** *unat-arith*

lemmas *no-olen-add-nat* =
no-plus-overflow-unat-size [*unfolded word-size*]

lemmas *unat-plus-simple* =
trans [*OF no-olen-add-nat unat-add-lem*]

lemma *word-div-mult*: ⟦*0* < *y*; *unat x* * *unat y* < *2* ^ *LENGTH*(*'a*)⟧ ⟹ *x* * *y*
div y = *x*

for *x y* :: *'a*::*len word*
by (*simp add*: *unat-eq-zero unat-mult-lem word-arith-nat-div*)

lemma *div-lt'*: $i \leq k \; div \; x \Longrightarrow unat \; i * unat \; x < 2 \; \hat{} \; LENGTH('a)$
for *i k x* :: *'a*::*len word*
by *unat-arith* (*meson le-less-trans less-mult-imp-div-less not-le unsigned-less*)

lemmas *div-lt''* = *order-less-imp-le* [*THEN div-lt'*]

lemma *div-lt-mult*: $[\![i < k \; div \; x; \; 0 < x]\!] \Longrightarrow i * x < k$
for *i k x* :: *'a*::*len word*
by (*metis div-le-mono div-lt'' not-le unat-div word-div-mult word-less-iff-unsigned*)

lemma *div-le-mult*: $[\![i \leq k \; div \; x; \; 0 < x]\!] \Longrightarrow i * x \leq k$
for *i k x* :: *'a*::*len word*
by (*metis div-lt' less-mult-imp-div-less not-less unat-arith-simps*(2) *unat-div unat-mult-lem*)

lemma *div-lt-uint'*: $i \leq k \; div \; x \Longrightarrow uint \; i * uint \; x < 2 \; \hat{} \; LENGTH('a)$
for *i k x* :: *'a*::*len word*
unfolding *uint-nat*
by (*metis div-lt' int-ops*(7) *of-nat-unat uint-mult-lem unat-mult-lem*)

lemmas *div-lt-uint''* = *order-less-imp-le* [*THEN div-lt-uint'*]

lemma *word-le-exists'*: $x \leq y \Longrightarrow \exists z. \; y = x + z \wedge uint \; x + uint \; z < 2 \; \hat{} \; LENGTH('a)$
for *x y z* :: *'a*::*len word*
by (*metis add.commute diff-add-cancel no-olen-add*)

lemmas *plus-minus-not-NULL* = *order-less-imp-le* [*THEN plus-minus-not-NULL-ab*]

lemmas *plus-minus-no-overflow* =
order-less-imp-le [*THEN plus-minus-no-overflow-ab*]

lemmas *mcs* = *word-less-minus-cancel word-less-minus-mono-left*
word-le-minus-cancel word-le-minus-mono-left

lemmas *word-l-diffs* = *mcs* [**where** *y* = *w* + *x*, *unfolded add-diff-cancel*] **for** *w x*
lemmas *word-diff-ls* = *mcs* [**where** *z* = *w* + *x*, *unfolded add-diff-cancel*] **for** *w x*
lemmas *word-plus-mcs* = *word-diff-ls* [**where** *y* = *v* + *x*, *unfolded add-diff-cancel*]
for *v x*

lemma *le-unat-uoi*:
‹*y* ≤ *unat z* ⟹ *unat* (*word-of-nat y* :: *'a word*) = *y*›
for *z* :: ‹*'a*::*len word*›
by *transfer* (*simp add*: *nat-take-bit-eq take-bit-nat-eq-self-iff le-less-trans*)

lemmas *thd* = *times-div-less-eq-dividend*

lemmas *uno-simps* [*THEN le-unat-uoi*] = *mod-le-divisor div-le-dividend*

lemma *word-mod-div-equality*: (*n div b*) * *b* + (*n mod b*) = *n*
for *n b* :: *'a*::*len word*
by (*fact div-mult-mod-eq*)

lemma *word-div-mult-le*: *a div b* * *b* ≤ *a*
for *a b* :: *'a*::*len word*
by (*metis div-le-mult mult-not-zero order.not-eq-order-implies-strict order-refl word-zero-le*)

lemma *word-mod-less-divisor*: *0* < *n* ⟹ *m mod n* < *n*
for *m n* :: *'a*::*len word*
by (*simp add*: *unat-arith-simps*)

lemma *word-of-int-power-hom*: *word-of-int a* ^ *n* = (*word-of-int* (*a* ^ *n*) :: *'a*::*len word*)
by (*induct n*) (*simp-all add*: *wi-hom-mult* [*symmetric*])

lemma *word-arith-power-alt*: *a* ^ *n* = (*word-of-int* (*uint a* ^ *n*) :: *'a*::*len word*)
by (*simp add* : *word-of-int-power-hom* [*symmetric*])

lemma *unatSuc*: *1* + *n* ≠ *0* ⟹ *unat* (*1* + *n*) = *Suc* (*unat n*)
for *n* :: *'a*::*len word*
by *unat-arith*

98.22 Cardinality, finiteness of set of words

lemma *inj-on-word-of-int*: ‹*inj-on* (*word-of-int* :: *int* ⇒ *'a word*) {*0*..<*2* ^ *LENGTH*(*'a*::*len*)}›
unfolding *inj-on-def*
by (*metis atLeastLessThan-iff word-of-int-inverse*)

lemma *range-uint*: ‹*range* (*uint* :: *'a word* ⇒ *int*) = {*0*..<*2* ^ *LENGTH*(*'a*::*len*)}›
apply *transfer*
apply (*auto simp add*: *image-iff*)
apply (*metis take-bit-int-eq-self-iff*)
done

lemma *UNIV-eq*: ‹(*UNIV* :: *'a word set*) = *word-of-int* ' {*0*..<*2* ^ *LENGTH*(*'a*::*len*)}›
by (*auto simp add*: *image-iff*) (*metis atLeastLessThan-iff linorder-not-le uint-split*)

lemma *card-word*: *CARD*(*'a word*) = *2* ^ *LENGTH*(*'a*::*len*)
by (*simp add*: *UNIV-eq card-image inj-on-word-of-int*)

lemma *card-word-size*: *CARD*(*'a word*) = *2* ^ *size x*
for *x* :: *'a*::*len word*
unfolding *word-size* **by** (*rule card-word*)

end

instance *word* :: (*len*) *finite*
by *standard* (*simp add*: *UNIV-eq*)

98.23 Bitwise Operations on Words

context
includes *bit-operations-syntax*
begin

lemma *word-wi-log-defs*:
NOT (*word-of-int a*) = *word-of-int* (*NOT a*)
word-of-int a AND word-of-int b = *word-of-int* (*a AND b*)
word-of-int a OR word-of-int b = *word-of-int* (*a OR b*)
word-of-int a XOR word-of-int b = *word-of-int* (*a XOR b*)
by (*transfer*, *rule refl*)+

lemma *word-no-log-defs* [*simp*]:
NOT (*numeral a*) = *word-of-int* (*NOT* (*numeral a*))
NOT (− *numeral a*) = *word-of-int* (*NOT* (− *numeral a*))
numeral a AND numeral b = *word-of-int* (*numeral a AND numeral b*)
numeral a AND − *numeral b* = *word-of-int* (*numeral a AND* − *numeral b*)
− *numeral a AND numeral b* = *word-of-int* (− *numeral a AND numeral b*)
− *numeral a AND* − *numeral b* = *word-of-int* (− *numeral a AND* − *numeral b*)
numeral a OR numeral b = *word-of-int* (*numeral a OR numeral b*)
numeral a OR − *numeral b* = *word-of-int* (*numeral a OR* − *numeral b*)
− *numeral a OR numeral b* = *word-of-int* (− *numeral a OR numeral b*)
− *numeral a OR* − *numeral b* = *word-of-int* (− *numeral a OR* − *numeral b*)
numeral a XOR numeral b = *word-of-int* (*numeral a XOR numeral b*)
numeral a XOR − *numeral b* = *word-of-int* (*numeral a XOR* − *numeral b*)
− *numeral a XOR numeral b* = *word-of-int* (− *numeral a XOR numeral b*)
− *numeral a XOR* − *numeral b* = *word-of-int* (− *numeral a XOR* − *numeral b*)
by (*transfer*, *rule refl*)+

Special cases for when one of the arguments equals 1.

lemma *word-bitwise-1-simps* [*simp*]:
NOT (*1*::*'a*::*len word*) = −*2*
1 AND numeral b = *word-of-int* (*1 AND numeral b*)
1 AND − *numeral b* = *word-of-int* (*1 AND* − *numeral b*)
numeral a AND 1 = *word-of-int* (*numeral a AND 1*)
− *numeral a AND 1* = *word-of-int* (− *numeral a AND 1*)
1 OR numeral b = *word-of-int* (*1 OR numeral b*)
1 OR − *numeral b* = *word-of-int* (*1 OR* − *numeral b*)
numeral a OR 1 = *word-of-int* (*numeral a OR 1*)
− *numeral a OR 1* = *word-of-int* (− *numeral a OR 1*)
1 XOR numeral b = *word-of-int* (*1 XOR numeral b*)
1 XOR − *numeral b* = *word-of-int* (*1 XOR* − *numeral b*)
numeral a XOR 1 = *word-of-int* (*numeral a XOR 1*)
− *numeral a XOR 1* = *word-of-int* (− *numeral a XOR 1*)

apply (*simp-all add*: *word-uint-eq-iff unsigned-not-eq unsigned-and-eq unsigned-or-eq*
unsigned-xor-eq of-nat-take-bit ac-simps unsigned-of-int)
apply (*simp-all add*: *minus-numeral-eq-not-sub-one*)
apply (*simp-all only*: *sub-one-eq-not-neg bit.xor-compl-right take-bit-xor bit.double-compl*)
apply *simp-all*
done

Special cases for when one of the arguments equals -1.

lemma *word-bitwise-m1-simps* [*simp*]:
NOT (*−1*::*'a*::*len word*) = *0*
(*−1*::*'a*::*len word*) *AND x* = *x*
x AND (*−1*::*'a*::*len word*) = *x*
(*−1*::*'a*::*len word*) *OR x* = *−1*
x OR (*−1*::*'a*::*len word*) = *−1*
(*−1*::*'a*::*len word*) *XOR x* = *NOT x*
x XOR (*−1*::*'a*::*len word*) = *NOT x*
by (*transfer*, *simp*)+

lemma *word-of-int-not-numeral-eq* [*simp*]:
‹(*word-of-int* (*NOT* (*numeral bin*)) :: *'a*::*len word*) = *−* *numeral bin − 1*›
by *transfer* (*simp add*: *not-eq-complement*)

lemma *uint-and*:
‹*uint* (*x AND y*) = *uint x AND uint y*›
by *transfer simp*

lemma *uint-or*:
‹*uint* (*x OR y*) = *uint x OR uint y*›
by *transfer simp*

lemma *uint-xor*:
‹*uint* (*x XOR y*) = *uint x XOR uint y*›
by *transfer simp*

— get from commutativity, associativity etc of *int-and* etc to same for *word-and etc*
lemmas *bwsimps* =
wi-hom-add
word-wi-log-defs

lemma *word-bw-assocs*:
(*x AND y*) *AND z* = *x AND y AND z*
(*x OR y*) *OR z* = *x OR y OR z*
(*x XOR y*) *XOR z* = *x XOR y XOR z*
for *x* :: *'a*::*len word*
by (*fact ac-simps*)+

lemma *word-bw-comms*:

$x\ AND\ y = y\ AND\ x$
$x\ OR\ y = y\ OR\ x$
$x\ XOR\ y = y\ XOR\ x$
for $x :: 'a::len\ word$
by (*fact ac-simps*)+

lemma *word-bw-lcs*:
$y\ AND\ x\ AND\ z = x\ AND\ y\ AND\ z$
$y\ OR\ x\ OR\ z = x\ OR\ y\ OR\ z$
$y\ XOR\ x\ XOR\ z = x\ XOR\ y\ XOR\ z$
for $x :: 'a::len\ word$
by (*fact ac-simps*)+

lemma *word-log-esimps*:
$x\ AND\ 0 = 0$
$x\ AND\ -1 = x$
$x\ OR\ 0 = x$
$x\ OR\ -1 = -1$
$x\ XOR\ 0 = x$
$x\ XOR\ -1 = NOT\ x$
$0\ AND\ x = 0$
$-1\ AND\ x = x$
$0\ OR\ x = x$
$-1\ OR\ x = -1$
$0\ XOR\ x = x$
$-1\ XOR\ x = NOT\ x$
for $x :: 'a::len\ word$
by *simp-all*

lemma *word-not-dist*:
$NOT\ (x\ OR\ y) = NOT\ x\ AND\ NOT\ y$
$NOT\ (x\ AND\ y) = NOT\ x\ OR\ NOT\ y$
for $x :: 'a::len\ word$
by *simp-all*

lemma *word-bw-same*:
$x\ AND\ x = x$
$x\ OR\ x = x$
$x\ XOR\ x = 0$
for $x :: 'a::len\ word$
by *simp-all*

lemma *word-ao-absorbs* [*simp*]:
$x\ AND\ (y\ OR\ x) = x$
$x\ OR\ y\ AND\ x = x$
$x\ AND\ (x\ OR\ y) = x$
$y\ AND\ x\ OR\ x = x$
$(y\ OR\ x)\ AND\ x = x$
$x\ OR\ x\ AND\ y = x$

(*x OR y*) *AND x* = *x*
x AND y OR x = *x*
for *x* :: *'a*::*len word*
by (*auto intro*: *bit-eqI simp add*: *bit-and-iff bit-or-iff*)

lemma *word-not-not* [*simp*]: *NOT* (*NOT x*) = *x*
for *x* :: *'a*::*len word*
by (*fact bit.double-compl*)

lemma *word-ao-dist*: (*x OR y*) *AND z* = *x AND z OR y AND z*
for *x* :: *'a*::*len word*
by (*fact bit.conj-disj-distrib2*)

lemma *word-oa-dist*: *x AND y OR z* = (*x OR z*) *AND* (*y OR z*)
for *x* :: *'a*::*len word*
by (*fact bit.disj-conj-distrib2*)

lemma *word-add-not* [*simp*]: *x* + *NOT x* = −*1*
for *x* :: *'a*::*len word*
by (*simp add*: *not-eq-complement*)

lemma *word-plus-and-or* [*simp*]: (*x AND y*) + (*x OR y*) = *x* + *y*
for *x* :: *'a*::*len word*
by *transfer* (*simp add*: *plus-and-or*)

lemma *leoa*: *w* = *x OR y* ⟹ *y* = *w AND y*
for *x* :: *'a*::*len word*
by *auto*

lemma *leao*: *w'* = *x' AND y'* ⟹ *x'* = *x' OR w'*
for *x'* :: *'a*::*len word*
by *auto*

lemma *word-ao-equiv*: *w* = *w OR w'* ⟷ *w'* = *w AND w'*
for *w w'* :: *'a*::*len word*
by (*auto intro*: *leoa leao*)

lemma *le-word-or2*: *x* ≤ *x OR y*
for *x y* :: *'a*::*len word*
by (*simp add*: *or-greater-eq uint-or word-le-def*)

lemmas *le-word-or1* = *xtrans*(*3*) [*OF word-bw-comms* (*2*) *le-word-or2*]
lemmas *word-and-le1* = *xtrans*(*3*) [*OF word-ao-absorbs* (*4*) [*symmetric*] *le-word-or2*]
lemmas *word-and-le2* = *xtrans*(*3*) [*OF word-ao-absorbs* (*8*) [*symmetric*] *le-word-or2*]

lemma *bit-horner-sum-bit-word-iff* [*bit-simps*]:
‹*bit* (*horner-sum of-bool* (*2* :: *'a*::*len word*) *bs*) *n*
⟷ *n* < *min LENGTH*('*a*) (*length bs*) ∧ *bs* ! *n*›
by *transfer* (*simp add*: *bit-horner-sum-bit-iff*)

definition *word-reverse* :: ‹*'a*::*len word* ⇒ *'a word*›
where ‹*word-reverse w* = *horner-sum of-bool 2* (*rev* (*map* (*bit w*) [*0*..<*LENGTH*(*'a*)]))›

lemma *bit-word-reverse-iff* [*bit-simps*]:
‹*bit* (*word-reverse w*) *n* ⟷ *n* < *LENGTH*(*'a*) ∧ *bit w* (*LENGTH*(*'a*) − *Suc n*)›
for *w* :: ‹*'a*::*len word*›
by (*cases* ‹*n* < *LENGTH*(*'a*)›)
(*simp-all add*: *word-reverse-def bit-horner-sum-bit-word-iff rev-nth*)

lemma *word-rev-rev* [*simp*] : *word-reverse* (*word-reverse w*) = *w*
by (*rule bit-word-eqI*)
(*auto simp add*: *bit-word-reverse-iff bit-imp-le-length Suc-diff-Suc*)

lemma *word-rev-gal*: *word-reverse w* = *u* ⟹ *word-reverse u* = *w*
by (*metis word-rev-rev*)

lemma *word-rev-gal'*: *u* = *word-reverse w* ⟹ *w* = *word-reverse u*
by *simp*

lemma *uint-2p*: (*0*::*'a*::*len word*) < *2* ^ *n* ⟹ *uint* (*2* ^ *n*::*'a*::*len word*) = *2* ^ *n*
by (*cases* ‹*n* < *LENGTH*(*'a*)›; *transfer*; *force*)

lemma *word-of-int-2p*: (*word-of-int* (*2* ^ *n*) :: *'a*::*len word*) = *2* ^ *n*
by (*induct n*) (*simp-all add*: *wi-hom-syms*)

98.23.1 shift functions in terms of lists of bools

lemma *drop-bit-word-numeral* [*simp*]:
‹*drop-bit* (*numeral n*) (*numeral k*) =
(*word-of-int* (*drop-bit* (*numeral n*) (*take-bit LENGTH*(*'a*) (*numeral k*))) :: *'a*::*len word*)›
by *transfer simp*

lemma *drop-bit-word-Suc-numeral* [*simp*]:
‹*drop-bit* (*Suc n*) (*numeral k*) =
(*word-of-int* (*drop-bit* (*Suc n*) (*take-bit LENGTH*(*'a*) (*numeral k*))) :: *'a*::*len word*)›
by *transfer simp*

lemma *drop-bit-word-minus-numeral* [*simp*]:
‹*drop-bit* (*numeral n*) (− *numeral k*) =
(*word-of-int* (*drop-bit* (*numeral n*) (*take-bit LENGTH*(*'a*) (− *numeral k*))) :: *'a*::*len word*)›
by *transfer simp*

lemma *drop-bit-word-Suc-minus-numeral* [*simp*]:
‹*drop-bit* (*Suc n*) (− *numeral k*) =
(*word-of-int* (*drop-bit* (*Suc n*) (*take-bit LENGTH*(*'a*) (− *numeral k*))) :: *'a*::*len*

word)›
by *transfer simp*

lemma *signed-drop-bit-word-numeral* [*simp*]:
‹*signed-drop-bit* (*numeral n*) (*numeral k*) =
(*word-of-int* (*drop-bit* (*numeral n*) (*signed-take-bit* (*LENGTH*('*a*) − *1*) (*numeral k*))) :: '*a*::*len word*)›
by *transfer simp*

lemma *signed-drop-bit-word-Suc-numeral* [*simp*]:
‹*signed-drop-bit* (*Suc n*) (*numeral k*) =
(*word-of-int* (*drop-bit* (*Suc n*) (*signed-take-bit* (*LENGTH*('*a*) − *1*) (*numeral k*))) :: '*a*::*len word*)›
by *transfer simp*

lemma *signed-drop-bit-word-minus-numeral* [*simp*]:
‹*signed-drop-bit* (*numeral n*) (− *numeral k*) =
(*word-of-int* (*drop-bit* (*numeral n*) (*signed-take-bit* (*LENGTH*('*a*) − *1*) (− *numeral k*))) :: '*a*::*len word*)›
by *transfer simp*

lemma *signed-drop-bit-word-Suc-minus-numeral* [*simp*]:
‹*signed-drop-bit* (*Suc n*) (− *numeral k*) =
(*word-of-int* (*drop-bit* (*Suc n*) (*signed-take-bit* (*LENGTH*('*a*) − *1*) (− *numeral k*))) :: '*a*::*len word*)›
by *transfer simp*

lemma *take-bit-word-numeral* [*simp*]:
‹*take-bit* (*numeral n*) (*numeral k*) =
(*word-of-int* (*take-bit* (*min LENGTH*('*a*) (*numeral n*)) (*numeral k*)) :: '*a*::*len word*)›
by *transfer rule*

lemma *take-bit-word-Suc-numeral* [*simp*]:
‹*take-bit* (*Suc n*) (*numeral k*) =
(*word-of-int* (*take-bit* (*min LENGTH*('*a*) (*Suc n*)) (*numeral k*)) :: '*a*::*len word*)›
by *transfer rule*

lemma *take-bit-word-minus-numeral* [*simp*]:
‹*take-bit* (*numeral n*) (− *numeral k*) =
(*word-of-int* (*take-bit* (*min LENGTH*('*a*) (*numeral n*)) (− *numeral k*)) :: '*a*::*len word*)›
by *transfer rule*

lemma *take-bit-word-Suc-minus-numeral* [*simp*]:
‹*take-bit* (*Suc n*) (− *numeral k*) =
(*word-of-int* (*take-bit* (*min LENGTH*('*a*) (*Suc n*)) (− *numeral k*)) :: '*a*::*len word*)›
by *transfer rule*

lemma *signed-take-bit-word-numeral* [*simp*]:
‹*signed-take-bit* (*numeral n*) (*numeral k*) =
(*word-of-int* (*signed-take-bit* (*numeral n*) (*take-bit LENGTH*('*a*) (*numeral k*)))
:: '*a*::*len word*)›
by *transfer rule*

lemma *signed-take-bit-word-Suc-numeral* [*simp*]:
‹*signed-take-bit* (*Suc n*) (*numeral k*) =
(*word-of-int* (*signed-take-bit* (*Suc n*) (*take-bit LENGTH*('*a*) (*numeral k*))) ::
'*a*::*len word*)›
by *transfer rule*

lemma *signed-take-bit-word-minus-numeral* [*simp*]:
‹*signed-take-bit* (*numeral n*) (− *numeral k*) =
(*word-of-int* (*signed-take-bit* (*numeral n*) (*take-bit LENGTH*('*a*) (− *numeral* *k*))) :: '*a*::*len word*)›
by *transfer rule*

lemma *signed-take-bit-word-Suc-minus-numeral* [*simp*]:
‹*signed-take-bit* (*Suc n*) (− *numeral k*) =
(*word-of-int* (*signed-take-bit* (*Suc n*) (*take-bit LENGTH*('*a*) (− *numeral k*))) ::
'*a*::*len word*)›
by *transfer rule*

lemma *False-map2-or*: ⟦*set xs* ⊆ {*False*}; *length ys* = *length xs*⟧ ⟹ *map2* (∨) *xs* *ys* = *ys*
by (*induction xs arbitrary*: *ys*) (*auto simp*: *length-Suc-conv*)

lemma *align-lem-or*:
assumes *length xs* = *n* + *m length ys* = *n* + *m*
and *drop m xs* = *replicate n False take m ys* = *replicate m False*
shows *map2* (∨) *xs ys* = *take m xs* @ *drop m ys*
using *assms*
proof (*induction xs arbitrary*: *ys m*)
case (*Cons a xs*)
then show *?case*
by (*cases m*) (*auto simp*: *length-Suc-conv False-map2-or*)
qed *auto*

lemma *False-map2-and*: ⟦*set xs* ⊆ {*False*}; *length ys* = *length xs*⟧ ⟹ *map2* (∧) *xs ys* = *xs*
by (*induction xs arbitrary*: *ys*) (*auto simp*: *length-Suc-conv*)

lemma *align-lem-and*:
assumes *length xs* = *n* + *m length ys* = *n* + *m*
and *drop m xs* = *replicate n False take m ys* = *replicate m False*
shows *map2* (∧) *xs ys* = *replicate* (*n* + *m*) *False*
using *assms*

proof (*induction xs arbitrary*: *ys m*)
case (*Cons a xs*)
then show *?case*
by (*cases m*) (*auto simp*: *length-Suc-conv set-replicate-conv-if False-map2-and*)
qed *auto*

98.23.2 Mask

lemma *minus-1-eq-mask*:
‹*− 1 = (mask LENGTH('a) :: 'a::len word)*›
by (*rule bit-eqI*) (*simp add*: *bit-exp-iff bit-mask-iff*)

lemma *mask-eq-decr-exp*:
‹*mask n = 2 ^ n − (1 :: 'a::len word)*›
by (*fact mask-eq-exp-minus-1*)

lemma *mask-Suc-rec*:
‹*mask (Suc n) = 2 * mask n + (1 :: 'a::len word)*›
by (*simp add*: *mask-eq-exp-minus-1*)

context
begin

qualified lemma *bit-mask-iff* [*bit-simps*]:
‹*bit (mask m :: 'a::len word) n ⟷ n < min LENGTH('a) m*›
by (*simp add*: *bit-mask-iff not-le*)

end

lemma *mask-bin*: *mask n = word-of-int (take-bit n (− 1))*
by *transfer simp*

lemma *and-mask-bintr*: *w AND mask n = word-of-int (take-bit n (uint w))*
by *transfer* (*simp add*: *ac-simps take-bit-eq-mask*)

lemma *and-mask-wi*: *word-of-int i AND mask n = word-of-int (take-bit n i)*
by (*simp add*: *take-bit-eq-mask of-int-and-eq of-int-mask-eq*)

lemma *and-mask-wi'*:
word-of-int i AND mask n = (word-of-int (take-bit (min LENGTH('a) n) i) :: 'a::len word)
by (*auto simp add*: *and-mask-wi min-def wi-bintr*)

lemma *and-mask-no*: *numeral i AND mask n = word-of-int (take-bit n (numeral i))*
unfolding *word-numeral-alt* **by** (*rule and-mask-wi*)

lemma *and-mask-mod-2p*: *w AND mask n = word-of-int (uint w mod 2 ^ n)*
by (*simp only*: *and-mask-bintr take-bit-eq-mod*)

lemma *uint-mask-eq*:
‹uint (mask n :: 'a::len word) = mask (min LENGTH('a) n)›
by *transfer simp*

lemma *and-mask-lt-2p*: *uint (w AND mask n) < 2 ^ n*
by (*metis take-bit-eq-mask take-bit-int-less-exp unsigned-take-bit-eq*)

lemma *mask-eq-iff*: *w AND mask n = w ⟷ uint w < 2 ^ n*
apply (*auto simp flip*: *take-bit-eq-mask*)
apply (*metis take-bit-int-eq-self-iff uint-take-bit-eq*)
apply (*simp add*: *take-bit-int-eq-self unsigned-take-bit-eq word-uint-eqI*)
done

lemma *and-mask-dvd*: *2 ^ n dvd uint w ⟷ w AND mask n = 0*
by (*simp flip*: *take-bit-eq-mask take-bit-eq-mod unsigned-take-bit-eq add*: *dvd-eq-mod-eq-0 uint-0-iff*)

lemma *and-mask-dvd-nat*: *2 ^ n dvd unat w ⟷ w AND mask n = 0*
by (*simp flip*: *take-bit-eq-mask take-bit-eq-mod unsigned-take-bit-eq add*: *dvd-eq-mod-eq-0 unat-0-iff uint-0-iff*)

lemma *word-2p-lem*: *n < size w ⟹ w < 2 ^ n = (uint w < 2 ^ n)*
for *w :: 'a::len word*
by *transfer simp*

lemma *less-mask-eq*:
fixes *x :: 'a::len word*
assumes *x < 2 ^ n* **shows** *x AND mask n = x*
by (*metis* (*no-types*) *assms lt2p-lem mask-eq-iff not-less word-2p-lem word-size*)

lemmas *mask-eq-iff-w2p = trans* [*OF mask-eq-iff word-2p-lem* [*symmetric*]]

lemmas *and-mask-less' = iffD2* [*OF word-2p-lem and-mask-lt-2p*, *simplified word-size*]

lemma *and-mask-less-size*: *n < size x ⟹ x AND mask n < 2 ^ n*
for *x :: ‹'a::len word›*
unfolding *word-size* **by** (*erule and-mask-less'*)

lemma *word-mod-2p-is-mask* [*OF refl*]: *c = 2 ^ n ⟹ c > 0 ⟹ x mod c = x AND mask n*
for *c x :: 'a::len word*
by (*auto simp*: *word-mod-def uint-2p and-mask-mod-2p*)

lemma *mask-eqs*:
(a AND mask n) + b AND mask n = a + b AND mask n
a + (b AND mask n) AND mask n = a + b AND mask n
(a AND mask n) − b AND mask n = a − b AND mask n
a − (b AND mask n) AND mask n = a − b AND mask n

a ∗ (b AND mask n) AND mask n = a ∗ b AND mask n
(b AND mask n) ∗ a AND mask n = b ∗ a AND mask n
(a AND mask n) + (b AND mask n) AND mask n = a + b AND mask n
(a AND mask n) − (b AND mask n) AND mask n = a − b AND mask n
(a AND mask n) ∗ (b AND mask n) AND mask n = a ∗ b AND mask n
− (a AND mask n) AND mask n = − a AND mask n
word-succ (a AND mask n) AND mask n = word-succ a AND mask n
word-pred (a AND mask n) AND mask n = word-pred a AND mask n
using *word-of-int-Ex* [**where** *x=a*] *word-of-int-Ex* [**where** *x=b*]
unfolding *take-bit-eq-mask* [*symmetric*]
by (*transfer*; *simp add*: *take-bit-eq-mod mod-simps*)+

lemma *mask-power-eq*: *(x AND mask n) ^ k AND mask n = x ^ k AND mask n*
for *x* :: ‹*'a*::*len word*›
using *word-of-int-Ex* [**where** *x=x*]
unfolding *take-bit-eq-mask* [*symmetric*]
by (*transfer*; *simp add*: *take-bit-eq-mod mod-simps*)+

lemma *mask-full* [*simp*]: *mask LENGTH('a) = (− 1 :: 'a::len word)*
by *transfer simp*

98.23.3 Slices

definition *slice1* :: ‹*nat ⇒ 'a::len word ⇒ 'b::len word*›
where ‹*slice1 n w = (if n < LENGTH('a)*
then ucast (drop-bit (LENGTH('a) − n) w)
else push-bit (n − LENGTH('a)) (ucast w))›

lemma *bit-slice1-iff* [*bit-simps*]:
‹*bit (slice1 m w :: 'b::len word) n ⟷ m − LENGTH('a) ≤ n ∧ n < min LENGTH('b) m*
∧ bit w (n + (LENGTH('a) − m) − (m − LENGTH('a)))›
for *w* :: ‹*'a::len word*›
by (*auto simp add*: *slice1-def bit-ucast-iff bit-drop-bit-eq bit-push-bit-iff not-less not-le ac-simps*
dest: *bit-imp-le-length*)

definition *slice* :: ‹*nat ⇒ 'a::len word ⇒ 'b::len word*›
where ‹*slice n = slice1 (LENGTH('a) − n)*›

lemma *bit-slice-iff* [*bit-simps*]:
‹*bit (slice m w :: 'b::len word) n ⟷ n < min LENGTH('b) (LENGTH('a) − m) ∧ bit w (n + LENGTH('a) − (LENGTH('a) − m))*›
for *w* :: ‹*'a::len word*›
by (*simp add*: *slice-def word-size bit-slice1-iff*)

lemma *slice1-0* [*simp*] : *slice1 n 0 = 0*
unfolding *slice1-def* **by** *simp*

lemma *slice-0* [*simp*] : *slice n 0 = 0*
unfolding *slice-def* **by** *auto*

lemma *ucast-slice1*: *ucast w = slice1* (*size w*) *w*
unfolding *slice1-def* **by** (*simp add*: *size-word.rep-eq*)

lemma *ucast-slice*: *ucast w = slice 0 w*
by (*simp add*: *slice-def slice1-def*)

lemma *slice-id*: *slice 0 t = t*
by (*simp only*: *ucast-slice* [*symmetric*] *ucast-id*)

lemma *rev-slice1*:
‹*slice1 n* (*word-reverse w* :: *'b*::*len word*) = *word-reverse* (*slice1 k w* :: *'a*::*len word*)›
if ‹*n + k = LENGTH*('*a*) + *LENGTH*('*b*)›
proof (*rule bit-word-eqI*)
fix *m*
assume *∗*: ‹*m < LENGTH*('*a*)›
from *that* **have** *∗∗*: ‹*LENGTH*('*b*) = *n + k − LENGTH*('*a*)›
by *simp*
show ‹*bit* (*slice1 n* (*word-reverse w* :: *'b word*) :: *'a word*) *m* ⟷ *bit* (*word-reverse* (*slice1 k w* :: *'a word*)) *m*›
unfolding *bit-slice1-iff bit-word-reverse-iff*
using *∗ ∗∗*
by (*cases* ‹*n ≤ LENGTH*('*a*)›; *cases* ‹*k ≤ LENGTH*('*a*)›) *auto*
qed

lemma *rev-slice*:
n + k + LENGTH('*a*::*len*) = *LENGTH*('*b*::*len*) ⟹
slice n (*word-reverse* (*w*::*'b word*)) = *word-reverse* (*slice k w* :: *'a word*)
unfolding *slice-def word-size*
by (*simp add*: *rev-slice1*)

98.23.4 Revcast

definition *revcast* :: ‹*'a*::*len word* ⇒ *'b*::*len word*›
where ‹*revcast = slice1 LENGTH*('*b*)›

lemma *bit-revcast-iff* [*bit-simps*]:
‹*bit* (*revcast w* :: *'b*::*len word*) *n* ⟷ *LENGTH*('*b*) − *LENGTH*('*a*) ≤ *n* ∧ *n* < *LENGTH*('*b*)
∧ *bit w* (*n* + (*LENGTH*('*a*) − *LENGTH*('*b*)) − (*LENGTH*('*b*) − *LENGTH*('*a*)))›
for *w* :: ‹*'a*::*len word*›
by (*simp add*: *revcast-def bit-slice1-iff*)

lemma *revcast-slice1* [*OF refl*]: *rc = revcast w* ⟹ *slice1* (*size rc*) *w = rc*
by (*simp add*: *revcast-def word-size*)

lemma *revcast-rev-ucast* [*OF refl refl refl*]:
cs = [*rc*, *uc*] ⟹ *rc* = *revcast* (*word-reverse w*) ⟹ *uc* = *ucast w* ⟹
rc = *word-reverse uc*
by (*metis rev-slice1 revcast-slice1 ucast-slice1 word-size*)

lemma *revcast-ucast*: *revcast w* = *word-reverse* (*ucast* (*word-reverse w*))
using *revcast-rev-ucast* [*of word-reverse w*] **by** *simp*

lemma *ucast-revcast*: *ucast w* = *word-reverse* (*revcast* (*word-reverse w*))
by (*fact revcast-rev-ucast* [*THEN word-rev-gal'*])

lemma *ucast-rev-revcast*: *ucast* (*word-reverse w*) = *word-reverse* (*revcast w*)
by (*fact revcast-ucast* [*THEN word-rev-gal'*])

linking revcast and cast via shift

lemmas *wsst-TYs* = *source-size target-size word-size*

lemmas *sym-notr* =
not-iff [*THEN iffD2*, *THEN not-sym*, *THEN not-iff* [*THEN iffD1*]]

98.24 Split and cat

lemmas *word-split-bin'* = *word-split-def*
lemmas *word-cat-bin'* = *word-cat-eq*

— this odd result is analogous to *ucast-id*, result to the length given by the result type

lemma *word-cat-id*: *word-cat a b* = *b*
by *transfer* (*simp add*: *take-bit-concat-bit-eq*)

lemma *word-cat-split-alt*: ⟦*size w* ≤ *size u* + *size v*; *word-split w* = (*u*,*v*)⟧ ⟹
word-cat u v = *w*
unfolding *word-split-def*
by (*rule bit-word-eqI*) (*auto simp add*: *bit-word-cat-iff not-less word-size bit-ucast-iff bit-drop-bit-eq*)

lemmas *word-cat-split-size* = *sym* [*THEN* [*2*] *word-cat-split-alt* [*symmetric*]]

98.24.1 Split and slice

lemma *split-slices*:
assumes *word-split w* = (*u*, *v*)
shows *u* = *slice* (*size v*) *w* ∧ *v* = *slice 0 w*
unfolding *word-size*
proof (*intro conjI*)
have §: ⋀*n*. ⟦*ucast* (*drop-bit LENGTH*('*b*) *w*) = *u*; *LENGTH*('*c*) < *LENGTH*('*b*)⟧
⟹ ¬ *bit u n*
by (*metis bit-take-bit-iff bit-word-of-int-iff diff-is-0-eq' drop-bit-take-bit less-imp-le less-nat-zero-code of-int-uint unsigned-drop-bit-eq*)

show *u = slice LENGTH('b) w*
proof (*rule bit-word-eqI*)
show *bit u n = bit ((slice LENGTH('b) w)::'a word) n* **if** *n < LENGTH('a)*
for *n*
using *assms bit-imp-le-length*
unfolding *word-split-def bit-slice-iff*
by (*fastforce simp add*: § *ac-simps word-size bit-ucast-iff bit-drop-bit-eq*)
qed
show *v = slice 0 w*
by (*metis Pair-inject assms ucast-slice word-split-bin'*)
qed

lemma *slice-cat1* [*OF refl*]:
⟦*wc = word-cat a b*; *size a + size b ≤ size wc*⟧ ⟹ *slice (size b) wc = a*
by (*rule bit-word-eqI*) (*auto simp add*: *bit-slice-iff bit-word-cat-iff word-size*)

lemmas *slice-cat2 = trans* [*OF slice-id word-cat-id*]

lemma *cat-slices*:
⟦*a = slice n c*; *b = slice 0 c*; *n = size b*; *size c ≤ size a + size b*⟧ ⟹ *word-cat a b = c*
by (*rule bit-word-eqI*) (*auto simp add*: *bit-slice-iff bit-word-cat-iff word-size*)

lemma *word-split-cat-alt*:
assumes *w = word-cat u v* **and** *size*: *size u + size v ≤ size w*
shows *word-split w = (u,v)*
proof −
have *ucast ((drop-bit LENGTH('c) (word-cat u v))::'a word) = u ucast ((word-cat u v)::'a word) = v*
using *assms*
by (*auto simp add*: *word-size bit-ucast-iff bit-drop-bit-eq bit-word-cat-iff intro*: *bit-eqI*)
then show *?thesis*
by (*simp add*: *assms*(*1*) *word-split-bin'*)
qed

lemma *horner-sum-uint-exp-Cons-eq*:
‹*horner-sum uint (2 ^ LENGTH('a)) (w # ws) =*
concat-bit LENGTH('a) (uint w) (horner-sum uint (2 ^ LENGTH('a)) ws)›
for *ws* :: ‹'*a*::*len word list*›
by (*simp add*: *bintr-uint concat-bit-eq push-bit-eq-mult*)

lemma *bit-horner-sum-uint-exp-iff*:
‹*bit (horner-sum uint (2 ^ LENGTH('a)) ws) n* ⟷
n div LENGTH('a) < length ws ∧ bit (ws ! (n div LENGTH('a))) (n mod LENGTH('a))›
for *ws* :: ‹'*a*::*len word list*›
proof (*induction ws arbitrary*: *n*)

case *Nil*
then show *?case*
by *simp*
next
case (*Cons w ws*)
then show *?case*
by (*cases ‹n ≥ LENGTH('a)›*)
(*simp-all only*: *horner-sum-uint-exp-Cons-eq*, *simp-all add*: *bit-concat-bit-iff le-div-geq le-mod-geq bit-uint-iff Cons*)
qed

98.25 Rotation

lemma *word-rotr-word-rotr-eq*: *‹word-rotr m (word-rotr n w) = word-rotr (m + n) w›*
by (*rule bit-word-eqI*) (*simp add*: *bit-word-rotr-iff ac-simps mod-add-right-eq*)

lemma *word-rot-lem*: *⟦l + k = d + k mod l; n < l⟧ ⟹ ((d + n) mod l) = n* **for** *l*::*nat*
by (*metis* (*no-types*, *lifting*) *add.commute add.right-neutral add-diff-cancel-left' mod-if mod-mult-div-eq mod-mult-self2 mod-self*)

lemma *word-rot-rl* [*simp*]: *‹word-rotl k (word-rotr k v) = v›*
proof (*rule bit-word-eqI*)
show *bit (word-rotl k (word-rotr k v)) n = bit v n* **if** *n < LENGTH('a)* **for** *n*
using *that*
by (*auto simp*: *word-rot-lem word-rotl-eq-word-rotr word-rotr-word-rotr-eq bit-word-rotr-iff algebra-simps split*: *nat-diff-split*)
qed

lemma *word-rot-lr* [*simp*]: *‹word-rotr k (word-rotl k v) = v›*
proof (*rule bit-word-eqI*)
show *bit (word-rotr k (word-rotl k v)) n = bit v n* **if** *n < LENGTH('a)* **for** *n*
using *that*
by (*auto simp add*: *word-rot-lem word-rotl-eq-word-rotr word-rotr-word-rotr-eq bit-word-rotr-iff algebra-simps split*: *nat-diff-split*)
qed

lemma *word-rot-gal*:
‹word-rotr n v = w ⟷ word-rotl n w = v›
by *auto*

lemma *word-rot-gal'*:
‹w = word-rotr n v ⟷ v = word-rotl n w›
by *auto*

lemma *word-rotr-rev*:
‹word-rotr n w = word-reverse (word-rotl n (word-reverse w))›
proof (*rule bit-word-eqI*)

fix *m*
assume ‹*m* < *LENGTH*('*a*)›
moreover have ‹*1* +
((*int m* + *int n mod int LENGTH*('*a*)) *mod int LENGTH*('*a*) +
((*int LENGTH*('*a*) * *2*) *mod int LENGTH*('*a*) − (*1* + (*int m* + *int n mod int LENGTH*('*a*)))) *mod int LENGTH*('*a*)) =
int LENGTH('*a*)›
apply (*cases* ‹(*1* + (*int m* + *int n mod int LENGTH*('*a*))) *mod int LENGTH*('*a*) = *0*›)
using *zmod-zminus1-eq-if* [*of* ‹*1* + (*int m* + *int n mod int LENGTH*('*a*))› ‹*int LENGTH*('*a*)›]
apply *simp-all*
apply (*auto simp add*: *algebra-simps*)
apply (*metis* (*mono-tags*, *opaque-lifting*) *Abs-fnat-hom-add mod-Suc mod-mult-self2-is-0 of-nat-Suc of-nat-mod semiring-char-0-class.of-nat-neq-0*)
apply (*metis* (*no-types*, *opaque-lifting*) *Abs-fnat-hom-add less-not-refl mod-Suc of-nat-Suc of-nat-gt-0 of-nat-mod*)
done
then have ‹*int* ((*m* + *n*) *mod LENGTH*('*a*)) =
int (*LENGTH*('*a*) − *Suc* ((*LENGTH*('*a*) − *Suc m* + *LENGTH*('*a*) − *n mod LENGTH*('*a*)) *mod LENGTH*('*a*)))›
using ‹*m* < *LENGTH*('*a*)›
by (*simp only*: *of-nat-mod mod-simps*)
(*simp add*: *of-nat-diff of-nat-mod Suc-le-eq add-less-mono algebra-simps mod-simps*)
then have ‹(*m* + *n*) *mod LENGTH*('*a*) =
LENGTH('*a*) − *Suc* ((*LENGTH*('*a*) − *Suc m* + *LENGTH*('*a*) − *n mod LENGTH*('*a*)) *mod LENGTH*('*a*))›
by *simp*
ultimately show ‹*bit* (*word-rotr n w*) *m* ⟷ *bit* (*word-reverse* (*word-rotl n* (*word-reverse w*))) *m*›
by (*simp add*: *word-rotl-eq-word-rotr bit-word-rotr-iff bit-word-reverse-iff*)
qed

lemma *word-roti-0* [*simp*]: *word-roti 0 w* = *w*
by *transfer simp*

lemma *word-roti-add*: *word-roti* (*m* + *n*) *w* = *word-roti m* (*word-roti n w*)
by (*rule bit-word-eqI*)
(*simp add*: *bit-word-roti-iff nat-less-iff mod-simps ac-simps*)

lemma *word-roti-conv-mod'*:
word-roti n w = *word-roti* (*n mod int* (*size w*)) *w*
by *transfer simp*

lemmas *word-roti-conv-mod* = *word-roti-conv-mod'* [*unfolded word-size*]

end

98.25.1 "Word rotation commutes with bit-wise operations

locale *word-rotate*
begin

context
includes *bit-operations-syntax*
begin

lemma *word-rot-logs*:
word-rotl n (NOT v) = NOT (word-rotl n v)
word-rotr n (NOT v) = NOT (word-rotr n v)
word-rotl n (x AND y) = word-rotl n x AND word-rotl n y
word-rotr n (x AND y) = word-rotr n x AND word-rotr n y
word-rotl n (x OR y) = word-rotl n x OR word-rotl n y
word-rotr n (x OR y) = word-rotr n x OR word-rotr n y
word-rotl n (x XOR y) = word-rotl n x XOR word-rotl n y
word-rotr n (x XOR y) = word-rotr n x XOR word-rotr n y
by (*rule bit-word-eqI*, *auto simp add*: *bit-word-rotl-iff bit-word-rotr-iff bit-and-iff bit-or-iff bit-xor-iff bit-not-iff algebra-simps not-le*)+

end

end

lemmas *word-rot-logs = word-rotate.word-rot-logs*

lemma *word-rotx-0* [*simp*] : *word-rotr i 0 = 0 ∧ word-rotl i 0 = 0*
by *transfer simp-all*

lemma *word-roti-0'* [*simp*] : *word-roti n 0 = 0*
by *transfer simp*

declare *word-roti-eq-word-rotr-word-rotl* [*simp*]

98.26 Maximum machine word

context
includes *bit-operations-syntax*
begin

lemma *word-int-cases*:
fixes *x* :: *'a*::*len word*
obtains *n* **where** *x = word-of-int n* **and** *0 ≤ n* **and** *n < 2^LENGTH('a)*
by (*rule that* [*of* ‹*uint x*›]) *simp-all*

lemma *word-nat-cases* [*cases type*: *word*]:
fixes *x* :: *'a*::*len word*
obtains *n* **where** *x = of-nat n* **and** *n < 2^LENGTH('a)*
by (*rule that* [*of* ‹*unat x*›]) *simp-all*

lemma *max-word-max* [*intro*!]:
‹$n \leq -1$› **for** n :: ‹$'a$::*len word*›
by (*fact word-order.extremum*)

lemma *word-of-int-2p-len*: *word-of-int* ($2 \hat{} LENGTH('a)$) = (0::$'a$::*len word*)
by *simp*

lemma *word-pow-0*: (2::$'a$::*len word*) $\hat{}$ $LENGTH('a) = 0$
by (*fact word-exp-length-eq-0*)

lemma *max-word-wrap*:
‹$x + 1 = 0 \Longrightarrow x = -1$› **for** x :: ‹$'a$::*len word*›
by (*simp add*: *eq-neg-iff-add-eq-0*)

lemma *word-and-max*:
‹x *AND* $-1 = x$› **for** x :: ‹$'a$::*len word*›
by (*fact word-log-esimps*)

lemma *word-or-max*:
‹x *OR* $-1 = -1$› **for** x :: ‹$'a$::*len word*›
by (*fact word-log-esimps*)

lemma *word-ao-dist2*: x *AND* (y *OR* z) = x *AND* y *OR* x *AND* z
for x y z :: $'a$::*len word*
by (*fact bit.conj-disj-distrib*)

lemma *word-oa-dist2*: x *OR* y *AND* z = (x *OR* y) *AND* (x *OR* z)
for x y z :: $'a$::*len word*
by (*fact bit.disj-conj-distrib*)

lemma *word-and-not* [*simp*]: x *AND NOT* $x = 0$
for x :: $'a$::*len word*
by (*fact bit.conj-cancel-right*)

lemma *word-or-not* [*simp*]:
‹x *OR NOT* $x = -1$› **for** x :: ‹$'a$::*len word*›
by (*fact bit.disj-cancel-right*)

lemma *word-xor-and-or*: x *XOR* y = x *AND NOT* y *OR NOT* x *AND* y
for x y :: $'a$::*len word*
by (*fact bit.xor-def*)

lemma *uint-lt-0* [*simp*]: *uint* $x < 0$ = *False*
by (*simp add*: *linorder-not-less*)

lemma *word-less-1* [*simp*]: $x < 1 \longleftrightarrow x = 0$
for x :: $'a$::*len word*
by (*simp add*: *word-less-nat-alt unat-0-iff*)

lemma *uint-plus-if-size*:
uint (*x* + *y*) =
(*if uint x* + *uint y* < *2^size x*
then uint x + *uint y*
else uint x + *uint y* − *2^size x*)
by (*simp add*: *take-bit-eq-mod word-size uint-word-of-int-eq uint-plus-if'*)

lemma *unat-plus-if-size*:
unat (*x* + *y*) =
(*if unat x* + *unat y* < *2^size x*
then unat x + *unat y*
else unat x + *unat y* − *2^size x*)
for *x y* :: *'a*::*len word*
by (*simp add*: *size-word.rep-eq unat-arith-simps*)

lemma *word-neq-0-conv*: *w* ≠ *0* ⟷ *0* < *w*
for *w* :: *'a*::*len word*
by (*fact word-coorder.not-eq-extremum*)

lemma *max-lt*: *unat* (*max a b div c*) = *unat* (*max a b*) *div unat c*
for *c* :: *'a*::*len word*
by (*fact unat-div*)

lemma *uint-sub-if-size*:
uint (*x* − *y*) =
(*if uint y* ≤ *uint x*
then uint x − *uint y*
else uint x − *uint y* + *2^size x*)
by (*simp add*: *size-word.rep-eq uint-sub-if'*)

lemma *unat-sub*:
‹*unat* (*a* − *b*) = *unat a* − *unat b*›
if ‹*b* ≤ *a*›
by (*meson that unat-sub-if-size word-le-nat-alt*)

lemmas *word-less-sub1-numberof* [*simp*] = *word-less-sub1* [*of numeral w*] **for** *w*
lemmas *word-le-sub1-numberof* [*simp*] = *word-le-sub1* [*of numeral w*] **for** *w*

lemma *word-of-int-minus*: *word-of-int* (*2^LENGTH('a) − i*) = (*word-of-int* (*−i*)::*'a*::*len word*)
by *simp*

lemma *word-of-int-inj*:
‹(*word-of-int x* :: *'a*::*len word*) = *word-of-int y* ⟷ *x* = *y*›
if ‹*0* ≤ *x* ∧ *x* < *2* ^ *LENGTH('a)*› ‹*0* ≤ *y* ∧ *y* < *2* ^ *LENGTH('a)*›
using *that* **by** (*transfer fixing*: *x y*) (*simp add*: *take-bit-int-eq-self*)

lemma *word-le-less-eq*: *x* ≤ *y* ⟷ *x* = *y* ∨ *x* < *y*

```
  for x y :: 'z::len word
  by (auto simp add: order-class.le-less)

lemma mod-plus-cong:
  fixes b b' :: int
  assumes 1: b = b'
    and 2: x mod b' = x' mod b'
    and 3: y mod b' = y' mod b'
    and 4: x' + y' = z'
  shows (x + y) mod b = z' mod b'
proof -
  from 1 2[symmetric] 3[symmetric] have (x + y) mod b = (x' mod b' + y' mod
b') mod b'
    by (simp add: mod-add-eq)
  also have ... = (x' + y') mod b'
    by (simp add: mod-add-eq)
  finally show ?thesis
    by (simp add: 4)
qed

lemma mod-minus-cong:
  fixes b b' :: int
  assumes b = b'
    and x mod b' = x' mod b'
    and y mod b' = y' mod b'
    and x' - y' = z'
  shows (x - y) mod b = z' mod b'
  using assms [symmetric] by (auto intro: mod-diff-cong)

lemma word-induct-less [case-names zero less]:
  ‹P m› if zero: ‹P 0› and less: ‹⋀n. n < m ⟹ P n ⟹ P (1 + n)›
  for m :: ‹'a::len word›
proof -
  define q where ‹q = unat m›
  with less have ‹⋀n. n < word-of-nat q ⟹ P n ⟹ P (1 + n)›
    by simp
  then have ‹P (word-of-nat q :: 'a word)›
  proof (induction q)
    case 0
    show ?case
      by (simp add: zero)
  next
    case (Suc q)
    show ?case
    proof (cases ‹1 + word-of-nat q = (0 :: 'a word)›)
      case True
      then show ?thesis
        by (simp add: zero)
    next
```

```
      case False
      then have ∗: ‹word-of-nat q < (word-of-nat (Suc q) :: 'a word)›
        by (simp add: unatSuc word-less-nat-alt)
      then have ∗∗: ‹n < (1 + word-of-nat q :: 'a word) ⟷ n ≤ (word-of-nat q
:: 'a word)› for n
          by (metis (no-types, lifting) add.commute inc-le le-less-trans not-less
of-nat-Suc)
      have ‹P (word-of-nat q)›
        by (simp add: ∗∗ Suc.IH Suc.prems)
      with ∗ have ‹P (1 + word-of-nat q)›
        by (rule Suc.prems)
      then show ?thesis
        by simp
    qed
  qed
  with ‹q = unat m› show ?thesis
    by simp
qed

lemma word-induct: P 0 ⟹ (⋀n. P n ⟹ P (1 + n)) ⟹ P m
  for P :: 'a::len word ⇒ bool
  by (rule word-induct-less)

lemma word-induct2 [case-names zero suc, induct type]: P 0 ⟹ (⋀n. 1 + n ≠
0 ⟹ P n ⟹ P (1 + n)) ⟹ P n
  for P :: 'b::len word ⇒ bool
by (induction rule: word-induct-less; force)
```

98.27 Recursion combinator for words

```
definition word-rec :: 'a ⇒ ('b::len word ⇒ 'a ⇒ 'a) ⇒ 'b word ⇒ 'a
  where word-rec forZero forSuc n = rec-nat forZero (forSuc ∘ of-nat) (unat n)

lemma word-rec-0 [simp]: word-rec z s 0 = z
  by (simp add: word-rec-def)

lemma word-rec-Suc [simp]: 1 + n ≠ 0 ⟹ word-rec z s (1 + n) = s n (word-rec
z s n)
  for n :: 'a::len word
  by (simp add: unatSuc word-rec-def)

lemma word-rec-Pred: n ≠ 0 ⟹ word-rec z s n = s (n − 1) (word-rec z s (n −
1))
  by (metis add.commute diff-add-cancel word-rec-Suc)

lemma word-rec-in: f (word-rec z (λ-. f) n) = word-rec (f z) (λ-. f) n
  by (induct n) simp-all

lemma word-rec-in2: f n (word-rec z f n) = word-rec (f 0 z) (f ∘ (+) 1) n
```

by (*induct n*) *simp-all*

lemma *word-rec-twice*:
$m \leq n \Longrightarrow$ *word-rec z f n* = *word-rec* (*word-rec z f* (*n* − *m*)) (*f* ∘ (+) (*n* − *m*)) *m*
proof (*induction n arbitrary*: *z f*)
case *zero*
then show *?case*
by (*metis diff-0-right word-le-0-iff word-rec-0*)
next
case (*suc n z f*)
show *?case*
proof (*cases 1* + (*n* − *m*) = *0*)
case *True*
then show *?thesis*
by (*simp add*: *add-diff-eq*)
next
case *False*
then have *eq*: *1* + *n* − *m* = *1* + (*n* − *m*)
by *simp*
with *False* **have** *m* ≤ *n*
by (*metis suc.prems add.commute dual-order.antisym eq-iff-diff-eq-0 inc-le leI*)
with *False suc.hyps* **show** *?thesis*
using *suc.IH* [*of f 0 z f* ∘ (+) *1*]
by (*simp add*: *word-rec-in2 eq add.assoc o-def*)
qed
qed

lemma *word-rec-id*: *word-rec z* (λ-. *id*) *n* = *z*
by (*induct n*) *auto*

lemma *word-rec-id-eq*: (⋀*m*. *m* < *n* ⟹ *f m* = *id*) ⟹ *word-rec z f n* = *z*
by (*induction n*) (*auto simp add*: *unatSuc unat-arith-simps*(*2*))

lemma *word-rec-max*:
assumes ∀ *m*≥*n*. *m* ≠ − *1* ⟶ *f m* = *id*
shows *word-rec z f* (− *1*) = *word-rec z f n*
proof −
have §: ⋀*m*. ⟦*m* < − *1* − *n*⟧ ⟹ (*f* ∘ (+) *n*) *m* = *id*
using *assms*
by (*metis* (*mono-tags*, *lifting*) *add.commute add-diff-cancel-left' comp-apply less-le olen-add-eqv plus-minus-no-overflow word-n1-ge*)
have *word-rec z f* (− *1*) = *word-rec* (*word-rec z f* (− *1* − (− *1* − *n*))) (*f* ∘ (+) (− *1* − (− *1* − *n*))) (− *1* − *n*)
by (*meson word-n1-ge word-rec-twice*)
also have ... = *word-rec z f n*
by (*metis* (*no-types*, *lifting*) § *diff-add-cancel minus-diff-eq uminus-add-conv-diff word-rec-id-eq*)

finally show *?thesis* **.**
qed

end

98.28 Tool support

ML-file ‹*Tools/smt-word.ML*›

end

99 The Field of Integers mod 2

theory *Z2*
imports *Main*
begin

Note that in most cases *bool* is appropriate when a binary type is needed; the type provided here, for historical reasons named *bit*, is only needed if proper field operations are required.

