Isabelle/HOL-NSA — Non-Standard Analysis

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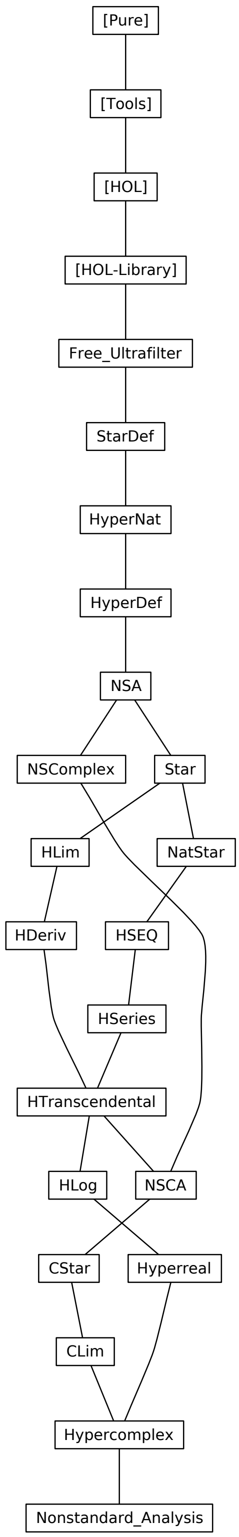
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1 Filters and Ultrafilters

theory *Free-Ultrafilter*
imports *HOL−Library.Infinite-Set*
begin

1.1 Definitions and basic properties

1.1.1 Ultrafilters

locale *ultrafilter* =
fixes F :: *'a filter*
assumes *proper*: $F \neq bot$
assumes *ultra*: *eventually* P F $\vee$ *eventually* $(\lambda x.\ \neg\ P\ x)$ F
begin

lemma *eventually-imp-frequently*: *frequently* P $F \Longrightarrow$ *eventually* P F
⟨*proof*⟩

lemma *frequently-eq-eventually*: *frequently* P F = *eventually* P F
⟨*proof*⟩

lemma *eventually-disj-iff*: *eventually* $(\lambda x.\ P\ x \vee Q\ x)$ $F \longleftrightarrow$ *eventually* P F $\vee$ *eventually* Q F
⟨*proof*⟩

lemma *eventually-all-iff*: *eventually* $(\lambda x.\ \forall y.\ P\ x\ y)$ $F = (\forall Y.$ *eventually* $(\lambda x.\ P\ x\ (Y\ x))$ $F)$
⟨*proof*⟩

lemma *eventually-imp-iff*: *eventually* $(\lambda x.\ P\ x \longrightarrow Q\ x)$ $F \longleftrightarrow$ (*eventually* P F $\longrightarrow$ *eventually* Q F)
⟨*proof*⟩

lemma *eventually-iff-iff*: *eventually* $(\lambda x.\ P\ x \longleftrightarrow Q\ x)$ $F \longleftrightarrow$ (*eventually* P F $\longleftrightarrow$ *eventually* Q F)
⟨*proof*⟩

lemma *eventually-not-iff*: *eventually* $(\lambda x.\ \neg\ P\ x)$ $F \longleftrightarrow \neg$ *eventually* P F
⟨*proof*⟩

end

1.2 Maximal filter = Ultrafilter

A filter F is an ultrafilter iff it is a maximal filter, i.e. whenever G is a filter and $F \subseteq G$ then $F = G$

Lemma that shows existence of an extension to what was assumed to be a maximal filter. Will be used to derive contradiction in proof of property of ultrafilter.

lemma *extend-filter*: *frequently* P $F \Longrightarrow$ *inf* F (*principal* $\{x.\ P\ x\}$) $\neq$ *bot*
⟨*proof*⟩

lemma *max-filter-ultrafilter*:
assumes $F \neq$ *bot*
assumes *max*: $\bigwedge G.\ G \neq$ *bot* $\Longrightarrow G \leq F \Longrightarrow F = G$
shows *ultrafilter* F
⟨*proof*⟩

lemma *le-filter-frequently*: $F \leq G \longleftrightarrow (\forall P.$ *frequently* $P\ F \longrightarrow$ *frequently* $P\ G)$
⟨*proof*⟩

lemma (**in** *ultrafilter*) *max-filter*:
assumes G: $G \neq$ *bot*
and *sub*: $G \leq F$
shows $F = G$
⟨*proof*⟩

1.3 Ultrafilter Theorem

lemma *ex-max-ultrafilter*:
fixes F :: *'a filter*
assumes F: $F \neq$ *bot*
shows $\exists\, U \leq F.$ *ultrafilter* U
⟨*proof*⟩

1.3.1 Free Ultrafilters

There exists a free ultrafilter on any infinite set.

locale *freeultrafilter* = *ultrafilter* +
assumes *infinite*: *eventually* $P\ F \Longrightarrow$ *infinite* $\{x.\ P\ x\}$
begin

lemma *finite*: *finite* $\{x.\ P\ x\} \Longrightarrow \neg$ *eventually* $P\ F$
⟨*proof*⟩

lemma *finite'*: *finite* $\{x.\ \neg\ P\ x\} \Longrightarrow$ *eventually* $P\ F$
⟨*proof*⟩

lemma *le-cofinite*: $F \leq$ *cofinite*
⟨*proof*⟩

lemma *singleton*: $\neg$ *eventually* $(\lambda x.\ x = a)\ F$
⟨*proof*⟩

lemma *singleton'*: $\neg$ *eventually* $((=)\ a)\ F$
⟨*proof*⟩

lemma *ultrafilter*: *ultrafilter F* ⟨*proof*⟩

end

lemma *freeultrafilter-Ex*:
  assumes [*simp*]: *infinite* (*UNIV* :: *'a set*)
  shows $\exists$ *U*::*'a filter*. *freeultrafilter U*
⟨*proof*⟩

end

2 Construction of Star Types Using Ultrafilters

theory *StarDef*
  imports *Free-Ultrafilter*
begin

2.1 A Free Ultrafilter over the Naturals

definition *FreeUltrafilterNat* :: *nat filter* (‹$\mathcal{U}$›)
  where $\mathcal{U}$ = (*SOME U*. *freeultrafilter U*)

lemma *freeultrafilter-FreeUltrafilterNat*: *freeultrafilter* $\mathcal{U}$
  ⟨*proof*⟩

interpretation *FreeUltrafilterNat*: *freeultrafilter* $\mathcal{U}$
  ⟨*proof*⟩

2.2 Definition of *star* type constructor

definition *starrel* :: ((*nat* ⇒ *'a*) × (*nat* ⇒ *'a*)) *set*
  where *starrel* = {(*X*, *Y*). *eventually* (λ*n*. *X n* = *Y n*) $\mathcal{U}$}

definition *star* = (*UNIV* :: (*nat* ⇒ *'a*) *set*) // *starrel*

typedef *'a star* = *star* :: (*nat* ⇒ *'a*) *set set*
  ⟨*proof*⟩

definition *star-n* :: (*nat* ⇒ *'a*) ⇒ *'a star*
  where *star-n X* = *Abs-star* (*starrel* `` {*X*})

theorem *star-cases* [*case-names star-n*, *cases type*: *star*]:
  obtains *X* **where** *x* = *star-n X*
  ⟨*proof*⟩

lemma *all-star-eq*: (∀ *x*. *P x*) ⟷ (∀ *X*. *P* (*star-n X*))
  ⟨*proof*⟩

lemma *ex-star-eq*: $(\exists\, x.\ P\ x) \longleftrightarrow (\exists\, X.\ P\ (star\text{-}n\ X))$
⟨*proof*⟩

Proving that *starrel* is an equivalence relation.

lemma *starrel-iff* [*iff*]: $(X,\ Y) \in starrel \longleftrightarrow eventually\ (\lambda n.\ X\ n = Y\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *equiv-starrel*: *equiv UNIV starrel*
⟨*proof*⟩

lemmas *equiv-starrel-iff* = *eq-equiv-class-iff* [*OF equiv-starrel UNIV-I UNIV-I*]

lemma *starrel-in-star*: $starrel``\{x\} \in star$
⟨*proof*⟩

lemma *star-n-eq-iff*: $star\text{-}n\ X = star\text{-}n\ Y \longleftrightarrow eventually\ (\lambda n.\ X\ n = Y\ n)\ \mathcal{U}$
⟨*proof*⟩

2.3 Transfer principle

This introduction rule starts each transfer proof.

lemma *transfer-start*: $P \equiv eventually\ (\lambda n.\ Q)\ \mathcal{U} \Longrightarrow Trueprop\ P \equiv Trueprop\ Q$
⟨*proof*⟩

Standard principles that play a central role in the transfer tactic.

definition *Ifun* :: $('a \Rightarrow 'b)\ star \Rightarrow 'a\ star \Rightarrow 'b\ star$ (‹(- ⋆/ -)› [*300*, *301*] *300*)
where *Ifun f* ≡
$\lambda x.\ Abs\text{-}star\ (\bigcup F{\in}Rep\text{-}star\ f.\ \bigcup X{\in}Rep\text{-}star\ x.\ starrel``\{\lambda n.\ F\ n\ (X\ n)\})$

lemma *Ifun-congruent2*: $congruent2\ starrel\ starrel\ (\lambda F\ X.\ starrel``\{\lambda n.\ F\ n\ (X\ n)\})$
⟨*proof*⟩

lemma *Ifun-star-n*: $star\text{-}n\ F \star star\text{-}n\ X = star\text{-}n\ (\lambda n.\ F\ n\ (X\ n))$
⟨*proof*⟩

lemma *transfer-Ifun*: $f \equiv star\text{-}n\ F \Longrightarrow x \equiv star\text{-}n\ X \Longrightarrow f \star x \equiv star\text{-}n\ (\lambda n.\ F\ n\ (X\ n))$
⟨*proof*⟩

definition *star-of* :: $'a \Rightarrow 'a\ star$
where *star-of* $x \equiv star\text{-}n\ (\lambda n.\ x)$

Initialize transfer tactic.

⟨*ML*⟩

Transfer introduction rules.

lemma *transfer-ex* [*transfer-intro*]:

$(\bigwedge X.\ p\ (\textit{star-n}\ X) \equiv \textit{eventually}\ (\lambda n.\ P\ n\ (X\ n))\ \mathcal{U}) \Longrightarrow$
$\exists x{::}'a\ \textit{star}.\ p\ x \equiv \textit{eventually}\ (\lambda n.\ \exists x.\ P\ n\ x)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-all* [*transfer-intro*]:
$(\bigwedge X.\ p\ (\textit{star-n}\ X) \equiv \textit{eventually}\ (\lambda n.\ P\ n\ (X\ n))\ \mathcal{U}) \Longrightarrow$
$\forall x{::}'a\ \textit{star}.\ p\ x \equiv \textit{eventually}\ (\lambda n.\ \forall x.\ P\ n\ x)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-not* [*transfer-intro*]: $p \equiv \textit{eventually}\ P\ \mathcal{U} \Longrightarrow \neg\ p \equiv \textit{eventually}\ (\lambda n.\ \neg\ P\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-conj* [*transfer-intro*]:
$p \equiv \textit{eventually}\ P\ \mathcal{U} \Longrightarrow q \equiv \textit{eventually}\ Q\ \mathcal{U} \Longrightarrow p \wedge q \equiv \textit{eventually}\ (\lambda n.\ P\ n \wedge Q\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-disj* [*transfer-intro*]:
$p \equiv \textit{eventually}\ P\ \mathcal{U} \Longrightarrow q \equiv \textit{eventually}\ Q\ \mathcal{U} \Longrightarrow p \vee q \equiv \textit{eventually}\ (\lambda n.\ P\ n \vee Q\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-imp* [*transfer-intro*]:
$p \equiv \textit{eventually}\ P\ \mathcal{U} \Longrightarrow q \equiv \textit{eventually}\ Q\ \mathcal{U} \Longrightarrow p \longrightarrow q \equiv \textit{eventually}\ (\lambda n.\ P\ n \longrightarrow Q\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-iff* [*transfer-intro*]:
$p \equiv \textit{eventually}\ P\ \mathcal{U} \Longrightarrow q \equiv \textit{eventually}\ Q\ \mathcal{U} \Longrightarrow p = q \equiv \textit{eventually}\ (\lambda n.\ P\ n = Q\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-if-bool* [*transfer-intro*]:
$p \equiv \textit{eventually}\ P\ \mathcal{U} \Longrightarrow x \equiv \textit{eventually}\ X\ \mathcal{U} \Longrightarrow y \equiv \textit{eventually}\ Y\ \mathcal{U} \Longrightarrow$
$(\textit{if}\ p\ \textit{then}\ x\ \textit{else}\ y) \equiv \textit{eventually}\ (\lambda n.\ \textit{if}\ P\ n\ \textit{then}\ X\ n\ \textit{else}\ Y\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-eq* [*transfer-intro*]:
$x \equiv \textit{star-n}\ X \Longrightarrow y \equiv \textit{star-n}\ Y \Longrightarrow x = y \equiv \textit{eventually}\ (\lambda n.\ X\ n = Y\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-if* [*transfer-intro*]:
$p \equiv \textit{eventually}\ (\lambda n.\ P\ n)\ \mathcal{U} \Longrightarrow x \equiv \textit{star-n}\ X \Longrightarrow y \equiv \textit{star-n}\ Y \Longrightarrow$
$(\textit{if}\ p\ \textit{then}\ x\ \textit{else}\ y) \equiv \textit{star-n}\ (\lambda n.\ \textit{if}\ P\ n\ \textit{then}\ X\ n\ \textit{else}\ Y\ n)$
⟨*proof*⟩

lemma *transfer-fun-eq* [*transfer-intro*]:
$(\bigwedge X.\ f\ (\textit{star-n}\ X) = g\ (\textit{star-n}\ X) \equiv \textit{eventually}\ (\lambda n.\ F\ n\ (X\ n) = G\ n\ (X\ n))$

$\mathcal{U}) \Longrightarrow$
  $f = g \equiv$ *eventually* $(\lambda n.\ F\ n = G\ n)\ \mathcal{U}$
  ⟨*proof*⟩

lemma *transfer-star-n* [*transfer-intro*]: *star-n* $X \equiv$ *star-n* $(\lambda n.\ X\ n)$
  ⟨*proof*⟩

lemma *transfer-bool* [*transfer-intro*]: $p \equiv$ *eventually* $(\lambda n.\ p)\ \mathcal{U}$
  ⟨*proof*⟩

2.4 Standard elements

definition *Standard* :: *'a star set*
  where *Standard* = *range star-of*

Transfer tactic should remove occurrences of *star-of*.

⟨*ML*⟩

lemma *star-of-inject*: *star-of* x = *star-of* $y \longleftrightarrow x = y$
  ⟨*proof*⟩

lemma *Standard-star-of* [*simp*]: *star-of* $x \in$ *Standard*
  ⟨*proof*⟩

2.5 Internal functions

Transfer tactic should remove occurrences of *Ifun*.

⟨*ML*⟩

lemma *Ifun-star-of* [*simp*]: *star-of* $f \star$ *star-of* x = *star-of* $(f\ x)$
  ⟨*proof*⟩

lemma *Standard-Ifun* [*simp*]: $f \in$ *Standard* $\Longrightarrow x \in$ *Standard* $\Longrightarrow f \star x \in$ *Standard*
  ⟨*proof*⟩

Nonstandard extensions of functions.

definition *starfun* :: $('a \Rightarrow 'b) \Rightarrow$ *'a star* $\Rightarrow$ *'b star* (‹$*f*$ -› [*80*] *80*)
  where *starfun* $f \equiv \lambda x.$ *star-of* $f \star x$

definition *starfun2* :: $('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow$ *'a star* $\Rightarrow$ *'b star* $\Rightarrow$ *'c star* (‹$*f2*$ -› [*80*] *80*)
  where *starfun2* $f \equiv \lambda x\ y.$ *star-of* $f \star x \star y$

declare *starfun-def* [*transfer-unfold*]
declare *starfun2-def* [*transfer-unfold*]

lemma *starfun-star-n*: $(\ *f*\ f)$ (*star-n* X) = *star-n* $(\lambda n.\ f\ (X\ n))$
  ⟨*proof*⟩

lemma *starfun2-star-n*: $(*f2* \ f) \ (star\text{-}n \ X) \ (star\text{-}n \ Y) = star\text{-}n \ (\lambda n. \ f \ (X \ n) \ (Y \ n))$
⟨*proof*⟩

lemma *starfun-star-of* [*simp*]: $(*f* \ f) \ (star\text{-}of \ x) = star\text{-}of \ (f \ x)$
⟨*proof*⟩

lemma *starfun2-star-of* [*simp*]: $(*f2* \ f) \ (star\text{-}of \ x) = *f* \ f \ x$
⟨*proof*⟩

lemma *Standard-starfun* [*simp*]: $x \in Standard \Longrightarrow starfun \ f \ x \in Standard$
⟨*proof*⟩

lemma *Standard-starfun2* [*simp*]: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow starfun2 \ f \ x \ y \in Standard$
⟨*proof*⟩

lemma *Standard-starfun-iff*:
assumes *inj*: $\bigwedge x \ y. \ f \ x = f \ y \Longrightarrow x = y$
shows $starfun \ f \ x \in Standard \longleftrightarrow x \in Standard$
⟨*proof*⟩

lemma *Standard-starfun2-iff*:
assumes *inj*: $\bigwedge a \ b \ a' \ b'. \ f \ a \ b = f \ a' \ b' \Longrightarrow a = a' \wedge b = b'$
shows $starfun2 \ f \ x \ y \in Standard \longleftrightarrow x \in Standard \wedge y \in Standard$
⟨*proof*⟩

2.6 Internal predicates

definition $unstar :: bool \ star \Rightarrow bool$
where $unstar \ b \longleftrightarrow b = star\text{-}of \ True$

lemma *unstar-star-n*: $unstar \ (star\text{-}n \ P) \longleftrightarrow eventually \ P \ \mathcal{U}$
⟨*proof*⟩

lemma *unstar-star-of* [*simp*]: $unstar \ (star\text{-}of \ p) = p$
⟨*proof*⟩

Transfer tactic should remove occurrences of *unstar*.

⟨*ML*⟩

lemma *transfer-unstar* [*transfer-intro*]: $p \equiv star\text{-}n \ P \Longrightarrow unstar \ p \equiv eventually \ P \ \mathcal{U}$
⟨*proof*⟩

definition $starP :: ('a \Rightarrow bool) \Rightarrow 'a \ star \Rightarrow bool$ (‹*p* -› [80] 80)
where $*p* \ P = (\lambda x. \ unstar \ (star\text{-}of \ P \star x))$

definition *starP2* :: ($'a \Rightarrow 'b \Rightarrow bool$) $\Rightarrow$ $'a$ *star* $\Rightarrow$ $'b$ *star* $\Rightarrow$ *bool* (‹$*p2*$ -› [*80*] *80*)
  where $*p2*$ P = ($\lambda x\ y.$ *unstar* (*star-of* $P \star x \star y$))

declare *starP-def* [*transfer-unfold*]
declare *starP2-def* [*transfer-unfold*]

lemma *starP-star-n*: ($*p*$ P) (*star-n* X) = *eventually* ($\lambda n.\ P\ (X\ n)$) $\mathcal{U}$
  ⟨*proof*⟩

lemma *starP2-star-n*: ($*p2*$ P) (*star-n* X) (*star-n* Y) = (*eventually* ($\lambda n.\ P\ (X\ n)\ (Y\ n)$) $\mathcal{U}$)
  ⟨*proof*⟩

lemma *starP-star-of* [*simp*]: ($*p*$ P) (*star-of* x) = $P\ x$
  ⟨*proof*⟩

lemma *starP2-star-of* [*simp*]: ($*p2*$ P) (*star-of* x) = $*p*$ $P\ x$
  ⟨*proof*⟩

2.7 Internal sets

definition *Iset* :: $'a$ *set star* $\Rightarrow$ $'a$ *star set*
  where *Iset* $A = \{x.$ ($*p2*$ $(\in)$) $x\ A\}$

lemma *Iset-star-n*: (*star-n* $X \in$ *Iset* (*star-n* A)) = (*eventually* ($\lambda n.\ X\ n \in A\ n$) $\mathcal{U}$)
  ⟨*proof*⟩

Transfer tactic should remove occurrences of *Iset*.

⟨*ML*⟩

lemma *transfer-mem* [*transfer-intro*]:
  $x \equiv$ *star-n* $X \Longrightarrow a \equiv$ *Iset* (*star-n* A) $\Longrightarrow x \in a \equiv$ *eventually* ($\lambda n.\ X\ n \in A\ n$) $\mathcal{U}$
  ⟨*proof*⟩

lemma *transfer-Collect* [*transfer-intro*]:
  ($\bigwedge X.\ p$ (*star-n* X) $\equiv$ *eventually* ($\lambda n.\ P\ n\ (X\ n)$) $\mathcal{U}$) $\Longrightarrow$
    Collect $p \equiv$ *Iset* (*star-n* ($\lambda n.$ *Collect* ($P\ n$)))
  ⟨*proof*⟩

lemma *transfer-set-eq* [*transfer-intro*]:
  $a \equiv$ *Iset* (*star-n* A) $\Longrightarrow b \equiv$ *Iset* (*star-n* B) $\Longrightarrow a = b \equiv$ *eventually* ($\lambda n.\ A\ n = B\ n$) $\mathcal{U}$
  ⟨*proof*⟩

lemma *transfer-ball* [*transfer-intro*]:
  $a \equiv$ *Iset* (*star-n* A) $\Longrightarrow$ ($\bigwedge X.\ p$ (*star-n* X) $\equiv$ *eventually* ($\lambda n.\ P\ n\ (X\ n)$) $\mathcal{U}$) $\Longrightarrow$

$\forall x{\in}a.\ p\ x \equiv eventually\ (\lambda n.\ \forall x{\in}A\ n.\ P\ n\ x)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-bex* [*transfer-intro*]:
$a \equiv Iset\ (star\text{-}n\ A) \Longrightarrow (\bigwedge X.\ p\ (star\text{-}n\ X) \equiv eventually\ (\lambda n.\ P\ n\ (X\ n))\ \mathcal{U}) \Longrightarrow$
$\exists x{\in}a.\ p\ x \equiv eventually\ (\lambda n.\ \exists x{\in}A\ n.\ P\ n\ x)\ \mathcal{U}$
⟨*proof*⟩

lemma *transfer-Iset* [*transfer-intro*]: $a \equiv star\text{-}n\ A \Longrightarrow Iset\ a \equiv Iset\ (star\text{-}n\ (\lambda n.\ A\ n))$
⟨*proof*⟩

Nonstandard extensions of sets.

definition *starset* :: $'a\ set \Rightarrow 'a\ star\ set$ (‹∗s∗ -› [80] 80)
where $starset\ A = Iset\ (star\text{-}of\ A)$

declare *starset-def* [*transfer-unfold*]

lemma *starset-mem*: $star\text{-}of\ x \in {*s*}\ A \longleftrightarrow x \in A$
⟨*proof*⟩

lemma *starset-UNIV*: ${*s*}\ (UNIV{::}'a\ set) = (UNIV{::}'a\ star\ set)$
⟨*proof*⟩

lemma *starset-empty*: ${*s*}\ \{\} = \{\}$
⟨*proof*⟩

lemma *starset-insert*: ${*s*}\ (insert\ x\ A) = insert\ (star\text{-}of\ x)\ (\ {*s*}\ A)$
⟨*proof*⟩

lemma *starset-Un*: ${*s*}\ (A \cup B) = {*s*}\ A \cup {*s*}\ B$
⟨*proof*⟩

lemma *starset-Int*: ${*s*}\ (A \cap B) = {*s*}\ A \cap {*s*}\ B$
⟨*proof*⟩

lemma *starset-Compl*: ${*s*}\ -A = -(\ {*s*}\ A)$
⟨*proof*⟩

lemma *starset-diff*: ${*s*}\ (A - B) = {*s*}\ A - {*s*}\ B$
⟨*proof*⟩

lemma *starset-image*: ${*s*}\ (f\ `\ A) = (\ {*f*}\ f)\ `\ (\ {*s*}\ A)$
⟨*proof*⟩

lemma *starset-vimage*: ${*s*}\ (f\ -`\ A) = (\ {*f*}\ f)\ -`\ (\ {*s*}\ A)$
⟨*proof*⟩

lemma *starset-subset*: $(\ {*s*}\ A \subseteq {*s*}\ B) \longleftrightarrow A \subseteq B$

⟨*proof*⟩

lemma *starset-eq*: ($*s*$ A = $*s*$ B) $\longleftrightarrow$ $A = B$
⟨*proof*⟩

lemmas *starset-simps* [*simp*] =
starset-mem *starset-UNIV*
starset-empty *starset-insert*
starset-Un *starset-Int*
starset-Compl *starset-diff*
starset-image *starset-vimage*
starset-subset *starset-eq*

2.8 Syntactic classes

instantiation *star* :: (*zero*) *zero*
begin
definition *star-zero-def*: *0* ≡ *star-of 0*
instance ⟨*proof*⟩
end

instantiation *star* :: (*one*) *one*
begin
definition *star-one-def*: *1* ≡ *star-of 1*
instance ⟨*proof*⟩
end

instantiation *star* :: (*plus*) *plus*
begin
definition *star-add-def*: (+) ≡ $*f2*$ (+)
instance ⟨*proof*⟩
end

instantiation *star* :: (*times*) *times*
begin
definition *star-mult-def*: ((∗)) ≡ $*f2*$ ((∗))
instance ⟨*proof*⟩
end

instantiation *star* :: (*uminus*) *uminus*
begin
definition *star-minus-def*: *uminus* ≡ $*f*$ *uminus*
instance ⟨*proof*⟩
end

instantiation *star* :: (*minus*) *minus*
begin
definition *star-diff-def*: (−) ≡ $*f2*$ (−)
instance ⟨*proof*⟩

end

instantiation *star* :: (*abs*) *abs*
begin
  definition *star-abs-def*: *abs* ≡ ∗*f*∗ *abs*
  instance ⟨*proof*⟩
end

instantiation *star* :: (*sgn*) *sgn*
begin
  definition *star-sgn-def*: *sgn* ≡ ∗*f*∗ *sgn*
  instance ⟨*proof*⟩
end

instantiation *star* :: (*divide*) *divide*
begin
  definition *star-divide-def*: *divide* ≡ ∗*f2*∗ *divide*
  instance ⟨*proof*⟩
end

instantiation *star* :: (*inverse*) *inverse*
begin
  definition *star-inverse-def*: *inverse* ≡ ∗*f*∗ *inverse*
  instance ⟨*proof*⟩
end

instance *star* :: (*Rings.dvd*) *Rings.dvd* ⟨*proof*⟩

instantiation *star* :: (*modulo*) *modulo*
begin
  definition *star-mod-def*: (*mod*) ≡ ∗*f2*∗ (*mod*)
  instance ⟨*proof*⟩
end

instantiation *star* :: (*ord*) *ord*
begin
  definition *star-le-def*: (≤) ≡ ∗*p2*∗ (≤)
  definition *star-less-def*: (<) ≡ ∗*p2*∗ (<)
  instance ⟨*proof*⟩
end

lemmas *star-class-defs* [*transfer-unfold*] =
  star-zero-def *star-one-def*
  star-add-def *star-diff-def* *star-minus-def*
  star-mult-def *star-divide-def* *star-inverse-def*
  star-le-def *star-less-def* *star-abs-def* *star-sgn-def*
  star-mod-def

Class operations preserve standard elements.

lemma *Standard-zero*: $0 \in Standard$
⟨*proof*⟩

lemma *Standard-one*: $1 \in Standard$
⟨*proof*⟩

lemma *Standard-add*: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow x + y \in Standard$
⟨*proof*⟩

lemma *Standard-diff*: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow x - y \in Standard$
⟨*proof*⟩

lemma *Standard-minus*: $x \in Standard \Longrightarrow - x \in Standard$
⟨*proof*⟩

lemma *Standard-mult*: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow x * y \in Standard$
⟨*proof*⟩

lemma *Standard-divide*: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow x / y \in Standard$
⟨*proof*⟩

lemma *Standard-inverse*: $x \in Standard \Longrightarrow inverse\ x \in Standard$
⟨*proof*⟩

lemma *Standard-abs*: $x \in Standard \Longrightarrow |x| \in Standard$
⟨*proof*⟩

lemma *Standard-mod*: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow x\ mod\ y \in Standard$
⟨*proof*⟩

lemmas *Standard-simps* [*simp*] =
Standard-zero *Standard-one*
Standard-add *Standard-diff* *Standard-minus*
Standard-mult *Standard-divide* *Standard-inverse*
Standard-abs *Standard-mod*

star-of preserves class operations.

lemma *star-of-add*: $star\text{-}of\ (x + y) = star\text{-}of\ x + star\text{-}of\ y$
⟨*proof*⟩

lemma *star-of-diff*: $star\text{-}of\ (x - y) = star\text{-}of\ x - star\text{-}of\ y$
⟨*proof*⟩

lemma *star-of-minus*: $star\text{-}of\ (-x) = -\ star\text{-}of\ x$
⟨*proof*⟩

lemma *star-of-mult*: $star\text{-}of\ (x * y) = star\text{-}of\ x * star\text{-}of\ y$
⟨*proof*⟩

lemma *star-of-divide*: *star-of* $(x\ /\ y) = $ *star-of* x / *star-of* y
⟨*proof*⟩

lemma *star-of-inverse*: *star-of* (*inverse* x) = *inverse* (*star-of* x)
⟨*proof*⟩

lemma *star-of-mod*: *star-of* (x *mod* y) = *star-of* x *mod* *star-of* y
⟨*proof*⟩

lemma *star-of-abs*: *star-of* $|x|$ = |*star-of* x|
⟨*proof*⟩

star-of preserves numerals.

lemma *star-of-zero*: *star-of* $0 = 0$
⟨*proof*⟩

lemma *star-of-one*: *star-of* $1 = 1$
⟨*proof*⟩

star-of preserves orderings.

lemma *star-of-less*: (*star-of* x < *star-of* y) = ($x < y$)
⟨*proof*⟩

lemma *star-of-le*: (*star-of* x ≤ *star-of* y) = ($x \leq y$)
⟨*proof*⟩

lemma *star-of-eq*: (*star-of* x = *star-of* y) = ($x = y$)
⟨*proof*⟩

As above, for *0*.

lemmas *star-of-0-less* = *star-of-less* [*of 0*, *simplified star-of-zero*]
lemmas *star-of-0-le* = *star-of-le* [*of 0*, *simplified star-of-zero*]
lemmas *star-of-0-eq* = *star-of-eq* [*of 0*, *simplified star-of-zero*]

lemmas *star-of-less-0* = *star-of-less* [*of - 0*, *simplified star-of-zero*]
lemmas *star-of-le-0* = *star-of-le* [*of - 0*, *simplified star-of-zero*]
lemmas *star-of-eq-0* = *star-of-eq* [*of - 0*, *simplified star-of-zero*]

As above, for *1*.

lemmas *star-of-1-less* = *star-of-less* [*of 1*, *simplified star-of-one*]
lemmas *star-of-1-le* = *star-of-le* [*of 1*, *simplified star-of-one*]
lemmas *star-of-1-eq* = *star-of-eq* [*of 1*, *simplified star-of-one*]

lemmas *star-of-less-1* = *star-of-less* [*of - 1*, *simplified star-of-one*]
lemmas *star-of-le-1* = *star-of-le* [*of - 1*, *simplified star-of-one*]
lemmas *star-of-eq-1* = *star-of-eq* [*of - 1*, *simplified star-of-one*]

lemmas *star-of-simps* [*simp*] =

star-of-add *star-of-diff* *star-of-minus*
star-of-mult *star-of-divide* *star-of-inverse*
star-of-mod *star-of-abs*
star-of-zero *star-of-one*
star-of-less *star-of-le* *star-of-eq*
star-of-0-less *star-of-0-le* *star-of-0-eq*
star-of-less-0 *star-of-le-0* *star-of-eq-0*
star-of-1-less *star-of-1-le* *star-of-1-eq*
star-of-less-1 *star-of-le-1* *star-of-eq-1*