typedef *bit* = ‹*UNIV* :: *bool set*› **..**

instantiation *bit* :: *zero-neq-one*
begin

definition *zero-bit* :: *bit*
where ‹*0* = *Abs-bit False*›

definition *one-bit* :: *bit*
where ‹*1* = *Abs-bit True*›

instance
by *standard* (*simp add*: *zero-bit-def one-bit-def Abs-bit-inject*)

end

free-constructors *case-bit* **for** ‹*0*::*bit*› | ‹*1*::*bit*›
proof −
fix *P* :: *bool*
fix *a* :: *bit*
assume ‹*a* = *0* ⟹ *P*› **and** ‹*a* = *1* ⟹ *P*›
then show *P*
by (*cases a*) (*auto simp add*: *zero-bit-def one-bit-def Abs-bit-inject*)
qed *simp*

lemma *bit-not-zero-iff* [*simp*]:
‹*a* ≠ *0* ⟷ *a* = *1*› **for** *a* :: *bit*
by (*cases a*) *simp-all*

lemma *bit-not-one-iff* [*simp*]:
‹$a \neq 1 \longleftrightarrow a = 0$› **for** a :: *bit*
by (*cases a*) *simp-all*

instantiation *bit* :: *semidom-modulo*
begin

definition *plus-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
where ‹$a + b$ = *Abs-bit* (*Rep-bit a* ≠ *Rep-bit b*)›

definition *minus-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
where [*simp*]: ‹*minus-bit* = *plus*›

definition *times-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
where ‹$a * b$ = *Abs-bit* (*Rep-bit a* ∧ *Rep-bit b*)›

definition *divide-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
where [*simp*]: ‹*divide-bit* = *times*›

definition *modulo-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
where ‹*a mod b* = *Abs-bit* (*Rep-bit a* ∧ ¬ *Rep-bit b*)›

instance
by *standard*
(*auto simp flip*: *Rep-bit-inject*
simp add: *zero-bit-def one-bit-def plus-bit-def times-bit-def modulo-bit-def Abs-bit-inverse Rep-bit-inverse*)

end

lemma *bit-2-eq-0* [*simp*]:
‹2 = (0::*bit*)›
by (*simp flip*: *one-add-one add*: *zero-bit-def plus-bit-def*)

instance *bit* :: *semiring-parity*
apply *standard*
apply (*auto simp flip*: *Rep-bit-inject simp add*: *modulo-bit-def Abs-bit-inverse Rep-bit-inverse*)
apply (*auto simp add*: *zero-bit-def one-bit-def Abs-bit-inverse Rep-bit-inverse*)
done

lemma *Abs-bit-eq-of-bool* [*code-abbrev*]:
‹*Abs-bit* = *of-bool*›
by (*simp add*: *fun-eq-iff zero-bit-def one-bit-def*)

lemma *Rep-bit-eq-odd*:
‹*Rep-bit* = *odd*›
proof −
have ‹¬ *Rep-bit 0*›

by (*simp only*: *zero-bit-def*) (*subst Abs-bit-inverse*, *auto*)
then show *?thesis*
by (*auto simp flip*: *Rep-bit-inject simp add*: *fun-eq-iff*)
qed

lemma *Rep-bit-iff-odd* [*code-abbrev*]:
‹*Rep-bit b* ⟷ *odd b*›
by (*simp add*: *Rep-bit-eq-odd*)

lemma *Not-Rep-bit-iff-even* [*code-abbrev*]:
‹¬ *Rep-bit b* ⟷ *even b*›
by (*simp add*: *Rep-bit-eq-odd*)

lemma *Not-Not-Rep-bit* [*code-unfold*]:
‹¬ ¬ *Rep-bit b* ⟷ *Rep-bit b*›
by *simp*

code-datatype ‹*0*::*bit*› ‹*1*::*bit*›

lemma *Abs-bit-code* [*code*]:
‹*Abs-bit False* = *0*›
‹*Abs-bit True* = *1* ›
by (*simp-all add*: *Abs-bit-eq-of-bool*)

lemma *Rep-bit-code* [*code*]:
‹*Rep-bit 0* ⟷ *False*›
‹*Rep-bit 1* ⟷ *True*›
by (*simp-all add*: *Rep-bit-eq-odd*)

context *zero-neq-one*
begin

abbreviation *of-bit* :: ‹*bit* ⇒ *'a*›
where ‹*of-bit b* ≡ *of-bool* (*odd b*)›

end

context
begin

qualified lemma *bit-eq-iff*:
‹*a* = *b* ⟷ (*even a* ⟷ *even b*)› **for** *a b* :: *bit*
by (*cases a*; *cases b*) *simp-all*

end

lemma *modulo-bit-unfold* [*simp*, *code*]:
‹*a mod b* = *of-bool* (*odd a* ∧ *even b*)› **for** *a b* :: *bit*
by (*simp add*: *modulo-bit-def Abs-bit-eq-of-bool Rep-bit-eq-odd*)

lemma *power-bit-unfold* [*simp*, *code*]:
  ‹*a* ^ *n* = *of-bool* (*odd a* ∨ *n* = *0*)› **for** *a* :: *bit*
  by (*cases a*) *simp-all*

instantiation *bit* :: *field*
begin

definition *uminus-bit* :: ‹*bit* ⇒ *bit*›
  where [*simp*]: ‹*uminus-bit* = *id*›

definition *inverse-bit* :: ‹*bit* ⇒ *bit*›
  where [*simp*]: ‹*inverse-bit* = *id*›

instance
  apply *standard*
  apply *simp-all*
  apply (*simp only*: *Z2.bit-eq-iff even-add*)
  apply *simp*
  done

end

instantiation *bit* :: *semiring-bits*
begin

definition *bit-bit* :: ‹*bit* ⇒ *nat* ⇒ *bool*›
  where [*simp*]: ‹*bit-bit b n* ⟷ *odd b* ∧ *n* = *0*›

instance
  apply *standard*
        apply *auto*
  apply (*metis bit.exhaust of-bool-eq*(*2*))
  done

end

instantiation *bit* :: *ring-bit-operations*
begin

context
  includes *bit-operations-syntax*
begin

definition *not-bit* :: ‹*bit* ⇒ *bit*›
  where [*simp*]: ‹*NOT b* = *of-bool* (*even b*)› **for** *b* :: *bit*

definition *and-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*b AND c* = *of-bool* (*odd b* ∧ *odd c*)› **for** *b c* :: *bit*

definition *or-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*b OR c* = *of-bool* (*odd b* ∨ *odd c*)› **for** *b c* :: *bit*

definition *xor-bit* :: ‹*bit* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*b XOR c* = *of-bool* (*odd b* ≠ *odd c*)› **for** *b c* :: *bit*

definition *mask-bit* :: ‹*nat* ⇒ *bit*›
  where [*simp*]: ‹*mask n* = (*of-bool* (*n* > *0*) :: *bit*)›

definition *set-bit-bit* :: ‹*nat* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*set-bit n b* = *of-bool* (*n* = *0* ∨ *odd b*)› **for** *b* :: *bit*

definition *unset-bit-bit* :: ‹*nat* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*unset-bit n b* = *of-bool* (*n* > *0* ∧ *odd b*)› **for** *b* :: *bit*

definition *flip-bit-bit* :: ‹*nat* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*flip-bit n b* = *of-bool* ((*n* = *0*) ≠ *odd b*)› **for** *b* :: *bit*

definition *push-bit-bit* :: ‹*nat* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*push-bit n b* = *of-bool* (*odd b* ∧ *n* = *0*)› **for** *b* :: *bit*

definition *drop-bit-bit* :: ‹*nat* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*drop-bit n b* = *of-bool* (*odd b* ∧ *n* = *0*)› **for** *b* :: *bit*

definition *take-bit-bit* :: ‹*nat* ⇒ *bit* ⇒ *bit*›
  where [*simp*]: ‹*take-bit n b* = *of-bool* (*odd b* ∧ *n* > *0*)› **for** *b* :: *bit*

end

instance
  apply *standard*
        apply *auto*
  apply (*simp only*: *Z2.bit-eq-iff even-add even-zero*)
  done

end

lemma *add-bit-eq-xor* [*simp*, *code*]:
  ‹(+) = (*Bit-Operations.xor* :: *bit* ⇒ -)›
  by (*auto simp add*: *fun-eq-iff*)

lemma *mult-bit-eq-and* [*simp*, *code*]:
  ‹(∗) = (*Bit-Operations.and* :: *bit* ⇒ -)›
  by (*simp add*: *fun-eq-iff*)

lemma *bit-numeral-even* [*simp*]: *numeral* (*Num.Bit0 w*) = (*0* :: *bit*)
  by (*simp only*: *Z2.bit-eq-iff even-numeral*) *simp*

lemma *bit-numeral-odd* [*simp*]: *numeral* (*Num.Bit1 w*) = (*1* :: *bit*)
by (*simp only*: *Z2.bit-eq-iff odd-numeral*) *simp*

end

100 Pointwise order on product types

theory *Product-Order*
imports *Product-Plus*
begin

100.1 Pointwise ordering

instantiation *prod* :: (*ord*, *ord*) *ord*
begin

definition
$x \leq y \longleftrightarrow fst\ x \leq fst\ y \wedge snd\ x \leq snd\ y$

definition
$(x::'a \times 'b) < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$

instance ..

end

lemma *fst-mono*: $x \leq y \Longrightarrow fst\ x \leq fst\ y$
unfolding *less-eq-prod-def* **by** *simp*

lemma *snd-mono*: $x \leq y \Longrightarrow snd\ x \leq snd\ y$
unfolding *less-eq-prod-def* **by** *simp*

lemma *Pair-mono*: $x \leq x' \Longrightarrow y \leq y' \Longrightarrow (x, y) \leq (x', y')$
unfolding *less-eq-prod-def* **by** *simp*

lemma *Pair-le* [*simp*]: $(a, b) \leq (c, d) \longleftrightarrow a \leq c \wedge b \leq d$
unfolding *less-eq-prod-def* **by** *simp*

instance *prod* :: (*preorder*, *preorder*) *preorder*
proof
fix $x\ y\ z :: 'a \times 'b$
show $x < y \longleftrightarrow x \leq y \wedge \neg\ y \leq x$
by (*rule less-prod-def*)
show $x \leq x$
unfolding *less-eq-prod-def*
by *fast*
assume $x \leq y$ **and** $y \leq z$ **thus** $x \leq z$
unfolding *less-eq-prod-def*

by (*fast elim*: *order-trans*)
qed

instance *prod* :: (*order*, *order*) *order*
by *standard auto*

100.2 Binary infimum and supremum

instantiation *prod* :: (*inf*, *inf*) *inf*
begin

definition *inf x y* = (*inf* (*fst x*) (*fst y*), *inf* (*snd x*) (*snd y*))

lemma *inf-Pair-Pair* [*simp*]: *inf* (*a*, *b*) (*c*, *d*) = (*inf a c*, *inf b d*)
unfolding *inf-prod-def* **by** *simp*

lemma *fst-inf* [*simp*]: *fst* (*inf x y*) = *inf* (*fst x*) (*fst y*)
unfolding *inf-prod-def* **by** *simp*

lemma *snd-inf* [*simp*]: *snd* (*inf x y*) = *inf* (*snd x*) (*snd y*)
unfolding *inf-prod-def* **by** *simp*

instance ..

end

instance *prod* :: (*semilattice-inf*, *semilattice-inf*) *semilattice-inf*
by *standard auto*

instantiation *prod* :: (*sup*, *sup*) *sup*
begin

definition
sup x y = (*sup* (*fst x*) (*fst y*), *sup* (*snd x*) (*snd y*))

lemma *sup-Pair-Pair* [*simp*]: *sup* (*a*, *b*) (*c*, *d*) = (*sup a c*, *sup b d*)
unfolding *sup-prod-def* **by** *simp*

lemma *fst-sup* [*simp*]: *fst* (*sup x y*) = *sup* (*fst x*) (*fst y*)
unfolding *sup-prod-def* **by** *simp*

lemma *snd-sup* [*simp*]: *snd* (*sup x y*) = *sup* (*snd x*) (*snd y*)
unfolding *sup-prod-def* **by** *simp*

instance ..

end

instance *prod* :: (*semilattice-sup*, *semilattice-sup*) *semilattice-sup*
by *standard auto*

instance *prod* :: (*lattice*, *lattice*) *lattice* **..**

instance *prod* :: (*distrib-lattice*, *distrib-lattice*) *distrib-lattice*
by *standard* (*auto simp add*: *sup-inf-distrib1*)

100.3 Top and bottom elements

instantiation *prod* :: (*top*, *top*) *top*
begin

definition
top = (*top*, *top*)

instance ..

end

lemma *fst-top* [*simp*]: *fst top* = *top*
unfolding *top-prod-def* **by** *simp*

lemma *snd-top* [*simp*]: *snd top* = *top*
unfolding *top-prod-def* **by** *simp*

lemma *Pair-top-top*: (*top*, *top*) = *top*
unfolding *top-prod-def* **by** *simp*

instance *prod* :: (*order-top*, *order-top*) *order-top*
by *standard* (*auto simp add*: *top-prod-def*)

instantiation *prod* :: (*bot*, *bot*) *bot*
begin

definition
bot = (*bot*, *bot*)

instance ..

end

lemma *fst-bot* [*simp*]: *fst bot* = *bot*
unfolding *bot-prod-def* **by** *simp*

lemma *snd-bot* [*simp*]: *snd bot* = *bot*
unfolding *bot-prod-def* **by** *simp*

lemma *Pair-bot-bot*: (*bot*, *bot*) = *bot*

unfolding *bot-prod-def* **by** *simp*

instance *prod* :: (*order-bot*, *order-bot*) *order-bot*
by *standard* (*auto simp add*: *bot-prod-def*)

instance *prod* :: (*bounded-lattice*, *bounded-lattice*) *bounded-lattice* **..**

instance *prod* :: (*boolean-algebra*, *boolean-algebra*) *boolean-algebra*
by *standard* (*auto simp add*: *prod-eqI diff-eq*)

100.4 Complete lattice operations

instantiation *prod* :: (*Inf*, *Inf*) *Inf*
begin

definition *Inf A* = (*INF x*∈*A*. *fst x*, *INF x*∈*A*. *snd x*)

instance ..

end

instantiation *prod* :: (*Sup*, *Sup*) *Sup*
begin

definition *Sup A* = (*SUP x*∈*A*. *fst x*, *SUP x*∈*A*. *snd x*)

instance ..

end

instance *prod* :: (*conditionally-complete-lattice*, *conditionally-complete-lattice*)
conditionally-complete-lattice
by *standard* (*force simp*: *less-eq-prod-def Inf-prod-def Sup-prod-def bdd-below-def bdd-above-def*
intro!: *cInf-lower cSup-upper cInf-greatest cSup-least*)+

instance *prod* :: (*complete-lattice*, *complete-lattice*) *complete-lattice*
by *standard* (*simp-all add*: *less-eq-prod-def Inf-prod-def Sup-prod-def*
INF-lower SUP-upper le-INF-iff SUP-le-iff bot-prod-def top-prod-def)

lemma *fst-Inf*: *fst* (*Inf A*) = (*INF x*∈*A*. *fst x*)
by (*simp add*: *Inf-prod-def*)

lemma *fst-INF*: *fst* (*INF x*∈*A*. *f x*) = (*INF x*∈*A*. *fst* (*f x*))
by (*simp add*: *fst-Inf image-image*)

lemma *fst-Sup*: *fst* (*Sup A*) = (*SUP x*∈*A*. *fst x*)
by (*simp add*: *Sup-prod-def*)

lemma *fst-SUP*: *fst* (*SUP* $x{\in}A.$ *f x*) = (*SUP* $x{\in}A.$ *fst* (*f x*))
by (*simp add*: *fst-Sup image-image*)

lemma *snd-Inf*: *snd* (*Inf A*) = (*INF* $x{\in}A.$ *snd x*)
by (*simp add*: *Inf-prod-def*)

lemma *snd-INF*: *snd* (*INF* $x{\in}A.$ *f x*) = (*INF* $x{\in}A.$ *snd* (*f x*))
by (*simp add*: *snd-Inf image-image*)

lemma *snd-Sup*: *snd* (*Sup A*) = (*SUP* $x{\in}A.$ *snd x*)
by (*simp add*: *Sup-prod-def*)

lemma *snd-SUP*: *snd* (*SUP* $x{\in}A.$ *f x*) = (*SUP* $x{\in}A.$ *snd* (*f x*))
by (*simp add*: *snd-Sup image-image*)

lemma *INF-Pair*: (*INF* $x{\in}A.$ (*f x*, *g x*)) = (*INF* $x{\in}A.$ *f x*, *INF* $x{\in}A.$ *g x*)
by (*simp add*: *Inf-prod-def image-image*)

lemma *SUP-Pair*: (*SUP* $x{\in}A.$ (*f x*, *g x*)) = (*SUP* $x{\in}A.$ *f x*, *SUP* $x{\in}A.$ *g x*)
by (*simp add*: *Sup-prod-def image-image*)

Alternative formulations for set infima and suprema over the product of two complete lattices:

lemma *INF-prod-alt-def*:
Inf (*f* ‘ *A*) = (*Inf* ((*fst* ∘ *f*) ‘ *A*), *Inf* ((*snd* ∘ *f*) ‘ *A*))
by (*simp add*: *Inf-prod-def image-image*)

lemma *SUP-prod-alt-def*:
Sup (*f* ‘ *A*) = (*Sup* ((*fst* ∘ *f*) ‘ *A*), *Sup*((*snd* ∘ *f*) ‘ *A*))
by (*simp add*: *Sup-prod-def image-image*)

100.5 Complete distributive lattices

instance *prod* :: (*complete-distrib-lattice*, *complete-distrib-lattice*) *complete-distrib-lattice*

proof
fix *A*::(*'a*×*'b*) *set set*
show *Inf* (*Sup* ‘ *A*) ≤ *Sup* (*Inf* ‘ {*f* ‘ *A* |*f*. ∀ *Y*∈*A*. *f Y* ∈ *Y*})
by (*simp add*: *Inf-prod-def Sup-prod-def INF-SUP-set image-image*)
qed

100.6 Bekic's Theorem

Simultaneous fixed points over pairs can be written in terms of separate fixed points. Transliterated from HOLCF.Fix by Peter Gammie

lemma *lfp-prod*:
fixes *F* :: *'a*::*complete-lattice* × *'b*::*complete-lattice* ⇒ *'a* × *'b*
assumes *mono F*
shows *lfp F* = (*lfp* (λ*x*. *fst* (*F* (*x*, *lfp* (λ*y*. *snd* (*F* (*x*, *y*)))))),

```
            (lfp (λy. snd (F (lfp (λx. fst (F (x, lfp (λy. snd (F (x, y))))))), y)))))
  (is lfp F = (?x, ?y))
proof(rule lfp-eqI[OF assms])
  have 1: fst (F (?x, ?y)) = ?x
    by (rule trans [symmetric, OF lfp-unfold])
       (blast intro!: monoI monoD[OF assms(1)] fst-mono snd-mono Pair-mono
lfp-mono)+
  have 2: snd (F (?x, ?y)) = ?y
    by (rule trans [symmetric, OF lfp-unfold])
       (blast intro!: monoI monoD[OF assms(1)] fst-mono snd-mono Pair-mono
lfp-mono)+
  from 1 2 show F (?x, ?y) = (?x, ?y) by (simp add: prod-eq-iff)
next
  fix z assume F-z: F z = z
  obtain x y where z: z = (x, y) by (rule prod.exhaust)
  from F-z z have F-x: fst (F (x, y)) = x by simp
  from F-z z have F-y: snd (F (x, y)) = y by simp
  let ?y1 = lfp (λy. snd (F (x, y)))
  have ?y1 ≤ y by (rule lfp-lowerbound, simp add: F-y)
  hence fst (F (x, ?y1)) ≤ fst (F (x, y))
    by (simp add: assms fst-mono monoD)
  hence fst (F (x, ?y1)) ≤ x using F-x by simp
  hence 1: ?x ≤ x by (simp add: lfp-lowerbound)
  hence snd (F (?x, y)) ≤ snd (F (x, y))
    by (simp add: assms snd-mono monoD)
  hence snd (F (?x, y)) ≤ y using F-y by simp
  hence 2: ?y ≤ y by (simp add: lfp-lowerbound)
  show (?x, ?y) ≤ z using z 1 2 by simp
qed

lemma gfp-prod:
  fixes F :: 'a::complete-lattice × 'b::complete-lattice ⇒ 'a × 'b
  assumes mono F
  shows gfp F = (gfp (λx. fst (F (x, gfp (λy. snd (F (x, y)))))),
            (gfp (λy. snd (F (gfp (λx. fst (F (x, gfp (λy. snd (F (x, y))))))), y)))))
  (is gfp F = (?x, ?y))
proof(rule gfp-eqI[OF assms])
  have 1: fst (F (?x, ?y)) = ?x
    by (rule trans [symmetric, OF gfp-unfold])
       (blast intro!: monoI monoD[OF assms(1)] fst-mono snd-mono Pair-mono
gfp-mono)+
  have 2: snd (F (?x, ?y)) = ?y
    by (rule trans [symmetric, OF gfp-unfold])
       (blast intro!: monoI monoD[OF assms(1)] fst-mono snd-mono Pair-mono
gfp-mono)+
  from 1 2 show F (?x, ?y) = (?x, ?y) by (simp add: prod-eq-iff)
next
  fix z assume F-z: F z = z
  obtain x y where z: z = (x, y) by (rule prod.exhaust)
```

from *F-z z* **have** *F-x*: *fst* (*F* (*x*, *y*)) = *x* **by** *simp*
from *F-z z* **have** *F-y*: *snd* (*F* (*x*, *y*)) = *y* **by** *simp*
let *?y1* = *gfp* (λ*y*. *snd* (*F* (*x*, *y*)))
have *y* ≤ *?y1* **by** (*rule gfp-upperbound*, *simp add*: *F-y*)
hence *fst* (*F* (*x*, *y*)) ≤ *fst* (*F* (*x*, *?y1*))
by (*simp add*: *assms fst-mono monoD*)
hence *x* ≤ *fst* (*F* (*x*, *?y1*)) **using** *F-x* **by** *simp*
hence *1*: *x* ≤ *?x* **by** (*simp add*: *gfp-upperbound*)
hence *snd* (*F* (*x*, *y*)) ≤ *snd* (*F* (*?x*, *y*))
by (*simp add*: *assms snd-mono monoD*)
hence *y* ≤ *snd* (*F* (*?x*, *y*)) **using** *F-y* **by** *simp*
hence *2*: *y* ≤ *?y* **by** (*simp add*: *gfp-upperbound*)
show *z* ≤ (*?x*, *?y*) **using** *z 1 2* **by** *simp*
qed

end

theory *Finite-Lattice*
imports *Product-Order*
begin

A non-empty finite lattice is a complete lattice. Since types are never empty in Isabelle/HOL, a type of classes *finite* and *lattice* should also have class *complete-lattice*. A type class is defined that extends classes *finite* and *lattice* with the operators *bot*, *top*, *Inf*, and *Sup*, along with assumptions that define these operators in terms of the ones of classes *finite* and *lattice*. The resulting class is a subclass of *complete-lattice*.

class *finite-lattice-complete* = *finite* + *lattice* + *bot* + *top* + *Inf* + *Sup* +
assumes *bot-def*: *bot* = *Inf-fin UNIV*
assumes *top-def*: *top* = *Sup-fin UNIV*
assumes *Inf-def*: *Inf A* = *Finite-Set.fold inf top A*
assumes *Sup-def*: *Sup A* = *Finite-Set.fold sup bot A*

The definitional assumptions on the operators *bot* and *top* of class *finite-lattice-complete* ensure that they yield bottom and top.

lemma *finite-lattice-complete-bot-least*: (*bot*::'*a*::*finite-lattice-complete*) ≤ *x*
by (*auto simp*: *bot-def intro*: *Inf-fin.coboundedI*)

instance *finite-lattice-complete* ⊆ *order-bot*
by *standard* (*auto simp*: *finite-lattice-complete-bot-least*)

lemma *finite-lattice-complete-top-greatest*: (*top*::'*a*::*finite-lattice-complete*) ≥ *x*
by (*auto simp*: *top-def Sup-fin.coboundedI*)

instance *finite-lattice-complete* ⊆ *order-top*
by *standard* (*auto simp*: *finite-lattice-complete-top-greatest*)

instance *finite-lattice-complete* ⊆ *bounded-lattice* **..**

The definitional assumptions on the operators *Inf* and *Sup* of class *finite-lattice-complete* ensure that they yield infimum and supremum.

lemma *finite-lattice-complete-Inf-empty*: *Inf* {} = (*top* :: 'a::*finite-lattice-complete*)
by (*simp add*: *Inf-def*)

lemma *finite-lattice-complete-Sup-empty*: *Sup* {} = (*bot* :: 'a::*finite-lattice-complete*)
by (*simp add*: *Sup-def*)

lemma *finite-lattice-complete-Inf-insert*:
fixes *A* :: 'a::*finite-lattice-complete set*
shows *Inf* (*insert x A*) = *inf x* (*Inf A*)
proof −
interpret *comp-fun-idem inf* :: 'a ⇒ -
by (*fact comp-fun-idem-inf*)
show *?thesis* **by** (*simp add*: *Inf-def*)
qed

lemma *finite-lattice-complete-Sup-insert*:
fixes *A* :: 'a::*finite-lattice-complete set*
shows *Sup* (*insert x A*) = *sup x* (*Sup A*)
proof −
interpret *comp-fun-idem sup* :: 'a ⇒ -
by (*fact comp-fun-idem-sup*)
show *?thesis* **by** (*simp add*: *Sup-def*)
qed

lemma *finite-lattice-complete-Inf-lower*:
(*x*::'a::*finite-lattice-complete*) ∈ *A* ⟹ *Inf A* ≤ *x*
using *finite* [*of A*]
by (*induct A*) (*auto simp add*: *finite-lattice-complete-Inf-insert intro*: *le-infI2*)

lemma *finite-lattice-complete-Inf-greatest*:
∀ *x*::'a::*finite-lattice-complete* ∈ *A*. *z* ≤ *x* ⟹ *z* ≤ *Inf A*
using *finite* [*of A*]
by (*induct A*) (*auto simp add*: *finite-lattice-complete-Inf-empty finite-lattice-complete-Inf-insert*)

lemma *finite-lattice-complete-Sup-upper*:
(*x*::'a::*finite-lattice-complete*) ∈ *A* ⟹ *Sup A* ≥ *x*
using *finite* [*of A*]
by (*induct A*) (*auto simp add*: *finite-lattice-complete-Sup-insert intro*: *le-supI2*)

lemma *finite-lattice-complete-Sup-least*:
∀ *x*::'a::*finite-lattice-complete* ∈ *A*. *z* ≥ *x* ⟹ *z* ≥ *Sup A*
using *finite* [*of A*]
by (*induct A*) (*auto simp add*: *finite-lattice-complete-Sup-empty finite-lattice-complete-Sup-insert*)

instance *finite-lattice-complete* ⊆ *complete-lattice*
proof
qed (*auto simp*:

finite-lattice-complete-Inf-lower
finite-lattice-complete-Inf-greatest
finite-lattice-complete-Sup-upper
finite-lattice-complete-Sup-least
finite-lattice-complete-Inf-empty
finite-lattice-complete-Sup-empty)

The product of two finite lattices is already a finite lattice.

lemma *finite-bot-prod*:
(*bot* :: (*'a*::*finite-lattice-complete* × *'b*::*finite-lattice-complete*)) =
Inf-fin UNIV
by (*metis Inf-fin.coboundedI UNIV-I bot.extremum-uniqueI finite-UNIV*)

lemma *finite-top-prod*:
(*top* :: (*'a*::*finite-lattice-complete* × *'b*::*finite-lattice-complete*)) =
Sup-fin UNIV
by (*metis Sup-fin.coboundedI UNIV-I top.extremum-uniqueI finite-UNIV*)

lemma *finite-Inf-prod*:
Inf(*A* :: (*'a*::*finite-lattice-complete* × *'b*::*finite-lattice-complete*) *set*) =
Finite-Set.fold inf top A
by (*metis Inf-fold-inf finite*)

lemma *finite-Sup-prod*:
Sup (*A* :: (*'a*::*finite-lattice-complete* × *'b*::*finite-lattice-complete*) *set*) =
Finite-Set.fold sup bot A
by (*metis Sup-fold-sup finite*)

instance *prod* :: (*finite-lattice-complete*, *finite-lattice-complete*) *finite-lattice-complete*
by *standard* (*auto simp*: *finite-bot-prod finite-top-prod finite-Inf-prod finite-Sup-prod*)

Functions with a finite domain and with a finite lattice as codomain already form a finite lattice.

lemma *finite-bot-fun*: (*bot* :: (*'a*::*finite* ⇒ *'b*::*finite-lattice-complete*)) = *Inf-fin UNIV*
by (*metis Inf-UNIV Inf-fin-Inf empty-not-UNIV finite*)

lemma *finite-top-fun*: (*top* :: (*'a*::*finite* ⇒ *'b*::*finite-lattice-complete*)) = *Sup-fin UNIV*
by (*metis Sup-UNIV Sup-fin-Sup empty-not-UNIV finite*)

lemma *finite-Inf-fun*:
Inf (*A*::(*'a*::*finite* ⇒ *'b*::*finite-lattice-complete*) *set*) =
Finite-Set.fold inf top A
by (*metis Inf-fold-inf finite*)

lemma *finite-Sup-fun*:
Sup (*A*::(*'a*::*finite* ⇒ *'b*::*finite-lattice-complete*) *set*) =
Finite-Set.fold sup bot A
by (*metis Sup-fold-sup finite*)

instance *fun* :: (*finite*, *finite-lattice-complete*) *finite-lattice-complete*
by *standard* (*auto simp*: *finite-bot-fun finite-top-fun finite-Inf-fun finite-Sup-fun*)

100.7 Finite Distributive Lattices

A finite distributive lattice is a complete lattice whose *inf* and *sup* operators distribute over *Sup* and *Inf*.

class *finite-distrib-lattice-complete* =
distrib-lattice + *finite-lattice-complete*

lemma *finite-distrib-lattice-complete-sup-Inf*:
sup (*x*::'*a*::*finite-distrib-lattice-complete*) (*Inf A*) = (*INF y*∈*A*. *sup x y*)
using *finite*
by (*induct A rule*: *finite-induct*) (*simp-all add*: *sup-inf-distrib1*)

lemma *finite-distrib-lattice-complete-inf-Sup*:
inf (*x*::'*a*::*finite-distrib-lattice-complete*) (*Sup A*) = (*SUP y*∈*A*. *inf x y*)
using *finite* [*of A*] **by** *induct* (*simp-all add*: *inf-sup-distrib1*)

context *finite-distrib-lattice-complete*
begin
subclass *finite-distrib-lattice*
proof −
show *class.finite-distrib-lattice Inf Sup inf* (≤) (<) *sup bot top*
proof
show *bot* = *Inf UNIV*
unfolding *bot-def top-def Inf-def*
using *Inf-fin.eq-fold Inf-fin.insert inf.absorb2* **by** *force*
next
show *top* = *Sup UNIV*
unfolding *bot-def top-def Sup-def*
using *Sup-fin.eq-fold Sup-fin.insert* **by** *force*
next
show *Inf* {} = *Sup UNIV*
unfolding *Inf-def Sup-def bot-def top-def*
using *Sup-fin.eq-fold Sup-fin.insert* **by** *force*
next
show *Sup* {} = *Inf UNIV*
unfolding *Inf-def Sup-def bot-def top-def*
using *Inf-fin.eq-fold Inf-fin.insert inf.absorb2* **by** *force*
next
interpret *comp-fun-idem-inf*: *comp-fun-idem inf*
by (*fact comp-fun-idem-inf*)
show *Inf* (*insert a A*) = *inf a* (*Inf A*) **for** *a A*
using *comp-fun-idem-inf.fold-insert-idem Inf-def finite* **by** *simp*
next
interpret *comp-fun-idem-sup*: *comp-fun-idem sup*
by (*fact comp-fun-idem-sup*)

show *Sup* (*insert a A*) = *sup a* (*Sup A*) **for** *a A*
using *comp-fun-idem-sup.fold-insert-idem Sup-def finite* **by** *simp*
qed
qed
end

instance *finite-distrib-lattice-complete* ⊆ *complete-distrib-lattice* **..**

The product of two finite distributive lattices is already a finite distributive lattice.

instance *prod* ::
(*finite-distrib-lattice-complete*, *finite-distrib-lattice-complete*)
finite-distrib-lattice-complete
..

Functions with a finite domain and with a finite distributive lattice as codomain already form a finite distributive lattice.

instance *fun* ::
(*finite*, *finite-distrib-lattice-complete*) *finite-distrib-lattice-complete*
..

100.8 Linear Orders

A linear order is a distributive lattice. A type class is defined that extends class *linorder* with the operators *inf* and *sup*, along with assumptions that define these operators in terms of the ones of class *linorder*. The resulting class is a subclass of *distrib-lattice*.

class *linorder-lattice* = *linorder* + *inf* + *sup* +
assumes *inf-def*: *inf x y* = (*if x* ≤ *y then x else y*)
assumes *sup-def*: *sup x y* = (*if x* ≥ *y then x else y*)

The definitional assumptions on the operators *inf* and *sup* of class *linorder-lattice* ensure that they yield infimum and supremum and that they distribute over each other.

lemma *linorder-lattice-inf-le1*: *inf* (*x*::'*a*::*linorder-lattice*) *y* ≤ *x*
unfolding *inf-def* **by** (*metis* (*full-types*) *linorder-linear*)

lemma *linorder-lattice-inf-le2*: *inf* (*x*::'*a*::*linorder-lattice*) *y* ≤ *y*
unfolding *inf-def* **by** (*metis* (*full-types*) *linorder-linear*)

lemma *linorder-lattice-inf-greatest*:
(*x*::'*a*::*linorder-lattice*) ≤ *y* ⟹ *x* ≤ *z* ⟹ *x* ≤ *inf y z*
unfolding *inf-def* **by** (*metis* (*full-types*))

lemma *linorder-lattice-sup-ge1*: *sup* (*x*::'*a*::*linorder-lattice*) *y* ≥ *x*
unfolding *sup-def* **by** (*metis* (*full-types*) *linorder-linear*)

lemma *linorder-lattice-sup-ge2*: *sup* (*x*::'*a*::*linorder-lattice*) *y* ≥ *y*

unfolding *sup-def* **by** (*metis* (*full-types*) *linorder-linear*)

lemma *linorder-lattice-sup-least*:
(*x*::'*a*::*linorder-lattice*) ≥ *y* ⟹ *x* ≥ *z* ⟹ *x* ≥ *sup y z*
by (*auto simp*: *sup-def*)

lemma *linorder-lattice-sup-inf-distrib1*:
sup (*x*::'*a*::*linorder-lattice*) (*inf y z*) = *inf* (*sup x y*) (*sup x z*)
by (*auto simp*: *inf-def sup-def*)

instance *linorder-lattice* ⊆ *distrib-lattice*
proof
qed (*auto simp*:
linorder-lattice-inf-le1
linorder-lattice-inf-le2
linorder-lattice-inf-greatest
linorder-lattice-sup-ge1
linorder-lattice-sup-ge2
linorder-lattice-sup-least
linorder-lattice-sup-inf-distrib1)

100.9 Finite Linear Orders

A (non-empty) finite linear order is a complete linear order.

class *finite-linorder-complete* = *linorder-lattice* + *finite-lattice-complete*

instance *finite-linorder-complete* ⊆ *complete-linorder* **..**

A (non-empty) finite linear order is a complete lattice whose *inf* and *sup* operators distribute over *Sup* and *Inf*.

instance *finite-linorder-complete* ⊆ *finite-distrib-lattice-complete* **..**

end

101 Lexicographic order on lists

theory *List-Lexorder*
imports *Main*
begin

instantiation *list* :: (*ord*) *ord*
begin

definition
list-less-def: *xs* < *ys* ⟷ (*xs*, *ys*) ∈ *lexord* {(*u*, *v*). *u* < *v*}

definition
list-le-def: (*xs* :: - *list*) ≤ *ys* ⟷ *xs* < *ys* ∨ *xs* = *ys*

instance ..

end

instance *list* :: (*order*) *order*
proof
let *?r* = {(*u*, *v*::'*a*). *u* < *v*}
have *tr*: *trans ?r*
using *trans-def* **by** *fastforce*
have §: *False*
if (*xs*,*ys*) ∈ *lexord ?r* (*ys*,*xs*) ∈ *lexord ?r* **for** *xs ys* :: '*a list*
proof −
have (*xs*,*xs*) ∈ *lexord ?r*
using *that transD* [*OF lexord-transI* [*OF tr*]] **by** *blast*
then show *False*
by (*meson case-prodD lexord-irreflexive less-irrefl mem-Collect-eq*)
qed
show *xs* ≤ *xs* **for** *xs* :: '*a list* **by** (*simp add*: *list-le-def*)
show *xs* ≤ *zs* **if** *xs* ≤ *ys* **and** *ys* ≤ *zs* **for** *xs ys zs* :: '*a list*
using *that transD* [*OF lexord-transI* [*OF tr*]] **by** (*auto simp add*: *list-le-def list-less-def*)
show *xs* = *ys* **if** *xs* ≤ *ys ys* ≤ *xs* **for** *xs ys* :: '*a list*
using § *that list-le-def list-less-def* **by** *blast*
show *xs* < *ys* ⟷ *xs* ≤ *ys* ∧ ¬ *ys* ≤ *xs* **for** *xs ys* :: '*a list*
by (*auto simp add*: *list-less-def list-le-def dest*: §)
qed

instance *list* :: (*linorder*) *linorder*
proof
fix *xs ys* :: '*a list*
have *total* (*lexord* {(*u*, *v*::'*a*). *u* < *v*})
by (*rule total-lexord*) (*auto simp*: *total-on-def*)
then show *xs* ≤ *ys* ∨ *ys* ≤ *xs*
by (*auto simp add*: *total-on-def list-le-def list-less-def*)
qed

instantiation *list* :: (*linorder*) *distrib-lattice*
begin

definition (*inf* :: '*a list* ⇒ -) = *min*

definition (*sup* :: '*a list* ⇒ -) = *max*

instance
by *standard* (*auto simp add*: *inf-list-def sup-list-def max-min-distrib2*)

end

lemma *not-less-Nil* [*simp*]: $\neg\ x < []$
by (*simp add*: *list-less-def*)

lemma *Nil-less-Cons* [*simp*]: $[] < a \# x$
by (*simp add*: *list-less-def*)

lemma *Cons-less-Cons* [*simp*]: $a \# x < b \# y \longleftrightarrow a < b \vee a = b \wedge x < y$
by (*simp add*: *list-less-def*)

lemma *le-Nil* [*simp*]: $x \leq [] \longleftrightarrow x = []$
unfolding *list-le-def* **by** (*cases x*) *auto*

lemma *Nil-le-Cons* [*simp*]: $[] \leq x$
unfolding *list-le-def* **by** (*cases x*) *auto*

lemma *Cons-le-Cons* [*simp*]: $a \# x \leq b \# y \longleftrightarrow a < b \vee a = b \wedge x \leq y$
unfolding *list-le-def* **by** *auto*

instantiation *list* :: (*order*) *order-bot*
begin

definition *bot* = []

instance
by *standard* (*simp add*: *bot-list-def*)

end

lemma *less-list-code* [*code*]:
$xs < ([]::'a::\{equal, order\}\ list) \longleftrightarrow False$
$[] < (x::'a::\{equal, order\}) \# xs \longleftrightarrow True$
$(x::'a::\{equal, order\}) \# xs < y \# ys \longleftrightarrow x < y \vee x = y \wedge xs < ys$
by *simp-all*

lemma *less-eq-list-code* [*code*]:
$x \# xs \leq ([]::'a::\{equal, order\}\ list) \longleftrightarrow False$
$[] \leq (xs::'a::\{equal, order\}\ list) \longleftrightarrow True$
$(x::'a::\{equal, order\}) \# xs \leq y \# ys \longleftrightarrow x < y \vee x = y \wedge xs \leq ys$
by *simp-all*

end

102 Lexicographic order on lists

theory *List-Lenlexorder*
imports *Main*
begin

instantiation *list* :: (*ord*) *ord*
begin

definition
list-less-def: $xs < ys \longleftrightarrow (xs, ys) \in lenlex\ \{(u, v).\ u < v\}$

definition
list-le-def: $(xs :: - list) \leq ys \longleftrightarrow xs < ys \vee xs = ys$

instance ..

end

instance *list* :: (*order*) *order*
proof
have *tr*: *trans* $\{(u, v::'a).\ u < v\}$
using *trans-def* **by** *fastforce*
have §: *False*
if $(xs,ys) \in lenlex\ \{(u, v).\ u < v\}$ $(ys,xs) \in lenlex\ \{(u, v).\ u < v\}$ **for** *xs ys* :: *'a list*
proof −
have $(xs,xs) \in lenlex\ \{(u, v).\ u < v\}$
using *that transD* [*OF lenlex-transI* [*OF tr*]] **by** *blast*
then show *False*
by (*meson case-prodD lenlex-irreflexive less-irrefl mem-Collect-eq*)
qed
show $xs \leq xs$ **for** *xs* :: *'a list* **by** (*simp add*: *list-le-def*)
show $xs \leq zs$ **if** $xs \leq ys$ **and** $ys \leq zs$ **for** *xs ys zs* :: *'a list*
using *that transD* [*OF lenlex-transI* [*OF tr*]] **by** (*auto simp add*: *list-le-def list-less-def*)
show $xs = ys$ **if** $xs \leq ys$ $ys \leq xs$ **for** *xs ys* :: *'a list*
using § *that list-le-def list-less-def* **by** *blast*
show $xs < ys \longleftrightarrow xs \leq ys \wedge \neg\ ys \leq xs$ **for** *xs ys* :: *'a list*
by (*auto simp add*: *list-less-def list-le-def dest*: §)
qed

instance *list* :: (*linorder*) *linorder*
proof
fix *xs ys* :: *'a list*
have *total* (*lenlex* $\{(u, v::'a).\ u < v\}$)
by (*rule total-lenlex*) (*auto simp*: *total-on-def*)
then show $xs \leq ys \vee ys \leq xs$
by (*auto simp add*: *total-on-def list-le-def list-less-def*)
qed

instantiation *list* :: (*linorder*) *distrib-lattice*
begin

definition (*inf* :: *'a list* ⇒ -) = *min*

definition (*sup* :: *'a list* ⇒ -) = *max*

instance
by *standard* (*auto simp add*: *inf-list-def sup-list-def max-min-distrib2*)

end

lemma *not-less-Nil* [*simp*]: ¬ *x* < []
by (*simp add*: *list-less-def*)

lemma *Nil-less-Cons* [*simp*]: [] < *a* # *x*
by (*simp add*: *list-less-def*)

lemma *Cons-less-Cons*: *a* # *x* < *b* # *y* ⟷ *length x* < *length y* ∨ *length x* = *length y* ∧ (*a* < *b* ∨ *a* = *b* ∧ *x* < *y*)
using *lenlex-length*
by (*fastforce simp*: *list-less-def Cons-lenlex-iff*)

lemma *le-Nil* [*simp*]: *x* ≤ [] ⟷ *x* = []
unfolding *list-le-def* **by** (*cases x*) *auto*

lemma *Nil-le-Cons* [*simp*]: [] ≤ *x*
unfolding *list-le-def* **by** (*cases x*) *auto*

lemma *Cons-le-Cons*: *a* # *x* ≤ *b* # *y* ⟷ *length x* < *length y* ∨ *length x* = *length y* ∧ (*a* < *b* ∨ *a* = *b* ∧ *x* ≤ *y*)
by (*auto simp*: *list-le-def Cons-less-Cons*)

instantiation *list* :: (*order*) *order-bot*
begin

definition *bot* = []

instance
by *standard* (*simp add*: *bot-list-def*)

end

end

103 Prefix order on lists as order class instance

theory *Prefix-Order*
imports *Sublist*
begin

instantiation *list* :: (*type*) *order*
begin

definition $xs \leq ys \equiv prefix\ xs\ ys$ **for** $xs\ ys :: 'a\ list$
definition $xs < ys \equiv xs \leq ys \wedge \neg\ (ys \leq xs)$ **for** $xs\ ys :: 'a\ list$

instance
by *standard* (*auto simp*: *less-eq-list-def less-list-def*)

end

lemma *less-list-def'*: $xs < ys \longleftrightarrow strict\text{-}prefix\ xs\ ys$ **for** $xs\ ys :: 'a\ list$
by (*simp add*: *less-eq-list-def order.strict-iff-order prefix-order.less-le*)

lemmas *prefixI* [*intro?*] = *prefixI* [*folded less-eq-list-def*]
lemmas *prefixE* [*elim?*] = *prefixE* [*folded less-eq-list-def*]
lemmas *strict-prefixI'* [*intro?*] = *strict-prefixI'* [*folded less-list-def'*]
lemmas *strict-prefixE'* [*elim?*] = *strict-prefixE'* [*folded less-list-def'*]
lemmas *strict-prefixI* [*intro?*] = *strict-prefixI* [*folded less-list-def'*]
lemmas *strict-prefixE* [*elim?*] = *strict-prefixE* [*folded less-list-def'*]
lemmas *Nil-prefix* [*iff*] = *Nil-prefix* [*folded less-eq-list-def*]
lemmas *prefix-Nil* [*simp*] = *prefix-Nil* [*folded less-eq-list-def*]
lemmas *prefix-snoc* [*simp*] = *prefix-snoc* [*folded less-eq-list-def*]
lemmas *Cons-prefix-Cons* [*simp*] = *Cons-prefix-Cons* [*folded less-eq-list-def*]
lemmas *same-prefix-prefix* [*simp*] = *same-prefix-prefix* [*folded less-eq-list-def*]
lemmas *same-prefix-nil* [*iff*] = *same-prefix-nil* [*folded less-eq-list-def*]
lemmas *prefix-prefix* [*simp*] = *prefix-prefix* [*folded less-eq-list-def*]
lemmas *prefix-Cons* = *prefix-Cons* [*folded less-eq-list-def*]
lemmas *prefix-length-le* = *prefix-length-le* [*folded less-eq-list-def*]
lemmas *strict-prefix-simps* [*simp*, *code*] = *strict-prefix-simps* [*folded less-list-def'*]
lemmas *not-prefix-induct* [*consumes 1*, *case-names Nil Neq Eq*] =
not-prefix-induct [*folded less-eq-list-def*]

end

104 Lexicographic order on product types

theory *Product-Lexorder*
imports *Main*
begin

instantiation *prod* :: (*ord*, *ord*) *ord*
begin

definition
$x \leq y \longleftrightarrow fst\ x < fst\ y \vee fst\ x \leq fst\ y \wedge snd\ x \leq snd\ y$

definition
$x < y \longleftrightarrow fst\ x < fst\ y \vee fst\ x \leq fst\ y \wedge snd\ x < snd\ y$

instance ..

end

lemma *less-eq-prod-simp* [*simp*, *code*]:
(*x1*, *y1*) ≤ (*x2*, *y2*) ⟷ *x1* < *x2* ∨ *x1* ≤ *x2* ∧ *y1* ≤ *y2*
by (*simp add*: *less-eq-prod-def*)

lemma *less-prod-simp* [*simp*, *code*]:
(*x1*, *y1*) < (*x2*, *y2*) ⟷ *x1* < *x2* ∨ *x1* ≤ *x2* ∧ *y1* < *y2*
by (*simp add*: *less-prod-def*)

A stronger version for partial orders.

lemma *less-prod-def'*:
fixes *x y* :: 'a::*order* × 'b::*ord*
shows *x* < *y* ⟷ *fst x* < *fst y* ∨ *fst x* = *fst y* ∧ *snd x* < *snd y*
by (*auto simp add*: *less-prod-def le-less*)

instance *prod* :: (*preorder*, *preorder*) *preorder*
by *standard* (*auto simp*: *less-eq-prod-def less-prod-def less-le-not-le intro*: *order-trans*)

instance *prod* :: (*order*, *order*) *order*
by *standard* (*auto simp add*: *less-eq-prod-def*)

instance *prod* :: (*linorder*, *linorder*) *linorder*
by *standard* (*auto simp*: *less-eq-prod-def*)

instantiation *prod* :: (*linorder*, *linorder*) *distrib-lattice*
begin

definition
(*inf* :: 'a × 'b ⇒ - ⇒ -) = *min*

definition
(*sup* :: 'a × 'b ⇒ - ⇒ -) = *max*

instance
by *standard* (*auto simp add*: *inf-prod-def sup-prod-def max-min-distrib2*)

end

instantiation *prod* :: (*bot*, *bot*) *bot*
begin

definition
bot = (*bot*, *bot*)

instance ..

```
end

instance prod :: (order-bot, order-bot) order-bot
  by standard (auto simp add: bot-prod-def)

instantiation prod :: (top, top) top
begin

definition
  top = (top, top)

instance ..

end

instance prod :: (order-top, order-top) order-top
  by standard (auto simp add: top-prod-def)

instance prod :: (wellorder, wellorder) wellorder
proof
  fix P :: 'a × 'b ⇒ bool and z :: 'a × 'b
  assume P: ⋀x. (⋀y. y < x ⟹ P y) ⟹ P x
  show P z
  proof (induct z)
    case (Pair a b)
    show P (a, b)
    proof (induct a arbitrary: b rule: less-induct)
      case (less a₁) note a₁ = this
      show P (a₁, b)
      proof (induct b rule: less-induct)
        case (less b₁) note b₁ = this
        show P (a₁, b₁)
        proof (rule P)
          fix p assume p: p < (a₁, b₁)
          show P p
          proof (cases fst p < a₁)
            case True
            then have P (fst p, snd p) by (rule a₁)
            then show ?thesis by simp
          next
            case False
            with p have 1: a₁ = fst p and 2: snd p < b₁
              by (simp-all add: less-prod-def')
            from 2 have P (a₁, snd p) by (rule b₁)
            with 1 show ?thesis by simp
          qed
        qed
      qed
    qed
```

qed
qed

Legacy lemma bindings

lemmas *prod-le-def* = *less-eq-prod-def*
lemmas *prod-less-def* = *less-prod-def*
lemmas *prod-less-eq* = *less-prod-def'*

end

105 Subsequence Ordering

theory *Subseq-Order*
imports *Sublist*
begin

This theory defines subsequence ordering on lists. A list *ys* is a subsequence of a list *xs*, iff one obtains *ys* by erasing some elements from *xs*.

105.1 Definitions and basic lemmas

instantiation *list* :: (*type*) *ord*
begin

definition $xs \leq ys \longleftrightarrow$ *subseq xs ys* **for** *xs ys* :: *'a list*
definition $xs < ys \longleftrightarrow xs \leq ys \wedge \neg\ ys \leq xs$ **for** *xs ys* :: *'a list*

instance ..

end

instance *list* :: (*type*) *order*
proof
  fix *xs ys zs* :: *'a list*
  show $xs < ys \longleftrightarrow xs \leq ys \wedge \neg\ ys \leq xs$
    unfolding *less-list-def* **..**
  show $xs \leq xs$
    by (*simp add*: *less-eq-list-def*)
  show $xs = ys$ **if** $xs \leq ys$ **and** $ys \leq xs$
    using *that* **unfolding** *less-eq-list-def*
    by (*rule subseq-order.antisym*)
  show $xs \leq zs$ **if** $xs \leq ys$ **and** $ys \leq zs$
    using *that* **unfolding** *less-eq-list-def*
    by (*rule subseq-order.order-trans*)
qed

lemmas *less-eq-list-induct* [*consumes 1*, *case-names empty drop take*] =
  list-emb.induct [*of* (=), *folded less-eq-list-def*]
lemmas *less-eq-list-drop* = *list-emb.list-emb-Cons* [*of* (=), *folded less-eq-list-def*]

lemmas *le-list-Cons2-iff* [*simp*, *code*] = *subseq-Cons2-iff* [*folded less-eq-list-def*]
lemmas *le-list-map* = *subseq-map* [*folded less-eq-list-def*]
lemmas *le-list-filter* = *subseq-filter* [*folded less-eq-list-def*]
lemmas *le-list-length* = *list-emb-length* [*of* (=), *folded less-eq-list-def*]

lemma *less-list-length*: $xs < ys \Longrightarrow length\ xs < length\ ys$
by (*metis list-emb-length subseq-same-length le-neq-implies-less less-list-def less-eq-list-def*)

lemma *less-list-empty* [*simp*]: $[] < xs \longleftrightarrow xs \neq []$
by (*metis less-eq-list-def list-emb-Nil order-less-le*)

lemma *less-list-below-empty* [*simp*]: $xs < [] \longleftrightarrow False$
by (*metis list-emb-Nil less-eq-list-def less-list-def*)

lemma *less-list-drop*: $xs < ys \Longrightarrow xs < x \# ys$
by (*unfold less-le less-eq-list-def*) (*auto*)

lemma *less-list-take-iff*: $x \# xs < x \# ys \longleftrightarrow xs < ys$
by (*metis subseq-Cons2-iff less-list-def less-eq-list-def*)

lemma *less-list-drop-many*: $xs < ys \Longrightarrow xs < zs\ @\ ys$
by (*metis subseq-append-le-same-iff subseq-drop-many order-less-le self-append-conv2 less-eq-list-def*)

lemma *less-list-take-many-iff*: $zs\ @\ xs < zs\ @\ ys \longleftrightarrow xs < ys$
by (*metis less-list-def less-eq-list-def subseq-append'*)

lemma *less-list-rev-take*: $xs\ @\ zs < ys\ @\ zs \longleftrightarrow xs < ys$
by (*unfold less-le less-eq-list-def*) *auto*

end

106 Records based on BNF/datatype machinery

theory *Datatype-Records*
imports *Main*
keywords *datatype-record* :: *thy-defn*
begin

This theory provides an alternative, stripped-down implementation of records based on the machinery of the **datatype** package.