2.9 Ordering and lattice classes

instance *star* :: (*order*) *order*
⟨*proof*⟩

instantiation *star* :: (*semilattice-inf*) *semilattice-inf*
begin
definition *star-inf-def* [*transfer-unfold*]: *inf* ≡ ∗*f2*∗ *inf*
instance ⟨*proof*⟩
end

instantiation *star* :: (*semilattice-sup*) *semilattice-sup*
begin
definition *star-sup-def* [*transfer-unfold*]: *sup* ≡ ∗*f2*∗ *sup*
instance ⟨*proof*⟩
end

instance *star* :: (*lattice*) *lattice* ⟨*proof*⟩

instance *star* :: (*distrib-lattice*) *distrib-lattice*
⟨*proof*⟩

lemma *Standard-inf* [*simp*]: *x* ∈ *Standard* ⟹ *y* ∈ *Standard* ⟹ *inf x y* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-sup* [*simp*]: *x* ∈ *Standard* ⟹ *y* ∈ *Standard* ⟹ *sup x y* ∈ *Standard*
⟨*proof*⟩

lemma *star-of-inf* [*simp*]: *star-of* (*inf x y*) = *inf* (*star-of x*) (*star-of y*)
⟨*proof*⟩

lemma *star-of-sup* [*simp*]: *star-of* (*sup x y*) = *sup* (*star-of x*) (*star-of y*)
⟨*proof*⟩

instance *star* :: (*linorder*) *linorder*
⟨*proof*⟩

lemma *star-max-def* [*transfer-unfold*]: *max* = ∗*f2*∗ *max*

⟨*proof*⟩

lemma *star-min-def* [*transfer-unfold*]: *min* = *∗f2∗ min*
⟨*proof*⟩

lemma *Standard-max* [*simp*]: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow max\ x\ y \in Standard$
⟨*proof*⟩

lemma *Standard-min* [*simp*]: $x \in Standard \Longrightarrow y \in Standard \Longrightarrow min\ x\ y \in Standard$
⟨*proof*⟩

lemma *star-of-max* [*simp*]: *star-of* (*max x y*) = *max* (*star-of x*) (*star-of y*)
⟨*proof*⟩

lemma *star-of-min* [*simp*]: *star-of* (*min x y*) = *min* (*star-of x*) (*star-of y*)
⟨*proof*⟩

2.10 Ordered group classes

instance *star* :: (*semigroup-add*) *semigroup-add*
⟨*proof*⟩

instance *star* :: (*ab-semigroup-add*) *ab-semigroup-add*
⟨*proof*⟩

instance *star* :: (*semigroup-mult*) *semigroup-mult*
⟨*proof*⟩

instance *star* :: (*ab-semigroup-mult*) *ab-semigroup-mult*
⟨*proof*⟩

instance *star* :: (*comm-monoid-add*) *comm-monoid-add*
⟨*proof*⟩

instance *star* :: (*monoid-mult*) *monoid-mult*
⟨*proof*⟩

instance *star* :: (*power*) *power* ⟨*proof*⟩

instance *star* :: (*comm-monoid-mult*) *comm-monoid-mult*
⟨*proof*⟩

instance *star* :: (*cancel-semigroup-add*) *cancel-semigroup-add*
⟨*proof*⟩

instance *star* :: (*cancel-ab-semigroup-add*) *cancel-ab-semigroup-add*
⟨*proof*⟩

instance *star* :: (*cancel-comm-monoid-add*) *cancel-comm-monoid-add* ⟨*proof*⟩

instance *star* :: (*ab-group-add*) *ab-group-add*
⟨*proof*⟩

instance *star* :: (*ordered-ab-semigroup-add*) *ordered-ab-semigroup-add*
⟨*proof*⟩

instance *star* :: (*ordered-cancel-ab-semigroup-add*) *ordered-cancel-ab-semigroup-add*
⟨*proof*⟩

instance *star* :: (*ordered-ab-semigroup-add-imp-le*) *ordered-ab-semigroup-add-imp-le*
⟨*proof*⟩

instance *star* :: (*ordered-comm-monoid-add*) *ordered-comm-monoid-add* ⟨*proof*⟩
instance *star* :: (*ordered-ab-semigroup-monoid-add-imp-le*) *ordered-ab-semigroup-monoid-add-imp-le*
⟨*proof*⟩
instance *star* :: (*ordered-cancel-comm-monoid-add*) *ordered-cancel-comm-monoid-add*
⟨*proof*⟩
instance *star* :: (*ordered-ab-group-add*) *ordered-ab-group-add* ⟨*proof*⟩

instance *star* :: (*ordered-ab-group-add-abs*) *ordered-ab-group-add-abs*
⟨*proof*⟩

instance *star* :: (*linordered-cancel-ab-semigroup-add*) *linordered-cancel-ab-semigroup-add*
⟨*proof*⟩

2.11 Ring and field classes

instance *star* :: (*semiring*) *semiring*
⟨*proof*⟩

instance *star* :: (*semiring-0*) *semiring-0*
⟨*proof*⟩

instance *star* :: (*semiring-0-cancel*) *semiring-0-cancel* ⟨*proof*⟩

instance *star* :: (*comm-semiring*) *comm-semiring*
⟨*proof*⟩

instance *star* :: (*comm-semiring-0*) *comm-semiring-0* ⟨*proof*⟩
instance *star* :: (*comm-semiring-0-cancel*) *comm-semiring-0-cancel* ⟨*proof*⟩

instance *star* :: (*zero-neq-one*) *zero-neq-one*
⟨*proof*⟩

instance *star* :: (*semiring-1*) *semiring-1* ⟨*proof*⟩
instance *star* :: (*comm-semiring-1*) *comm-semiring-1* ⟨*proof*⟩

declare *dvd-def* [*transfer-refold*]

instance *star* :: (*comm-semiring-1-cancel*) *comm-semiring-1-cancel*
⟨*proof*⟩

instance *star* :: (*semiring-no-zero-divisors*) *semiring-no-zero-divisors*
⟨*proof*⟩

instance *star* :: (*semiring-1-no-zero-divisors*) *semiring-1-no-zero-divisors* ⟨*proof*⟩

instance *star* :: (*semiring-no-zero-divisors-cancel*) *semiring-no-zero-divisors-cancel*
⟨*proof*⟩

instance *star* :: (*semiring-1-cancel*) *semiring-1-cancel* ⟨*proof*⟩
instance *star* :: (*ring*) *ring* ⟨*proof*⟩
instance *star* :: (*comm-ring*) *comm-ring* ⟨*proof*⟩
instance *star* :: (*ring-1*) *ring-1* ⟨*proof*⟩
instance *star* :: (*comm-ring-1*) *comm-ring-1* ⟨*proof*⟩
instance *star* :: (*semidom*) *semidom* ⟨*proof*⟩

instance *star* :: (*semidom-divide*) *semidom-divide*
⟨*proof*⟩

instance *star* :: (*ring-no-zero-divisors*) *ring-no-zero-divisors* ⟨*proof*⟩
instance *star* :: (*ring-1-no-zero-divisors*) *ring-1-no-zero-divisors* ⟨*proof*⟩
instance *star* :: (*idom*) *idom* ⟨*proof*⟩
instance *star* :: (*idom-divide*) *idom-divide* ⟨*proof*⟩

instance *star* :: (*division-ring*) *division-ring*
⟨*proof*⟩

instance *star* :: (*field*) *field*
⟨*proof*⟩

instance *star* :: (*ordered-semiring*) *ordered-semiring*
⟨*proof*⟩

instance *star* :: (*ordered-cancel-semiring*) *ordered-cancel-semiring* ⟨*proof*⟩

instance *star* :: (*linordered-semiring-strict*) *linordered-semiring-strict*
⟨*proof*⟩

instance *star* :: (*ordered-comm-semiring*) *ordered-comm-semiring*
⟨*proof*⟩

instance *star* :: (*ordered-cancel-comm-semiring*) *ordered-cancel-comm-semiring* ⟨*proof*⟩

instance *star* :: (*linordered-comm-semiring-strict*) *linordered-comm-semiring-strict*

$\langle proof \rangle$

instance *star* :: (*ordered-ring*) *ordered-ring* $\langle proof \rangle$

instance *star* :: (*ordered-ring-abs*) *ordered-ring-abs*
$\langle proof \rangle$

instance *star* :: (*abs-if*) *abs-if*
$\langle proof \rangle$

instance *star* :: (*linordered-ring-strict*) *linordered-ring-strict* $\langle proof \rangle$
instance *star* :: (*ordered-comm-ring*) *ordered-comm-ring* $\langle proof \rangle$

instance *star* :: (*linordered-semidom*) *linordered-semidom*
$\langle proof \rangle$

instance *star* :: (*linordered-idom*) *linordered-idom*
$\langle proof \rangle$

instance *star* :: (*linordered-field*) *linordered-field* $\langle proof \rangle$

instance *star* :: (*algebraic-semidom*) *algebraic-semidom* $\langle proof \rangle$

instantiation *star* :: (*normalization-semidom*) *normalization-semidom*
begin

definition *unit-factor-star* :: $'a\ star \Rightarrow 'a\ star$
where [*transfer-unfold*]: *unit-factor-star* $=$ $*f*$ *unit-factor*

definition *normalize-star* :: $'a\ star \Rightarrow 'a\ star$
where [*transfer-unfold*]: *normalize-star* $=$ $*f*$ *normalize*

instance
$\langle proof \rangle$

end

instance *star* :: (*semidom-modulo*) *semidom-modulo*
$\langle proof \rangle$

2.12 Power

lemma *star-power-def* [*transfer-unfold*]: $(\widehat{\ }) \equiv \lambda x\ n.\ (\ *f*\ (\lambda x.\ x \widehat{\ } n))\ x$
$\langle proof \rangle$

lemma *Standard-power* [*simp*]: $x \in Standard \Longrightarrow x \widehat{\ } n \in Standard$
$\langle proof \rangle$

lemma *star-of-power* [*simp*]: *star-of* $(x \widehat{\ } n) =$ *star-of* $x \widehat{\ } n$

⟨*proof*⟩

2.13 Number classes

instance *star* :: (*numeral*) *numeral* ⟨*proof*⟩

lemma *star-numeral-def* [*transfer-unfold*]: *numeral k* = *star-of* (*numeral k*)
⟨*proof*⟩

lemma *Standard-numeral* [*simp*]: *numeral k* ∈ *Standard*
⟨*proof*⟩

lemma *star-of-numeral* [*simp*]: *star-of* (*numeral k*) = *numeral k*
⟨*proof*⟩

lemma *star-of-nat-def* [*transfer-unfold*]: *of-nat n* = *star-of* (*of-nat n*)
⟨*proof*⟩

lemmas *star-of-compare-numeral* [*simp*] =
star-of-less [*of numeral k*, *simplified star-of-numeral*]
star-of-le [*of numeral k*, *simplified star-of-numeral*]
star-of-eq [*of numeral k*, *simplified star-of-numeral*]
star-of-less [*of - numeral k*, *simplified star-of-numeral*]
star-of-le [*of - numeral k*, *simplified star-of-numeral*]
star-of-eq [*of - numeral k*, *simplified star-of-numeral*]
star-of-less [*of* − *numeral k*, *simplified star-of-numeral*]
star-of-le [*of* − *numeral k*, *simplified star-of-numeral*]
star-of-eq [*of* − *numeral k*, *simplified star-of-numeral*]
star-of-less [*of* - − *numeral k*, *simplified star-of-numeral*]
star-of-le [*of* - − *numeral k*, *simplified star-of-numeral*]
star-of-eq [*of* - − *numeral k*, *simplified star-of-numeral*] **for** *k*

lemma *Standard-of-nat* [*simp*]: *of-nat n* ∈ *Standard*
⟨*proof*⟩

lemma *star-of-of-nat* [*simp*]: *star-of* (*of-nat n*) = *of-nat n*
⟨*proof*⟩

lemma *star-of-int-def* [*transfer-unfold*]: *of-int z* = *star-of* (*of-int z*)
⟨*proof*⟩

lemma *Standard-of-int* [*simp*]: *of-int z* ∈ *Standard*
⟨*proof*⟩

lemma *star-of-of-int* [*simp*]: *star-of* (*of-int z*) = *of-int z*
⟨*proof*⟩

instance *star* :: (*semiring-char-0*) *semiring-char-0*
⟨*proof*⟩

instance *star* :: (*ring-char-0*) *ring-char-0* ⟨*proof*⟩

2.14 Finite class

lemma *starset-finite*: *finite* $A \Longrightarrow *s* \ A = star\text{-}of \ ` \ A$
⟨*proof*⟩

instance *star* :: (*finite*) *finite*
⟨*proof*⟩

end

3 Hypernatural numbers

theory *HyperNat*
imports *StarDef*
begin

type-synonym *hypnat* = *nat star*

abbreviation *hypnat-of-nat* :: *nat* ⇒ *nat star*
where *hypnat-of-nat* ≡ *star-of*

definition *hSuc* :: *hypnat* ⇒ *hypnat*
where *hSuc-def* [*transfer-unfold*]: $hSuc = *f* \ Suc$

3.1 Properties Transferred from Naturals

lemma *hSuc-not-zero* [*iff*]: $\bigwedge m. \ hSuc \ m \neq 0$
⟨*proof*⟩

lemma *zero-not-hSuc* [*iff*]: $\bigwedge m. \ 0 \neq hSuc \ m$
⟨*proof*⟩

lemma *hSuc-hSuc-eq* [*iff*]: $\bigwedge m \ n. \ hSuc \ m = hSuc \ n \longleftrightarrow m = n$
⟨*proof*⟩

lemma *zero-less-hSuc* [*iff*]: $\bigwedge n. \ 0 < hSuc \ n$
⟨*proof*⟩

lemma *hypnat-minus-zero* [*simp*]: $\bigwedge z::hypnat. \ z - z = 0$
⟨*proof*⟩

lemma *hypnat-diff-0-eq-0* [*simp*]: $\bigwedge n::hypnat. \ 0 - n = 0$
⟨*proof*⟩

lemma *hypnat-add-is-0* [*iff*]: $\bigwedge m \ n::hypnat. \ m + n = 0 \longleftrightarrow m = 0 \land n = 0$
⟨*proof*⟩

lemma *hypnat-diff-diff-left*: $\bigwedge i\ j\ k::hypnat.\ i - j - k = i - (j + k)$
⟨*proof*⟩

lemma *hypnat-diff-commute*: $\bigwedge i\ j\ k::hypnat.\ i - j - k = i - k - j$
⟨*proof*⟩

lemma *hypnat-diff-add-inverse* [*simp*]: $\bigwedge m\ n::hypnat.\ n + m - n = m$
⟨*proof*⟩

lemma *hypnat-diff-add-inverse2* [*simp*]: $\bigwedge m\ n::hypnat.\ m + n - n = m$
⟨*proof*⟩

lemma *hypnat-diff-cancel* [*simp*]: $\bigwedge k\ m\ n::hypnat.\ (k + m) - (k + n) = m - n$
⟨*proof*⟩

lemma *hypnat-diff-cancel2* [*simp*]: $\bigwedge k\ m\ n::hypnat.\ (m + k) - (n + k) = m - n$
⟨*proof*⟩

lemma *hypnat-diff-add-0* [*simp*]: $\bigwedge m\ n::hypnat.\ n - (n + m) = 0$
⟨*proof*⟩

lemma *hypnat-diff-mult-distrib*: $\bigwedge k\ m\ n::hypnat.\ (m - n) * k = (m * k) - (n * k)$
⟨*proof*⟩

lemma *hypnat-diff-mult-distrib2*: $\bigwedge k\ m\ n::hypnat.\ k * (m - n) = (k * m) - (k * n)$
⟨*proof*⟩

lemma *hypnat-le-zero-cancel* [*iff*]: $\bigwedge n::hypnat.\ n \leq 0 \longleftrightarrow n = 0$
⟨*proof*⟩

lemma *hypnat-mult-is-0* [*simp*]: $\bigwedge m\ n::hypnat.\ m * n = 0 \longleftrightarrow m = 0 \vee n = 0$
⟨*proof*⟩

lemma *hypnat-diff-is-0-eq* [*simp*]: $\bigwedge m\ n::hypnat.\ m - n = 0 \longleftrightarrow m \leq n$
⟨*proof*⟩

lemma *hypnat-not-less0* [*iff*]: $\bigwedge n::hypnat.\ \neg\ n < 0$
⟨*proof*⟩

lemma *hypnat-less-one* [*iff*]: $\bigwedge n::hypnat.\ n < 1 \longleftrightarrow n = 0$
⟨*proof*⟩

lemma *hypnat-add-diff-inverse*: $\bigwedge m\ n::hypnat.\ \neg\ m < n \Longrightarrow n + (m - n) = m$
⟨*proof*⟩

lemma *hypnat-le-add-diff-inverse* [*simp*]: $\bigwedge m\ n::hypnat.\ n \leq m \Longrightarrow n + (m - n)$

$=$ *m*
⟨*proof*⟩

lemma *hypnat-le-add-diff-inverse2* [*simp*]: $\bigwedge m\ n$::*hypnat*. $n \le m \Longrightarrow (m - n) + n = m$
⟨*proof*⟩

declare *hypnat-le-add-diff-inverse2* [*OF order-less-imp-le*]

lemma *hypnat-le0* [*iff*]: $\bigwedge n$::*hypnat*. $0 \le n$
⟨*proof*⟩

lemma *hypnat-le-add1* [*simp*]: $\bigwedge x\ n$::*hypnat*. $x \le x + n$
⟨*proof*⟩

lemma *hypnat-add-self-le* [*simp*]: $\bigwedge x\ n$::*hypnat*. $x \le n + x$
⟨*proof*⟩

lemma *hypnat-add-one-self-less* [*simp*]: $x < x + 1$ **for** *x* :: *hypnat*
⟨*proof*⟩

lemma *hypnat-neq0-conv* [*iff*]: $\bigwedge n$::*hypnat*. $n \ne 0 \longleftrightarrow 0 < n$
⟨*proof*⟩

lemma *hypnat-gt-zero-iff*: $0 < n \longleftrightarrow 1 \le n$ **for** *n* :: *hypnat*
⟨*proof*⟩

lemma *hypnat-gt-zero-iff2*: $0 < n \longleftrightarrow (\exists\, m.\ n = m + 1)$ **for** *n* :: *hypnat*
⟨*proof*⟩

lemma *hypnat-add-self-not-less*: $\neg\ x + y < x$ **for** *x y* :: *hypnat*
⟨*proof*⟩

lemma *hypnat-diff-split*: $P\ (a - b) \longleftrightarrow (a < b \longrightarrow P\ 0) \wedge (\forall\, d.\ a = b + d \longrightarrow P\ d)$
for *a b* :: *hypnat*
— elimination of $-$ on *hypnat*
⟨*proof*⟩

3.2 Properties of the set of embedded natural numbers

lemma *of-nat-eq-star-of* [*simp*]: *of-nat* = *star-of*
⟨*proof*⟩

lemma *Nats-eq-Standard*: (*Nats* :: *nat star set*) = *Standard*
⟨*proof*⟩

lemma *hypnat-of-nat-mem-Nats* [*simp*]: *hypnat-of-nat* $n \in$ *Nats*
⟨*proof*⟩

lemma *hypnat-of-nat-one* [*simp*]: *hypnat-of-nat* (*Suc 0*) = *1*
⟨*proof*⟩

lemma *hypnat-of-nat-Suc* [*simp*]: *hypnat-of-nat* (*Suc n*) = *hypnat-of-nat n* + *1*
⟨*proof*⟩

lemma *of-nat-eq-add*:
fixes *d*::*hypnat*
shows *of-nat m* = *of-nat n* + *d* $\Longrightarrow$ *d* $\in$ *range of-nat*
⟨*proof*⟩

lemma *Nats-diff* [*simp*]: *a* $\in$ *Nats* $\Longrightarrow$ *b* $\in$ *Nats* $\Longrightarrow$ *a* − *b* $\in$ *Nats* **for** *a b* :: *hypnat*
⟨*proof*⟩

3.3 Infinite Hypernatural Numbers – *HNatInfinite*

The set of infinite hypernatural numbers.

definition *HNatInfinite* :: *hypnat set*
where *HNatInfinite* = {*n*. *n* $\notin$ *Nats*}

lemma *Nats-not-HNatInfinite-iff*: *x* $\in$ *Nats* $\longleftrightarrow$ *x* $\notin$ *HNatInfinite*
⟨*proof*⟩

lemma *HNatInfinite-not-Nats-iff*: *x* $\in$ *HNatInfinite* $\longleftrightarrow$ *x* $\notin$ *Nats*
⟨*proof*⟩

lemma *star-of-neq-HNatInfinite*: *N* $\in$ *HNatInfinite* $\Longrightarrow$ *star-of n* $\neq$ *N*
⟨*proof*⟩

lemma *star-of-Suc-lessI*: $\bigwedge$*N*. *star-of n* < *N* $\Longrightarrow$ *star-of* (*Suc n*) $\neq$ *N* $\Longrightarrow$ *star-of* (*Suc n*) < *N*
⟨*proof*⟩

lemma *star-of-less-HNatInfinite*:
assumes *N*: *N* $\in$ *HNatInfinite*
shows *star-of n* < *N*
⟨*proof*⟩

lemma *star-of-le-HNatInfinite*: *N* $\in$ *HNatInfinite* $\Longrightarrow$ *star-of n* $\leq$ *N*
⟨*proof*⟩

3.3.1 Closure Rules

lemma *Nats-less-HNatInfinite*: *x* $\in$ *Nats* $\Longrightarrow$ *y* $\in$ *HNatInfinite* $\Longrightarrow$ *x* < *y*
⟨*proof*⟩

lemma *Nats-le-HNatInfinite*: *x* $\in$ *Nats* $\Longrightarrow$ *y* $\in$ *HNatInfinite* $\Longrightarrow$ *x* $\leq$ *y*

⟨*proof*⟩

lemma *zero-less-HNatInfinite*: $x \in HNatInfinite \Longrightarrow 0 < x$
⟨*proof*⟩

lemma *one-less-HNatInfinite*: $x \in HNatInfinite \Longrightarrow 1 < x$
⟨*proof*⟩

lemma *one-le-HNatInfinite*: $x \in HNatInfinite \Longrightarrow 1 \leq x$
⟨*proof*⟩

lemma *zero-not-mem-HNatInfinite* [*simp*]: $0 \notin HNatInfinite$
⟨*proof*⟩

lemma *Nats-downward-closed*: $x \in Nats \Longrightarrow y \leq x \Longrightarrow y \in Nats$ **for** $x\ y :: hypnat$
⟨*proof*⟩

lemma *HNatInfinite-upward-closed*: $x \in HNatInfinite \Longrightarrow x \leq y \Longrightarrow y \in HNatInfinite$
⟨*proof*⟩

lemma *HNatInfinite-add*: $x \in HNatInfinite \Longrightarrow x + y \in HNatInfinite$
⟨*proof*⟩

lemma *HNatInfinite-diff*: $[\![x \in HNatInfinite;\ y \in Nats]\!] \Longrightarrow x - y \in HNatInfinite$
⟨*proof*⟩

lemma *HNatInfinite-is-Suc*: $x \in HNatInfinite \Longrightarrow \exists\, y.\ x = y + 1$ **for** $x :: hypnat$
⟨*proof*⟩

3.4 Existence of an infinite hypernatural number

ω is in fact an infinite hypernatural number = $[<1,\ 2,\ 3,\ \ldots>]$

definition *whn* :: *hypnat*
where *hypnat-omega-def*: $whn = star\text{-}n\ (\lambda n{::}nat.\ n)$

lemma *hypnat-of-nat-neq-whn*: $hypnat\text{-}of\text{-}nat\ n \neq whn$
⟨*proof*⟩

lemma *whn-neq-hypnat-of-nat*: $whn \neq hypnat\text{-}of\text{-}nat\ n$
⟨*proof*⟩

lemma *whn-not-Nats* [*simp*]: $whn \notin Nats$
⟨*proof*⟩

lemma *HNatInfinite-whn* [*simp*]: $whn \in HNatInfinite$
⟨*proof*⟩

lemma *lemma-unbounded-set* [*simp*]: $eventually\ (\lambda n{::}nat.\ m < n)\ \mathcal{U}$

⟨*proof*⟩

lemma *hypnat-of-nat-eq*: $\textit{hypnat-of-nat}\ m\ = \textit{star-n}\ (\lambda n{::}nat.\ m)$
⟨*proof*⟩

lemma *SHNat-eq*: $\textit{Nats} = \{n.\ \exists N.\ n = \textit{hypnat-of-nat}\ N\}$
⟨*proof*⟩

lemma *Nats-less-whn*: $n \in \textit{Nats} \Longrightarrow n < \textit{whn}$
⟨*proof*⟩

lemma *Nats-le-whn*: $n \in \textit{Nats} \Longrightarrow n \leq \textit{whn}$
⟨*proof*⟩

lemma *hypnat-of-nat-less-whn* [*simp*]: $\textit{hypnat-of-nat}\ n < \textit{whn}$
⟨*proof*⟩

lemma *hypnat-of-nat-le-whn* [*simp*]: $\textit{hypnat-of-nat}\ n \leq \textit{whn}$
⟨*proof*⟩

lemma *hypnat-zero-less-hypnat-omega* [*simp*]: $0 < \textit{whn}$
⟨*proof*⟩

lemma *hypnat-one-less-hypnat-omega* [*simp*]: $1 < \textit{whn}$
⟨*proof*⟩

3.4.1 Alternative characterization of the set of infinite hypernaturals

$\textit{HNatInfinite} = \{N.\ \forall n{\in}\mathbb{N}.\ n < N\}$

unused, but possibly interesting

lemma *HNatInfinite-FreeUltrafilterNat-eventually*:
assumes $\bigwedge k{::}nat.\ \textit{eventually}\ (\lambda n.\ f\ n \neq k)\ \mathcal{U}$
shows $\textit{eventually}\ (\lambda n.\ m < f\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *HNatInfinite-iff*: $\textit{HNatInfinite} = \{N.\ \forall n \in \textit{Nats}.\ n < N\}$
⟨*proof*⟩

3.4.2 Alternative Characterization of *HNatInfinite* using Free Ultrafilter

lemma *HNatInfinite-FreeUltrafilterNat*:
$\textit{star-n}\ X \in \textit{HNatInfinite} \Longrightarrow \forall u.\ \textit{eventually}\ (\lambda n.\ u < X\ n)\ \mathcal{U}$
⟨*proof*⟩

lemma *FreeUltrafilterNat-HNatInfinite*:
$\forall u.\ \textit{eventually}\ (\lambda n.\ u < X\ n)\ \mathcal{U} \Longrightarrow \textit{star-n}\ X \in \textit{HNatInfinite}$

⟨*proof*⟩

lemma *HNatInfinite-FreeUltrafilterNat-iff*:
$(star\text{-}n\ X \in HNatInfinite) = (\forall\, u.\ eventually\ (\lambda n.\ u < X\ n)\ \mathcal{U})$
⟨*proof*⟩

3.5 Embedding of the Hypernaturals into other types

definition *of-hypnat* :: *hypnat* ⇒ *'a::semiring-1-cancel star*
where *of-hypnat-def* [*transfer-unfold*]: $of\text{-}hypnat = {*}f{*}\ of\text{-}nat$

lemma *of-hypnat-0* [*simp*]: $of\text{-}hypnat\ 0 = 0$
⟨*proof*⟩

lemma *of-hypnat-1* [*simp*]: $of\text{-}hypnat\ 1 = 1$
⟨*proof*⟩

lemma *of-hypnat-hSuc*: $\bigwedge m.\ of\text{-}hypnat\ (hSuc\ m) = 1 + of\text{-}hypnat\ m$
⟨*proof*⟩

lemma *of-hypnat-add* [*simp*]: $\bigwedge m\ n.\ of\text{-}hypnat\ (m + n) = of\text{-}hypnat\ m + of\text{-}hypnat\ n$
⟨*proof*⟩

lemma *of-hypnat-mult* [*simp*]: $\bigwedge m\ n.\ of\text{-}hypnat\ (m * n) = of\text{-}hypnat\ m * of\text{-}hypnat\ n$
⟨*proof*⟩

lemma *of-hypnat-less-iff* [*simp*]:
$\bigwedge m\ n.\ of\text{-}hypnat\ m < (of\text{-}hypnat\ n{::}'a{::}linordered\text{-}semidom\ star) \longleftrightarrow m < n$
⟨*proof*⟩

lemma *of-hypnat-0-less-iff* [*simp*]:
$\bigwedge n.\ 0 < (of\text{-}hypnat\ n{::}'a{::}linordered\text{-}semidom\ star) \longleftrightarrow 0 < n$
⟨*proof*⟩

lemma *of-hypnat-less-0-iff* [*simp*]: $\bigwedge m.\ \neg\ (of\text{-}hypnat\ m{::}'a{::}linordered\text{-}semidom\ star) < 0$
⟨*proof*⟩

lemma *of-hypnat-le-iff* [*simp*]:
$\bigwedge m\ n.\ of\text{-}hypnat\ m \le (of\text{-}hypnat\ n{::}'a{::}linordered\text{-}semidom\ star) \longleftrightarrow m \le n$
⟨*proof*⟩

lemma *of-hypnat-0-le-iff* [*simp*]: $\bigwedge n.\ 0 \le (of\text{-}hypnat\ n{::}'a{::}linordered\text{-}semidom\ star)$
⟨*proof*⟩

lemma *of-hypnat-le-0-iff* [*simp*]: $\bigwedge m.\ (of\text{-}hypnat\ m{::}'a{::}linordered\text{-}semidom\ star)$

$\leq$ *0* $\longleftrightarrow$ $m = 0$
⟨*proof*⟩

lemma *of-hypnat-eq-iff* [*simp*]:
$\bigwedge m\ n.$ *of-hypnat* m = (*of-hypnat* n::'*a*::*linordered-semidom star*) $\longleftrightarrow$ $m = n$
⟨*proof*⟩

lemma *of-hypnat-eq-0-iff* [*simp*]: $\bigwedge m.$ (*of-hypnat* m::'*a*::*linordered-semidom star*) = *0* $\longleftrightarrow$ $m = 0$
⟨*proof*⟩

lemma *HNatInfinite-of-hypnat-gt-zero*:
$N \in$ *HNatInfinite* $\Longrightarrow$ (*0*::'*a*::*linordered-semidom star*) < *of-hypnat* N
⟨*proof*⟩

end

4 Construction of Hyperreals Using Ultrafilters

theory *HyperDef*
imports *Complex-Main HyperNat*
begin

type-synonym *hypreal* = *real star*

abbreviation *hypreal-of-real* :: *real* $\Rightarrow$ *real star*
where *hypreal-of-real* $\equiv$ *star-of*

abbreviation *hypreal-of-hypnat* :: *hypnat* $\Rightarrow$ *hypreal*
where *hypreal-of-hypnat* $\equiv$ *of-hypnat*

definition *omega* :: *hypreal* (ω)
where ω = *star-n* ($\lambda n.$ *real* (*Suc* n))
— an infinite number = [<*1*, *2*, *3*, ...>]

definition *epsilon* :: *hypreal* (ε)
where ε = *star-n* ($\lambda n.$ *inverse* (*real* (*Suc* n)))
— an infinitesimal number = [<*1*, *1/2*, *1/3*, ...>]