It supports:

- similar declaration syntax as records
- record creation and update syntax (using ⦇ ... ⦈ brackets)
- regular datatype features (e.g. dead type variables etc.)
- "after-the-fact" registration of single-free-constructor types as records

Caveats:

- there is no compatibility layer; importing this theory will disrupt existing syntax
- extensible records are not supported

no-syntax
```
  -constify           :: id => ident                          (-)
  -constify           :: longid => ident                       (-)

  -field-type         :: ident => type => field-type          ((2- ::/ -))
                      :: field-type => field-types            (-)
  -field-types        :: field-type => field-types => field-types   (-,/ -)
  -record-type        :: field-types => type                  ((3⦇-⦈))
  -record-type-scheme :: field-types => type => type          ((3⦇-,/ (2... ::/ -)⦈))

  -field              :: ident => 'a => field                 ((2- =/ -))
                      :: field => fields                      (-)
  -fields             :: field => fields => fields            (-,/ -)
  -record             :: fields => 'a                         ((3⦇-⦈))
  -record-scheme      :: fields => 'a => 'a                   ((3⦇-,/ (2... =/ -)⦈))

  -field-update       :: ident => 'a => field-update          ((2- :=/ -))
                      :: field-update => field-updates        (-)
  -field-updates      :: field-update => field-updates => field-updates  (-,/ -)
  -record-update      :: 'a => field-updates => 'b            (-/(3⦇-⦈) [900, 0] 900)
```

no-syntax (*ASCII*)
```
  -record-type        :: field-types => type                  ((3'(| - |')))
  -record-type-scheme :: field-types => type => type          ((3'(| -,/ (2... ::/ -) |')))
  -record             :: fields => 'a                         ((3'(| - |')))
  -record-scheme      :: fields => 'a => 'a                   ((3'(| -,/ (2... =/ -) |')))
  -record-update      :: 'a => field-updates => 'b            (-/(3'(| - |')) [900, 0] 900)
```

nonterminal
field **and**
fields **and**
field-update **and**
field-updates

syntax
```
  -constify           :: id => ident                          (-)
  -constify           :: longid => ident                       (-)

  -datatype-field     :: ident => 'a => field                 ((2- =/ -))
                      :: field => fields                      (-)
```

-datatype-fields :: *field* => *fields* => *fields* (-,/ -)
-datatype-record :: *fields* => *'a* ((*3*⦇-⦈))
-datatype-field-update :: *ident* => *'a* => *field-update* ((*2*- :=/ -))
:: *field-update* => *field-updates* (-)
-datatype-field-updates :: *field-update* => *field-updates* => *field-updates* (-,/ -)
-datatype-record-update :: *'a* => *field-updates* => *'b* (-/(*3*⦇-⦈) [*900*, *0*] *900*)

syntax (*ASCII*)
-datatype-record :: *fields* => *'a* ((*3'*(| - |')))
-datatype-record-scheme :: *fields* => *'a* => *'a* ((*3'*(| -,/ (*2*... =/ -) |')))
-datatype-record-update :: *'a* => *field-updates* => *'b* (-/(*3'*(| - |')) [*900*, *0*] *900*)

named-theorems *datatype-record-update*

ML-file ‹*datatype-records.ML*›
setup ‹*Datatype-Records.setup*›

end

107 Implementation of mappings with Association Lists

theory *AList-Mapping*
imports *AList Mapping*
begin

lift-definition *Mapping* :: (*'a* × *'b*) *list* ⇒ (*'a*, *'b*) *mapping* **is** *map-of* **.**

code-datatype *Mapping*

lemma *lookup-Mapping* [*simp*, *code*]: *Mapping.lookup* (*Mapping xs*) = *map-of xs*
by *transfer rule*

lemma *keys-Mapping* [*simp*, *code*]: *Mapping.keys* (*Mapping xs*) = *set* (*map fst xs*)
by *transfer* (*simp add*: *dom-map-of-conv-image-fst*)

lemma *empty-Mapping* [*code*]: *Mapping.empty* = *Mapping* []
by *transfer simp*

lemma *is-empty-Mapping* [*code*]: *Mapping.is-empty* (*Mapping xs*) ⟷ *List.null xs*
by (*cases xs*) (*simp-all add*: *is-empty-def null-def*)

lemma *update-Mapping* [*code*]: *Mapping.update k v* (*Mapping xs*) = *Mapping* (*AList.update k v xs*)
by *transfer* (*simp add*: *update-conv'*)

lemma *delete-Mapping* [*code*]: *Mapping.delete k* (*Mapping xs*) = *Mapping* (*AList.delete k xs*)
by *transfer* (*simp add*: *delete-conv'*)

lemma *ordered-keys-Mapping* [*code*]:
Mapping.ordered-keys (*Mapping xs*) = *sort* (*remdups* (*map fst xs*))
by (*simp only*: *ordered-keys-def keys-Mapping sorted-list-of-set-sort-remdups*) *simp*

lemma *entries-Mapping* [*code*]:
Mapping.entries (*Mapping xs*) = *set* (*AList.clearjunk xs*)
by *transfer* (*fact graph-map-of*)

lemma *ordered-entries-Mapping* [*code*]:
Mapping.ordered-entries (*Mapping xs*) = *sort-key fst* (*AList.clearjunk xs*)
proof −
have *distinct*: *distinct* (*sort-key fst* (*AList.clearjunk xs*))
using *distinct-clearjunk distinct-map distinct-sort* **by** *blast*
note *folding-Map-graph.idem-if-sorted-distinct*[**where** *?m*=*map-of xs*, *OF - sorted-sort-key distinct*]
then show *?thesis*
unfolding *ordered-entries-def*
by (*transfer fixing*: *xs*) (*auto simp*: *graph-map-of*)
qed

lemma *fold-Mapping* [*code*]:
Mapping.fold f (*Mapping xs*) *a* = *List.fold* (*case-prod f*) (*sort-key fst* (*AList.clearjunk xs*)) *a*
by (*simp add*: *Mapping.fold-def ordered-entries-Mapping*)

lemma *size-Mapping* [*code*]: *Mapping.size* (*Mapping xs*) = *length* (*remdups* (*map fst xs*))
by (*simp add*: *size-def length-remdups-card-conv dom-map-of-conv-image-fst*)

lemma *tabulate-Mapping* [*code*]: *Mapping.tabulate ks f* = *Mapping* (*map* (λ*k*. (*k*, *f k*)) *ks*)
by *transfer* (*simp add*: *map-of-map-restrict*)

lemma *bulkload-Mapping* [*code*]:
Mapping.bulkload vs = *Mapping* (*map* (λ*n*. (*n*, *vs* ! *n*)) [*0*..<*length vs*])
by *transfer* (*simp add*: *map-of-map-restrict fun-eq-iff*)

lemma *equal-Mapping* [*code*]:
HOL.equal (*Mapping xs*) (*Mapping ys*) ⟷
(*let ks* = *map fst xs*; *ls* = *map fst ys*
in (∀ *l*∈*set ls*. *l* ∈ *set ks*) ∧ (∀ *k*∈*set ks*. *k* ∈ *set ls* ∧ *map-of xs k* = *map-of ys k*))
proof −
have ∗: (*a*, *b*) ∈ *set xs* ⟹ *a* ∈ *fst* ‘ *set xs* **for** *a b xs*

```
    by (auto simp add: image-def intro!: bexI)
  show ?thesis
    apply transfer
    apply (auto intro!: map-of-eqI)
     apply (auto dest!: map-of-eq-dom intro: *)
    done
qed

lemma map-values-Mapping [code]:
  Mapping.map-values f (Mapping xs) = Mapping (map (λ(x,y). (x, f x y)) xs)
  for f :: 'c ⇒ 'a ⇒ 'b and xs :: ('c × 'a) list
  apply transfer
  apply (rule ext)
  subgoal for f xs x by (induct xs) auto
  done

lemma combine-with-key-code [code]:
  Mapping.combine-with-key f (Mapping xs) (Mapping ys) =
     Mapping.tabulate (remdups (map fst xs @ map fst ys))
       (λx. the (combine-options (f x) (map-of xs x) (map-of ys x)))
  apply transfer
  apply (rule ext)
  apply (rule sym)
  subgoal for f xs ys x
    apply (cases map-of xs x; cases map-of ys x; simp)
        apply (force simp: map-of-eq-None-iff combine-options-def option.the-def
o-def image-iff
        dest: map-of-SomeD split: option.splits)+
    done
  done

lemma combine-code [code]:
  Mapping.combine f (Mapping xs) (Mapping ys) =
     Mapping.tabulate (remdups (map fst xs @ map fst ys))
       (λx. the (combine-options f (map-of xs x) (map-of ys x)))
  apply transfer
  apply (rule ext)
  apply (rule sym)
  subgoal for f xs ys x
    apply (cases map-of xs x; cases map-of ys x; simp)
        apply (force simp: map-of-eq-None-iff combine-options-def option.the-def
o-def image-iff
        dest: map-of-SomeD split: option.splits)+
    done
  done

lemma map-of-filter-distinct:
  assumes distinct (map fst xs)
  shows map-of (filter P xs) x =
```

(*case map-of xs x of*
None ⇒ None
| Some y ⇒ if P (x,y) then Some y else None)
using *assms*
by (*auto simp*: *map-of-eq-None-iff filter-map distinct-map-filter dest*: *map-of-SomeD*
simp del: *map-of-eq-Some-iff intro*!: *map-of-is-SomeI split*: *option.splits*)

lemma *filter-Mapping* [*code*]:
Mapping.filter P (*Mapping xs*) = *Mapping* (*filter* (λ(*k*,*v*). *P k v*) (*AList.clearjunk xs*))
apply *transfer*
apply (*rule ext*)
apply (*subst map-of-filter-distinct*)
apply (*simp-all add*: *map-of-clearjunk split*: *option.split*)
done

lemma [*code nbe*]: *HOL.equal* (*x* :: ('*a*, '*b*) *mapping*) *x* ⟷ *True*
by (*fact equal-refl*)

end

108 Avoidance of pattern matching on natural numbers

theory *Code-Abstract-Nat*
imports *Main*
begin

When natural numbers are implemented in another than the conventional inductive *0*/*Suc* representation, it is necessary to avoid all pattern matching on natural numbers altogether. This is accomplished by this theory (up to a certain extent).

108.1 Case analysis

Case analysis on natural numbers is rephrased using a conditional expression:

lemma [*code*, *code-unfold*]:
case-nat = (λ*f g n*. *if n* = *0 then f else g* (*n* − *1*))
by (*auto simp add*: *fun-eq-iff dest*!: *gr0-implies-Suc*)

108.2 Preprocessors

The term *Suc n* is no longer a valid pattern. Therefore, all occurrences of this term in a position where a pattern is expected (i.e. on the left-hand side of a code equation) must be eliminated. This can be accomplished – as far as possible – by applying the following transformation rule:

lemma *Suc-if-eq*:
assumes $\bigwedge n.\ f\ (Suc\ n) \equiv h\ n$
assumes $f\ 0 \equiv g$
shows $f\ n \equiv if\ n = 0\ then\ g\ else\ h\ (n - 1)$
by (*rule eq-reflection*) (*cases n*, *insert assms*, *simp-all*)

The rule above is built into a preprocessor that is plugged into the code generator.

```
setup ‹
let

val Suc-if-eq = Thm.incr-indexes 1 @{thm Suc-if-eq};

fun remove-suc ctxt thms =
  let
    val vname = singleton (Name.variant-list (map fst
      (fold (Term.add-var-names o Thm.full-prop-of) thms []))) n;
    val cv = Thm.cterm-of ctxt (Var ((vname, 0), HOLogic.natT));
    val lhs-of = snd o Thm.dest-comb o fst o Thm.dest-comb o Thm.cprop-of;
    val rhs-of = snd o Thm.dest-comb o Thm.cprop-of;
    fun find-vars ct = (case Thm.term-of ct of
        (Const (const-name‹Suc›, -) $ Var -) => [(cv, snd (Thm.dest-comb ct))]
      | - $ - =>
        let val (ct1, ct2) = Thm.dest-comb ct
        in
          map (apfst (fn ct => Thm.apply ct ct2)) (find-vars ct1) @
          map (apfst (Thm.apply ct1)) (find-vars ct2)
        end
      | - => []);
    val eqs = maps
      (fn thm => map (pair thm) (find-vars (lhs-of thm))) thms;
    fun mk-thms (thm, (ct, cv')) =
      let
        val thm' =
          Thm.implies-elim
           (Conv.fconv-rule (Thm.beta-conversion true)
             (Thm.instantiate'
               [SOME (Thm.ctyp-of-cterm ct)] [SOME (Thm.lambda cv ct),
                 SOME (Thm.lambda cv' (rhs-of thm)), NONE, SOME cv']
               Suc-if-eq)) (Thm.forall-intr cv' thm)
      in
        case map-filter (fn thm'' =>
            SOME (thm'', singleton
              (Variable.trade (K (fn [thm'''] => [thm''' RS thm']))
                (Variable.declare-thm thm'' ctxt)) thm'')
          handle THM - => NONE) thms of
            [] => NONE
          | thmps =>
              let val (thms1, thms2) = split-list thmps
```

```
          in SOME (subtract Thm.eq-thm (thm :: thms1) thms @ thms2) end
      end
  in get-first mk-thms eqs end;

fun eqn-suc-base-preproc ctxt thms =
  let
    val dest = fst o Logic.dest-equals o Thm.prop-of;
    val contains-suc = exists-Const (fn (c, -) => c = const-name ‹Suc›);
  in
    if forall (can dest) thms andalso exists (contains-suc o dest) thms
    then thms |> perhaps-loop (remove-suc ctxt) |> (Option.map o map) Drule.zero-var-indexes
      else NONE
  end;

val eqn-suc-preproc = Code-Preproc.simple-functrans eqn-suc-base-preproc;

in

  Code-Preproc.add-functrans (eqn-Suc, eqn-suc-preproc)

end
›
```

108.3 One candidate which needs special treatment

```
lemma take-bit-num-code [code]:
  ‹take-bit-num n Num.One =
    (case n of 0 ⇒ None | Suc n ⇒ Some Num.One)›
  ‹take-bit-num n (Num.Bit0 m) =
    (case n of 0 ⇒ None | Suc n ⇒ (case take-bit-num n m of None ⇒ None |
Some q ⇒ Some (Num.Bit0 q)))›
  ‹take-bit-num n (Num.Bit1 m) =
    (case n of 0 ⇒ None | Suc n ⇒ Some (case take-bit-num n m of None ⇒
Num.One | Some q ⇒ Num.Bit1 q))›
    apply (cases n; simp)+
  done

end
```

109 Implementation of natural numbers as binary numerals

```
theory Code-Binary-Nat
imports Code-Abstract-Nat
begin
```

When generating code for functions on natural numbers, the canonical representation using *0* and *Suc* is unsuitable for computations involving large numbers. This theory refines the representation of natural numbers

for code generation to use binary numerals, which do not grow linear in size but logarithmic.

109.1 Representation

code-datatype *0*::*nat nat-of-num*

lemma [*code*]:
num-of-nat 0 = Num.One
num-of-nat (*nat-of-num k*) = *k*
by (*simp-all add*: *nat-of-num-inverse*)

lemma [*code*]:
(*1*::*nat*) = *Numeral1*
by *simp*

lemma [*code-abbrev*]: *Numeral1* = (*1*::*nat*)
by *simp*

lemma [*code*]:
Suc n = *n* + *1*
by *simp*

109.2 Basic arithmetic

context
begin

declare [[*code drop*: *plus* :: *nat* ⇒ -]]

lemma *plus-nat-code* [*code*]:
nat-of-num k + *nat-of-num l* = *nat-of-num* (*k* + *l*)
m + *0* = (*m*::*nat*)
0 + *n* = (*n*::*nat*)
by (*simp-all add*: *nat-of-num-numeral*)

Bounded subtraction needs some auxiliary

qualified definition *dup* :: *nat* ⇒ *nat* **where**
dup n = *n* + *n*

lemma *dup-code* [*code*]:
dup 0 = *0*
dup (*nat-of-num k*) = *nat-of-num* (*Num.Bit0 k*)
by (*simp-all add*: *dup-def numeral-Bit0*)

qualified definition *sub* :: *num* ⇒ *num* ⇒ *nat option* **where**
sub k l = (*if k* ≥ *l then Some* (*numeral k* − *numeral l*) *else None*)

lemma *sub-code* [*code*]:

sub Num.One Num.One = Some 0
sub (Num.Bit0 m) Num.One = Some (nat-of-num (Num.BitM m))
sub (Num.Bit1 m) Num.One = Some (nat-of-num (Num.Bit0 m))
sub Num.One (Num.Bit0 n) = None
sub Num.One (Num.Bit1 n) = None
sub (Num.Bit0 m) (Num.Bit0 n) = map-option dup (sub m n)
sub (Num.Bit1 m) (Num.Bit1 n) = map-option dup (sub m n)
sub (Num.Bit1 m) (Num.Bit0 n) = map-option (λq. dup q + 1) (sub m n)
sub (Num.Bit0 m) (Num.Bit1 n) = (case sub m n of None ⇒ None
| Some q ⇒ if q = 0 then None else Some (dup q − 1))
apply (*auto simp add*: *nat-of-num-numeral*
Num.dbl-def Num.dbl-inc-def Num.dbl-dec-def
Let-def le-imp-diff-is-add BitM-plus-one sub-def dup-def)
apply (*simp-all add*: *sub-non-positive*)
apply (*simp-all add*: *sub-non-negative* [*symmetric*, **where** *?'a = int*])
done

declare [[*code drop*: *minus* :: *nat* ⇒ -]]

lemma *minus-nat-code* [*code*]:
nat-of-num k − nat-of-num l = (case sub k l of None ⇒ 0 | Some j ⇒ j)
m − 0 = (m::nat)
0 − n = (0::nat)
by (*simp-all add*: *nat-of-num-numeral sub-non-positive sub-def*)

declare [[*code drop*: *times* :: *nat* ⇒ -]]

lemma *times-nat-code* [*code*]:
nat-of-num k ∗ nat-of-num l = nat-of-num (k ∗ l)
m ∗ 0 = (0::nat)
0 ∗ n = (0::nat)
by (*simp-all add*: *nat-of-num-numeral*)

declare [[*code drop*: *HOL.equal* :: *nat* ⇒ -]]

lemma *equal-nat-code* [*code*]:
HOL.equal 0 (0::nat) ⟷ True
HOL.equal 0 (nat-of-num l) ⟷ False
HOL.equal (nat-of-num k) 0 ⟷ False
HOL.equal (nat-of-num k) (nat-of-num l) ⟷ HOL.equal k l
by (*simp-all add*: *nat-of-num-numeral equal*)

lemma *equal-nat-refl* [*code nbe*]:
HOL.equal (n::nat) n ⟷ True
by (*rule equal-refl*)

declare [[*code drop*: *less-eq* :: *nat* ⇒ -]]

lemma *less-eq-nat-code* [*code*]:

$0 \leq (n::nat) \longleftrightarrow True$
$nat\text{-}of\text{-}num\ k \leq 0 \longleftrightarrow False$
$nat\text{-}of\text{-}num\ k \leq nat\text{-}of\text{-}num\ l \longleftrightarrow k \leq l$
by (*simp-all add*: *nat-of-num-numeral*)

declare [[*code drop*: *less* :: *nat* ⇒ -]]

lemma *less-nat-code* [*code*]:
$(m::nat) < 0 \longleftrightarrow False$
$0 < nat\text{-}of\text{-}num\ l \longleftrightarrow True$
$nat\text{-}of\text{-}num\ k < nat\text{-}of\text{-}num\ l \longleftrightarrow k < l$
by (*simp-all add*: *nat-of-num-numeral*)

declare [[*code drop*: *Divides.divmod-nat*]]

lemma *divmod-nat-code* [*code*]:
$Divides.divmod\text{-}nat\ (nat\text{-}of\text{-}num\ k)\ (nat\text{-}of\text{-}num\ l) = divmod\ k\ l$
$Divides.divmod\text{-}nat\ m\ 0 = (0, m)$
$Divides.divmod\text{-}nat\ 0\ n = (0, 0)$
by (*simp-all add*: *prod-eq-iff nat-of-num-numeral*)

end

109.3 Conversions

declare [[*code drop*: *of-nat*]]

lemma *of-nat-code* [*code*]:
$of\text{-}nat\ 0 = 0$
$of\text{-}nat\ (nat\text{-}of\text{-}num\ k) = numeral\ k$
by (*simp-all add*: *nat-of-num-numeral*)

code-identifier
code-module *Code-Binary-Nat* ⇀
(*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

110 Code generation of prolog programs

theory *Code-Prolog*
imports *Main*
keywords *values-prolog* :: *diag*
begin

ML-file ‹~~/*src*/*HOL*/*Tools*/*Predicate-Compile*/*code-prolog.ML*›

111 Setup for Numerals

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*numeral*›]›

setup ‹*Predicate-Compile-Data.keep-functions* [***const-name*** ‹*numeral*›]›

end

112 Implementation of integer numbers by target-language integers

theory *Code-Target-Int*
imports *Main*
begin

code-datatype *int-of-integer*

declare [[*code drop*: *integer-of-int*]]

context
includes *integer.lifting*
begin

lemma [*code*]:
  integer-of-int (*int-of-integer k*) = *k*
  by *transfer rule*

lemma [*code*]:
  Int.Pos = *int-of-integer* ∘ *integer-of-num*
  by *transfer* (*simp add*: *fun-eq-iff*)

lemma [*code*]:
  Int.Neg = *int-of-integer* ∘ *uminus* ∘ *integer-of-num*
  by *transfer* (*simp add*: *fun-eq-iff*)

lemma [*code-abbrev*]:
  int-of-integer (*numeral k*) = *Int.Pos k*
  by *transfer simp*

lemma [*code-abbrev*]:
  int-of-integer (− *numeral k*) = *Int.Neg k*
  by *transfer simp*

context
begin

qualified definition *positive* :: *num* ⇒ *int*
  where [*simp*]: *positive* = *numeral*

qualified definition *negative* :: *num* ⇒ *int*
  where [*simp*]: *negative* = *uminus* ∘ *numeral*

lemma [*code-computation-unfold*]:
  numeral = *positive*
  Int.Pos = *positive*
  Int.Neg = *negative*
  by (*simp-all add*: *fun-eq-iff*)

end

lemma [*code*, *symmetric*, *code-post*]:
  0 = *int-of-integer 0*
  by *transfer simp*

lemma [*code*, *symmetric*, *code-post*]:
  1 = *int-of-integer 1*
  by *transfer simp*

lemma [*code-post*]:
  int-of-integer (− *1*) = − *1*
  by *simp*

lemma [*code*]:
  k + *l* = *int-of-integer* (*of-int k* + *of-int l*)
  by *transfer simp*

lemma [*code*]:
  − *k* = *int-of-integer* (− *of-int k*)
  by *transfer simp*

lemma [*code*]:
  k − *l* = *int-of-integer* (*of-int k* − *of-int l*)
  by *transfer simp*

lemma [*code*]:
  Int.dup k = *int-of-integer* (*Code-Numeral.dup* (*of-int k*))
  by *transfer simp*

declare [[*code drop*: *Int.sub*]]

lemma [*code*]:
  k ∗ *l* = *int-of-integer* (*of-int k* ∗ *of-int l*)
  by *simp*

lemma [*code*]:
  k div l = *int-of-integer* (*of-int k div of-int l*)
  by *simp*

lemma [*code*]:
k mod l = int-of-integer (of-int k mod of-int l)
by *simp*

lemma [*code*]:
divmod m n = map-prod int-of-integer int-of-integer (divmod m n)
unfolding *prod-eq-iff divmod-def map-prod-def case-prod-beta fst-conv snd-conv*
by *transfer simp*

lemma [*code*]:
HOL.equal k l = HOL.equal (of-int k :: integer) (of-int l)
by *transfer* (*simp add*: *equal*)

lemma [*code*]:
k ≤ l ⟷ (of-int k :: integer) ≤ of-int l
by *transfer rule*

lemma [*code*]:
k < l ⟷ (of-int k :: integer) < of-int l
by *transfer rule*

declare [[*code drop*: *gcd* :: *int* ⇒ - *lcm* :: *int* ⇒ -]]

lemma *gcd-int-of-integer* [*code*]:
gcd (int-of-integer x) (int-of-integer y) = int-of-integer (gcd x y)
by *transfer rule*

lemma *lcm-int-of-integer* [*code*]:
lcm (int-of-integer x) (int-of-integer y) = int-of-integer (lcm x y)
by *transfer rule*

end

lemma (**in** *ring-1*) *of-int-code-if*:
of-int k = (if k = 0 then 0
else if k < 0 then − of-int (− k)
else let
l = 2 ∗ of-int (k div 2);
j = k mod 2
in if j = 0 then l else l + 1)
proof −
from *div-mult-mod-eq* **have** ∗: *of-int k = of-int (k div 2 ∗ 2 + k mod 2)* **by** *simp*
show *?thesis*
by (*simp add*: *Let-def of-int-add* [*symmetric*]) (*simp add*: ∗ *mult.commute*)
qed

declare *of-int-code-if* [*code*]

lemma [*code*]:
nat = *nat-of-integer* ∘ *of-int*
including *integer.lifting* **by** *transfer* (*simp add*: *fun-eq-iff*)

definition *char-of-int* :: *int* ⇒ *char*
where [*code-abbrev*]: *char-of-int* = *char-of*

definition *int-of-char* :: *char* ⇒ *int*
where [*code-abbrev*]: *int-of-char* = *of-char*

lemma [*code*]:
char-of-int = *char-of-integer* ∘ *integer-of-int*
including *integer.lifting* **unfolding** *char-of-integer-def char-of-int-def*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma [*code*]:
int-of-char = *int-of-integer* ∘ *integer-of-char*
including *integer.lifting* **unfolding** *integer-of-char-def int-of-char-def*
by *transfer* (*simp add*: *fun-eq-iff*)

code-identifier
code-module *Code-Target-Int* ⇀
(*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

theory *Code-Real-Approx-By-Float*
imports *Complex-Main Code-Target-Int*
begin

WARNING! This theory implements mathematical reals by machine reals (floats). This is inconsistent. See the proof of False at the end of the theory, where an equality on mathematical reals is (incorrectly) disproved by mapping it to machine reals.

The **value** command cannot display real results yet.

The only legitimate use of this theory is as a tool for code generation purposes.

code-printing
type-constructor *real* ⇀
(*SML*) *real*
and (*OCaml*) *float*

code-printing
constant *Ratreal* ⇀
(*SML*) *error*/ *Bad constant*: *Ratreal*

code-printing
constant *0* :: *real* ⇀

(*SML*) *0.0*
and (*OCaml*) *0.0*

code-printing
constant *1* :: *real* ⇀
(*SML*) *1.0*
and (*OCaml*) *1.0*

code-printing
constant *HOL.equal* :: *real* ⇒ *real* ⇒ *bool* ⇀
(*SML*) *Real.==* ((-), (-))
and (*OCaml*) *Pervasives.*(=)

code-printing
constant *Orderings.less-eq* :: *real* ⇒ *real* ⇒ *bool* ⇀
(*SML*) *Real.<=* ((-), (-))
and (*OCaml*) *Pervasives.*(<=)

code-printing
constant *Orderings.less* :: *real* ⇒ *real* ⇒ *bool* ⇀
(*SML*) *Real.<* ((-), (-))
and (*OCaml*) *Pervasives.*(<)

code-printing
constant (+) :: *real* ⇒ *real* ⇒ *real* ⇀
(*SML*) *Real.+* ((-), (-))
and (*OCaml*) *Pervasives.*(+.)

code-printing
constant (∗) :: *real* ⇒ *real* ⇒ *real* ⇀
(*SML*) *Real.∗* ((-), (-))
and (*OCaml*) *Pervasives.*(∗.)

code-printing
constant (−) :: *real* ⇒ *real* ⇒ *real* ⇀
(*SML*) *Real.−* ((-), (-))
and (*OCaml*) *Pervasives.*(−.)

code-printing
constant *uminus* :: *real* ⇒ *real* ⇀
(*SML*) *Real.~*
and (*OCaml*) *Pervasives.*(~−.)

code-printing
constant (/) :: *real* ⇒ *real* ⇒ *real* ⇀
(*SML*) *Real.'/* ((-), (-))
and (*OCaml*) *Pervasives.*('/.)

code-printing

constant *HOL.equal* :: *real* ⇒ *real* ⇒ *bool* ⇀
(*SML*) *Real.==* ((-:*real*), (-))

code-printing
constant *sqrt* :: *real* ⇒ *real* ⇀
(*SML*) *Math.sqrt*
and (*OCaml*) *Pervasives.sqrt*
declare *sqrt-def*[*code del*]

context
begin

qualified definition *real-exp* :: *real* ⇒ *real*
where *real-exp* = *exp*

lemma *exp-eq-real-exp* [*code-unfold*]: *exp* = *real-exp*
unfolding *real-exp-def* **..**

end

code-printing
constant *Code-Real-Approx-By-Float.real-exp* ⇀
(*SML*) *Math.exp*
and (*OCaml*) *Pervasives.exp*
declare *Code-Real-Approx-By-Float.real-exp-def*[*code del*]
declare *exp-def*[*code del*]

code-printing
constant *ln* ⇀
(*SML*) *Math.ln*
and (*OCaml*) *Pervasives.ln*
declare *ln-real-def*[*code del*]

code-printing
constant *cos* ⇀
(*SML*) *Math.cos*
and (*OCaml*) *Pervasives.cos*
declare *cos-def*[*code del*]

code-printing
constant *sin* ⇀
(*SML*) *Math.sin*
and (*OCaml*) *Pervasives.sin*
declare *sin-def*[*code del*]

code-printing
constant *pi* ⇀
(*SML*) *Math.pi*
and (*OCaml*) *Pervasives.pi*

declare *pi-def*[*code del*]

code-printing
  constant *arctan* ⇀
    (*SML*) *Math.atan*
    and (*OCaml*) *Pervasives.atan*
declare *arctan-def*[*code del*]

code-printing
  constant *arccos* ⇀
    (*SML*) *Math.scos*
    and (*OCaml*) *Pervasives.acos*
declare *arccos-def*[*code del*]

code-printing
  constant *arcsin* ⇀
    (*SML*) *Math.asin*
    and (*OCaml*) *Pervasives.asin*
declare *arcsin-def*[*code del*]

definition *real-of-integer* :: *integer* ⇒ *real*
  where *real-of-integer* = *of-int* ∘ *int-of-integer*

code-printing
  constant *real-of-integer* ⇀
    (*SML*) *Real.fromInt*
    and (*OCaml*) *Pervasives.float*/ (*Big'-int.to'-int* (-))

context
begin

qualified definition *real-of-int* :: *int* ⇒ *real*
  where [*code-abbrev*]: *real-of-int* = *of-int*

lemma [*code*]: *real-of-int* = *real-of-integer* ∘ *integer-of-int*
  by (*simp add*: *fun-eq-iff real-of-integer-def real-of-int-def*)

lemma [*code-unfold del*]: *0* ≡ (*of-rat 0* :: *real*)
  by *simp*

lemma [*code-unfold del*]: *1* ≡ (*of-rat 1* :: *real*)
  by *simp*

lemma [*code-unfold del*]: *numeral k* ≡ (*of-rat* (*numeral k*) :: *real*)
  by *simp*

lemma [*code-unfold del*]: − *numeral k* ≡ (*of-rat* (− *numeral k*) :: *real*)
  by *simp*

end

code-printing
  constant *Ratreal* ⇀ (*SML*)

definition *Realfract* :: *int* ⇒ *int* ⇒ *real*
  where *Realfract p q* = *of-int p* / *of-int q*

code-datatype *Realfract*

code-printing
  constant *Realfract* ⇀ (*SML*) *Real.fromInt -*/ *'*// *Real.fromInt -*

lemma [*code*]: *Ratreal r* = *case-prod Realfract* (*quotient-of r*)
  by (*cases r*) (*simp add*: *Realfract-def quotient-of-Fract of-rat-rat*)

declare [[*code drop*: *HOL.equal* :: *real* ⇒ *real* ⇒ *bool*
  plus :: *real* ⇒ *real* ⇒ *real*
  uminus :: *real* ⇒ *real*
  minus :: *real* ⇒ *real* ⇒ *real*
  times :: *real* ⇒ *real* ⇒ *real*
  divide :: *real* ⇒ *real* ⇒ *real*
  (<) :: *real* ⇒ *real* ⇒ *bool*
  (≤) :: *real* ⇒ *real* ⇒ *bool*]]

lemma [*code*]: *inverse r* = *1* / *r* **for** *r* :: *real*
  by (*fact inverse-eq-divide*)

notepad
begin
  have *cos* (*pi*/*2*) = *0* **by** (*rule cos-pi-half*)
  moreover have *cos* (*pi*/*2*) ≠ *0* **by** *eval*
  ultimately have *False* **by** *blast*
end

end

113 Implementation of natural numbers by target-language integers

theory *Code-Target-Nat*
imports *Code-Abstract-Nat*
begin

113.1 Implementation for *nat*

context
includes *natural.lifting integer.lifting*

begin

lift-definition *Nat* :: *integer* ⇒ *nat*
is *nat*
.

lemma [*code-post*]:
Nat 0 = 0
Nat 1 = 1
Nat (*numeral k*) = *numeral k*
by (*transfer*, *simp*)+

lemma [*code-abbrev*]:
integer-of-nat = *of-nat*
by *transfer rule*

lemma [*code-unfold*]:
Int.nat (*int-of-integer k*) = *nat-of-integer k*
by *transfer rule*

lemma [*code abstype*]:
Code-Target-Nat.Nat (*integer-of-nat n*) = *n*
by *transfer simp*

lemma [*code abstract*]:
integer-of-nat (*nat-of-integer k*) = *max 0 k*
by *transfer auto*

lemma [*code-abbrev*]:
nat-of-integer (*numeral k*) = *nat-of-num k*
by *transfer* (*simp add*: *nat-of-num-numeral*)

context
begin

qualified definition *natural* :: *num* ⇒ *nat*
where [*simp*]: *natural* = *nat-of-num*

lemma [*code-computation-unfold*]:
numeral = *natural*
nat-of-num = *natural*
by (*simp-all add*: *nat-of-num-numeral*)

end

lemma [*code abstract*]:
integer-of-nat (*nat-of-num n*) = *integer-of-num n*
by (*simp add*: *nat-of-num-numeral integer-of-nat-numeral*)

lemma [*code abstract*]:
integer-of-nat 0 = 0
by *transfer simp*

lemma [*code abstract*]:
integer-of-nat 1 = 1
by *transfer simp*

lemma [*code*]:
Suc n = n + 1
by *simp*

lemma [*code abstract*]:
integer-of-nat (m + n) = of-nat m + of-nat n
by *transfer simp*

lemma [*code abstract*]:
integer-of-nat (m − n) = max 0 (of-nat m − of-nat n)
by *transfer simp*

lemma [*code abstract*]:
*integer-of-nat (m * n) = of-nat m * of-nat n*
by *transfer* (*simp add*: *of-nat-mult*)

lemma [*code abstract*]:
integer-of-nat (m div n) = of-nat m div of-nat n
by *transfer* (*simp add*: *zdiv-int*)

lemma [*code abstract*]:
integer-of-nat (m mod n) = of-nat m mod of-nat n
by *transfer* (*simp add*: *zmod-int*)

context
includes *integer.lifting*
begin

lemma *divmod-nat-code* [*code*]:
Divides.divmod-nat m n = (
let k = integer-of-nat m; *l = integer-of-nat n*
in map-prod nat-of-integer nat-of-integer
(*if k = 0 then* (*0*, *0*)
else if l = 0 then (*0*, *k*) *else*
Code-Numeral.divmod-abs k l))
by (*simp add*: *prod-eq-iff Let-def*; *transfer*)
(*simp add*: *nat-div-distrib nat-mod-distrib*)

end

lemma [*code*]:

divmod m n = map-prod nat-of-integer nat-of-integer (divmod m n)
by (*simp only*: *prod-eq-iff divmod-def map-prod-def case-prod-beta fst-conv snd-conv*; *transfer*)
(*simp-all only*: *nat-div-distrib nat-mod-distrib*
zero-le-numeral nat-numeral)

lemma [*code*]:
HOL.equal m n = HOL.equal (*of-nat m* :: *integer*) (*of-nat n*)
by *transfer* (*simp add*: *equal*)

lemma [*code*]:
$m \leq n \longleftrightarrow$ (*of-nat m* :: *integer*) $\leq$ *of-nat n*
by *simp*

lemma [*code*]:
$m < n \longleftrightarrow$ (*of-nat m* :: *integer*) $<$ *of-nat n*
by *simp*

lemma *num-of-nat-code* [*code*]:
num-of-nat = num-of-integer $\circ$ *of-nat*
by *transfer* (*simp add*: *fun-eq-iff*)

end

lemma (**in** *semiring-1*) *of-nat-code-if*:
of-nat n = (*if n = 0 then 0*
else let
(*m*, *q*) = *Divides.divmod-nat n 2*;
m' = *2* $*$ *of-nat m*
in if q = 0 then m' else m' + 1)
proof −
from *div-mult-mod-eq* **have** $*$: *of-nat n = of-nat* (*n div 2* $*$ *2 + n mod 2*) **by** *simp*
show *?thesis*
by (*simp add*: *Let-def divmod-nat-def of-nat-add* [*symmetric*])
(*simp add*: $*$ *mult.commute of-nat-mult add.commute*)
qed

declare *of-nat-code-if* [*code*]

definition *int-of-nat* :: *nat* $\Rightarrow$ *int* **where**
[*code-abbrev*]: *int-of-nat = of-nat*

lemma [*code*]:
int-of-nat n = int-of-integer (*of-nat n*)
by (*simp add*: *int-of-nat-def*)

lemma [*code abstract*]:
integer-of-nat (*nat k*) = *max 0* (*integer-of-int k*)

including *integer.lifting* **by** *transfer auto*

definition *char-of-nat* :: *nat* ⇒ *char*
where [*code-abbrev*]: *char-of-nat* = *char-of*

definition *nat-of-char* :: *char* ⇒ *nat*
where [*code-abbrev*]: *nat-of-char* = *of-char*

lemma [*code*]:
char-of-nat = *char-of-integer* ∘ *integer-of-nat*
including *integer.lifting* **unfolding** *char-of-integer-def char-of-nat-def*
by *transfer* (*simp add*: *fun-eq-iff*)

lemma [*code abstract*]:
integer-of-nat (*nat-of-char c*) = *integer-of-char c*
by (*cases c*) (*simp add*: *nat-of-char-def integer-of-char-def integer-of-nat-eq-of-nat*)

lemma *term-of-nat-code* [*code*]:
— Use *nat-of-integer* in term reconstruction instead of *Code-Target-Nat.Nat* such that reconstructed terms can be fed back to the code generator
term-of-class.term-of n =
Code-Evaluation.App
(*Code-Evaluation.Const* (*STR ''Code-Numeral.nat-of-integer''*)
(*typerep.Typerep* (*STR ''fun''*)
[*typerep.Typerep* (*STR ''Code-Numeral.integer''*) [],
typerep.Typerep (*STR ''Nat.nat''*) []]))
(*term-of-class.term-of* (*integer-of-nat n*))
by (*simp add*: *term-of-anything*)

lemma *nat-of-integer-code-post* [*code-post*]:
nat-of-integer 0 = *0*
nat-of-integer 1 = *1*
nat-of-integer (*numeral k*) = *numeral k*
including *integer.lifting* **by** (*transfer*, *simp*)+

code-identifier
code-module *Code-Target-Nat* ⇀
(*SML*) *Arith* **and** (*OCaml*) *Arith* **and** (*Haskell*) *Arith*

end

114 Implementation of natural and integer numbers by target-language integers

theory *Code-Target-Numeral*
imports *Code-Target-Int Code-Target-Nat*
begin

end

115 Preprocessor setup for floats implemented by target language numerals

theory *Code-Target-Numeral-Float*
imports *Float Code-Target-Numeral*
begin

lemma *numeral-float-computation-unfold* [*code-computation-unfold*]:
‹*numeral k = Float (int-of-integer (Code-Numeral.positive k)) 0*›
‹*− numeral k = Float (int-of-integer (Code-Numeral.negative k)) 0*›
by (*simp-all add*: *Float.compute-float-numeral Float.compute-float-neg-numeral*)

end

theory *Complex-Order*
imports *Complex-Main*
begin

instantiation *complex* :: *order* **begin**

definition ‹*x* < *y* ⟷ *Re x* < *Re y* ∧ *Im x* = *Im y*›
definition ‹*x* ≤ *y* ⟷ *Re x* ≤ *Re y* ∧ *Im x* = *Im y*›

instance
apply *standard*
by (*auto simp*: *less-complex-def less-eq-complex-def complex-eq-iff*)
end

lemma *nonnegative-complex-is-real*: ‹(*x*::*complex*) ≥ *0* ⟹ *x* ∈ ℝ›
by (*simp add*: *complex-is-Real-iff less-eq-complex-def*)

lemma *complex-is-real-iff-compare0*: ‹(*x*::*complex*) ∈ ℝ ⟷ *x* ≤ *0* ∨ *x* ≥ *0*›
using *complex-is-Real-iff less-eq-complex-def* **by** *auto*

instance *complex* :: *ordered-comm-ring*
apply *standard*
by (*auto simp*: *less-complex-def less-eq-complex-def complex-eq-iff mult-left-mono mult-right-mono*)

instance *complex* :: *ordered-real-vector*
apply *standard*
by (*auto simp*: *less-complex-def less-eq-complex-def mult-left-mono mult-right-mono*)

instance *complex* :: *ordered-cancel-comm-semiring*
by *standard*

end

116 Abstract type of association lists with unique keys

theory *DAList*
imports *AList*
begin

This was based on some existing fragments in the AFP-Collection framework.

116.1 Preliminaries

lemma *distinct-map-fst-filter*:
distinct (*map fst xs*) ⟹ *distinct* (*map fst* (*List.filter P xs*))
by (*induct xs*) *auto*

116.2 Type ('*key*, '*value*) *alist*

typedef ('*key*, '*value*) *alist* = {*xs* :: ('*key* × '*value*) *list*. (*distinct* ∘ *map fst*) *xs*}
morphisms *impl-of Alist*
proof
show [] ∈ {*xs*. (*distinct* ∘ *map fst*) *xs*}
by *simp*
qed

setup-lifting *type-definition-alist*

lemma *alist-ext*: *impl-of xs* = *impl-of ys* ⟹ *xs* = *ys*
by (*simp add*: *impl-of-inject*)

lemma *alist-eq-iff*: *xs* = *ys* ⟷ *impl-of xs* = *impl-of ys*
by (*simp add*: *impl-of-inject*)

lemma *impl-of-distinct* [*simp*, *intro*]: *distinct* (*map fst* (*impl-of xs*))
using *impl-of*[*of xs*] **by** *simp*

lemma *Alist-impl-of* [*code abstype*]: *Alist* (*impl-of xs*) = *xs*
by (*rule impl-of-inverse*)

116.3 Primitive operations

lift-definition *lookup* :: ('*key*, '*value*) *alist* ⇒ '*key* ⇒ '*value option* **is** *map-of* .

lift-definition *empty* :: ('*key*, '*value*) *alist* **is** []
by *simp*

lift-definition *update* :: *'key* ⇒ *'value* ⇒ (*'key*, *'value*) *alist* ⇒ (*'key*, *'value*) *alist*
is *AList.update*
by (*simp add*: *distinct-update*)

lift-definition *delete* :: *'key* ⇒ (*'key*, *'value*) *alist* ⇒ (*'key*, *'value*) *alist*
is *AList.delete*
by (*simp add*: *distinct-delete*)

lift-definition *map-entry* ::
'key ⇒ (*'value* ⇒ *'value*) ⇒ (*'key*, *'value*) *alist* ⇒ (*'key*, *'value*) *alist*
is *AList.map-entry*
by (*simp add*: *distinct-map-entry*)

lift-definition *filter* :: (*'key* × *'value* ⇒ *bool*) ⇒ (*'key*, *'value*) *alist* ⇒ (*'key*, *'value*) *alist*
is *List.filter*
by (*simp add*: *distinct-map-fst-filter*)

lift-definition *map-default* ::
'key ⇒ *'value* ⇒ (*'value* ⇒ *'value*) ⇒ (*'key*, *'value*) *alist* ⇒ (*'key*, *'value*) *alist*
is *AList.map-default*
by (*simp add*: *distinct-map-default*)

116.4 Abstract operation properties

lemma *lookup-empty* [*simp*]: *lookup empty k* = *None*
by (*simp add*: *empty-def lookup-def Alist-inverse*)

lemma *lookup-update*:
lookup (*update k1 v xs*) *k2* = (*if k1* = *k2 then Some v else lookup xs k2*)
by(*transfer*)(*simp add*: *update-conv'*)

lemma *lookup-update-eq* [*simp*]:
k1 = *k2* ⟹ *lookup* (*update k1 v xs*) *k2* = *Some v*
by(*simp add*: *lookup-update*)

lemma *lookup-update-neq* [*simp*]:
k1 ≠ *k2* ⟹ *lookup* (*update k1 v xs*) *k2* = *lookup xs k2*
by(*simp add*: *lookup-update*)

lemma *update-update-eq* [*simp*]:
k1 = *k2* ⟹ *update k2 v2* (*update k1 v1 xs*) = *update k2 v2 xs*
by(*transfer*)(*simp add*: *update-conv'*)

lemma *lookup-delete* [*simp*]: *lookup* (*delete k al*) = (*lookup al*)(*k* := *None*)
by (*simp add*: *lookup-def delete-def Alist-inverse distinct-delete delete-conv'*)

116.5 Further operations

116.5.1 Equality

instantiation *alist* :: (*equal*, *equal*) *equal*
begin

definition *HOL.equal* (*xs* :: ('*a*, '*b*) *alist*) *ys* == *impl-of xs* = *impl-of ys*

instance
by *standard* (*simp add*: *equal-alist-def impl-of-inject*)

end

116.5.2 Size

instantiation *alist* :: (*type*, *type*) *size*
begin

definition *size* (*al* :: ('*a*, '*b*) *alist*) = *length* (*impl-of al*)

instance ..

end

116.6 Quickcheck generators

context
includes *state-combinator-syntax term-syntax*
begin

definition
valterm-empty :: ('*key* :: *typerep*, '*value* :: *typerep*) *alist* × (*unit* ⇒ *Code-Evaluation.term*)
where *valterm-empty* = *Code-Evaluation.valtermify empty*

definition
valterm-update :: '*key* :: *typerep* × (*unit* ⇒ *Code-Evaluation.term*) ⇒
'*value* :: *typerep* × (*unit* ⇒ *Code-Evaluation.term*) ⇒
('*key*, '*value*) *alist* × (*unit* ⇒ *Code-Evaluation.term*) ⇒
('*key*, '*value*) *alist* × (*unit* ⇒ *Code-Evaluation.term*) **where**
[*code-unfold*]: *valterm-update k v a* = *Code-Evaluation.valtermify update* {·} *k* {·}
v {·}*a*

fun *random-aux-alist*
where
random-aux-alist i j =
(*if i* = *0 then Pair valterm-empty*
else Quickcheck-Random.collapse
(*Random.select-weight*
[(*i*, *Quickcheck-Random.random j* ∘→ (λ*k*. *Quickcheck-Random.random j*

```
∘→
        (λv. random-aux-alist (i − 1) j ∘→ (λa. Pair (valterm-update k v a))))),
       (1, Pair valterm-empty)]))

end

instantiation alist :: (random, random) random
begin

definition random-alist
where
  random-alist i = random-aux-alist i i

instance ..

end

instantiation alist :: (exhaustive, exhaustive) exhaustive
begin

fun exhaustive-alist ::
  (('a, 'b) alist ⇒ (bool × term list) option) ⇒ natural ⇒ (bool × term list) option
where
  exhaustive-alist f i =
    (if i = 0 then None
     else
      case f empty of
        Some ts ⇒ Some ts
      | None ⇒
          exhaustive-alist
            (λa. Quickcheck-Exhaustive.exhaustive
              (λk. Quickcheck-Exhaustive.exhaustive (λv. f (update k v a)) (i − 1))
(i − 1))
            (i − 1))

instance ..

end

instantiation alist :: (full-exhaustive, full-exhaustive) full-exhaustive
begin

fun full-exhaustive-alist ::
  (('a, 'b) alist × (unit ⇒ term) ⇒ (bool × term list) option) ⇒ natural ⇒
    (bool × term list) option
where
  full-exhaustive-alist f i =
    (if i = 0 then None
     else
```

case f valterm-empty of
Some ts ⇒ Some ts
| None ⇒
full-exhaustive-alist
(λa.
Quickcheck-Exhaustive.full-exhaustive
(λk. Quickcheck-Exhaustive.full-exhaustive (λv. f (valterm-update k v a)) (i − 1))
(i − 1))
(i − 1))

instance ..

end

117 alist is a BNF

lift-bnf (*dead 'k*, *set*: *'v*) *alist* [*wits*: [] :: (*'k* × *'v*) *list*] **for** *map*: *map rel*: *rel*
by *auto*

hide-const *valterm-empty valterm-update random-aux-alist*

hide-fact (**open**) *lookup-def empty-def update-def delete-def map-entry-def filter-def map-default-def*
hide-const (**open**) *impl-of lookup empty update delete map-entry filter map-default map set rel*

end

118 Multisets partially implemented by association lists

theory *DAList-Multiset*
imports *Multiset DAList*
begin

Delete prexisting code equations

declare [[*code drop*: {#} *Multiset.is-empty add-mset*
plus :: *'a multiset* ⇒ - *minus* :: *'a multiset* ⇒ -
inter-mset union-mset image-mset filter-mset count
size :: - *multiset* ⇒ *nat sum-mset prod-mset*
set-mset sorted-list-of-multiset subset-mset subseteq-mset
equal-multiset-inst.equal-multiset]]

Raw operations on lists

definition *join-raw* ::
(*'key* ⇒ *'val* × *'val* ⇒ *'val*) ⇒
(*'key* × *'val*) *list* ⇒ (*'key* × *'val*) *list* ⇒ (*'key* × *'val*) *list*

where *join-raw f xs ys* = *foldr* (λ(*k*, *v*). *map-default k v* (λ*v'*. *f k* (*v'*, *v*))) *ys xs*

lemma *join-raw-Nil* [*simp*]: *join-raw f xs* [] = *xs*
by (*simp add*: *join-raw-def*)

lemma *join-raw-Cons* [*simp*]:
join-raw f xs ((*k*, *v*) # *ys*) = *map-default k v* (λ*v'*. *f k* (*v'*, *v*)) (*join-raw f xs ys*)
by (*simp add*: *join-raw-def*)

lemma *map-of-join-raw*:
assumes *distinct* (*map fst ys*)
shows *map-of* (*join-raw f xs ys*) *x* =
(*case map-of xs x of*
None ⇒ *map-of ys x*
| *Some v* ⇒ (*case map-of ys x of None* ⇒ *Some v* | *Some v'* ⇒ *Some* (*f x* (*v*, *v'*))))
using *assms*
apply (*induct ys*)
apply (*auto simp add*: *map-of-map-default split*: *option.split*)
apply (*metis map-of-eq-None-iff option.simps*(*2*) *weak-map-of-SomeI*)
apply (*metis Some-eq-map-of-iff map-of-eq-None-iff option.simps*(*2*))
done

lemma *distinct-join-raw*:
assumes *distinct* (*map fst xs*)
shows *distinct* (*map fst* (*join-raw f xs ys*))
using *assms*
proof (*induct ys*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons y ys*)
then show *?case* **by** (*cases y*) (*simp add*: *distinct-map-default*)
qed

definition *subtract-entries-raw xs ys* = *foldr* (λ(*k*, *v*). *AList.map-entry k* (λ*v'*. *v'* − *v*)) *ys xs*

lemma *map-of-subtract-entries-raw*:
assumes *distinct* (*map fst ys*)
shows *map-of* (*subtract-entries-raw xs ys*) *x* =
(*case map-of xs x of*
None ⇒ *None*
| *Some v* ⇒ (*case map-of ys x of None* ⇒ *Some v* | *Some v'* ⇒ *Some* (*v* − *v'*)))
using *assms*
unfolding *subtract-entries-raw-def*
apply (*induct ys*)
apply *auto*
apply (*simp split*: *option.split*)

apply (*simp add*: *map-of-map-entry*)
apply (*auto split*: *option.split*)
apply (*metis map-of-eq-None-iff option.simps*(*3*) *option.simps*(*4*))
apply (*metis map-of-eq-None-iff option.simps*(*4*) *option.simps*(*5*))
done

lemma *distinct-subtract-entries-raw*:
assumes *distinct* (*map fst xs*)
shows *distinct* (*map fst* (*subtract-entries-raw xs ys*))
using *assms*
unfolding *subtract-entries-raw-def*
by (*induct ys*) (*auto simp add*: *distinct-map-entry*)

Operations on alists with distinct keys

lift-definition *join* :: ('*a* ⇒ '*b* × '*b* ⇒ '*b*) ⇒ ('*a*, '*b*) *alist* ⇒ ('*a*, '*b*) *alist* ⇒ ('*a*, '*b*) *alist*
is *join-raw*
by (*simp add*: *distinct-join-raw*)

lift-definition *subtract-entries* :: ('*a*, ('*b* :: *minus*)) *alist* ⇒ ('*a*, '*b*) *alist* ⇒ ('*a*, '*b*) *alist*
is *subtract-entries-raw*
by (*simp add*: *distinct-subtract-entries-raw*)

Implementing multisets by means of association lists

definition *count-of* :: ('*a* × *nat*) *list* ⇒ '*a* ⇒ *nat*
where *count-of xs x* = (*case map-of xs x of None* ⇒ *0* | *Some n* ⇒ *n*)

lemma *count-of-multiset*: *finite* {*x*. *0* < *count-of xs x*}
proof −
let *?A* = {*x*::'*a*. *0* < (*case map-of xs x of None* ⇒ *0*::*nat* | *Some n* ⇒ *n*)}
have *?A* ⊆ *dom* (*map-of xs*)
proof
fix *x*
assume *x* ∈ *?A*
then have *0* < (*case map-of xs x of None* ⇒ *0*::*nat* | *Some n* ⇒ *n*)
by *simp*
then have *map-of xs x* ≠ *None*
by (*cases map-of xs x*) *auto*
then show *x* ∈ *dom* (*map-of xs*)
by *auto*
qed
with *finite-dom-map-of* [*of xs*] **have** *finite ?A*
by (*auto intro*: *finite-subset*)
then show *?thesis*
by (*simp add*: *count-of-def fun-eq-iff*)
qed

lemma *count-simps* [*simp*]:

count-of [] = (λ-. *0*)
count-of ((*x*, *n*) # *xs*) = (λ*y*. *if x = y then n else count-of xs y*)
by (*simp-all add*: *count-of-def fun-eq-iff*)

lemma *count-of-empty*: *x* ∉ *fst* ' *set xs* ⟹ *count-of xs x = 0*
by (*induct xs*) (*simp-all add*: *count-of-def*)

lemma *count-of-filter*: *count-of* (*List.filter* (*P* ∘ *fst*) *xs*) *x* = (*if P x then count-of xs x else 0*)
by (*induct xs*) *auto*

lemma *count-of-map-default* [*simp*]:
count-of (*map-default x b* (λ*x*. *x* + *b*) *xs*) *y* =
(*if x = y then count-of xs x* + *b else count-of xs y*)
unfolding *count-of-def* **by** (*simp add*: *map-of-map-default split*: *option.split*)

lemma *count-of-join-raw*:
distinct (*map fst ys*) ⟹
count-of xs x + *count-of ys x = count-of* (*join-raw* (λ*x* (*x*, *y*). *x* + *y*) *xs ys*) *x*
unfolding *count-of-def* **by** (*simp add*: *map-of-join-raw split*: *option.split*)

lemma *count-of-subtract-entries-raw*:
distinct (*map fst ys*) ⟹
count-of xs x − *count-of ys x = count-of* (*subtract-entries-raw xs ys*) *x*
unfolding *count-of-def* **by** (*simp add*: *map-of-subtract-entries-raw split*: *option.split*)

Code equations for multiset operations

definition *Bag* :: ('*a*, *nat*) *alist* ⇒ '*a multiset*
where *Bag xs = Abs-multiset* (*count-of* (*DAList.impl-of xs*))

code-datatype *Bag*

lemma *count-Bag* [*simp*, *code*]: *count* (*Bag xs*) = *count-of* (*DAList.impl-of xs*)
by (*simp add*: *Bag-def count-of-multiset*)

lemma *Mempty-Bag* [*code*]: {#} = *Bag* (*DAList.empty*)
by (*simp add*: *multiset-eq-iff alist.Alist-inverse DAList.empty-def*)

lift-definition *is-empty-Bag-impl* :: ('*a*, *nat*) *alist* ⇒ *bool* **is**
λ*xs*. *list-all* (λ*x*. *snd x = 0*) *xs* .

lemma *is-empty-Bag* [*code*]: *Multiset.is-empty* (*Bag xs*) ⟷ *is-empty-Bag-impl xs*
proof −
have *Multiset.is-empty* (*Bag xs*) ⟷ (∀ *x*. *count* (*Bag xs*) *x = 0*)
unfolding *Multiset.is-empty-def multiset-eq-iff* **by** *simp*
also have ... ⟷ (∀ *x*∈*fst* ' *set* (*alist.impl-of xs*). *count* (*Bag xs*) *x = 0*)
proof (*intro iffI allI ballI*)
fix *x* **assume** *A*: ∀ *x*∈*fst* ' *set* (*alist.impl-of xs*). *count* (*Bag xs*) *x = 0*

```
    thus count (Bag xs) x = 0
    proof (cases x ∈ fst ` set (alist.impl-of xs))
      case False
      thus ?thesis by (force simp: count-of-def split: option.splits)
    qed (insert A, auto)
  qed simp-all
  also have ... ⟷ list-all (λx. snd x = 0) (alist.impl-of xs)
    by (auto simp: count-of-def list-all-def)
  finally show ?thesis by (simp add: is-empty-Bag-impl.rep-eq)
qed

lemma union-Bag [code]: Bag xs + Bag ys = Bag (join (λx (n1, n2). n1 + n2)
xs ys)
  by (rule multiset-eqI)
    (simp add: count-of-join-raw alist.Alist-inverse distinct-join-raw join-def)

lemma add-mset-Bag [code]: add-mset x (Bag xs) =
    Bag (join (λx (n1, n2). n1 + n2) (DAList.update x 1 DAList.empty) xs)
  unfolding add-mset-add-single[of x Bag xs] union-Bag[symmetric]
  by (simp add: multiset-eq-iff update.rep-eq empty.rep-eq)

lemma minus-Bag [code]: Bag xs − Bag ys = Bag (subtract-entries xs ys)
  by (rule multiset-eqI)
    (simp add: count-of-subtract-entries-raw alist.Alist-inverse
      distinct-subtract-entries-raw subtract-entries-def)

lemma filter-Bag [code]: filter-mset P (Bag xs) = Bag (DAList.filter (P ∘ fst) xs)
  by (rule multiset-eqI) (simp add: count-of-filter DAList.filter.rep-eq)


lemma mset-eq [code]: HOL.equal (m1::'a::equal multiset) m2 ⟷ m1 ⊆# m2 ∧
m2 ⊆# m1
  by (metis equal-multiset-def subset-mset.order-eq-iff)
```