4.1 Real vector class instances

instantiation *star* :: (*scaleR*) *scaleR*
begin
definition *star-scaleR-def* [*transfer-unfold*]: *scaleR* $r \equiv *f*$ (*scaleR* r)
instance ⟨*proof*⟩
end

lemma *Standard-scaleR* [*simp*]: $x \in$ *Standard* $\Longrightarrow$ *scaleR* r $x \in$ *Standard*
⟨*proof*⟩

lemma *star-of-scaleR* [*simp*]: *star-of* (*scaleR r x*) = *scaleR r* (*star-of x*)
⟨*proof*⟩

instance *star* :: (*real-vector*) *real-vector*
⟨*proof*⟩

instance *star* :: (*real-algebra*) *real-algebra*
⟨*proof*⟩

instance *star* :: (*real-algebra-1*) *real-algebra-1* ⟨*proof*⟩

instance *star* :: (*real-div-algebra*) *real-div-algebra* ⟨*proof*⟩

instance *star* :: (*field-char-0*) *field-char-0* ⟨*proof*⟩

instance *star* :: (*real-field*) *real-field* ⟨*proof*⟩

lemma *star-of-real-def* [*transfer-unfold*]: *of-real r* = *star-of* (*of-real r*)
⟨*proof*⟩

lemma *Standard-of-real* [*simp*]: *of-real r* ∈ *Standard*
⟨*proof*⟩

lemma *star-of-of-real* [*simp*]: *star-of* (*of-real r*) = *of-real r*
⟨*proof*⟩

lemma *of-real-eq-star-of* [*simp*]: *of-real* = *star-of*
⟨*proof*⟩

lemma *Reals-eq-Standard*: ($\mathbb{R}$ :: *hypreal set*) = *Standard*
⟨*proof*⟩

4.2 Injection from *hypreal*

definition *of-hypreal* :: *hypreal* ⇒ *'a::real-algebra-1 star*
where [*transfer-unfold*]: *of-hypreal* = **f* of-real*

lemma *Standard-of-hypreal* [*simp*]: *r* ∈ *Standard* ⟹ *of-hypreal r* ∈ *Standard*
⟨*proof*⟩

lemma *of-hypreal-0* [*simp*]: *of-hypreal 0* = *0*
⟨*proof*⟩

lemma *of-hypreal-1* [*simp*]: *of-hypreal 1* = *1*
⟨*proof*⟩

lemma *of-hypreal-add* [*simp*]: ⋀*x y*. *of-hypreal* (*x* + *y*) = *of-hypreal x* + *of-hypreal y*

$\langle proof \rangle$

lemma *of-hypreal-minus* [*simp*]: $\bigwedge x.$ *of-hypreal* $(- x) = -$ *of-hypreal* x
$\langle proof \rangle$

lemma *of-hypreal-diff* [*simp*]: $\bigwedge x\ y.$ *of-hypreal* $(x - y) =$ *of-hypreal* $x -$ *of-hypreal* y
$\langle proof \rangle$

lemma *of-hypreal-mult* [*simp*]: $\bigwedge x\ y.$ *of-hypreal* $(x * y) =$ *of-hypreal* $x *$ *of-hypreal* y
$\langle proof \rangle$

lemma *of-hypreal-inverse* [*simp*]:
$\bigwedge x.$ *of-hypreal* (*inverse* x) =
inverse (*of-hypreal* x :: $'a$::{*real-div-algebra*, *division-ring*} *star*)
$\langle proof \rangle$

lemma *of-hypreal-divide* [*simp*]:
$\bigwedge x\ y.$ *of-hypreal* $(x\ /\ y) =$
(*of-hypreal* x / *of-hypreal* y :: $'a$::{*real-field*, *field*} *star*)
$\langle proof \rangle$

lemma *of-hypreal-eq-iff* [*simp*]: $\bigwedge x\ y.$ (*of-hypreal* $x =$ *of-hypreal* y) $= (x = y)$
$\langle proof \rangle$

lemma *of-hypreal-eq-0-iff* [*simp*]: $\bigwedge x.$ (*of-hypreal* $x = 0$) $= (x = 0)$
$\langle proof \rangle$

4.3 Properties of *starrel*

lemma *lemma-starrel-refl* [*simp*]: $x \in$ *starrel* `` $\{x\}$
$\langle proof \rangle$

lemma *starrel-in-hypreal* [*simp*]: *starrel*``$\{x\}\in$*star*
$\langle proof \rangle$

declare *Abs-star-inject* [*simp*] *Abs-star-inverse* [*simp*]
declare *equiv-starrel* [*THEN eq-equiv-class-iff*, *simp*]

4.4 *hypreal-of-real*: the Injection from *real* to *hypreal*

lemma *inj-star-of*: *inj star-of*
$\langle proof \rangle$

lemma *mem-Rep-star-iff*: $X \in$ *Rep-star* $x \longleftrightarrow x =$ *star-n* X
$\langle proof \rangle$

lemma *Rep-star-star-n-iff* [*simp*]: $X \in$ *Rep-star* (*star-n* Y) $\longleftrightarrow$ *eventually* $(\lambda n.\ Y\ n = X\ n)\ \mathcal{U}$

⟨*proof*⟩

lemma *Rep-star-star-n*: $X \in$ *Rep-star* (*star-n* X)
⟨*proof*⟩

4.5 Properties of *star-n*

lemma *star-n-add*: *star-n* X + *star-n* Y = *star-n* ($\lambda n.\ X\ n + Y\ n$)
⟨*proof*⟩

lemma *star-n-minus*: $-$ *star-n* X = *star-n* ($\lambda n.\ -(X\ n)$)
⟨*proof*⟩

lemma *star-n-diff*: *star-n* X $-$ *star-n* Y = *star-n* ($\lambda n.\ X\ n - Y\ n$)
⟨*proof*⟩

lemma *star-n-mult*: *star-n* X $*$ *star-n* Y = *star-n* ($\lambda n.\ X\ n * Y\ n$)
⟨*proof*⟩

lemma *star-n-inverse*: *inverse* (*star-n* X) = *star-n* ($\lambda n.$ *inverse* ($X\ n$))
⟨*proof*⟩

lemma *star-n-le*: *star-n* $X \leq$ *star-n* Y = *eventually* ($\lambda n.\ X\ n \leq Y\ n$) $\mathcal{U}$
⟨*proof*⟩

lemma *star-n-less*: *star-n* $X <$ *star-n* Y = *eventually* ($\lambda n.\ X\ n < Y\ n$) $\mathcal{U}$
⟨*proof*⟩

lemma *star-n-zero-num*: 0 = *star-n* ($\lambda n.\ 0$)
⟨*proof*⟩

lemma *star-n-one-num*: 1 = *star-n* ($\lambda n.\ 1$)
⟨*proof*⟩

lemma *star-n-abs*: |*star-n* X| = *star-n* ($\lambda n.\ |X\ n|$)
⟨*proof*⟩

lemma *hypreal-omega-gt-zero* [*simp*]: $0 < \omega$
⟨*proof*⟩

4.6 Existence of Infinite Hyperreal Number

Existence of infinite number not corresponding to any real number. Use assumption that member $\mathcal{U}$ is not finite.

lemma *hypreal-of-real-not-eq-omega*: *hypreal-of-real* $x \neq \omega$
⟨*proof*⟩

Existence of infinitesimal number also not corresponding to any real number.

lemma *hypreal-of-real-not-eq-epsilon*: *hypreal-of-real* $x \neq \varepsilon$

⟨*proof*⟩

lemma *epsilon-ge-zero* [*simp*]: $0 \leq \varepsilon$
⟨*proof*⟩

lemma *epsilon-not-zero*: $\varepsilon \neq 0$
⟨*proof*⟩

lemma *epsilon-inverse-omega*: $\varepsilon = \mathit{inverse}\ \omega$
⟨*proof*⟩

lemma *epsilon-gt-zero*: $0 < \varepsilon$
⟨*proof*⟩

4.7 Embedding the Naturals into the Hyperreals

abbreviation *hypreal-of-nat* :: *nat* ⇒ *hypreal*
where *hypreal-of-nat* ≡ *of-nat*

lemma *SNat-eq*: $\mathit{Nats} = \{n.\ \exists N.\ n = \mathit{hypreal\text{-}of\text{-}nat}\ N\}$
⟨*proof*⟩

Naturals embedded in hyperreals: is a hyperreal c.f. NS extension.

lemma *hypreal-of-nat*: $\mathit{hypreal\text{-}of\text{-}nat}\ m = \mathit{star\text{-}n}\ (\lambda n.\ \mathit{real}\ m)$
⟨*proof*⟩

⟨*ML*⟩

4.8 Exponentials on the Hyperreals

lemma *hpowr-0* [*simp*]: $r \mathbin{\hat{}} 0 = (1{::}\mathit{hypreal})$
for *r* :: *hypreal*
⟨*proof*⟩

lemma *hpowr-Suc* [*simp*]: $r \mathbin{\hat{}} (\mathit{Suc}\ n) = r * (r \mathbin{\hat{}} n)$
for *r* :: *hypreal*
⟨*proof*⟩

lemma *hrealpow-two*: $r \mathbin{\hat{}} \mathit{Suc}\ (\mathit{Suc}\ 0) = r * r$
for *r* :: *hypreal*
⟨*proof*⟩

lemma *hrealpow-two-le* [*simp*]: $0 \leq r \mathbin{\hat{}} \mathit{Suc}\ (\mathit{Suc}\ 0)$
for *r* :: *hypreal*
⟨*proof*⟩

lemma *hrealpow-two-le-add-order* [*simp*]: $0 \leq u \mathbin{\hat{}} \mathit{Suc}\ (\mathit{Suc}\ 0) + v \mathbin{\hat{}} \mathit{Suc}\ (\mathit{Suc}\ 0)$
for *u v* :: *hypreal*
⟨*proof*⟩

lemma *hrealpow-two-le-add-order2* [*simp*]: $0 \le u$ ^ *Suc* (*Suc* 0) + v ^ *Suc* (*Suc* 0) + w ^ *Suc* (*Suc* 0)
  for $u\ v\ w$ :: *hypreal*
  ⟨*proof*⟩

lemma *hypreal-add-nonneg-eq-0-iff*: $0 \le x \Longrightarrow 0 \le y \Longrightarrow x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
  for $x\ y$ :: *hypreal*
  ⟨*proof*⟩

lemma *hypreal-three-squares-add-zero-iff*: $x * x + y * y + z * z = 0 \longleftrightarrow x = 0 \wedge y = 0 \wedge z = 0$
  for $x\ y\ z$ :: *hypreal*
  ⟨*proof*⟩

lemma *hrealpow-three-squares-add-zero-iff* [*simp*]:
  x ^ *Suc* (*Suc* 0) + y ^ *Suc* (*Suc* 0) + z ^ *Suc* (*Suc* 0) $= 0 \longleftrightarrow x = 0 \wedge y = 0 \wedge z = 0$
  for $x\ y\ z$ :: *hypreal*
  ⟨*proof*⟩

lemma *hrabs-hrealpow-two* [*simp*]: |x ^ *Suc* (*Suc* 0)| = x ^ *Suc* (*Suc* 0)
  for x :: *hypreal*
  ⟨*proof*⟩

lemma *two-hrealpow-ge-one* [*simp*]: (*1*::*hypreal*) ≤ *2* ^ n
  ⟨*proof*⟩

lemma *hrealpow*: *star-n X* ^ m = *star-n* (λ*n*. (*X n*::*real*) ^ m)
  ⟨*proof*⟩

lemma *hrealpow-sum-square-expand*:
  (x + y) ^ *Suc* (*Suc* 0) =
    x ^ *Suc* (*Suc* 0) + y ^ *Suc* (*Suc* 0) + (*hypreal-of-nat* (*Suc* (*Suc* 0))) $* x * y$
  for $x\ y$ :: *hypreal*
  ⟨*proof*⟩

lemma *power-hypreal-of-real-numeral*:
  (*numeral v* :: *hypreal*) ^ n = *hypreal-of-real* ((*numeral v*) ^ n)
  ⟨*proof*⟩
declare *power-hypreal-of-real-numeral* [*of - numeral w*, *simp*] **for** *w*

lemma *power-hypreal-of-real-neg-numeral*:
  (− *numeral v* :: *hypreal*) ^ n = *hypreal-of-real* ((− *numeral v*) ^ n)
  ⟨*proof*⟩

declare *power-hypreal-of-real-neg-numeral* [*of* - *numeral w*, *simp*] **for** *w*

4.9 Powers with Hypernatural Exponents

Hypernatural powers of hyperreals.

definition *pow* :: *'a::power star* ⇒ *nat star* ⇒ *'a star* (**infixr** *pow 80*)
where *hyperpow-def* [*transfer-unfold*]: *R pow N* = (*∗f2∗* (^)) *R N*

lemma *Standard-hyperpow* [*simp*]: *r* ∈ *Standard* ⟹ *n* ∈ *Standard* ⟹ *r pow n* ∈ *Standard*
⟨*proof*⟩

lemma *hyperpow*: *star-n X pow star-n Y* = *star-n* (λ*n*. *X n* ^ *Y n*)
⟨*proof*⟩

lemma *hyperpow-zero* [*simp*]: ⋀*n*. (*0*::*'a*::{*power*,*semiring-0*} *star*) *pow* (*n* + (*1*::*hypnat*)) = *0*
⟨*proof*⟩

lemma *hyperpow-not-zero*: ⋀*r n*. *r* ≠ (*0*::*'a*::{*field*} *star*) ⟹ *r pow n* ≠ *0*
⟨*proof*⟩

lemma *hyperpow-inverse*: ⋀*r n*. *r* ≠ (*0*::*'a*::*field star*) ⟹ *inverse* (*r pow n*) = (*inverse r*) *pow n*
⟨*proof*⟩

lemma *hyperpow-hrabs*: ⋀*r n*. |*r*::*'a*::{*linordered-idom*} *star*| *pow n* = |*r pow n*|
⟨*proof*⟩

lemma *hyperpow-add*: ⋀*r n m*. (*r*::*'a*::*monoid-mult star*) *pow* (*n* + *m*) = (*r pow n*) ∗ (*r pow m*)
⟨*proof*⟩

lemma *hyperpow-one* [*simp*]: ⋀*r*. (*r*::*'a*::*monoid-mult star*) *pow* (*1*::*hypnat*) = *r*
⟨*proof*⟩

lemma *hyperpow-two*: ⋀*r*. (*r*::*'a*::*monoid-mult star*) *pow* (*2*::*hypnat*) = *r* ∗ *r*
⟨*proof*⟩

lemma *hyperpow-gt-zero*: ⋀*r n*. (*0*::*'a*::{*linordered-semidom*} *star*) < *r* ⟹ *0* < *r pow n*
⟨*proof*⟩

lemma *hyperpow-ge-zero*: ⋀*r n*. (*0*::*'a*::{*linordered-semidom*} *star*) ≤ *r* ⟹ *0* ≤ *r pow n*
⟨*proof*⟩

lemma *hyperpow-le*: ⋀*x y n*. (*0*::*'a*::{*linordered-semidom*} *star*) < *x* ⟹ *x* ≤ *y* ⟹ *x pow n* ≤ *y pow n*

⟨*proof*⟩

lemma *hyperpow-eq-one* [*simp*]: $\bigwedge n.\ 1\ pow\ n = (1::'a::monoid\text{-}mult\ star)$
⟨*proof*⟩

lemma *hrabs-hyperpow-minus* [*simp*]: $\bigwedge(a::'a::linordered\text{-}idom\ star)\ n.\ |(-a)\ pow\ n| = |a\ pow\ n|$
⟨*proof*⟩

lemma *hyperpow-mult*: $\bigwedge r\ s\ n.\ (r * s::'a::comm\text{-}monoid\text{-}mult\ star)\ pow\ n = (r\ pow\ n) * (s\ pow\ n)$
⟨*proof*⟩

lemma *hyperpow-two-le* [*simp*]: $\bigwedge r.\ (0::'a::\{monoid\text{-}mult,linordered\text{-}ring\text{-}strict\}\ star) \le r\ pow\ 2$
⟨*proof*⟩

lemma *hyperpow-two-hrabs* [*simp*]: $|x::'a::linordered\text{-}idom\ star|\ pow\ 2 = x\ pow\ 2$
⟨*proof*⟩

lemma *hyperpow-two-gt-one*: $\bigwedge r::'a::linordered\text{-}semidom\ star.\ 1 < r \Longrightarrow 1 < r\ pow\ 2$
⟨*proof*⟩

lemma *hyperpow-two-ge-one*: $\bigwedge r::'a::linordered\text{-}semidom\ star.\ 1 \le r \Longrightarrow 1 \le r\ pow\ 2$
⟨*proof*⟩

lemma *two-hyperpow-ge-one* [*simp*]: $(1::hypreal) \le 2\ pow\ n$
⟨*proof*⟩

lemma *hyperpow-minus-one2* [*simp*]: $\bigwedge n.\ (-\ 1)\ pow\ (2 * n) = (1::hypreal)$
⟨*proof*⟩

lemma *hyperpow-less-le*: $\bigwedge r\ n\ N.\ (0::hypreal) \le r \Longrightarrow r \le 1 \Longrightarrow n < N \Longrightarrow r\ pow\ N \le r\ pow\ n$
⟨*proof*⟩

lemma *hyperpow-SHNat-le*:
$0 \le r \Longrightarrow r \le (1::hypreal) \Longrightarrow N \in HNatInfinite \Longrightarrow \forall n{\in}Nats.\ r\ pow\ N \le r\ pow\ n$
⟨*proof*⟩

lemma *hyperpow-realpow*: $(hypreal\text{-}of\text{-}real\ r)\ pow\ (hypnat\text{-}of\text{-}nat\ n) = hypreal\text{-}of\text{-}real\ (r\ \widehat{}\ n)$
⟨*proof*⟩

lemma *hyperpow-SReal* [*simp*]: $(hypreal\text{-}of\text{-}real\ r)\ pow\ (hypnat\text{-}of\text{-}nat\ n) \in \mathbb{R}$
⟨*proof*⟩

lemma *hyperpow-zero-HNatInfinite* [*simp*]: $N \in HNatInfinite \Longrightarrow (0::hypreal)\ pow\ N = 0$
⟨*proof*⟩

lemma *hyperpow-le-le*: $(0::hypreal) \le r \Longrightarrow r \le 1 \Longrightarrow n \le N \Longrightarrow r\ pow\ N \le r\ pow\ n$
⟨*proof*⟩

lemma *hyperpow-Suc-le-self2*: $(0::hypreal) \le r \Longrightarrow r < 1 \Longrightarrow r\ pow\ (n + (1::hypnat)) \le r$
⟨*proof*⟩

lemma *hyperpow-hypnat-of-nat*: $\bigwedge x.\ x\ pow\ hypnat\text{-}of\text{-}nat\ n = x\ \hat{}\ n$
⟨*proof*⟩

lemma *of-hypreal-hyperpow*:
$\bigwedge x\ n.\ of\text{-}hypreal\ (x\ pow\ n) = (of\text{-}hypreal\ x::'a::\{real\text{-}algebra\text{-}1\}\ star)\ pow\ n$
⟨*proof*⟩

end

5 Infinite Numbers, Infinitesimals, Infinitely Close Relation

theory *NSA*
imports *HyperDef HOL−Library.Lub-Glb*
begin

definition *hnorm* :: *'a::real-normed-vector star ⇒ real star*
where [*transfer-unfold*]: *hnorm = *f* norm*

definition *Infinitesimal* :: (*'a::real-normed-vector*) *star set*
where $Infinitesimal = \{x.\ \forall r \in Reals.\ 0 < r \longrightarrow hnorm\ x < r\}$

definition *HFinite* :: (*'a::real-normed-vector*) *star set*
where $HFinite = \{x.\ \exists r \in Reals.\ hnorm\ x < r\}$

definition *HInfinite* :: (*'a::real-normed-vector*) *star set*
where $HInfinite = \{x.\ \forall r \in Reals.\ r < hnorm\ x\}$

definition *approx* :: *'a::real-normed-vector star ⇒ 'a star ⇒ bool* (**infixl** ≈ *50*)
where $x \approx y \longleftrightarrow x - y \in Infinitesimal$
— the "infinitely close" relation

definition *st* :: *hypreal ⇒ hypreal*
where $st = (\lambda x.\ SOME\ r.\ x \in HFinite \wedge r \in \mathbb{R} \wedge r \approx x)$
— the standard part of a hyperreal

definition *monad* :: $'a::real\text{-}normed\text{-}vector\ star \Rightarrow\ 'a\ star\ set$
where $monad\ x = \{y.\ x \approx y\}$

definition *galaxy* :: $'a::real\text{-}normed\text{-}vector\ star \Rightarrow\ 'a\ star\ set$
where $galaxy\ x = \{y.\ (x + -y) \in HFinite\}$

lemma *SReal-def*: $\mathbb{R} \equiv \{x.\ \exists\, r.\ x = hypreal\text{-}of\text{-}real\ r\}$
⟨*proof*⟩

5.1 Nonstandard Extension of the Norm Function

definition *scaleHR* :: $real\ star \Rightarrow\ 'a\ star \Rightarrow\ 'a::real\text{-}normed\text{-}vector\ star$
where [*transfer-unfold*]: $scaleHR = starfun2\ scaleR$

lemma *Standard-hnorm* [*simp*]: $x \in Standard \Longrightarrow hnorm\ x \in Standard$
⟨*proof*⟩

lemma *star-of-norm* [*simp*]: $star\text{-}of\ (norm\ x) = hnorm\ (star\text{-}of\ x)$
⟨*proof*⟩

lemma *hnorm-ge-zero* [*simp*]: $\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ 0 \le hnorm\ x$
⟨*proof*⟩

lemma *hnorm-eq-zero* [*simp*]: $\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *hnorm-triangle-ineq*: $\bigwedge x\ y::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (x + y) \le hnorm\ x + hnorm\ y$
⟨*proof*⟩

lemma *hnorm-triangle-ineq3*: $\bigwedge x\ y::'a::real\text{-}normed\text{-}vector\ star.\ |hnorm\ x - hnorm\ y| \le hnorm\ (x - y)$
⟨*proof*⟩

lemma *hnorm-scaleR*: $\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (a *_R x) = |star\text{-}of\ a| * hnorm\ x$
⟨*proof*⟩

lemma *hnorm-scaleHR*: $\bigwedge a\ (x::'a::real\text{-}normed\text{-}vector\ star).\ hnorm\ (scaleHR\ a\ x) = |a| * hnorm\ x$
⟨*proof*⟩

lemma *hnorm-mult-ineq*: $\bigwedge x\ y::'a::real\text{-}normed\text{-}algebra\ star.\ hnorm\ (x * y) \le hnorm\ x * hnorm\ y$
⟨*proof*⟩

lemma *hnorm-mult*: $\bigwedge x\ y::'a::real\text{-}normed\text{-}div\text{-}algebra\ star.\ hnorm\ (x * y) = hnorm$

$x * hnorm\ y$
⟨*proof*⟩

lemma *hnorm-hyperpow*: $\bigwedge(x::'a::\{real\text{-}normed\text{-}div\text{-}algebra\}\ star)\ n.\ hnorm\ (x\ pow\ n) = hnorm\ x\ pow\ n$
⟨*proof*⟩

lemma *hnorm-one* [*simp*]: $hnorm\ (1::'a::real\text{-}normed\text{-}div\text{-}algebra\ star) = 1$
⟨*proof*⟩

lemma *hnorm-zero* [*simp*]: $hnorm\ (0::'a::real\text{-}normed\text{-}vector\ star) = 0$
⟨*proof*⟩

lemma *zero-less-hnorm-iff* [*simp*]: $\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ 0 < hnorm\ x \longleftrightarrow x \neq 0$
⟨*proof*⟩

lemma *hnorm-minus-cancel* [*simp*]: $\bigwedge x::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (-\ x) = hnorm\ x$
⟨*proof*⟩

lemma *hnorm-minus-commute*: $\bigwedge a\ b::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (a - b) = hnorm\ (b - a)$
⟨*proof*⟩

lemma *hnorm-triangle-ineq2*: $\bigwedge a\ b::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ a - hnorm\ b \leq hnorm\ (a - b)$
⟨*proof*⟩

lemma *hnorm-triangle-ineq4*: $\bigwedge a\ b::'a::real\text{-}normed\text{-}vector\ star.\ hnorm\ (a - b) \leq hnorm\ a + hnorm\ b$
⟨*proof*⟩

lemma *abs-hnorm-cancel* [*simp*]: $\bigwedge a::'a::real\text{-}normed\text{-}vector\ star.\ |hnorm\ a| = hnorm\ a$
⟨*proof*⟩

lemma *hnorm-of-hypreal* [*simp*]: $\bigwedge r.\ hnorm\ (of\text{-}hypreal\ r::'a::real\text{-}normed\text{-}algebra\text{-}1\ star) = |r|$
⟨*proof*⟩

lemma *nonzero-hnorm-inverse*:
$\bigwedge a::'a::real\text{-}normed\text{-}div\text{-}algebra\ star.\ a \neq 0 \Longrightarrow hnorm\ (inverse\ a) = inverse\ (hnorm\ a)$
⟨*proof*⟩

lemma *hnorm-inverse*:
$\bigwedge a::'a::\{real\text{-}normed\text{-}div\text{-}algebra,\ division\text{-}ring\}\ star.\ hnorm\ (inverse\ a) = inverse\ (hnorm\ a)$

⟨*proof*⟩

lemma *hnorm-divide*: $\bigwedge a\ b::'a::\{real\text{-}normed\text{-}field,\ field\}\ star.\ hnorm\ (a\ /\ b) = hnorm\ a\ /\ hnorm\ b$
⟨*proof*⟩

lemma *hypreal-hnorm-def* [*simp*]: $\bigwedge r::hypreal.\ hnorm\ r = |r|$
⟨*proof*⟩

lemma *hnorm-add-less*:
$\bigwedge(x::'a::real\text{-}normed\text{-}vector\ star)\ y\ r\ s.\ hnorm\ x < r \Longrightarrow hnorm\ y < s \Longrightarrow hnorm\ (x + y) < r + s$
⟨*proof*⟩

lemma *hnorm-mult-less*:
$\bigwedge(x::'a::real\text{-}normed\text{-}algebra\ star)\ y\ r\ s.\ hnorm\ x < r \Longrightarrow hnorm\ y < s \Longrightarrow hnorm\ (x * y) < r * s$
⟨*proof*⟩

lemma *hnorm-scaleHR-less*: $|x| < r \Longrightarrow hnorm\ y < s \Longrightarrow hnorm\ (scaleHR\ x\ y) < r * s$
⟨*proof*⟩

5.2 Closure Laws for the Standard Reals

lemma *Reals-add-cancel*: $x + y \in \mathbb{R} \Longrightarrow y \in \mathbb{R} \Longrightarrow x \in \mathbb{R}$
⟨*proof*⟩

lemma *SReal-hrabs*: $x \in \mathbb{R} \Longrightarrow |x| \in \mathbb{R}$
for x :: *hypreal*
⟨*proof*⟩

lemma *SReal-hypreal-of-real* [*simp*]: $hypreal\text{-}of\text{-}real\ x \in \mathbb{R}$
⟨*proof*⟩

lemma *SReal-divide-numeral*: $r \in \mathbb{R} \Longrightarrow r\ /\ (numeral\ w::hypreal) \in \mathbb{R}$
⟨*proof*⟩

ε is not in Reals because it is an infinitesimal

lemma *SReal-epsilon-not-mem*: $\varepsilon \notin \mathbb{R}$
⟨*proof*⟩

lemma *SReal-omega-not-mem*: $\omega \notin \mathbb{R}$
⟨*proof*⟩

lemma *SReal-UNIV-real*: $\{x.\ hypreal\text{-}of\text{-}real\ x \in \mathbb{R}\} = (UNIV::real\ set)$
⟨*proof*⟩

lemma *SReal-iff*: $x \in \mathbb{R} \longleftrightarrow (\exists\, y.\ x = hypreal\text{-}of\text{-}real\ y)$

⟨*proof*⟩

lemma *hypreal-of-real-image*: $hypreal\text{-}of\text{-}real\ `(UNIV::real\ set) = \mathbb{R}$
⟨*proof*⟩

lemma *inv-hypreal-of-real-image*: $inv\ hypreal\text{-}of\text{-}real\ `\ \mathbb{R} = UNIV$
⟨*proof*⟩

lemma *SReal-dense*: $x \in \mathbb{R} \Longrightarrow y \in \mathbb{R} \Longrightarrow x < y \Longrightarrow \exists\, r \in Reals.\ x < r \wedge r < y$
for $x\ y :: hypreal$
⟨*proof*⟩

5.3 Set of Finite Elements is a Subring of the Extended Reals

lemma *HFinite-add*: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow x + y \in HFinite$
⟨*proof*⟩

lemma *HFinite-mult*: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow x * y \in HFinite$
for $x\ y :: 'a::real\text{-}normed\text{-}algebra\ star$
⟨*proof*⟩

lemma *HFinite-scaleHR*: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow scaleHR\ x\ y \in HFinite$
⟨*proof*⟩

lemma *HFinite-minus-iff*: $-\ x \in HFinite \longleftrightarrow x \in HFinite$
⟨*proof*⟩

lemma *HFinite-star-of* [*simp*]: $star\text{-}of\ x \in HFinite$
⟨*proof*⟩

lemma *SReal-subset-HFinite*: $(\mathbb{R}::hypreal\ set) \subseteq HFinite$
⟨*proof*⟩

lemma *HFiniteD*: $x \in HFinite \Longrightarrow \exists\, t \in Reals.\ hnorm\ x < t$
⟨*proof*⟩

lemma *HFinite-hrabs-iff* [*iff*]: $|x| \in HFinite \longleftrightarrow x \in HFinite$
for $x :: hypreal$
⟨*proof*⟩

lemma *HFinite-hnorm-iff* [*iff*]: $hnorm\ x \in HFinite \longleftrightarrow x \in HFinite$
for $x :: hypreal$
⟨*proof*⟩

lemma *HFinite-numeral* [*simp*]: $numeral\ w \in HFinite$
⟨*proof*⟩

As always with numerals, *0* and *1* are special cases.

lemma *HFinite-0* [*simp*]: $0 \in HFinite$

⟨*proof*⟩

lemma *HFinite-1* [*simp*]: $1 \in HFinite$
⟨*proof*⟩

lemma *hrealpow-HFinite*: $x \in HFinite \Longrightarrow x \hat{} n \in HFinite$
for x :: $'a$::{*real-normed-algebra*,*monoid-mult*} *star*
⟨*proof*⟩

lemma *HFinite-bounded*:
fixes $x\ y$ :: *hypreal*
assumes $x \in HFinite$ **and** y: $y \le x$ $0 \le y$ **shows** $y \in HFinite$
⟨*proof*⟩

5.4 Set of Infinitesimals is a Subring of the Hyperreals

lemma *InfinitesimalI*: $(\bigwedge r.\ r \in \mathbb{R} \Longrightarrow 0 < r \Longrightarrow hnorm\ x < r) \Longrightarrow x \in Infinitesimal$
⟨*proof*⟩

lemma *InfinitesimalD*: $x \in Infinitesimal \Longrightarrow \forall\, r \in Reals.\ 0 < r \longrightarrow hnorm\ x < r$
⟨*proof*⟩

lemma *InfinitesimalI2*: $(\bigwedge r.\ 0 < r \Longrightarrow hnorm\ x < star\text{-}of\ r) \Longrightarrow x \in Infinitesimal$
⟨*proof*⟩

lemma *InfinitesimalD2*: $x \in Infinitesimal \Longrightarrow 0 < r \Longrightarrow hnorm\ x < star\text{-}of\ r$
⟨*proof*⟩

lemma *Infinitesimal-zero* [*iff*]: $0 \in Infinitesimal$
⟨*proof*⟩

lemma *Infinitesimal-add*:
assumes $x \in Infinitesimal$ $y \in Infinitesimal$
shows $x + y \in Infinitesimal$
⟨*proof*⟩

lemma *Infinitesimal-minus-iff* [*simp*]: $-\ x \in Infinitesimal \longleftrightarrow x \in Infinitesimal$
⟨*proof*⟩

lemma *Infinitesimal-hnorm-iff*: $hnorm\ x \in Infinitesimal \longleftrightarrow x \in Infinitesimal$
⟨*proof*⟩

lemma *Infinitesimal-hrabs-iff* [*iff*]: $|x| \in Infinitesimal \longleftrightarrow x \in Infinitesimal$
for x :: *hypreal*
⟨*proof*⟩

lemma *Infinitesimal-of-hypreal-iff* [*simp*]:
$(of\text{-}hypreal\ x{::}'a{::}real\text{-}normed\text{-}algebra\text{-}1\ star) \in Infinitesimal \longleftrightarrow x \in Infinitesimal$

⟨*proof*⟩

lemma *Infinitesimal-diff*: $x \in \mathit{Infinitesimal} \Longrightarrow y \in \mathit{Infinitesimal} \Longrightarrow x - y \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Infinitesimal-mult*:
fixes $x\ y :: {'a}::\mathit{real\text{-}normed\text{-}algebra\ star}$
assumes $x \in \mathit{Infinitesimal}\ y \in \mathit{Infinitesimal}$
shows $x * y \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Infinitesimal-HFinite-mult*:
fixes $x\ y :: {'a}::\mathit{real\text{-}normed\text{-}algebra\ star}$
assumes $x \in \mathit{Infinitesimal}\ y \in \mathit{HFinite}$
shows $x * y \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Infinitesimal-HFinite-scaleHR*:
assumes $x \in \mathit{Infinitesimal}\ y \in \mathit{HFinite}$
shows $\mathit{scaleHR}\ x\ y \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Infinitesimal-HFinite-mult2*:
fixes $x\ y :: {'a}::\mathit{real\text{-}normed\text{-}algebra\ star}$
assumes $x \in \mathit{Infinitesimal}\ y \in \mathit{HFinite}$
shows $y * x \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Infinitesimal-scaleR2*:
assumes $x \in \mathit{Infinitesimal}$ **shows** $a *_R x \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Compl-HFinite*: $-\ \mathit{HFinite} = \mathit{HInfinite}$
⟨*proof*⟩

lemma *HInfinite-inverse-Infinitesimal*:
$x \in \mathit{HInfinite} \Longrightarrow \mathit{inverse}\ x \in \mathit{Infinitesimal}$
for $x :: {'a}::\mathit{real\text{-}normed\text{-}div\text{-}algebra\ star}$
⟨*proof*⟩

lemma *inverse-Infinitesimal-iff-HInfinite*:
$x \neq 0 \Longrightarrow \mathit{inverse}\ x \in \mathit{Infinitesimal} \longleftrightarrow x \in \mathit{HInfinite}$
for $x :: {'a}::\mathit{real\text{-}normed\text{-}div\text{-}algebra\ star}$
⟨*proof*⟩

lemma *HInfiniteI*: $(\bigwedge r.\ r \in \mathbb{R} \Longrightarrow r < \mathit{hnorm}\ x) \Longrightarrow x \in \mathit{HInfinite}$
⟨*proof*⟩

lemma *HInfiniteD*: $x \in \mathit{HInfinite} \Longrightarrow r \in \mathbb{R} \Longrightarrow r < \mathit{hnorm}\ x$
⟨*proof*⟩

lemma *HInfinite-mult*:
fixes $x\ y$ :: *'a::real-normed-div-algebra star*
assumes $x \in \mathit{HInfinite}$ $y \in \mathit{HInfinite}$ **shows** $x * y \in \mathit{HInfinite}$
⟨*proof*⟩

lemma *hypreal-add-zero-less-le-mono*: $r < x \Longrightarrow 0 \leq y \Longrightarrow r < x + y$
for $r\ x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *HInfinite-add-ge-zero*: $x \in \mathit{HInfinite} \Longrightarrow 0 \leq y \Longrightarrow 0 \leq x \Longrightarrow x + y \in \mathit{HInfinite}$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *HInfinite-add-ge-zero2*: $x \in \mathit{HInfinite} \Longrightarrow 0 \leq y \Longrightarrow 0 \leq x \Longrightarrow y + x \in \mathit{HInfinite}$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *HInfinite-add-gt-zero*: $x \in \mathit{HInfinite} \Longrightarrow 0 < y \Longrightarrow 0 < x \Longrightarrow x + y \in \mathit{HInfinite}$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *HInfinite-minus-iff*: $- x \in \mathit{HInfinite} \longleftrightarrow x \in \mathit{HInfinite}$
⟨*proof*⟩

lemma *HInfinite-add-le-zero*: $x \in \mathit{HInfinite} \Longrightarrow y \leq 0 \Longrightarrow x \leq 0 \Longrightarrow x + y \in \mathit{HInfinite}$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *HInfinite-add-lt-zero*: $x \in \mathit{HInfinite} \Longrightarrow y < 0 \Longrightarrow x < 0 \Longrightarrow x + y \in \mathit{HInfinite}$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *not-Infinitesimal-not-zero*: $x \notin \mathit{Infinitesimal} \Longrightarrow x \neq 0$
⟨*proof*⟩

lemma *HFinite-diff-Infinitesimal-hrabs*:
$x \in \mathit{HFinite} - \mathit{Infinitesimal} \Longrightarrow |x| \in \mathit{HFinite} - \mathit{Infinitesimal}$
for x :: *hypreal*
⟨*proof*⟩

lemma *hnorm-le-Infinitesimal*: $e \in \mathit{Infinitesimal} \Longrightarrow \mathit{hnorm}\ x \leq e \Longrightarrow x \in$ *In-*

finitesimal
⟨*proof*⟩

lemma *hnorm-less-Infinitesimal*: $e \in \mathit{Infinitesimal} \Longrightarrow \mathit{hnorm}\ x < e \Longrightarrow x \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *hrabs-le-Infinitesimal*: $e \in \mathit{Infinitesimal} \Longrightarrow |x| \leq e \Longrightarrow x \in \mathit{Infinitesimal}$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *hrabs-less-Infinitesimal*: $e \in \mathit{Infinitesimal} \Longrightarrow |x| < e \Longrightarrow x \in \mathit{Infinitesimal}$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *Infinitesimal-interval*:
$e \in \mathit{Infinitesimal} \Longrightarrow e' \in \mathit{Infinitesimal} \Longrightarrow e' < x \Longrightarrow x < e \Longrightarrow x \in \mathit{Infinitesimal}$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *Infinitesimal-interval2*:
$e \in \mathit{Infinitesimal} \Longrightarrow e' \in \mathit{Infinitesimal} \Longrightarrow e' \leq x \Longrightarrow x \leq e \Longrightarrow x \in \mathit{Infinitesimal}$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *lemma-Infinitesimal-hyperpow*: $x \in \mathit{Infinitesimal} \Longrightarrow 0 < N \Longrightarrow |x\ \mathit{pow}\ N| \leq |x|$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *Infinitesimal-hyperpow*: $x \in \mathit{Infinitesimal} \Longrightarrow 0 < N \Longrightarrow x\ \mathit{pow}\ N \in \mathit{Infinitesimal}$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *hrealpow-hyperpow-Infinitesimal-iff*:
$(x \mathbin{\hat{}} n \in \mathit{Infinitesimal}) \longleftrightarrow x\ \mathit{pow}\ (\mathit{hypnat\text{-}of\text{-}nat}\ n) \in \mathit{Infinitesimal}$
⟨*proof*⟩

lemma *Infinitesimal-hrealpow*: $x \in \mathit{Infinitesimal} \Longrightarrow 0 < n \Longrightarrow x \mathbin{\hat{}} n \in \mathit{Infinitesimal}$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *not-Infinitesimal-mult*:
$x \notin \mathit{Infinitesimal} \Longrightarrow y \notin \mathit{Infinitesimal} \Longrightarrow x * y \notin \mathit{Infinitesimal}$
for $x\ y :: 'a{::}\mathit{real\text{-}normed\text{-}div\text{-}algebra}\ \mathit{star}$
⟨*proof*⟩

lemma *Infinitesimal-mult-disj*: $x * y \in \mathit{Infinitesimal} \Longrightarrow x \in \mathit{Infinitesimal} \lor y \in \mathit{Infinitesimal}$
for $x\ y$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *HFinite-Infinitesimal-not-zero*: $x \in \mathit{HFinite}-\mathit{Infinitesimal} \Longrightarrow x \neq 0$
⟨*proof*⟩

lemma *HFinite-Infinitesimal-diff-mult*:
$x \in \mathit{HFinite} - \mathit{Infinitesimal} \Longrightarrow y \in \mathit{HFinite} - \mathit{Infinitesimal} \Longrightarrow x * y \in \mathit{HFinite} - \mathit{Infinitesimal}$
for $x\ y$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *Infinitesimal-subset-HFinite*: $\mathit{Infinitesimal} \subseteq \mathit{HFinite}$
⟨*proof*⟩

lemma *Infinitesimal-star-of-mult*: $x \in \mathit{Infinitesimal} \Longrightarrow x * \mathit{star\text{-}of}\ r \in \mathit{Infinitesimal}$
for x :: $'a$::*real-normed-algebra star*
⟨*proof*⟩

lemma *Infinitesimal-star-of-mult2*: $x \in \mathit{Infinitesimal} \Longrightarrow \mathit{star\text{-}of}\ r * x \in \mathit{Infinitesimal}$
for x :: $'a$::*real-normed-algebra star*
⟨*proof*⟩

5.5 The Infinitely Close Relation

lemma *mem-infmal-iff*: $x \in \mathit{Infinitesimal} \longleftrightarrow x \approx 0$
⟨*proof*⟩

lemma *approx-minus-iff*: $x \approx y \longleftrightarrow x - y \approx 0$
⟨*proof*⟩

lemma *approx-minus-iff2*: $x \approx y \longleftrightarrow - y + x \approx 0$
⟨*proof*⟩

lemma *approx-refl* [*iff*]: $x \approx x$
⟨*proof*⟩

lemma *approx-sym*: $x \approx y \Longrightarrow y \approx x$
⟨*proof*⟩

lemma *approx-trans*:
assumes $x \approx y$ $y \approx z$ **shows** $x \approx z$
⟨*proof*⟩

lemma *approx-trans2*: $r \approx x \Longrightarrow s \approx x \Longrightarrow r \approx s$
⟨*proof*⟩

lemma *approx-trans3*: $x \approx r \Longrightarrow x \approx s \Longrightarrow r \approx s$
⟨*proof*⟩

lemma *approx-reorient*: $x \approx y \longleftrightarrow y \approx x$
⟨*proof*⟩

Reorientation simplification procedure: reorients (polymorphic) $0 = x$, $1 = x$, $nnn = x$ provided x isn't 0, 1 or a numeral.

⟨*ML*⟩

lemma *Infinitesimal-approx-minus*: $x - y \in Infinitesimal \longleftrightarrow x \approx y$
⟨*proof*⟩

lemma *approx-monad-iff*: $x \approx y \longleftrightarrow monad\ x = monad\ y$
⟨*proof*⟩

lemma *Infinitesimal-approx*: $x \in Infinitesimal \Longrightarrow y \in Infinitesimal \Longrightarrow x \approx y$
⟨*proof*⟩

lemma *approx-add*: $a \approx b \Longrightarrow c \approx d \Longrightarrow a + c \approx b + d$
⟨*proof*⟩

lemma *approx-minus*: $a \approx b \Longrightarrow - a \approx - b$
⟨*proof*⟩

lemma *approx-minus2*: $- a \approx - b \Longrightarrow a \approx b$
⟨*proof*⟩

lemma *approx-minus-cancel* [*simp*]: $- a \approx - b \longleftrightarrow a \approx b$
⟨*proof*⟩

lemma *approx-add-minus*: $a \approx b \Longrightarrow c \approx d \Longrightarrow a + - c \approx b + - d$
⟨*proof*⟩

lemma *approx-diff*: $a \approx b \Longrightarrow c \approx d \Longrightarrow a - c \approx b - d$
⟨*proof*⟩

lemma *approx-mult1*: $a \approx b \Longrightarrow c \in HFinite \Longrightarrow a * c \approx b * c$
for $a\ b\ c$ :: *'a::real-normed-algebra star*
⟨*proof*⟩

lemma *approx-mult2*: $a \approx b \Longrightarrow c \in HFinite \Longrightarrow c * a \approx c * b$
for $a\ b\ c$ :: *'a::real-normed-algebra star*
⟨*proof*⟩

lemma *approx-mult-subst*: $u \approx v * x \Longrightarrow x \approx y \Longrightarrow v \in HFinite \Longrightarrow u \approx v * y$

for $u\ v\ x\ y$ :: $'a$::*real-normed-algebra star*
⟨*proof*⟩

lemma *approx-mult-subst2*: $u \approx x * v \Longrightarrow x \approx y \Longrightarrow v \in \mathit{HFinite} \Longrightarrow u \approx y * v$
for $u\ v\ x\ y$ :: $'a$::*real-normed-algebra star*
⟨*proof*⟩

lemma *approx-mult-subst-star-of*: $u \approx x * \mathit{star\text{-}of}\ v \Longrightarrow x \approx y \Longrightarrow u \approx y * \mathit{star\text{-}of}\ v$
for $u\ x\ y$ :: $'a$::*real-normed-algebra star*
⟨*proof*⟩

lemma *approx-eq-imp*: $a = b \Longrightarrow a \approx b$
⟨*proof*⟩

lemma *Infinitesimal-minus-approx*: $x \in \mathit{Infinitesimal} \Longrightarrow -\ x \approx x$
⟨*proof*⟩

lemma *bex-Infinitesimal-iff*: $(\exists\, y \in \mathit{Infinitesimal}.\ x - z = y) \longleftrightarrow x \approx z$
⟨*proof*⟩

lemma *bex-Infinitesimal-iff2*: $(\exists\, y \in \mathit{Infinitesimal}.\ x = z + y) \longleftrightarrow x \approx z$
⟨*proof*⟩

lemma *Infinitesimal-add-approx*: $y \in \mathit{Infinitesimal} \Longrightarrow x + y = z \Longrightarrow x \approx z$
⟨*proof*⟩

lemma *Infinitesimal-add-approx-self*: $y \in \mathit{Infinitesimal} \Longrightarrow x \approx x + y$
⟨*proof*⟩

lemma *Infinitesimal-add-approx-self2*: $y \in \mathit{Infinitesimal} \Longrightarrow x \approx y + x$
⟨*proof*⟩

lemma *Infinitesimal-add-minus-approx-self*: $y \in \mathit{Infinitesimal} \Longrightarrow x \approx x + -\ y$
⟨*proof*⟩

lemma *Infinitesimal-add-cancel*: $y \in \mathit{Infinitesimal} \Longrightarrow x + y \approx z \Longrightarrow x \approx z$
⟨*proof*⟩

lemma *Infinitesimal-add-right-cancel*: $y \in \mathit{Infinitesimal} \Longrightarrow x \approx z + y \Longrightarrow x \approx z$
⟨*proof*⟩

lemma *approx-add-left-cancel*: $d + b \approx d + c \Longrightarrow b \approx c$
⟨*proof*⟩

lemma *approx-add-right-cancel*: $b + d \approx c + d \Longrightarrow b \approx c$
⟨*proof*⟩

lemma *approx-add-mono1*: $b \approx c \Longrightarrow d + b \approx d + c$

⟨*proof*⟩

lemma *approx-add-mono2*: $b \approx c \Longrightarrow b + a \approx c + a$
⟨*proof*⟩

lemma *approx-add-left-iff* [*simp*]: $a + b \approx a + c \longleftrightarrow b \approx c$
⟨*proof*⟩

lemma *approx-add-right-iff* [*simp*]: $b + a \approx c + a \longleftrightarrow b \approx c$
⟨*proof*⟩

lemma *approx-HFinite*: $x \in \mathit{HFinite} \Longrightarrow x \approx y \Longrightarrow y \in \mathit{HFinite}$
⟨*proof*⟩

lemma *approx-star-of-HFinite*: $x \approx \mathit{star\text{-}of}\ D \Longrightarrow x \in \mathit{HFinite}$
⟨*proof*⟩

lemma *approx-mult-HFinite*: $a \approx b \Longrightarrow c \approx d \Longrightarrow b \in \mathit{HFinite} \Longrightarrow d \in \mathit{HFinite} \Longrightarrow a * c \approx b * d$
for $a\ b\ c\ d$ :: *'a::real-normed-algebra star*
⟨*proof*⟩

lemma *scaleHR-left-diff-distrib*: $\bigwedge a\ b\ x.\ \mathit{scaleHR}\ (a - b)\ x = \mathit{scaleHR}\ a\ x - \mathit{scaleHR}\ b\ x$
⟨*proof*⟩

lemma *approx-scaleR1*: $a \approx \mathit{star\text{-}of}\ b \Longrightarrow c \in \mathit{HFinite} \Longrightarrow \mathit{scaleHR}\ a\ c \approx b *_R c$
⟨*proof*⟩

lemma *approx-scaleR2*: $a \approx b \Longrightarrow c *_R a \approx c *_R b$
⟨*proof*⟩

lemma *approx-scaleR-HFinite*: $a \approx \mathit{star\text{-}of}\ b \Longrightarrow c \approx d \Longrightarrow d \in \mathit{HFinite} \Longrightarrow \mathit{scaleHR}\ a\ c \approx b *_R d$
⟨*proof*⟩

lemma *approx-mult-star-of*: $a \approx \mathit{star\text{-}of}\ b \Longrightarrow c \approx \mathit{star\text{-}of}\ d \Longrightarrow a * c \approx \mathit{star\text{-}of}\ b * \mathit{star\text{-}of}\ d$
for $a\ c$ :: *'a::real-normed-algebra star*
⟨*proof*⟩

lemma *approx-SReal-mult-cancel-zero*:
fixes $a\ x$ :: *hypreal*
assumes $a \in \mathbb{R}$ $a \neq 0$ $a * x \approx 0$ **shows** $x \approx 0$
⟨*proof*⟩

lemma *approx-mult-SReal1*: $a \in \mathbb{R} \Longrightarrow x \approx 0 \Longrightarrow x * a \approx 0$
for $a\ x$ :: *hypreal*
⟨*proof*⟩

lemma *approx-mult-SReal2*: $a \in \mathbb{R} \Longrightarrow x \approx 0 \Longrightarrow a * x \approx 0$
for $a\ x$ :: *hypreal*
⟨*proof*⟩

lemma *approx-mult-SReal-zero-cancel-iff* [*simp*]: $a \in \mathbb{R} \Longrightarrow a \neq 0 \Longrightarrow a * x \approx 0 \longleftrightarrow x \approx 0$
for $a\ x$ :: *hypreal*
⟨*proof*⟩

lemma *approx-SReal-mult-cancel*:
fixes $a\ w\ z$ :: *hypreal*
assumes $a \in \mathbb{R}$ $a \neq 0$ $a * w \approx a * z$ **shows** $w \approx z$
⟨*proof*⟩

lemma *approx-SReal-mult-cancel-iff1* [*simp*]: $a \in \mathbb{R} \Longrightarrow a \neq 0 \Longrightarrow a * w \approx a * z \longleftrightarrow w \approx z$
for $a\ w\ z$ :: *hypreal*
⟨*proof*⟩

lemma *approx-le-bound*:
fixes z :: *hypreal*
assumes $z \leq f$ $f \approx g$ $g \leq z$ **shows** $f \approx z$
⟨*proof*⟩

lemma *approx-hnorm*: $x \approx y \Longrightarrow hnorm\ x \approx hnorm\ y$
for $x\ y$ :: *'a::real-normed-vector star*
⟨*proof*⟩

5.6 Zero is the Only Infinitesimal that is also a Real

lemma *Infinitesimal-less-SReal*: $x \in \mathbb{R} \Longrightarrow y \in Infinitesimal \Longrightarrow 0 < x \Longrightarrow y < x$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *Infinitesimal-less-SReal2*: $y \in Infinitesimal \Longrightarrow \forall r \in Reals.\ 0 < r \longrightarrow y < r$
for y :: *hypreal*
⟨*proof*⟩

lemma *SReal-not-Infinitesimal*: $0 < y \Longrightarrow y \in \mathbb{R} ==> y \notin Infinitesimal$
for y :: *hypreal*
⟨*proof*⟩

lemma *SReal-minus-not-Infinitesimal*: $y < 0 \Longrightarrow y \in \mathbb{R} \Longrightarrow y \notin Infinitesimal$
for y :: *hypreal*
⟨*proof*⟩

lemma *SReal-Int-Infinitesimal-zero*: $\mathbb{R}$ *Int Infinitesimal* = $\{0::hypreal\}$
⟨*proof*⟩

lemma *SReal-Infinitesimal-zero*: $x \in \mathbb{R} \Longrightarrow x \in Infinitesimal \Longrightarrow x = 0$
for x :: *hypreal*
⟨*proof*⟩

lemma *SReal-HFinite-diff-Infinitesimal*: $x \in \mathbb{R} \Longrightarrow x \neq 0 \Longrightarrow x \in HFinite - Infinitesimal$
for x :: *hypreal*
⟨*proof*⟩

lemma *hypreal-of-real-HFinite-diff-Infinitesimal*:
hypreal-of-real $x \neq 0 \Longrightarrow$ *hypreal-of-real* $x \in HFinite - Infinitesimal$
⟨*proof*⟩

lemma *star-of-Infinitesimal-iff-0* [*iff*]: *star-of* $x \in Infinitesimal \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *star-of-HFinite-diff-Infinitesimal*: $x \neq 0 \Longrightarrow$ *star-of* $x \in HFinite - Infinitesimal$
⟨*proof*⟩

lemma *numeral-not-Infinitesimal* [*simp*]:
numeral $w \neq (0::hypreal) \Longrightarrow$ (*numeral* w :: *hypreal*) $\notin$ *Infinitesimal*
⟨*proof*⟩

Again: *1* is a special case, but not *0* this time.

lemma *one-not-Infinitesimal* [*simp*]:
($1::'a::\{$*real-normed-vector*,*zero-neq-one*$\}$ *star*) $\notin$ *Infinitesimal*
⟨*proof*⟩

lemma *approx-SReal-not-zero*: $y \in \mathbb{R} \Longrightarrow x \approx y \Longrightarrow y \neq 0 \Longrightarrow x \neq 0$
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *HFinite-diff-Infinitesimal-approx*:
$x \approx y \Longrightarrow y \in HFinite - Infinitesimal \Longrightarrow x \in HFinite - Infinitesimal$
⟨*proof*⟩

The premise $y \neq 0$ is essential; otherwise $x / y = 0$ and we lose the *HFinite* premise.

lemma *Infinitesimal-ratio*:
$y \neq 0 \Longrightarrow y \in Infinitesimal \Longrightarrow x / y \in HFinite \Longrightarrow x \in Infinitesimal$
for $x\ y$ :: $'a::\{$*real-normed-div-algebra*,*field*$\}$ *star*
⟨*proof*⟩

lemma *Infinitesimal-SReal-divide*: $x \in Infinitesimal \Longrightarrow y \in \mathbb{R} \Longrightarrow x / y \in Infinitesimal$

for $x\ y :: hypreal$
⟨*proof*⟩

6 Standard Part Theorem

Every finite $x \in R*$ is infinitely close to a unique real number (i.e. a member of *Reals*).

6.1 Uniqueness: Two Infinitely Close Reals are Equal

lemma *star-of-approx-iff* [*simp*]: $star\text{-}of\ x \approx star\text{-}of\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *SReal-approx-iff*: $x \in \mathbb{R} \Longrightarrow y \in \mathbb{R} \Longrightarrow x \approx y \longleftrightarrow x = y$
for $x\ y :: hypreal$
⟨*proof*⟩

lemma *numeral-approx-iff* [*simp*]:
$(numeral\ v \approx (numeral\ w :: 'a::\{numeral,real\text{-}normed\text{-}vector\}\ star)) = (numeral\ v = (numeral\ w :: 'a))$
⟨*proof*⟩

And also for $0 \approx \#nn$ and $1 \approx \#nn$, $\#nn \approx 0$ and $\#nn \approx 1$.

lemma [*simp*]:
$(numeral\ w \approx (0::'a::\{numeral,real\text{-}normed\text{-}vector\}\ star)) = (numeral\ w = (0::'a))$
$((0::'a::\{numeral,real\text{-}normed\text{-}vector\}\ star) \approx numeral\ w) = (numeral\ w = (0::'a))$
$(numeral\ w \approx (1::'b::\{numeral,one,real\text{-}normed\text{-}vector\}\ star)) = (numeral\ w = (1::'b))$
$((1::'b::\{numeral,one,real\text{-}normed\text{-}vector\}\ star) \approx numeral\ w) = (numeral\ w = (1::'b))$
$\neg\ (0 \approx (1::'c::\{zero\text{-}neq\text{-}one,real\text{-}normed\text{-}vector\}\ star))$
$\neg\ (1 \approx (0::'c::\{zero\text{-}neq\text{-}one,real\text{-}normed\text{-}vector\}\ star))$
⟨*proof*⟩

lemma *star-of-approx-numeral-iff* [*simp*]: $star\text{-}of\ k \approx numeral\ w \longleftrightarrow k = numeral\ w$
⟨*proof*⟩

lemma *star-of-approx-zero-iff* [*simp*]: $star\text{-}of\ k \approx 0 \longleftrightarrow k = 0$
⟨*proof*⟩

lemma *star-of-approx-one-iff* [*simp*]: $star\text{-}of\ k \approx 1 \longleftrightarrow k = 1$
⟨*proof*⟩

lemma *approx-unique-real*: $r \in \mathbb{R} \Longrightarrow s \in \mathbb{R} \Longrightarrow r \approx x \Longrightarrow s \approx x \Longrightarrow r = s$
for $r\ s :: hypreal$
⟨*proof*⟩

6.2 Existence of Unique Real Infinitely Close

6.2.1 Lifting of the Ub and Lub Properties

lemma *hypreal-of-real-isUb-iff*: *isUb* $\mathbb{R}$ (*hypreal-of-real* ‘ Q) (*hypreal-of-real* Y) = *isUb UNIV* Q Y
  for Q :: *real set* **and** Y :: *real*
  ⟨*proof*⟩

lemma *hypreal-of-real-isLub-iff*:
  isLub $\mathbb{R}$ (*hypreal-of-real* ‘ Q) (*hypreal-of-real* Y) = *isLub* (*UNIV* :: *real set*) Q Y
(**is** *?lhs* = *?rhs*)
  for Q :: *real set* **and** Y :: *real*
⟨*proof*⟩

lemma *lemma-isUb-hypreal-of-real*: *isUb* $\mathbb{R}$ P Y $\Longrightarrow$ $\exists$ *Yo*. *isUb* $\mathbb{R}$ P (*hypreal-of-real* *Yo*)
  ⟨*proof*⟩

lemma *lemma-isLub-hypreal-of-real*: *isLub* $\mathbb{R}$ P Y $\Longrightarrow$ $\exists$ *Yo*. *isLub* $\mathbb{R}$ P (*hypreal-of-real* *Yo*)
  ⟨*proof*⟩

lemma *SReal-complete*:
  fixes P :: *hypreal set*
  assumes *isUb* $\mathbb{R}$ P Y $P \subseteq \mathbb{R}$ $P \neq \{\}$
    shows $\exists\, t$. *isLub* $\mathbb{R}$ P t
⟨*proof*⟩

Lemmas about lubs.

lemma *lemma-st-part-lub*:
  fixes x :: *hypreal*
  assumes $x \in$ *HFinite*
  shows $\exists\, t$. *isLub* $\mathbb{R}$ $\{s.\ s \in \mathbb{R} \wedge s < x\}$ t
⟨*proof*⟩

lemma *hypreal-setle-less-trans*: S $*$<= x $\Longrightarrow$ $x < y$ $\Longrightarrow$ S $*$<= y
  for x y :: *hypreal*
  ⟨*proof*⟩

lemma *hypreal-gt-isUb*: *isUb* R S x $\Longrightarrow$ $x < y$ $\Longrightarrow$ $y \in R$ $\Longrightarrow$ *isUb* R S y
  for x y :: *hypreal*
  ⟨*proof*⟩

lemma *lemma-SReal-ub*: $x \in \mathbb{R}$ $\Longrightarrow$ *isUb* $\mathbb{R}$ $\{s.\ s \in \mathbb{R} \wedge s < x\}$ x
  for x :: *hypreal*
  ⟨*proof*⟩

lemma *lemma-SReal-lub*:
  fixes x :: *hypreal*

assumes $x \in \mathbb{R}$ **shows** *isLub* $\mathbb{R}$ $\{s.\ s \in \mathbb{R} \wedge s < x\}$ x
⟨*proof*⟩

lemma *lemma-st-part-major*:
fixes $x\ r\ t :: hypreal$
assumes x: $x \in HFinite$ **and** r: $r \in \mathbb{R}$ $0 < r$ **and** t: *isLub* $\mathbb{R}$ $\{s.\ s \in \mathbb{R} \wedge s < x\}$ t
shows $|x - t| < r$
⟨*proof*⟩

lemma *lemma-st-part-major2*:
$x \in HFinite \Longrightarrow isLub\ \mathbb{R}\ \{s.\ s \in \mathbb{R} \wedge s < x\}\ t \Longrightarrow \forall r \in Reals.\ 0 < r \longrightarrow |x - t| < r$
for $x\ t :: hypreal$
⟨*proof*⟩

Existence of real and Standard Part Theorem.

lemma *lemma-st-part-Ex*: $x \in HFinite \Longrightarrow \exists t{\in}Reals.\ \forall r \in Reals.\ 0 < r \longrightarrow |x - t| < r$
for $x :: hypreal$
⟨*proof*⟩

lemma *st-part-Ex*: $x \in HFinite \Longrightarrow \exists t{\in}Reals.\ x \approx t$
for $x :: hypreal$
⟨*proof*⟩

There is a unique real infinitely close.

lemma *st-part-Ex1*: $x \in HFinite \Longrightarrow \exists!t{::}hypreal.\ t \in \mathbb{R} \wedge x \approx t$
⟨*proof*⟩

6.3 Finite, Infinite and Infinitesimal

lemma *HFinite-Int-HInfinite-empty* [*simp*]: $HFinite\ Int\ HInfinite = \{\}$
⟨*proof*⟩

lemma *HFinite-not-HInfinite*:
assumes x: $x \in HFinite$ **shows** $x \notin HInfinite$
⟨*proof*⟩

lemma *not-HFinite-HInfinite*: $x \notin HFinite \Longrightarrow x \in HInfinite$
⟨*proof*⟩

lemma *HInfinite-HFinite-disj*: $x \in HInfinite \vee x \in HFinite$
⟨*proof*⟩

lemma *HInfinite-HFinite-iff*: $x \in HInfinite \longleftrightarrow x \notin HFinite$
⟨*proof*⟩

lemma *HFinite-HInfinite-iff*: $x \in HFinite \longleftrightarrow x \notin HInfinite$