By default the code for $<$ is $(xs < ys) = (xs \leq ys \wedge xs \neq ys)$. With equality implemented by $\leq$, this leads to three calls of $\leq$. Here is a more efficient version:

```
lemma mset-less[code]: xs ⊂# (ys :: 'a multiset) ⟷ xs ⊆# ys ∧ ¬ ys ⊆# xs
  by (rule subset-mset.less-le-not-le)

lemma mset-less-eq-Bag0:
  Bag xs ⊆# A ⟷ (∀(x, n) ∈ set (DAList.impl-of xs). count-of (DAList.impl-of
xs) x ≤ count A x)
    (is ?lhs ⟷ ?rhs)
proof
  assume ?lhs
  then show ?rhs by (auto simp add: subseteq-mset-def)
next
  assume ?rhs
```

```
  show ?lhs
  proof (rule mset-subset-eqI)
    fix x
    from ‹?rhs› have count-of (DAList.impl-of xs) x ≤ count A x
      by (cases x ∈ fst ‘ set (DAList.impl-of xs)) (auto simp add: count-of-empty)
    then show count (Bag xs) x ≤ count A x by (simp add: subset-mset-def)
  qed
qed

lemma mset-less-eq-Bag [code]:
  Bag xs ⊆# (A :: 'a multiset) ⟷ (∀(x, n) ∈ set (DAList.impl-of xs). n ≤ count
A x)
proof −
  {
    fix x n
    assume (x,n) ∈ set (DAList.impl-of xs)
    then have count-of (DAList.impl-of xs) x = n
    proof transfer
      fix x n
      fix xs :: ('a × nat) list
      show (distinct ∘ map fst) xs ⟹ (x, n) ∈ set xs ⟹ count-of xs x = n
      proof (induct xs)
        case Nil
        then show ?case by simp
      next
        case (Cons ym ys)
        obtain y m where ym: ym = (y,m) by force
        note Cons = Cons[unfolded ym]
        show ?case
        proof (cases x = y)
          case False
          with Cons show ?thesis
            unfolding ym by auto
        next
          case True
          with Cons(2−3) have m = n by force
          with True show ?thesis
            unfolding ym by auto
        qed
      qed
    qed
  }
  then show ?thesis
    unfolding mset-less-eq-Bag0 by auto
qed

declare inter-mset-def [code]
declare union-mset-def [code]
declare mset.simps [code]
```

fun *fold-impl* :: $('a \Rightarrow nat \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow ('a \times nat)\ list \Rightarrow 'b$
where
  fold-impl fn e $((a,n) \# ms) =$ (*fold-impl fn* ((*fn a n*) *e*) *ms*)
| *fold-impl fn e* $[] = e$

context
begin

qualified definition *fold* :: $('a \Rightarrow nat \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow ('a,\ nat)\ alist \Rightarrow 'b$
  where *fold f e al* = *fold-impl f e* (*DAList.impl-of al*)

end

context *comp-fun-commute*
begin

lemma *DAList-Multiset-fold*:
  assumes *fn*: $\bigwedge a\ n\ x.$ *fn a n x* = (*f a* $\frown\frown$ *n*) *x*
  shows *fold-mset f e* (*Bag al*) = *DAList-Multiset.fold fn e al*
  unfolding *DAList-Multiset.fold-def*
proof (*induct al*)
  fix *ys*
  let *?inv* = $\{xs :: ('a \times nat)\ list.$ (*distinct* $\circ$ *map fst*) *xs*$\}$
  note *cs*[*simp del*] = *count-simps*
  have *count*[*simp*]: $\bigwedge x.$ *count* (*Abs-multiset* (*count-of x*)) = *count-of x*
    by (*rule Abs-multiset-inverse*) (*simp add*: *count-of-multiset*)
  assume *ys*: *ys* $\in$ *?inv*
  then show *fold-mset f e* (*Bag* (*Alist ys*)) = *fold-impl fn e* (*DAList.impl-of* (*Alist ys*))
    unfolding *Bag-def* **unfolding** *Alist-inverse*[*OF ys*]
  proof (*induct ys arbitrary*: *e rule*: *list.induct*)
    case *Nil*
    show *?case*
      by (*rule trans*[*OF arg-cong*[*of* - $\{\#\}$ *fold-mset f e*, *OF multiset-eqI*]])
        (*auto*, *simp add*: *cs*)
  next
    case (*Cons pair ys e*)
    obtain *a n* **where** *pair*: *pair* = (*a*,*n*)
      by *force*
    from *fn*[*of a n*] **have** [*simp*]: *fn a n* = (*f a* $\frown\frown$ *n*)
      by *auto*
    have *inv*: *ys* $\in$ *?inv*
      using *Cons*(*2*) **by** *auto*
    note *IH* = *Cons*(*1*)[*OF inv*]
    define *Ys* **where** *Ys* = *Abs-multiset* (*count-of ys*)
    have *id*: *Abs-multiset* (*count-of* ((*a*, *n*) # *ys*)) = (((+) $\{\#$ *a* $\#\}$) $\frown\frown$ *n*) *Ys*
      unfolding *Ys-def*

```
    proof (rule multiset-eqI, unfold count)
      fix c
      show count-of ((a, n) # ys) c =
        count (((+) {#a#} ^^ n) (Abs-multiset (count-of ys))) c (is ?l = ?r)
      proof (cases c = a)
        case False
        then show ?thesis
          unfolding cs by (induct n) auto
      next
        case True
        then have ?l = n by (simp add: cs)
        also have n = ?r unfolding True
        proof (induct n)
          case 0
          from Cons(2)[unfolded pair] have a ∉ fst ` set ys by auto
          then show ?case by (induct ys) (simp, auto simp: cs)
        next
          case Suc
          then show ?case by simp
        qed
        finally show ?thesis .
      qed
    qed
    show ?case
      unfolding pair
      apply (simp add: IH[symmetric])
      unfolding id Ys-def[symmetric]
      apply (induct n)
      apply (auto simp: fold-mset-fun-left-comm[symmetric])
      done
  qed
qed

end

context
begin

private lift-definition single-alist-entry :: 'a ⇒ 'b ⇒ ('a, 'b) alist is λa b. [(a,
b)]
  by auto

lemma image-mset-Bag [code]:
  image-mset f (Bag ms) =
    DAList-Multiset.fold (λa n m. Bag (single-alist-entry (f a) n) + m) {#} ms
  unfolding image-mset-def
proof (rule comp-fun-commute.DAList-Multiset-fold, unfold-locales, (auto simp:
ac-simps)[1])
  fix a n m
```

show *Bag* (*single-alist-entry* (*f a*) *n*) + *m* = ((*add-mset* ∘ *f*) *a* ^^ *n*) *m* (**is** *?l* = *?r*)
proof (*rule multiset-eqI*)
fix *x*
have *count ?r x* = (*if x* = *f a then n* + *count m x else count m x*)
by (*induct n*) *auto*
also have ... = *count ?l x*
by (*simp add*: *single-alist-entry.rep-eq*)
finally show *count ?l x* = *count ?r x* **..**
qed
qed

end

— we cannot use λ*a n*. (+) (*a* ∗ *n*) for folding, since (∗) is not defined in *comm-monoid-add*
lemma *sum-mset-Bag*[*code*]: *sum-mset* (*Bag ms*) = *DAList-Multiset.fold* (λ*a n*. (((+) *a*) ^^ *n*)) *0 ms*
unfolding *sum-mset.eq-fold*
apply (*rule comp-fun-commute.DAList-Multiset-fold*)
apply *unfold-locales*
apply (*auto simp*: *ac-simps*)
done

— we cannot use λ*a n*. (∗) (*a* ^ *n*) for folding, since (^) is not defined in *comm-monoid-mult*
lemma *prod-mset-Bag*[*code*]: *prod-mset* (*Bag ms*) = *DAList-Multiset.fold* (λ*a n*. (((∗) *a*) ^^ *n*)) *1 ms*
unfolding *prod-mset.eq-fold*
apply (*rule comp-fun-commute.DAList-Multiset-fold*)
apply *unfold-locales*
apply (*auto simp*: *ac-simps*)
done

lemma *size-fold*: *size A* = *fold-mset* (λ-. *Suc*) *0 A* (**is** - = *fold-mset ?f* - -)
proof −
interpret *comp-fun-commute ?f* **by** *standard auto*
show *?thesis* **by** (*induct A*) *auto*
qed

lemma *size-Bag*[*code*]: *size* (*Bag ms*) = *DAList-Multiset.fold* (λ*a n*. (+) *n*) *0 ms*
unfolding *size-fold*
proof (*rule comp-fun-commute.DAList-Multiset-fold*, *unfold-locales*, *simp*)
fix *a n x*
show *n* + *x* = (*Suc* ^^ *n*) *x*
by (*induct n*) *auto*
qed

lemma *set-mset-fold*: *set-mset A* = *fold-mset insert* {} *A* (**is** - = *fold-mset ?f* - -)
proof −

interpret *comp-fun-commute ?f* **by** *standard auto*
show *?thesis* **by** (*induct A*) *auto*
qed

lemma *set-mset-Bag*[*code*]:
set-mset (*Bag ms*) = *DAList-Multiset.fold* (λ*a n*. (*if n* = *0 then* (λ*m*. *m*) *else insert a*)) {} *ms*
unfolding *set-mset-fold*
proof (*rule comp-fun-commute.DAList-Multiset-fold*, *unfold-locales*, (*auto simp*: *ac-simps*)[*1*])
fix *a n x*
show (*if n* = *0 then* λ*m*. *m else insert a*) *x* = (*insert a* ^^ *n*) *x* (**is** *?l n* = *?r n*)
proof (*cases n*)
case *0*
then show *?thesis* **by** *simp*
next
case (*Suc m*)
then have *?l n* = *insert a x* **by** *simp*
moreover have *?r n* = *insert a x* **unfolding** *Suc* **by** (*induct m*) *auto*
ultimately show *?thesis* **by** *auto*
qed
qed

instantiation *multiset* :: (*exhaustive*) *exhaustive*
begin

definition *exhaustive-multiset* ::
('*a multiset* ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where *exhaustive-multiset f i* = *Quickcheck-Exhaustive.exhaustive* (λ*xs*. *f* (*Bag xs*)) *i*

instance ..

end

end

119 Implementation of Red-Black Trees

theory *RBT-Impl*
imports *Main*
begin

For applications, you should use theory *RBT* which defines an abstract type of red-black tree obeying the invariant.

119.1 Datatype of RB trees

datatype *color* = *R* | *B*
datatype ('*a*, '*b*) *rbt* = *Empty* | *Branch color* ('*a*, '*b*) *rbt* '*a* '*b* ('*a*, '*b*) *rbt*

lemma *rbt-cases*:
  obtains (*Empty*) *t* = *Empty*
  | (*Red*) *l k v r* **where** *t* = *Branch R l k v r*
  | (*Black*) *l k v r* **where** *t* = *Branch B l k v r*
proof (*cases t*)
  case *Empty* **with** *that* **show** *thesis* **by** *blast*
next
  case (*Branch c*) **with** *that* **show** *thesis* **by** (*cases c*) *blast+*
qed

119.2 Tree properties

119.2.1 Content of a tree

primrec *entries* :: ('*a*, '*b*) *rbt* ⇒ ('*a* × '*b*) *list*
where
  entries Empty = []
| *entries* (*Branch* - *l k v r*) = *entries l* @ (*k*,*v*) # *entries r*

abbreviation (*input*) *entry-in-tree* :: '*a* ⇒ '*b* ⇒ ('*a*, '*b*) *rbt* ⇒ *bool*
where
  entry-in-tree k v t ≡ (*k*, *v*) ∈ *set* (*entries t*)

definition *keys* :: ('*a*, '*b*) *rbt* ⇒ '*a list* **where**
  keys t = *map fst* (*entries t*)

lemma *keys-simps* [*simp*, *code*]:
  keys Empty = []
  keys (*Branch c l k v r*) = *keys l* @ *k* # *keys r*
  by (*simp-all add*: *keys-def*)

lemma *entry-in-tree-keys*:
  assumes (*k*, *v*) ∈ *set* (*entries t*)
  shows *k* ∈ *set* (*keys t*)
proof −
  from *assms* **have** *fst* (*k*, *v*) ∈ *fst* ' *set* (*entries t*) **by** (*rule imageI*)
  then show *?thesis* **by** (*simp add*: *keys-def*)
qed

lemma *keys-entries*:
  k ∈ *set* (*keys t*) ⟷ (∃ *v*. (*k*, *v*) ∈ *set* (*entries t*))
  by (*auto intro*: *entry-in-tree-keys*) (*auto simp add*: *keys-def*)

lemma *non-empty-rbt-keys*:
  t ≠ *rbt.Empty* ⟹ *keys t* ≠ []

by (*cases t*) *simp-all*

119.2.2 Search tree properties

context *ord* **begin**

definition *rbt-less* :: $'a \Rightarrow ('a, 'b)\ rbt \Rightarrow bool$
where
rbt-less-prop: *rbt-less k t* $\longleftrightarrow$ $(\forall x{\in}set\ (keys\ t).\ x < k)$

abbreviation *rbt-less-symbol* (**infix** $|\!\ll$ *50*)
where $t\ |\!\ll\ x \equiv$ *rbt-less x t*

definition *rbt-greater* :: $'a \Rightarrow ('a, 'b)\ rbt \Rightarrow bool$ (**infix** $\ll\!|$ *50*)
where
rbt-greater-prop: *rbt-greater k t* $= (\forall x{\in}set\ (keys\ t).\ k < x)$

lemma *rbt-less-simps* [*simp*]:
Empty $|\!\ll\ k =$ *True*
Branch c lt kt v rt $|\!\ll\ k \longleftrightarrow kt < k \wedge lt\ |\!\ll\ k \wedge rt\ |\!\ll\ k$
by (*auto simp add*: *rbt-less-prop*)

lemma *rbt-greater-simps* [*simp*]:
$k \ll\!|$ *Empty* $=$ *True*
$k \ll\!|$ (*Branch c lt kt v rt*) $\longleftrightarrow k < kt \wedge k \ll\!| lt \wedge k \ll\!| rt$
by (*auto simp add*: *rbt-greater-prop*)

lemmas *rbt-ord-props* $=$ *rbt-less-prop rbt-greater-prop*

lemmas *rbt-greater-nit* $=$ *rbt-greater-prop entry-in-tree-keys*
lemmas *rbt-less-nit* $=$ *rbt-less-prop entry-in-tree-keys*

lemma (**in** *order*)
shows *rbt-less-eq-trans*: $l\ |\!\ll\ u \Longrightarrow u \leq v \Longrightarrow l\ |\!\ll\ v$
and *rbt-less-trans*: $t\ |\!\ll\ x \Longrightarrow x < y \Longrightarrow t\ |\!\ll\ y$
and *rbt-greater-eq-trans*: $u \leq v \Longrightarrow v \ll\!|\ r \Longrightarrow u \ll\!|\ r$
and *rbt-greater-trans*: $x < y \Longrightarrow y \ll\!|\ t \Longrightarrow x \ll\!|\ t$
by (*auto simp*: *rbt-ord-props*)

primrec *rbt-sorted* :: $('a, 'b)\ rbt \Rightarrow bool$
where
rbt-sorted Empty $=$ *True*
| *rbt-sorted* (*Branch c l k v r*) $= (l\ |\!\ll\ k \wedge k \ll\!|\ r \wedge$ *rbt-sorted l* $\wedge$ *rbt-sorted r*$)$

end

context *linorder* **begin**

lemma *rbt-sorted-entries*:

rbt-sorted t ⟹ *List.sorted* (*map fst* (*entries t*))
by (*induct t*) (*force simp*: *sorted-append rbt-ord-props dest*!: *entry-in-tree-keys*)+

lemma *distinct-entries*:
rbt-sorted t ⟹ *distinct* (*map fst* (*entries t*))
by (*induct t*) (*force simp*: *sorted-append rbt-ord-props dest*!: *entry-in-tree-keys*)+

lemma *distinct-keys*:
rbt-sorted t ⟹ *distinct* (*keys t*)
by (*simp add*: *distinct-entries keys-def*)

119.2.3 Tree lookup

primrec (**in** *ord*) *rbt-lookup* :: ('*a*, '*b*) *rbt* ⇒ '*a* ⇀ '*b*
where
rbt-lookup Empty k = *None*
| *rbt-lookup* (*Branch - l x y r*) *k* =
(*if k* < *x then rbt-lookup l k else if x* < *k then rbt-lookup r k else Some y*)

lemma *rbt-lookup-keys*: *rbt-sorted t* ⟹ *dom* (*rbt-lookup t*) = *set* (*keys t*)
by (*induct t*) (*auto simp*: *dom-def rbt-greater-prop rbt-less-prop*)

lemma *dom-rbt-lookup-Branch*:
rbt-sorted (*Branch c t1 k v t2*) ⟹
dom (*rbt-lookup* (*Branch c t1 k v t2*))
= *Set.insert k* (*dom* (*rbt-lookup t1*) ∪ *dom* (*rbt-lookup t2*))
proof −
assume *rbt-sorted* (*Branch c t1 k v t2*)
then show *?thesis* **by** (*simp add*: *rbt-lookup-keys*)
qed

lemma *finite-dom-rbt-lookup* [*simp*, *intro*!]: *finite* (*dom* (*rbt-lookup t*))
proof (*induct t*)
case *Empty* **then show** *?case* **by** *simp*
next
case (*Branch color t1 a b t2*)
let *?A* = *Set.insert a* (*dom* (*rbt-lookup t1*) ∪ *dom* (*rbt-lookup t2*))
have *dom* (*rbt-lookup* (*Branch color t1 a b t2*)) ⊆ *?A* **by** (*auto split*: *if-split-asm*)
moreover from *Branch* **have** *finite* (*insert a* (*dom* (*rbt-lookup t1*) ∪ *dom* (*rbt-lookup t2*))) **by** *simp*
ultimately show *?case* **by** (*rule finite-subset*)
qed

end

context *ord* **begin**

lemma *rbt-lookup-rbt-less*[*simp*]: *t* |« *k* ⟹ *rbt-lookup t k* = *None*
by (*induct t*) *auto*

lemma *rbt-lookup-rbt-greater*[*simp*]: $k \ll\!| \ t \Longrightarrow$ *rbt-lookup t k* = *None*
by (*induct t*) *auto*

lemma *rbt-lookup-Empty*: *rbt-lookup Empty* = *Map.empty*
by (*rule ext*) *simp*

end

context *linorder* **begin**

lemma *map-of-entries*:
rbt-sorted t $\Longrightarrow$ *map-of* (*entries t*) = *rbt-lookup t*
proof (*induct t*)
case *Empty* **thus** *?case* **by** (*simp add*: *rbt-lookup-Empty*)
next
case (*Branch c t1 k v t2*)
have *rbt-lookup* (*Branch c t1 k v t2*) = *rbt-lookup t2* ++ [$k \mapsto v$] ++ *rbt-lookup t1*
proof (*rule ext*)
fix *x*
from *Branch* **have** *RBT-SORTED*: *rbt-sorted* (*Branch c t1 k v t2*) **by** *simp*
let *?thesis* = *rbt-lookup* (*Branch c t1 k v t2*) *x* = (*rbt-lookup t2* ++ [$k \mapsto v$] ++ *rbt-lookup t1*) *x*

have *DOM-T1*: $!!k'.\ k' \in dom$ (*rbt-lookup t1*) $\Longrightarrow k > k'$
proof −
fix k'
from *RBT-SORTED* **have** *t1* $\ll\!|$ *k* **by** *simp*
with *rbt-less-prop* **have** $\forall k' \in set$ (*keys t1*). $k > k'$ **by** *auto*
moreover assume $k' \in dom$ (*rbt-lookup t1*)
ultimately show $k > k'$ **using** *rbt-lookup-keys RBT-SORTED* **by** *auto*
qed

have *DOM-T2*: $!!k'.\ k' \in dom$ (*rbt-lookup t2*) $\Longrightarrow k < k'$
proof −
fix k'
from *RBT-SORTED* **have** *k* $\ll\!|$ *t2* **by** *simp*
with *rbt-greater-prop* **have** $\forall k' \in set$ (*keys t2*). $k < k'$ **by** *auto*
moreover assume $k' \in dom$ (*rbt-lookup t2*)
ultimately show $k < k'$ **using** *rbt-lookup-keys RBT-SORTED* **by** *auto*
qed

{
assume *C*: $x < k$
hence *rbt-lookup* (*Branch c t1 k v t2*) *x* = *rbt-lookup t1 x* **by** *simp*
moreover from *C* **have** $x \notin dom$ [$k \mapsto v$] **by** *simp*
moreover have $x \notin$ *dom* (*rbt-lookup t2*)
proof

assume $x \in dom\ (rbt\text{-}lookup\ t2)$
with *DOM-T2* **have** $k<x$ **by** *blast*
with *C* **show** *False* **by** *simp*
qed
ultimately have *?thesis* **by** (*simp add*: *map-add-upd-left map-add-dom-app-simps*)
} moreover {
assume [*simp*]: $x=k$
hence $rbt\text{-}lookup\ (Branch\ c\ t1\ k\ v\ t2)\ x = [k \mapsto v]\ x$ **by** *simp*
moreover have $x \notin dom\ (rbt\text{-}lookup\ t1)$
proof
assume $x \in dom\ (rbt\text{-}lookup\ t1)$
with *DOM-T1* **have** $k>x$ **by** *blast*
thus *False* **by** *simp*
qed
ultimately have *?thesis* **by** (*simp add*: *map-add-upd-left map-add-dom-app-simps*)
} moreover {
assume *C*: $x>k$
hence $rbt\text{-}lookup\ (Branch\ c\ t1\ k\ v\ t2)\ x = rbt\text{-}lookup\ t2\ x$ **by** (*simp add*: *less-not-sym*[*of k x*])
moreover from *C* **have** $x \notin dom\ [k \mapsto v]$ **by** *simp*
moreover have $x \notin dom\ (rbt\text{-}lookup\ t1)$ **proof**
assume $x \in dom\ (rbt\text{-}lookup\ t1)$
with *DOM-T1* **have** $k>x$ **by** *simp*
with *C* **show** *False* **by** *simp*
qed
ultimately have *?thesis* **by** (*simp add*: *map-add-upd-left map-add-dom-app-simps*)
} ultimately show *?thesis* **using** *less-linear* **by** *blast*
qed
also from *Branch*
have $rbt\text{-}lookup\ t2 +\!\!+ [k \mapsto v] +\!\!+ rbt\text{-}lookup\ t1 = map\text{-}of\ (entries\ (Branch\ c\ t1\ k\ v\ t2))$ **by** *simp*
finally show *?case* **by** *simp*
qed

lemma *rbt-lookup-in-tree*: $rbt\text{-}sorted\ t \Longrightarrow rbt\text{-}lookup\ t\ k = Some\ v \longleftrightarrow (k, v) \in set\ (entries\ t)$
by (*simp add*: *map-of-entries* [*symmetric*] *distinct-entries*)

lemma *set-entries-inject*:
assumes *rbt-sorted*: *rbt-sorted t1 rbt-sorted t2*
shows $set\ (entries\ t1) = set\ (entries\ t2) \longleftrightarrow entries\ t1 = entries\ t2$
proof −
from *rbt-sorted* **have** $distinct\ (map\ fst\ (entries\ t1))$
$distinct\ (map\ fst\ (entries\ t2))$
by (*auto intro*: *distinct-entries*)
with *rbt-sorted* **show** *?thesis*
by (*auto intro*: *map-sorted-distinct-set-unique rbt-sorted-entries simp add*: *distinct-map*)
qed

lemma *entries-eqI*:
assumes *rbt-sorted*: *rbt-sorted t1 rbt-sorted t2*
assumes *rbt-lookup*: *rbt-lookup t1 = rbt-lookup t2*
shows *entries t1 = entries t2*
proof −
from *rbt-sorted rbt-lookup* **have** *map-of* (*entries t1*) = *map-of* (*entries t2*)
by (*simp add*: *map-of-entries*)
with *rbt-sorted* **have** *set* (*entries t1*) = *set* (*entries t2*)
by (*simp add*: *map-of-inject-set distinct-entries*)
with *rbt-sorted* **show** *?thesis* **by** (*simp add*: *set-entries-inject*)
qed

lemma *entries-rbt-lookup*:
assumes *rbt-sorted t1 rbt-sorted t2*
shows *entries t1 = entries t2* ⟷ *rbt-lookup t1 = rbt-lookup t2*
using *assms* **by** (*auto intro*: *entries-eqI simp add*: *map-of-entries* [*symmetric*])

lemma *rbt-lookup-from-in-tree*:
assumes *rbt-sorted t1 rbt-sorted t2*
and ⋀*v*. (*k*, *v*) ∈ *set* (*entries t1*) ⟷ (*k*, *v*) ∈ *set* (*entries t2*)
shows *rbt-lookup t1 k = rbt-lookup t2 k*
proof −
from *assms* **have** *k* ∈ *dom* (*rbt-lookup t1*) ⟷ *k* ∈ *dom* (*rbt-lookup t2*)
by (*simp add*: *keys-entries rbt-lookup-keys*)
with *assms* **show** *?thesis* **by** (*auto simp add*: *rbt-lookup-in-tree* [*symmetric*])
qed

end

119.2.4 Red-black properties

primrec *color-of* :: (*'a*, *'b*) *rbt* ⇒ *color*
where
color-of Empty = B
| *color-of* (*Branch c - - - -*) = *c*

primrec *bheight* :: (*'a*,*'b*) *rbt* ⇒ *nat*
where
bheight Empty = 0
| *bheight* (*Branch c lt k v rt*) = (*if c = B then Suc* (*bheight lt*) *else bheight lt*)

primrec *inv1* :: (*'a*, *'b*) *rbt* ⇒ *bool*
where
inv1 Empty = True
| *inv1* (*Branch c lt k v rt*) ⟷ *inv1 lt* ∧ *inv1 rt* ∧ (*c = B* ∨ *color-of lt = B* ∧ *color-of rt = B*)

primrec *inv1l* :: (*'a*, *'b*) *rbt* ⇒ *bool* — Weaker version

where
inv1l Empty = *True*
| *inv1l* (*Branch c l k v r*) = (*inv1 l* ∧ *inv1 r*)
lemma [*simp*]: *inv1 t* ⟹ *inv1l t* **by** (*cases t*) *simp+*

primrec *inv2* :: ('*a*, '*b*) *rbt* ⇒ *bool*
where
inv2 Empty = *True*
| *inv2* (*Branch c lt k v rt*) = (*inv2 lt* ∧ *inv2 rt* ∧ *bheight lt* = *bheight rt*)

context *ord* **begin**

definition *is-rbt* :: ('*a*, '*b*) *rbt* ⇒ *bool* **where**
is-rbt t ⟷ *inv1 t* ∧ *inv2 t* ∧ *color-of t* = *B* ∧ *rbt-sorted t*

lemma *is-rbt-rbt-sorted* [*simp*]:
is-rbt t ⟹ *rbt-sorted t* **by** (*simp add*: *is-rbt-def*)

theorem *Empty-is-rbt* [*simp*]:
is-rbt Empty **by** (*simp add*: *is-rbt-def*)

end

119.3 Insertion

The function definitions are based on the book by Okasaki.

fun
balance :: ('*a*,'*b*) *rbt* ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*
where
balance (*Branch R a w x b*) *s t* (*Branch R c y z d*) = *Branch R* (*Branch B a w x b*) *s t* (*Branch B c y z d*) |
balance (*Branch R* (*Branch R a w x b*) *s t c*) *y z d* = *Branch R* (*Branch B a w x b*) *s t* (*Branch B c y z d*) |
balance (*Branch R a w x* (*Branch R b s t c*)) *y z d* = *Branch R* (*Branch B a w x b*) *s t* (*Branch B c y z d*) |
balance a w x (*Branch R b s t* (*Branch R c y z d*)) = *Branch R* (*Branch B a w x b*) *s t* (*Branch B c y z d*) |
balance a w x (*Branch R* (*Branch R b s t c*) *y z d*) = *Branch R* (*Branch B a w x b*) *s t* (*Branch B c y z d*) |
balance a s t b = *Branch B a s t b*

lemma *balance-inv1*: ⟦*inv1l l*; *inv1l r*⟧ ⟹ *inv1* (*balance l k v r*)
by (*induct l k v r rule*: *balance.induct*) *auto*

lemma *balance-bheight*: *bheight l* = *bheight r* ⟹ *bheight* (*balance l k v r*) = *Suc* (*bheight l*)
by (*induct l k v r rule*: *balance.induct*) *auto*

lemma *balance-inv2*:

assumes *inv2 l inv2 r bheight l = bheight r*
shows *inv2* (*balance l k v r*)
using *assms*
by (*induct l k v r rule*: *balance.induct*) *auto*

context *ord* **begin**

lemma *balance-rbt-greater*[*simp*]: (*v* «| *balance a k x b*) = (*v* «| *a* ∧ *v* «| *b* ∧ *v* < *k*)
by (*induct a k x b rule*: *balance.induct*) *auto*

lemma *balance-rbt-less*[*simp*]: (*balance a k x b* |« *v*) = (*a* |« *v* ∧ *b* |« *v* ∧ *k* < *v*)
by (*induct a k x b rule*: *balance.induct*) *auto*

end

lemma (**in** *linorder*) *balance-rbt-sorted*:
fixes *k* :: *'a*
assumes *rbt-sorted l rbt-sorted r l* |« *k k* «| *r*
shows *rbt-sorted* (*balance l k v r*)
using *assms* **proof** (*induct l k v r rule*: *balance.induct*)
case (*2-2 a x w b y t c z s va vb vd vc*)
hence *y* < *z* ∧ *z* «| *Branch B va vb vd vc*
by (*auto simp add*: *rbt-ord-props*)
hence *y* «| (*Branch B va vb vd vc*) **by** (*blast dest*: *rbt-greater-trans*)
with *2-2* **show** *?case* **by** *simp*
next
case (*3-2 va vb vd vc x w b y s c z*)
from *3-2* **have** *x* < *y* ∧ *Branch B va vb vd vc* |« *x*
by *simp*
hence *Branch B va vb vd vc* |« *y* **by** (*blast dest*: *rbt-less-trans*)
with *3-2* **show** *?case* **by** *simp*
next
case (*3-3 x w b y s c z t va vb vd vc*)
from *3-3* **have** *y* < *z* ∧ *z* «| *Branch B va vb vd vc* **by** *simp*
hence *y* «| *Branch B va vb vd vc* **by** (*blast dest*: *rbt-greater-trans*)
with *3-3* **show** *?case* **by** *simp*
next
case (*3-4 vd ve vg vf x w b y s c z t va vb vii vc*)
hence *x* < *y* ∧ *Branch B vd ve vg vf* |« *x* **by** *simp*
hence *1*: *Branch B vd ve vg vf* |« *y* **by** (*blast dest*: *rbt-less-trans*)
from *3-4* **have** *y* < *z* ∧ *z* «| *Branch B va vb vii vc* **by** *simp*
hence *y* «| *Branch B va vb vii vc* **by** (*blast dest*: *rbt-greater-trans*)
with *1 3-4* **show** *?case* **by** *simp*
next
case (*4-2 va vb vd vc x w b y s c z t dd*)
hence *x* < *y* ∧ *Branch B va vb vd vc* |« *x* **by** *simp*
hence *Branch B va vb vd vc* |« *y* **by** (*blast dest*: *rbt-less-trans*)
with *4-2* **show** *?case* **by** *simp*

next
case (*5-2 x w b y s c z t va vb vd vc*)
hence $y < z \wedge z \mathrel{«|} \mathit{Branch}\ B\ va\ vb\ vd\ vc$ **by** *simp*
hence $y \mathrel{«|} \mathit{Branch}\ B\ va\ vb\ vd\ vc$ **by** (*blast dest*: *rbt-greater-trans*)
with *5-2* **show** *?case* **by** *simp*
next
case (*5-3 va vb vd vc x w b y s c z t*)
hence $x < y \wedge \mathit{Branch}\ B\ va\ vb\ vd\ vc \mathrel{|«} x$ **by** *simp*
hence $\mathit{Branch}\ B\ va\ vb\ vd\ vc \mathrel{|«} y$ **by** (*blast dest*: *rbt-less-trans*)
with *5-3* **show** *?case* **by** *simp*
next
case (*5-4 va vb vg vc x w b y s c z t vd ve vii vf*)
hence $x < y \wedge \mathit{Branch}\ B\ va\ vb\ vg\ vc \mathrel{|«} x$ **by** *simp*
hence *1*: $\mathit{Branch}\ B\ va\ vb\ vg\ vc \mathrel{|«} y$ **by** (*blast dest*: *rbt-less-trans*)
from *5-4* **have** $y < z \wedge z \mathrel{«|} \mathit{Branch}\ B\ vd\ ve\ vii\ vf$ **by** *simp*
hence $y \mathrel{«|} \mathit{Branch}\ B\ vd\ ve\ vii\ vf$ **by** (*blast dest*: *rbt-greater-trans*)
with *1 5-4* **show** *?case* **by** *simp*
qed *simp+*

lemma *entries-balance* [*simp*]:
$\mathit{entries}\ (\mathit{balance}\ l\ k\ v\ r) = \mathit{entries}\ l\ @\ (k,\ v)\ \#\ \mathit{entries}\ r$
by (*induct l k v r rule*: *balance.induct*) *auto*

lemma *keys-balance* [*simp*]:
$\mathit{keys}\ (\mathit{balance}\ l\ k\ v\ r) = \mathit{keys}\ l\ @\ k\ \#\ \mathit{keys}\ r$
by (*simp add*: *keys-def*)

lemma *balance-in-tree*:
$\mathit{entry\text{-}in\text{-}tree}\ k\ x\ (\mathit{balance}\ l\ v\ y\ r) \longleftrightarrow \mathit{entry\text{-}in\text{-}tree}\ k\ x\ l \vee k = v \wedge x = y \vee \mathit{entry\text{-}in\text{-}tree}\ k\ x\ r$
by (*auto simp add*: *keys-def*)

lemma (**in** *linorder*) *rbt-lookup-balance*[*simp*]:
fixes $k :: 'a$
assumes *rbt-sorted l rbt-sorted r l* $\mathrel{|«}$ *k k* $\mathrel{«|}$ *r*
shows *rbt-lookup* (*balance l k v r*) *x* = *rbt-lookup* (*Branch B l k v r*) *x*
by (*rule rbt-lookup-from-in-tree*) (*auto simp*:*assms balance-in-tree balance-rbt-sorted*)

primrec *paint* :: *color* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*
where
paint c Empty = *Empty*
| *paint c* (*Branch* - *l k v r*) = *Branch c l k v r*

lemma *paint-inv1l*[*simp*]: *inv1l t* ⟹ *inv1l* (*paint c t*) **by** (*cases t*) *auto*
lemma *paint-inv1*[*simp*]: *inv1l t* ⟹ *inv1* (*paint B t*) **by** (*cases t*) *auto*
lemma *paint-inv2*[*simp*]: *inv2 t* ⟹ *inv2* (*paint c t*) **by** (*cases t*) *auto*
lemma *paint-color-of*[*simp*]: *color-of* (*paint B t*) = *B* **by** (*cases t*) *auto*
lemma *paint-in-tree*[*simp*]: *entry-in-tree k x* (*paint c t*) = *entry-in-tree k x t* **by** (*cases t*) *auto*

context *ord* **begin**

lemma *paint-rbt-sorted*[*simp*]: *rbt-sorted t* ⟹ *rbt-sorted* (*paint c t*) **by** (*cases t*) *auto*
lemma *paint-rbt-lookup*[*simp*]: *rbt-lookup* (*paint c t*) = *rbt-lookup t* **by** (*rule ext*) (*cases t*, *auto*)
lemma *paint-rbt-greater*[*simp*]: (*v* «| *paint c t*) = (*v* «| *t*) **by** (*cases t*) *auto*
lemma *paint-rbt-less*[*simp*]: (*paint c t* |« *v*) = (*t* |« *v*) **by** (*cases t*) *auto*

fun
rbt-ins :: ('*a* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*
where
rbt-ins f k v Empty = *Branch R Empty k v Empty* |
rbt-ins f k v (*Branch B l x y r*) = (*if k* < *x then balance* (*rbt-ins f k v l*) *x y r*
else if k > *x then balance l x y* (*rbt-ins f k v r*)
else Branch B l x (*f k y v*) *r*) |
rbt-ins f k v (*Branch R l x y r*) = (*if k* < *x then Branch R* (*rbt-ins f k v l*) *x y r*
else if k > *x then Branch R l x y* (*rbt-ins f k v r*)
else Branch R l x (*f k y v*) *r*)

lemma *ins-inv1-inv2*:
assumes *inv1 t inv2 t*
shows *inv2* (*rbt-ins f k x t*) *bheight* (*rbt-ins f k x t*) = *bheight t*
color-of t = *B* ⟹ *inv1* (*rbt-ins f k x t*) *inv1l* (*rbt-ins f k x t*)
using *assms*
by (*induct f k x t rule*: *rbt-ins.induct*) (*auto simp*: *balance-inv1 balance-inv2 balance-bheight*)

end

context *linorder* **begin**

lemma *ins-rbt-greater*[*simp*]: (*v* «| *rbt-ins f* (*k* :: '*a*) *x t*) = (*v* «| *t* ∧ *k* > *v*)
by (*induct f k x t rule*: *rbt-ins.induct*) *auto*
lemma *ins-rbt-less*[*simp*]: (*rbt-ins f k x t* |« *v*) = (*t* |« *v* ∧ *k* < *v*)
by (*induct f k x t rule*: *rbt-ins.induct*) *auto*
lemma *ins-rbt-sorted*[*simp*]: *rbt-sorted t* ⟹ *rbt-sorted* (*rbt-ins f k x t*)
by (*induct f k x t rule*: *rbt-ins.induct*) (*auto simp*: *balance-rbt-sorted*)

lemma *keys-ins*: *set* (*keys* (*rbt-ins f k v t*)) = { *k* } ∪ *set* (*keys t*)
by (*induct f k v t rule*: *rbt-ins.induct*) *auto*

lemma *rbt-lookup-ins*:
fixes *k* :: '*a*
assumes *rbt-sorted t*
shows *rbt-lookup* (*rbt-ins f k v t*) *x* = ((*rbt-lookup t*)(*k* |−> *case rbt-lookup t k of None* ⇒ *v*
| *Some w* ⇒ *f k w v*)) *x*

using *assms* **by** (*induct f k v t rule*: *rbt-ins.induct*) *auto*

end

context *ord* **begin**

definition *rbt-insert-with-key* :: ('*a* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*
where *rbt-insert-with-key f k v t* = *paint B* (*rbt-ins f k v t*)

definition *rbt-insertw-def*: *rbt-insert-with f* = *rbt-insert-with-key* (λ-. *f*)

definition *rbt-insert* :: '*a* ⇒ '*b* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* **where**
rbt-insert = *rbt-insert-with-key* (λ- - *nv*. *nv*)

end

context *linorder* **begin**

lemma *rbt-insertwk-rbt-sorted*: *rbt-sorted t* ⟹ *rbt-sorted* (*rbt-insert-with-key f* (*k* :: '*a*) *x t*)
by (*auto simp*: *rbt-insert-with-key-def*)

theorem *rbt-insertwk-is-rbt*:
assumes *inv*: *is-rbt t*
shows *is-rbt* (*rbt-insert-with-key f k x t*)
using *assms*
unfolding *rbt-insert-with-key-def is-rbt-def*
by (*auto simp*: *ins-inv1-inv2*)

lemma *rbt-lookup-rbt-insertwk*:
assumes *rbt-sorted t*
shows *rbt-lookup* (*rbt-insert-with-key f k v t*) *x* = ((*rbt-lookup t*)(*k* |–> *case rbt-lookup t k of None* ⇒ *v*
| *Some w* ⇒ *f k w v*)) *x*
unfolding *rbt-insert-with-key-def* **using** *assms*
by (*simp add*:*rbt-lookup-ins*)

lemma *rbt-insertw-rbt-sorted*: *rbt-sorted t* ⟹ *rbt-sorted* (*rbt-insert-with f k v t*)
by (*simp add*: *rbt-insertwk-rbt-sorted rbt-insertw-def*)
theorem *rbt-insertw-is-rbt*: *is-rbt t* ⟹ *is-rbt* (*rbt-insert-with f k v t*)
by (*simp add*: *rbt-insertwk-is-rbt rbt-insertw-def*)

lemma *rbt-lookup-rbt-insertw*:
is-rbt t ⟹
rbt-lookup (*rbt-insert-with f k v t*) =
(*rbt-lookup t*)(*k* ↦ (*if k* ∈ *dom* (*rbt-lookup t*) *then f* (*the* (*rbt-lookup t k*)) *v else v*))
by (*rule ext*, *cases rbt-lookup t k*) (*auto simp*: *rbt-lookup-rbt-insertwk dom-def*

rbt-insertw-def)

lemma *rbt-insert-rbt-sorted*: *rbt-sorted t* ⟹ *rbt-sorted* (*rbt-insert k v t*)
by (*simp add*: *rbt-insertwk-rbt-sorted rbt-insert-def*)
theorem *rbt-insert-is-rbt* [*simp*]: *is-rbt t* ⟹ *is-rbt* (*rbt-insert k v t*)
by (*simp add*: *rbt-insertwk-is-rbt rbt-insert-def*)

lemma *rbt-lookup-rbt-insert*: *is-rbt t* ⟹ *rbt-lookup* (*rbt-insert k v t*) = (*rbt-lookup t*)(*k*↦*v*)
by (*rule ext*) (*simp add*: *rbt-insert-def rbt-lookup-rbt-insertwk split*: *option.split*)

end

119.4 Deletion

lemma *bheight-paintR'*[*simp*]: *color-of t* = *B* ⟹ *bheight* (*paint R t*) = *bheight t* − *1*
by (*cases t rule*: *rbt-cases*) *auto*

The function definitions are based on the Haskell code by Stefan Kahrs at http://www.cs.ukc.ac.uk/people/staff/smk/redblack/rb.html.

fun
balance-left :: ('*a*,'*b*) *rbt* ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*
where
balance-left (*Branch R a k x b*) *s y c* = *Branch R* (*Branch B a k x b*) *s y c* |
balance-left bl k x (*Branch B a s y b*) = *balance bl k x* (*Branch R a s y b*) |
balance-left bl k x (*Branch R* (*Branch B a s y b*) *t z c*) = *Branch R* (*Branch B bl k x a*) *s y* (*balance b t z* (*paint R c*)) |
balance-left t k x s = *Empty*

lemma *balance-left-inv2-with-inv1*:
assumes *inv2 lt inv2 rt bheight lt* + *1* = *bheight rt inv1 rt*
shows *bheight* (*balance-left lt k v rt*) = *bheight lt* + *1*
and *inv2* (*balance-left lt k v rt*)
using *assms*
by (*induct lt k v rt rule*: *balance-left.induct*) (*auto simp*: *balance-inv2 balance-bheight*)

lemma *balance-left-inv2-app*:
assumes *inv2 lt inv2 rt bheight lt* + *1* = *bheight rt color-of rt* = *B*
shows *inv2* (*balance-left lt k v rt*)
bheight (*balance-left lt k v rt*) = *bheight rt*
using *assms*
by (*induct lt k v rt rule*: *balance-left.induct*) (*auto simp add*: *balance-inv2 balance-bheight*)+

lemma *balance-left-inv1*: ⟦*inv1l a*; *inv1 b*; *color-of b* = *B*⟧ ⟹ *inv1* (*balance-left a k x b*)
by (*induct a k x b rule*: *balance-left.induct*) (*simp add*: *balance-inv1*)+

lemma *balance-left-inv1l*: ⟦ *inv1l lt*; *inv1 rt* ⟧ ⟹ *inv1l* (*balance-left lt k x rt*)
by (*induct lt k x rt rule*: *balance-left.induct*) (*auto simp*: *balance-inv1*)

lemma (**in** *linorder*) *balance-left-rbt-sorted*:
⟦ *rbt-sorted l*; *rbt-sorted r*; *rbt-less k l*; *k* «| *r* ⟧ ⟹ *rbt-sorted* (*balance-left l k v r*)
apply (*induct l k v r rule*: *balance-left.induct*)
apply (*auto simp*: *balance-rbt-sorted*)
apply (*unfold rbt-greater-prop rbt-less-prop*)
by *force*+

context *order* **begin**

lemma *balance-left-rbt-greater*:
fixes *k* :: *'a*
assumes *k* «| *a k* «| *b k* < *x*
shows *k* «| *balance-left a x t b*
using *assms*
by (*induct a x t b rule*: *balance-left.induct*) *auto*

lemma *balance-left-rbt-less*:
fixes *k* :: *'a*
assumes *a* |« *k b* |« *k x* < *k*
shows *balance-left a x t b* |« *k*
using *assms*
by (*induct a x t b rule*: *balance-left.induct*) *auto*

end

lemma *balance-left-in-tree*:
assumes *inv1l l inv1 r bheight l* + *1* = *bheight r*
shows *entry-in-tree k v* (*balance-left l a b r*) = (*entry-in-tree k v l* ∨ *k* = *a* ∧ *v* = *b* ∨ *entry-in-tree k v r*)
using *assms*
by (*induct l k v r rule*: *balance-left.induct*) (*auto simp*: *balance-in-tree*)

fun
balance-right :: (*'a*,*'b*) *rbt* ⇒ *'a* ⇒ *'b* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
where
balance-right a k x (*Branch R b s y c*) = *Branch R a k x* (*Branch B b s y c*) |
balance-right (*Branch B a k x b*) *s y bl* = *balance* (*Branch R a k x b*) *s y bl* |
balance-right (*Branch R a k x* (*Branch B b s y c*)) *t z bl* = *Branch R* (*balance* (*paint R a*) *k x b*) *s y* (*Branch B c t z bl*) |
balance-right t k x s = *Empty*

lemma *balance-right-inv2-with-inv1*:
assumes *inv2 lt inv2 rt bheight lt* = *bheight rt* + *1 inv1 lt*
shows *inv2* (*balance-right lt k v rt*) ∧ *bheight* (*balance-right lt k v rt*) = *bheight lt*

using *assms*
by (*induct lt k v rt rule*: *balance-right.induct*) (*auto simp*: *balance-inv2 balance-bheight*)

lemma *balance-right-inv1*: ⟦*inv1 a*; *inv1l b*; *color-of a* = *B*⟧ ⟹ *inv1* (*balance-right a k x b*)
by (*induct a k x b rule*: *balance-right.induct*) (*simp add*: *balance-inv1*)+

lemma *balance-right-inv1l*: ⟦ *inv1 lt*; *inv1l rt* ⟧ ⟹*inv1l* (*balance-right lt k x rt*)
by (*induct lt k x rt rule*: *balance-right.induct*) (*auto simp*: *balance-inv1*)

lemma (**in** *linorder*) *balance-right-rbt-sorted*:
⟦ *rbt-sorted l*; *rbt-sorted r*; *rbt-less k l*; *k* «| *r* ⟧ ⟹ *rbt-sorted* (*balance-right l k v r*)
apply (*induct l k v r rule*: *balance-right.induct*)
apply (*auto simp*:*balance-rbt-sorted*)
apply (*unfold rbt-less-prop rbt-greater-prop*)
by *force*+

context *order* **begin**

lemma *balance-right-rbt-greater*:
fixes *k* :: *'a*
assumes *k* «| *a k* «| *b k* < *x*
shows *k* «| *balance-right a x t b*
using *assms* **by** (*induct a x t b rule*: *balance-right.induct*) *auto*

lemma *balance-right-rbt-less*:
fixes *k* :: *'a*
assumes *a* |« *k b* |« *k x* < *k*
shows *balance-right a x t b* |« *k*
using *assms* **by** (*induct a x t b rule*: *balance-right.induct*) *auto*

end

lemma *balance-right-in-tree*:
assumes *inv1 l inv1l r bheight l* = *bheight r* + *1 inv2 l inv2 r*
shows *entry-in-tree x y* (*balance-right l k v r*) = (*entry-in-tree x y l* ∨ *x* = *k* ∧ *y* = *v* ∨ *entry-in-tree x y r*)
using *assms* **by** (*induct l k v r rule*: *balance-right.induct*) (*auto simp*: *balance-in-tree*)

fun
combine :: (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
where
combine Empty x = *x*
| *combine x Empty* = *x*
| *combine* (*Branch R a k x b*) (*Branch R c s y d*) = (*case* (*combine b c*) *of*
Branch R b2 t z c2 ⇒ (*Branch R* (*Branch R a k x b2*) *t z* (*Branch R c2 s y d*)) |
bc ⇒ *Branch R a k x* (*Branch R bc s y d*))

| *combine* (*Branch B a k x b*) (*Branch B c s y d*) = (*case* (*combine b c*) *of*
Branch R b2 t z c2 ⇒ *Branch R* (*Branch B a k x b2*)
t z (*Branch B c2 s y d*) |
bc ⇒ *balance-left a k x* (*Branch B bc s y d*))
| *combine a* (*Branch R b k x c*) = *Branch R* (*combine a b*) *k x c*
| *combine* (*Branch R a k x b*) *c* = *Branch R a k x* (*combine b c*)

lemma *combine-inv2*:
assumes *inv2 lt inv2 rt bheight lt* = *bheight rt*
shows *bheight* (*combine lt rt*) = *bheight lt inv2* (*combine lt rt*)
using *assms*
by (*induct lt rt rule*: *combine.induct*)
(*auto simp*: *balance-left-inv2-app split*: *rbt.splits color.splits*)

lemma *combine-inv1*:
assumes *inv1 lt inv1 rt*
shows *color-of lt* = *B* ⟹ *color-of rt* = *B* ⟹ *inv1* (*combine lt rt*)
inv1l (*combine lt rt*)
using *assms*
by (*induct lt rt rule*: *combine.induct*)
(*auto simp*: *balance-left-inv1 split*: *rbt.splits color.splits*)

context *linorder* **begin**

lemma *combine-rbt-greater*[*simp*]:
fixes *k* :: *'a*
assumes *k* «| *l k* «| *r*
shows *k* «| *combine l r*
using *assms*
by (*induct l r rule*: *combine.induct*)
(*auto simp*: *balance-left-rbt-greater split*:*rbt.splits color.splits*)

lemma *combine-rbt-less*[*simp*]:
fixes *k* :: *'a*
assumes *l* |« *k r* |« *k*
shows *combine l r* |« *k*
using *assms*
by (*induct l r rule*: *combine.induct*)
(*auto simp*: *balance-left-rbt-less split*:*rbt.splits color.splits*)

lemma *combine-rbt-sorted*:
fixes *k* :: *'a*
assumes *rbt-sorted l rbt-sorted r l* |« *k k* «| *r*
shows *rbt-sorted* (*combine l r*)
using *assms* **proof** (*induct l r rule*: *combine.induct*)
case (*3 a x v b c y w d*)
hence *ineqs*: *a* |« *x x* «| *b b* |« *k k* «| *c c* |« *y y* «| *d*
by *auto*
with *3*

```
  show ?case
    by (cases combine b c rule: rbt-cases)
     (auto, (metis combine-rbt-greater combine-rbt-less ineqs ineqs rbt-less-simps(2)
rbt-greater-simps(2) rbt-greater-trans rbt-less-trans)+)
next
  case (4 a x v b c y w d)
  hence x < k ∧ rbt-greater k c by simp
  hence rbt-greater x c by (blast dest: rbt-greater-trans)
  with 4 have 2: rbt-greater x (combine b c) by (simp add: combine-rbt-greater)
  from 4 have k < y ∧ rbt-less k b by simp
  hence rbt-less y b by (blast dest: rbt-less-trans)
  with 4 have 3: rbt-less y (combine b c) by (simp add: combine-rbt-less)
  show ?case
  proof (cases combine b c rule: rbt-cases)
    case Empty
    from 4 have x < y ∧ rbt-greater y d by auto
    hence rbt-greater x d by (blast dest: rbt-greater-trans)
    with 4 Empty have rbt-sorted a and rbt-sorted (Branch B Empty y w d)
      and rbt-less x a and rbt-greater x (Branch B Empty y w d) by auto
    with Empty show ?thesis by (simp add: balance-left-rbt-sorted)
  next
    case (Red lta va ka rta)
    with 2 4 have x < va ∧ rbt-less x a by simp
    hence 5: rbt-less va a by (blast dest: rbt-less-trans)
    from Red 3 4 have va < y ∧ rbt-greater y d by simp
    hence rbt-greater va d by (blast dest: rbt-greater-trans)
    with Red 2 3 4 5 show ?thesis by simp
  next
    case (Black lta va ka rta)
    from 4 have x < y ∧ rbt-greater y d by auto
    hence rbt-greater x d by (blast dest: rbt-greater-trans)
    with Black 2 3 4 have rbt-sorted a and rbt-sorted (Branch B (combine b c) y
w d)
      and rbt-less x a and rbt-greater x (Branch B (combine b c) y w d) by auto
    with Black show ?thesis by (simp add: balance-left-rbt-sorted)
  qed
next
  case (5 va vb vd vc b x w c)
  hence k < x ∧ rbt-less k (Branch B va vb vd vc) by simp
  hence rbt-less x (Branch B va vb vd vc) by (blast dest: rbt-less-trans)
  with 5 show ?case by (simp add: combine-rbt-less)
next
  case (6 a x v b va vb vd vc)
  hence x < k ∧ rbt-greater k (Branch B va vb vd vc) by simp
  hence rbt-greater x (Branch B va vb vd vc) by (blast dest: rbt-greater-trans)
  with 6 show ?case by (simp add: combine-rbt-greater)
qed simp+

end
```

lemma *combine-in-tree*:
assumes *inv2 l inv2 r bheight l = bheight r inv1 l inv1 r*
shows *entry-in-tree k v (combine l r) = (entry-in-tree k v l ∨ entry-in-tree k v r)*
using *assms*
proof (*induct l r rule*: *combine.induct*)
case (*4 - - - b c*)
hence *a*: *bheight* (*combine b c*) = *bheight b* **by** (*simp add*: *combine-inv2*)
from *4* **have** *b*: *inv1l* (*combine b c*) **by** (*simp add*: *combine-inv1*)

show *?case*
proof (*cases combine b c rule*: *rbt-cases*)
case *Empty*
with *4 a* **show** *?thesis* **by** (*auto simp*: *balance-left-in-tree*)
next
case (*Red lta ka va rta*)
with *4* **show** *?thesis* **by** *auto*
next
case (*Black lta ka va rta*)
with *a b 4* **show** *?thesis* **by** (*auto simp*: *balance-left-in-tree*)
qed
qed (*auto split*: *rbt.splits color.splits*)

context *ord* **begin**

fun
rbt-del-from-left :: *'a* ⇒ (*'a*,*'b*) *rbt* ⇒ *'a* ⇒ *'b* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt* **and**
rbt-del-from-right :: *'a* ⇒ (*'a*,*'b*) *rbt* ⇒ *'a* ⇒ *'b* ⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt* **and**
rbt-del :: *'a*⇒ (*'a*,*'b*) *rbt* ⇒ (*'a*,*'b*) *rbt*
where
rbt-del x Empty = Empty |
rbt-del x (*Branch c a y s b*) =
(*if x < y then rbt-del-from-left x a y s b*
else (*if x > y then rbt-del-from-right x a y s b else combine a b*)) |
rbt-del-from-left x (*Branch B lt z v rt*) *y s b = balance-left* (*rbt-del x* (*Branch B lt z v rt*)) *y s b* |
rbt-del-from-left x a y s b = Branch R (*rbt-del x a*) *y s b* |
rbt-del-from-right x a y s (*Branch B lt z v rt*) = *balance-right a y s* (*rbt-del x* (*Branch B lt z v rt*)) |
rbt-del-from-right x a y s b = Branch R a y s (*rbt-del x b*)

end

context *linorder* **begin**

lemma
assumes *inv2 lt inv1 lt*
shows
⟦*inv2 rt*; *bheight lt = bheight rt*; *inv1 rt*⟧ ⟹

inv2 (*rbt-del-from-left x lt k v rt*) ∧
bheight (*rbt-del-from-left x lt k v rt*) = *bheight lt* ∧
(*color-of lt* = *B* ∧ *color-of rt* = *B* ∧ *inv1* (*rbt-del-from-left x lt k v rt*) ∨
(*color-of lt* ≠ *B* ∨ *color-of rt* ≠ *B*) ∧ *inv1l* (*rbt-del-from-left x lt k v rt*))
and ⟦*inv2 rt*; *bheight lt* = *bheight rt*; *inv1 rt*⟧ ⟹
inv2 (*rbt-del-from-right x lt k v rt*) ∧
bheight (*rbt-del-from-right x lt k v rt*) = *bheight lt* ∧
(*color-of lt* = *B* ∧ *color-of rt* = *B* ∧ *inv1* (*rbt-del-from-right x lt k v rt*) ∨
(*color-of lt* ≠ *B* ∨ *color-of rt* ≠ *B*) ∧ *inv1l* (*rbt-del-from-right x lt k v rt*))
and *rbt-del-inv1-inv2*: *inv2* (*rbt-del x lt*) ∧ (*color-of lt* = *R* ∧ *bheight* (*rbt-del x lt*) = *bheight lt* ∧ *inv1* (*rbt-del x lt*)
∨ *color-of lt* = *B* ∧ *bheight* (*rbt-del x lt*) = *bheight lt* − *1* ∧ *inv1l* (*rbt-del x lt*))
using *assms*
proof (*induct x lt k v rt* **and** *x lt k v rt* **and** *x lt rule*: *rbt-del-from-left-rbt-del-from-right-rbt-del.induct*)
case (*2 y c - y'*)
have *y* = *y'* ∨ *y* < *y'* ∨ *y* > *y'* **by** *auto*
thus *?case* **proof** (*elim disjE*)
assume *y* = *y'*
with *2* **show** *?thesis* **by** (*cases c*) (*simp add*: *combine-inv2 combine-inv1*)+
next
assume *y* < *y'*
with *2* **show** *?thesis* **by** (*cases c*) *auto*
next
assume *y'* < *y*
with *2* **show** *?thesis* **by** (*cases c*) *auto*
qed
next
case (*3 y lt z v rta y' ss bb*)
thus *?case* **by** (*cases color-of* (*Branch B lt z v rta*) = *B* ∧ *color-of bb* = *B*) (*simp add*: *balance-left-inv2-with-inv1 balance-left-inv1 balance-left-inv1l*)+
next
case (*5 y a y' ss lt z v rta*)
thus *?case* **by** (*cases color-of a* = *B* ∧ *color-of* (*Branch B lt z v rta*) = *B*) (*simp add*: *balance-right-inv2-with-inv1 balance-right-inv1 balance-right-inv1l*)+
next
case (*6-1 y a y' ss*) **thus** *?case* **by** (*cases color-of a* = *B* ∧ *color-of Empty* = *B*) *simp*+
qed *auto*

lemma
rbt-del-from-left-rbt-less: ⟦ *lt* |« *v*; *rt* |« *v*; *k* < *v*⟧ ⟹ *rbt-del-from-left x lt k y rt* |« *v*
and *rbt-del-from-right-rbt-less*: ⟦*lt* |« *v*; *rt* |« *v*; *k* < *v*⟧ ⟹ *rbt-del-from-right x lt k y rt* |« *v*
and *rbt-del-rbt-less*: *lt* |« *v* ⟹ *rbt-del x lt* |« *v*
by (*induct x lt k y rt* **and** *x lt k y rt* **and** *x lt rule*: *rbt-del-from-left-rbt-del-from-right-rbt-del.induct*)