⟨*proof*⟩

lemma *HInfinite-diff-HFinite-Infinitesimal-disj*:
$x \notin Infinitesimal \Longrightarrow x \in HInfinite \vee x \in HFinite - Infinitesimal$
⟨*proof*⟩

lemma *HFinite-inverse*: $x \in HFinite \Longrightarrow x \notin Infinitesimal \Longrightarrow inverse\ x \in HFinite$
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *HFinite-inverse2*: $x \in HFinite - Infinitesimal \Longrightarrow inverse\ x \in HFinite$
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

Stronger statement possible in fact.

lemma *Infinitesimal-inverse-HFinite*: $x \notin Infinitesimal \Longrightarrow inverse\ x \in HFinite$
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *HFinite-not-Infinitesimal-inverse*:
$x \in HFinite - Infinitesimal \Longrightarrow inverse\ x \in HFinite - Infinitesimal$
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *approx-inverse*:
fixes $x\ y$ :: $'a$::*real-normed-div-algebra star*
assumes $x \approx y$ **and** y: $y \in HFinite - Infinitesimal$ **shows** $inverse\ x \approx inverse\ y$
⟨*proof*⟩

lemmas *star-of-approx-inverse* = *star-of-HFinite-diff-Infinitesimal* [*THEN* [2] *approx-inverse*]
lemmas *hypreal-of-real-approx-inverse* = *hypreal-of-real-HFinite-diff-Infinitesimal* [*THEN* [2] *approx-inverse*]

lemma *inverse-add-Infinitesimal-approx*:
$x \in HFinite - Infinitesimal \Longrightarrow h \in Infinitesimal \Longrightarrow inverse\ (x + h) \approx inverse\ x$
for $x\ h$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *inverse-add-Infinitesimal-approx2*:
$x \in HFinite - Infinitesimal \Longrightarrow h \in Infinitesimal \Longrightarrow inverse\ (h + x) \approx inverse\ x$
for $x\ h$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *inverse-add-Infinitesimal-approx-Infinitesimal*:
$x \in$ *HFinite* $-$ *Infinitesimal* $\Longrightarrow h \in$ *Infinitesimal* $\Longrightarrow$ *inverse* $(x + h) -$ *inverse* $x \approx h$
for $x\ h$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *Infinitesimal-square-iff*: $x \in$ *Infinitesimal* $\longleftrightarrow x * x \in$ *Infinitesimal*
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩
declare *Infinitesimal-square-iff* [*symmetric*, *simp*]

lemma *HFinite-square-iff* [*simp*]: $x * x \in$ *HFinite* $\longleftrightarrow x \in$ *HFinite*
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *HInfinite-square-iff* [*simp*]: $x * x \in$ *HInfinite* $\longleftrightarrow x \in$ *HInfinite*
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *approx-HFinite-mult-cancel*: $a \in$ *HFinite* $-$ *Infinitesimal* $\Longrightarrow a * w \approx a * z \Longrightarrow w \approx z$
for $a\ w\ z$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *approx-HFinite-mult-cancel-iff1*: $a \in$ *HFinite* $-$ *Infinitesimal* $\Longrightarrow a * w \approx a * z \longleftrightarrow w \approx z$
for $a\ w\ z$ :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *HInfinite-HFinite-add-cancel*: $x + y \in$ *HInfinite* $\Longrightarrow y \in$ *HFinite* $\Longrightarrow x \in$ *HInfinite*
⟨*proof*⟩

lemma *HInfinite-HFinite-add*: $x \in$ *HInfinite* $\Longrightarrow y \in$ *HFinite* $\Longrightarrow x + y \in$ *HInfinite*
⟨*proof*⟩

lemma *HInfinite-ge-HInfinite*: $x \in$ *HInfinite* $\Longrightarrow x \le y \Longrightarrow 0 \le x \Longrightarrow y \in$ *HInfinite*
for $x\ y$ :: *hypreal*
⟨*proof*⟩

lemma *Infinitesimal-inverse-HInfinite*: $x \in$ *Infinitesimal* $\Longrightarrow x \ne 0 \Longrightarrow$ *inverse* $x \in$ *HInfinite*
for x :: $'a$::*real-normed-div-algebra star*
⟨*proof*⟩

lemma *HInfinite-HFinite-not-Infinitesimal-mult*:
$x \in$ *HInfinite* $\Longrightarrow y \in$ *HFinite* $-$ *Infinitesimal* $\Longrightarrow x * y \in$ *HInfinite*

for $x\ y :: {}'a::real\text{-}normed\text{-}div\text{-}algebra\ star$
⟨*proof*⟩

lemma *HInfinite-HFinite-not-Infinitesimal-mult2*:
$x \in HInfinite \Longrightarrow y \in HFinite - Infinitesimal \Longrightarrow y * x \in HInfinite$
for $x\ y :: {}'a::real\text{-}normed\text{-}div\text{-}algebra\ star$
⟨*proof*⟩

lemma *HInfinite-gt-SReal*: $x \in HInfinite \Longrightarrow 0 < x \Longrightarrow y \in \mathbb{R} \Longrightarrow y < x$
for $x\ y :: hypreal$
⟨*proof*⟩

lemma *HInfinite-gt-zero-gt-one*: $x \in HInfinite \Longrightarrow 0 < x \Longrightarrow 1 < x$
for $x :: hypreal$
⟨*proof*⟩

lemma *not-HInfinite-one* [*simp*]: $1 \notin HInfinite$
⟨*proof*⟩

lemma *approx-hrabs-disj*: $|x| \approx x \lor |x| \approx -x$
for $x :: hypreal$
⟨*proof*⟩

6.4 Theorems about Monads

lemma *monad-hrabs-Un-subset*: $monad\ |x| \le monad\ x \cup monad\ (-\ x)$
for $x :: hypreal$
⟨*proof*⟩

lemma *Infinitesimal-monad-eq*: $e \in Infinitesimal \Longrightarrow monad\ (x + e) = monad\ x$
⟨*proof*⟩

lemma *mem-monad-iff*: $u \in monad\ x \longleftrightarrow -\ u \in monad\ (-\ x)$
⟨*proof*⟩

lemma *Infinitesimal-monad-zero-iff*: $x \in Infinitesimal \longleftrightarrow x \in monad\ 0$
⟨*proof*⟩

lemma *monad-zero-minus-iff*: $x \in monad\ 0 \longleftrightarrow -\ x \in monad\ 0$
⟨*proof*⟩

lemma *monad-zero-hrabs-iff*: $x \in monad\ 0 \longleftrightarrow |x| \in monad\ 0$
for $x :: hypreal$
⟨*proof*⟩

lemma *mem-monad-self* [*simp*]: $x \in monad\ x$
⟨*proof*⟩

6.5 Proof that $x \approx y$ implies $|x| \approx |y|$

lemma *approx-subset-monad*: $x \approx y \Longrightarrow \{x, y\} \leq \mathit{monad}\ x$
⟨*proof*⟩

lemma *approx-subset-monad2*: $x \approx y \Longrightarrow \{x, y\} \leq \mathit{monad}\ y$
⟨*proof*⟩

lemma *mem-monad-approx*: $u \in \mathit{monad}\ x \Longrightarrow x \approx u$
⟨*proof*⟩

lemma *approx-mem-monad*: $x \approx u \Longrightarrow u \in \mathit{monad}\ x$
⟨*proof*⟩

lemma *approx-mem-monad2*: $x \approx u \Longrightarrow x \in \mathit{monad}\ u$
⟨*proof*⟩

lemma *approx-mem-monad-zero*: $x \approx y \Longrightarrow x \in \mathit{monad}\ 0 \Longrightarrow y \in \mathit{monad}\ 0$
⟨*proof*⟩

lemma *Infinitesimal-approx-hrabs*: $x \approx y \Longrightarrow x \in \mathit{Infinitesimal} \Longrightarrow |x| \approx |y|$
for $x\ y :: \mathit{hypreal}$
⟨*proof*⟩

lemma *less-Infinitesimal-less*: $0 < x \Longrightarrow x \notin \mathit{Infinitesimal} \Longrightarrow e \in \mathit{Infinitesimal} \Longrightarrow e < x$
for $x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *Ball-mem-monad-gt-zero*: $0 < x \Longrightarrow x \notin \mathit{Infinitesimal} \Longrightarrow u \in \mathit{monad}\ x \Longrightarrow 0 < u$
for $u\ x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *Ball-mem-monad-less-zero*: $x < 0 \Longrightarrow x \notin \mathit{Infinitesimal} \Longrightarrow u \in \mathit{monad}\ x \Longrightarrow u < 0$
for $u\ x :: \mathit{hypreal}$
⟨*proof*⟩

lemma *lemma-approx-gt-zero*: $0 < x \Longrightarrow x \notin \mathit{Infinitesimal} \Longrightarrow x \approx y \Longrightarrow 0 < y$
for $x\ y :: \mathit{hypreal}$
⟨*proof*⟩

lemma *lemma-approx-less-zero*: $x < 0 \Longrightarrow x \notin \mathit{Infinitesimal} \Longrightarrow x \approx y \Longrightarrow y < 0$
for $x\ y :: \mathit{hypreal}$
⟨*proof*⟩

lemma *approx-hrabs*: $x \approx y \Longrightarrow |x| \approx |y|$
for $x\ y :: \mathit{hypreal}$

$\langle proof \rangle$

lemma *approx-hrabs-zero-cancel*: $|x| \approx 0 \Longrightarrow x \approx 0$
for x :: *hypreal*
$\langle proof \rangle$

lemma *approx-hrabs-add-Infinitesimal*: $e \in$ *Infinitesimal* $\Longrightarrow |x| \approx |x + e|$
for $e\ x$ :: *hypreal*
$\langle proof \rangle$

lemma *approx-hrabs-add-minus-Infinitesimal*: $e \in$ *Infinitesimal* $==> |x| \approx |x + -e|$
for $e\ x$ :: *hypreal*
$\langle proof \rangle$

lemma *hrabs-add-Infinitesimal-cancel*:
$e \in$ *Infinitesimal* $\Longrightarrow e' \in$ *Infinitesimal* $\Longrightarrow |x + e| = |y + e'| \Longrightarrow |x| \approx |y|$
for $e\ e'\ x\ y$ :: *hypreal*
$\langle proof \rangle$

lemma *hrabs-add-minus-Infinitesimal-cancel*:
$e \in$ *Infinitesimal* $\Longrightarrow e' \in$ *Infinitesimal* $\Longrightarrow |x + -e| = |y + -e'| \Longrightarrow |x| \approx |y|$
for $e\ e'\ x\ y$ :: *hypreal*
$\langle proof \rangle$

6.6 More *HFinite* and *Infinitesimal* Theorems

Interesting slightly counterintuitive theorem: necessary for proving that an open interval is an NS open set.

lemma *Infinitesimal-add-hypreal-of-real-less*:
assumes $x < y$ **and** u: $u \in$ *Infinitesimal*
shows *hypreal-of-real* $x + u <$ *hypreal-of-real* y
$\langle proof \rangle$

lemma *Infinitesimal-add-hrabs-hypreal-of-real-less*:
$x \in$ *Infinitesimal* $\Longrightarrow$ |*hypreal-of-real* r| $<$ *hypreal-of-real* $y \Longrightarrow$
|*hypreal-of-real* $r + x$| $<$ *hypreal-of-real* y
$\langle proof \rangle$

lemma *Infinitesimal-add-hrabs-hypreal-of-real-less2*:
$x \in$ *Infinitesimal* $\Longrightarrow$ |*hypreal-of-real* r| $<$ *hypreal-of-real* $y \Longrightarrow$
$|x +$ *hypreal-of-real* r| $<$ *hypreal-of-real* y
$\langle proof \rangle$

lemma *hypreal-of-real-le-add-Infininitesimal-cancel*:
assumes *le*: *hypreal-of-real* $x + u \leq$ *hypreal-of-real* $y + v$
and u: $u \in$ *Infinitesimal* **and** v: $v \in$ *Infinitesimal*
shows *hypreal-of-real* $x \leq$ *hypreal-of-real* y
$\langle proof \rangle$

lemma *hypreal-of-real-le-add-Infininitesimal-cancel2*:
$u \in$ *Infinitesimal* $\Longrightarrow v \in$ *Infinitesimal* $\Longrightarrow$
hypreal-of-real $x + u \le$ *hypreal-of-real* $y + v \Longrightarrow x \le y$
⟨*proof*⟩

lemma *hypreal-of-real-less-Infinitesimal-le-zero*:
hypreal-of-real $x < e \Longrightarrow e \in$ *Infinitesimal* $\Longrightarrow$ *hypreal-of-real* $x \le 0$
⟨*proof*⟩

lemma *Infinitesimal-add-not-zero*: $h \in$ *Infinitesimal* $\Longrightarrow x \neq 0 \Longrightarrow$ *star-of* $x + h \neq 0$
⟨*proof*⟩

lemma *monad-hrabs-less*: $y \in$ *monad* $x \Longrightarrow 0 <$ *hypreal-of-real* $e \Longrightarrow |y - x| <$ *hypreal-of-real* e
⟨*proof*⟩

lemma *mem-monad-SReal-HFinite*: $x \in$ *monad* (*hypreal-of-real* a) $\Longrightarrow x \in$ *HFinite*
⟨*proof*⟩

6.7 Theorems about Standard Part

lemma *st-approx-self*: $x \in$ *HFinite* $\Longrightarrow$ *st* $x \approx x$
⟨*proof*⟩

lemma *st-SReal*: $x \in$ *HFinite* $\Longrightarrow$ *st* $x \in \mathbb{R}$
⟨*proof*⟩

lemma *st-HFinite*: $x \in$ *HFinite* $\Longrightarrow$ *st* $x \in$ *HFinite*
⟨*proof*⟩

lemma *st-unique*: $r \in \mathbb{R} \Longrightarrow r \approx x \Longrightarrow$ *st* $x = r$
⟨*proof*⟩

lemma *st-SReal-eq*: $x \in \mathbb{R} \Longrightarrow$ *st* $x = x$
⟨*proof*⟩

lemma *st-hypreal-of-real* [*simp*]: *st* (*hypreal-of-real* x) = *hypreal-of-real* x
⟨*proof*⟩

lemma *st-eq-approx*: $x \in$ *HFinite* $\Longrightarrow y \in$ *HFinite* $\Longrightarrow$ *st* $x =$ *st* $y \Longrightarrow x \approx y$
⟨*proof*⟩

lemma *approx-st-eq*:
assumes x: $x \in$ *HFinite* **and** y: $y \in$ *HFinite* **and** xy: $x \approx y$
shows *st* $x =$ *st* y
⟨*proof*⟩

lemma *st-eq-approx-iff*: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow x \approx y \longleftrightarrow st\ x = st\ y$
⟨*proof*⟩

lemma *st-Infinitesimal-add-SReal*: $x \in \mathbb{R} \Longrightarrow e \in Infinitesimal \Longrightarrow st\ (x + e) = x$
⟨*proof*⟩

lemma *st-Infinitesimal-add-SReal2*: $x \in \mathbb{R} \Longrightarrow e \in Infinitesimal \Longrightarrow st\ (e + x) = x$
⟨*proof*⟩

lemma *HFinite-st-Infinitesimal-add*: $x \in HFinite \Longrightarrow \exists\, e \in Infinitesimal.\ x = st(x) + e$
⟨*proof*⟩

lemma *st-add*: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow st\ (x + y) = st\ x + st\ y$
⟨*proof*⟩

lemma *st-numeral* [*simp*]: $st\ (numeral\ w) = numeral\ w$
⟨*proof*⟩

lemma *st-neg-numeral* [*simp*]: $st\ (-\ numeral\ w) = -\ numeral\ w$
⟨*proof*⟩

lemma *st-0* [*simp*]: $st\ 0 = 0$
⟨*proof*⟩

lemma *st-1* [*simp*]: $st\ 1 = 1$
⟨*proof*⟩

lemma *st-neg-1* [*simp*]: $st\ (-\ 1) = -\ 1$
⟨*proof*⟩

lemma *st-minus*: $x \in HFinite \Longrightarrow st\ (-\ x) = -\ st\ x$
⟨*proof*⟩

lemma *st-diff*: $[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow st\ (x - y) = st\ x - st\ y$
⟨*proof*⟩

lemma *st-mult*: $[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow st\ (x * y) = st\ x * st\ y$
⟨*proof*⟩

lemma *st-Infinitesimal*: $x \in Infinitesimal \Longrightarrow st\ x = 0$
⟨*proof*⟩

lemma *st-not-Infinitesimal*: $st(x) \neq 0 \Longrightarrow x \notin Infinitesimal$
⟨*proof*⟩

lemma *st-inverse*: $x \in HFinite \Longrightarrow st\ x \neq 0 \Longrightarrow st\ (inverse\ x) = inverse\ (st\ x)$
⟨*proof*⟩

lemma *st-divide* [*simp*]: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow st\ y \neq 0 \Longrightarrow st\ (x\ /\ y) = st\ x\ /\ st\ y$
⟨*proof*⟩

lemma *st-idempotent* [*simp*]: $x \in HFinite \Longrightarrow st\ (st\ x) = st\ x$
⟨*proof*⟩

lemma *Infinitesimal-add-st-less*:
$x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow u \in Infinitesimal \Longrightarrow st\ x < st\ y \Longrightarrow st\ x + u < st\ y$
⟨*proof*⟩

lemma *Infinitesimal-add-st-le-cancel*:
$x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow u \in Infinitesimal \Longrightarrow$
$st\ x \leq st\ y + u \Longrightarrow st\ x \leq st\ y$
⟨*proof*⟩

lemma *st-le*: $x \in HFinite \Longrightarrow y \in HFinite \Longrightarrow x \leq y \Longrightarrow st\ x \leq st\ y$
⟨*proof*⟩

lemma *st-zero-le*: $0 \leq x \Longrightarrow x \in HFinite \Longrightarrow 0 \leq st\ x$
⟨*proof*⟩

lemma *st-zero-ge*: $x \leq 0 \Longrightarrow x \in HFinite \Longrightarrow st\ x \leq 0$
⟨*proof*⟩

lemma *st-hrabs*: $x \in HFinite \Longrightarrow |st\ x| = st\ |x|$
⟨*proof*⟩

6.8 Alternative Definitions using Free Ultrafilter

6.8.1 *HFinite*

lemma *HFinite-FreeUltrafilterNat*:
assumes $star\text{-}n\ X \in HFinite$
shows $\exists u.\ eventually\ (\lambda n.\ norm\ (X\ n) < u)\ \mathcal{U}$
⟨*proof*⟩

lemma *FreeUltrafilterNat-HFinite*:
assumes $eventually\ (\lambda n.\ norm\ (X\ n) < u)\ \mathcal{U}$
shows $star\text{-}n\ X \in HFinite$
⟨*proof*⟩

lemma *HFinite-FreeUltrafilterNat-iff*:
$star\text{-}n\ X \in HFinite \longleftrightarrow (\exists u.\ eventually\ (\lambda n.\ norm\ (X\ n) < u)\ \mathcal{U})$
⟨*proof*⟩

6.8.2 *HInfinite*

Exclude this type of sets from free ultrafilter for Infinite numbers!

lemma *FreeUltrafilterNat-const-Finite*:
$eventually\ (\lambda n.\ norm\ (X\ n) = u)\ \mathcal{U} \Longrightarrow star\text{-}n\ X \in HFinite$
⟨*proof*⟩

lemma *HInfinite-FreeUltrafilterNat*:
$star\text{-}n\ X \in HInfinite \Longrightarrow eventually\ (\lambda n.\ u < norm\ (X\ n))\ \mathcal{U}$
⟨*proof*⟩

lemma *FreeUltrafilterNat-HInfinite*:
assumes $\bigwedge u.\ eventually\ (\lambda n.\ u < norm\ (X\ n))\ \mathcal{U}$
shows $star\text{-}n\ X \in HInfinite$
⟨*proof*⟩

lemma *HInfinite-FreeUltrafilterNat-iff*:
$star\text{-}n\ X \in HInfinite \longleftrightarrow (\forall u.\ eventually\ (\lambda n.\ u < norm\ (X\ n))\ \mathcal{U})$
⟨*proof*⟩

6.8.3 *Infinitesimal*

lemma *ball-SReal-eq*: $(\forall x{::}hypreal \in Reals.\ P\ x) \longleftrightarrow (\forall x{::}real.\ P\ (star\text{-}of\ x))$
⟨*proof*⟩

lemma *Infinitesimal-FreeUltrafilterNat-iff*:
$(star\text{-}n\ X \in Infinitesimal) = (\forall u{>}0.\ eventually\ (\lambda n.\ norm\ (X\ n) < u)\ \mathcal{U})$ (**is** $?lhs = ?rhs$)
⟨*proof*⟩

Infinitesimals as smaller than $1/n$ for all $n{::}nat$ (> 0).

lemma *lemma-Infinitesimal*: $(\forall r.\ 0 < r \longrightarrow x < r) \longleftrightarrow (\forall n.\ x < inverse\ (real\ (Suc\ n)))$
⟨*proof*⟩

lemma *lemma-Infinitesimal2*:
$(\forall r \in Reals.\ 0 < r \longrightarrow x < r) \longleftrightarrow (\forall n.\ x < inverse(hypreal\text{-}of\text{-}nat\ (Suc\ n)))$
⟨*proof*⟩

lemma *Infinitesimal-hypreal-of-nat-iff*:
$Infinitesimal = \{x.\ \forall n.\ hnorm\ x < inverse\ (hypreal\text{-}of\text{-}nat\ (Suc\ n))\}$
⟨*proof*⟩

6.9 Proof that ω is an infinite number

It will follow that ε is an infinitesimal number.

lemma *Suc-Un-eq*: $\{n.\ n < Suc\ m\} = \{n.\ n < m\}\ Un\ \{n.\ n = m\}$

⟨*proof*⟩

Prove that any segment is finite and hence cannot belong to $\mathcal{U}$.

lemma *finite-real-of-nat-segment*: $finite\ \{n::nat.\ real\ n < real\ (m::nat)\}$
⟨*proof*⟩

lemma *finite-real-of-nat-less-real*: $finite\ \{n::nat.\ real\ n < u\}$
⟨*proof*⟩

lemma *finite-real-of-nat-le-real*: $finite\ \{n::nat.\ real\ n \le u\}$
⟨*proof*⟩

lemma *finite-rabs-real-of-nat-le-real*: $finite\ \{n::nat.\ |real\ n| \le u\}$
⟨*proof*⟩

lemma *rabs-real-of-nat-le-real-FreeUltrafilterNat*:
$\neg\ eventually\ (\lambda n.\ |real\ n| \le u)\ \mathcal{U}$
⟨*proof*⟩

lemma *FreeUltrafilterNat-nat-gt-real*: $eventually\ (\lambda n.\ u < real\ n)\ \mathcal{U}$
⟨*proof*⟩

The complement of $\{n.\ |real\ n| \le u\} = \{n.\ u < |real\ n|\}$ is in $\mathcal{U}$ by property of (free) ultrafilters.

ω is a member of *HInfinite*.

theorem *HInfinite-omega* [*simp*]: $\omega \in HInfinite$
⟨*proof*⟩

Epsilon is a member of Infinitesimal.

lemma *Infinitesimal-epsilon* [*simp*]: $\varepsilon \in Infinitesimal$
⟨*proof*⟩

lemma *HFinite-epsilon* [*simp*]: $\varepsilon \in HFinite$
⟨*proof*⟩

lemma *epsilon-approx-zero* [*simp*]: $\varepsilon \approx 0$
⟨*proof*⟩

Needed for proof that we define a hyperreal $[<X(n)] \approx hypreal\text{-}of\text{-}real\ a$ given that $\forall\, n.\ |X\ n - a| < 1/n$. Used in proof of $NSLIM \Rightarrow LIM$.

lemma *real-of-nat-less-inverse-iff*: $0 < u \Longrightarrow u < inverse\ (real(Suc\ n)) \longleftrightarrow real(Suc\ n) < inverse\ u$
⟨*proof*⟩

lemma *finite-inverse-real-of-posnat-gt-real*: $0 < u \Longrightarrow finite\ \{n.\ u < inverse\ (real\ (Suc\ n))\}$
⟨*proof*⟩

lemma *finite-inverse-real-of-posnat-ge-real*:
  assumes $0 < u$
  shows *finite* $\{n.\ u \leq inverse\ (real\ (Suc\ n))\}$
⟨*proof*⟩

lemma *inverse-real-of-posnat-ge-real-FreeUltrafilterNat*:
  $0 < u \Longrightarrow \neg\ eventually\ (\lambda n.\ u \leq inverse(real(Suc\ n)))\ \mathcal{U}$
  ⟨*proof*⟩

lemma *FreeUltrafilterNat-inverse-real-of-posnat*:
  $0 < u \Longrightarrow eventually\ (\lambda n.\ inverse(real(Suc\ n)) < u)\ \mathcal{U}$
  ⟨*proof*⟩

Example of an hypersequence (i.e. an extended standard sequence) whose term with an hypernatural suffix is an infinitesimal i.e. the whn'nth term of the hypersequence is a member of Infinitesimal

lemma *SEQ-Infinitesimal*: $(\ {*}f{*}\ (\lambda n{::}nat.\ inverse(real(Suc\ n))))\ whn \in Infinitesimal$
  ⟨*proof*⟩

Example where we get a hyperreal from a real sequence for which a particular property holds. The theorem is used in proofs about equivalence of nonstandard and standard neighbourhoods. Also used for equivalence of nonstandard ans standard definitions of pointwise limit.

$|X(n) - x| < 1/n \Longrightarrow [<X\ n>] - hypreal\text{-}of\text{-}real\ x| \in Infinitesimal$

lemma *real-seq-to-hypreal-Infinitesimal*:
  $\forall n.\ norm\ (X\ n - x) < inverse\ (real\ (Suc\ n)) \Longrightarrow star\text{-}n\ X - star\text{-}of\ x \in Infinitesimal$
  ⟨*proof*⟩

lemma *real-seq-to-hypreal-approx*:
  $\forall n.\ norm\ (X\ n - x) < inverse\ (real\ (Suc\ n)) \Longrightarrow star\text{-}n\ X \approx star\text{-}of\ x$
  ⟨*proof*⟩

lemma *real-seq-to-hypreal-approx2*:
  $\forall n.\ norm\ (x - X\ n) < inverse(real(Suc\ n)) \Longrightarrow star\text{-}n\ X \approx star\text{-}of\ x$
  ⟨*proof*⟩

lemma *real-seq-to-hypreal-Infinitesimal2*:
  $\forall n.\ norm(X\ n - Y\ n) < inverse(real(Suc\ n)) \Longrightarrow star\text{-}n\ X - star\text{-}n\ Y \in Infinitesimal$
  ⟨*proof*⟩

end

7 Nonstandard Complex Numbers

theory *NSComplex*
imports *NSA*
begin

type-synonym *hcomplex* = *complex star*

abbreviation *hcomplex-of-complex* :: *complex* ⇒ *complex star*
where *hcomplex-of-complex* ≡ *star-of*

abbreviation *hcmod* :: *complex star* ⇒ *real star*
where *hcmod* ≡ *hnorm*

7.0.1 Real and Imaginary parts

definition *hRe* :: *hcomplex* ⇒ *hypreal*
where *hRe* = ∗*f*∗ *Re*

definition *hIm* :: *hcomplex* ⇒ *hypreal*
where *hIm* = ∗*f*∗ *Im*

7.0.2 Imaginary unit

definition *iii* :: *hcomplex*
where *iii* = *star-of* i

7.0.3 Complex conjugate

definition *hcnj* :: *hcomplex* ⇒ *hcomplex*
where *hcnj* = ∗*f*∗ *cnj*

7.0.4 Argand

definition *hsgn* :: *hcomplex* ⇒ *hcomplex*
where *hsgn* = ∗*f*∗ *sgn*

definition *harg* :: *hcomplex* ⇒ *hypreal*
where *harg* = ∗*f*∗ *Arg*

definition — abbreviation for *cos a* + *i sin a*
hcis :: *hypreal* ⇒ *hcomplex*
where *hcis* = ∗*f*∗ *cis*

7.0.5 Injection from hyperreals

abbreviation *hcomplex-of-hypreal* :: *hypreal* ⇒ *hcomplex*
where *hcomplex-of-hypreal* ≡ *of-hypreal*

definition — abbreviation for *r* ∗ (*cos a* + *i sin a*)

hrcis :: *hypreal* ⇒ *hypreal* ⇒ *hcomplex*
where *hrcis* = *∗f2∗ rcis*

7.0.6 $e \widehat{\ } (x + iy)$

definition *hExp* :: *hcomplex* ⇒ *hcomplex*
where *hExp* = *∗f∗ exp*

definition *HComplex* :: *hypreal* ⇒ *hypreal* ⇒ *hcomplex*
where *HComplex* = *∗f2∗ Complex*

lemmas *hcomplex-defs* [*transfer-unfold*] =
hRe-def hIm-def iii-def hcnj-def hsgn-def harg-def hcis-def
hrcis-def hExp-def HComplex-def

lemma *Standard-hRe* [*simp*]: *x* ∈ *Standard* ⟹ *hRe x* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-hIm* [*simp*]: *x* ∈ *Standard* ⟹ *hIm x* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-iii* [*simp*]: *iii* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-hcnj* [*simp*]: *x* ∈ *Standard* ⟹ *hcnj x* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-hsgn* [*simp*]: *x* ∈ *Standard* ⟹ *hsgn x* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-harg* [*simp*]: *x* ∈ *Standard* ⟹ *harg x* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-hcis* [*simp*]: *r* ∈ *Standard* ⟹ *hcis r* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-hExp* [*simp*]: *x* ∈ *Standard* ⟹ *hExp x* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-hrcis* [*simp*]: *r* ∈ *Standard* ⟹ *s* ∈ *Standard* ⟹ *hrcis r s* ∈ *Standard*
⟨*proof*⟩

lemma *Standard-HComplex* [*simp*]: *r* ∈ *Standard* ⟹ *s* ∈ *Standard* ⟹ *HComplex r s* ∈ *Standard*
⟨*proof*⟩

lemma *hcmod-def*: *hcmod* = *∗f∗ cmod*
⟨*proof*⟩

7.1 Properties of Nonstandard Real and Imaginary Parts

lemma *hcomplex-hRe-hIm-cancel-iff*: $\bigwedge w\ z.\ w = z \longleftrightarrow hRe\ w = hRe\ z \wedge hIm\ w = hIm\ z$
⟨*proof*⟩

lemma *hcomplex-equality* [*intro?*]: $\bigwedge z\ w.\ hRe\ z = hRe\ w \Longrightarrow hIm\ z = hIm\ w \Longrightarrow z = w$
⟨*proof*⟩

lemma *hcomplex-hRe-zero* [*simp*]: $hRe\ 0 = 0$
⟨*proof*⟩

lemma *hcomplex-hIm-zero* [*simp*]: $hIm\ 0 = 0$
⟨*proof*⟩

lemma *hcomplex-hRe-one* [*simp*]: $hRe\ 1 = 1$
⟨*proof*⟩

lemma *hcomplex-hIm-one* [*simp*]: $hIm\ 1 = 0$
⟨*proof*⟩

7.2 Addition for Nonstandard Complex Numbers

lemma *hRe-add*: $\bigwedge x\ y.\ hRe\ (x + y) = hRe\ x + hRe\ y$
⟨*proof*⟩

lemma *hIm-add*: $\bigwedge x\ y.\ hIm\ (x + y) = hIm\ x + hIm\ y$
⟨*proof*⟩

7.3 More Minus Laws

lemma *hRe-minus*: $\bigwedge z.\ hRe\ (-\ z) = -\ hRe\ z$
⟨*proof*⟩

lemma *hIm-minus*: $\bigwedge z.\ hIm\ (-\ z) = -\ hIm\ z$
⟨*proof*⟩

lemma *hcomplex-add-minus-eq-minus*: $x + y = 0 \Longrightarrow x = -\ y$
for $x\ y$ :: *hcomplex*
⟨*proof*⟩

lemma *hcomplex-i-mult-eq* [*simp*]: $iii * iii = -\ 1$
⟨*proof*⟩

lemma *hcomplex-i-mult-left* [*simp*]: $\bigwedge z.\ iii * (iii * z) = -\ z$
⟨*proof*⟩

lemma *hcomplex-i-not-zero* [*simp*]: $iii \neq 0$
⟨*proof*⟩

7.4 More Multiplication Laws

lemma *hcomplex-mult-minus-one*: $- 1 * z = - z$
  for z :: *hcomplex*
  ⟨*proof*⟩

lemma *hcomplex-mult-minus-one-right*: $z * - 1 = - z$
  for z :: *hcomplex*
  ⟨*proof*⟩

lemma *hcomplex-mult-left-cancel*: $c \neq 0 \Longrightarrow c * a = c * b \longleftrightarrow a = b$
  for $a\ b\ c$ :: *hcomplex*
  ⟨*proof*⟩

lemma *hcomplex-mult-right-cancel*: $c \neq 0 \Longrightarrow a * c = b * c \longleftrightarrow a = b$
  for $a\ b\ c$ :: *hcomplex*
  ⟨*proof*⟩