(*auto simp*: *balance-left-rbt-less balance-right-rbt-less*)

lemma *rbt-del-from-left-rbt-greater*: ⟦*v «| lt*; *v «| rt*; *k > v*⟧ ⟹ *v «| rbt-del-from-left x lt k y rt*
and *rbt-del-from-right-rbt-greater*: ⟦*v «| lt*; *v «| rt*; *k > v*⟧ ⟹ *v «| rbt-del-from-right x lt k y rt*
and *rbt-del-rbt-greater*: *v «| lt* ⟹ *v «| rbt-del x lt*
by (*induct x lt k y rt* **and** *x lt k y rt* **and** *x lt rule*: *rbt-del-from-left-rbt-del-from-right-rbt-del.induct*)
(*auto simp*: *balance-left-rbt-greater balance-right-rbt-greater*)

lemma ⟦*rbt-sorted lt*; *rbt-sorted rt*; *lt |« k*; *k «| rt*⟧ ⟹ *rbt-sorted* (*rbt-del-from-left x lt k y rt*)
and ⟦*rbt-sorted lt*; *rbt-sorted rt*; *lt |« k*; *k «| rt*⟧ ⟹ *rbt-sorted* (*rbt-del-from-right x lt k y rt*)
and *rbt-del-rbt-sorted*: *rbt-sorted lt* ⟹ *rbt-sorted* (*rbt-del x lt*)
proof (*induct x lt k y rt* **and** *x lt k y rt* **and** *x lt rule*: *rbt-del-from-left-rbt-del-from-right-rbt-del.induct*)
case (*3 x lta zz v rta yy ss bb*)
from *3* **have** *Branch B lta zz v rta |« yy* **by** *simp*
hence *rbt-del x* (*Branch B lta zz v rta*) *|« yy* **by** (*rule rbt-del-rbt-less*)
with *3* **show** *?case* **by** (*simp add*: *balance-left-rbt-sorted*)
next
case (*4-2 x vaa vbb vdd vc yy ss bb*)
hence *Branch R vaa vbb vdd vc |« yy* **by** *simp*
hence *rbt-del x* (*Branch R vaa vbb vdd vc*) *|« yy* **by** (*rule rbt-del-rbt-less*)
with *4-2* **show** *?case* **by** *simp*
next
case (*5 x aa yy ss lta zz v rta*)
hence *yy «| Branch B lta zz v rta* **by** *simp*
hence *yy «| rbt-del x* (*Branch B lta zz v rta*) **by** (*rule rbt-del-rbt-greater*)
with *5* **show** *?case* **by** (*simp add*: *balance-right-rbt-sorted*)
next
case (*6-2 x aa yy ss vaa vbb vdd vc*)
hence *yy «| Branch R vaa vbb vdd vc* **by** *simp*
hence *yy «| rbt-del x* (*Branch R vaa vbb vdd vc*) **by** (*rule rbt-del-rbt-greater*)
with *6-2* **show** *?case* **by** *simp*
qed (*auto simp*: *combine-rbt-sorted*)

lemma ⟦*rbt-sorted lt*; *rbt-sorted rt*; *lt |« kt*; *kt «| rt*; *inv1 lt*; *inv1 rt*; *inv2 lt*; *inv2 rt*; *bheight lt = bheight rt*; *x < kt*⟧ ⟹ *entry-in-tree k v* (*rbt-del-from-left x lt kt y rt*) = (*False* ∨ (*x ≠ k* ∧ *entry-in-tree k v* (*Branch c lt kt y rt*)))
and ⟦*rbt-sorted lt*; *rbt-sorted rt*; *lt |« kt*; *kt «| rt*; *inv1 lt*; *inv1 rt*; *inv2 lt*; *inv2 rt*; *bheight lt = bheight rt*; *x > kt*⟧ ⟹ *entry-in-tree k v* (*rbt-del-from-right x lt kt y rt*) = (*False* ∨ (*x ≠ k* ∧ *entry-in-tree k v* (*Branch c lt kt y rt*)))
and *rbt-del-in-tree*: ⟦*rbt-sorted t*; *inv1 t*; *inv2 t*⟧ ⟹ *entry-in-tree k v* (*rbt-del x t*) = (*False* ∨ (*x ≠ k* ∧ *entry-in-tree k v t*))
proof (*induct x lt kt y rt* **and** *x lt kt y rt* **and** *x t rule*: *rbt-del-from-left-rbt-del-from-right-rbt-del.induct*)
case (*2 xx c aa yy ss bb*)
have *xx = yy* ∨ *xx < yy* ∨ *xx > yy* **by** *auto*
from *this 2* **show** *?case* **proof** (*elim disjE*)
assume *xx = yy*
with *2* **show** *?thesis* **proof** (*cases xx = k*)

case *True*
from *2* ‹*xx* = *yy*› ‹*xx* = *k*› **have** *rbt-sorted* (*Branch c aa yy ss bb*) ∧ *k* = *yy* **by** *simp*
hence ¬ *entry-in-tree k v aa* ¬ *entry-in-tree k v bb* **by** (*auto simp*: *rbt-less-nit rbt-greater-prop*)
with ‹*xx* = *yy*› *2* ‹*xx* = *k*› **show** *?thesis* **by** (*simp add*: *combine-in-tree*)
qed (*simp add*: *combine-in-tree*)
qed *simp+*
next
case (*3 xx lta zz vv rta yy ss bb*)
define *mt* **where** [*simp*]: *mt* = *Branch B lta zz vv rta*
from *3* **have** *inv2 mt* ∧ *inv1 mt* **by** *simp*
hence *inv2* (*rbt-del xx mt*) ∧ (*color-of mt* = *R* ∧ *bheight* (*rbt-del xx mt*) = *bheight mt* ∧ *inv1* (*rbt-del xx mt*) ∨ *color-of mt* = *B* ∧ *bheight* (*rbt-del xx mt*) = *bheight mt* − *1* ∧ *inv1l* (*rbt-del xx mt*)) **by** (*blast dest*: *rbt-del-inv1-inv2*)
with *3* **have** *4*: *entry-in-tree k v* (*rbt-del-from-left xx mt yy ss bb*) = (*False* ∨ *xx* ≠ *k* ∧ *entry-in-tree k v mt* ∨ (*k* = *yy* ∧ *v* = *ss*) ∨ *entry-in-tree k v bb*) **by** (*simp add*: *balance-left-in-tree*)
thus *?case* **proof** (*cases xx* = *k*)
case *True*
from *3 True* **have** *yy* «| *bb* ∧ *yy* > *k* **by** *simp*
hence *k* «| *bb* **by** (*blast dest*: *rbt-greater-trans*)
with *3 4 True* **show** *?thesis* **by** (*auto simp*: *rbt-greater-nit*)
qed *auto*
next
case (*4-1 xx yy ss bb*)
show *?case* **proof** (*cases xx* = *k*)
case *True*
with *4-1* **have** *yy* «| *bb* ∧ *k* < *yy* **by** *simp*
hence *k* «| *bb* **by** (*blast dest*: *rbt-greater-trans*)
with *4-1* ‹*xx* = *k*›
have *entry-in-tree k v* (*Branch R Empty yy ss bb*) = *entry-in-tree k v Empty* **by** (*auto simp*: *rbt-greater-nit*)
thus *?thesis* **by** *auto*
qed *simp+*
next
case (*4-2 xx vaa vbb vdd vc yy ss bb*)
thus *?case* **proof** (*cases xx* = *k*)
case *True*
with *4-2* **have** *k* < *yy* ∧ *yy* «| *bb* **by** *simp*
hence *k* «| *bb* **by** (*blast dest*: *rbt-greater-trans*)
with *True 4-2* **show** *?thesis* **by** (*auto simp*: *rbt-greater-nit*)
qed *auto*
next
case (*5 xx aa yy ss lta zz vv rta*)
define *mt* **where** [*simp*]: *mt* = *Branch B lta zz vv rta*
from *5* **have** *inv2 mt* ∧ *inv1 mt* **by** *simp*
hence *inv2* (*rbt-del xx mt*) ∧ (*color-of mt* = *R* ∧ *bheight* (*rbt-del xx mt*) = *bheight mt* ∧ *inv1* (*rbt-del xx mt*) ∨ *color-of mt* = *B* ∧ *bheight* (*rbt-del xx mt*) = *bheight*

mt − 1 ∧ inv1l (rbt-del xx mt)) **by** (*blast dest*: *rbt-del-inv1-inv2*)
with *5* **have** *3*: *entry-in-tree k v* (*rbt-del-from-right xx aa yy ss mt*) = (*entry-in-tree k v aa ∨ (k = yy ∧ v = ss) ∨ False ∨ xx ≠ k ∧ entry-in-tree k v mt*) **by** (*simp add*: *balance-right-in-tree*)
thus *?case* **proof** (*cases xx = k*)
case *True*
from *5 True* **have** *aa* |« *yy ∧ yy < k* **by** *simp*
hence *aa* |« *k* **by** (*blast dest*: *rbt-less-trans*)
with *3 5 True* **show** *?thesis* **by** (*auto simp*: *rbt-less-nit*)
qed *auto*
next
case (*6-1 xx aa yy ss*)
show *?case* **proof** (*cases xx = k*)
case *True*
with *6-1* **have** *aa* |« *yy ∧ k > yy* **by** *simp*
hence *aa* |« *k* **by** (*blast dest*: *rbt-less-trans*)
with *6-1* ‹*xx = k*› **show** *?thesis* **by** (*auto simp*: *rbt-less-nit*)
qed *simp*
next
case (*6-2 xx aa yy ss vaa vbb vdd vc*)
thus *?case* **proof** (*cases xx = k*)
case *True*
with *6-2* **have** *k > yy ∧ aa* |« *yy* **by** *simp*
hence *aa* |« *k* **by** (*blast dest*: *rbt-less-trans*)
with *True 6-2* **show** *?thesis* **by** (*auto simp*: *rbt-less-nit*)
qed *auto*
qed *simp*

definition (**in** *ord*) *rbt-delete* **where**
rbt-delete k t = paint B (*rbt-del k t*)

theorem *rbt-delete-is-rbt* [*simp*]: **assumes** *is-rbt t* **shows** *is-rbt* (*rbt-delete k t*)
proof −
from *assms* **have** *inv2 t* **and** *inv1 t* **unfolding** *is-rbt-def* **by** *auto*
hence *inv2* (*rbt-del k t*) ∧ (*color-of t = R ∧ bheight* (*rbt-del k t*) = *bheight t ∧ inv1* (*rbt-del k t*) ∨ *color-of t = B ∧ bheight* (*rbt-del k t*) = *bheight t − 1 ∧ inv1l* (*rbt-del k t*)) **by** (*rule rbt-del-inv1-inv2*)
hence *inv2* (*rbt-del k t*) ∧ *inv1l* (*rbt-del k t*) **by** (*cases color-of t*) *auto*
with *assms* **show** *?thesis*
unfolding *is-rbt-def rbt-delete-def*
by (*auto intro*: *paint-rbt-sorted rbt-del-rbt-sorted*)
qed

lemma *rbt-delete-in-tree*:
assumes *is-rbt t*
shows *entry-in-tree k v* (*rbt-delete x t*) = (*x ≠ k ∧ entry-in-tree k v t*)
using *assms* **unfolding** *is-rbt-def rbt-delete-def*
by (*auto simp*: *rbt-del-in-tree*)

lemma *rbt-lookup-rbt-delete*:
assumes *is-rbt*: *is-rbt t*
shows *rbt-lookup* (*rbt-delete k t*) = (*rbt-lookup t*)|`(−{*k*})
proof
fix *x*
show *rbt-lookup* (*rbt-delete k t*) *x* = (*rbt-lookup t* |` (−{*k*})) *x*
proof (*cases x* = *k*)
assume *x* = *k*
with *is-rbt* **show** *?thesis*
by (*cases rbt-lookup* (*rbt-delete k t*) *k*) (*auto simp*: *rbt-lookup-in-tree rbt-delete-in-tree*)
next
assume *x* ≠ *k*
thus *?thesis*
by *auto* (*metis is-rbt rbt-delete-is-rbt rbt-delete-in-tree is-rbt-rbt-sorted rbt-lookup-from-in-tree*)
qed
qed

end

119.5 Modifying existing entries

context *ord* **begin**

primrec
rbt-map-entry :: *'a* ⇒ (*'b* ⇒ *'b*) ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt*
where
rbt-map-entry k f Empty = *Empty*
| *rbt-map-entry k f* (*Branch c lt x v rt*) =
(*if k* < *x then Branch c* (*rbt-map-entry k f lt*) *x v rt*
else if k > *x then* (*Branch c lt x v* (*rbt-map-entry k f rt*))
else Branch c lt x (*f v*) *rt*)

lemma *rbt-map-entry-color-of*: *color-of* (*rbt-map-entry k f t*) = *color-of t* **by** (*induct t*) *simp+*
lemma *rbt-map-entry-inv1*: *inv1* (*rbt-map-entry k f t*) = *inv1 t* **by** (*induct t*) (*simp add*: *rbt-map-entry-color-of*)+
lemma *rbt-map-entry-inv2*: *inv2* (*rbt-map-entry k f t*) = *inv2 t bheight* (*rbt-map-entry k f t*) = *bheight t* **by** (*induct t*) *simp+*
lemma *rbt-map-entry-rbt-greater*: *rbt-greater a* (*rbt-map-entry k f t*) = *rbt-greater a t* **by** (*induct t*) *simp+*
lemma *rbt-map-entry-rbt-less*: *rbt-less a* (*rbt-map-entry k f t*) = *rbt-less a t* **by** (*induct t*) *simp+*
lemma *rbt-map-entry-rbt-sorted*: *rbt-sorted* (*rbt-map-entry k f t*) = *rbt-sorted t*
by (*induct t*) (*simp-all add*: *rbt-map-entry-rbt-less rbt-map-entry-rbt-greater*)

theorem *rbt-map-entry-is-rbt* [*simp*]: *is-rbt* (*rbt-map-entry k f t*) = *is-rbt t*
unfolding *is-rbt-def* **by** (*simp add*: *rbt-map-entry-inv2 rbt-map-entry-color-of rbt-map-entry-rbt-sorted rbt-map-entry-inv1*)

end

theorem (**in** *linorder*) *rbt-lookup-rbt-map-entry*:
rbt-lookup (*rbt-map-entry k f t*) = (*rbt-lookup t*)(*k* := *map-option f* (*rbt-lookup t k*))
by (*induct t*) (*auto split*: *option.splits simp add*: *fun-eq-iff*)

119.6 Mapping all entries

primrec
map :: ('*a* ⇒ '*b* ⇒ '*c*) ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*c*) *rbt*
where
map f Empty = *Empty*
| *map f* (*Branch c lt k v rt*) = *Branch c* (*map f lt*) *k* (*f k v*) (*map f rt*)

lemma *map-entries* [*simp*]: *entries* (*map f t*) = *List.map* (λ(*k*, *v*). (*k*, *f k v*)) (*entries t*)
by (*induct t*) *auto*
lemma *map-keys* [*simp*]: *keys* (*map f t*) = *keys t* **by** (*simp add*: *keys-def split-def*)
lemma *map-color-of*: *color-of* (*map f t*) = *color-of t* **by** (*induct t*) *simp+*
lemma *map-inv1*: *inv1* (*map f t*) = *inv1 t* **by** (*induct t*) (*simp add*: *map-color-of*)+
lemma *map-inv2*: *inv2* (*map f t*) = *inv2 t bheight* (*map f t*) = *bheight t* **by** (*induct t*) *simp+*

context *ord* **begin**

lemma *map-rbt-greater*: *rbt-greater k* (*map f t*) = *rbt-greater k t* **by** (*induct t*) *simp+*
lemma *map-rbt-less*: *rbt-less k* (*map f t*) = *rbt-less k t* **by** (*induct t*) *simp+*
lemma *map-rbt-sorted*: *rbt-sorted* (*map f t*) = *rbt-sorted t* **by** (*induct t*) (*simp add*: *map-rbt-less map-rbt-greater*)+
theorem *map-is-rbt* [*simp*]: *is-rbt* (*map f t*) = *is-rbt t*
unfolding *is-rbt-def* **by** (*simp add*: *map-inv1 map-inv2 map-rbt-sorted map-color-of*)

end

theorem (**in** *linorder*) *rbt-lookup-map*: *rbt-lookup* (*map f t*) *x* = *map-option* (*f x*) (*rbt-lookup t x*)
by (*induct t*) (*auto simp*: *antisym-conv3*)

hide-const (**open**) *map*

119.7 Folding over entries

definition *fold* :: ('*a* ⇒ '*b* ⇒ '*c* ⇒ '*c*) ⇒ ('*a*, '*b*) *rbt* ⇒ '*c* ⇒ '*c* **where**
fold f t = *List.fold* (*case-prod f*) (*entries t*)

lemma *fold-simps* [*simp*]:

```
  fold f Empty = id
  fold f (Branch c lt k v rt) = fold f rt ∘ f k v ∘ fold f lt
  by (simp-all add: fold-def fun-eq-iff)

lemma fold-code [code]:
  fold f Empty x = x
  fold f (Branch c lt k v rt) x = fold f rt (f k v (fold f lt x))
by(simp-all)

— fold with continuation predicate
fun foldi :: ('c ⇒ bool) ⇒ ('a ⇒ 'b ⇒ 'c ⇒ 'c) ⇒ ('a :: linorder, 'b) rbt ⇒ 'c ⇒
'c
  where
  foldi c f Empty s = s |
  foldi c f (Branch col l k v r) s = (
    if (c s) then
      let s' = foldi c f l s in
        if (c s') then
          foldi c f r (f k v s')
        else s'
    else
      s
  )
```

119.8 Bulkloading a tree

```
definition (in ord) rbt-bulkload :: ('a × 'b) list ⇒ ('a, 'b) rbt where
  rbt-bulkload xs = foldr (λ(k, v). rbt-insert k v) xs Empty

context linorder begin

lemma rbt-bulkload-is-rbt [simp, intro]:
  is-rbt (rbt-bulkload xs)
  unfolding rbt-bulkload-def by (induct xs) auto

lemma rbt-lookup-rbt-bulkload:
  rbt-lookup (rbt-bulkload xs) = map-of xs
proof −
  obtain ys where ys = rev xs by simp
  have ⋀t. is-rbt t ⟹
    rbt-lookup (List.fold (case-prod rbt-insert) ys t) = rbt-lookup t ++ map-of (rev
ys)
    by (induct ys) (simp-all add: rbt-bulkload-def rbt-lookup-rbt-insert case-prod-beta)
  from this Empty-is-rbt have
    rbt-lookup (List.fold (case-prod rbt-insert) (rev xs) Empty) = rbt-lookup Empty
++ map-of xs
    by (simp add: ‹ys = rev xs›)
 then show ?thesis by (simp add: rbt-bulkload-def rbt-lookup-Empty foldr-conv-fold)
qed
```

end

119.9 Building a RBT from a sorted list

These functions have been adapted from Andrew W. Appel, Efficient Verified Red-Black Trees (September 2011)

```
fun rbtreeify-f :: nat ⇒ ('a × 'b) list ⇒ ('a, 'b) rbt × ('a × 'b) list
  and rbtreeify-g :: nat ⇒ ('a × 'b) list ⇒ ('a, 'b) rbt × ('a × 'b) list
where
  rbtreeify-f n kvs =
   (if n = 0 then (Empty, kvs)
    else if n = 1 then
      case kvs of (k, v) # kvs' ⇒ (Branch R Empty k v Empty, kvs')
    else if (n mod 2 = 0) then
      case rbtreeify-f (n div 2) kvs of (t1, (k, v) # kvs') ⇒
        apfst (Branch B t1 k v) (rbtreeify-g (n div 2) kvs')
    else case rbtreeify-f (n div 2) kvs of (t1, (k, v) # kvs') ⇒
        apfst (Branch B t1 k v) (rbtreeify-f (n div 2) kvs'))

| rbtreeify-g n kvs =
   (if n = 0 ∨ n = 1 then (Empty, kvs)
    else if n mod 2 = 0 then
      case rbtreeify-g (n div 2) kvs of (t1, (k, v) # kvs') ⇒
        apfst (Branch B t1 k v) (rbtreeify-g (n div 2) kvs')
    else case rbtreeify-f (n div 2) kvs of (t1, (k, v) # kvs') ⇒
        apfst (Branch B t1 k v) (rbtreeify-g (n div 2) kvs'))

definition rbtreeify :: ('a × 'b) list ⇒ ('a, 'b) rbt
where rbtreeify kvs = fst (rbtreeify-g (Suc (length kvs)) kvs)

declare rbtreeify-f.simps [simp del] rbtreeify-g.simps [simp del]

lemma rbtreeify-f-code [code]:
  rbtreeify-f n kvs =
   (if n = 0 then (Empty, kvs)
    else if n = 1 then
      case kvs of (k, v) # kvs' ⇒
        (Branch R Empty k v Empty, kvs')
    else let (n', r) = Divides.divmod-nat n 2 in
      if r = 0 then
        case rbtreeify-f n' kvs of (t1, (k, v) # kvs') ⇒
          apfst (Branch B t1 k v) (rbtreeify-g n' kvs')
      else case rbtreeify-f n' kvs of (t1, (k, v) # kvs') ⇒
          apfst (Branch B t1 k v) (rbtreeify-f n' kvs'))
by (subst rbtreeify-f.simps) (simp only: Let-def divmod-nat-def prod.case)

lemma rbtreeify-g-code [code]:
  rbtreeify-g n kvs =
```

```
  (if n = 0 ∨ n = 1 then (Empty, kvs)
   else let (n', r) = Divides.divmod-nat n 2 in
     if r = 0 then
       case rbtreeify-g n' kvs of (t1, (k, v) # kvs') ⇒
         apfst (Branch B t1 k v) (rbtreeify-g n' kvs')
     else case rbtreeify-f n' kvs of (t1, (k, v) # kvs') ⇒
         apfst (Branch B t1 k v) (rbtreeify-g n' kvs'))
by(subst rbtreeify-g.simps)(simp only: Let-def divmod-nat-def prod.case)

lemma Suc-double-half: Suc (2 * n) div 2 = n
by simp

lemma div2-plus-div2: n div 2 + n div 2 = (n :: nat) - n mod 2
by arith

lemma rbtreeify-f-rec-aux-lemma:
  ⟦k - n div 2 = Suc k'; n ≤ k; n mod 2 = Suc 0⟧
  ⟹ k' - n div 2 = k - n
apply(rule add-right-imp-eq[where a = n - n div 2])
apply(subst add-diff-assoc2, arith)
apply(simp add: div2-plus-div2)
done

lemma rbtreeify-f-simps:
  rbtreeify-f 0 kvs = (Empty, kvs)
  rbtreeify-f (Suc 0) ((k, v) # kvs) =
  (Branch R Empty k v Empty, kvs)
  0 < n ⟹ rbtreeify-f (2 * n) kvs =
   (case rbtreeify-f n kvs of (t1, (k, v) # kvs') ⇒
     apfst (Branch B t1 k v) (rbtreeify-g n kvs'))
  0 < n ⟹ rbtreeify-f (Suc (2 * n)) kvs =
   (case rbtreeify-f n kvs of (t1, (k, v) # kvs') ⇒
     apfst (Branch B t1 k v) (rbtreeify-f n kvs'))
by(subst (1) rbtreeify-f.simps, simp add: Suc-double-half)+

lemma rbtreeify-g-simps:
  rbtreeify-g 0 kvs = (Empty, kvs)
  rbtreeify-g (Suc 0) kvs = (Empty, kvs)
  0 < n ⟹ rbtreeify-g (2 * n) kvs =
   (case rbtreeify-g n kvs of (t1, (k, v) # kvs') ⇒
     apfst (Branch B t1 k v) (rbtreeify-g n kvs'))
  0 < n ⟹ rbtreeify-g (Suc (2 * n)) kvs =
   (case rbtreeify-f n kvs of (t1, (k, v) # kvs') ⇒
     apfst (Branch B t1 k v) (rbtreeify-g n kvs'))
by(subst (1) rbtreeify-g.simps, simp add: Suc-double-half)+

declare rbtreeify-f-simps[simp] rbtreeify-g-simps[simp]

lemma length-rbtreeify-f: n ≤ length kvs
```

```
  ⟹ length (snd (rbtreeify-f n kvs)) = length kvs − n
  and length-rbtreeify-g:⟦ 0 < n; n ≤ Suc (length kvs) ⟧
  ⟹ length (snd (rbtreeify-g n kvs)) = Suc (length kvs) − n
proof(induction n kvs and n kvs rule: rbtreeify-f-rbtreeify-g.induct)
  case (1 n kvs)
  show ?case
  proof(cases n ≤ 1)
    case True thus ?thesis using 1.prems
      by(cases n kvs rule: nat.exhaust[case-product list.exhaust]) auto
  next
    case False
    hence n ≠ 0 n ≠ 1 by simp-all
    note IH = 1.IH[OF this]
    show ?thesis
    proof(cases n mod 2 = 0)
      case True
      hence length (snd (rbtreeify-f n kvs)) =
        length (snd (rbtreeify-f (2 * (n div 2)) kvs))
        by(metis minus-nat.diff-0 minus-mod-eq-mult-div [symmetric])
      also from 1.prems False obtain k v kvs'
        where kvs: kvs = (k, v) # kvs' by(cases kvs) auto
      also have 0 < n div 2 using False by(simp)
      note rbtreeify-f-simps(3)[OF this]
      also note kvs[symmetric]
      also let ?rest1 = snd (rbtreeify-f (n div 2) kvs)
      from 1.prems have n div 2 ≤ length kvs by simp
      with True have len: length ?rest1 = length kvs − n div 2 by(rule IH)
      with 1.prems False obtain t1 k' v' kvs''
        where kvs'': rbtreeify-f (n div 2) kvs = (t1, (k', v') # kvs'')
         by(cases ?rest1)(auto simp add: snd-def split: prod.split-asm)
      note this also note prod.case also note list.simps(5)
      also note prod.case also note snd-apfst
      also have 0 < n div 2 n div 2 ≤ Suc (length kvs'')
        using len 1.prems False unfolding kvs'' by simp-all
      with True kvs''[symmetric] refl refl
      have length (snd (rbtreeify-g (n div 2) kvs'')) =
        Suc (length kvs'') − n div 2 by(rule IH)
      finally show ?thesis using len[unfolded kvs''] 1.prems True
      by(simp add: Suc-diff-le[symmetric] mult-2[symmetric] minus-mod-eq-mult-div
[symmetric])
    next
      case False
      hence length (snd (rbtreeify-f n kvs)) =
        length (snd (rbtreeify-f (Suc (2 * (n div 2))) kvs))
        by (simp add: mod-eq-0-iff-dvd)
      also from 1.prems ‹¬ n ≤ 1› obtain k v kvs'
        where kvs: kvs = (k, v) # kvs' by(cases kvs) auto
      also have 0 < n div 2 using ‹¬ n ≤ 1› by(simp)
      note rbtreeify-f-simps(4)[OF this]
```

```
    also note kvs[symmetric]
    also let ?rest1 = snd (rbtreeify-f (n div 2) kvs)
    from 1.prems have n div 2 ≤ length kvs by simp
    with False have len: length ?rest1 = length kvs − n div 2 by(rule IH)
    with 1.prems ‹¬ n ≤ 1› obtain t1 k' v' kvs''
      where kvs'': rbtreeify-f (n div 2) kvs = (t1, (k', v') # kvs'')
      by(cases ?rest1)(auto simp add: snd-def split: prod.split-asm)
    note this also note prod.case also note list.simps(5)
    also note prod.case also note snd-apfst
    also have n div 2 ≤ length kvs''
      using len 1.prems False unfolding kvs'' by simp arith
    with False kvs''[symmetric] refl refl
    have length (snd (rbtreeify-f (n div 2) kvs'')) = length kvs'' − n div 2
      by(rule IH)
    finally show ?thesis using len[unfolded kvs''] 1.prems False
      by simp(rule rbtreeify-f-rec-aux-lemma[OF sym])
  qed
 qed
next
 case (2 n kvs)
 show ?case
 proof(cases n > 1)
  case False with ‹0 < n› show ?thesis
   by(cases n kvs rule: nat.exhaust[case-product list.exhaust]) simp-all
 next
  case True
  hence ¬ (n = 0 ∨ n = 1) by simp
  note IH = 2.IH[OF this]
  show ?thesis
  proof(cases n mod 2 = 0)
   case True
   hence length (snd (rbtreeify-g n kvs)) =
     length (snd (rbtreeify-g (2 * (n div 2)) kvs))
     by(metis minus-nat.diff-0 minus-mod-eq-mult-div [symmetric])
   also from 2.prems True obtain k v kvs'
     where kvs: kvs = (k, v) # kvs' by(cases kvs) auto
   also have 0 < n div 2 using ‹1 < n› by(simp)
   note rbtreeify-g-simps(3)[OF this]
   also note kvs[symmetric]
   also let ?rest1 = snd (rbtreeify-g (n div 2) kvs)
   from 2.prems ‹1 < n›
   have 0 < n div 2 n div 2 ≤ Suc (length kvs) by simp-all
   with True have len: length ?rest1 = Suc (length kvs) − n div 2 by(rule IH)
   with 2.prems obtain t1 k' v' kvs''
     where kvs'': rbtreeify-g (n div 2) kvs = (t1, (k', v') # kvs'')
     by(cases ?rest1)(auto simp add: snd-def split: prod.split-asm)
   note this also note prod.case also note list.simps(5)
   also note prod.case also note snd-apfst
   also have n div 2 ≤ Suc (length kvs'')
```

```
      using len 2.prems unfolding kvs'' by simp
    with True kvs''[symmetric] refl refl ‹0 < n div 2›
    have length (snd (rbtreeify-g (n div 2) kvs'')) = Suc (length kvs'') − n div 2
      by(rule IH)
    finally show ?thesis using len[unfolded kvs''] 2.prems True
    by(simp add: Suc-diff-le[symmetric] mult-2[symmetric] minus-mod-eq-mult-div
[symmetric])
  next
    case False
    hence length (snd (rbtreeify-g n kvs)) =
      length (snd (rbtreeify-g (Suc (2 ∗ (n div 2))) kvs))
      by (simp add: mod-eq-0-iff-dvd)
    also from 2.prems ‹1 < n› obtain k v kvs'
      where kvs: kvs = (k, v) # kvs' by(cases kvs) auto
    also have 0 < n div 2 using ‹1 < n› by(simp)
    note rbtreeify-g-simps(4)[OF this]
    also note kvs[symmetric]
    also let ?rest1 = snd (rbtreeify-f (n div 2) kvs)
    from 2.prems have n div 2 ≤ length kvs by simp
    with False have len: length ?rest1 = length kvs − n div 2 by(rule IH)
    with 2.prems ‹1 < n› False obtain t1 k' v' kvs''
      where kvs'': rbtreeify-f (n div 2) kvs = (t1, (k', v') # kvs'')
      by(cases ?rest1)(auto simp add: snd-def split: prod.split-asm, arith)
    note this also note prod.case also note list.simps(5)
    also note prod.case also note snd-apfst
    also have n div 2 ≤ Suc (length kvs'')
      using len 2.prems False unfolding kvs'' by simp arith
    with False kvs''[symmetric] refl refl ‹0 < n div 2›
    have length (snd (rbtreeify-g (n div 2) kvs'')) = Suc (length kvs'') − n div 2
      by(rule IH)
    finally show ?thesis using len[unfolded kvs''] 2.prems False
      by(simp add: div2-plus-div2)
  qed
 qed
qed

lemma rbtreeify-induct [consumes 1, case-names f-0 f-1 f-even f-odd g-0 g-1 g-even
g-odd]:
 fixes P Q
 defines f0 == (⋀kvs. P 0 kvs)
 and f1 == (⋀k v kvs. P (Suc 0) ((k, v) # kvs))
 and feven ==
  (⋀n kvs t k v kvs'. ⟦ n > 0; n ≤ length kvs; P n kvs;
     rbtreeify-f n kvs = (t, (k, v) # kvs'); n ≤ Suc (length kvs'); Q n kvs' ⟧
   ⟹ P (2 ∗ n) kvs)
 and fodd ==
  (⋀n kvs t k v kvs'. ⟦ n > 0; n ≤ length kvs; P n kvs;
     rbtreeify-f n kvs = (t, (k, v) # kvs'); n ≤ length kvs'; P n kvs' ⟧
  ⟹ P (Suc (2 ∗ n)) kvs)
```

```
  and g0 == (⋀kvs. Q 0 kvs)
  and g1 == (⋀kvs. Q (Suc 0) kvs)
  and geven ==
    (⋀n kvs t k v kvs'. ⟦ n > 0; n ≤ Suc (length kvs); Q n kvs;
       rbtreeify-g n kvs = (t, (k, v) # kvs'); n ≤ Suc (length kvs'); Q n kvs' ⟧
    ⟹ Q (2 * n) kvs)
  and godd ==
    (⋀n kvs t k v kvs'. ⟦ n > 0; n ≤ length kvs; P n kvs;
       rbtreeify-f n kvs = (t, (k, v) # kvs'); n ≤ Suc (length kvs'); Q n kvs' ⟧
    ⟹ Q (Suc (2 * n)) kvs)
  shows ⟦ n ≤ length kvs;
          PROP f0; PROP f1; PROP feven; PROP fodd;
          PROP g0; PROP g1; PROP geven; PROP godd ⟧
        ⟹ P n kvs
  and ⟦ n ≤ Suc (length kvs);
          PROP f0; PROP f1; PROP feven; PROP fodd;
          PROP g0; PROP g1; PROP geven; PROP godd ⟧
      ⟹ Q n kvs
proof −
  assume f0: PROP f0 and f1: PROP f1 and feven: PROP feven and fodd:
PROP fodd
    and g0: PROP g0 and g1: PROP g1 and geven: PROP geven and godd:
PROP godd
  show n ≤ length kvs ⟹ P n kvs and n ≤ Suc (length kvs) ⟹ Q n kvs
  proof(induction rule: rbtreeify-f-rbtreeify-g.induct)
    case (1 n kvs)
    show ?case
    proof(cases n ≤ 1)
      case True thus ?thesis using 1.prems
        by(cases n kvs rule: nat.exhaust[case-product list.exhaust])
          (auto simp add: f0[unfolded f0-def] f1[unfolded f1-def])
    next
      case False
      hence ns: n ≠ 0 n ≠ 1 by simp-all
      hence ge0: n div 2 > 0 by simp
      note IH = 1.IH[OF ns]
      show ?thesis
      proof(cases n mod 2 = 0)
        case True note ge0
        moreover from 1.prems have n2: n div 2 ≤ length kvs by simp
        moreover from True n2 have P (n div 2) kvs by(rule IH)
        moreover from length-rbtreeify-f[OF n2] ge0 1.prems obtain t k v kvs'
          where kvs': rbtreeify-f (n div 2) kvs = (t, (k, v) # kvs')
          by(cases snd (rbtreeify-f (n div 2) kvs))
            (auto simp add: snd-def split: prod.split-asm)
        moreover from 1.prems length-rbtreeify-f[OF n2] ge0
        have n2': n div 2 ≤ Suc (length kvs') by(simp add: kvs')
        moreover from True kvs'[symmetric] refl refl n2'
        have Q (n div 2) kvs' by(rule IH)
```

```
      moreover note feven[unfolded feven-def]

      ultimately have P (2 * (n div 2)) kvs by -
        thus ?thesis using True by (metis minus-mod-eq-div-mult [symmetric]
minus-nat.diff-0 mult.commute)
    next
      case False note ge0
      moreover from 1.prems have n2: n div 2 ≤ length kvs by simp
      moreover from False n2 have P (n div 2) kvs by(rule IH)
      moreover from length-rbtreeify-f[OF n2] ge0 1.prems obtain t k v kvs'
        where kvs': rbtreeify-f (n div 2) kvs = (t, (k, v) # kvs')
        by(cases snd (rbtreeify-f (n div 2) kvs))
          (auto simp add: snd-def split: prod.split-asm)
      moreover from 1.prems length-rbtreeify-f[OF n2] ge0 False
      have n2': n div 2 ≤ length kvs' by(simp add: kvs') arith
       moreover from False kvs'[symmetric] refl refl n2' have P (n div 2) kvs'
by(rule IH)
      moreover note fodd[unfolded fodd-def]
      ultimately have P (Suc (2 * (n div 2))) kvs by -
      thus ?thesis using False
     by simp (metis One-nat-def Suc-eq-plus1-left le-add-diff-inverse mod-less-eq-dividend
minus-mod-eq-mult-div [symmetric])
    qed
  qed
 next
  case (2 n kvs)
  show ?case
  proof(cases n ≤ 1)
    case True thus ?thesis using 2.prems
     by(cases n kvs rule: nat.exhaust[case-product list.exhaust])
       (auto simp add: g0[unfolded g0-def] g1[unfolded g1-def])
  next
    case False
    hence ns: ¬ (n = 0 ∨ n = 1) by simp
    hence ge0: n div 2 > 0 by simp
    note IH = 2.IH[OF ns]
    show ?thesis
    proof(cases n mod 2 = 0)
      case True note ge0
      moreover from 2.prems have n2: n div 2 ≤ Suc (length kvs) by simp
      moreover from True n2 have Q (n div 2) kvs by(rule IH)
     moreover from length-rbtreeify-g[OF ge0 n2] ge0 2.prems obtain t k v kvs'

       where kvs': rbtreeify-g (n div 2) kvs = (t, (k, v) # kvs')
       by(cases snd (rbtreeify-g (n div 2) kvs))
         (auto simp add: snd-def split: prod.split-asm)
      moreover from 2.prems length-rbtreeify-g[OF ge0 n2] ge0
      have n2': n div 2 ≤ Suc (length kvs') by(simp add: kvs')
      moreover from True kvs'[symmetric] refl refl  n2'
```

```
      have Q (n div 2) kvs' by(rule IH)
      moreover note geven[unfolded geven-def]
      ultimately have Q (2 * (n div 2)) kvs by −
      thus ?thesis using True
     by(metis minus-mod-eq-div-mult [symmetric] minus-nat.diff-0 mult.commute)
    next
      case False note ge0
      moreover from 2.prems have n2: n div 2 ≤ length kvs by simp
      moreover from False n2 have P (n div 2) kvs by(rule IH)
       moreover from length-rbtreeify-f[OF n2] ge0 2.prems False obtain t k v
kvs'
        where kvs': rbtreeify-f (n div 2) kvs = (t, (k, v) # kvs')
        by(cases snd (rbtreeify-f (n div 2) kvs))
          (auto simp add: snd-def split: prod.split-asm, arith)
      moreover from 2.prems length-rbtreeify-f[OF n2] ge0 False
      have n2': n div 2 ≤ Suc (length kvs') by(simp add: kvs') arith
      moreover from False kvs'[symmetric] refl refl n2'
      have Q (n div 2) kvs' by(rule IH)
      moreover note godd[unfolded godd-def]
      ultimately have Q (Suc (2 * (n div 2))) kvs by −
      thus ?thesis using False
     by simp (metis One-nat-def Suc-eq-plus1-left le-add-diff-inverse mod-less-eq-dividend
minus-mod-eq-mult-div [symmetric])
   qed
  qed
 qed
qed

lemma inv1-rbtreeify-f: n ≤ length kvs
 ⟹ inv1 (fst (rbtreeify-f n kvs))
 and inv1-rbtreeify-g: n ≤ Suc (length kvs)
 ⟹ inv1 (fst (rbtreeify-g n kvs))
by(induct n kvs and n kvs rule: rbtreeify-induct) simp-all

fun plog2 :: nat ⇒ nat
where plog2 n = (if n ≤ 1 then 0 else plog2 (n div 2) + 1)

declare plog2.simps [simp del]

lemma plog2-simps [simp]:
 plog2 0 = 0 plog2 (Suc 0) = 0
 0 < n ⟹ plog2 (2 * n) = 1 + plog2 n
 0 < n ⟹ plog2 (Suc (2 * n)) = 1 + plog2 n
by(subst plog2.simps, simp add: Suc-double-half)+

lemma bheight-rbtreeify-f: n ≤ length kvs
 ⟹ bheight (fst (rbtreeify-f n kvs)) = plog2 n
 and bheight-rbtreeify-g: n ≤ Suc (length kvs)
 ⟹ bheight (fst (rbtreeify-g n kvs)) = plog2 n
```

by(*induct n kvs* **and** *n kvs rule*: *rbtreeify-induct*) *simp-all*

lemma *bheight-rbtreeify-f-eq-plog2I*:
⟦ *rbtreeify-f n kvs* = (*t*, *kvs'*); *n* ≤ *length kvs* ⟧
⟹ *bheight t* = *plog2 n*
using *bheight-rbtreeify-f*[*of n kvs*] **by** *simp*

lemma *bheight-rbtreeify-g-eq-plog2I*:
⟦ *rbtreeify-g n kvs* = (*t*, *kvs'*); *n* ≤ *Suc* (*length kvs*) ⟧
⟹ *bheight t* = *plog2 n*
using *bheight-rbtreeify-g*[*of n kvs*] **by** *simp*

hide-const (**open**) *plog2*

lemma *inv2-rbtreeify-f*: *n* ≤ *length kvs*
⟹ *inv2* (*fst* (*rbtreeify-f n kvs*))
and *inv2-rbtreeify-g*: *n* ≤ *Suc* (*length kvs*)
⟹ *inv2* (*fst* (*rbtreeify-g n kvs*))
by(*induct n kvs* **and** *n kvs rule*: *rbtreeify-induct*)
(*auto simp add*: *bheight-rbtreeify-f bheight-rbtreeify-g*
intro: *bheight-rbtreeify-f-eq-plog2I bheight-rbtreeify-g-eq-plog2I*)

lemma *n* ≤ *length kvs* ⟹ *True*
and *color-of-rbtreeify-g*:
⟦ *n* ≤ *Suc* (*length kvs*); *0* < *n* ⟧
⟹ *color-of* (*fst* (*rbtreeify-g n kvs*)) = *B*
by(*induct n kvs* **and** *n kvs rule*: *rbtreeify-induct*) *simp-all*

lemma *entries-rbtreeify-f-append*:
n ≤ *length kvs*
⟹ *entries* (*fst* (*rbtreeify-f n kvs*)) @ *snd* (*rbtreeify-f n kvs*) = *kvs*
and *entries-rbtreeify-g-append*:
n ≤ *Suc* (*length kvs*)
⟹ *entries* (*fst* (*rbtreeify-g n kvs*)) @ *snd* (*rbtreeify-g n kvs*) = *kvs*
by(*induction rule*: *rbtreeify-induct*) *simp-all*

lemma *length-entries-rbtreeify-f*:
n ≤ *length kvs* ⟹ *length* (*entries* (*fst* (*rbtreeify-f n kvs*))) = *n*
and *length-entries-rbtreeify-g*:
n ≤ *Suc* (*length kvs*) ⟹ *length* (*entries* (*fst* (*rbtreeify-g n kvs*))) = *n* − *1*
by(*induct rule*: *rbtreeify-induct*) *simp-all*

lemma *rbtreeify-f-conv-drop*:
n ≤ *length kvs* ⟹ *snd* (*rbtreeify-f n kvs*) = *drop n kvs*
using *entries-rbtreeify-f-append*[*of n kvs*]
by(*simp add*: *append-eq-conv-conj length-entries-rbtreeify-f*)

lemma *rbtreeify-g-conv-drop*:
n ≤ *Suc* (*length kvs*) ⟹ *snd* (*rbtreeify-g n kvs*) = *drop* (*n* − *1*) *kvs*

using *entries-rbtreeify-g-append*[*of n kvs*]
by(*simp add*: *append-eq-conv-conj length-entries-rbtreeify-g*)

lemma *entries-rbtreeify-f* [*simp*]:
n ≤ *length kvs* ⟹ *entries* (*fst* (*rbtreeify-f n kvs*)) = *take n kvs*
using *entries-rbtreeify-f-append*[*of n kvs*]
by(*simp add*: *append-eq-conv-conj length-entries-rbtreeify-f*)

lemma *entries-rbtreeify-g* [*simp*]:
n ≤ *Suc* (*length kvs*) ⟹
entries (*fst* (*rbtreeify-g n kvs*)) = *take* (*n* − *1*) *kvs*
using *entries-rbtreeify-g-append*[*of n kvs*]
by(*simp add*: *append-eq-conv-conj length-entries-rbtreeify-g*)

lemma *keys-rbtreeify-f* [*simp*]: *n* ≤ *length kvs*
⟹ *keys* (*fst* (*rbtreeify-f n kvs*)) = *take n* (*map fst kvs*)
by(*simp add*: *keys-def take-map*)

lemma *keys-rbtreeify-g* [*simp*]: *n* ≤ *Suc* (*length kvs*)
⟹ *keys* (*fst* (*rbtreeify-g n kvs*)) = *take* (*n* − *1*) (*map fst kvs*)
by(*simp add*: *keys-def take-map*)

lemma *rbtreeify-fD*:
⟦ *rbtreeify-f n kvs* = (*t*, *kvs'*); *n* ≤ *length kvs* ⟧
⟹ *entries t* = *take n kvs* ∧ *kvs'* = *drop n kvs*
using *rbtreeify-f-conv-drop*[*of n kvs*] *entries-rbtreeify-f*[*of n kvs*] **by** *simp*

lemma *rbtreeify-gD*:
⟦ *rbtreeify-g n kvs* = (*t*, *kvs'*); *n* ≤ *Suc* (*length kvs*) ⟧
⟹ *entries t* = *take* (*n* − *1*) *kvs* ∧ *kvs'* = *drop* (*n* − *1*) *kvs*
using *rbtreeify-g-conv-drop*[*of n kvs*] *entries-rbtreeify-g*[*of n kvs*] **by** *simp*

lemma *entries-rbtreeify* [*simp*]: *entries* (*rbtreeify kvs*) = *kvs*
by(*simp add*: *rbtreeify-def entries-rbtreeify-g*)

context *linorder* **begin**

lemma *rbt-sorted-rbtreeify-f*:
⟦ *n* ≤ *length kvs*; *sorted* (*map fst kvs*); *distinct* (*map fst kvs*) ⟧
⟹ *rbt-sorted* (*fst* (*rbtreeify-f n kvs*))
and *rbt-sorted-rbtreeify-g*:
⟦ *n* ≤ *Suc* (*length kvs*); *sorted* (*map fst kvs*); *distinct* (*map fst kvs*) ⟧
⟹ *rbt-sorted* (*fst* (*rbtreeify-g n kvs*))
proof(*induction n kvs* **and** *n kvs rule*: *rbtreeify-induct*)
case (*f-even n kvs t k v kvs'*)
from *rbtreeify-fD*[*OF* ‹*rbtreeify-f n kvs* = (*t*, (*k*, *v*) # *kvs'*)› ‹*n* ≤ *length kvs*›]
have *entries t* = *take n kvs*
and *kvs'*: *drop n kvs* = (*k*, *v*) # *kvs'* **by** *simp-all*
hence *unfold*: *kvs* = *take n kvs* @ (*k*, *v*) # *kvs'* **by**(*metis append-take-drop-id*)

from ‹*sorted* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* ≤ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* ≤ *x*)
by(*subst* (*asm*) *unfold*)(*auto simp add*: *sorted-append*)
moreover from ‹*distinct* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* ≠ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *x* ≠ *k*)
by(*subst* (*asm*) *unfold*)(*auto intro*: *rev-image-eqI*)
ultimately have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* < *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* < *x*)
by *fastforce*
hence *fst* (*rbtreeify-f n kvs*) |« *k k* «| *fst* (*rbtreeify-g n kvs'*)
using ‹*n* ≤ *Suc* (*length kvs'*)› ‹*n* ≤ *length kvs*› *set-take-subset*[*of n* − *1 kvs'*]
by(*auto simp add*: *ord.rbt-greater-prop ord.rbt-less-prop take-map split-def*)
moreover from ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
have *rbt-sorted* (*fst* (*rbtreeify-f n kvs*)) **by**(*rule f-even.IH*)
moreover have *sorted* (*map fst kvs'*) *distinct* (*map fst kvs'*)
using ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
by(*subst* (*asm*) (*1 2*) *unfold*, *simp add*: *sorted-append*)+
hence *rbt-sorted* (*fst* (*rbtreeify-g n kvs'*)) **by**(*rule f-even.IH*)
ultimately show *?case*
using ‹*0* < *n*› ‹*rbtreeify-f n kvs* = (*t*, (*k*, *v*) # *kvs'*)› **by** *simp*
next
case (*f-odd n kvs t k v kvs'*)
from *rbtreeify-fD*[*OF* ‹*rbtreeify-f n kvs* = (*t*, (*k*, *v*) # *kvs'*)› ‹*n* ≤ *length kvs*›]
have *entries t* = *take n kvs*
and *kvs'*: *drop n kvs* = (*k*, *v*) # *kvs'* **by** *simp-all*
hence *unfold*: *kvs* = *take n kvs* @ (*k*, *v*) # *kvs'* **by**(*metis append-take-drop-id*)
from ‹*sorted* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* ≤ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* ≤ *x*)
by(*subst* (*asm*) *unfold*)(*auto simp add*: *sorted-append*)
moreover from ‹*distinct* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* ≠ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *x* ≠ *k*)
by(*subst* (*asm*) *unfold*)(*auto intro*: *rev-image-eqI*)
ultimately have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* < *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* < *x*)
by *fastforce*
hence *fst* (*rbtreeify-f n kvs*) |« *k k* «| *fst* (*rbtreeify-f n kvs'*)
using ‹*n* ≤ *length kvs'*› ‹*n* ≤ *length kvs*› *set-take-subset*[*of n kvs'*]
by(*auto simp add*: *rbt-greater-prop rbt-less-prop take-map split-def*)
moreover from ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
have *rbt-sorted* (*fst* (*rbtreeify-f n kvs*)) **by**(*rule f-odd.IH*)
moreover have *sorted* (*map fst kvs'*) *distinct* (*map fst kvs'*)
using ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
by(*subst* (*asm*) (*1 2*) *unfold*, *simp add*: *sorted-append*)+
hence *rbt-sorted* (*fst* (*rbtreeify-f n kvs'*)) **by**(*rule f-odd.IH*)
ultimately show *?case*
using ‹*0* < *n*› ‹*rbtreeify-f n kvs* = (*t*, (*k*, *v*) # *kvs'*)› **by** *simp*
next
case (*g-even n kvs t k v kvs'*)
from *rbtreeify-gD*[*OF* ‹*rbtreeify-g n kvs* = (*t*, (*k*, *v*) # *kvs'*)› ‹*n* ≤ *Suc* (*length*

kvs)›]
have *t*: *entries t* = *take* (*n* − *1*) *kvs*
and *kvs'*: *drop* (*n* − *1*) *kvs* = (*k*, *v*) # *kvs'* **by** *simp-all*
hence *unfold*: *kvs* = *take* (*n* − *1*) *kvs* @ (*k*, *v*) # *kvs'* **by**(*metis append-take-drop-id*)
from ‹*sorted* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take* (*n* − *1*) *kvs*). *x* ≤ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* ≤ *x*)
by(*subst* (*asm*) *unfold*)(*auto simp add*: *sorted-append*)
moreover from ‹*distinct* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take* (*n* − *1*) *kvs*). *x* ≠ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *x* ≠ *k*)
by(*subst* (*asm*) *unfold*)(*auto intro*: *rev-image-eqI*)
ultimately have (∀(*x*, *y*) ∈ *set* (*take* (*n* − *1*) *kvs*). *x* < *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* < *x*)
by *fastforce*
hence *fst* (*rbtreeify-g n kvs*) |« *k k* «| *fst* (*rbtreeify-g n kvs'*)
using ‹*n* ≤ *Suc* (*length kvs'*)› ‹*n* ≤ *Suc* (*length kvs*)› *set-take-subset*[*of n* − *1 kvs'*]
by(*auto simp add*: *rbt-greater-prop rbt-less-prop take-map split-def*)
moreover from ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
have *rbt-sorted* (*fst* (*rbtreeify-g n kvs*)) **by**(*rule g-even.IH*)
moreover have *sorted* (*map fst kvs'*) *distinct* (*map fst kvs'*)
using ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
by(*subst* (*asm*) (*1 2*) *unfold*, *simp add*: *sorted-append*)+
hence *rbt-sorted* (*fst* (*rbtreeify-g n kvs'*)) **by**(*rule g-even.IH*)
ultimately show *?case* **using** ‹*0* < *n*› ‹*rbtreeify-g n kvs* = (*t*, (*k*, *v*) # *kvs'*)› **by** *simp*
next
case (*g-odd n kvs t k v kvs'*)
from *rbtreeify-fD*[*OF* ‹*rbtreeify-f n kvs* = (*t*, (*k*, *v*) # *kvs'*)› ‹*n* ≤ *length kvs*›]
have *entries t* = *take n kvs*
and *kvs'*: *drop n kvs* = (*k*, *v*) # *kvs'* **by** *simp-all*
hence *unfold*: *kvs* = *take n kvs* @ (*k*, *v*) # *kvs'* **by**(*metis append-take-drop-id*)
from ‹*sorted* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* ≤ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* ≤ *x*)
by(*subst* (*asm*) *unfold*)(*auto simp add*: *sorted-append*)
moreover from ‹*distinct* (*map fst kvs*)› *kvs'*
have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* ≠ *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *x* ≠ *k*)
by(*subst* (*asm*) *unfold*)(*auto intro*: *rev-image-eqI*)
ultimately have (∀(*x*, *y*) ∈ *set* (*take n kvs*). *x* < *k*) ∧ (∀(*x*, *y*) ∈ *set kvs'*. *k* < *x*)
by *fastforce*
hence *fst* (*rbtreeify-f n kvs*) |« *k k* «| *fst* (*rbtreeify-g n kvs'*)
using ‹*n* ≤ *Suc* (*length kvs'*)› ‹*n* ≤ *length kvs*› *set-take-subset*[*of n* − *1 kvs'*]
by(*auto simp add*: *rbt-greater-prop rbt-less-prop take-map split-def*)
moreover from ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
have *rbt-sorted* (*fst* (*rbtreeify-f n kvs*)) **by**(*rule g-odd.IH*)
moreover have *sorted* (*map fst kvs'*) *distinct* (*map fst kvs'*)
using ‹*sorted* (*map fst kvs*)› ‹*distinct* (*map fst kvs*)›
by(*subst* (*asm*) (*1 2*) *unfold*, *simp add*: *sorted-append*)+
hence *rbt-sorted* (*fst* (*rbtreeify-g n kvs'*)) **by**(*rule g-odd.IH*)

ultimately show *?case*
using ‹*0 < n*› ‹*rbtreeify-f n kvs = (t, (k, v) # kvs')*› **by** *simp*
qed *simp-all*

lemma *rbt-sorted-rbtreeify*:
⟦ *sorted* (*map fst kvs*); *distinct* (*map fst kvs*) ⟧ ⟹ *rbt-sorted* (*rbtreeify kvs*)
by(*simp add*: *rbtreeify-def rbt-sorted-rbtreeify-g*)

lemma *is-rbt-rbtreeify*:
⟦ *sorted* (*map fst kvs*); *distinct* (*map fst kvs*) ⟧
⟹ *is-rbt* (*rbtreeify kvs*)
by(*simp add*: *is-rbt-def rbtreeify-def inv1-rbtreeify-g inv2-rbtreeify-g rbt-sorted-rbtreeify-g color-of-rbtreeify-g*)

lemma *rbt-lookup-rbtreeify*:
⟦ *sorted* (*map fst kvs*); *distinct* (*map fst kvs*) ⟧ ⟹
rbt-lookup (*rbtreeify kvs*) = *map-of kvs*
by(*simp add*: *map-of-entries*[*symmetric*] *rbt-sorted-rbtreeify*)

end

Functions to compare the height of two rbt trees, taken from Andrew W. Appel, Efficient Verified Red-Black Trees (September 2011)

fun *skip-red* :: ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt*
where
skip-red (*Branch color.R l k v r*) = *l*
| *skip-red t* = *t*

definition *skip-black* :: ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt*
where
skip-black t = (*let t'* = *skip-red t in case t' of Branch color.B l k v r* ⇒ *l* | - ⇒ *t'*)

datatype *compare* = *LT* | *GT* | *EQ*

partial-function (*tailrec*) *compare-height* :: ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* ⇒ *compare*
where
compare-height sx s t tx =
(*case* (*skip-red sx*, *skip-red s*, *skip-red t*, *skip-red tx*) *of*
(*Branch* - *sx'* - - -, *Branch* - *s'* - - -, *Branch* - *t'* - - -, *Branch* - *tx'* - - -) ⇒
compare-height (*skip-black sx'*) *s' t'* (*skip-black tx'*)
| (-, *rbt.Empty*, -, *Branch* - - - - -) ⇒ *LT*
| (*Branch* - - - - -, -, *rbt.Empty*, -) ⇒ *GT*
| (*Branch* - *sx'* - - -, *Branch* - *s'* - - -, *Branch* - *t'* - - -, *rbt.Empty*) ⇒
compare-height (*skip-black sx'*) *s' t' rbt.Empty*
| (*rbt.Empty*, *Branch* - *s'* - - -, *Branch* - *t'* - - -, *Branch* - *tx'* - - -) ⇒
compare-height rbt.Empty s' t' (*skip-black tx'*)
| - ⇒ *EQ*)

declare *compare-height.simps* [*code*]

hide-type (**open**) *compare*
hide-const (**open**)
compare-height skip-black skip-red LT GT EQ case-compare rec-compare
Abs-compare Rep-compare
hide-fact (**open**)
Abs-compare-cases Abs-compare-induct Abs-compare-inject Abs-compare-inverse
Rep-compare Rep-compare-cases Rep-compare-induct Rep-compare-inject Rep-compare-inverse
compare.simps compare.exhaust compare.induct compare.rec compare.simps
compare.size compare.case-cong compare.case-cong-weak compare.case
compare.nchotomy compare.split compare.split-asm compare.eq.refl compare.eq.simps
equal-compare-def
skip-red.simps skip-red.cases skip-red.induct
skip-black-def
compare-height.simps

119.10 union and intersection of sorted associative lists

context *ord* **begin**

function *sunion-with* :: $('a \Rightarrow 'b \Rightarrow 'b \Rightarrow 'b) \Rightarrow ('a \times 'b)\ list \Rightarrow ('a \times 'b)\ list \Rightarrow ('a \times 'b)\ list$
where
sunion-with f ((k, v) # as) ((k', v') # bs) =
(if k > k' then (k', v') # sunion-with f ((k, v) # as) bs
else if k < k' then (k, v) # sunion-with f as ((k', v') # bs)
else (k, f k v v') # sunion-with f as bs)
| *sunion-with f [] bs = bs*
| *sunion-with f as [] = as*
by *pat-completeness auto*
termination by *lexicographic-order*

function *sinter-with* :: $('a \Rightarrow 'b \Rightarrow 'b \Rightarrow 'b) \Rightarrow ('a \times 'b)\ list \Rightarrow ('a \times 'b)\ list \Rightarrow ('a \times 'b)\ list$
where
sinter-with f ((k, v) # as) ((k', v') # bs) =
(if k > k' then sinter-with f ((k, v) # as) bs
else if k < k' then sinter-with f as ((k', v') # bs)
else (k, f k v v') # sinter-with f as bs)
| *sinter-with f [] - = []*
| *sinter-with f - [] = []*
by *pat-completeness auto*
termination by *lexicographic-order*

end

declare *ord.sunion-with.simps* [*code*] *ord.sinter-with.simps*[*code*]

context *linorder* **begin**

lemma *set-fst-sunion-with*:
set (*map fst* (*sunion-with f xs ys*)) = *set* (*map fst xs*) ∪ *set* (*map fst ys*)
by(*induct f xs ys rule*: *sunion-with.induct*) *auto*

lemma *sorted-sunion-with* [*simp*]:
⟦ *sorted* (*map fst xs*); *sorted* (*map fst ys*) ⟧
⟹ *sorted* (*map fst* (*sunion-with f xs ys*))
by(*induct f xs ys rule*: *sunion-with.induct*)
(*auto simp add*: *set-fst-sunion-with simp del*: *set-map*)

lemma *distinct-sunion-with* [*simp*]:
⟦ *distinct* (*map fst xs*); *distinct* (*map fst ys*); *sorted* (*map fst xs*); *sorted* (*map fst ys*) ⟧
⟹ *distinct* (*map fst* (*sunion-with f xs ys*))
proof(*induct f xs ys rule*: *sunion-with.induct*)
case (*1 f k v xs k' v' ys*)
have ⟦ ¬ *k* < *k'*; ¬ *k'* < *k* ⟧ ⟹ *k* = *k'* **by** *simp*
thus *?case* **using** *1*
by(*auto simp add*: *set-fst-sunion-with simp del*: *set-map*)
qed *simp-all*

lemma *map-of-sunion-with*:
⟦ *sorted* (*map fst xs*); *sorted* (*map fst ys*) ⟧
⟹ *map-of* (*sunion-with f xs ys*) *k* =
(*case map-of xs k of None* ⇒ *map-of ys k*
| *Some v* ⇒ *case map-of ys k of None* ⇒ *Some v*
| *Some w* ⇒ *Some* (*f k v w*))
by(*induct f xs ys rule*: *sunion-with.induct*)(*auto split*: *option.split dest*: *map-of-SomeD bspec*)

lemma *set-fst-sinter-with* [*simp*]:
⟦ *sorted* (*map fst xs*); *sorted* (*map fst ys*) ⟧
⟹ *set* (*map fst* (*sinter-with f xs ys*)) = *set* (*map fst xs*) ∩ *set* (*map fst ys*)
by(*induct f xs ys rule*: *sinter-with.induct*)(*auto simp del*: *set-map*)

lemma *set-fst-sinter-with-subset1*:
set (*map fst* (*sinter-with f xs ys*)) ⊆ *set* (*map fst xs*)
by(*induct f xs ys rule*: *sinter-with.induct*) *auto*

lemma *set-fst-sinter-with-subset2*:
set (*map fst* (*sinter-with f xs ys*)) ⊆ *set* (*map fst ys*)
by(*induct f xs ys rule*: *sinter-with.induct*)(*auto simp del*: *set-map*)

lemma *sorted-sinter-with* [*simp*]:
⟦ *sorted* (*map fst xs*); *sorted* (*map fst ys*) ⟧
⟹ *sorted* (*map fst* (*sinter-with f xs ys*))

by(*induct f xs ys rule*: *sinter-with.induct*)(*auto simp del*: *set-map*)

lemma *distinct-sinter-with* [*simp*]:
⟦ *distinct* (*map fst xs*); *distinct* (*map fst ys*) ⟧
⟹ *distinct* (*map fst* (*sinter-with f xs ys*))
proof(*induct f xs ys rule*: *sinter-with.induct*)
case (*1 f k v as k' v' bs*)
have ⟦ ¬ *k* < *k'*; ¬ *k'* < *k* ⟧ ⟹ *k* = *k'* **by** *simp*
thus *?case* **using** *1 set-fst-sinter-with-subset1*[*of f as bs*]
set-fst-sinter-with-subset2[*of f as bs*]
by(*auto simp del*: *set-map*)
qed *simp-all*

lemma *map-of-sinter-with*:
⟦ *sorted* (*map fst xs*); *sorted* (*map fst ys*) ⟧
⟹ *map-of* (*sinter-with f xs ys*) *k* =
(*case map-of xs k of None* ⇒ *None* | *Some v* ⇒ *map-option* (*f k v*) (*map-of ys k*))
apply(*induct f xs ys rule*: *sinter-with.induct*)
apply(*auto simp add*: *map-option-case split*: *option.splits dest*: *map-of-SomeD bspec*)
done

end

lemma *distinct-map-of-rev*: *distinct* (*map fst xs*) ⟹ *map-of* (*rev xs*) = *map-of xs*
by(*induct xs*)(*auto 4 3 simp add*: *map-add-def intro*!: *ext split*: *option.split intro*: *rev-image-eqI*)

lemma *map-map-filter*:
map f (*List.map-filter g xs*) = *List.map-filter* (*map-option f* ∘ *g*) *xs*
by(*auto simp add*: *List.map-filter-def*)

lemma *map-filter-map-option-const*:
List.map-filter (λ*x*. *map-option* (λ*y*. *f x*) (*g* (*f x*))) *xs* = *filter* (λ*x*. *g x* ≠ *None*) (*map f xs*)
by(*auto simp add*: *map-filter-def filter-map o-def*)

lemma *set-map-filter*: *set* (*List.map-filter P xs*) = *the* ‘ (*P* ‘ *set xs* − {*None*})
by(*auto simp add*: *List.map-filter-def intro*: *rev-image-eqI*)

definition *is-rbt-empty* :: ('*a*, '*b*) *rbt* ⇒ *bool* **where**
is-rbt-empty t ⟷ (*case t of RBT-Impl.Empty* ⇒ *True* | - ⇒ *False*)

lemma *is-rbt-empty-prop*[*simp*]: *is-rbt-empty t* ⟷ *t* = *RBT-Impl.Empty*
by (*auto simp*: *is-rbt-empty-def split*: *RBT-Impl.rbt.splits*)

definition *small-rbt* :: ('*a*, '*b*) *rbt* ⇒ *bool* **where**

small-rbt t $\longleftrightarrow$ *bheight t* $<$ *4*

definition *flip-rbt* :: (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ *bool* **where**
flip-rbt t1 t2 $\longleftrightarrow$ *bheight t2* $<$ *bheight t1*

abbreviation (*input*) *MR* **where** *MR l a b r* $\equiv$ *Branch RBT-Impl.R l a b r*
abbreviation (*input*) *MB* **where** *MB l a b r* $\equiv$ *Branch RBT-Impl.B l a b r*

fun *rbt-baliL* :: (*'a*, *'b*) *rbt* $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**
rbt-baliL (*MR* (*MR t1 a b t2*) *a' b' t3*) *a'' b'' t4* = *MR* (*MB t1 a b t2*) *a' b'* (*MB t3 a'' b'' t4*)
| *rbt-baliL* (*MR t1 a b* (*MR t2 a' b' t3*)) *a'' b'' t4* = *MR* (*MB t1 a b t2*) *a' b'* (*MB t3 a'' b'' t4*)
| *rbt-baliL t1 a b t2* = *MB t1 a b t2*

fun *rbt-baliR* :: (*'a*, *'b*) *rbt* $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**
rbt-baliR t1 a b (*MR t2 a' b'* (*MR t3 a'' b'' t4*)) = *MR* (*MB t1 a b t2*) *a' b'* (*MB t3 a'' b'' t4*)
| *rbt-baliR t1 a b* (*MR* (*MR t2 a' b' t3*) *a'' b'' t4*) = *MR* (*MB t1 a b t2*) *a' b'* (*MB t3 a'' b'' t4*)
| *rbt-baliR t1 a b t2* = *MB t1 a b t2*

fun *rbt-baldL* :: (*'a*, *'b*) *rbt* $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**
rbt-baldL (*MR t1 a b t2*) *a' b' t3* = *MR* (*MB t1 a b t2*) *a' b' t3*
| *rbt-baldL t1 a b* (*MB t2 a' b' t3*) = *rbt-baliR t1 a b* (*MR t2 a' b' t3*)
| *rbt-baldL t1 a b* (*MR* (*MB t2 a' b' t3*) *a'' b'' t4*) =
MR (*MB t1 a b t2*) *a' b'* (*rbt-baliR t3 a'' b''* (*paint RBT-Impl.R t4*))
| *rbt-baldL t1 a b t2* = *MR t1 a b t2*

fun *rbt-baldR* :: (*'a*, *'b*) *rbt* $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**
rbt-baldR t1 a b (*MR t2 a' b' t3*) = *MR t1 a b* (*MB t2 a' b' t3*)
| *rbt-baldR* (*MB t1 a b t2*) *a' b' t3* = *rbt-baliL* (*MR t1 a b t2*) *a' b' t3*
| *rbt-baldR* (*MR t1 a b* (*MB t2 a' b' t3*)) *a'' b'' t4* =
MR (*rbt-baliL* (*paint RBT-Impl.R t1*) *a b t2*) *a' b'* (*MB t3 a'' b'' t4*)
| *rbt-baldR t1 a b t2* = *MR t1 a b t2*

fun *rbt-app* :: (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**
rbt-app RBT-Impl.Empty t = *t*
| *rbt-app t RBT-Impl.Empty* = *t*
| *rbt-app* (*MR t1 a b t2*) (*MR t3 a'' b'' t4*) = (*case rbt-app t2 t3 of*
MR u2 a' b' u3 $\Rightarrow$ (*MR* (*MR t1 a b u2*) *a' b'* (*MR u3 a'' b'' t4*))
| *t23* $\Rightarrow$ *MR t1 a b* (*MR t23 a'' b'' t4*))
| *rbt-app* (*MB t1 a b t2*) (*MB t3 a'' b'' t4*) = (*case rbt-app t2 t3 of*
MR u2 a' b' u3 $\Rightarrow$ *MR* (*MB t1 a b u2*) *a' b'* (*MB u3 a'' b'' t4*)
| *t23* $\Rightarrow$ *rbt-baldL t1 a b* (*MB t23 a'' b'' t4*))
| *rbt-app t1* (*MR t2 a b t3*) = *MR* (*rbt-app t1 t2*) *a b t3*
| *rbt-app* (*MR t1 a b t2*) *t3* = *MR t1 a b* (*rbt-app t2 t3*)

fun *rbt-joinL* :: (*'a*, *'b*) *rbt* $\Rightarrow$ *'a* $\Rightarrow$ *'b* $\Rightarrow$ (*'a*, *'b*) *rbt* $\Rightarrow$ (*'a*, *'b*) *rbt* **where**

rbt-joinL l a b r = (if bheight l ≥ bheight r then MR l a b r
else case r of MB l′ a′ b′ r′ ⇒ rbt-baliL (rbt-joinL l a b l′) a′ b′ r′
| MR l′ a′ b′ r′ ⇒ MR (rbt-joinL l a b l′) a′ b′ r′)

declare *rbt-joinL.simps*[*simp del*]

fun *rbt-joinR* :: ('*a*, '*b*) *rbt* ⇒ '*a* ⇒ '*b* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* **where**
rbt-joinR l a b r = (if bheight l ≤ bheight r then MR l a b r
else case l of MB l′ a′ b′ r′ ⇒ rbt-baliR l′ a′ b′ (rbt-joinR r′ a b r)
| MR l′ a′ b′ r′ ⇒ MR l′ a′ b′ (rbt-joinR r′ a b r))

declare *rbt-joinR.simps*[*simp del*]

definition *rbt-join* :: ('*a*, '*b*) *rbt* ⇒ '*a* ⇒ '*b* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* **where**
rbt-join l a b r =
(let bhl = bheight l; *bhr = bheight r*
in if bhl > bhr
then paint RBT-Impl.B (rbt-joinR l a b r)
else if bhl < bhr
then paint RBT-Impl.B (rbt-joinL l a b r)
else MB l a b r)

lemma *size-paint*[*simp*]: *size (paint c t) = size t*
by (*cases t*) *auto*

lemma *size-baliL*[*simp*]: *size (rbt-baliL t1 a b t2) = Suc (size t1 + size t2)*
by (*induction t1 a b t2 rule*: *rbt-baliL.induct*) *auto*

lemma *size-baliR*[*simp*]: *size (rbt-baliR t1 a b t2) = Suc (size t1 + size t2)*
by (*induction t1 a b t2 rule*: *rbt-baliR.induct*) *auto*

lemma *size-baldL*[*simp*]: *size (rbt-baldL t1 a b t2) = Suc (size t1 + size t2)*
by (*induction t1 a b t2 rule*: *rbt-baldL.induct*) *auto*

lemma *size-baldR*[*simp*]: *size (rbt-baldR t1 a b t2) = Suc (size t1 + size t2)*
by (*induction t1 a b t2 rule*: *rbt-baldR.induct*) *auto*

lemma *size-rbt-app*[*simp*]: *size (rbt-app t1 t2) = size t1 + size t2*
by (*induction t1 t2 rule*: *rbt-app.induct*)
(*auto split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *size-rbt-joinL*[*simp*]: *size (rbt-joinL t1 a b t2) = Suc (size t1 + size t2)*
by (*induction t1 a b t2 rule*: *rbt-joinL.induct*)
(*auto simp*: *rbt-joinL.simps split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *size-rbt-joinR*[*simp*]: *size (rbt-joinR t1 a b t2) = Suc (size t1 + size t2)*
by (*induction t1 a b t2 rule*: *rbt-joinR.induct*)
(*auto simp*: *rbt-joinR.simps split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *size-rbt-join*[*simp*]: *size* (*rbt-join t1 a b t2*) = *Suc* (*size t1* + *size t2*)
by (*auto simp*: *rbt-join-def Let-def*)

definition *inv-12 t* ⟷ *inv1 t* ∧ *inv2 t*

lemma *rbt-Node*: *inv-12* (*RBT-Impl.Branch c l a b r*) ⟹ *inv-12 l* ∧ *inv-12 r*
by (*auto simp*: *inv-12-def*)

lemma *paint2*: *paint c2* (*paint c1 t*) = *paint c2 t*
by (*cases t*) *auto*

lemma *inv1-rbt-baliL*: *inv1l l* ⟹ *inv1 r* ⟹ *inv1* (*rbt-baliL l a b r*)
by (*induct l a b r rule*: *rbt-baliL.induct*) *auto*

lemma *inv1-rbt-baliR*: *inv1 l* ⟹ *inv1l r* ⟹ *inv1* (*rbt-baliR l a b r*)
by (*induct l a b r rule*: *rbt-baliR.induct*) *auto*

lemma *rbt-bheight-rbt-baliL*: *bheight l* = *bheight r* ⟹ *bheight* (*rbt-baliL l a b r*) = *Suc* (*bheight l*)
by (*induct l a b r rule*: *rbt-baliL.induct*) *auto*

lemma *rbt-bheight-rbt-baliR*: *bheight l* = *bheight r* ⟹ *bheight* (*rbt-baliR l a b r*) = *Suc* (*bheight l*)
by (*induct l a b r rule*: *rbt-baliR.induct*) *auto*

lemma *inv2-rbt-baliL*: *inv2 l* ⟹ *inv2 r* ⟹ *bheight l* = *bheight r* ⟹ *inv2* (*rbt-baliL l a b r*)
by (*induct l a b r rule*: *rbt-baliL.induct*) *auto*

lemma *inv2-rbt-baliR*: *inv2 l* ⟹ *inv2 r* ⟹ *bheight l* = *bheight r* ⟹ *inv2* (*rbt-baliR l a b r*)
by (*induct l a b r rule*: *rbt-baliR.induct*) *auto*

lemma *inv-rbt-baliR*: *inv2 l* ⟹ *inv2 r* ⟹ *inv1 l* ⟹ *inv1l r* ⟹ *bheight l* = *bheight r* ⟹
inv1 (*rbt-baliR l a b r*) ∧ *inv2* (*rbt-baliR l a b r*) ∧ *bheight* (*rbt-baliR l a b r*) = *Suc* (*bheight l*)
by (*induct l a b r rule*: *rbt-baliR.induct*) *auto*

lemma *inv-rbt-baliL*: *inv2 l* ⟹ *inv2 r* ⟹ *inv1l l* ⟹ *inv1 r* ⟹ *bheight l* = *bheight r* ⟹
inv1 (*rbt-baliL l a b r*) ∧ *inv2* (*rbt-baliL l a b r*) ∧ *bheight* (*rbt-baliL l a b r*) = *Suc* (*bheight l*)
by (*induct l a b r rule*: *rbt-baliL.induct*) *auto*

lemma *inv2-rbt-baldL-inv1*: *inv2 l* ⟹ *inv2 r* ⟹ *bheight l* + *1* = *bheight r* ⟹ *inv1 r* ⟹
inv2 (*rbt-baldL l a b r*) ∧ *bheight* (*rbt-baldL l a b r*) = *bheight r*
by (*induct l a b r rule*: *rbt-baldL.induct*) (*auto simp*: *inv2-rbt-baliR rbt-bheight-rbt-baliR*)

lemma *inv2-rbt-baldL-B*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l + 1 = bheight r* $\Longrightarrow$ *color-of r = RBT-Impl.B* $\Longrightarrow$
inv2 (*rbt-baldL l a b r*) $\wedge$ *bheight* (*rbt-baldL l a b r*) = *bheight r*
by (*induct l a b r rule*: *rbt-baldL.induct*) (*auto simp add*: *inv2-rbt-baliR rbt-bheight-rbt-baliR*)

lemma *inv1-rbt-baldL*: *inv1l l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$ *color-of r = RBT-Impl.B* $\Longrightarrow$ *inv1* (*rbt-baldL l a b r*)
by (*induct l a b r rule*: *rbt-baldL.induct*) (*simp-all add*: *inv1-rbt-baliR*)

lemma *inv1lI*: *inv1 t* $\Longrightarrow$ *inv1l t*
by (*cases t*) *auto*

lemma *neq-Black*[*simp*]: (*c* $\neq$ *RBT-Impl.B*) = (*c = RBT-Impl.R*)
by (*cases c*) *auto*

lemma *inv1l-rbt-baldL*: *inv1l l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$ *inv1l* (*rbt-baldL l a b r*)
by (*induct l a b r rule*: *rbt-baldL.induct*) (*auto simp*: *inv1-rbt-baliR paint2*)

lemma *inv2-rbt-baldR-inv1*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l = bheight r + 1* $\Longrightarrow$ *inv1 l* $\Longrightarrow$
inv2 (*rbt-baldR l a b r*) $\wedge$ *bheight* (*rbt-baldR l a b r*) = *bheight l*
by (*induct l a b r rule*: *rbt-baldR.induct*) (*auto simp*: *inv2-rbt-baliL rbt-bheight-rbt-baliL*)

lemma *inv1-rbt-baldR*: *inv1 l* $\Longrightarrow$ *inv1l r* $\Longrightarrow$ *color-of l = RBT-Impl.B* $\Longrightarrow$ *inv1* (*rbt-baldR l a b r*)
by (*induct l a b r rule*: *rbt-baldR.induct*) (*simp-all add*: *inv1-rbt-baliL*)

lemma *inv1l-rbt-baldR*: *inv1 l* $\Longrightarrow$ *inv1l r* $\Longrightarrow$*inv1l* (*rbt-baldR l a b r*)
by (*induct l a b r rule*: *rbt-baldR.induct*) (*auto simp*: *inv1-rbt-baliL paint2*)

lemma *inv2-rbt-app*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l = bheight r* $\Longrightarrow$
inv2 (*rbt-app l r*) $\wedge$ *bheight* (*rbt-app l r*) = *bheight l*
by (*induct l r rule*: *rbt-app.induct*)
(*auto simp*: *inv2-rbt-baldL-B split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *inv1-rbt-app*: *inv1 l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$ (*color-of l = RBT-Impl.B* $\wedge$
color-of r = RBT-Impl.B $\longrightarrow$ *inv1* (*rbt-app l r*)) $\wedge$ *inv1l* (*rbt-app l r*)
by (*induct l r rule*: *rbt-app.induct*)
(*auto simp*: *inv1-rbt-baldL split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *inv-rbt-baldL*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l + 1 = bheight r* $\Longrightarrow$ *inv1l l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$
inv2 (*rbt-baldL l a b r*) $\wedge$ *bheight* (*rbt-baldL l a b r*) = *bheight r* $\wedge$
inv1l (*rbt-baldL l a b r*) $\wedge$ (*color-of r = RBT-Impl.B* $\longrightarrow$ *inv1* (*rbt-baldL l a b r*))
by (*induct l a b r rule*: *rbt-baldL.induct*) (*auto simp*: *inv-rbt-baliR rbt-bheight-rbt-baliR paint2*)

lemma *inv-rbt-baldR*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l* = *bheight r* + *1* $\Longrightarrow$ *inv1 l* $\Longrightarrow$ *inv1l r* $\Longrightarrow$
inv2 (*rbt-baldR l a b r*) $\wedge$ *bheight* (*rbt-baldR l a b r*) = *bheight l* $\wedge$
inv1l (*rbt-baldR l a b r*) $\wedge$ (*color-of l* = *RBT-Impl.B* $\longrightarrow$ *inv1* (*rbt-baldR l a b r*))
by (*induct l a b r rule*: *rbt-baldR.induct*) (*auto simp*: *inv-rbt-baliL rbt-bheight-rbt-baliL paint2*)

lemma *inv-rbt-app*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l* = *bheight r* $\Longrightarrow$ *inv1 l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$
inv2 (*rbt-app l r*) $\wedge$ *bheight* (*rbt-app l r*) = *bheight l* $\wedge$
inv1l (*rbt-app l r*) $\wedge$ (*color-of l* = *RBT-Impl.B* $\wedge$ *color-of r* = *RBT-Impl.B* $\longrightarrow$ *inv1* (*rbt-app l r*))
by (*induct l r rule*: *rbt-app.induct*)
(*auto simp*: *inv2-rbt-baldL-B inv-rbt-baldL split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *inv1l-rbt-joinL*: *inv1 l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$ *bheight l* $\leq$ *bheight r* $\Longrightarrow$
inv1l (*rbt-joinL l a b r*) $\wedge$
(*bheight l* $\neq$ *bheight r* $\wedge$ *color-of r* = *RBT-Impl.B* $\longrightarrow$ *inv1* (*rbt-joinL l a b r*))
proof (*induct l a b r rule*: *rbt-joinL.induct*)
case (*1 l a b r*)
then show *?case*
by (*auto simp*: *inv1-rbt-baliL rbt-joinL.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *inv1l-rbt-joinR*: *inv1 l* $\Longrightarrow$ *inv2 l* $\Longrightarrow$ *inv1 r* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l* $\geq$ *bheight r* $\Longrightarrow$
inv1l (*rbt-joinR l a b r*) $\wedge$
(*bheight l* $\neq$ *bheight r* $\wedge$ *color-of l* = *RBT-Impl.B* $\longrightarrow$ *inv1* (*rbt-joinR l a b r*))
proof (*induct l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
then show *?case*
by (*fastforce simp*: *inv1-rbt-baliR rbt-joinR.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *bheight-rbt-joinL*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l* $\leq$ *bheight r* $\Longrightarrow$
bheight (*rbt-joinL l a b r*) = *bheight r*
proof (*induct l a b r rule*: *rbt-joinL.induct*)
case (*1 l a b r*)
then show *?case*
by (*auto simp*: *rbt-bheight-rbt-baliL rbt-joinL.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *inv2-rbt-joinL*: *inv2 l* $\Longrightarrow$ *inv2 r* $\Longrightarrow$ *bheight l* $\leq$ *bheight r* $\Longrightarrow$ *inv2* (*rbt-joinL l a b r*)
proof (*induct l a b r rule*: *rbt-joinL.induct*)

case (*1 l a b r*)
then show *?case*
by (*auto simp*: *inv2-rbt-baliL bheight-rbt-joinL rbt-joinL.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *bheight-rbt-joinR*: *inv2 l* ⟹ *inv2 r* ⟹ *bheight l* ≥ *bheight r* ⟹
bheight (*rbt-joinR l a b r*) = *bheight l*
proof (*induct l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
then show *?case*
by (*fastforce simp*: *rbt-bheight-rbt-baliR rbt-joinR.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *inv2-rbt-joinR*: *inv2 l* ⟹ *inv2 r* ⟹ *bheight l* ≥ *bheight r* ⟹ *inv2*
(*rbt-joinR l a b r*)
proof (*induct l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
then show *?case*
by (*fastforce simp*: *inv2-rbt-baliR bheight-rbt-joinR rbt-joinR.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *keys-paint*[*simp*]: *RBT-Impl.keys* (*paint c t*) = *RBT-Impl.keys t*
by (*cases t*) *auto*

lemma *keys-rbt-baliL*: *RBT-Impl.keys* (*rbt-baliL l a b r*) = *RBT-Impl.keys l* @ *a*
RBT-Impl.keys r
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliL.cases*) *auto*

lemma *keys-rbt-baliR*: *RBT-Impl.keys* (*rbt-baliR l a b r*) = *RBT-Impl.keys l* @ *a*
RBT-Impl.keys r
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliR.cases*) *auto*

lemma *keys-rbt-baldL*: *RBT-Impl.keys* (*rbt-baldL l a b r*) = *RBT-Impl.keys l* @ *a*
RBT-Impl.keys r
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baldL.cases*) (*auto simp*: *keys-rbt-baliL keys-rbt-baliR*)

lemma *keys-rbt-baldR*: *RBT-Impl.keys* (*rbt-baldR l a b r*) = *RBT-Impl.keys l* @ *a*
RBT-Impl.keys r
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baldR.cases*) (*auto simp*: *keys-rbt-baliL keys-rbt-baliR*)

lemma *keys-rbt-app*: *RBT-Impl.keys* (*rbt-app l r*) = *RBT-Impl.keys l* @ *RBT-Impl.keys*
r
by (*induction l r rule*: *rbt-app.induct*)
(*auto simp*: *keys-rbt-baldL keys-rbt-baldR split*: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)

lemma *keys-rbt-joinL*: *bheight l* ≤ *bheight r* ⟹

RBT-Impl.keys (rbt-joinL l a b r) = RBT-Impl.keys l @ a # RBT-Impl.keys r
proof (*induction l a b r rule*: *rbt-joinL.induct*)
case (*1 l a b r*)
thus *?case*
by (*auto simp*: *keys-rbt-baliL rbt-joinL.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *keys-rbt-joinR*: *RBT-Impl.keys (rbt-joinR l a b r) = RBT-Impl.keys l @ a # RBT-Impl.keys r*
proof (*induction l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
thus *?case*
by (*force simp*: *keys-rbt-baliR rbt-joinR.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *rbt-set-rbt-baliL*: *set (RBT-Impl.keys (rbt-baliL l a b r)) =*
set (RBT-Impl.keys l) ∪ {a} ∪ set (RBT-Impl.keys r)
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliL.cases*) *auto*

lemma *set-rbt-joinL*: *set (RBT-Impl.keys (rbt-joinL l a b r)) =*
set (RBT-Impl.keys l) ∪ {a} ∪ set (RBT-Impl.keys r)
proof (*induction l a b r rule*: *rbt-joinL.induct*)
case (*1 l a b r*)
thus *?case*
by (*auto simp*: *rbt-set-rbt-baliL rbt-joinL.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *rbt-set-rbt-baliR*: *set (RBT-Impl.keys (rbt-baliR l a b r)) =*
set (RBT-Impl.keys l) ∪ {a} ∪ set (RBT-Impl.keys r)
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliR.cases*) *auto*

lemma *set-rbt-joinR*: *set (RBT-Impl.keys (rbt-joinR l a b r)) =*
set (RBT-Impl.keys l) ∪ {a} ∪ set (RBT-Impl.keys r)
proof (*induction l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
thus *?case*
by (*force simp*: *rbt-set-rbt-baliR rbt-joinR.simps*[*of l a b r*]
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *set-keys-paint*: *set (RBT-Impl.keys (paint c t)) = set (RBT-Impl.keys t)*
by (*cases t*) *auto*

lemma *set-rbt-join*: *set (RBT-Impl.keys (rbt-join l a b r)) =*
set (RBT-Impl.keys l) ∪ {a} ∪ set (RBT-Impl.keys r)
by (*simp add*: *set-rbt-joinL set-rbt-joinR set-keys-paint rbt-join-def Let-def*)

lemma *inv-rbt-join*: *inv-12 l* $\Longrightarrow$ *inv-12 r* $\Longrightarrow$ *inv-12* (*rbt-join l a b r*)
by (*auto simp*: *rbt-join-def Let-def inv1l-rbt-joinL inv1l-rbt-joinR inv2-rbt-joinL inv2-rbt-joinR inv-12-def*)

fun *rbt-recolor* :: ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* **where**
rbt-recolor (*Branch RBT-Impl.R t1 k v t2*) =
(*if color-of t1* = *RBT-Impl.B* $\wedge$ *color-of t2* = *RBT-Impl.B then Branch RBT-Impl.B t1 k v t2*
else Branch RBT-Impl.R t1 k v t2)
| *rbt-recolor t* = *t*

lemma *rbt-recolor*: *inv-12 t* $\Longrightarrow$ *inv-12* (*rbt-recolor t*)
by (*induction t rule*: *rbt-recolor.induct*) (*auto simp*: *inv-12-def*)

fun *rbt-split-min* :: ('*a*, '*b*) *rbt* $\Rightarrow$ '*a* $\times$ '*b* $\times$ ('*a*, '*b*) *rbt* **where**
rbt-split-min RBT-Impl.Empty = *undefined*
| *rbt-split-min* (*RBT-Impl.Branch - l a b r*) =
(*if is-rbt-empty l then* (*a*,*b*,*r*) *else let* (*a*',*b*',*l*') = *rbt-split-min l in* (*a*',*b*',*rbt-join l*' *a b r*))

lemma *rbt-split-min-set*:
rbt-split-min t = (*a*,*b*,*t*') $\Longrightarrow$ *t* $\neq$ *RBT-Impl.Empty* $\Longrightarrow$
a $\in$ *set* (*RBT-Impl.keys t*) $\wedge$ *set* (*RBT-Impl.keys t*) = {*a*} $\cup$ *set* (*RBT-Impl.keys t*')
by (*induction t arbitrary*: *t*') (*auto simp*: *set-rbt-join split*: *prod.splits if-splits*)

lemma *rbt-split-min-inv*: *rbt-split-min t* = (*a*,*b*,*t*') $\Longrightarrow$ *inv-12 t* $\Longrightarrow$ *t* $\neq$ *RBT-Impl.Empty* $\Longrightarrow$ *inv-12 t*'
by (*induction t arbitrary*: *t*')
(*auto simp*: *inv-rbt-join split*: *if-splits prod.splits dest*: *rbt-Node*)

definition *rbt-join2* :: ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* **where**
rbt-join2 l r = (*if is-rbt-empty r then l else let* (*a*,*b*,*r*') = *rbt-split-min r in rbt-join l a b r*')

lemma *set-rbt-join2*[*simp*]: *set* (*RBT-Impl.keys* (*rbt-join2 l r*)) =
set (*RBT-Impl.keys l*) $\cup$ *set* (*RBT-Impl.keys r*)
by (*simp add*: *rbt-join2-def rbt-split-min-set set-rbt-join split*: *prod.split*)

lemma *inv-rbt-join2*: *inv-12 l* $\Longrightarrow$ *inv-12 r* $\Longrightarrow$ *inv-12* (*rbt-join2 l r*)
by (*simp add*: *rbt-join2-def inv-rbt-join rbt-split-min-set rbt-split-min-inv split*: *prod.split*)

context *ord* **begin**

fun *rbt-split* :: ('*a*, '*b*) *rbt* $\Rightarrow$ '*a* $\Rightarrow$ ('*a*, '*b*) *rbt* $\times$ '*b option* $\times$ ('*a*, '*b*) *rbt* **where**
rbt-split RBT-Impl.Empty k = (*RBT-Impl.Empty*, *None*, *RBT-Impl.Empty*)
| *rbt-split* (*RBT-Impl.Branch - l a b r*) *x* =

(*if* $x < a$ *then* (*case rbt-split l x of* ($l1$, β, $l2$) $\Rightarrow$ ($l1$, β, *rbt-join l2 a b r*))
else if $a < x$ *then* (*case rbt-split r x of* ($r1$, β, $r2$) $\Rightarrow$ (*rbt-join l a b r1*, β, $r2$))
else (l, *Some b*, r))

lemma *rbt-split*: *rbt-split t x* = (l,β,r) $\Longrightarrow$ *inv-12 t* $\Longrightarrow$ *inv-12 l* $\wedge$ *inv-12 r*
by (*induction t arbitrary*: *l r*)
(*auto simp*: *set-rbt-join inv-rbt-join rbt-greater-prop rbt-less-prop*
split: *if-splits prod.splits dest*!: *rbt-Node*)

lemma *rbt-split-size*: ($l2$,β,$r2$) = *rbt-split t2 a* $\Longrightarrow$ *size l2* + *size r2* $\leq$ *size t2*
by (*induction t2 a arbitrary*: *l2 r2 rule*: *rbt-split.induct*) (*auto split*: *if-splits prod.splits*)

function *rbt-union-rec* :: ('*a* $\Rightarrow$ '*b* $\Rightarrow$ '*b* $\Rightarrow$ '*b*) $\Rightarrow$ ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* **where**
rbt-union-rec f t1 t2 = (*let* (f, *t2*, *t1*) =
if flip-rbt t2 t1 then ($\lambda k\ v\ v'$. $f\ k\ v'\ v$, *t1*, *t2*) *else* (f, *t2*, *t1*) *in*
if small-rbt t2 then RBT-Impl.fold (*rbt-insert-with-key f*) *t2 t1*
else (*case t1 of RBT-Impl.Empty* $\Rightarrow$ *t2*
| *RBT-Impl.Branch - l1 a b r1* $\Rightarrow$
case rbt-split t2 a of ($l2$, β, $r2$) $\Rightarrow$
rbt-join (*rbt-union-rec f l1 l2*) *a* (*case* β *of None* $\Rightarrow$ *b* | *Some b'* $\Rightarrow$ *f a b b'*) (*rbt-union-rec f r1 r2*)))
by *pat-completeness auto*
termination
using *rbt-split-size*
by (*relation measure* ($\lambda(f,t1,t2)$. *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-union-rec.simps*[*simp del*]

function *rbt-union-swap-rec* :: ('*a* $\Rightarrow$ '*b* $\Rightarrow$ '*b* $\Rightarrow$ '*b*) $\Rightarrow$ *bool* $\Rightarrow$ ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* $\Rightarrow$ ('*a*, '*b*) *rbt* **where**
rbt-union-swap-rec f γ *t1 t2* = (*let* (γ, *t2*, *t1*) =
if flip-rbt t2 t1 then ($\neg\gamma$, *t1*, *t2*) *else* (γ, *t2*, *t1*);
f' = (*if* γ *then* ($\lambda k\ v\ v'$. $f\ k\ v'\ v$) *else f*) *in*
if small-rbt t2 then RBT-Impl.fold (*rbt-insert-with-key f'*) *t2 t1*
else (*case t1 of RBT-Impl.Empty* $\Rightarrow$ *t2*
| *RBT-Impl.Branch - l1 a b r1* $\Rightarrow$
case rbt-split t2 a of ($l2$, β, $r2$) $\Rightarrow$
rbt-join (*rbt-union-swap-rec f* γ *l1 l2*) *a* (*case* β *of None* $\Rightarrow$ *b* | *Some b'* $\Rightarrow$ *f' a b b'*) (*rbt-union-swap-rec f* γ *r1 r2*)))
by *pat-completeness auto*
termination
using *rbt-split-size*
by (*relation measure* ($\lambda(f,\gamma,t1,t2)$. *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-union-swap-rec.simps*[*simp del*]

lemma *rbt-union-swap-rec*: *rbt-union-swap-rec f* γ *t1 t2* =

```
  rbt-union-rec (if γ then (λk v v'. f k v' v) else f) t1 t2
proof (induction f γ t1 t2 rule: rbt-union-swap-rec.induct)
  case (1 f γ t1 t2)
  show ?case
    using 1[OF refl - refl refl - refl - refl]
    unfolding rbt-union-swap-rec.simps[of - - t1] rbt-union-rec.simps[of - t1]
    by (auto simp: Let-def split: rbt.splits prod.splits option.splits)
qed

lemma rbt-fold-rbt-insert:
  assumes inv-12 t2
  shows inv-12 (RBT-Impl.fold (rbt-insert-with-key f) t1 t2)
proof −
  define xs where xs = RBT-Impl.entries t1
  from assms show ?thesis
    unfolding RBT-Impl.fold-def xs-def[symmetric]
    by (induct xs rule: rev-induct)
       (auto simp: inv-12-def rbt-insert-with-key-def ins-inv1-inv2)
qed

lemma rbt-union-rec: inv-12 t1 ⟹ inv-12 t2 ⟹ inv-12 (rbt-union-rec f t1 t2)
proof (induction f t1 t2 rule: rbt-union-rec.induct)
  case (1 t1 t2)
  thus ?case
   by (auto simp: rbt-union-rec.simps[of t1 t2] inv-rbt-join rbt-split rbt-fold-rbt-insert
         split!: RBT-Impl.rbt.splits RBT-Impl.color.splits prod.split if-splits dest:
rbt-Node)
qed

definition map-filter-inter f t1 t2 = List.map-filter (λ(k, v).
  case rbt-lookup t1 k of None ⇒ None
  | Some v' ⇒ Some (k, f k v' v)) (RBT-Impl.entries t2)

function rbt-inter-rec :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a, 'b) rbt ⇒ ('a, 'b) rbt ⇒ ('a,
'b) rbt where
  rbt-inter-rec f t1 t2 = (let (f, t2, t1) =
    if flip-rbt t2 t1 then (λk v v'. f k v' v, t1, t2) else (f, t2, t1) in
    if small-rbt t2 then rbtreeify (map-filter-inter f t1 t2)
    else case t1 of RBT-Impl.Empty ⇒ RBT-Impl.Empty
    | RBT-Impl.Branch - l1 a b r1 ⇒
     case rbt-split t2 a of (l2, β, r2) ⇒ let l' = rbt-inter-rec f l1 l2; r' = rbt-inter-rec
f r1 r2 in
      (case β of None ⇒ rbt-join2 l' r' | Some b' ⇒ rbt-join l' a (f a b b') r'))
  by pat-completeness auto
termination
  using rbt-split-size
  by (relation measure (λ(f,t1,t2). size t1 + size t2)) (fastforce split: if-splits)+

declare rbt-inter-rec.simps[simp del]
```

function *rbt-inter-swap-rec* :: ('*a* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ *bool* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* **where**
rbt-inter-swap-rec f γ t1 t2 = (*let* (*γ*, *t2*, *t1*) =
if flip-rbt t2 t1 then (¬*γ*, *t1*, *t2*) *else* (*γ*, *t2*, *t1*);
f' = (*if γ then* (λ*k v v'*. *f k v' v*) *else f*) *in*
if small-rbt t2 then rbtreeify (*map-filter-inter f' t1 t2*)
else case t1 of RBT-Impl.Empty ⇒ *RBT-Impl.Empty*
| *RBT-Impl.Branch - l1 a b r1* ⇒
case rbt-split t2 a of (*l2*, *β*, *r2*) ⇒ *let l'* = *rbt-inter-swap-rec f γ l1 l2*; *r'* = *rbt-inter-swap-rec f γ r1 r2 in*
(*case β of None* ⇒ *rbt-join2 l' r'* | *Some b'* ⇒ *rbt-join l' a* (*f' a b b'*) *r'*))
by *pat-completeness auto*
termination
using *rbt-split-size*
by (*relation measure* (λ(*f*,*γ*,*t1*,*t2*). *size t1* + *size t2*)) (*fastforce split*: *if-splits*)+

declare *rbt-inter-swap-rec.simps*[*simp del*]

lemma *rbt-inter-swap-rec*: *rbt-inter-swap-rec f γ t1 t2* =
rbt-inter-rec (*if γ then* (λ*k v v'*. *f k v' v*) *else f*) *t1 t2*
proof (*induction f γ t1 t2 rule*: *rbt-inter-swap-rec.induct*)
case (*1 f γ t1 t2*)
show *?case*
using *1*[*OF refl - refl refl - refl - refl*]
unfolding *rbt-inter-swap-rec.simps*[*of - - t1*] *rbt-inter-rec.simps*[*of - t1*]
by (*auto simp add*: *Let-def split*: *rbt.splits prod.splits option.splits*)
qed

lemma *rbt-rbtreeify*[*simp*]: *inv-12* (*rbtreeify kvs*)
by (*simp add*: *inv-12-def rbtreeify-def inv1-rbtreeify-g inv2-rbtreeify-g*)

lemma *rbt-inter-rec*: *inv-12 t1* ⟹ *inv-12 t2* ⟹ *inv-12* (*rbt-inter-rec f t1 t2*)
proof(*induction f t1 t2 rule*: *rbt-inter-rec.induct*)
case (*1 t1 t2*)
thus *?case*
by (*auto simp*: *rbt-inter-rec.simps*[*of t1 t2*] *inv-rbt-join inv-rbt-join2 rbt-split Let-def*
split!: *RBT-Impl.rbt.splits RBT-Impl.color.splits prod.split if-splits*
option.splits dest!: *rbt-Node*)
qed

definition *filter-minus t1 t2* = *filter* (λ(*k*, -). *rbt-lookup t2 k* = *None*) (*RBT-Impl.entries t1*)

fun *rbt-minus-rec* :: ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* **where**
rbt-minus-rec t1 t2 = (*if small-rbt t2 then RBT-Impl.fold* (λ*k* - *t*. *rbt-delete k t*) *t2 t1*
else if small-rbt t1 then rbtreeify (*filter-minus t1 t2*)

else case t2 of RBT-Impl.Empty ⇒ t1
| RBT-Impl.Branch - l2 a b r2 ⇒
case rbt-split t1 a of (l1, -, r1) ⇒ rbt-join2 (rbt-minus-rec l1 l2) (rbt-minus-rec r1 r2))

declare *rbt-minus-rec.simps*[*simp del*]

end

context *linorder* **begin**

lemma *rbt-sorted-entries-right-unique*:
⟦ (*k*, *v*) ∈ *set* (*entries t*); (*k*, *v′*) ∈ *set* (*entries t*);
rbt-sorted t ⟧ ⟹ *v* = *v′*
by(*auto dest*!: *distinct-entries inj-onD*[**where** *x*=(*k*, *v*) **and** *y*=(*k*, *v′*)] *simp add*: *distinct-map*)

lemma *rbt-sorted-fold-rbt-insertwk*:
rbt-sorted t ⟹ *rbt-sorted* (*List.fold* (λ(*k*, *v*). *rbt-insert-with-key f k v*) *xs t*)
by(*induct xs rule*: *rev-induct*)(*auto simp add*: *rbt-insertwk-rbt-sorted*)

lemma *is-rbt-fold-rbt-insertwk*:
assumes *is-rbt t1*
shows *is-rbt* (*fold* (*rbt-insert-with-key f*) *t2 t1*)
proof −
define *xs* **where** *xs* = *entries t2*
from *assms* **show** *?thesis* **unfolding** *fold-def xs-def*[*symmetric*]
by(*induct xs rule*: *rev-induct*)(*auto simp add*: *rbt-insertwk-is-rbt*)
qed

lemma *rbt-delete*: *inv-12 t* ⟹ *inv-12* (*rbt-delete x t*)
using *rbt-del-inv1-inv2*[*of t x*]
by (*auto simp*: *inv-12-def rbt-delete-def rbt-del-inv1-inv2*)

lemma *rbt-sorted-delete*: *rbt-sorted t* ⟹ *rbt-sorted* (*rbt-delete x t*)
by (*auto simp*: *rbt-delete-def rbt-del-rbt-sorted*)

lemma *rbt-fold-rbt-delete*:
assumes *inv-12 t2*
shows *inv-12* (*RBT-Impl.fold* (λ*k* - *t*. *rbt-delete k t*) *t1 t2*)
proof −
define *xs* **where** *xs* = *RBT-Impl.entries t1*
from *assms* **show** *?thesis*
unfolding *RBT-Impl.fold-def xs-def*[*symmetric*]
by (*induct xs rule*: *rev-induct*) (*auto simp*: *rbt-delete*)
qed

lemma *rbt-minus-rec*: *inv-12 t1* ⟹ *inv-12 t2* ⟹ *inv-12* (*rbt-minus-rec t1 t2*)
proof(*induction t1 t2 rule*: *rbt-minus-rec.induct*)

case (*1 t1 t2*)
thus *?case*
by (*auto simp*: *rbt-minus-rec.simps*[*of t1 t2*] *inv-rbt-join inv-rbt-join2 rbt-split rbt-fold-rbt-delete split*!: *RBT-Impl.rbt.splits RBT-Impl.color.splits prod.split if-splits dest*: *rbt-Node*)
qed

end

context *linorder* **begin**

lemma *rbt-sorted-rbt-baliL*: *rbt-sorted l* $\Longrightarrow$ *rbt-sorted r* $\Longrightarrow$ *l* |« *a* $\Longrightarrow$ *a* «| *r* $\Longrightarrow$ *rbt-sorted* (*rbt-baliL l a b r*)
using *rbt-greater-trans rbt-less-trans*
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliL.cases*) *fastforce*+

lemma *rbt-lookup-rbt-baliL*: *rbt-sorted l* $\Longrightarrow$ *rbt-sorted r* $\Longrightarrow$ *l* |« *a* $\Longrightarrow$ *a* «| *r* $\Longrightarrow$ *rbt-lookup* (*rbt-baliL l a b r*) *k* = (*if k* < *a then rbt-lookup l k else if k* = *a then Some b else rbt-lookup r k*)
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliL.cases*) (*auto split*!: *if-splits*)

lemma *rbt-sorted-rbt-baliR*: *rbt-sorted l* $\Longrightarrow$ *rbt-sorted r* $\Longrightarrow$ *l* |« *a* $\Longrightarrow$ *a* «| *r* $\Longrightarrow$ *rbt-sorted* (*rbt-baliR l a b r*)
using *rbt-greater-trans rbt-less-trans*
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliR.cases*) *fastforce*+

lemma *rbt-lookup-rbt-baliR*: *rbt-sorted l* $\Longrightarrow$ *rbt-sorted r* $\Longrightarrow$ *l* |« *a* $\Longrightarrow$ *a* «| *r* $\Longrightarrow$ *rbt-lookup* (*rbt-baliR l a b r*) *k* = (*if k* < *a then rbt-lookup l k else if k* = *a then Some b else rbt-lookup r k*)
by (*cases* (*l*,*a*,*b*,*r*) *rule*: *rbt-baliR.cases*) (*auto split*!: *if-splits*)

lemma *rbt-sorted-rbt-joinL*: *rbt-sorted* (*RBT-Impl.Branch c l a b r*) $\Longrightarrow$ *bheight l* $\leq$ *bheight r* $\Longrightarrow$ *rbt-sorted* (*rbt-joinL l a b r*)
proof (*induction l a b r arbitrary*: *c rule*: *rbt-joinL.induct*)
case (*1 l a b r*)
thus *?case*
by (*auto simp*: *rbt-set-rbt-baliL rbt-joinL.simps*[*of l a b r*] *set-rbt-joinL rbt-less-prop intro*!: *rbt-sorted-rbt-baliL split*!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *rbt-lookup-rbt-joinL*: *rbt-sorted l* $\Longrightarrow$ *rbt-sorted r* $\Longrightarrow$ *l* |« *a* $\Longrightarrow$ *a* «| *r* $\Longrightarrow$ *rbt-lookup* (*rbt-joinL l a b r*) *k* = (*if k* < *a then rbt-lookup l k else if k* = *a then Some b else rbt-lookup r k*)
proof (*induction l a b r rule*: *rbt-joinL.induct*)
case (*1 l a b r*)
have *less-rbt-joinL*:
rbt-sorted r1 $\Longrightarrow$ *r1* |« *x* $\Longrightarrow$ *a* «| *r1* $\Longrightarrow$ *a* < *x* $\Longrightarrow$ *rbt-joinL l a b r1* |« *x* **for**

x r1
using *1(5)*
by (*auto simp*: *rbt-less-prop rbt-greater-prop set-rbt-joinL*)
show *?case*
using *1 less-rbt-joinL rbt-lookup-rbt-baliL*[*OF rbt-sorted-rbt-joinL*[*of - l a b*],
where *?k=k*]
by (*auto simp*: *rbt-joinL.simps*[*of l a b r*] *split*!: *if-splits rbt.splits color.splits*)
qed

lemma *rbt-sorted-rbt-joinR*: *rbt-sorted l* ⟹ *rbt-sorted r* ⟹ *l* |« *a* ⟹ *a* «| *r* ⟹
rbt-sorted (*rbt-joinR l a b r*)
proof (*induction l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
thus *?case*
by (*auto simp*: *rbt-set-rbt-baliR rbt-joinR.simps*[*of l a b r*] *set-rbt-joinR rbt-greater-prop*
intro!: *rbt-sorted-rbt-baliR split*!: *RBT-Impl.rbt.splits RBT-Impl.color.splits*)
qed

lemma *rbt-lookup-rbt-joinR*: *rbt-sorted l* ⟹ *rbt-sorted r* ⟹ *l* |« *a* ⟹ *a* «| *r*
⟹
rbt-lookup (*rbt-joinR l a b r*) *k* =
(*if k* < *a then rbt-lookup l k else if k* = *a then Some b else rbt-lookup r k*)
proof (*induction l a b r rule*: *rbt-joinR.induct*)
case (*1 l a b r*)
have *less-rbt-joinR*:
rbt-sorted l1 ⟹ *x* «| *l1* ⟹ *l1* |« *a* ⟹ *x* < *a* ⟹ *x* «| *rbt-joinR l1 a b r* **for**
x l1
using *1(6)*
by (*auto simp*: *rbt-less-prop rbt-greater-prop set-rbt-joinR*)
show *?case*
using *1 less-rbt-joinR rbt-lookup-rbt-baliR*[*OF - rbt-sorted-rbt-joinR*[*of - r a b*],
where *?k=k*]
by (*auto simp*: *rbt-joinR.simps*[*of l a b r*] *split*!: *if-splits rbt.splits color.splits*)
qed

lemma *rbt-sorted-paint*: *rbt-sorted* (*paint c t*) = *rbt-sorted t*
by (*cases t*) *auto*

lemma *rbt-sorted-rbt-join*: *rbt-sorted* (*RBT-Impl.Branch c l a b r*) ⟹
rbt-sorted (*rbt-join l a b r*)
by (*auto simp*: *rbt-sorted-paint rbt-sorted-rbt-joinL rbt-sorted-rbt-joinR rbt-join-def*
Let-def)

lemma *rbt-lookup-rbt-join*: *rbt-sorted l* ⟹ *rbt-sorted r* ⟹ *l* |« *a* ⟹ *a* «| *r* ⟹
rbt-lookup (*rbt-join l a b r*) *k* =
(*if k* < *a then rbt-lookup l k else if k* = *a then Some b else rbt-lookup r k*)
by (*auto simp*: *rbt-join-def Let-def rbt-lookup-rbt-joinL rbt-lookup-rbt-joinR*)

lemma *rbt-split-min-rbt-sorted*: *rbt-split-min t* = (*a*,*b*,*t'*) ⟹ *rbt-sorted t* ⟹ *t* ≠