7.5 Subtraction and Division

lemma *hcomplex-diff-eq-eq* [*simp*]: $x - y = z \longleftrightarrow x = z + y$
  for $x\ y\ z$ :: *hcomplex*
  ⟨*proof*⟩

7.6 Embedding Properties for *hcomplex-of-hypreal* Map

lemma *hRe-hcomplex-of-hypreal* [*simp*]: $\bigwedge z.$ *hRe* (*hcomplex-of-hypreal* z) = z
  ⟨*proof*⟩

lemma *hIm-hcomplex-of-hypreal* [*simp*]: $\bigwedge z.$ *hIm* (*hcomplex-of-hypreal* z) = 0
  ⟨*proof*⟩

lemma *hcomplex-of-epsilon-not-zero* [*simp*]: *hcomplex-of-hypreal* $\varepsilon \neq 0$
  ⟨*proof*⟩

7.7 *HComplex* theorems

lemma *hRe-HComplex* [*simp*]: $\bigwedge x\ y.$ *hRe* (*HComplex* $x\ y$) = x
  ⟨*proof*⟩

lemma *hIm-HComplex* [*simp*]: $\bigwedge x\ y.$ *hIm* (*HComplex* $x\ y$) = y
  ⟨*proof*⟩

lemma *hcomplex-surj* [*simp*]: $\bigwedge z.$ *HComplex* (*hRe* z) (*hIm* z) = z
  ⟨*proof*⟩

lemma *hcomplex-induct* [*case-names rect*]:
  ($\bigwedge x\ y.$ P (*HComplex* $x\ y$)) $\Longrightarrow P\ z$
  ⟨*proof*⟩

7.8 Modulus (Absolute Value) of Nonstandard Complex Number

lemma *hcomplex-of-hypreal-abs*:
$hcomplex\text{-}of\text{-}hypreal\ |x| = hcomplex\text{-}of\text{-}hypreal\ (hcmod\ (hcomplex\text{-}of\text{-}hypreal\ x))$
⟨*proof*⟩

lemma *HComplex-inject* [*simp*]: $\bigwedge x\ y\ x'\ y'.\ HComplex\ x\ y = HComplex\ x'\ y' \longleftrightarrow x = x' \wedge y = y'$
⟨*proof*⟩

lemma *HComplex-add* [*simp*]:
$\bigwedge x1\ y1\ x2\ y2.\ HComplex\ x1\ y1 + HComplex\ x2\ y2 = HComplex\ (x1 + x2)\ (y1 + y2)$
⟨*proof*⟩

lemma *HComplex-minus* [*simp*]: $\bigwedge x\ y.\ -\ HComplex\ x\ y = HComplex\ (-\ x)\ (-\ y)$
⟨*proof*⟩

lemma *HComplex-diff* [*simp*]:
$\bigwedge x1\ y1\ x2\ y2.\ HComplex\ x1\ y1 - HComplex\ x2\ y2 = HComplex\ (x1 - x2)\ (y1 - y2)$
⟨*proof*⟩

lemma *HComplex-mult* [*simp*]:
$\bigwedge x1\ y1\ x2\ y2.\ HComplex\ x1\ y1 * HComplex\ x2\ y2 = HComplex\ (x1*x2 - y1*y2)\ (x1*y2 + y1*x2)$
⟨*proof*⟩

HComplex-inverse is proved below.

lemma *hcomplex-of-hypreal-eq*: $\bigwedge r.\ hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ r\ 0$
⟨*proof*⟩

lemma *HComplex-add-hcomplex-of-hypreal* [*simp*]:
$\bigwedge x\ y\ r.\ HComplex\ x\ y + hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ (x + r)\ y$
⟨*proof*⟩

lemma *hcomplex-of-hypreal-add-HComplex* [*simp*]:
$\bigwedge r\ x\ y.\ hcomplex\text{-}of\text{-}hypreal\ r + HComplex\ x\ y = HComplex\ (r + x)\ y$
⟨*proof*⟩

lemma *HComplex-mult-hcomplex-of-hypreal*:
$\bigwedge x\ y\ r.\ HComplex\ x\ y * hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ (x * r)\ (y * r)$
⟨*proof*⟩

lemma *hcomplex-of-hypreal-mult-HComplex*:
$\bigwedge r\ x\ y.\ hcomplex\text{-}of\text{-}hypreal\ r * HComplex\ x\ y = HComplex\ (r * x)\ (r * y)$
⟨*proof*⟩

lemma *i-hcomplex-of-hypreal* [*simp*]: $\bigwedge r.\ iii * hcomplex\text{-}of\text{-}hypreal\ r = HComplex\ 0\ r$
⟨*proof*⟩

lemma *hcomplex-of-hypreal-i* [*simp*]: $\bigwedge r.\ hcomplex\text{-}of\text{-}hypreal\ r * iii = HComplex\ 0\ r$
⟨*proof*⟩

7.9 Conjugation

lemma *hcomplex-hcnj-cancel-iff* [*iff*]: $\bigwedge x\ y.\ hcnj\ x = hcnj\ y \longleftrightarrow x = y$
⟨*proof*⟩

lemma *hcomplex-hcnj-hcnj* [*simp*]: $\bigwedge z.\ hcnj\ (hcnj\ z) = z$
⟨*proof*⟩

lemma *hcomplex-hcnj-hcomplex-of-hypreal* [*simp*]:
$\bigwedge x.\ hcnj\ (hcomplex\text{-}of\text{-}hypreal\ x) = hcomplex\text{-}of\text{-}hypreal\ x$
⟨*proof*⟩

lemma *hcomplex-hmod-hcnj* [*simp*]: $\bigwedge z.\ hcmod\ (hcnj\ z) = hcmod\ z$
⟨*proof*⟩

lemma *hcomplex-hcnj-minus*: $\bigwedge z.\ hcnj\ (-\ z) = -\ hcnj\ z$
⟨*proof*⟩

lemma *hcomplex-hcnj-inverse*: $\bigwedge z.\ hcnj\ (inverse\ z) = inverse\ (hcnj\ z)$
⟨*proof*⟩

lemma *hcomplex-hcnj-add*: $\bigwedge w\ z.\ hcnj\ (w + z) = hcnj\ w + hcnj\ z$
⟨*proof*⟩

lemma *hcomplex-hcnj-diff*: $\bigwedge w\ z.\ hcnj\ (w - z) = hcnj\ w - hcnj\ z$
⟨*proof*⟩

lemma *hcomplex-hcnj-mult*: $\bigwedge w\ z.\ hcnj\ (w * z) = hcnj\ w * hcnj\ z$
⟨*proof*⟩

lemma *hcomplex-hcnj-divide*: $\bigwedge w\ z.\ hcnj\ (w\ /\ z) = hcnj\ w\ /\ hcnj\ z$
⟨*proof*⟩

lemma *hcnj-one* [*simp*]: $hcnj\ 1 = 1$
⟨*proof*⟩

lemma *hcomplex-hcnj-zero* [*simp*]: $hcnj\ 0 = 0$
⟨*proof*⟩

lemma *hcomplex-hcnj-zero-iff* [*iff*]: $\bigwedge z.\ hcnj\ z = 0 \longleftrightarrow z = 0$
⟨*proof*⟩

lemma *hcomplex-mult-hcnj*: $\bigwedge z.\ z * hcnj\ z = hcomplex\text{-}of\text{-}hypreal\ ((hRe\ z)^2 + (hIm\ z)^2)$
⟨*proof*⟩

7.10 More Theorems about the Function *hcmod*

lemma *hcmod-hcomplex-of-hypreal-of-nat* [*simp*]:
$hcmod\ (hcomplex\text{-}of\text{-}hypreal\ (hypreal\text{-}of\text{-}nat\ n)) = hypreal\text{-}of\text{-}nat\ n$
⟨*proof*⟩

lemma *hcmod-hcomplex-of-hypreal-of-hypnat* [*simp*]:
$hcmod\ (hcomplex\text{-}of\text{-}hypreal(hypreal\text{-}of\text{-}hypnat\ n)) = hypreal\text{-}of\text{-}hypnat\ n$
⟨*proof*⟩

lemma *hcmod-mult-hcnj*: $\bigwedge z.\ hcmod\ (z * hcnj\ z) = (hcmod\ z)^2$
⟨*proof*⟩

lemma *hcmod-triangle-ineq2* [*simp*]: $\bigwedge a\ b.\ hcmod\ (b + a) - hcmod\ b \leq hcmod\ a$
⟨*proof*⟩

lemma *hcmod-diff-ineq* [*simp*]: $\bigwedge a\ b.\ hcmod\ a - hcmod\ b \leq hcmod\ (a + b)$
⟨*proof*⟩

7.11 Exponentiation

lemma *hcomplexpow-0* [*simp*]: $z\ \hat{}\ 0 = 1$
for $z :: hcomplex$
⟨*proof*⟩

lemma *hcomplexpow-Suc* [*simp*]: $z\ \hat{}\ (Suc\ n) = z * (z\ \hat{}\ n)$
for $z :: hcomplex$
⟨*proof*⟩

lemma *hcomplexpow-i-squared* [*simp*]: $iii^2 = -1$
⟨*proof*⟩

lemma *hcomplex-of-hypreal-pow*: $\bigwedge x.\ hcomplex\text{-}of\text{-}hypreal\ (x\ \hat{}\ n) = hcomplex\text{-}of\text{-}hypreal\ x\ \hat{}\ n$
⟨*proof*⟩

lemma *hcomplex-hcnj-pow*: $\bigwedge z.\ hcnj\ (z\ \hat{}\ n) = hcnj\ z\ \hat{}\ n$
⟨*proof*⟩

lemma *hcmod-hcomplexpow*: $\bigwedge x.\ hcmod\ (x\ \hat{}\ n) = hcmod\ x\ \hat{}\ n$
⟨*proof*⟩

lemma *hcpow-minus*:
$\bigwedge x\ n.\ (-\ x :: hcomplex)\ pow\ n = (if\ (\ *p*\ even)\ n\ then\ (x\ pow\ n)\ else - (x\ pow\ n))$

⟨*proof*⟩

lemma *hcpow-mult*: $(r * s)$ *pow* $n = (r$ *pow* $n) * (s$ *pow* $n)$
for $r\ s$ :: *hcomplex*
⟨*proof*⟩

lemma *hcpow-zero2* [*simp*]: $\bigwedge n.\ 0$ *pow* (*hSuc* n) = (0::′*a*::*semiring-1 star*)
⟨*proof*⟩

lemma *hcpow-not-zero* [*simp,intro*]: $\bigwedge r\ n.\ r \neq 0 \Longrightarrow r$ *pow* $n \neq (0$::*hcomplex*)
⟨*proof*⟩

lemma *hcpow-zero-zero*: r *pow* $n = 0 \Longrightarrow r = 0$
for r :: *hcomplex*
⟨*proof*⟩

7.12 The Function *hsgn*

lemma *hsgn-zero* [*simp*]: *hsgn* $0 = 0$
⟨*proof*⟩

lemma *hsgn-one* [*simp*]: *hsgn* $1 = 1$
⟨*proof*⟩

lemma *hsgn-minus*: $\bigwedge z.$ *hsgn* $(- z) = -$ *hsgn* z
⟨*proof*⟩

lemma *hsgn-eq*: $\bigwedge z.$ *hsgn* $z = z$ / *hcomplex-of-hypreal* (*hcmod* z)
⟨*proof*⟩

lemma *hcmod-i*: $\bigwedge x\ y.$ *hcmod* (*HComplex* $x\ y$) = ($*f*$ *sqrt*) $(x^2 + y^2)$
⟨*proof*⟩

lemma *hcomplex-eq-cancel-iff1* [*simp*]:
hcomplex-of-hypreal xa = *HComplex* $x\ y \longleftrightarrow xa = x \wedge y = 0$
⟨*proof*⟩

lemma *hcomplex-eq-cancel-iff2* [*simp*]:
HComplex $x\ y$ = *hcomplex-of-hypreal* $xa \longleftrightarrow x = xa \wedge y = 0$
⟨*proof*⟩

lemma *HComplex-eq-0* [*simp*]: $\bigwedge x\ y.$ *HComplex* $x\ y = 0 \longleftrightarrow x = 0 \wedge y = 0$
⟨*proof*⟩

lemma *HComplex-eq-1* [*simp*]: $\bigwedge x\ y.$ *HComplex* $x\ y = 1 \longleftrightarrow x = 1 \wedge y = 0$
⟨*proof*⟩

lemma *i-eq-HComplex-0-1*: *iii* = *HComplex* $0\ 1$
⟨*proof*⟩

lemma *HComplex-eq-i* [*simp*]: $\bigwedge x\ y.\ HComplex\ x\ y = iii \longleftrightarrow x = 0 \wedge y = 1$
⟨*proof*⟩

lemma *hRe-hsgn* [*simp*]: $\bigwedge z.\ hRe\ (hsgn\ z) = hRe\ z\ /\ hcmod\ z$
⟨*proof*⟩

lemma *hIm-hsgn* [*simp*]: $\bigwedge z.\ hIm\ (hsgn\ z) = hIm\ z\ /\ hcmod\ z$
⟨*proof*⟩

lemma *HComplex-inverse*: $\bigwedge x\ y.\ inverse\ (HComplex\ x\ y) = HComplex\ (x\ /\ (x^2 + y^2))\ (-\ y\ /\ (x^2 + y^2))$
⟨*proof*⟩

lemma *hRe-mult-i-eq*[*simp*]: $\bigwedge y.\ hRe\ (iii * hcomplex\text{-}of\text{-}hypreal\ y) = 0$
⟨*proof*⟩

lemma *hIm-mult-i-eq* [*simp*]: $\bigwedge y.\ hIm\ (iii * hcomplex\text{-}of\text{-}hypreal\ y) = y$
⟨*proof*⟩

lemma *hcmod-mult-i* [*simp*]: $\bigwedge y.\ hcmod\ (iii * hcomplex\text{-}of\text{-}hypreal\ y) = |y|$
⟨*proof*⟩

lemma *hcmod-mult-i2* [*simp*]: $\bigwedge y.\ hcmod\ (hcomplex\text{-}of\text{-}hypreal\ y * iii) = |y|$
⟨*proof*⟩

7.12.1 *harg*

lemma *cos-harg-i-mult-zero* [*simp*]: $\bigwedge y.\ y \neq 0 \Longrightarrow (\ *f*\ cos)\ (harg\ (HComplex\ 0\ y)) = 0$
⟨*proof*⟩

7.13 Polar Form for Nonstandard Complex Numbers

lemma *complex-split-polar2*: $\forall\, n.\ \exists\, r\ a.\ (z\ n) = complex\text{-}of\text{-}real\ r * Complex\ (cos\ a)\ (sin\ a)$
⟨*proof*⟩

lemma *hcomplex-split-polar*:
$\bigwedge z.\ \exists\, r\ a.\ z = hcomplex\text{-}of\text{-}hypreal\ r * (HComplex\ ((\ *f*\ cos)\ a)\ ((\ *f*\ sin)\ a))$
⟨*proof*⟩

lemma *hcis-eq*:
$\bigwedge a.\ hcis\ a = hcomplex\text{-}of\text{-}hypreal\ ((\ *f*\ cos)\ a) + iii * hcomplex\text{-}of\text{-}hypreal\ ((\ *f*\ sin)\ a)$
⟨*proof*⟩

lemma *hrcis-Ex*: $\bigwedge z.\ \exists\, r\ a.\ z = hrcis\ r\ a$
⟨*proof*⟩

lemma *hRe-hcomplex-polar* [*simp*]:
$\bigwedge r\ a.\ hRe\ (hcomplex\text{-}of\text{-}hypreal\ r * HComplex\ ((\ *f*\ cos)\ a)\ ((\ *f*\ sin)\ a)) = r * (\ *f*\ cos)\ a$
⟨*proof*⟩

lemma *hRe-hrcis* [*simp*]: $\bigwedge r\ a.\ hRe\ (hrcis\ r\ a) = r * (\ *f*\ cos)\ a$
⟨*proof*⟩

lemma *hIm-hcomplex-polar* [*simp*]:
$\bigwedge r\ a.\ hIm\ (hcomplex\text{-}of\text{-}hypreal\ r * HComplex\ ((\ *f*\ cos)\ a)\ ((\ *f*\ sin)\ a)) = r * (\ *f*\ sin)\ a$
⟨*proof*⟩

lemma *hIm-hrcis* [*simp*]: $\bigwedge r\ a.\ hIm\ (hrcis\ r\ a) = r * (\ *f*\ sin)\ a$
⟨*proof*⟩

lemma *hcmod-unit-one* [*simp*]: $\bigwedge a.\ hcmod\ (HComplex\ ((\ *f*\ cos)\ a)\ ((\ *f*\ sin)\ a)) = 1$
⟨*proof*⟩

lemma *hcmod-complex-polar* [*simp*]:
$\bigwedge r\ a.\ hcmod\ (hcomplex\text{-}of\text{-}hypreal\ r * HComplex\ ((\ *f*\ cos)\ a)\ ((\ *f*\ sin)\ a)) = |r|$
⟨*proof*⟩

lemma *hcmod-hrcis* [*simp*]: $\bigwedge r\ a.\ hcmod(hrcis\ r\ a) = |r|$
⟨*proof*⟩

$(r1 * hrcis\ a) * (r2 * hrcis\ b) = r1 * r2 * hrcis\ (a + b)$

lemma *hcis-hrcis-eq*: $\bigwedge a.\ hcis\ a = hrcis\ 1\ a$
⟨*proof*⟩
declare *hcis-hrcis-eq* [*symmetric*, *simp*]

lemma *hrcis-mult*: $\bigwedge a\ b\ r1\ r2.\ hrcis\ r1\ a * hrcis\ r2\ b = hrcis\ (r1 * r2)\ (a + b)$
⟨*proof*⟩

lemma *hcis-mult*: $\bigwedge a\ b.\ hcis\ a * hcis\ b = hcis\ (a + b)$
⟨*proof*⟩

lemma *hcis-zero* [*simp*]: $hcis\ 0 = 1$
⟨*proof*⟩

lemma *hrcis-zero-mod* [*simp*]: $\bigwedge a.\ hrcis\ 0\ a = 0$
⟨*proof*⟩

lemma *hrcis-zero-arg* [*simp*]: $\bigwedge r.\ hrcis\ r\ 0 = hcomplex\text{-}of\text{-}hypreal\ r$
⟨*proof*⟩

lemma *hcomplex-i-mult-minus* [*simp*]: $\bigwedge x.\ iii * (iii * x) = -\ x$

⟨*proof*⟩

lemma *hcomplex-i-mult-minus2* [*simp*]: $iii * iii * x = - x$
⟨*proof*⟩

lemma *hcis-hypreal-of-nat-Suc-mult*:
$\bigwedge a.\ hcis\ (hypreal\text{-}of\text{-}nat\ (Suc\ n) * a) = hcis\ a * hcis\ (hypreal\text{-}of\text{-}nat\ n * a)$
⟨*proof*⟩

lemma *NSDeMoivre*: $\bigwedge a.\ (hcis\ a)\ \widehat{}\ n = hcis\ (hypreal\text{-}of\text{-}nat\ n * a)$
⟨*proof*⟩

lemma *hcis-hypreal-of-hypnat-Suc-mult*:
$\bigwedge a\ n.\ hcis\ (hypreal\text{-}of\text{-}hypnat\ (n + 1) * a) = hcis\ a * hcis\ (hypreal\text{-}of\text{-}hypnat\ n * a)$
⟨*proof*⟩

lemma *NSDeMoivre-ext*: $\bigwedge a\ n.\ (hcis\ a)\ pow\ n = hcis\ (hypreal\text{-}of\text{-}hypnat\ n * a)$
⟨*proof*⟩

lemma *NSDeMoivre2*: $\bigwedge a\ r.\ (hrcis\ r\ a)\ \widehat{}\ n = hrcis\ (r\ \widehat{}\ n)\ (hypreal\text{-}of\text{-}nat\ n * a)$
⟨*proof*⟩

lemma *DeMoivre2-ext*: $\bigwedge a\ r\ n.\ (hrcis\ r\ a)\ pow\ n = hrcis\ (r\ pow\ n)\ (hypreal\text{-}of\text{-}hypnat\ n * a)$
⟨*proof*⟩

lemma *hcis-inverse* [*simp*]: $\bigwedge a.\ inverse\ (hcis\ a) = hcis\ (- a)$
⟨*proof*⟩

lemma *hrcis-inverse*: $\bigwedge a\ r.\ inverse\ (hrcis\ r\ a) = hrcis\ (inverse\ r)\ (- a)$
⟨*proof*⟩

lemma *hRe-hcis* [*simp*]: $\bigwedge a.\ hRe\ (hcis\ a) = (\ *f*\ cos)\ a$
⟨*proof*⟩

lemma *hIm-hcis* [*simp*]: $\bigwedge a.\ hIm\ (hcis\ a) = (\ *f*\ sin)\ a$
⟨*proof*⟩

lemma *cos-n-hRe-hcis-pow-n*: $(\ *f*\ cos)\ (hypreal\text{-}of\text{-}nat\ n * a) = hRe\ (hcis\ a\ \widehat{}\ n)$
⟨*proof*⟩

lemma *sin-n-hIm-hcis-pow-n*: $(\ *f*\ sin)\ (hypreal\text{-}of\text{-}nat\ n * a) = hIm\ (hcis\ a\ \widehat{}\ n)$
⟨*proof*⟩

lemma *cos-n-hRe-hcis-hcpow-n*: $(\ *f*\ cos)\ (hypreal\text{-}of\text{-}hypnat\ n * a) = hRe\ (hcis\ a\ pow\ n)$
⟨*proof*⟩

lemma *sin-n-hIm-hcis-hcpow-n*: $(*f* \ sin) \ (hypreal\text{-}of\text{-}hypnat \ n * a) = hIm \ (hcis \ a \ pow \ n)$
⟨*proof*⟩

lemma *hExp-add*: $\bigwedge a \ b. \ hExp \ (a + b) = hExp \ a * hExp \ b$
⟨*proof*⟩

7.14 *hcomplex-of-complex*: the Injection from type *complex* to to *hcomplex*

lemma *hcomplex-of-complex-i*: $iii = hcomplex\text{-}of\text{-}complex \ \mathrm{i}$
⟨*proof*⟩

lemma *hRe-hcomplex-of-complex*: $hRe \ (hcomplex\text{-}of\text{-}complex \ z) = hypreal\text{-}of\text{-}real \ (Re \ z)$
⟨*proof*⟩

lemma *hIm-hcomplex-of-complex*: $hIm \ (hcomplex\text{-}of\text{-}complex \ z) = hypreal\text{-}of\text{-}real \ (Im \ z)$
⟨*proof*⟩

lemma *hcmod-hcomplex-of-complex*: $hcmod \ (hcomplex\text{-}of\text{-}complex \ x) = hypreal\text{-}of\text{-}real \ (cmod \ x)$
⟨*proof*⟩

7.15 Numerals and Arithmetic

lemma *hcomplex-of-hypreal-eq-hcomplex-of-complex*:
$hcomplex\text{-}of\text{-}hypreal \ (hypreal\text{-}of\text{-}real \ x) = hcomplex\text{-}of\text{-}complex \ (complex\text{-}of\text{-}real \ x)$
⟨*proof*⟩

lemma *hcomplex-hypreal-numeral*:
$hcomplex\text{-}of\text{-}complex \ (numeral \ w) = hcomplex\text{-}of\text{-}hypreal(numeral \ w)$
⟨*proof*⟩

lemma *hcomplex-hypreal-neg-numeral*:
$hcomplex\text{-}of\text{-}complex \ (- \ numeral \ w) = hcomplex\text{-}of\text{-}hypreal(- \ numeral \ w)$
⟨*proof*⟩

lemma *hcomplex-numeral-hcnj* [*simp*]: $hcnj \ (numeral \ v :: hcomplex) = numeral \ v$
⟨*proof*⟩

lemma *hcomplex-numeral-hcmod* [*simp*]: $hcmod \ (numeral \ v :: hcomplex) = (numeral \ v :: hypreal)$
⟨*proof*⟩

lemma *hcomplex-neg-numeral-hcmod* [*simp*]: $hcmod \ (- \ numeral \ v :: hcomplex) = (numeral \ v :: hypreal)$
⟨*proof*⟩

lemma *hcomplex-numeral-hRe* [*simp*]: *hRe* (*numeral v* :: *hcomplex*) = *numeral v*
⟨*proof*⟩

lemma *hcomplex-numeral-hIm* [*simp*]: *hIm* (*numeral v* :: *hcomplex*) = *0*
⟨*proof*⟩

end

8 Star-Transforms in Non-Standard Analysis

theory *Star*
imports *NSA*
begin

definition — internal sets
starset-n :: (*nat* ⇒ ′*a set*) ⇒ ′*a star set* (∗*sn*∗ - [*80*] *80*)
where ∗*sn*∗ *As* = *Iset* (*star-n As*)

definition *InternalSets* :: ′*a star set set*
where *InternalSets* = {*X*. ∃ *As*. *X* = ∗*sn*∗ *As*}

definition — nonstandard extension of function
is-starext :: (′*a star* ⇒ ′*a star*) ⇒ (′*a* ⇒ ′*a*) ⇒ *bool*
where *is-starext F f* ⟷
(∀ *x y*. ∃ *X* ∈ *Rep-star x*. ∃ *Y* ∈ *Rep-star y*. *y* = *F x* ⟷ *eventually* (λ*n*. *Y n* = *f*(*X n*)) $\mathcal{U}$)

definition — internal functions
starfun-n :: (*nat* ⇒ ′*a* ⇒ ′*b*) ⇒ ′*a star* ⇒ ′*b star* (∗*fn*∗ - [*80*] *80*)
where ∗*fn*∗ *F* = *Ifun* (*star-n F*)

definition *InternalFuns* :: (′*a star* => ′*b star*) *set*
where *InternalFuns* = {*X*. ∃ *F*. *X* = ∗*fn*∗ *F*}

8.1 Preamble - Pulling ∃ over ∀

This proof does not need AC and was suggested by the referee for the JCM Paper: let *f x* be least *y* such that *Q x y*.

lemma *no-choice*: ∀ *x*. ∃ *y*. *Q x y* ⟹ ∃ *f* :: ′*a* ⇒ *nat*. ∀ *x*. *Q x* (*f x*)
⟨*proof*⟩

8.2 Properties of the Star-transform Applied to Sets of Reals

lemma *STAR-star-of-image-subset*: *star-of* ‘ *A* ⊆ ∗*s*∗ *A*
⟨*proof*⟩

lemma *STAR-hypreal-of-real-Int*: ∗*s*∗ *X* ∩ ℝ = *hypreal-of-real* ‘ *X*

⟨*proof*⟩

lemma *STAR-star-of-Int*: $*s*\ X \cap Standard = star\text{-}of\ `\ X$
⟨*proof*⟩

lemma *lemma-not-hyprealA*: $x \notin hypreal\text{-}of\text{-}real\ `\ A \Longrightarrow \forall\, y \in A.\ x \neq hypreal\text{-}of\text{-}real\ y$
⟨*proof*⟩

lemma *lemma-not-starA*: $x \notin star\text{-}of\ `\ A \Longrightarrow \forall\, y \in A.\ x \neq star\text{-}of\ y$
⟨*proof*⟩

lemma *STAR-real-seq-to-hypreal*: $\forall\, n.\ (X\ n) \notin M \Longrightarrow star\text{-}n\ X \notin *s*\ M$
⟨*proof*⟩

lemma *STAR-singleton*: $*s*\ \{x\} = \{star\text{-}of\ x\}$
⟨*proof*⟩

lemma *STAR-not-mem*: $x \notin F \Longrightarrow star\text{-}of\ x \notin *s*\ F$
⟨*proof*⟩

lemma *STAR-subset-closed*: $x \in *s*\ A \Longrightarrow A \subseteq B \Longrightarrow x \in *s*\ B$
⟨*proof*⟩

Nonstandard extension of a set (defined using a constant sequence) as a special case of an internal set.

lemma *starset-n-starset*: $\forall\, n.\ As\ n = A \Longrightarrow *sn*\ As = *s*\ A$
⟨*proof*⟩

8.3 Theorems about nonstandard extensions of functions

Nonstandard extension of a function (defined using a constant sequence) as a special case of an internal function.

lemma *starfun-n-starfun*: $F = (\lambda n.\ f) \Longrightarrow *fn*\ F = *f*\ f$
⟨*proof*⟩

Prove that *abs* for hypreal is a nonstandard extension of abs for real w/o use of congruence property (proved after this for general nonstandard extensions of real valued functions).