RBT-Impl.Empty $\Longrightarrow$
rbt-sorted $t' \wedge (\forall x \in set\ (RBT\text{-}Impl.keys\ t').\ a < x)$
by (*induction t arbitrary*: t')
(*fastforce simp*: *rbt-split-min-set rbt-sorted-rbt-join set-rbt-join rbt-less-prop rbt-greater-prop*
split: *if-splits prod.splits*)+

lemma *rbt-split-min-rbt-lookup*: *rbt-split-min* $t = (a,b,t') \Longrightarrow$ *rbt-sorted* $t \Longrightarrow t \neq$ *RBT-Impl.Empty* $\Longrightarrow$
rbt-lookup $t\ k =$ (*if* $k < a$ *then None else if* $k = a$ *then Some* b *else rbt-lookup* t' k)
apply (*induction t arbitrary*: $a\ b\ t'$)
apply(*simp-all split*: *if-splits prod.splits*)
apply(*auto simp*: *rbt-less-prop rbt-split-min-set rbt-lookup-rbt-join rbt-split-min-rbt-sorted*)
done

lemma *rbt-sorted-rbt-join2*: *rbt-sorted* $l \Longrightarrow$ *rbt-sorted* $r \Longrightarrow$
$\forall x \in set\ (RBT\text{-}Impl.keys\ l).\ \forall y \in set\ (RBT\text{-}Impl.keys\ r).\ x < y \Longrightarrow$ *rbt-sorted* (*rbt-join2* $l\ r$)
by (*simp add*: *rbt-join2-def rbt-sorted-rbt-join rbt-split-min-set rbt-split-min-rbt-sorted set-rbt-join*
rbt-greater-prop rbt-less-prop split: *prod.split*)

lemma *rbt-lookup-rbt-join2*: *rbt-sorted* $l \Longrightarrow$ *rbt-sorted* $r \Longrightarrow$
$\forall x \in set\ (RBT\text{-}Impl.keys\ l).\ \forall y \in set\ (RBT\text{-}Impl.keys\ r).\ x < y \Longrightarrow$
rbt-lookup (*rbt-join2* $l\ r$) $k =$ (*case rbt-lookup* $l\ k$ *of None* $\Rightarrow$ *rbt-lookup* $r\ k$ | *Some* $v \Rightarrow$ *Some* v)
using *rbt-lookup-keys*
by (*fastforce simp*: *rbt-join2-def rbt-greater-prop rbt-less-prop rbt-lookup-rbt-join*
rbt-split-min-rbt-lookup rbt-split-min-rbt-sorted rbt-split-min-set split: *option.splits prod.splits*)

lemma *rbt-split-props*: *rbt-split* $t\ x = (l,\beta,r) \Longrightarrow$ *rbt-sorted* $t \Longrightarrow$
$set\ (RBT\text{-}Impl.keys\ l) = \{a \in set\ (RBT\text{-}Impl.keys\ t).\ a < x\} \wedge$
$set\ (RBT\text{-}Impl.keys\ r) = \{a \in set\ (RBT\text{-}Impl.keys\ t).\ x < a\} \wedge$
rbt-sorted $l \wedge$ *rbt-sorted* r
apply (*induction t arbitrary*: $l\ r$)
apply(*simp-all split*!: *prod.splits if-splits*)
apply(*force simp*: *set-rbt-join rbt-greater-prop rbt-less-prop*
intro: *rbt-sorted-rbt-join*)+
done

lemma *rbt-split-lookup*: *rbt-split* $t\ x = (l,\beta,r) \Longrightarrow$ *rbt-sorted* $t \Longrightarrow$
rbt-lookup $t\ k =$ (*if* $k < x$ *then rbt-lookup* $l\ k$ *else if* $k = x$ *then* β *else rbt-lookup* $r\ k$)
proof (*induction t arbitrary*: $x\ l\ \beta\ r$)
case (*Branch c t1 a b t2*)
have *rbt-sorted r1 r1* |« a **if** *rbt-split t1* $x = (l,\ \beta,\ r1)$ **for** *r1*
using *rbt-split-props Branch*(*4*) *that*

```
    by (fastforce simp: rbt-less-prop)+
  moreover have rbt-sorted l1 a «| l1 if rbt-split t2 x = (l1, β, r) for l1
    using rbt-split-props Branch(4) that
    by (fastforce simp: rbt-greater-prop)+
  ultimately show ?case
    using Branch rbt-lookup-rbt-join[of t1 - a b k] rbt-lookup-rbt-join[of - t2 a b k]
    by (auto split!: if-splits prod.splits)
qed simp

lemma rbt-sorted-fold-insertwk: rbt-sorted t ⟹
  rbt-sorted (RBT-Impl.fold (rbt-insert-with-key f) t' t)
  by (induct t' arbitrary: t)
     (simp-all add: rbt-insertwk-rbt-sorted)

lemma rbt-lookup-iff-keys:
  rbt-sorted t ⟹ set (RBT-Impl.keys t) = {k. ∃ v. rbt-lookup t k = Some v}
  rbt-sorted t ⟹ rbt-lookup t k = None ⟷ k ∉ set (RBT-Impl.keys t)
  rbt-sorted t ⟹ (∃ v. rbt-lookup t k = Some v) ⟷ k ∈ set (RBT-Impl.keys t)
  using entry-in-tree-keys rbt-lookup-keys[of t]
  by force+

lemma rbt-lookup-fold-rbt-insertwk:
  assumes t1: rbt-sorted t1 and t2: rbt-sorted t2
  shows rbt-lookup (fold (rbt-insert-with-key f) t1 t2) k =
  (case rbt-lookup t1 k of None ⇒ rbt-lookup t2 k
   | Some v ⇒ case rbt-lookup t2 k of None ⇒ Some v
               | Some w ⇒ Some (f k w v))
proof −
  define xs where xs = entries t1
  hence dt1: distinct (map fst xs) using t1 by(simp add: distinct-entries)
  with t2 show ?thesis
    unfolding fold-def map-of-entries[OF t1, symmetric]
      xs-def[symmetric] distinct-map-of-rev[OF dt1, symmetric]
    apply(induct xs rule: rev-induct)
    apply(auto simp add: rbt-lookup-rbt-insertwk rbt-sorted-fold-rbt-insertwk split:
option.splits)
    apply(auto simp add: distinct-map-of-rev intro: rev-image-eqI)
    done
qed

lemma rbt-lookup-union-rec: rbt-sorted t1 ⟹ rbt-sorted t2 ⟹
  rbt-sorted (rbt-union-rec f t1 t2) ∧ rbt-lookup (rbt-union-rec f t1 t2) k =
  (case rbt-lookup t1 k of None ⇒ rbt-lookup t2 k
  | Some v ⇒ (case rbt-lookup t2 k of None ⇒ Some v
              | Some w ⇒ Some (f k v w)))
proof(induction f t1 t2 arbitrary: k rule: rbt-union-rec.induct)
  case (1 f t1 t2)
  obtain f' t1' t2' where flip: (f', t2', t1') =
    (if flip-rbt t2 t1 then (λk v v'. f k v' v, t1, t2) else (f, t2, t1))
```

```
    by fastforce
  have rbt-sorted': rbt-sorted t1' rbt-sorted t2'
    using 1(3,4) flip
    by (auto split: if-splits)
  show ?case
  proof (cases t1')
    case Empty
    show ?thesis
      unfolding rbt-union-rec.simps[of - t1] flip[symmetric]
      using flip rbt-sorted' rbt-split-props[of t2]
      by (auto simp: Empty rbt-lookup-fold-rbt-insertwk
          intro!: rbt-sorted-fold-insertwk split: if-splits option.splits)
  next
    case (Branch c l1 a b r1)
    {
      assume not-small: ¬small-rbt t2'
      obtain l2 β r2 where rbt-split-t2': rbt-split t2' a = (l2, β, r2)
        by (cases rbt-split t2' a) auto
      have rbt-sort: rbt-sorted l1 rbt-sorted r1
        using 1(3,4) flip
        by (auto simp: Branch split: if-splits)
      note rbt-split-t2'-props = rbt-split-props[OF rbt-split-t2' rbt-sorted'(2)]
      have union-l1-l2: rbt-sorted (rbt-union-rec f' l1 l2) rbt-lookup (rbt-union-rec
f' l1 l2) k =
        (case rbt-lookup l1 k of None ⇒ rbt-lookup l2 k
        | Some v ⇒ (case rbt-lookup l2 k of None ⇒ Some v | Some w ⇒ Some (f'
k v w))) for k
          using 1(1)[OF flip refl refl - Branch rbt-split-t2'[symmetric]] rbt-sort
rbt-split-t2'-props
        by (auto simp: not-small)
      have union-r1-r2: rbt-sorted (rbt-union-rec f' r1 r2) rbt-lookup (rbt-union-rec
f' r1 r2) k =
        (case rbt-lookup r1 k of None ⇒ rbt-lookup r2 k
        | Some v ⇒ (case rbt-lookup r2 k of None ⇒ Some v | Some w ⇒ Some (f'
k v w))) for k
          using 1(2)[OF flip refl refl - Branch rbt-split-t2'[symmetric]] rbt-sort
rbt-split-t2'-props
        by (auto simp: not-small)
      have union-l1-l2-keys: set (RBT-Impl.keys (rbt-union-rec f' l1 l2)) =
       set (RBT-Impl.keys l1) ∪ set (RBT-Impl.keys l2)
        using rbt-sorted'(1) rbt-split-t2'-props
        by (auto simp: Branch rbt-lookup-iff-keys(1) union-l1-l2 split: option.splits)
      have union-r1-r2-keys: set (RBT-Impl.keys (rbt-union-rec f' r1 r2)) =
        set (RBT-Impl.keys r1) ∪ set (RBT-Impl.keys r2)
        using rbt-sorted'(1) rbt-split-t2'-props
       by (auto simp: Branch rbt-lookup-iff-keys(1) union-r1-r2 split: option.splits)
      have union-l1-l2-less: rbt-union-rec f' l1 l2 |« a
        using rbt-sorted'(1) rbt-split-t2'-props
        by (auto simp: Branch rbt-less-prop union-l1-l2-keys)
```

```
    have union-r1-r2-greater: a «| rbt-union-rec f' r1 r2
      using rbt-sorted'(1) rbt-split-t2'-props
      by (auto simp: Branch rbt-greater-prop union-r1-r2-keys)
    have rbt-lookup (rbt-union-rec f t1 t2) k =
     (case rbt-lookup t1' k of None ⇒ rbt-lookup t2' k
     | Some v ⇒ (case rbt-lookup t2' k of None ⇒ Some v | Some w ⇒ Some (f'
k v w)))
      using rbt-sorted' union-l1-l2 union-r1-r2 rbt-split-t2'-props
        union-l1-l2-less union-r1-r2-greater not-small
      by (auto simp: rbt-union-rec.simps[of - t1] flip[symmetric] Branch
            rbt-split-t2' rbt-lookup-rbt-join rbt-split-lookup[OF rbt-split-t2'] split:
option.splits)
    moreover have rbt-sorted (rbt-union-rec f t1 t2)
      using rbt-sorted' rbt-split-t2'-props not-small
    by (auto simp: rbt-union-rec.simps[of - t1] flip[symmetric] Branch rbt-split-t2'
        union-l1-l2 union-r1-r2 union-l1-l2-keys union-r1-r2-keys rbt-less-prop
        rbt-greater-prop intro!: rbt-sorted-rbt-join)
    ultimately have ?thesis
      using flip
      by (auto split: if-splits option.splits)
  }
  then show ?thesis
    unfolding rbt-union-rec.simps[of - t1] flip[symmetric]
    using rbt-sorted' flip
    by (auto simp: rbt-sorted-fold-insertwk rbt-lookup-fold-rbt-insertwk split: op-
tion.splits)
 qed
qed

lemma rbtreeify-map-filter-inter:
 fixes f :: 'a ⇒ 'b ⇒ 'b ⇒ 'b
 assumes rbt-sorted t2
 shows rbt-sorted (rbtreeify (map-filter-inter f t1 t2))
   rbt-lookup (rbtreeify (map-filter-inter f t1 t2)) k =
   (case rbt-lookup t1 k of None ⇒ None
   | Some v ⇒ (case rbt-lookup t2 k of None ⇒ None | Some w ⇒ Some (f k v
w)))
proof −
 have map-of-map-filter: map-of (List.map-filter (λ(k, v).
   case rbt-lookup t1 k of None ⇒ None | Some v' ⇒ Some (k, f k v' v)) xs) k =
   (case rbt-lookup t1 k of None ⇒ None
   | Some v ⇒ (case map-of xs k of None ⇒ None | Some w ⇒ Some (f k v w)))
for xs k
   by (induction xs) (auto simp: List.map-filter-def split: option.splits)
 have map-fst-map-filter: map fst (List.map-filter (λ(k, v).
   case rbt-lookup t1 k of None ⇒ None | Some v' ⇒ Some (k, f k v' v)) xs) =
   filter (λk. rbt-lookup t1 k ≠ None) (map fst xs) for xs
   by (induction xs) (auto simp: List.map-filter-def split: option.splits)
 have sorted (map fst (map-filter-inter f t1 t2))
```

```
    using sorted-filter[of id] rbt-sorted-entries[OF assms]
    by (auto simp: map-filter-inter-def map-fst-map-filter)
  moreover have distinct (map fst (map-filter-inter f t1 t2))
    using distinct-filter distinct-entries[OF assms]
    by (auto simp: map-filter-inter-def map-fst-map-filter)
  ultimately show
    rbt-sorted (rbtreeify (map-filter-inter f t1 t2))
    rbt-lookup (rbtreeify (map-filter-inter f t1 t2)) k =
      (case rbt-lookup t1 k of None ⇒ None
      | Some v ⇒ (case rbt-lookup t2 k of None ⇒ None | Some w ⇒ Some (f k v
w)))
    using rbt-sorted-rbtreeify
    by (auto simp: rbt-lookup-rbtreeify map-filter-inter-def map-of-map-filter
        map-of-entries[OF assms] split: option.splits)
qed

lemma rbt-lookup-inter-rec: rbt-sorted t1 ⟹ rbt-sorted t2 ⟹
  rbt-sorted (rbt-inter-rec f t1 t2) ∧ rbt-lookup (rbt-inter-rec f t1 t2) k =
  (case rbt-lookup t1 k of None ⇒ None
  | Some v ⇒ (case rbt-lookup t2 k of None ⇒ None | Some w ⇒ Some (f k v w)))
proof(induction f t1 t2 arbitrary: k rule: rbt-inter-rec.induct)
  case (1 f t1 t2)
  obtain f' t1' t2' where flip: (f', t2', t1') =
    (if flip-rbt t2 t1 then (λk v v'. f k v' v, t1, t2) else (f, t2, t1))
    by fastforce
  have rbt-sorted': rbt-sorted t1' rbt-sorted t2'
    using 1(3,4) flip
    by (auto split: if-splits)
  show ?case
  proof (cases t1')
    case Empty
    show ?thesis
      unfolding rbt-inter-rec.simps[of - t1] flip[symmetric]
     using flip rbt-sorted' rbt-split-props[of t2] rbtreeify-map-filter-inter[OF rbt-sorted'(2)]
      by (auto simp: Empty split: option.splits)
  next
    case (Branch c l1 a b r1)
    {
      assume not-small: ¬small-rbt t2'
      obtain l2 β r2 where rbt-split-t2': rbt-split t2' a = (l2, β, r2)
        by (cases rbt-split t2' a) auto
      note rbt-split-t2'-props = rbt-split-props[OF rbt-split-t2' rbt-sorted'(2)]
      have rbt-sort: rbt-sorted l1 rbt-sorted r1 rbt-sorted l2 rbt-sorted r2
        using 1(3,4) flip
        by (auto simp: Branch rbt-split-t2'-props split: if-splits)
      have inter-l1-l2: rbt-sorted (rbt-inter-rec f' l1 l2) rbt-lookup (rbt-inter-rec f'
l1 l2) k =
        (case rbt-lookup l1 k of None ⇒ None
        | Some v ⇒ (case rbt-lookup l2 k of None ⇒ None | Some w ⇒ Some (f' k
```

v w))) **for** *k*
using *1(1)*[*OF flip refl refl - Branch rbt-split-t2'*[*symmetric*]] *rbt-sort rbt-split-t2'-props*
by (*auto simp*: *not-small*)
have *inter-r1-r2*: *rbt-sorted* (*rbt-inter-rec f' r1 r2*) *rbt-lookup* (*rbt-inter-rec f' r1 r2*) *k* =
(*case rbt-lookup r1 k of None ⇒ None*
| *Some v* ⇒ (*case rbt-lookup r2 k of None ⇒ None* | *Some w* ⇒ *Some* (*f' k v w*))) **for** *k*
using *1(2)*[*OF flip refl refl - Branch rbt-split-t2'*[*symmetric*]] *rbt-sort rbt-split-t2'-props*
by (*auto simp*: *not-small*)
have *inter-l1-l2-keys*: *set* (*RBT-Impl.keys* (*rbt-inter-rec f' l1 l2*)) =
set (*RBT-Impl.keys l1*) ∩ *set* (*RBT-Impl.keys l2*)
using *inter-l1-l2(1)*
by (*auto simp*: *rbt-lookup-iff-keys(1) inter-l1-l2(2) rbt-sort split*: *option.splits*)
have *inter-r1-r2-keys*: *set* (*RBT-Impl.keys* (*rbt-inter-rec f' r1 r2*)) =
set (*RBT-Impl.keys r1*) ∩ *set* (*RBT-Impl.keys r2*)
using *inter-r1-r2(1)*
by (*auto simp*: *rbt-lookup-iff-keys(1) inter-r1-r2(2) rbt-sort split*: *option.splits*)
have *inter-l1-l2-less*: *rbt-inter-rec f' l1 l2* |« *a*
using *rbt-sorted'(1) rbt-split-t2'-props*
by (*auto simp*: *Branch rbt-less-prop inter-l1-l2-keys*)
have *inter-r1-r2-greater*: *a* «| *rbt-inter-rec f' r1 r2*
using *rbt-sorted'(1) rbt-split-t2'-props*
by (*auto simp*: *Branch rbt-greater-prop inter-r1-r2-keys*)
have *rbt-lookup-join2*: *rbt-lookup* (*rbt-join2* (*rbt-inter-rec f' l1 l2*) (*rbt-inter-rec f' r1 r2*)) *k* =
(*case rbt-lookup* (*rbt-inter-rec f' l1 l2*) *k of None ⇒ rbt-lookup* (*rbt-inter-rec f' r1 r2*) *k*
| *Some v* ⇒ *Some v*) **for** *k*
using *rbt-lookup-rbt-join2*[*OF inter-l1-l2(1) inter-r1-r2(1)*] *rbt-sorted'(1)*
by (*fastforce simp*: *Branch inter-l1-l2-keys inter-r1-r2-keys rbt-less-prop rbt-greater-prop*)
have *rbt-lookup-l1-k*: *rbt-lookup l1 k* = *Some v* ⟹ *k* < *a* **for** *k v*
using *rbt-sorted'(1) rbt-lookup-iff-keys(3)*
by (*auto simp*: *Branch rbt-less-prop*)
have *rbt-lookup-r1-k*: *rbt-lookup r1 k* = *Some v* ⟹ *a* < *k* **for** *k v*
using *rbt-sorted'(1) rbt-lookup-iff-keys(3)*
by (*auto simp*: *Branch rbt-greater-prop*)
have *rbt-lookup* (*rbt-inter-rec f t1 t2*) *k* =
(*case rbt-lookup t1' k of None ⇒ None*
| *Some v* ⇒ (*case rbt-lookup t2' k of None ⇒ None* | *Some w* ⇒ *Some* (*f' k v w*)))
by (*auto simp*: *Let-def rbt-inter-rec.simps*[*of - t1*] *flip*[*symmetric*] *not-small Branch*
rbt-split-t2' rbt-lookup-join2 rbt-lookup-rbt-join inter-l1-l2-less inter-r1-r2-greater

```
        rbt-split-lookup[OF rbt-split-t2' rbt-sorted'(2)] inter-l1-l2 inter-r1-r2
        split!: if-splits option.splits dest: rbt-lookup-l1-k rbt-lookup-r1-k)
    moreover have rbt-sorted (rbt-inter-rec f t1 t2)
     using rbt-sorted' inter-l1-l2 inter-r1-r2 rbt-split-t2'-props not-small
      by (auto simp: Let-def rbt-inter-rec.simps[of - t1] flip[symmetric] Branch
rbt-split-t2'
        rbt-less-prop rbt-greater-prop inter-l1-l2-less inter-r1-r2-greater
       inter-l1-l2-keys inter-r1-r2-keys intro!: rbt-sorted-rbt-join rbt-sorted-rbt-join2
        split: if-splits option.splits dest!: bspec)
    ultimately have ?thesis
     using flip
     by (auto split: if-splits split: option.splits)
  }
  then show ?thesis
    unfolding rbt-inter-rec.simps[of - t1] flip[symmetric]
    using rbt-sorted' flip rbtreeify-map-filter-inter[OF rbt-sorted'(2)]
    by (auto split: option.splits)
 qed
qed

lemma rbt-lookup-delete:
 assumes inv-12 t rbt-sorted t
 shows rbt-lookup (rbt-delete x t) k = (if x = k then None else rbt-lookup t k)
proof −
 note rbt-sorted-del = rbt-del-rbt-sorted[OF assms(2), of x]
 show ?thesis
   using assms rbt-sorted-del rbt-del-in-tree rbt-lookup-from-in-tree[OF assms(2)
rbt-sorted-del]
   by (fastforce simp: inv-12-def rbt-delete-def rbt-lookup-iff-keys(2) keys-entries)
qed

lemma fold-rbt-delete:
 assumes inv-12 t1 rbt-sorted t1 rbt-sorted t2
 shows inv-12 (RBT-Impl.fold (λk - t. rbt-delete k t) t2 t1) ∧
   rbt-sorted (RBT-Impl.fold (λk - t. rbt-delete k t) t2 t1) ∧
   rbt-lookup (RBT-Impl.fold (λk - t. rbt-delete k t) t2 t1) k =
   (case rbt-lookup t1 k of None ⇒ None
   | Some v ⇒ (case rbt-lookup t2 k of None ⇒ Some v | - ⇒ None))
proof −
 define xs where xs = RBT-Impl.entries t2
 show inv-12 (RBT-Impl.fold (λk - t. rbt-delete k t) t2 t1) ∧
   rbt-sorted (RBT-Impl.fold (λk - t. rbt-delete k t) t2 t1) ∧
   rbt-lookup (RBT-Impl.fold (λk - t. rbt-delete k t) t2 t1) k =
   (case rbt-lookup t1 k of None ⇒ None
   | Some v ⇒ (case rbt-lookup t2 k of None ⇒ Some v | - ⇒ None))
   using assms(1,2)
  unfolding map-of-entries[OF assms(3), symmetric] RBT-Impl.fold-def xs-def[symmetric]
   by (induction xs arbitrary: t1 rule: rev-induct)
     (auto simp: rbt-delete rbt-sorted-delete rbt-lookup-delete split!: option.splits)
```

qed

lemma *rbtreeify-filter-minus*:
assumes *rbt-sorted t1*
shows *rbt-sorted* (*rbtreeify* (*filter-minus t1 t2*)) ∧
rbt-lookup (*rbtreeify* (*filter-minus t1 t2*)) *k* =
(*case rbt-lookup t1 k of None* ⇒ *None*
| *Some v* ⇒ (*case rbt-lookup t2 k of None* ⇒ *Some v* | - ⇒ *None*))
proof −
have *map-of-filter*: *map-of* (*filter* (λ(*k*, -). *rbt-lookup t2 k* = *None*) *xs*) *k* =
(*case map-of xs k of None* ⇒ *None*
| *Some v* ⇒ (*case rbt-lookup t2 k of None* ⇒ *Some v* | *Some x* ⇒ *Map.empty x*))
for *xs* :: (′*a* × ′*b*) *list*
by (*induction xs*) (*auto split*: *option.splits*)
have *map-fst-filter-minus*: *map fst* (*filter-minus t1 t2*) =
filter (λ*k*. *rbt-lookup t2 k* = *None*) (*map fst* (*RBT-Impl.entries t1*))
by (*auto simp*: *filter-minus-def filter-map comp-def case-prod-unfold*)
have *sorted* (*map fst* (*filter-minus t1 t2*)) *distinct* (*map fst* (*filter-minus t1 t2*))
using *distinct-filter distinct-entries*[*OF assms*]
sorted-filter[*of id*] *rbt-sorted-entries*[*OF assms*]
by (*auto simp*: *map-fst-filter-minus intro*!: *rbt-sorted-rbtreeify*)
then show *?thesis*
by (*auto simp*: *rbt-lookup-rbtreeify filter-minus-def map-of-filter map-of-entries*[*OF assms*]
intro!: *rbt-sorted-rbtreeify*)
qed

lemma *rbt-lookup-minus-rec*: *inv-12 t1* ⟹ *rbt-sorted t1* ⟹ *rbt-sorted t2* ⟹
rbt-sorted (*rbt-minus-rec t1 t2*) ∧ *rbt-lookup* (*rbt-minus-rec t1 t2*) *k* =
(*case rbt-lookup t1 k of None* ⇒ *None*
| *Some v* ⇒ (*case rbt-lookup t2 k of None* ⇒ *Some v* | - ⇒ *None*))
proof(*induction t1 t2 arbitrary*: *k rule*: *rbt-minus-rec.induct*)
case (*1 t1 t2*)
show *?case*
proof (*cases t2*)
case *Empty*
show *?thesis*
using *rbtreeify-filter-minus*[*OF 1*(*4*)] *1*(*4*)
by (*auto simp*: *rbt-minus-rec.simps*[*of t1*] *Empty split*: *option.splits*)
next
case (*Branch c l2 a b r2*)
{
assume *not-small*: ¬*small-rbt t2* ¬*small-rbt t1*
obtain *l1 β r1* **where** *rbt-split-t1*: *rbt-split t1 a* = (*l1*, *β*, *r1*)
by (*cases rbt-split t1 a*) *auto*
note *rbt-split-t1-props* = *rbt-split-props*[*OF rbt-split-t1 1*(*4*)]
have *minus-l1-l2*: *rbt-sorted* (*rbt-minus-rec l1 l2*)
rbt-lookup (*rbt-minus-rec l1 l2*) *k* =

(*case rbt-lookup l1 k of None ⇒ None*
| *Some v ⇒ (case rbt-lookup l2 k of None ⇒ Some v | Some x ⇒ None)*)
for *k*
using *1(1)*[*OF not-small Branch rbt-split-t1*[*symmetric*] *refl*] *1(5) rbt-split-t1-props*
rbt-split[*OF rbt-split-t1 1(3)*]
by (*auto simp*: *Branch*)
have *minus-r1-r2*: *rbt-sorted* (*rbt-minus-rec r1 r2*)
rbt-lookup (*rbt-minus-rec r1 r2*) *k* =
(*case rbt-lookup r1 k of None ⇒ None*
| *Some v ⇒ (case rbt-lookup r2 k of None ⇒ Some v | Some x ⇒ None)*)
for *k*
using *1(2)*[*OF not-small Branch rbt-split-t1*[*symmetric*] *refl*] *1(5) rbt-split-t1-props*
rbt-split[*OF rbt-split-t1 1(3)*]
by (*auto simp*: *Branch*)
have *minus-l1-l2-keys*: *set* (*RBT-Impl.keys* (*rbt-minus-rec l1 l2*)) =
set (*RBT-Impl.keys l1*) − *set* (*RBT-Impl.keys l2*)
using *minus-l1-l2(1) 1(5) rbt-lookup-iff-keys(3) rbt-split-t1-props*
by (*auto simp*: *Branch rbt-lookup-iff-keys(1) minus-l1-l2(2) split*: *option.splits*)
have *minus-r1-r2-keys*: *set* (*RBT-Impl.keys* (*rbt-minus-rec r1 r2*)) =
set (*RBT-Impl.keys r1*) − *set* (*RBT-Impl.keys r2*)
using *minus-r1-r2(1) 1(5) rbt-lookup-iff-keys(3) rbt-split-t1-props*
by (*auto simp*: *Branch rbt-lookup-iff-keys(1) minus-r1-r2(2) split*: *option.splits*)
have *rbt-lookup-join2*: *rbt-lookup* (*rbt-join2* (*rbt-minus-rec l1 l2*) (*rbt-minus-rec r1 r2*)) *k* =
(*case rbt-lookup* (*rbt-minus-rec l1 l2*) *k of None ⇒ rbt-lookup* (*rbt-minus-rec r1 r2*) *k*
| *Some v ⇒ Some v*) **for** *k*
using *rbt-lookup-rbt-join2*[*OF minus-l1-l2(1) minus-r1-r2(1)*] *rbt-split-t1-props*
by (*fastforce simp*: *minus-l1-l2-keys minus-r1-r2-keys*)
have *lookup-l1-r1-a*: *rbt-lookup l1 a* = *None rbt-lookup r1 a* = *None*
using *rbt-split-t1-props*
by (*auto simp*: *rbt-lookup-iff-keys(2)*)
have *rbt-lookup* (*rbt-minus-rec t1 t2*) *k* =
(*case rbt-lookup t1 k of None ⇒ None*
| *Some v ⇒ (case rbt-lookup t2 k of None ⇒ Some v | - ⇒ None)*)
using *not-small rbt-lookup-iff-keys(2)*[*of l1*] *rbt-lookup-iff-keys(3)*[*of l1*]
rbt-lookup-iff-keys(3)[*of r1*] *rbt-split-t1-props*
using [[*simp-depth-limit* = *2*]]
by (*auto simp*: *rbt-minus-rec.simps*[*of t1*] *Branch rbt-split-t1 rbt-lookup-join2*
minus-l1-l2(2) minus-r1-r2(2) rbt-split-lookup[*OF rbt-split-t1 1(4)*]
lookup-l1-r1-a
split: *option.splits*)
moreover have *rbt-sorted* (*rbt-minus-rec t1 t2*)
using *not-small minus-l1-l2(1) minus-r1-r2(1) rbt-split-t1-props rbt-sorted-rbt-join2*
by (*fastforce simp*: *rbt-minus-rec.simps*[*of t1*] *Branch rbt-split-t1 minus-l1-l2-keys minus-r1-r2-keys*)
ultimately have *?thesis*

```
      by (auto split: if-splits split: option.splits)
   }
   then show ?thesis
    using fold-rbt-delete[OF 1(3,4,5)] rbtreeify-filter-minus[OF 1(4)]
    by (auto simp: rbt-minus-rec.simps[of t1])
 qed
qed

end

context ord begin

definition rbt-union-with-key :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a, 'b) rbt ⇒ ('a, 'b) rbt
⇒ ('a, 'b) rbt
where
  rbt-union-with-key f t1 t2 = paint B (rbt-union-swap-rec f False t1 t2)

definition rbt-union-with where
  rbt-union-with f = rbt-union-with-key (λ-. f)

definition rbt-union where
  rbt-union = rbt-union-with-key (%- - rv. rv)

definition rbt-inter-with-key :: ('a ⇒ 'b ⇒ 'b ⇒ 'b) ⇒ ('a, 'b) rbt ⇒ ('a, 'b) rbt
⇒ ('a, 'b) rbt
where
  rbt-inter-with-key f t1 t2 = paint B (rbt-inter-swap-rec f False t1 t2)

definition rbt-inter-with where
  rbt-inter-with f = rbt-inter-with-key (λ-. f)

definition rbt-inter where
  rbt-inter = rbt-inter-with-key (λ- - rv. rv)

definition rbt-minus where
  rbt-minus t1 t2 = paint B (rbt-minus-rec t1 t2)

end

context linorder begin

lemma is-rbt-rbt-unionwk [simp]:
  ⟦ is-rbt t1; is-rbt t2 ⟧ ⟹ is-rbt (rbt-union-with-key f t1 t2)
  using rbt-union-rec rbt-lookup-union-rec
 by (fastforce simp: rbt-union-with-key-def rbt-union-swap-rec is-rbt-def inv-12-def)

lemma rbt-lookup-rbt-unionwk:
  ⟦ rbt-sorted t1; rbt-sorted t2 ⟧
  ⟹ rbt-lookup (rbt-union-with-key f t1 t2) k =
```

(*case rbt-lookup t1 k of None ⇒ rbt-lookup t2 k*
| *Some v ⇒ case rbt-lookup t2 k of None ⇒ Some v*
| *Some w ⇒ Some (f k v w)*)
using *rbt-lookup-union-rec*
by (*auto simp*: *rbt-union-with-key-def rbt-union-swap-rec*)

lemma *rbt-unionw-is-rbt*: ⟦ *is-rbt lt*; *is-rbt rt* ⟧ ⟹ *is-rbt* (*rbt-union-with f lt rt*)
by(*simp add*: *rbt-union-with-def*)

lemma *rbt-union-is-rbt*: ⟦ *is-rbt lt*; *is-rbt rt* ⟧ ⟹ *is-rbt* (*rbt-union lt rt*)
by(*simp add*: *rbt-union-def*)

lemma *rbt-lookup-rbt-union*:
⟦ *rbt-sorted s*; *rbt-sorted t* ⟧ ⟹
rbt-lookup (*rbt-union s t*) = *rbt-lookup s* ++ *rbt-lookup t*
by(*rule ext*)(*simp add*: *rbt-lookup-rbt-unionwk rbt-union-def map-add-def split*: *option.split*)

lemma *rbt-interwk-is-rbt* [*simp*]:
⟦ *is-rbt t1*; *is-rbt t2* ⟧ ⟹ *is-rbt* (*rbt-inter-with-key f t1 t2*)
using *rbt-inter-rec rbt-lookup-inter-rec*
by (*fastforce simp*: *rbt-inter-with-key-def rbt-inter-swap-rec is-rbt-def inv-12-def rbt-sorted-paint*)

lemma *rbt-interw-is-rbt*:
⟦ *is-rbt t1*; *is-rbt t2* ⟧ ⟹ *is-rbt* (*rbt-inter-with f t1 t2*)
by(*simp add*: *rbt-inter-with-def*)

lemma *rbt-inter-is-rbt*:
⟦ *is-rbt t1*; *is-rbt t2* ⟧ ⟹ *is-rbt* (*rbt-inter t1 t2*)
by(*simp add*: *rbt-inter-def*)

lemma *rbt-lookup-rbt-interwk*:
⟦ *rbt-sorted t1*; *rbt-sorted t2* ⟧
⟹ *rbt-lookup* (*rbt-inter-with-key f t1 t2*) *k* =
(*case rbt-lookup t1 k of None ⇒ None*
| *Some v ⇒ case rbt-lookup t2 k of None ⇒ None*
| *Some w ⇒ Some (f k v w)*)
using *rbt-lookup-inter-rec*
by (*auto simp*: *rbt-inter-with-key-def rbt-inter-swap-rec*)

lemma *rbt-lookup-rbt-inter*:
⟦ *rbt-sorted t1*; *rbt-sorted t2* ⟧
⟹ *rbt-lookup* (*rbt-inter t1 t2*) = *rbt-lookup t2* |‘ *dom* (*rbt-lookup t1*)
by(*auto simp add*: *rbt-inter-def rbt-lookup-rbt-interwk restrict-map-def split*: *option.split*)

lemma *rbt-minus-is-rbt*:
⟦ *is-rbt t1*; *is-rbt t2* ⟧ ⟹ *is-rbt* (*rbt-minus t1 t2*)

using *rbt-minus-rec*[*of t1 t2*] *rbt-lookup-minus-rec*[*of t1 t2*]
by (*auto simp*: *rbt-minus-def is-rbt-def inv-12-def*)

lemma *rbt-lookup-rbt-minus*:
⟦ *is-rbt t1*; *is-rbt t2* ⟧
⟹ *rbt-lookup* (*rbt-minus t1 t2*) = *rbt-lookup t1* |` (− *dom* (*rbt-lookup t2*))
by (*rule ext*)
(*auto simp*: *rbt-minus-def is-rbt-def inv-12-def restrict-map-def rbt-lookup-minus-rec*
split: *option.splits*)

end

119.11 Code generator setup

lemmas [*code*] =
ord.rbt-less-prop
ord.rbt-greater-prop
ord.rbt-sorted.simps
ord.rbt-lookup.simps
ord.is-rbt-def
ord.rbt-ins.simps
ord.rbt-insert-with-key-def
ord.rbt-insertw-def
ord.rbt-insert-def
ord.rbt-del-from-left.simps
ord.rbt-del-from-right.simps
ord.rbt-del.simps
ord.rbt-delete-def
ord.rbt-split.simps
ord.rbt-union-swap-rec.simps
ord.map-filter-inter-def
ord.rbt-inter-swap-rec.simps
ord.filter-minus-def
ord.rbt-minus-rec.simps
ord.rbt-union-with-key-def
ord.rbt-union-with-def
ord.rbt-union-def
ord.rbt-inter-with-key-def
ord.rbt-inter-with-def
ord.rbt-inter-def
ord.rbt-minus-def
ord.rbt-map-entry.simps
ord.rbt-bulkload-def

More efficient implementations for *entries* and *keys*

definition *gen-entries* ::
(('*a* × '*b*) × ('*a*, '*b*) *rbt*) *list* ⇒ ('*a*, '*b*) *rbt* ⇒ ('*a* × '*b*) *list*
where
gen-entries kvts t = *entries t* @ *concat* (*map* (λ(*kv*, *t*). *kv* # *entries t*) *kvts*)

lemma *gen-entries-simps* [*simp*, *code*]:
gen-entries [] *Empty* = []
gen-entries ((*kv*, *t*) # *kvts*) *Empty* = *kv* # *gen-entries kvts t*
gen-entries kvts (*Branch c l k v r*) = *gen-entries* (((*k*, *v*), *r*) # *kvts*) *l*
by(*simp-all add*: *gen-entries-def*)

lemma *entries-code* [*code*]:
entries = *gen-entries* []
by(*simp add*: *gen-entries-def fun-eq-iff*)

definition *gen-keys* :: ('*a* × ('*a*, '*b*) *rbt*) *list* ⇒ ('*a*, '*b*) *rbt* ⇒ '*a list*
where *gen-keys kts t* = *RBT-Impl.keys t* @ *concat* (*List.map* (λ(*k*, *t*). *k* # *keys t*) *kts*)

lemma *gen-keys-simps* [*simp*, *code*]:
gen-keys [] *Empty* = []
gen-keys ((*k*, *t*) # *kts*) *Empty* = *k* # *gen-keys kts t*
gen-keys kts (*Branch c l k v r*) = *gen-keys* ((*k*, *r*) # *kts*) *l*
by(*simp-all add*: *gen-keys-def*)

lemma *keys-code* [*code*]:
keys = *gen-keys* []
by(*simp add*: *gen-keys-def fun-eq-iff*)

Restore original type constraints for constants

setup ‹
fold Sign.add-const-constraint
[(***const-name*** ‹*rbt-less*›, *SOME* ***typ*** ‹('*a* :: *order*) ⇒ ('*a*, '*b*) *rbt* ⇒ *bool*›),
(***const-name*** ‹*rbt-greater*›, *SOME* ***typ*** ‹('*a* :: *order*) ⇒ ('*a*, '*b*) *rbt* ⇒ *bool*›),
(***const-name*** ‹*rbt-sorted*›, *SOME* ***typ*** ‹('*a* :: *linorder*, '*b*) *rbt* ⇒ *bool*›),
(***const-name*** ‹*rbt-lookup*›, *SOME* ***typ*** ‹('*a* :: *linorder*, '*b*) *rbt* ⇒ '*a* ⇀ '*b*›),
(***const-name*** ‹*is-rbt*›, *SOME* ***typ*** ‹('*a* :: *linorder*, '*b*) *rbt* ⇒ *bool*›),
(***const-name*** ‹*rbt-ins*›, *SOME* ***typ*** ‹('*a*::*linorder* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-insert-with-key*›, *SOME* ***typ*** ‹('*a*::*linorder* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-insert-with*›, *SOME* ***typ*** ‹('*b* ⇒ '*b* ⇒ '*b*) ⇒ ('*a* :: *linorder*) ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-insert*›, *SOME* ***typ*** ‹('*a* :: *linorder*) ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-del-from-left*›, *SOME* ***typ*** ‹('*a*::*linorder*) ⇒ ('*a*,'*b*) *rbt* ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-del-from-right*›, *SOME* ***typ*** ‹('*a*::*linorder*) ⇒ ('*a*,'*b*) *rbt* ⇒ '*a* ⇒ '*b* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-del*›, *SOME* ***typ*** ‹('*a*::*linorder*) ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-delete*›, *SOME* ***typ*** ‹('*a*::*linorder*) ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-union-with-key*›, *SOME* ***typ*** ‹('*a*::*linorder* ⇒ '*b* ⇒ '*b* ⇒ '*b*) ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),

(***const-name*** ‹*rbt-union-with*›, *SOME* ***typ*** ‹('*b* ⇒ '*b* ⇒ '*b*) ⇒ ('*a*::*linorder*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-union*›, *SOME* ***typ*** ‹('*a*::*linorder*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-map-entry*›, *SOME* ***typ*** ‹'*a*::*linorder* ⇒ ('*b* ⇒ '*b*) ⇒ ('*a*,'*b*) *rbt* ⇒ ('*a*,'*b*) *rbt*›),
(***const-name*** ‹*rbt-bulkload*›, *SOME* ***typ*** ‹('*a* × '*b*) *list* ⇒ ('*a*::*linorder*,'*b*) *rbt*›)]
›

hide-const (**open**) *MR MB R B Empty entries keys fold gen-keys gen-entries*

end

120 Abstract type of RBT trees

theory *RBT*
imports *Main RBT-Impl*
begin

120.1 Type definition

typedef (**overloaded**) ('*a*, '*b*) *rbt* = {*t* :: ('*a*::*linorder*, '*b*) *RBT-Impl.rbt*. *is-rbt t*}
morphisms *impl-of RBT*
proof −
have *RBT-Impl.Empty* ∈ *?rbt* **by** *simp*
then show *?thesis* **..**
qed

lemma *rbt-eq-iff*:
t1 = *t2* ⟷ *impl-of t1* = *impl-of t2*
by (*simp add*: *impl-of-inject*)

lemma *rbt-eqI*:
impl-of t1 = *impl-of t2* ⟹ *t1* = *t2*
by (*simp add*: *rbt-eq-iff*)

lemma *is-rbt-impl-of* [*simp*, *intro*]:
is-rbt (*impl-of t*)
using *impl-of* [*of t*] **by** *simp*

lemma *RBT-impl-of* [*simp*, *code abstype*]:
RBT (*impl-of t*) = *t*
by (*simp add*: *impl-of-inverse*)

120.2 Primitive operations

setup-lifting *type-definition-rbt*

lift-definition *lookup* :: (*'a*::*linorder*, *'b*) *rbt* ⇒ *'a* ⇀ *'b* **is** *rbt-lookup* **.**

lift-definition *empty* :: (*'a*::*linorder*, *'b*) *rbt* **is** *RBT-Impl.Empty*
by (*simp add*: *empty-def*)

lift-definition *insert* :: *'a*::*linorder* ⇒ *'b* ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* **is** *rbt-insert*

by *simp*

lift-definition *delete* :: *'a*::*linorder* ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* **is** *rbt-delete*
by *simp*

lift-definition *entries* :: (*'a*::*linorder*, *'b*) *rbt* ⇒ (*'a* × *'b*) *list* **is** *RBT-Impl.entries*
.

lift-definition *keys* :: (*'a*::*linorder*, *'b*) *rbt* ⇒ *'a list* **is** *RBT-Impl.keys* **.**

lift-definition *bulkload* :: (*'a*::*linorder* × *'b*) *list* ⇒ (*'a*, *'b*) *rbt* **is** *rbt-bulkload* **..**

lift-definition *map-entry* :: *'a* ⇒ (*'b* ⇒ *'b*) ⇒ (*'a*::*linorder*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt*
is *rbt-map-entry*
by *simp*

lift-definition *map* :: (*'a* ⇒ *'b* ⇒ *'c*) ⇒ (*'a*::*linorder*, *'b*) *rbt* ⇒ (*'a*, *'c*) *rbt* **is**
RBT-Impl.map
by *simp*

lift-definition *fold* :: (*'a* ⇒ *'b* ⇒ *'c* ⇒ *'c*) ⇒ (*'a*::*linorder*, *'b*) *rbt* ⇒ *'c* ⇒ *'c* **is**
RBT-Impl.fold **.**

lift-definition *union* :: (*'a*::*linorder*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* **is**
rbt-union
by (*simp add*: *rbt-union-is-rbt*)

lift-definition *foldi* :: (*'c* ⇒ *bool*) ⇒ (*'a* ⇒ *'b* ⇒ *'c* ⇒ *'c*) ⇒ (*'a* :: *linorder*, *'b*)
rbt ⇒ *'c* ⇒ *'c*
is *RBT-Impl.foldi* **.**

lift-definition *combine-with-key* :: (*'a* ⇒ *'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a*::*linorder*, *'b*) *rbt* ⇒
(*'a*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt*
is *RBT-Impl.rbt-union-with-key* **by** (*rule is-rbt-rbt-unionwk*)

lift-definition *combine* :: (*'b* ⇒ *'b* ⇒ *'b*) ⇒ (*'a*::*linorder*, *'b*) *rbt* ⇒ (*'a*, *'b*) *rbt* ⇒
(*'a*, *'b*) *rbt*
is *RBT-Impl.rbt-union-with* **by** (*rule rbt-unionw-is-rbt*)

120.3 Derived operations

definition *is-empty* :: (*'a*::*linorder*, *'b*) *rbt* ⇒ *bool* **where**

[*code*]: *is-empty t* = (*case impl-of t of RBT-Impl.Empty* ⇒ *True* | - ⇒ *False*)

definition *filter* :: ('*a* ⇒ '*b* ⇒ *bool*) ⇒ ('*a*::*linorder*, '*b*) *rbt* ⇒ ('*a*, '*b*) *rbt* **where**
[*code*]: *filter P t* = *fold* (λ*k v t*. *if P k v then insert k v t else t*) *t empty*

120.4 Abstract lookup properties

lemma *lookup-RBT*:
is-rbt t ⟹ *lookup* (*RBT t*) = *rbt-lookup t*
by (*simp add*: *lookup-def RBT-inverse*)

lemma *lookup-impl-of*:
rbt-lookup (*impl-of t*) = *lookup t*
by *transfer* (*rule refl*)

lemma *entries-impl-of*:
RBT-Impl.entries (*impl-of t*) = *entries t*
by *transfer* (*rule refl*)

lemma *keys-impl-of*:
RBT-Impl.keys (*impl-of t*) = *keys t*
by *transfer* (*rule refl*)

lemma *lookup-keys*:
dom (*lookup t*) = *set* (*keys t*)
by *transfer* (*simp add*: *rbt-lookup-keys*)

lemma *lookup-empty* [*simp*]:
lookup empty = *Map.empty*
by (*simp add*: *empty-def lookup-RBT fun-eq-iff*)

lemma *lookup-insert* [*simp*]:
lookup (*insert k v t*) = (*lookup t*)(*k* ↦ *v*)
by *transfer* (*rule rbt-lookup-rbt-insert*)

lemma *lookup-delete* [*simp*]:
lookup (*delete k t*) = (*lookup t*)(*k* := *None*)
by *transfer* (*simp add*: *rbt-lookup-rbt-delete restrict-complement-singleton-eq*)

lemma *map-of-entries* [*simp*]:
map-of (*entries t*) = *lookup t*
by *transfer* (*simp add*: *map-of-entries*)

lemma *entries-lookup*:
entries t1 = *entries t2* ⟷ *lookup t1* = *lookup t2*
by *transfer* (*simp add*: *entries-rbt-lookup*)

lemma *lookup-bulkload* [*simp*]:

lookup (*bulkload xs*) = *map-of xs*
by *transfer* (*rule rbt-lookup-rbt-bulkload*)

lemma *lookup-map-entry* [*simp*]:
lookup (*map-entry k f t*) = (*lookup t*)(*k* := *map-option f* (*lookup t k*))
by *transfer* (*rule rbt-lookup-rbt-map-entry*)

lemma *lookup-map* [*simp*]:
lookup (*map f t*) *k* = *map-option* (*f k*) (*lookup t k*)
by *transfer* (*rule rbt-lookup-map*)

lemma *lookup-combine-with-key* [*simp*]:
lookup (*combine-with-key f t1 t2*) *k* = *combine-options* (*f k*) (*lookup t1 k*) (*lookup t2 k*)
by *transfer* (*simp-all add*: *combine-options-def rbt-lookup-rbt-unionwk*)

lemma *combine-altdef*: *combine f t1 t2* = *combine-with-key* (λ-. *f*) *t1 t2*
by *transfer* (*simp add*: *rbt-union-with-def*)

lemma *lookup-combine* [*simp*]:
lookup (*combine f t1 t2*) *k* = *combine-options f* (*lookup t1 k*) (*lookup t2 k*)
by (*simp add*: *combine-altdef*)

lemma *fold-fold*:
fold f t = *List.fold* (*case-prod f*) (*entries t*)
by *transfer* (*rule RBT-Impl.fold-def*)

lemma *impl-of-empty*:
impl-of empty = *RBT-Impl.Empty*
by *transfer* (*rule refl*)

lemma *is-empty-empty* [*simp*]:
is-empty t ⟷ *t* = *empty*
unfolding *is-empty-def* **by** *transfer* (*simp split*: *rbt.split*)

lemma *RBT-lookup-empty* [*simp*]:
rbt-lookup t = *Map.empty* ⟷ *t* = *RBT-Impl.Empty*
by (*cases t*) (*auto simp add*: *fun-eq-iff*)

lemma *lookup-empty-empty* [*simp*]:
lookup t = *Map.empty* ⟷ *t* = *empty*
by *transfer* (*rule RBT-lookup-empty*)

lemma *sorted-keys* [*iff*]:
sorted (*keys t*)
by *transfer* (*simp add*: *RBT-Impl.keys-def rbt-sorted-entries*)

lemma *distinct-keys* [*iff*]:
distinct (*keys t*)

by *transfer* (*simp add*: *RBT-Impl.keys-def distinct-entries*)

lemma *finite-dom-lookup* [*simp*, *intro*!]: *finite* (*dom* (*lookup t*))
by *transfer simp*

lemma *lookup-union*: *lookup* (*union s t*) = *lookup s* ++ *lookup t*
by *transfer* (*simp add*: *rbt-lookup-rbt-union*)

lemma *lookup-in-tree*: (*lookup t k* = *Some v*) = ((*k*, *v*) ∈ *set* (*entries t*))
by *transfer* (*simp add*: *rbt-lookup-in-tree*)

lemma *keys-entries*: (*k* ∈ *set* (*keys t*)) = (∃ *v*. (*k*, *v*) ∈ *set* (*entries t*))
by *transfer* (*simp add*: *keys-entries*)

lemma *fold-def-alt*:
fold f t = *List.fold* (*case-prod f*) (*entries t*)
by *transfer* (*auto simp*: *RBT-Impl.fold-def*)

lemma *distinct-entries*: *distinct* (*List.map fst* (*entries t*))
by *transfer* (*simp add*: *distinct-entries*)

lemma *sorted-entries*: *sorted* (*List.map fst* (*entries t*))
by (*transfer*) (*simp add*: *rbt-sorted-entries*)

lemma *non-empty-keys*: *t* ≠ *empty* ⟹ *keys t* ≠ []
by *transfer* (*simp add*: *non-empty-rbt-keys*)

lemma *keys-def-alt*:
keys t = *List.map fst* (*entries t*)
by *transfer* (*simp add*: *RBT-Impl.keys-def*)

context
begin

private lemma *lookup-filter-aux*:
assumes *distinct* (*List.map fst xs*)
shows *lookup* (*List.fold* (λ(*k*, *v*) *t*. *if P k v then insert k v t else t*) *xs t*) *k* =
(*case map-of xs k of*
None ⇒ *lookup t k*
| *Some v* ⇒ *if P k v then Some v else lookup t k*)
using *assms* **by** (*induction xs arbitrary*: *t*) (*force split*: *option.splits*)+

lemma *lookup-filter*:
lookup (*filter P t*) *k* =
(*case lookup t k of None* ⇒ *None* | *Some v* ⇒ *if P k v then Some v else None*)
unfolding *filter-def* **using** *lookup-filter-aux*[*of entries t P empty k*]
by (*simp add*: *fold-fold distinct-entries split*: *option.splits*)

end

120.5 Quickcheck generators

quickcheck-generator *rbt predicate*: *is-rbt constructors*: *empty*, *insert*

120.6 Hide implementation details

lifting-update *rbt.lifting*
lifting-forget *rbt.lifting*

hide-const (**open**) *impl-of empty lookup keys entries bulkload delete map fold union insert map-entry foldi*
is-empty filter
hide-fact (**open**) *empty-def lookup-def keys-def entries-def bulkload-def delete-def map-def fold-def*
union-def insert-def map-entry-def foldi-def is-empty-def filter-def

end

121 Implementation of mappings with Red-Black Trees

This theory defines abstract red-black trees as an efficient representation of finite maps, backed by the implementation in *HOL−Library.RBT-Impl*.

121.1 Data type and invariant

The type ('*k*, '*v*) *RBT-Impl.rbt* denotes red-black trees with keys of type '*k* and values of type '*v*. To function properly, the key type musorted belong to the *linorder* class.

A value *t* of this type is a valid red-black tree if it satisfies the invariant *is-rbt t*. The abstract type ('*k*, '*v*) *RBT.rbt* always obeys this invariant, and for this reason you should only use this in our application. Going back to ('*k*, '*v*) *RBT-Impl.rbt* may be necessary in proofs if not yet proven properties about the operations must be established.

The interpretation function *RBT.lookup* returns the partial map represented by a red-black tree:

RBT.lookup::('*a*, '*b*) *RBT.rbt* ⇒ '*a* ⇒ '*b option*

This function should be used for reasoning about the semantics of the RBT operations. Furthermore, it implements the lookup functionality for the data structure: It is executable and the lookup is performed in $O(\log n)$.

121.2 Operations

Currently, the following operations are supported:

RBT.empty::('a, 'b) *RBT.rbt*

Returns the empty tree. $O(1)$

RBT.insert::'a ⇒ 'b ⇒ ('a, 'b) *RBT.rbt* ⇒ ('a, 'b) *RBT.rbt*

Updates the map at a given position. $O(\log n)$

RBT.delete::'a ⇒ ('a, 'b) *RBT.rbt* ⇒ ('a, 'b) *RBT.rbt*

Deletes a map entry at a given position. $O(\log n)$

RBT.entries::('a, 'b) *RBT.rbt* ⇒ ('a × 'b) *list*

Return a corresponding key-value list for a tree.

RBT.bulkload::('a × 'b) *list* ⇒ ('a, 'b) *RBT.rbt*

Builds a tree from a key-value list.

RBT.map-entry::'a ⇒ ('b ⇒ 'b) ⇒ ('a, 'b) *RBT.rbt* ⇒ ('a, 'b) *RBT.rbt*

Maps a single entry in a tree.

RBT.map::('a ⇒ 'b ⇒ 'c) ⇒ ('a, 'b) *RBT.rbt* ⇒ ('a, 'c) *RBT.rbt*

Maps all values in a tree. $O(n)$

RBT.fold::('a ⇒ 'b ⇒ 'c ⇒ 'c) ⇒ ('a, 'b) *RBT.rbt* ⇒ 'c ⇒ 'c

Folds over all entries in a tree. $O(n)$

121.3 Invariant preservation

is-rbt rbt.Empty (*Empty-is-rbt*)
is-rbt ?t ⟹ *is-rbt* (*rbt-insert ?k ?v ?t*) (*rbt-insert-is-rbt*)
is-rbt ?t ⟹ *is-rbt* (*rbt-delete ?k ?t*) (*delete-is-rbt*)
is-rbt (*rbt-bulkload ?xs*) (*bulkload-is-rbt*)
is-rbt (*rbt-map-entry ?k ?f ?t*) = *is-rbt ?t* (*map-entry-is-rbt*)
is-rbt (*RBT-Impl.map ?f ?t*) = *is-rbt ?t* (*map-is-rbt*)
⟦*is-rbt ?lt*; *is-rbt ?rt*⟧ ⟹ *is-rbt* (*rbt-union ?lt ?rt*) (*union-is-rbt*)

121.4 Map Semantics

lookup-empty

Mapping.lookup Mapping.empty ?k = None

lookup-insert

RBT.lookup (RBT.insert ?k ?v ?t) = RBT.lookup ?t(?k ↦ ?v)

lookup-delete

Mapping.lookup (Mapping.delete ?k ?m) ?k = None

lookup-bulkload

RBT.lookup (RBT.bulkload ?xs) = map-of ?xs

lookup-map

RBT.lookup (RBT.map ?f ?t) ?k = map-option (?f ?k) (RBT.lookup ?t ?k)

end

122 Implementation of sets using RBT trees

theory *RBT-Set*
imports *RBT Product-Lexorder*
begin

123 Definition of code datatype constructors

definition *Set* :: (*'a*::*linorder*, *unit*) *rbt* ⇒ *'a set*
  where *Set t* = {*x* . *RBT.lookup t x* = *Some* ()}

definition *Coset* :: (*'a*::*linorder*, *unit*) *rbt* ⇒ *'a set*
  where [*simp*]: *Coset t* = − *Set t*

124 Deletion of already existing code equations

declare [[*code drop*: *Set.empty Set.is-empty uminus-set-inst.uminus-set*
  Set.member Set.insert Set.remove UNIV Set.filter image
  Set.subset-eq Ball Bex can-select Set.union minus-set-inst.minus-set Set.inter
  card the-elem Pow sum prod Product-Type.product Id-on
  Image trancl relcomp wf Min Inf-fin Max Sup-fin
  (*Inf* :: *'a set set* ⇒ *'a set*) (*Sup* :: *'a set set* ⇒ *'a set*)
  sorted-list-of-set List.map-project List.Bleast]]

125 Lemmas

125.1 Auxiliary lemmas

lemma [*simp*]: $x \neq Some\ () \longleftrightarrow x = None$
by (*auto simp*: *not-Some-eq*[*THEN iffD1*])

lemma *Set-set-keys*: $Set\ x = dom\ (RBT.lookup\ x)$
by (*auto simp*: *Set-def*)

lemma *finite-Set* [*simp*, *intro*!]: *finite* (*Set x*)
by (*simp add*: *Set-set-keys*)

lemma *set-keys*: $Set\ t = set(RBT.keys\ t)$
by (*simp add*: *Set-set-keys lookup-keys*)

125.2 fold and filter

lemma *finite-fold-rbt-fold-eq*:
assumes *comp-fun-commute f*
shows *Finite-Set.fold f A* (*set* (*RBT.entries t*)) = *RBT.fold* (*curry f*) *t A*
proof −
interpret *comp-fun-commute*: *comp-fun-commute f*
by (*fact assms*)
have ∗: *remdups* (*RBT.entries t*) = *RBT.entries t*
using *distinct-entries distinct-map* **by** (*auto intro*: *distinct-remdups-id*)
show *?thesis* **using** *assms* **by** (*auto simp*: *fold-def-alt comp-fun-commute.fold-set-fold-remdups* ∗)
qed

definition *fold-keys* :: ('*a* :: *linorder* ⇒ '*b* ⇒ '*b*) ⇒ ('*a*, -) *rbt* ⇒ '*b* ⇒ '*b*
where [*code-unfold*]:*fold-keys f t A* = *RBT.fold* (λ*k* - *t*. *f k t*) *t A*

lemma *fold-keys-def-alt*:
fold-keys f t s = *List.fold f* (*RBT.keys t*) *s*
by (*auto simp*: *fold-map o-def split-def fold-def-alt keys-def-alt fold-keys-def*)

lemma *finite-fold-fold-keys*:
assumes *comp-fun-commute f*
shows *Finite-Set.fold f A* (*Set t*) = *fold-keys f t A*
using *assms*
proof −
interpret *comp-fun-commute f* **by** *fact*
have *set* (*RBT.keys t*) = *fst* ' (*set* (*RBT.entries t*)) **by** (*auto simp*: *fst-eq-Domain keys-entries*)
moreover have *inj-on fst* (*set* (*RBT.entries t*)) **using** *distinct-entries distinct-map* **by** *auto*
ultimately show *?thesis*
by (*auto simp add*: *set-keys fold-keys-def curry-def fold-image finite-fold-rbt-fold-eq*

comp-comp-fun-commute)
qed

definition *rbt-filter* :: (*'a* :: *linorder* ⇒ *bool*) ⇒ (*'a*, *'b*) *rbt* ⇒ *'a set* **where**
rbt-filter P t = *RBT.fold* (λ*k* - *A'*. *if P k then Set.insert k A' else A'*) *t* {}

lemma *Set-filter-rbt-filter*:
Set.filter P (*Set t*) = *rbt-filter P t*
by (*simp add*: *fold-keys-def Set-filter-fold rbt-filter-def*
finite-fold-fold-keys[*OF comp-fun-commute-filter-fold*])

125.3 foldi and Ball

lemma *Ball-False*: *RBT-Impl.fold* (λ*k v s*. *s* ∧ *P k*) *t False* = *False*
by (*induction t*) *auto*

lemma *rbt-foldi-fold-conj*:
RBT-Impl.foldi (λ*s*. *s* = *True*) (λ*k v s*. *s* ∧ *P k*) *t val* = *RBT-Impl.fold* (λ*k v s*. *s* ∧ *P k*) *t val*
proof (*induction t arbitrary*: *val*)
case (*Branch c t1*) **then show** *?case*
by (*cases RBT-Impl.fold* (λ*k v s*. *s* ∧ *P k*) *t1 True*) (*simp-all add*: *Ball-False*)
qed *simp*

lemma *foldi-fold-conj*: *RBT.foldi* (λ*s*. *s* = *True*) (λ*k v s*. *s* ∧ *P k*) *t val* = *fold-keys* (λ*k s*. *s* ∧ *P k*) *t val*
unfolding *fold-keys-def* **including** *rbt.lifting* **by** *transfer* (*rule rbt-foldi-fold-conj*)

125.4 foldi and Bex

lemma *Bex-True*: *RBT-Impl.fold* (λ*k v s*. *s* ∨ *P k*) *t True* = *True*
by (*induction t*) *auto*

lemma *rbt-foldi-fold-disj*:
RBT-Impl.foldi (λ*s*. *s* = *False*) (λ*k v s*. *s* ∨ *P k*) *t val* = *RBT-Impl.fold* (λ*k v s*. *s* ∨ *P k*) *t val*
proof (*induction t arbitrary*: *val*)
case (*Branch c t1*) **then show** *?case*
by (*cases RBT-Impl.fold* (λ*k v s*. *s* ∨ *P k*) *t1 False*) (*simp-all add*: *Bex-True*)
qed *simp*

lemma *foldi-fold-disj*: *RBT.foldi* (λ*s*. *s* = *False*) (λ*k v s*. *s* ∨ *P k*) *t val* = *fold-keys* (λ*k s*. *s* ∨ *P k*) *t val*
unfolding *fold-keys-def* **including** *rbt.lifting* **by** *transfer* (*rule rbt-foldi-fold-disj*)

125.5 folding over non empty trees and selecting the minimal and maximal element

125.5.1 concrete

The concrete part is here because it's probably not general enough to be moved to *RBT-Impl*

definition *rbt-fold1-keys* :: ('*a* ⇒ '*a* ⇒ '*a*) ⇒ ('*a*::*linorder*, '*b*) *RBT-Impl.rbt* ⇒ '*a*
where *rbt-fold1-keys f t* = *List.fold f* (*tl*(*RBT-Impl.keys t*)) (*hd*(*RBT-Impl.keys t*))

minimum **definition** *rbt-min* :: ('*a*::*linorder*, *unit*) *RBT-Impl.rbt* ⇒ '*a*
where *rbt-min t* = *rbt-fold1-keys min t*

lemma *key-le-right*: *rbt-sorted* (*Branch c lt k v rt*) ⟹ (⋀*x*. *x* ∈*set* (*RBT-Impl.keys rt*) ⟹ *k* ≤ *x*)
by (*auto simp*: *rbt-greater-prop less-imp-le*)

lemma *left-le-key*: *rbt-sorted* (*Branch c lt k v rt*) ⟹ (⋀*x*. *x* ∈*set* (*RBT-Impl.keys lt*) ⟹ *x* ≤ *k*)
by (*auto simp*: *rbt-less-prop less-imp-le*)

lemma *fold-min-triv*:
fixes *k* :: - :: *linorder*
shows (∀ *x*∈*set xs*. *k* ≤ *x*) ⟹ *List.fold min xs k* = *k*
by (*induct xs*) (*auto simp add*: *min-def*)

lemma *rbt-min-simps*:
is-rbt (*Branch c RBT-Impl.Empty k v rt*) ⟹ *rbt-min* (*Branch c RBT-Impl.Empty k v rt*) = *k*
by (*auto intro*: *fold-min-triv dest*: *key-le-right is-rbt-rbt-sorted simp*: *rbt-fold1-keys-def rbt-min-def*)

fun *rbt-min-opt* **where**
rbt-min-opt (*Branch c RBT-Impl.Empty k v rt*) = *k* |
rbt-min-opt (*Branch c* (*Branch lc llc lk lv lrt*) *k v rt*) = *rbt-min-opt* (*Branch lc llc lk lv lrt*)

lemma *rbt-min-opt-Branch*:
t1 ≠ *rbt.Empty* ⟹ *rbt-min-opt* (*Branch c t1 k* () *t2*) = *rbt-min-opt t1*
by (*cases t1*) *auto*

lemma *rbt-min-opt-induct* [*case-names empty left-empty left-non-empty*]:
fixes *t* :: ('*a* :: *linorder*, *unit*) *RBT-Impl.rbt*
assumes *P rbt.Empty*
assumes ⋀*color t1 a b t2*. *P t1* ⟹ *P t2* ⟹ *t1* = *rbt.Empty* ⟹ *P* (*Branch color t1 a b t2*)
assumes ⋀*color t1 a b t2*. *P t1* ⟹ *P t2* ⟹ *t1* ≠ *rbt.Empty* ⟹ *P* (*Branch*

color t1 a b t2)
shows *P t*
using *assms*
proof (*induct t*)
case *Empty*
then show *?case* **by** *simp*
next
case (*Branch x1 t1 x3 x4 t2*)
then show *?case* **by** (*cases t1 = rbt.Empty*) *simp-all*
qed

lemma *rbt-min-opt-in-set*:
fixes *t* :: ('*a* :: *linorder*, *unit*) *RBT-Impl.rbt*
assumes *t* ≠ *rbt.Empty*
shows *rbt-min-opt t* ∈ *set* (*RBT-Impl.keys t*)
using *assms* **by** (*induction t rule*: *rbt-min-opt.induct*) (*auto*)

lemma *rbt-min-opt-is-min*:
fixes *t* :: ('*a* :: *linorder*, *unit*) *RBT-Impl.rbt*
assumes *rbt-sorted t*
assumes *t* ≠ *rbt.Empty*
shows ⋀*y*. *y* ∈ *set* (*RBT-Impl.keys t*) ⟹ *y* ≥ *rbt-min-opt t*
using *assms*
proof (*induction t rule*: *rbt-min-opt-induct*)
case *empty*
then show *?case* **by** *simp*
next
case *left-empty*
then show *?case* **by** (*auto intro*: *key-le-right simp del*: *rbt-sorted.simps*)
next
case (*left-non-empty c t1 k v t2 y*)
then consider *y* = *k* | *y* ∈ *set* (*RBT-Impl.keys t1*) | *y* ∈ *set* (*RBT-Impl.keys t2*)
by *auto*
then show *?case*
proof *cases*
case *1*
with *left-non-empty* **show** *?thesis*
by (*auto simp add*: *rbt-min-opt-Branch intro*: *left-le-key rbt-min-opt-in-set*)
next
case *2*
with *left-non-empty* **show** *?thesis*
by (*auto simp add*: *rbt-min-opt-Branch*)
next
case *y*: *3*
have *rbt-min-opt t1* ≤ *k*
using *left-non-empty* **by** (*simp add*: *left-le-key rbt-min-opt-in-set*)
moreover have *k* ≤ *y*
using *left-non-empty y* **by** (*simp add*: *key-le-right*)

ultimately show *?thesis*
using *left-non-empty y* **by** (*simp add*: *rbt-min-opt-Branch*)
qed
qed

lemma *rbt-min-eq-rbt-min-opt*:
assumes $t \neq$ *RBT-Impl.Empty*
assumes *is-rbt t*
shows *rbt-min t* = *rbt-min-opt t*
proof −
from *assms* **have** *hd* (*RBT-Impl.keys t*) # *tl* (*RBT-Impl.keys t*) = *RBT-Impl.keys t* **by** (*cases t*) *simp-all*
with *assms* **show** *?thesis*
by (*simp add*: *rbt-min-def rbt-fold1-keys-def rbt-min-opt-is-min Min.set-eq-fold* [*symmetric*] *Min-eqI rbt-min-opt-in-set*)
qed

maximum definition *rbt-max* :: ('*a*::*linorder*, *unit*) *RBT-Impl.rbt* ⇒ '*a*
where *rbt-max t* = *rbt-fold1-keys max t*

lemma *fold-max-triv*:
fixes *k* :: - :: *linorder*
shows (∀ *x*∈*set xs*. *x* ≤ *k*) ⟹ *List.fold max xs k* = *k*
by (*induct xs*) (*auto simp add*: *max-def*)

lemma *fold-max-rev-eq*:
fixes *xs* :: ('*a* :: *linorder*) *list*
assumes *xs* ≠ []
shows *List.fold max* (*tl xs*) (*hd xs*) = *List.fold max* (*tl* (*rev xs*)) (*hd* (*rev xs*))
using *assms* **by** (*simp add*: *Max.set-eq-fold* [*symmetric*])

lemma *rbt-max-simps*:
assumes *is-rbt* (*Branch c lt k v RBT-Impl.Empty*)
shows *rbt-max* (*Branch c lt k v RBT-Impl.Empty*) = *k*
proof −
have *List.fold max* (*tl* (*rev*(*RBT-Impl.keys lt* @ [*k*]))) (*hd* (*rev*(*RBT-Impl.keys lt* @ [*k*]))) = *k*
using *assms* **by** (*auto intro*!: *fold-max-triv dest*!: *left-le-key is-rbt-rbt-sorted*)
then show *?thesis* **by** (*auto simp add*: *rbt-max-def rbt-fold1-keys-def fold-max-rev-eq*)
qed

fun *rbt-max-opt* **where**
rbt-max-opt (*Branch c lt k v RBT-Impl.Empty*) = *k* |
rbt-max-opt (*Branch c lt k v* (*Branch rc rlc rk rv rrt*)) = *rbt-max-opt* (*Branch rc rlc rk rv rrt*)

lemma *rbt-max-opt-Branch*:
t2 ≠ *rbt.Empty* ⟹ *rbt-max-opt* (*Branch c t1 k* () *t2*) = *rbt-max-opt t2*
by (*cases t2*) *auto*

lemma *rbt-max-opt-induct* [*case-names empty right-empty right-non-empty*]:
fixes *t* :: ('*a* :: *linorder*, *unit*) *RBT-Impl.rbt*
assumes *P rbt.Empty*
assumes $\bigwedge$*color t1 a b t2*. *P t1* $\Longrightarrow$ *P t2* $\Longrightarrow$ *t2* = *rbt.Empty* $\Longrightarrow$ *P* (*Branch color t1 a b t2*)
assumes $\bigwedge$*color t1 a b t2*. *P t1* $\Longrightarrow$ *P t2* $\Longrightarrow$ *t2* $\neq$ *rbt.Empty* $\Longrightarrow$ *P* (*Branch color t1 a b t2*)
shows *P t*
using *assms*
proof (*induct t*)
case *Empty*
then show *?case* **by** *simp*
next
case (*Branch x1 t1 x3 x4 t2*)
then show *?case* **by** (*cases t2* = *rbt.Empty*) *simp-all*
qed

lemma *rbt-max-opt-in-set*:
fixes *t* :: ('*a* :: *linorder*, *unit*) *RBT-Impl.rbt*
assumes *t* $\neq$ *rbt.Empty*
shows *rbt-max-opt t* $\in$ *set* (*RBT-Impl.keys t*)
using *assms* **by** (*induction t rule*: *rbt-max-opt.induct*) (*auto*)

lemma *rbt-max-opt-is-max*:
fixes *t* :: ('*a* :: *linorder*, *unit*) *RBT-Impl.rbt*
assumes *rbt-sorted t*
assumes *t* $\neq$ *rbt.Empty*
shows $\bigwedge$*y*. *y* $\in$ *set* (*RBT-Impl.keys t*) $\Longrightarrow$ *y* $\leq$ *rbt-max-opt t*
using *assms*
proof (*induction t rule*: *rbt-max-opt-induct*)
case *empty*
then show *?case* **by** *simp*
next
case *right-empty*
then show *?case* **by** (*auto intro*: *left-le-key simp del*: *rbt-sorted.simps*)
next
case (*right-non-empty c t1 k v t2 y*)
then consider *y* = *k* | *y* $\in$ *set* (*RBT-Impl.keys t2*) | *y* $\in$ *set* (*RBT-Impl.keys t1*)
by *auto*
then show *?case*
proof *cases*
case *1*
with *right-non-empty* **show** *?thesis*
by (*auto simp add*: *rbt-max-opt-Branch intro*: *key-le-right rbt-max-opt-in-set*)
next
case *2*
with *right-non-empty* **show** *?thesis*

```
    by (auto simp add: rbt-max-opt-Branch)
  next
    case y: 3
    have rbt-max-opt t2 ≥ k
      using right-non-empty by (simp add: key-le-right rbt-max-opt-in-set)
    moreover have y ≤ k
      using right-non-empty y by (simp add: left-le-key)
    ultimately show ?thesis
      using right-non-empty by (simp add: rbt-max-opt-Branch)
  qed
qed

lemma rbt-max-eq-rbt-max-opt:
  assumes t ≠ RBT-Impl.Empty
  assumes is-rbt t
  shows rbt-max t = rbt-max-opt t
proof −
  from assms have hd (RBT-Impl.keys t) # tl (RBT-Impl.keys t) = RBT-Impl.keys
t by (cases t) simp-all
  with assms show ?thesis
    by (simp add: rbt-max-def rbt-fold1-keys-def rbt-max-opt-is-max
      Max.set-eq-fold [symmetric] Max-eqI rbt-max-opt-in-set)
qed
```

125.5.2 abstract

```
context includes rbt.lifting begin
lift-definition fold1-keys :: ('a ⇒ 'a ⇒ 'a) ⇒ ('a::linorder, 'b) rbt ⇒ 'a
  is rbt-fold1-keys .

lemma fold1-keys-def-alt:
  fold1-keys f t = List.fold f (tl (RBT.keys t)) (hd (RBT.keys t))
  by transfer (simp add: rbt-fold1-keys-def)

lemma finite-fold1-fold1-keys:
  assumes semilattice f
  assumes ¬ RBT.is-empty t
  shows semilattice-set.F f (Set t) = fold1-keys f t
proof −
  from ‹semilattice f› interpret semilattice-set f by (rule semilattice-set.intro)
  show ?thesis using assms
    by (auto simp: fold1-keys-def-alt set-keys fold-def-alt non-empty-keys set-eq-fold
[symmetric])
qed
```

minimum **lift-definition** *r-min* :: ('*a* :: *linorder*, *unit*) *rbt* ⇒ '*a* **is** *rbt-min* .

lift-definition *r-min-opt* :: ('*a* :: *linorder*, *unit*) *rbt* ⇒ '*a* **is** *rbt-min-opt* .

lemma *r-min-alt-def*: *r-min t = fold1-keys min t*
by *transfer* (*simp add*: *rbt-min-def*)

lemma *r-min-eq-r-min-opt*:
assumes ¬ (*RBT.is-empty t*)
shows *r-min t = r-min-opt t*
using *assms* **unfolding** *is-empty-empty* **by** *transfer* (*auto intro*: *rbt-min-eq-rbt-min-opt*)

lemma *fold-keys-min-top-eq*:
fixes *t* :: ('*a*::{*linorder*,*bounded-lattice-top*}, *unit*) *rbt*
assumes ¬ (*RBT.is-empty t*)
shows *fold-keys min t top = fold1-keys min t*
proof −
have ∗: ⋀*t*. *RBT-Impl.keys t* ≠ [] ⟹ *List.fold min* (*RBT-Impl.keys t*) *top* =
List.fold min (*hd* (*RBT-Impl.keys t*) # *tl* (*RBT-Impl.keys t*)) *top*
by (*simp add*: *hd-Cons-tl*[*symmetric*])
have ∗∗: *List.fold min* (*x* # *xs*) *top* = *List.fold min xs x* **for** *x* :: '*a* **and** *xs*
by (*simp add*: *inf-min*[*symmetric*])
show *?thesis*
using *assms*
unfolding *fold-keys-def-alt fold1-keys-def-alt is-empty-empty*
apply *transfer*
apply (*case-tac t*)
apply *simp*
apply (*subst* ∗)
apply *simp*
apply (*subst* ∗∗)
apply *simp*
done
qed

maximum lift-definition *r-max* :: ('*a* :: *linorder*, *unit*) *rbt* ⇒ '*a* **is** *rbt-max* **.**

lift-definition *r-max-opt* :: ('*a* :: *linorder*, *unit*) *rbt* ⇒ '*a* **is** *rbt-max-opt* **.**

lemma *r-max-alt-def*: *r-max t = fold1-keys max t*
by *transfer* (*simp add*: *rbt-max-def*)

lemma *r-max-eq-r-max-opt*:
assumes ¬ (*RBT.is-empty t*)
shows *r-max t = r-max-opt t*
using *assms* **unfolding** *is-empty-empty* **by** *transfer* (*auto intro*: *rbt-max-eq-rbt-max-opt*)

lemma *fold-keys-max-bot-eq*:
fixes *t* :: ('*a*::{*linorder*,*bounded-lattice-bot*}, *unit*) *rbt*
assumes ¬ (*RBT.is-empty t*)
shows *fold-keys max t bot = fold1-keys max t*
proof −
have ∗: ⋀*t*. *RBT-Impl.keys t* ≠ [] ⟹ *List.fold max* (*RBT-Impl.keys t*) *bot* =

List.fold max (*hd*(*RBT-Impl.keys t*) # *tl*(*RBT-Impl.keys t*)) *bot*
by (*simp add*: *hd-Cons-tl*[*symmetric*])
have ∗∗: *List.fold max* (*x* # *xs*) *bot* = *List.fold max xs x* **for** *x* :: *'a* **and** *xs*
by (*simp add*: *sup-max*[*symmetric*])
show *?thesis*
using *assms*
unfolding *fold-keys-def-alt fold1-keys-def-alt is-empty-empty*
apply *transfer*
apply (*case-tac t*)
apply *simp*
apply (*subst* ∗)
apply *simp*
apply (*subst* ∗∗)
apply *simp*
done
qed

end

126 Code equations

code-datatype *Set Coset*

declare *list.set*[*code*]

lemma *empty-Set* [*code*]:
Set.empty = *Set RBT.empty*
by (*auto simp*: *Set-def*)

lemma *UNIV-Coset* [*code*]:
UNIV = *Coset RBT.empty*
by (*auto simp*: *Set-def*)

lemma *is-empty-Set* [*code*]:
Set.is-empty (*Set t*) = *RBT.is-empty t*
unfolding *Set.is-empty-def* **by** (*auto simp*: *fun-eq-iff Set-def intro*: *lookup-empty-empty*[*THEN iffD1*])

lemma *compl-code* [*code*]:
− *Set xs* = *Coset xs*
− *Coset xs* = *Set xs*
by (*simp-all add*: *Set-def*)

lemma *member-code* [*code*]:
x ∈ (*Set t*) = (*RBT.lookup t x* = *Some* ())
x ∈ (*Coset t*) = (*RBT.lookup t x* = *None*)
by (*simp-all add*: *Set-def*)

lemma *insert-code* [*code*]:

Set.insert x (Set t) = Set (RBT.insert x () t)
Set.insert x (Coset t) = Coset (RBT.delete x t)
by (*auto simp*: *Set-def*)

lemma *remove-code* [*code*]:
Set.remove x (Set t) = Set (RBT.delete x t)
Set.remove x (Coset t) = Coset (RBT.insert x () t)
by (*auto simp*: *Set-def*)

lemma *union-Set* [*code*]:
Set t ∪ A = fold-keys Set.insert t A
proof −
interpret *comp-fun-idem Set.insert*
by (*fact comp-fun-idem-insert*)
from *finite-fold-fold-keys*[*OF comp-fun-commute-axioms*]
show *?thesis* **by** (*auto simp add*: *union-fold-insert*)
qed

lemma *inter-Set* [*code*]:
A ∩ Set t = rbt-filter (λk. k ∈ A) t
by (*simp add*: *inter-Set-filter Set-filter-rbt-filter*)

lemma *minus-Set* [*code*]:
A − Set t = fold-keys Set.remove t A
proof −
interpret *comp-fun-idem Set.remove*
by (*fact comp-fun-idem-remove*)
from *finite-fold-fold-keys*[*OF comp-fun-commute-axioms*]
show *?thesis* **by** (*auto simp add*: *minus-fold-remove*)
qed

lemma *union-Coset* [*code*]:
Coset t ∪ A = − rbt-filter (λk. k ∉ A) t
proof −
have ∗: $\bigwedge A\ B.\ (-A \cup B) = -(-B \cap A)$ **by** *blast*
show *?thesis* **by** (*simp del*: *boolean-algebra-class.compl-inf add*: ∗ *inter-Set*)
qed

lemma *union-Set-Set* [*code*]:
Set t1 ∪ Set t2 = Set (RBT.union t1 t2)
by (*auto simp add*: *lookup-union map-add-Some-iff Set-def*)

lemma *inter-Coset* [*code*]:
A ∩ Coset t = fold-keys Set.remove t A
by (*simp add*: *Diff-eq* [*symmetric*] *minus-Set*)

lemma *inter-Coset-Coset* [*code*]:
Coset t1 ∩ Coset t2 = Coset (RBT.union t1 t2)
by (*auto simp add*: *lookup-union map-add-Some-iff Set-def*)

lemma *minus-Coset* [*code*]:
A − Coset t = rbt-filter (λ*k*. *k* ∈ *A*) *t*
by (*simp add*: *inter-Set*[*simplified Int-commute*])

lemma *filter-Set* [*code*]:
Set.filter P (*Set t*) = (*rbt-filter P t*)
by (*auto simp add*: *Set-filter-rbt-filter*)

lemma *image-Set* [*code*]:
image f (*Set t*) = *fold-keys* (λ*k A*. *Set.insert* (*f k*) *A*) *t* {}
proof −
have *comp-fun-commute* (λ*k*. *Set.insert* (*f k*))
by *standard auto*
then show *?thesis*
by (*auto simp add*: *image-fold-insert intro*!: *finite-fold-fold-keys*)
qed

lemma *Ball-Set* [*code*]:
Ball (*Set t*) *P* ⟷ *RBT.foldi* (λ*s*. *s* = *True*) (λ*k v s*. *s* ∧ *P k*) *t True*
proof −
have *comp-fun-commute* (λ*k s*. *s* ∧ *P k*)
by *standard auto*
then show *?thesis*
by (*simp add*: *foldi-fold-conj*[*symmetric*] *Ball-fold finite-fold-fold-keys*)
qed

lemma *Bex-Set* [*code*]:
Bex (*Set t*) *P* ⟷ *RBT.foldi* (λ*s*. *s* = *False*) (λ*k v s*. *s* ∨ *P k*) *t False*
proof −
have *comp-fun-commute* (λ*k s*. *s* ∨ *P k*)
by *standard auto*
then show *?thesis*
by (*simp add*: *foldi-fold-disj*[*symmetric*] *Bex-fold finite-fold-fold-keys*)
qed

lemma *subset-code* [*code*]:
Set t ≤ *B* ⟷ (∀ *x*∈*Set t*. *x* ∈ *B*)
A ≤ *Coset t* ⟷ (∀ *y*∈*Set t*. *y* ∉ *A*)
by *auto*

lemma *subset-Coset-empty-Set-empty* [*code*]:
Coset t1 ≤ *Set t2* ⟷ (*case* (*RBT.impl-of t1*, *RBT.impl-of t2*) *of*
(*rbt.Empty*, *rbt.Empty*) ⇒ *False* |
(-, -) ⇒ *Code.abort* (*STR* ''*non-empty-trees*'') (λ-. *Coset t1* ≤ *Set t2*))
proof −
have ∗: ⋀*t*. *RBT.impl-of t* = *rbt.Empty* ⟹ *t* = *RBT rbt.Empty*
by (*subst*(*asm*) *RBT-inverse*[*symmetric*]) (*auto simp*: *impl-of-inject*)
have ∗∗: *eq-onp is-rbt rbt.Empty rbt.Empty* **unfolding** *eq-onp-def* **by** *simp*

```
  show ?thesis
    by (auto simp: Set-def lookup.abs-eq[OF **] dest!: * split: rbt.split)
qed
```

A frequent case – avoid intermediate sets

```
lemma [code-unfold]:
  Set t1 ⊆ Set t2 ⟷ RBT.foldi (λs. s = True) (λk v s. s ∧ k ∈ Set t2) t1 True
by (simp add: subset-code Ball-Set)

lemma card-Set [code]:
  card (Set t) = fold-keys (λ- n. n + 1) t 0
 by (auto simp add: card.eq-fold intro: finite-fold-fold-keys comp-fun-commute-const)

lemma sum-Set [code]:
  sum f (Set xs) = fold-keys (plus ∘ f) xs 0
proof −
  have comp-fun-commute (λx. (+) (f x))
    by standard (auto simp: ac-simps)
  then show ?thesis
    by (auto simp add: sum.eq-fold finite-fold-fold-keys o-def)
qed

lemma the-elem-set [code]:
  fixes t :: ('a :: linorder, unit) rbt
  shows the-elem (Set t) = (case RBT.impl-of t of
    (Branch RBT-Impl.B RBT-Impl.Empty x () RBT-Impl.Empty) ⇒ x
    | - ⇒ Code.abort (STR ''not-a-singleton-tree'') (λ-. the-elem (Set t)))
proof −
  {
    fix x :: 'a :: linorder
    let ?t = Branch RBT-Impl.B RBT-Impl.Empty x () RBT-Impl.Empty
    have *:?t ∈ {t. is-rbt t} unfolding is-rbt-def by auto
    then have **:eq-onp is-rbt ?t ?t unfolding eq-onp-def by auto

    have RBT.impl-of t = ?t ⟹ the-elem (Set t) = x
      by (subst(asm) RBT-inverse[symmetric, OF *])
        (auto simp: Set-def the-elem-def lookup.abs-eq[OF **] impl-of-inject)
  }
  then show ?thesis
    by(auto split: rbt.split unit.split color.split)
qed

lemma Pow-Set [code]: Pow (Set t) = fold-keys (λx A. A ∪ Set.insert x ` A) t
{{}}
  by (simp add: Pow-fold finite-fold-fold-keys[OF comp-fun-commute-Pow-fold])

lemma product-Set [code]:
  Product-Type.product (Set t1) (Set t2) =
    fold-keys (λx A. fold-keys (λy. Set.insert (x, y)) t2 A) t1 {}
```

proof −
have ∗: *comp-fun-commute* (λ*y*. *Set.insert* (*x*, *y*)) **for** *x*
by *standard auto*
show *?thesis* **using** *finite-fold-fold-keys*[*OF comp-fun-commute-product-fold*, *of Set t2* {} *t1*]
by (*simp add*: *product-fold Product-Type.product-def finite-fold-fold-keys*[*OF* ∗])
qed

lemma *Id-on-Set* [*code*]: *Id-on* (*Set t*) = *fold-keys* (λ*x*. *Set.insert* (*x*, *x*)) *t* {}
proof −
have *comp-fun-commute* (λ*x*. *Set.insert* (*x*, *x*))
by *standard auto*
then show *?thesis*
by (*auto simp add*: *Id-on-fold intro*!: *finite-fold-fold-keys*)
qed

lemma *Image-Set* [*code*]:
(*Set t*) `` *S* = *fold-keys* (λ(*x*,*y*) *A*. *if x* ∈ *S then Set.insert y A else A*) *t* {}
by (*auto simp add*: *Image-fold finite-fold-fold-keys*[*OF comp-fun-commute-Image-fold*])

lemma *trancl-set-ntrancl* [*code*]:
trancl (*Set t*) = *ntrancl* (*card* (*Set t*) − *1*) (*Set t*)
by (*simp add*: *finite-trancl-ntranl*)

lemma *relcomp-Set*[*code*]:
(*Set t1*) *O* (*Set t2*) = *fold-keys*
(λ(*x*,*y*) *A*. *fold-keys* (λ(*w*,*z*) *A*′. *if y* = *w then Set.insert* (*x*,*z*) *A*′ *else A*′) *t2 A*)
t1 {}
proof −
interpret *comp-fun-idem Set.insert*
by (*fact comp-fun-idem-insert*)
have ∗: ⋀*x y*. *comp-fun-commute* (λ(*w*, *z*) *A*′. *if y* = *w then Set.insert* (*x*, *z*) *A*′ *else A*′)
by *standard* (*auto simp add*: *fun-eq-iff*)
show *?thesis*
using *finite-fold-fold-keys*[*OF comp-fun-commute-relcomp-fold*, *of Set t2* {} *t1*]
by (*simp add*: *relcomp-fold finite-fold-fold-keys*[*OF* ∗])
qed

lemma *wf-set* [*code*]:
wf (*Set t*) = *acyclic* (*Set t*)
by (*simp add*: *wf-iff-acyclic-if-finite*)

lemma *Min-fin-set-fold* [*code*]:
Min (*Set t*) =
(*if RBT.is-empty t*
then Code.abort (*STR* ''*not-non-empty-tree*'') (λ-. *Min* (*Set t*))
else r-min-opt t)
proof −

have $*$: *semilattice* (*min* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a*) **..**
with *finite-fold1-fold1-keys* [*OF* $*$, *folded Min-def*]
show *?thesis*
by (*simp add*: *r-min-alt-def r-min-eq-r-min-opt* [*symmetric*])
qed

lemma *Inf-fin-set-fold* [*code*]:
Inf-fin (*Set t*) = *Min* (*Set t*)
by (*simp add*: *inf-min Inf-fin-def Min-def*)

lemma *Inf-Set-fold*:
fixes *t* :: (*'a* :: {*linorder*, *complete-lattice*}, *unit*) *rbt*
shows *Inf* (*Set t*) = (*if RBT.is-empty t then top else r-min-opt t*)
proof −
have *comp-fun-commute* (*min* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a*)
by *standard* (*simp add*: *fun-eq-iff ac-simps*)
then have *t* $\neq$ *RBT.empty* $\Longrightarrow$ *Finite-Set.fold min top* (*Set t*) = *fold1-keys min t*
by (*simp add*: *finite-fold-fold-keys fold-keys-min-top-eq*)
then show *?thesis*
by (*auto simp add*: *Inf-fold-inf inf-min empty-Set*[*symmetric*]
r-min-eq-r-min-opt[*symmetric*] *r-min-alt-def*)
qed

lemma *Max-fin-set-fold* [*code*]:
Max (*Set t*) =
(*if RBT.is-empty t*
then Code.abort (*STR ''not-non-empty-tree''*) (λ-. *Max* (*Set t*))
else r-max-opt t)
proof −
have $*$: *semilattice* (*max* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a*) **..**
with *finite-fold1-fold1-keys* [*OF* $*$, *folded Max-def*]
show *?thesis*
by (*simp add*: *r-max-alt-def r-max-eq-r-max-opt* [*symmetric*])
qed

lemma *Sup-fin-set-fold* [*code*]:
Sup-fin (*Set t*) = *Max* (*Set t*)
by (*simp add*: *sup-max Sup-fin-def Max-def*)

lemma *Sup-Set-fold*:
fixes *t* :: (*'a* :: {*linorder*, *complete-lattice*}, *unit*) *rbt*
shows *Sup* (*Set t*) = (*if RBT.is-empty t then bot else r-max-opt t*)
proof −
have *comp-fun-commute* (*max* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a*)
by *standard* (*simp add*: *fun-eq-iff ac-simps*)
then have *t* $\neq$ *RBT.empty* $\Longrightarrow$ *Finite-Set.fold max bot* (*Set t*) = *fold1-keys max t*
by (*simp add*: *finite-fold-fold-keys fold-keys-max-bot-eq*)

then show *?thesis*
by (*auto simp add*: *Sup-fold-sup sup-max empty-Set*[*symmetric*]
r-max-eq-r-max-opt[*symmetric*] *r-max-alt-def*)
qed

context
begin

declare [[*code drop*: *Gcd-fin Lcm-fin* ‹*Gcd* :: - ⇒ *nat*› ‹*Gcd* :: - ⇒ *int*› ‹*Lcm* :: - ⇒ *nat*› ‹*Lcm* :: - ⇒ *int*›]]

lemma [*code*]:
Gcd_{fin} (*Set t*) = *fold-keys gcd t* (*0*::*'a*::{*semiring-gcd*, *linorder*})
proof −
have *comp-fun-commute* (*gcd* :: *'a* ⇒ -)
by *standard* (*simp add*: *fun-eq-iff ac-simps*)
with *finite-fold-fold-keys* [*of* - *0 t*]
have *Finite-Set.fold gcd 0* (*Set t*) = *fold-keys gcd t 0*
by *blast*
then show *?thesis*
by (*simp add*: *Gcd-fin.eq-fold*)
qed

lemma [*code*]:
Gcd (*Set t*) = (Gcd_{fin} (*Set t*) :: *nat*)
by *simp*

lemma [*code*]:
Gcd (*Set t*) = (Gcd_{fin} (*Set t*) :: *int*)
by *simp*

lemma [*code*]:
Lcm_{fin} (*Set t*) = *fold-keys lcm t* (*1*::*'a*::{*semiring-gcd*, *linorder*})
proof −
have *comp-fun-commute* (*lcm* :: *'a* ⇒ -)
by *standard* (*simp add*: *fun-eq-iff ac-simps*)
with *finite-fold-fold-keys* [*of* - *1 t*]
have *Finite-Set.fold lcm 1* (*Set t*) = *fold-keys lcm t 1*
by *blast*
then show *?thesis*
by (*simp add*: *Lcm-fin.eq-fold*)
qed

lemma [*code drop*: *Lcm* :: - ⇒ *nat*, *code*]:
Lcm (*Set t*) = (Lcm_{fin} (*Set t*) :: *nat*)
by *simp*

lemma [*code drop*: *Lcm* :: - ⇒ *int*, *code*]:
Lcm (*Set t*) = (Lcm_{fin} (*Set t*) :: *int*)

by *simp*

qualified definition *Inf'* :: *'a* :: {*linorder*, *complete-lattice*} *set* ⇒ *'a*
where [*code-abbrev*]: *Inf' = Inf*

lemma *Inf'-Set-fold* [*code*]:
Inf' (Set t) = (if RBT.is-empty t then top else r-min-opt t)
by (*simp add*: *Inf'-def Inf-Set-fold*)

qualified definition *Sup'* :: *'a* :: {*linorder*, *complete-lattice*} *set* ⇒ *'a*
where [*code-abbrev*]: *Sup' = Sup*

lemma *Sup'-Set-fold* [*code*]:
Sup' (Set t) = (if RBT.is-empty t then bot else r-max-opt t)
by (*simp add*: *Sup'-def Sup-Set-fold*)

end

lemma *sorted-list-set*[*code*]: *sorted-list-of-set (Set t) = RBT.keys t*
by (*auto simp add*: *set-keys intro*: *sorted-distinct-set-unique*)

lemma *Bleast-code* [*code*]:
Bleast (Set t) P =
(*case List.filter P (RBT.keys t) of*
x # xs ⇒ *x*
| [] ⇒ *abort-Bleast (Set t) P*)
proof (*cases List.filter P (RBT.keys t)*)
case *Nil*
thus *?thesis* **by** (*simp add*: *Bleast-def abort-Bleast-def*)
next
case (*Cons x ys*)
have (*LEAST x. x ∈ Set t ∧ P x*) = *x*
proof (*rule Least-equality*)
show *x ∈ Set t ∧ P x*
using *Cons*[*symmetric*]
by (*auto simp add*: *set-keys Cons-eq-filter-iff*)
next
fix *y*
assume *y ∈ Set t ∧ P y*
then show *x ≤ y*
using *Cons*[*symmetric*]
by(*auto simp add*: *set-keys Cons-eq-filter-iff*)
(*metis sorted-wrt.simps*(*2*) *sorted-append sorted-keys*)
qed
thus *?thesis* **using** *Cons* **by** (*simp add*: *Bleast-def*)
qed

hide-const (**open**) *RBT-Set.Set RBT-Set.Coset*

end

theory *Predicate-Compile-Alternative-Defs*
imports *Main*
begin

127 Common constants

declare *HOL.if-bool-eq-disj*[*code-pred-inline*]

declare *bool-diff-def*[*code-pred-inline*]
declare *inf-bool-def*[*abs-def*, *code-pred-inline*]
declare *less-bool-def*[*abs-def*, *code-pred-inline*]
declare *le-bool-def*[*abs-def*, *code-pred-inline*]

lemma *min-bool-eq* [*code-pred-inline*]: (*min* :: *bool* => *bool* => *bool*) == (∧)
by (*rule eq-reflection*) (*auto simp add*: *fun-eq-iff min-def*)

lemma [*code-pred-inline*]:
((*A*::*bool*) ≠ (*B*::*bool*)) = ((*A* ∧ ¬ *B*) ∨ (*B* ∧ ¬ *A*))
by *fast*

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*Let*›]›

128 Pairs

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*fst*›, ***const-name*** ‹*snd*›, ***const-name*** ‹*case-prod*›]›

129 Filters

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*Abs-filter*›, ***const-name*** ‹*Rep-filter*›]›

130 Bounded quantifiers

declare *Ball-def*[*code-pred-inline*]
declare *Bex-def*[*code-pred-inline*]

131 Operations on Predicates

lemma *Diff*[*code-pred-inline*]:
(*A* − *B*) = (%*x*. *A x* ∧ ¬ *B x*)
by (*simp add*: *fun-eq-iff*)

lemma *subset-eq*[*code-pred-inline*]:
(*P* :: *'a* ⇒ *bool*) < (*Q* :: *'a* ⇒ *bool*) ≡ ((∃ *x*. *Q x* ∧ (¬ *P x*)) ∧ (∀ *x*. *P x* ⟶ *Q x*))
by (*rule eq-reflection*) (*auto simp add*: *less-fun-def le-fun-def*)

lemma *set-equality*[*code-pred-inline*]:
A = *B* ⟷ (∀ *x*. *A x* ⟶ *B x*) ∧ (∀ *x*. *B x* ⟶ *A x*)
by (*auto simp add*: *fun-eq-iff*)

132 Setup for Numerals

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*numeral*›]›
setup ‹*Predicate-Compile-Data.keep-functions* [***const-name*** ‹*numeral*›]›
setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*Char*›]›
setup ‹*Predicate-Compile-Data.keep-functions* [***const-name*** ‹*Char*›]›

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*divide*›, ***const-name*** ‹*modulo*›, ***const-name*** ‹*times*›]›

133 Arithmetic operations

133.1 Arithmetic on naturals and integers

definition *plus-eq-nat* :: *nat* => *nat* => *nat* => *bool*
where
plus-eq-nat x y z = (*x* + *y* = *z*)

definition *minus-eq-nat* :: *nat* => *nat* => *nat* => *bool*
where
minus-eq-nat x y z = (*x* − *y* = *z*)

definition *plus-eq-int* :: *int* => *int* => *int* => *bool*
where
plus-eq-int x y z = (*x* + *y* = *z*)

definition *minus-eq-int* :: *int* => *int* => *int* => *bool*
where
minus-eq-int x y z = (*x* − *y* = *z*)

definition *subtract*
where
[*code-unfold*]: *subtract x y* = *y* − *x*

setup ‹
let
val Fun = *Predicate-Compile-Aux.Fun*
val Input = *Predicate-Compile-Aux.Input*

```
  val Output = Predicate-Compile-Aux.Output
  val Bool = Predicate-Compile-Aux.Bool
  val iio = Fun (Input, Fun (Input, Fun (Output, Bool)))
  val ioi = Fun (Input, Fun (Output, Fun (Input, Bool)))
  val oii = Fun (Output, Fun (Input, Fun (Input, Bool)))
  val ooi = Fun (Output, Fun (Output, Fun (Input, Bool)))
  val plus-nat = Core-Data.functional-compilation const-name‹plus› iio
  val minus-nat = Core-Data.functional-compilation const-name‹minus› iio
  fun subtract-nat compfuns (- : typ) =
    let
      val T = Predicate-Compile-Aux.mk-monadT compfuns typ‹nat›
    in
      absdummy typ‹nat› (absdummy typ‹nat›
        (Const (const-name‹If›, typ‹bool› --> T --> T --> T) $
          (term‹(>) :: nat => nat => bool› $ Bound 1 $ Bound 0) $
          Predicate-Compile-Aux.mk-empty compfuns typ‹nat› $
          Predicate-Compile-Aux.mk-single compfuns
          (term‹(-) :: nat => nat => nat› $ Bound 0 $ Bound 1)))
    end
  fun enumerate-addups-nat compfuns (- : typ) =
    absdummy typ‹nat› (Predicate-Compile-Aux.mk-iterate-upto compfuns typ‹nat
* nat›
    (absdummy typ‹natural› (term‹Pair :: nat => nat => nat * nat› $
      (term‹nat-of-natural› $ Bound 0) $
       (term‹(-) :: nat => nat => nat› $ Bound 1 $ (term‹nat-of-natural› $
Bound 0))),
      term‹0 :: natural›, term‹natural-of-nat› $ Bound 0))
  fun enumerate-nats compfuns  (- : typ) =
    let
      val (single-const, -) = strip-comb (Predicate-Compile-Aux.mk-single compfuns
term‹0 :: nat›)
      val T = Predicate-Compile-Aux.mk-monadT compfuns typ‹nat›
    in
      absdummy typ‹nat› (absdummy typ‹nat›
        (Const (const-name‹If›, typ‹bool› --> T --> T --> T) $
          (term‹(=) :: nat => nat => bool› $ Bound 0 $ term‹0::nat›) $
        (Predicate-Compile-Aux.mk-iterate-upto compfuns typ‹nat› (term‹nat-of-natural›,
            term‹0::natural›, term‹natural-of-nat› $ Bound 1)) $
            (single-const $ (term‹(+) :: nat => nat => nat› $ Bound 1 $ Bound
0))))
    end
in
  Core-Data.force-modes-and-compilations const-name‹plus-eq-nat›
    [(iio, (plus-nat, false)), (oii, (subtract-nat, false)), (ioi, (subtract-nat, false)),
     (ooi, (enumerate-addups-nat, false))]
  #> Predicate-Compile-Fun.add-function-predicate-translation
       (term‹plus :: nat => nat => nat›, term‹plus-eq-nat›)
  #> Core-Data.force-modes-and-compilations const-name‹minus-eq-nat›
       [(iio, (minus-nat, false)), (oii, (enumerate-nats, false))]
```

```
  #> Predicate-Compile-Fun.add-function-predicate-translation
      (term ‹minus :: nat => nat => nat›, term ‹minus-eq-nat›)
  #> Core-Data.force-modes-and-functions const-name ‹plus-eq-int›
    [(iio, (const-name ‹plus›, false)), (ioi, (const-name ‹subtract›, false)),
     (oii, (const-name ‹subtract›, false))]
  #> Predicate-Compile-Fun.add-function-predicate-translation
       (term ‹plus :: int => int => int›, term ‹plus-eq-int›)
  #> Core-Data.force-modes-and-functions const-name ‹minus-eq-int›
    [(iio, (const-name ‹minus›, false)), (oii, (const-name ‹plus›, false)),
     (ioi, (const-name ‹minus›, false))]
  #> Predicate-Compile-Fun.add-function-predicate-translation
      (term ‹minus :: int => int => int›, term ‹minus-eq-int›)
end
›
```

133.2 Inductive definitions for ordering on naturals

```
inductive less-nat
where
  less-nat 0 (Suc y)
| less-nat x y ==> less-nat (Suc x) (Suc y)

lemma less-nat[code-pred-inline]:
  x < y = less-nat x y
apply (rule iffI)
apply (induct x arbitrary: y)
apply (case-tac y) apply (auto intro: less-nat.intros)
apply (case-tac y)
apply (auto intro: less-nat.intros)
apply (induct rule: less-nat.induct)
apply auto
done

inductive less-eq-nat
where
  less-eq-nat 0 y
| less-eq-nat x y ==> less-eq-nat (Suc x) (Suc y)

lemma [code-pred-inline]:
x <= y = less-eq-nat x y
apply (rule iffI)
apply (induct x arbitrary: y)
apply (auto intro: less-eq-nat.intros)
apply (case-tac y) apply (auto intro: less-eq-nat.intros)
apply (induct rule: less-eq-nat.induct)
apply auto done
```

134 Alternative list definitions

134.1 Alternative rules for *length*

definition *size-list'* :: *'a list => nat*
where *size-list' = size*

lemma *size-list'-simps*:
size-list' [] = 0
size-list' (x # xs) = Suc (size-list' xs)
by (*auto simp add*: *size-list'-def*)

declare *size-list'-simps*[*code-pred-def*]
declare *size-list'-def*[*symmetric*, *code-pred-inline*]

134.2 Alternative rules for *list-all2*

lemma *list-all2-NilI* [*code-pred-intro*]: *list-all2 P [] []*
by *auto*

lemma *list-all2-ConsI* [*code-pred-intro*]: *list-all2 P xs ys ==> P x y ==> list-all2 P (x#xs) (y#ys)*
by *auto*

code-pred [*skip-proof*] *list-all2*
proof −
case *list-all2*
from *this* **show** *thesis*
apply −
apply (*case-tac xb*)
apply (*case-tac xc*)
apply *auto*
apply (*case-tac xc*)
apply *auto*
done
qed

134.3 Alternative rules for membership in lists

declare *in-set-member*[*code-pred-inline*]

lemma *member-intros* [*code-pred-intro*]:
List.member (x#xs) x
List.member xs x ⟹ List.member (y#xs) x
by(*simp-all add*: *List.member-def*)

code-pred *List.member*
by(*auto simp add*: *List.member-def elim*: *list.set-cases*)

code-identifier constant *member-i-i*

⇀ (*SML*) *List.member-i-i*
and (*OCaml*) *List.member-i-i*
and (*Haskell*) *List.member-i-i*
and (*Scala*) *List.member-i-i*

code-identifier constant *member-i-o*
⇀ (*SML*) *List.member-i-o*
and (*OCaml*) *List.member-i-o*
and (*Haskell*) *List.member-i-o*
and (*Scala*) *List.member-i-o*

135 Setup for String.literal

setup ‹*Predicate-Compile-Data.ignore-consts* [***const-name*** ‹*String.Literal*›]›

136 Simplification rules for optimisation

lemma [*code-pred-simp*]: ¬ *False* == *True*
by *auto*

lemma [*code-pred-simp*]: ¬ *True* == *False*
by *auto*

lemma *less-nat-k-0* [*code-pred-simp*]: *less-nat k 0* == *False*
unfolding *less-nat*[*symmetric*] **by** *auto*

end

137 A Prototype of Quickcheck based on the Predicate Compiler

theory *Predicate-Compile-Quickcheck*
imports *Predicate-Compile-Alternative-Defs*
begin

ML-file ‹../*Tools*/*Predicate-Compile*/*predicate-compile-quickcheck.ML*›

end

138 TFL: recursive function definitions

theory *Old-Recdef*
imports *Main*
keywords
recdef :: *thy-defn* **and**
permissive congs hints
begin

138.1 Lemmas for TFL

lemma *tfl-wf-induct*: $\forall R.\ wf\ R \longrightarrow$
$(\forall P.\ (\forall x.\ (\forall y.\ (y,x)\in R \longrightarrow P\ y) \longrightarrow P\ x) \longrightarrow (\forall x.\ P\ x))$
apply *clarify*
apply (*rule-tac* $r = R$ **and** $P = P$ **and** $a = x$ **in** *wf-induct*, *assumption*, *blast*)
done

lemma *tfl-cut-def*: $cut\ f\ r\ x \equiv (\lambda y.\ if\ (y,x) \in r\ then\ f\ y\ else\ undefined)$
unfolding *cut-def* **.**

lemma *tfl-cut-apply*: $\forall f\ R.\ (x,a)\in R \longrightarrow (cut\ f\ R\ a)(x) = f(x)$
apply *clarify*
apply (*rule cut-apply*, *assumption*)
done

lemma *tfl-wfrec*:
$\forall M\ R\ f.\ (f{=}wfrec\ R\ M) \longrightarrow wf\ R \longrightarrow (\forall x.\ f\ x = M\ (cut\ f\ R\ x)\ x)$
apply *clarify*
apply (*erule wfrec*)
done

lemma *tfl-eq-True*: $(x = True) \longrightarrow x$
by *blast*

lemma *tfl-rev-eq-mp*: $(x = y) \longrightarrow y \longrightarrow x$
by *blast*

lemma *tfl-simp-thm*: $(x \longrightarrow y) \longrightarrow (x = x') \longrightarrow (x' \longrightarrow y)$
by *blast*

lemma *tfl-P-imp-P-iff-True*: $P \Longrightarrow P = True$
by *blast*

lemma *tfl-imp-trans*: $(A \longrightarrow B) \Longrightarrow (B \longrightarrow C) \Longrightarrow (A \longrightarrow C)$
by *blast*

lemma *tfl-disj-assoc*: $(a \vee b) \vee c \equiv a \vee (b \vee c)$
by *simp*

lemma *tfl-disjE*: $P \vee Q \Longrightarrow P \longrightarrow R \Longrightarrow Q \longrightarrow R \Longrightarrow R$
by *blast*

lemma *tfl-exE*: $\exists x.\ P\ x \Longrightarrow \forall x.\ P\ x \longrightarrow Q \Longrightarrow Q$
by *blast*

ML-file ‹*old-recdef.ML*›

138.2 Rule setup

lemmas [*recdef-simp*] =
inv-image-def
measure-def
lex-prod-def
same-fst-def
less-Suc-eq [*THEN iffD2*]

lemmas [*recdef-cong*] =
if-cong let-cong image-cong INF-cong SUP-cong bex-cong ball-cong imp-cong
map-cong filter-cong takeWhile-cong dropWhile-cong foldl-cong foldr-cong

lemmas [*recdef-wf*] =
wf-trancl
wf-less-than
wf-lex-prod
wf-inv-image
wf-measure
wf-measures
wf-pred-nat
wf-same-fst
wf-empty

end

139 Program extraction from proofs involving datatypes and inductive predicates

theory *Realizers*
imports *Main*
begin

ML-file ‹~~/*src*/*HOL*/*Tools*/*datatype-realizer.ML*›
ML-file ‹~~/*src*/*HOL*/*Tools*/*inductive-realizer.ML*›

end

140 Refute

theory *Refute*
imports *Main*
keywords
refute :: *diag* **and**
refute-params :: *thy-decl*
begin

ML-file ‹*refute.ML*›

refute-params
[*itself* = *1*,
minsize = *1*,
maxsize = *8*,
maxvars = *10000*,
maxtime = *60*,
satsolver = *auto*,
no-assms = *false*]

```
(* ------------------------------------------------------------------------ *)
(* REFUTE                                                                   *)
(*                                                                          *)
(* We use a SAT solver to search for a (finite) model that refutes a given  *)
(* HOL formula.                                                             *)
(* ------------------------------------------------------------------------ *)

(* ------------------------------------------------------------------------ *)
(* NOTE                                                                     *)
(*                                                                          *)
(* I strongly recommend that you install a stand-alone SAT solver if you    *)
(* want to use 'refute'.  For details see 'HOL/Tools/sat_solver.ML'.  If you *)
(* have installed (a supported version of) zChaff, simply set 'ZCHAFF_HOME' *)
(* in 'etc/settings'.                                                       *)
(* ------------------------------------------------------------------------ *)

(* ------------------------------------------------------------------------ *)
(* USAGE                                                                    *)
(*                                                                          *)
(* See the file 'HOL/ex/Refute_Examples.thy' for examples.  The supported   *)
(* parameters are explained below.                                          *)
(* ------------------------------------------------------------------------ *)

(* ------------------------------------------------------------------------ *)
(* CURRENT LIMITATIONS                                                      *)
(*                                                                          *)
(* 'refute' currently accepts formulas of higher-order predicate logic (with *)
(* equality), including free/bound/schematic variables, lambda abstractions, *)
(* sets and set membership, "arbitrary", "The", "Eps", records and          *)
(* inductively defined sets.  Constants are unfolded automatically, and sort *)
(* axioms are added as well.  Other, user-asserted axioms however are       *)
(* ignored.  Inductive datatypes and recursive functions are supported, but *)
(* may lead to spurious countermodels.                                      *)
(*                                                                          *)
(* The (space) complexity of the algorithm is non-elementary.               *)
(*                                                                          *)
(* Schematic type variables are not supported.                              *)
(* ------------------------------------------------------------------------ *)
```

```
(* ------------------------------------------------------------------------ *)
(* PARAMETERS                                                               *)
(*                                                                          *)
(* The following global parameters are currently supported (and required,   *)
(* except for "expect"):                                                    *)
(*                                                                          *)
(* Name          Type    Description                                        *)
(*                                                                          *)
(* "minsize"     int     Only search for models with size at least          *)
(*                       'minsize'.                                         *)
(* "maxsize"     int     If >0, only search for models with size at most    *)
(*                       'maxsize'.                                         *)
(* "maxvars"     int     If >0, use at most 'maxvars' boolean variables     *)
(*                       when transforming the term into a propositional    *)
(*                       formula.                                           *)
(* "maxtime"     int     If >0, terminate after at most 'maxtime' seconds.  *)
(*                       This value is ignored under some ML compilers.     *)
(* "satsolver"   string  Name of the SAT solver to be used.                 *)
(* "no_assms"    bool    If "true", assumptions in structured proofs are    *)
(*                       not considered.                                    *)
(* "expect"      string  Expected result ("genuine", "potential", "none", or *)
(*                       "unknown").                                        *)
(*                                                                          *)
(* The size of particular types can be specified in the form type=size      *)
(* (where 'type' is a string, and 'size' is an int).  Examples:             *)
(* "'a"=1                                                                   *)
(* "List.list"=2                                                            *)
(* ------------------------------------------------------------------------ *)

(* ------------------------------------------------------------------------ *)
(* FILES                                                                    *)
(*                                                                          *)
(* HOL/Tools/prop_logic.ML     Propositional logic                          *)
(* HOL/Tools/sat_solver.ML     SAT solvers                                  *)
(* HOL/Tools/refute.ML         Translation HOL -> propositional logic and   *)
(*                             Boolean assignment -> HOL model              *)
(* HOL/Refute.thy              This file: loads the ML files, basic setup,  *)
(*                             documentation                                *)
(* HOL/SAT.thy                 Sets default parameters                      *)
(* HOL/ex/Refute_Examples.thy  Examples                                     *)
(* ------------------------------------------------------------------------ *)
```

end

References

[1] J. Avigad and K. Donnelly. Formalizing O notation in Isabelle/HOL. In D. Basin and M. Rusiowitch, editors, *Automated Reasoning: second*

international conference, IJCAR 2004, pages 357–371. Springer, 2004.

[2] A. Lochbihler and P. Stoop. Lazy algebraic types in Isabelle/HOL. In *Isabelle Workshop 2018*, 2018.

[3] A. Podelski and A. Rybalchenko. Transition invariants. In *19th Annual IEEE Symposium on Logic in Computer Science (LICS'04)*, pages 32–41, 2004.