Proof now Uses the ultrafilter tactic!

lemma *hrabs-is-starext-rabs*: *is-starext abs abs*
⟨*proof*⟩

Nonstandard extension of functions.

lemma *starfun*: $(\ *f*\ f)\ (star\text{-}n\ X) = star\text{-}n\ (\lambda n.\ f\ (X\ n))$
⟨*proof*⟩

lemma *starfun-if-eq*: $\bigwedge w.\ w \neq \textit{star-of}\ x \Longrightarrow (\ {*}f{*}\ (\lambda z.\ \textit{if}\ z = x\ \textit{then}\ a\ \textit{else}\ g\ z))\ w = (\ {*}f{*}\ g)\ w$
⟨*proof*⟩

Multiplication: $(\ {*}f)\ x\ (\ {*}g) = {*}(f\ x\ g)$

lemma *starfun-mult*: $\bigwedge x.\ (\ {*}f{*}\ f)\ x * (\ {*}f{*}\ g)\ x = (\ {*}f{*}\ (\lambda x.\ f\ x * g\ x))\ x$
⟨*proof*⟩
declare *starfun-mult* [*symmetric*, *simp*]

Addition: $(\ {*}f) + (\ {*}g) = {*}(f + g)$

lemma *starfun-add*: $\bigwedge x.\ (\ {*}f{*}\ f)\ x + (\ {*}f{*}\ g)\ x = (\ {*}f{*}\ (\lambda x.\ f\ x + g\ x))\ x$
⟨*proof*⟩
declare *starfun-add* [*symmetric*, *simp*]

Subtraction: $(\ {*}f) + -(\ {*}g) = {*}(f + -g)$

lemma *starfun-minus*: $\bigwedge x.\ -\ (\ {*}f{*}\ f)\ x = (\ {*}f{*}\ (\lambda x.\ -\ f\ x))\ x$
⟨*proof*⟩
declare *starfun-minus* [*symmetric*, *simp*]

lemma *starfun-add-minus*: $\bigwedge x.\ (\ {*}f{*}\ f)\ x + -(\ {*}f{*}\ g)\ x = (\ {*}f{*}\ (\lambda x.\ f\ x + -g\ x))\ x$
⟨*proof*⟩
declare *starfun-add-minus* [*symmetric*, *simp*]

lemma *starfun-diff*: $\bigwedge x.\ (\ {*}f{*}\ f)\ x\ -\ (\ {*}f{*}\ g)\ x = (\ {*}f{*}\ (\lambda x.\ f\ x - g\ x))\ x$
⟨*proof*⟩
declare *starfun-diff* [*symmetric*, *simp*]

Composition: $(\ {*}f) \circ (\ {*}g) = {*}(f \circ g)$

lemma *starfun-o2*: $(\lambda x.\ (\ {*}f{*}\ f)\ ((\ {*}f{*}\ g)\ x)) = {*}f{*}\ (\lambda x.\ f\ (g\ x))$
⟨*proof*⟩

lemma *starfun-o*: $(\ {*}f{*}\ f) \circ (\ {*}f{*}\ g) = (\ {*}f{*}\ (f \circ g))$
⟨*proof*⟩

NS extension of constant function.

lemma *starfun-const-fun* [*simp*]: $\bigwedge x.\ (\ {*}f{*}\ (\lambda x.\ k))\ x = \textit{star-of}\ k$
⟨*proof*⟩

The NS extension of the identity function.

lemma *starfun-Id* [*simp*]: $\bigwedge x.\ (\ {*}f{*}\ (\lambda x.\ x))\ x = x$
⟨*proof*⟩

The Star-function is a (nonstandard) extension of the function.

lemma *is-starext-starfun*: *is-starext* $(\ {*}f{*}\ f)\ f$
⟨*proof*⟩

Any nonstandard extension is in fact the Star-function.

lemma *is-starfun-starext*:
assumes *is-starext* $F\ f$
shows $F = {*f*}\ f$
⟨*proof*⟩

lemma *is-starext-starfun-iff*: *is-starext* $F\ f \longleftrightarrow F = {*f*}\ f$
⟨*proof*⟩

Extended function has same solution as its standard version for real arguments. i.e they are the same for all real arguments.

lemma *starfun-eq*: $(\ {*f*}\ f)\ (\mathit{star\text{-}of}\ a) = \mathit{star\text{-}of}\ (f\ a)$
⟨*proof*⟩

lemma *starfun-approx*: $(\ {*f*}\ f)\ (\mathit{star\text{-}of}\ a) \approx \mathit{star\text{-}of}\ (f\ a)$
⟨*proof*⟩

Useful for NS definition of derivatives.

lemma *starfun-lambda-cancel*: $\bigwedge x'.\ (\ {*f*}\ (\lambda h.\ f\ (x + h)))\ x' = (\ {*f*}\ f)\ (\mathit{star\text{-}of}\ x + x')$
⟨*proof*⟩

lemma *starfun-lambda-cancel2*: $(\ {*f*}\ (\lambda h.\ f\ (g\ (x + h))))\ x' = (\ {*f*}\ (f \circ g))\ (\mathit{star\text{-}of}\ x + x')$
⟨*proof*⟩

lemma *starfun-mult-HFinite-approx*:
$(\ {*f*}\ f)\ x \approx l \Longrightarrow (\ {*f*}\ g)\ x \approx m \Longrightarrow l \in \mathit{HFinite} \Longrightarrow m \in \mathit{HFinite} \Longrightarrow$
$(\ {*f*}\ (\lambda x.\ f\ x * g\ x))\ x \approx l * m$
for $l\ m :: {'a}{::}\mathit{real\text{-}normed\text{-}algebra}\ \mathit{star}$
⟨*proof*⟩

lemma *starfun-add-approx*: $(\ {*f*}\ f)\ x \approx l \Longrightarrow (\ {*f*}\ g)\ x \approx m \Longrightarrow (\ {*f*}\ (\%x.\ f\ x + g\ x))\ x \approx l + m$
⟨*proof*⟩

Examples: *hrabs* is nonstandard extension of *rabs*, *inverse* is nonstandard extension of *inverse*.

Can be proved easily using theorem *starfun* and properties of ultrafilter as for inverse below we use the theorem we proved above instead.

lemma *starfun-rabs-hrabs*: ${*f*}\ \mathit{abs} = \mathit{abs}$
⟨*proof*⟩

lemma *starfun-inverse-inverse* [*simp*]: $(\ {*f*}\ \mathit{inverse})\ x = \mathit{inverse}\ x$
⟨*proof*⟩

lemma *starfun-inverse*: $\bigwedge x.\ \mathit{inverse}\ ((\ {*f*}\ f)\ x) = (\ {*f*}\ (\lambda x.\ \mathit{inverse}\ (f\ x)))\ x$

⟨*proof*⟩
declare *starfun-inverse* [*symmetric*, *simp*]

lemma *starfun-divide*: $\bigwedge x.\ (\ {*}f{*}\ f)\ x\ /\ (\ {*}f{*}\ g)\ x = (\ {*}f{*}\ (\lambda x.\ f\ x\ /\ g\ x))\ x$
⟨*proof*⟩
declare *starfun-divide* [*symmetric*, *simp*]

lemma *starfun-inverse2*: $\bigwedge x.\ inverse\ ((\ {*}f{*}\ f)\ x) = (\ {*}f{*}\ (\lambda x.\ inverse\ (f\ x)))\ x$
⟨*proof*⟩

General lemma/theorem needed for proofs in elementary topology of the reals.

lemma *starfun-mem-starset*: $\bigwedge x.\ (\ {*}f{*}\ f)\ x \in {*}s{*}\ A \Longrightarrow x \in {*}s{*}\ \{x.\ f\ x \in A\}$
⟨*proof*⟩

Alternative definition for *hrabs* with *rabs* function applied entrywise to equivalence class representative. This is easily proved using *starfun* and ns extension thm.

lemma *hypreal-hrabs*: $|star\text{-}n\ X| = star\text{-}n\ (\lambda n.\ |X\ n|)$
⟨*proof*⟩

Nonstandard extension of set through nonstandard extension of *rabs* function i.e. *hrabs*. A more general result should be where we replace *rabs* by some arbitrary function *f* and *hrabs* by its NS extenson. See second NS set extension below.

lemma *STAR-rabs-add-minus*: ${*}s{*}\ \{x.\ |x + -\ y| < r\} = \{x.\ |x + -star\text{-}of\ y| < star\text{-}of\ r\}$
⟨*proof*⟩

lemma *STAR-starfun-rabs-add-minus*:
${*}s{*}\ \{x.\ |f\ x + -\ y| < r\} = \{x.\ |(\ {*}f{*}\ f)\ x + -star\text{-}of\ y| < star\text{-}of\ r\}$
⟨*proof*⟩

Another characterization of Infinitesimal and one of ≈ relation. In this theory since *hypreal-hrabs* proved here. Maybe move both theorems??

lemma *Infinitesimal-FreeUltrafilterNat-iff2*:
$star\text{-}n\ X \in Infinitesimal \longleftrightarrow (\forall m.\ eventually\ (\lambda n.\ norm\ (X\ n) < inverse\ (real\ (Suc\ m)))\ \mathcal{U})$
⟨*proof*⟩

lemma *HNatInfinite-inverse-Infinitesimal* [*simp*]:
assumes $n \in HNatInfinite$
shows $inverse\ (hypreal\text{-}of\text{-}hypnat\ n) \in Infinitesimal$
⟨*proof*⟩

lemma *approx-FreeUltrafilterNat-iff*:
$star\text{-}n\ X \approx star\text{-}n\ Y \longleftrightarrow (\forall r{>}0.\ eventually\ (\lambda n.\ norm\ (X\ n -\ Y\ n) < r)\ \mathcal{U})$

(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *approx-FreeUltrafilterNat-iff2*:
$star\text{-}n\ X \approx star\text{-}n\ Y \longleftrightarrow (\forall m.\ eventually\ (\lambda n.\ norm\ (X\ n - Y\ n) < inverse\ (real\ (Suc\ m)))\ \mathcal{U})$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *inj-starfun*: *inj starfun*
⟨*proof*⟩

end

9 Star-transforms for the Hypernaturals

theory *NatStar*
imports *Star*
begin

lemma *star-n-eq-starfun-whn*: $star\text{-}n\ X = (\ {*f*}\ X)\ whn$
⟨*proof*⟩

lemma *starset-n-Un*: $*sn*\ (\lambda n.\ (A\ n) \cup (B\ n)) = *sn*\ A \cup *sn*\ B$
⟨*proof*⟩

lemma *InternalSets-Un*: $X \in InternalSets \Longrightarrow Y \in InternalSets \Longrightarrow X \cup Y \in InternalSets$
⟨*proof*⟩

lemma *starset-n-Int*: $*sn*\ (\lambda n.\ A\ n \cap B\ n) = *sn*\ A \cap *sn*\ B$
⟨*proof*⟩

lemma *InternalSets-Int*: $X \in InternalSets \Longrightarrow Y \in InternalSets \Longrightarrow X \cap Y \in InternalSets$
⟨*proof*⟩

lemma *starset-n-Compl*: $*sn*\ ((\lambda n.\ - A\ n)) = - (\ *sn*\ A)$
⟨*proof*⟩

lemma *InternalSets-Compl*: $X \in InternalSets \Longrightarrow - X \in InternalSets$
⟨*proof*⟩

lemma *starset-n-diff*: $*sn*\ (\lambda n.\ (A\ n) - (B\ n)) = *sn*\ A - *sn*\ B$
⟨*proof*⟩

lemma *InternalSets-diff*: $X \in InternalSets \Longrightarrow Y \in InternalSets \Longrightarrow X - Y \in InternalSets$
⟨*proof*⟩

lemma *NatStar-SHNat-subset*: $Nats \leq *s* \ (UNIV{::}\ nat\ set)$
⟨*proof*⟩

lemma *NatStar-hypreal-of-real-Int*: $*s* \ X \ Int \ Nats = hypnat\text{-}of\text{-}nat \ ` \ X$
⟨*proof*⟩

lemma *starset-starset-n-eq*: $*s* \ X = *sn* \ (\lambda n. \ X)$
⟨*proof*⟩

lemma *InternalSets-starset-n* [*simp*]: $(*s* \ X) \in InternalSets$
⟨*proof*⟩

lemma *InternalSets-UNIV-diff*: $X \in InternalSets \Longrightarrow UNIV - X \in InternalSets$
⟨*proof*⟩

9.1 Nonstandard Extensions of Functions

Example of transfer of a property from reals to hyperreals — used for limit comparison of sequences.

lemma *starfun-le-mono*: $\forall n. \ N \leq n \longrightarrow f\ n \leq g\ n \Longrightarrow$
$\forall n. \ hypnat\text{-}of\text{-}nat \ N \leq n \longrightarrow (*f* \ f) \ n \leq (*f* \ g) \ n$
⟨*proof*⟩

And another:

lemma *starfun-less-mono*:
$\forall n. \ N \leq n \longrightarrow f\ n < g\ n \Longrightarrow \forall n. \ hypnat\text{-}of\text{-}nat \ N \leq n \longrightarrow (*f* \ f) \ n < (*f* \ g) \ n$
⟨*proof*⟩

Nonstandard extension when we increment the argument by one.

lemma *starfun-shift-one*: $\bigwedge N. \ (*f* \ (\lambda n. \ f \ (Suc \ n))) \ N = (*f* \ f) \ (N + (1{::}hypnat))$
⟨*proof*⟩

Nonstandard extension with absolute value.

lemma *starfun-abs*: $\bigwedge N. \ (*f* \ (\lambda n. \ |f \ n|)) \ N = |(*f* \ f) \ N|$
⟨*proof*⟩

The *hyperpow* function as a nonstandard extension of *realpow*.

lemma *starfun-pow*: $\bigwedge N. \ (*f* \ (\lambda n. \ r \ \widehat{} \ n)) \ N = hypreal\text{-}of\text{-}real \ r \ pow \ N$
⟨*proof*⟩

lemma *starfun-pow2*: $\bigwedge N. \ (*f* \ (\lambda n. \ X \ n \ \widehat{} \ m)) \ N = (*f* \ X) \ N \ pow \ hypnat\text{-}of\text{-}nat \ m$
⟨*proof*⟩

lemma *starfun-pow3*: $\bigwedge R. \ (*f* \ (\lambda r. \ r \ \widehat{} \ n)) \ R = R \ pow \ hypnat\text{-}of\text{-}nat \ n$
⟨*proof*⟩

The *hypreal-of-hypnat* function as a nonstandard extension of *real*.

lemma *starfunNat-real-of-nat*: $(*f* \; real) = hypreal\text{-}of\text{-}hypnat$
⟨*proof*⟩

lemma *starfun-inverse-real-of-nat-eq*:
$N \in HNatInfinite \Longrightarrow (*f* (\lambda x{::}nat.\; inverse\; (real\; x)))\; N = inverse\; (hypreal\text{-}of\text{-}hypnat\; N)$
⟨*proof*⟩

Internal functions – some redundancy with $*f*$ now.

lemma *starfun-n*: $(*fn* \; f)\; (star\text{-}n\; X) = star\text{-}n\; (\lambda n.\; f\; n\; (X\; n))$
⟨*proof*⟩

Multiplication: $(*fn)\; x\; (*gn) = *(fn\; x\; gn)$

lemma *starfun-n-mult*: $(*fn* \; f)\; z * (*fn* \; g)\; z = (*fn* (\lambda i\; x.\; f\; i\; x * g\; i\; x))\; z$
⟨*proof*⟩

Addition: $(*fn) + (*gn) = *(fn + gn)$

lemma *starfun-n-add*: $(*fn* \; f)\; z + (*fn* \; g)\; z = (*fn* (\lambda i\; x.\; f\; i\; x + g\; i\; x))\; z$
⟨*proof*⟩

Subtraction: $(*fn) - (*gn) = *(fn + - \; gn)$

lemma *starfun-n-add-minus*: $(*fn* \; f)\; z + -(*fn* \; g)\; z = (*fn* (\lambda i\; x.\; f\; i\; x + -g\; i\; x))\; z$
⟨*proof*⟩

Composition: $(*fn) \circ (*gn) = *(fn \circ gn)$

lemma *starfun-n-const-fun* [*simp*]: $(*fn* (\lambda i\; x.\; k))\; z = star\text{-}of\; k$
⟨*proof*⟩

lemma *starfun-n-minus*: $- (*fn* \; f)\; x = (*fn* (\lambda i\; x.\; - (f\; i)\; x))\; x$
⟨*proof*⟩

lemma *starfun-n-eq* [*simp*]: $(*fn* \; f)\; (star\text{-}of\; n) = star\text{-}n\; (\lambda i.\; f\; i\; n)$
⟨*proof*⟩

lemma *starfun-eq-iff*: $((*f* \; f) = (*f* \; g)) \longleftrightarrow f = g$
⟨*proof*⟩

lemma *starfunNat-inverse-real-of-nat-Infinitesimal* [*simp*]:
$N \in HNatInfinite \Longrightarrow (*f* (\lambda x.\; inverse\; (real\; x)))\; N \in Infinitesimal$
⟨*proof*⟩

9.2 Nonstandard Characterization of Induction

lemma *hypnat-induct-obj*:
$\bigwedge n.\; ((*p* \; P)\; (0{::}hypnat) \wedge (\forall n.\; (*p* \; P)\; n \longrightarrow (*p* \; P)\; (n + 1))) \longrightarrow (*p* \; P)\; n$

⟨*proof*⟩

lemma *hypnat-induct*:
$\bigwedge n.\ (\ {*p*}\ P)\ (0{::}hypnat) \Longrightarrow (\bigwedge n.\ (\ {*p*}\ P)\ n \Longrightarrow (\ {*p*}\ P)\ (n + 1)) \Longrightarrow (\ {*p*}\ P)\ n$
⟨*proof*⟩

lemma *starP2-eq-iff*: $(\ {*p2*}\ (=)) = (=)$
⟨*proof*⟩

lemma *starP2-eq-iff2*: $(\ {*p2*}\ (\lambda x\ y.\ x = y))\ X\ Y \longleftrightarrow X = Y$
⟨*proof*⟩

lemma *nonempty-set-star-has-least-lemma*:
$\exists\, n{\in}S.\ \forall\, m{\in}S.\ n \le m$ **if** $S \neq \{\}$ **for** $S :: nat\ set$
⟨*proof*⟩

lemma *nonempty-set-star-has-least*:
$\bigwedge S{::}nat\ set\ star.\ Iset\ S \neq \{\} \Longrightarrow \exists\, n \in Iset\ S.\ \forall\, m \in Iset\ S.\ n \le m$
⟨*proof*⟩

lemma *nonempty-InternalNatSet-has-least*: $S \in InternalSets \Longrightarrow S \neq \{\} \Longrightarrow \exists\, n \in S.\ \forall\, m \in S.\ n \le m$
for $S :: hypnat\ set$
⟨*proof*⟩

Goldblatt, page 129 Thm 11.3.2.

lemma *internal-induct-lemma*:
$\bigwedge X{::}nat\ set\ star.$
$(0{::}hypnat) \in Iset\ X \Longrightarrow \forall\, n.\ n \in Iset\ X \longrightarrow n + 1 \in Iset\ X \Longrightarrow Iset\ X = (UNIV{::}\ hypnat\ set)$
⟨*proof*⟩

lemma *internal-induct*:
$X \in InternalSets \Longrightarrow (0{::}hypnat) \in X \Longrightarrow \forall\, n.\ n \in X \longrightarrow n + 1 \in X \Longrightarrow X = (UNIV{::}\ hypnat\ set)$
⟨*proof*⟩

end

10 Sequences and Convergence (Nonstandard)

theory *HSEQ*
imports *Complex-Main NatStar*
abbrevs $--->\ =\ \longrightarrow_{NS}$
begin

definition $NSLIMSEQ :: (nat \Rightarrow 'a{::}real\text{-}normed\text{-}vector) \Rightarrow 'a \Rightarrow bool$
$(((\text{-})/\ \longrightarrow_{NS}\ (\text{-}))\ [60,\ 60]\ 60)$ **where**

— Nonstandard definition of convergence of sequence
$X \longrightarrow_{NS} L \longleftrightarrow (\forall N \in \mathit{HNatInfinite}.\ (*f* X)\ N \approx \mathit{star\text{-}of}\ L)$

definition *nslim* :: $(nat \Rightarrow 'a::real\text{-}normed\text{-}vector) \Rightarrow 'a$
where *nslim* $X = (\mathit{THE}\ L.\ X \longrightarrow_{NS} L)$
— Nonstandard definition of limit using choice operator

definition *NSconvergent* :: $(nat \Rightarrow 'a::real\text{-}normed\text{-}vector) \Rightarrow bool$
where *NSconvergent* $X \longleftrightarrow (\exists L.\ X \longrightarrow_{NS} L)$
— Nonstandard definition of convergence

definition *NSBseq* :: $(nat \Rightarrow 'a::real\text{-}normed\text{-}vector) \Rightarrow bool$
where *NSBseq* $X \longleftrightarrow (\forall N \in \mathit{HNatInfinite}.\ (*f* X)\ N \in \mathit{HFinite})$
— Nonstandard definition for bounded sequence

definition *NSCauchy* :: $(nat \Rightarrow 'a::real\text{-}normed\text{-}vector) \Rightarrow bool$
where *NSCauchy* $X \longleftrightarrow (\forall M \in \mathit{HNatInfinite}.\ \forall N \in \mathit{HNatInfinite}.\ (*f* X)\ M \approx (*f* X)\ N)$
— Nonstandard definition

10.1 Limits of Sequences

lemma *NSLIMSEQ-I*: $(\bigwedge N.\ N \in \mathit{HNatInfinite} \Longrightarrow \mathit{starfun}\ X\ N \approx \mathit{star\text{-}of}\ L) \Longrightarrow X \longrightarrow_{NS} L$
⟨*proof*⟩

lemma *NSLIMSEQ-D*: $X \longrightarrow_{NS} L \Longrightarrow N \in \mathit{HNatInfinite} \Longrightarrow \mathit{starfun}\ X\ N \approx \mathit{star\text{-}of}\ L$
⟨*proof*⟩

lemma *NSLIMSEQ-const*: $(\lambda n.\ k) \longrightarrow_{NS} k$
⟨*proof*⟩

lemma *NSLIMSEQ-add*: $X \longrightarrow_{NS} a \Longrightarrow Y \longrightarrow_{NS} b \Longrightarrow (\lambda n.\ X\ n + Y\ n) \longrightarrow_{NS} a + b$
⟨*proof*⟩

lemma *NSLIMSEQ-add-const*: $f \longrightarrow_{NS} a \Longrightarrow (\lambda n.\ f\ n + b) \longrightarrow_{NS} a + b$
⟨*proof*⟩

lemma *NSLIMSEQ-mult*: $X \longrightarrow_{NS} a \Longrightarrow Y \longrightarrow_{NS} b \Longrightarrow (\lambda n.\ X\ n * Y\ n) \longrightarrow_{NS} a * b$
for $a\ b$ :: $'a::real\text{-}normed\text{-}algebra$
⟨*proof*⟩

lemma *NSLIMSEQ-minus*: $X \longrightarrow_{NS} a \Longrightarrow (\lambda n.\ -\ X\ n) \longrightarrow_{NS} -\ a$
⟨*proof*⟩

lemma *NSLIMSEQ-minus-cancel*: $(\lambda n. - X\ n) \longrightarrow_{NS} -a \Longrightarrow X \longrightarrow_{NS} a$
⟨*proof*⟩

lemma *NSLIMSEQ-diff*: $X \longrightarrow_{NS} a \Longrightarrow Y \longrightarrow_{NS} b \Longrightarrow (\lambda n.\ X\ n - Y\ n) \longrightarrow_{NS} a - b$
⟨*proof*⟩

lemma *NSLIMSEQ-diff-const*: $f \longrightarrow_{NS} a \Longrightarrow (\lambda n.\ f\ n - b) \longrightarrow_{NS} a - b$
⟨*proof*⟩

lemma *NSLIMSEQ-inverse*: $X \longrightarrow_{NS} a \Longrightarrow a \neq 0 \Longrightarrow (\lambda n.\ inverse\ (X\ n)) \longrightarrow_{NS} inverse\ a$
for a :: *'a::real-normed-div-algebra*
⟨*proof*⟩

lemma *NSLIMSEQ-mult-inverse*: $X \longrightarrow_{NS} a \Longrightarrow Y \longrightarrow_{NS} b \Longrightarrow b \neq 0 \Longrightarrow (\lambda n.\ X\ n\ /\ Y\ n) \longrightarrow_{NS} a\ /\ b$
for $a\ b$ :: *'a::real-normed-field*
⟨*proof*⟩

lemma *starfun-hnorm*: $\bigwedge x.\ hnorm\ ((\ *f*\ f)\ x) = (\ *f*\ (\lambda x.\ norm\ (f\ x)))\ x$
⟨*proof*⟩

lemma *NSLIMSEQ-norm*: $X \longrightarrow_{NS} a \Longrightarrow (\lambda n.\ norm\ (X\ n)) \longrightarrow_{NS} norm\ a$
⟨*proof*⟩

Uniqueness of limit.

lemma *NSLIMSEQ-unique*: $X \longrightarrow_{NS} a \Longrightarrow X \longrightarrow_{NS} b \Longrightarrow a = b$
⟨*proof*⟩

lemma *NSLIMSEQ-pow* [*rule-format*]: $(X \longrightarrow_{NS} a) \longrightarrow ((\lambda n.\ (X\ n)\ \hat{}\ m) \longrightarrow_{NS} a\ \hat{}\ m)$
for a :: *'a::{real-normed-algebra,power}*
⟨*proof*⟩

We can now try and derive a few properties of sequences, starting with the limit comparison property for sequences.

lemma *NSLIMSEQ-le*: $f \longrightarrow_{NS} l \Longrightarrow g \longrightarrow_{NS} m \Longrightarrow \exists N.\ \forall n \geq N.\ f\ n \leq g\ n \Longrightarrow l \leq m$
for $l\ m$ :: *real*
⟨*proof*⟩

lemma *NSLIMSEQ-le-const*: $X \longrightarrow_{NS} r \Longrightarrow \forall n.\ a \leq X\ n \Longrightarrow a \leq r$
for $a\ r$ :: *real*
⟨*proof*⟩

lemma *NSLIMSEQ-le-const2*: $X \longrightarrow_{NS} r \Longrightarrow \forall n.\ X\ n \leq a \Longrightarrow r \leq a$

for *a r* :: *real*
⟨*proof*⟩

Shift a convergent series by 1: By the equivalence between Cauchiness and convergence and because the successor of an infinite hypernatural is also infinite.

lemma *NSLIMSEQ-Suc-iff*: $((\lambda n.\ f\ (Suc\ n)) \longrightarrow_{NS} l) \longleftrightarrow (f \longrightarrow_{NS} l)$
⟨*proof*⟩

10.1.1 Equivalence of *LIMSEQ* and *NSLIMSEQ*

lemma *LIMSEQ-NSLIMSEQ*:
assumes *X*: $X \longrightarrow L$
shows $X \longrightarrow_{NS} L$
⟨*proof*⟩

lemma *NSLIMSEQ-LIMSEQ*:
assumes *X*: $X \longrightarrow_{NS} L$
shows $X \longrightarrow L$
⟨*proof*⟩

theorem *LIMSEQ-NSLIMSEQ-iff*: $f \longrightarrow L \longleftrightarrow f \longrightarrow_{NS} L$
⟨*proof*⟩

10.1.2 Derived theorems about *NSLIMSEQ*

We prove the NS version from the standard one, since the NS proof seems more complicated than the standard one above!

lemma *NSLIMSEQ-norm-zero*: $(\lambda n.\ norm\ (X\ n)) \longrightarrow_{NS} 0 \longleftrightarrow X \longrightarrow_{NS} 0$
⟨*proof*⟩

lemma *NSLIMSEQ-rabs-zero*: $(\lambda n.\ |f\ n|) \longrightarrow_{NS} 0 \longleftrightarrow f \longrightarrow_{NS} (0::real)$
⟨*proof*⟩

Generalization to other limits.

lemma *NSLIMSEQ-imp-rabs*: $f \longrightarrow_{NS} l \Longrightarrow (\lambda n.\ |f\ n|) \longrightarrow_{NS} |l|$
for *l* :: *real*
⟨*proof*⟩

lemma *NSLIMSEQ-inverse-zero*: $\forall y::real.\ \exists N.\ \forall n \geq N.\ y < f\ n \Longrightarrow (\lambda n.\ inverse\ (f\ n)) \longrightarrow_{NS} 0$
⟨*proof*⟩

lemma *NSLIMSEQ-inverse-real-of-nat*: $(\lambda n.\ inverse\ (real\ (Suc\ n))) \longrightarrow_{NS} 0$
⟨*proof*⟩

lemma *NSLIMSEQ-inverse-real-of-nat-add*: $(\lambda n.\ r + inverse\ (real\ (Suc\ n))) \longrightarrow_{NS} r$
⟨*proof*⟩

lemma *NSLIMSEQ-inverse-real-of-nat-add-minus*: $(\lambda n.\ r + -\ inverse\ (real\ (Suc\ n))) \longrightarrow_{NS} r$
⟨*proof*⟩

lemma *NSLIMSEQ-inverse-real-of-nat-add-minus-mult*:
$(\lambda n.\ r * (1 + -\ inverse\ (real\ (Suc\ n)))) \longrightarrow_{NS} r$
⟨*proof*⟩

10.2 Convergence

lemma *nslimI*: $X \longrightarrow_{NS} L \Longrightarrow nslim\ X = L$
⟨*proof*⟩

lemma *lim-nslim-iff*: $lim\ X = nslim\ X$
⟨*proof*⟩

lemma *NSconvergentD*: $NSconvergent\ X \Longrightarrow \exists L.\ X \longrightarrow_{NS} L$
⟨*proof*⟩

lemma *NSconvergentI*: $X \longrightarrow_{NS} L \Longrightarrow NSconvergent\ X$
⟨*proof*⟩

lemma *convergent-NSconvergent-iff*: $convergent\ X = NSconvergent\ X$
⟨*proof*⟩

lemma *NSconvergent-NSLIMSEQ-iff*: $NSconvergent\ X \longleftrightarrow X \longrightarrow_{NS} nslim\ X$
⟨*proof*⟩

10.3 Bounded Monotonic Sequences

lemma *NSBseqD*: $NSBseq\ X \Longrightarrow N \in HNatInfinite \Longrightarrow (\ *f*\ X)\ N \in HFinite$
⟨*proof*⟩

lemma *Standard-subset-HFinite*: $Standard \subseteq HFinite$
⟨*proof*⟩

lemma *NSBseqD2*: $NSBseq\ X \Longrightarrow (\ *f*\ X)\ N \in HFinite$
⟨*proof*⟩

lemma *NSBseqI*: $\forall N \in HNatInfinite.\ (\ *f*\ X)\ N \in HFinite \Longrightarrow NSBseq\ X$
⟨*proof*⟩

The standard definition implies the nonstandard definition.

lemma *Bseq-NSBseq*: $Bseq\ X \Longrightarrow NSBseq\ X$
⟨*proof*⟩

The nonstandard definition implies the standard definition.

lemma *SReal-less-omega*: $r \in \mathbb{R} \Longrightarrow r < \omega$
⟨*proof*⟩

lemma *NSBseq-Bseq*: *NSBseq X* $\Longrightarrow$ *Bseq X*
⟨*proof*⟩

Equivalence of nonstandard and standard definitions for a bounded sequence.

lemma *Bseq-NSBseq-iff*: *Bseq X* = *NSBseq X*
⟨*proof*⟩

A convergent sequence is bounded: Boundedness as a necessary condition for convergence. The nonstandard version has no existential, as usual.

lemma *NSconvergent-NSBseq*: *NSconvergent X* $\Longrightarrow$ *NSBseq X*
⟨*proof*⟩

Standard Version: easily now proved using equivalence of NS and standard definitions.

lemma *convergent-Bseq*: *convergent X* $\Longrightarrow$ *Bseq X*
for $X :: nat \Rightarrow {}'b::real\text{-}normed\text{-}vector$
⟨*proof*⟩

10.3.1 Upper Bounds and Lubs of Bounded Sequences

lemma *NSBseq-isUb*: *NSBseq X* $\Longrightarrow \exists\, U::real.\ isUb\ UNIV\ \{x.\ \exists\, n.\ X\ n = x\}\ U$
⟨*proof*⟩

lemma *NSBseq-isLub*: *NSBseq X* $\Longrightarrow \exists\, U::real.\ isLub\ UNIV\ \{x.\ \exists\, n.\ X\ n = x\}\ U$
⟨*proof*⟩

10.3.2 A Bounded and Monotonic Sequence Converges

The best of both worlds: Easier to prove this result as a standard theorem and then use equivalence to "transfer" it into the equivalent nonstandard form if needed!

lemma *Bmonoseq-NSLIMSEQ*: $\forall_F\ k$ *in sequentially*. $X\ k = X\ m \Longrightarrow X \longrightarrow_{NS} X\ m$
⟨*proof*⟩

lemma *NSBseq-mono-NSconvergent*: *NSBseq X* $\Longrightarrow \forall\, m.\ \forall\, n \geq m.\ X\ m \leq X\ n \Longrightarrow$ *NSconvergent X*
for $X :: nat \Rightarrow real$
⟨*proof*⟩

10.4 Cauchy Sequences

lemma *NSCauchyI*:
$(\bigwedge M\ N.\ M \in \mathit{HNatInfinite} \Longrightarrow N \in \mathit{HNatInfinite} \Longrightarrow \mathit{starfun}\ X\ M \approx \mathit{starfun}\ X\ N) \Longrightarrow \mathit{NSCauchy}\ X$
⟨*proof*⟩

lemma *NSCauchyD*:
$\mathit{NSCauchy}\ X \Longrightarrow M \in \mathit{HNatInfinite} \Longrightarrow N \in \mathit{HNatInfinite} \Longrightarrow \mathit{starfun}\ X\ M \approx \mathit{starfun}\ X\ N$
⟨*proof*⟩

10.4.1 Equivalence Between NS and Standard

lemma *Cauchy-NSCauchy*:
assumes *X*: *Cauchy X*
shows *NSCauchy X*
⟨*proof*⟩

lemma *NSCauchy-Cauchy*:
assumes *X*: *NSCauchy X*
shows *Cauchy X*
⟨*proof*⟩

theorem *NSCauchy-Cauchy-iff*: $\mathit{NSCauchy}\ X = \mathit{Cauchy}\ X$
⟨*proof*⟩

10.4.2 Cauchy Sequences are Bounded

A Cauchy sequence is bounded – nonstandard version.

lemma *NSCauchy-NSBseq*: $\mathit{NSCauchy}\ X \Longrightarrow \mathit{NSBseq}\ X$
⟨*proof*⟩

10.4.3 Cauchy Sequences are Convergent

Equivalence of Cauchy criterion and convergence: We will prove this using our NS formulation which provides a much easier proof than using the standard definition. We do not need to use properties of subsequences such as boundedness, monotonicity etc... Compare with Harrison's corresponding proof in HOL which is much longer and more complicated. Of course, we do not have problems which he encountered with guessing the right instantiations for his 'espsilon-delta' proof(s) in this case since the NS formulations do not involve existential quantifiers.

lemma *NSconvergent-NSCauchy*: $\mathit{NSconvergent}\ X \Longrightarrow \mathit{NSCauchy}\ X$
⟨*proof*⟩

lemma *real-NSCauchy-NSconvergent*:
fixes $X :: \mathit{nat} \Rightarrow \mathit{real}$

assumes *NSCauchy X* **shows** *NSconvergent X*
⟨*proof*⟩

lemma *NSCauchy-NSconvergent*: *NSCauchy X* $\Longrightarrow$ *NSconvergent X*
for X :: *nat* $\Rightarrow$ *'a::banach*
⟨*proof*⟩

lemma *NSCauchy-NSconvergent-iff*: *NSCauchy X* = *NSconvergent X*
for X :: *nat* $\Rightarrow$ *'a::banach*
⟨*proof*⟩

10.5 Power Sequences

The sequence x^n tends to 0 if $(0::'a) \leq x$ and $x < (1::'a)$. Proof will use (NS) Cauchy equivalence for convergence and also fact that bounded and monotonic sequence converges.

We now use NS criterion to bring proof of theorem through.

lemma *NSLIMSEQ-realpow-zero*:
fixes x :: *real*
assumes $0 \leq x$ $x < 1$ **shows** $(\lambda n.\ x \hat{}\ n) \longrightarrow_{NS} 0$
⟨*proof*⟩

lemma *NSLIMSEQ-abs-realpow-zero*: $|c| < 1 \Longrightarrow (\lambda n.\ |c| \hat{}\ n) \longrightarrow_{NS} 0$
for c :: *real*
⟨*proof*⟩

lemma *NSLIMSEQ-abs-realpow-zero2*: $|c| < 1 \Longrightarrow (\lambda n.\ c \hat{}\ n) \longrightarrow_{NS} 0$
for c :: *real*
⟨*proof*⟩

end

11 Finite Summation and Infinite Series for Hyperreals

theory *HSeries*
imports *HSEQ*
begin

definition *sumhr* :: *hypnat* × *hypnat* × (*nat* $\Rightarrow$ *real*) $\Rightarrow$ *hypreal*
where *sumhr* = $(\lambda(M,N,f).\ starfun2\ (\lambda m\ n.\ sum\ f\ \{m..<n\})\ M\ N)$

definition *NSsums* :: (*nat* $\Rightarrow$ *real*) $\Rightarrow$ *real* $\Rightarrow$ *bool* (**infixr** *NSsums 80*)
where *f NSsums s* = $(\lambda n.\ sum\ f\ \{..<n\}) \longrightarrow_{NS} s$

definition *NSsummable* :: (*nat* $\Rightarrow$ *real*) $\Rightarrow$ *bool*
where *NSsummable f* $\longleftrightarrow$ $(\exists s.\ f\ NSsums\ s)$

definition *NSsuminf* :: $(nat \Rightarrow real) \Rightarrow real$
where $NSsuminf\ f = (THE\ s.\ f\ NSsums\ s)$

lemma *sumhr-app*: $sumhr\ (M,\ N,\ f) = (\ {*}f2{*}\ (\lambda m\ n.\ sum\ f\ \{m..<n\}))\ M\ N$
⟨*proof*⟩

Base case in definition of *sumr*.

lemma *sumhr-zero* [*simp*]: $\bigwedge m.\ sumhr\ (m,\ 0,\ f) = 0$
⟨*proof*⟩

Recursive case in definition of *sumr*.

lemma *sumhr-if*:
$\bigwedge m\ n.\ sumhr\ (m,\ n + 1,\ f) = (if\ n + 1 \leq m\ then\ 0\ else\ sumhr\ (m,\ n,\ f) + (\ {*}f{*}\ f)\ n)$
⟨*proof*⟩

lemma *sumhr-Suc-zero* [*simp*]: $\bigwedge n.\ sumhr\ (n + 1,\ n,\ f) = 0$
⟨*proof*⟩

lemma *sumhr-eq-bounds* [*simp*]: $\bigwedge n.\ sumhr\ (n,\ n,\ f) = 0$
⟨*proof*⟩

lemma *sumhr-Suc* [*simp*]: $\bigwedge m.\ sumhr\ (m,\ m + 1,\ f) = (\ {*}f{*}\ f)\ m$
⟨*proof*⟩

lemma *sumhr-add-lbound-zero* [*simp*]: $\bigwedge k\ m.\ sumhr\ (m + k,\ k,\ f) = 0$
⟨*proof*⟩

lemma *sumhr-add*: $\bigwedge m\ n.\ sumhr\ (m,\ n,\ f) + sumhr\ (m,\ n,\ g) = sumhr\ (m,\ n,\ \lambda i.\ f\ i + g\ i)$
⟨*proof*⟩

lemma *sumhr-mult*: $\bigwedge m\ n.\ hypreal\text{-}of\text{-}real\ r * sumhr\ (m,\ n,\ f) = sumhr\ (m,\ n,\ \lambda n.\ r * f\ n)$
⟨*proof*⟩

lemma *sumhr-split-add*: $\bigwedge n\ p.\ n < p \Longrightarrow sumhr\ (0,\ n,\ f) + sumhr\ (n,\ p,\ f) = sumhr\ (0,\ p,\ f)$
⟨*proof*⟩

lemma *sumhr-split-diff*: $n < p \Longrightarrow sumhr\ (0,\ p,\ f) - sumhr\ (0,\ n,\ f) = sumhr\ (n,\ p,\ f)$
⟨*proof*⟩

lemma *sumhr-hrabs*: $\bigwedge m\ n.\ |sumhr\ (m,\ n,\ f)| \leq sumhr\ (m,\ n,\ \lambda i.\ |f\ i|)$
⟨*proof*⟩

Other general version also needed.

lemma *sumhr-fun-hypnat-eq*:
$(\forall r.\ m \le r \wedge r < n \longrightarrow f\ r = g\ r) \longrightarrow$
$\quad sumhr\ (hypnat\text{-}of\text{-}nat\ m,\ hypnat\text{-}of\text{-}nat\ n,\ f) =$
$\quad sumhr\ (hypnat\text{-}of\text{-}nat\ m,\ hypnat\text{-}of\text{-}nat\ n,\ g)$
⟨*proof*⟩

lemma *sumhr-const*: $\bigwedge n.\ sumhr\ (0,\ n,\ \lambda i.\ r) = hypreal\text{-}of\text{-}hypnat\ n * hypreal\text{-}of\text{-}real\ r$
⟨*proof*⟩

lemma *sumhr-less-bounds-zero* [*simp*]: $\bigwedge m\ n.\ n < m \Longrightarrow sumhr\ (m,\ n,\ f) = 0$
⟨*proof*⟩

lemma *sumhr-minus*: $\bigwedge m\ n.\ sumhr\ (m,\ n,\ \lambda i.\ -\ f\ i) = -\ sumhr\ (m,\ n,\ f)$
⟨*proof*⟩

lemma *sumhr-shift-bounds*:
$\bigwedge m\ n.\ sumhr\ (m + hypnat\text{-}of\text{-}nat\ k,\ n + hypnat\text{-}of\text{-}nat\ k,\ f) =$
$\quad sumhr\ (m,\ n,\ \lambda i.\ f\ (i + k))$
⟨*proof*⟩

11.1 Nonstandard Sums

Infinite sums are obtained by summing to some infinite hypernatural (such as *whn*).

lemma *sumhr-hypreal-of-hypnat-omega*: $sumhr\ (0,\ whn,\ \lambda i.\ 1) = hypreal\text{-}of\text{-}hypnat\ whn$
⟨*proof*⟩

lemma *sumhr-hypreal-omega-minus-one*: $sumhr(0,\ whn,\ \lambda i.\ 1) = \omega - 1$
⟨*proof*⟩

lemma *sumhr-minus-one-realpow-zero* [*simp*]: $\bigwedge N.\ sumhr\ (0,\ N + N,\ \lambda i.\ (-1)\ \hat{}\ (i + 1)) = 0$
⟨*proof*⟩

lemma *sumhr-interval-const*:
$(\forall n.\ m \le Suc\ n \longrightarrow f\ n = r) \wedge m \le na \Longrightarrow$
$\quad sumhr\ (hypnat\text{-}of\text{-}nat\ m,\ hypnat\text{-}of\text{-}nat\ na,\ f) = hypreal\text{-}of\text{-}nat\ (na - m) * hypreal\text{-}of\text{-}real\ r$
⟨*proof*⟩

lemma *starfunNat-sumr*: $\bigwedge N.\ (\ *f*\ (\lambda n.\ sum\ f\ \{0..<n\}))\ N = sumhr\ (0,\ N,\ f)$
⟨*proof*⟩

lemma *sumhr-hrabs-approx* [*simp*]: $sumhr\ (0,\ M,\ f) \approx sumhr\ (0,\ N,\ f) \Longrightarrow |sumhr\ (M,\ N,\ f)| \approx 0$
⟨*proof*⟩

11.2 Infinite sums: Standard and NS theorems

lemma *sums-NSsums-iff*: $f\ sums\ l \longleftrightarrow f\ NSsums\ l$
⟨*proof*⟩

lemma *summable-NSsummable-iff*: $summable\ f \longleftrightarrow NSsummable\ f$
⟨*proof*⟩

lemma *suminf-NSsuminf-iff*: $suminf\ f = NSsuminf\ f$
⟨*proof*⟩

lemma *NSsums-NSsummable*: $f\ NSsums\ l \Longrightarrow NSsummable\ f$
⟨*proof*⟩

lemma *NSsummable-NSsums*: $NSsummable\ f \Longrightarrow f\ NSsums\ (NSsuminf\ f)$
⟨*proof*⟩

lemma *NSsums-unique*: $f\ NSsums\ s \Longrightarrow s = NSsuminf\ f$
⟨*proof*⟩

lemma *NSseries-zero*: $\forall m.\ n \leq Suc\ m \longrightarrow f\ m = 0 \Longrightarrow f\ NSsums\ (sum\ f\ \{..<n\})$
⟨*proof*⟩

lemma *NSsummable-NSCauchy*:
$NSsummable\ f \longleftrightarrow (\forall M \in HNatInfinite.\ \forall N \in HNatInfinite.\ |sumhr\ (M,\ N,\ f)| \approx 0)$
⟨*proof*⟩

Terms of a convergent series tend to zero.

lemma *NSsummable-NSLIMSEQ-zero*: $NSsummable\ f \Longrightarrow f \longrightarrow_{NS} 0$
⟨*proof*⟩

Nonstandard comparison test.

lemma *NSsummable-comparison-test*: $\exists N.\ \forall n.\ N \leq n \longrightarrow |f\ n| \leq g\ n \Longrightarrow NSsummable\ g \Longrightarrow NSsummable\ f$
⟨*proof*⟩

lemma *NSsummable-rabs-comparison-test*:
$\exists N.\ \forall n.\ N \leq n \longrightarrow |f\ n| \leq g\ n \Longrightarrow NSsummable\ g \Longrightarrow NSsummable\ (\lambda k.\ |f\ k|)$
⟨*proof*⟩

end

12 Limits and Continuity (Nonstandard)

theory *HLim*
imports *Star*
abbrevs −−−> = −□→$_{NS}$
begin

Nonstandard Definitions.

definition *NSLIM* :: $('a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector) \Rightarrow 'a \Rightarrow 'b \Rightarrow bool$
$(((\text{-})/\ -(\text{-})/\to_{NS}\ (\text{-}))\ [60,\ 0,\ 60]\ 60)$
where $f\ -a\to_{NS}\ L \longleftrightarrow (\forall x.\ x \neq star\text{-}of\ a \wedge x \approx star\text{-}of\ a \longrightarrow (\ *f*\ f)\ x \approx star\text{-}of\ L)$

definition *isNSCont* :: $('a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector) \Rightarrow 'a \Rightarrow bool$
where — NS definition dispenses with limit notions
$isNSCont\ f\ a \longleftrightarrow (\forall y.\ y \approx star\text{-}of\ a \longrightarrow (\ *f*\ f)\ y \approx star\text{-}of\ (f\ a))$

definition *isNSUCont* :: $('a::real\text{-}normed\text{-}vector \Rightarrow 'b::real\text{-}normed\text{-}vector) \Rightarrow bool$
where $isNSUCont\ f \longleftrightarrow (\forall x\ y.\ x \approx y \longrightarrow (\ *f*\ f)\ x \approx (\ *f*\ f)\ y)$

12.1 Limits of Functions

lemma *NSLIM-I*: $(\bigwedge x.\ x \neq star\text{-}of\ a \Longrightarrow x \approx star\text{-}of\ a \Longrightarrow starfun\ f\ x \approx star\text{-}of\ L) \Longrightarrow f\ -a\to_{NS}\ L$
⟨*proof*⟩

lemma *NSLIM-D*: $f\ -a\to_{NS}\ L \Longrightarrow x \neq star\text{-}of\ a \Longrightarrow x \approx star\text{-}of\ a \Longrightarrow starfun\ f\ x \approx star\text{-}of\ L$
⟨*proof*⟩

Proving properties of limits using nonstandard definition. The properties hold for standard limits as well!

lemma *NSLIM-mult*: $f\ -x\to_{NS}\ l \Longrightarrow g\ -x\to_{NS}\ m \Longrightarrow (\lambda x.\ f\ x * g\ x)\ -x\to_{NS}\ (l * m)$
for $l\ m$ :: $'a::real\text{-}normed\text{-}algebra$
⟨*proof*⟩

lemma *starfun-scaleR* [*simp*]: $starfun\ (\lambda x.\ f\ x *_R\ g\ x) = (\lambda x.\ scaleHR\ (starfun\ f\ x)\ (starfun\ g\ x))$
⟨*proof*⟩

lemma *NSLIM-scaleR*: $f\ -x\to_{NS}\ l \Longrightarrow g\ -x\to_{NS}\ m \Longrightarrow (\lambda x.\ f\ x *_R\ g\ x)\ -x\to_{NS}\ (l *_R\ m)$
⟨*proof*⟩

lemma *NSLIM-add*: $f\ -x\to_{NS}\ l \Longrightarrow g\ -x\to_{NS}\ m \Longrightarrow (\lambda x.\ f\ x + g\ x)\ -x\to_{NS}\ (l + m)$
⟨*proof*⟩

lemma *NSLIM-const* [*simp*]: $(\lambda x.\ k)\ -x\to_{NS}\ k$
⟨*proof*⟩

lemma *NSLIM-minus*: $f\ -a\to_{NS}\ L \Longrightarrow (\lambda x.\ -\ f\ x)\ -a\to_{NS}\ -L$
⟨*proof*⟩

lemma *NSLIM-diff*: $f\ -x\rightarrow_{NS}\ l \Longrightarrow g\ -x\rightarrow_{NS}\ m \Longrightarrow (\lambda x.\ f\ x\ -\ g\ x)\ -x\rightarrow_{NS}\ (l\ -\ m)$
⟨*proof*⟩

lemma *NSLIM-add-minus*: $f\ -x\rightarrow_{NS}\ l \Longrightarrow g\ -x\rightarrow_{NS}\ m \Longrightarrow (\lambda x.\ f\ x\ +\ -\ g\ x)\ -x\rightarrow_{NS}\ (l\ +\ -m)$
⟨*proof*⟩

lemma *NSLIM-inverse*: $f\ -a\rightarrow_{NS}\ L \Longrightarrow L \neq 0 \Longrightarrow (\lambda x.\ inverse\ (f\ x))\ -a\rightarrow_{NS}\ (inverse\ L)$
for L :: *'a::real-normed-div-algebra*
⟨*proof*⟩

lemma *NSLIM-zero*:
assumes *f*: $f\ -a\rightarrow_{NS}\ l$
shows $(\lambda x.\ f(x)\ -\ l)\ -a\rightarrow_{NS}\ 0$
⟨*proof*⟩

lemma *NSLIM-zero-cancel*:
assumes $(\lambda x.\ f\ x\ -\ l)\ -x\rightarrow_{NS}\ 0$
shows $f\ -x\rightarrow_{NS}\ l$
⟨*proof*⟩

lemma *NSLIM-const-eq*:
fixes a :: *'a::real-normed-algebra-1*
assumes $(\lambda x.\ k)\ -a\rightarrow_{NS}\ l$
shows $k = l$
⟨*proof*⟩

lemma *NSLIM-unique*: $f\ -a\rightarrow_{NS}\ l \Longrightarrow f\ -a\rightarrow_{NS}\ M \Longrightarrow l = M$
for a :: *'a::real-normed-algebra-1*
⟨*proof*⟩

lemma *NSLIM-mult-zero*: $f\ -x\rightarrow_{NS}\ 0 \Longrightarrow g\ -x\rightarrow_{NS}\ 0 \Longrightarrow (\lambda x.\ f\ x\ *\ g\ x)\ -x\rightarrow_{NS}\ 0$
for *f g* :: *'a::real-normed-vector* ⇒ *'b::real-normed-algebra*
⟨*proof*⟩

lemma *NSLIM-self*: $(\lambda x.\ x)\ -a\rightarrow_{NS}\ a$
⟨*proof*⟩

12.1.1 Equivalence of *filterlim* and *NSLIM*

lemma *LIM-NSLIM*:
assumes *f*: $f\ -a\rightarrow\ L$
shows $f\ -a\rightarrow_{NS}\ L$
⟨*proof*⟩

lemma *NSLIM-LIM*:
  assumes f: $f\ -a\rightarrow_{NS}\ L$
  shows $f\ -a\rightarrow\ L$
$\langle proof \rangle$

theorem *LIM-NSLIM-iff*: $f\ -x\rightarrow\ L \longleftrightarrow f\ -x\rightarrow_{NS}\ L$
  $\langle proof \rangle$

12.2 Continuity

lemma *isNSContD*: $isNSCont\ f\ a \Longrightarrow y \approx star\text{-}of\ a \Longrightarrow (\ *f*\ f)\ y \approx star\text{-}of\ (f\ a)$
  $\langle proof \rangle$

lemma *isNSCont-NSLIM*: $isNSCont\ f\ a \Longrightarrow f\ -a\rightarrow_{NS}\ (f\ a)$
  $\langle proof \rangle$

lemma *NSLIM-isNSCont*: $f\ -a\rightarrow_{NS}\ (f\ a) \Longrightarrow isNSCont\ f\ a$
  $\langle proof \rangle$

NS continuity can be defined using NS Limit in similar fashion to standard definition of continuity.

lemma *isNSCont-NSLIM-iff*: $isNSCont\ f\ a \longleftrightarrow f\ -a\rightarrow_{NS}\ (f\ a)$
  $\langle proof \rangle$

Hence, NS continuity can be given in terms of standard limit.

lemma *isNSCont-LIM-iff*: $(isNSCont\ f\ a) = (f\ -a\rightarrow\ (f\ a))$
  $\langle proof \rangle$

Moreover, it's trivial now that NS continuity is equivalent to standard continuity.

lemma *isNSCont-isCont-iff*: $isNSCont\ f\ a \longleftrightarrow isCont\ f\ a$
  $\langle proof \rangle$

Standard continuity $\Longrightarrow$ NS continuity.

lemma *isCont-isNSCont*: $isCont\ f\ a \Longrightarrow isNSCont\ f\ a$
  $\langle proof \rangle$

NS continuity $\Longrightarrow$ Standard continuity.

lemma *isNSCont-isCont*: $isNSCont\ f\ a \Longrightarrow isCont\ f\ a$
  $\langle proof \rangle$

Alternative definition of continuity.

Prove equivalence between NS limits – seems easier than using standard definition.

lemma *NSLIM-at0-iff*: $f\ -a\rightarrow_{NS}\ L \longleftrightarrow (\lambda h.\ f\ (a + h))\ -0\rightarrow_{NS}\ L$
$\langle proof \rangle$

lemma *isNSCont-minus*: *isNSCont f a* $\Longrightarrow$ *isNSCont* ($\lambda x.$ $-$ *f x*) *a*
⟨*proof*⟩

lemma *isNSCont-inverse*: *isNSCont f x* $\Longrightarrow$ *f x* $\neq$ *0* $\Longrightarrow$ *isNSCont* ($\lambda x.$ *inverse* (*f x*)) *x*
for *f* :: *'a::real-normed-vector* $\Rightarrow$ *'b::real-normed-div-algebra*
⟨*proof*⟩

lemma *isNSCont-const* [*simp*]: *isNSCont* ($\lambda x.$ *k*) *a*
⟨*proof*⟩

lemma *isNSCont-abs* [*simp*]: *isNSCont abs a*
for *a* :: *real*
⟨*proof*⟩

12.3 Uniform Continuity

lemma *isNSUContD*: *isNSUCont f* $\Longrightarrow$ *x* $\approx$ *y* $\Longrightarrow$ (**f** *f*) *x* $\approx$ (**f** *f*) *y*
⟨*proof*⟩

lemma *isUCont-isNSUCont*:
fixes *f* :: *'a::real-normed-vector* $\Rightarrow$ *'b::real-normed-vector*
assumes *f*: *isUCont f*
shows *isNSUCont f*
⟨*proof*⟩

lemma *isNSUCont-isUCont*:
fixes *f* :: *'a::real-normed-vector* $\Rightarrow$ *'b::real-normed-vector*
assumes *f*: *isNSUCont f*
shows *isUCont f*
⟨*proof*⟩

end

13 Differentiation (Nonstandard)

theory *HDeriv*
imports *HLim*
begin

Nonstandard Definitions.

definition *nsderiv* :: [*'a::real-normed-field* $\Rightarrow$ *'a*, *'a*, *'a*] $\Rightarrow$ *bool*
((*NSDERIV* (-)/ (-)/ :> (-)) [*1000*, *1000*, *60*] *60*)
where *NSDERIV f x* :> *D* $\longleftrightarrow$
($\forall$ *h* $\in$ *Infinitesimal* $-$ {*0*}. ((**f** *f*)(*star-of x* + *h*) $-$ *star-of* (*f x*)) / *h* $\approx$ *star-of D*)

definition *NSdifferentiable* :: [*'a::real-normed-field* $\Rightarrow$ *'a*, *'a*] $\Rightarrow$ *bool*
(**infixl** *NSdifferentiable 60*)

where *f NSdifferentiable x* $\longleftrightarrow$ ($\exists$ *D*. *NSDERIV f x* :> *D*)

definition *increment* :: (*real* $\Rightarrow$ *real*) $\Rightarrow$ *real* $\Rightarrow$ *hypreal* $\Rightarrow$ *hypreal*
where *increment f x h* =
(*SOME inc. f NSdifferentiable x* $\wedge$ *inc* = (**f* f*) (*hypreal-of-real x* + *h*) − *hypreal-of-real* (*f x*))

13.1 Derivatives

lemma *DERIV-NS-iff*: (*DERIV f x* :> *D*) $\longleftrightarrow$ (λh. (*f* (*x* + *h*) − *f x*) / *h*) $-0\rightarrow_{NS}$ *D*
⟨*proof*⟩

lemma *NS-DERIV-D*: *DERIV f x* :> *D* $\Longrightarrow$ (λh. (*f* (*x* + *h*) − *f x*) / *h*) $-0\rightarrow_{NS}$ *D*
⟨*proof*⟩

lemma *Infinitesimal-of-hypreal*:
x $\in$ *Infinitesimal* $\Longrightarrow$ ((**f* of-real*) *x*::′*a*::*real-normed-div-algebra star*) $\in$ *Infinitesimal*
⟨*proof*⟩

lemma *of-hypreal-eq-0-iff*: $\bigwedge x$. ((**f* of-real*) *x* = (*0*::′*a*::*real-algebra-1 star*)) = (*x* = *0*)
⟨*proof*⟩

lemma *NSDeriv-unique*:
assumes *NSDERIV f x* :> *D NSDERIV f x* :> *E*
shows *NSDERIV f x* :> *D* $\Longrightarrow$ *NSDERIV f x* :> *E* $\Longrightarrow$ *D* = *E*
⟨*proof*⟩

First *NSDERIV* in terms of *NSLIM*.

First equivalence.

lemma *NSDERIV-NSLIM-iff*: (*NSDERIV f x* :> *D*) $\longleftrightarrow$ (λh. (*f* (*x* + *h*) − *f x*) / *h*) $-0\rightarrow_{NS}$ *D*
⟨*proof*⟩

Second equivalence.

lemma *NSDERIV-NSLIM-iff2*: (*NSDERIV f x* :> *D*) $\longleftrightarrow$ (λz. (*f z* − *f x*) / (*z* − *x*)) $-x\rightarrow_{NS}$ *D*
⟨*proof*⟩

While we're at it!

lemma *NSDERIV-iff2*:
(*NSDERIV f x* :> *D*) $\longleftrightarrow$
($\forall$ *w*. *w* $\neq$ *star-of x* $\wedge$ *w* $\approx$ *star-of x* $\longrightarrow$ (**f** (λz. (*f z* − *f x*) / (*z* − *x*))) *w* $\approx$ *star-of D*)
⟨*proof*⟩

lemma *NSDERIVD5*:
$[\![NSDERIV\ f\ x :> D;\ u \approx hypreal\text{-}of\text{-}real\ x]\!] \Longrightarrow$
$(\ {*}f{*}\ (\lambda z.\ f\ z - f\ x))\ u \approx hypreal\text{-}of\text{-}real\ D * (u - hypreal\text{-}of\text{-}real\ x)$
⟨*proof*⟩

lemma *NSDERIVD4*:
$[\![NSDERIV\ f\ x :> D;\ h \in Infinitesimal]\!]$
$\Longrightarrow (\ {*}f{*}\ f)(hypreal\text{-}of\text{-}real\ x + h) - hypreal\text{-}of\text{-}real\ (f\ x) \approx hypreal\text{-}of\text{-}real\ D * h$
⟨*proof*⟩

Differentiability implies continuity nice and simple "algebraic" proof.

lemma *NSDERIV-isNSCont*:
assumes *NSDERIV f x :> D* **shows** *isNSCont f x*
⟨*proof*⟩

Differentiation rules for combinations of functions follow from clear, straightforward, algebraic manipulations.

Constant function.

lemma *NSDERIV-const* [*simp*]: $NSDERIV\ (\lambda x.\ k)\ x :> 0$
⟨*proof*⟩

Sum of functions- proved easily.

lemma *NSDERIV-add*:
assumes *NSDERIV f x :> Da NSDERIV g x :> Db*
shows $NSDERIV\ (\lambda x.\ f\ x + g\ x)\ x :> Da + Db$
⟨*proof*⟩

Product of functions - Proof is simple.

lemma *NSDERIV-mult*:
assumes *NSDERIV g x :> Db NSDERIV f x :> Da*
shows $NSDERIV\ (\lambda x.\ f\ x * g\ x)\ x :> (Da * g\ x) + (Db * f\ x)$
⟨*proof*⟩

Multiplying by a constant.

lemma *NSDERIV-cmult*: $NSDERIV\ f\ x :> D \Longrightarrow NSDERIV\ (\lambda x.\ c * f\ x)\ x :> c * D$
⟨*proof*⟩

Negation of function.

lemma *NSDERIV-minus*: $NSDERIV\ f\ x :> D \Longrightarrow NSDERIV\ (\lambda x.\ -\ f\ x)\ x :> -\ D$
⟨*proof*⟩

Subtraction.

lemma *NSDERIV-add-minus*:

$NSDERIV\ f\ x :> Da \Longrightarrow NSDERIV\ g\ x :> Db \Longrightarrow NSDERIV\ (\lambda x.\ f\ x + -\ g\ x)\ x :> Da + -\ Db$
$\langle proof \rangle$

lemma *NSDERIV-diff*:
$NSDERIV\ f\ x :> Da \Longrightarrow NSDERIV\ g\ x :> Db \Longrightarrow NSDERIV\ (\lambda x.\ f\ x - g\ x)\ x :> Da - Db$
$\langle proof \rangle$

Similarly to the above, the chain rule admits an entirely straightforward derivation. Compare this with Harrison's HOL proof of the chain rule, which proved to be trickier and required an alternative characterisation of differentiability- the so-called Carathedory derivative. Our main problem is manipulation of terms.

13.2 Lemmas

lemma *NSDERIV-zero*:
$[\![NSDERIV\ g\ x :> D;\ (\ {*f*}\ g)\ (star\text{-}of\ x + y) = star\text{-}of\ (g\ x);\ y \in Infinitesimal;\ y \neq 0]\!]$
$\Longrightarrow D = 0$
$\langle proof \rangle$

Can be proved differently using *NSLIM-isCont-iff*.

lemma *NSDERIV-approx*:
$NSDERIV\ f\ x :> D \Longrightarrow h \in Infinitesimal \Longrightarrow h \neq 0 \Longrightarrow$
$(\ {*f*}\ f)\ (star\text{-}of\ x + h) - star\text{-}of\ (f\ x) \approx 0$
$\langle proof \rangle$

From one version of differentiability

$f\ x - f\ a$ −−−−−−−−−−−−−−− $\approx Db\ x - a$

lemma *NSDERIVD1*:
$[\![NSDERIV\ f\ (g\ x) :> Da;$
$(\ {*f*}\ g)\ (star\text{-}of\ x + y) \neq star\text{-}of\ (g\ x);$
$(\ {*f*}\ g)\ (star\text{-}of\ x + y) \approx star\text{-}of\ (g\ x)]\!]$
$\Longrightarrow ((\ {*f*}\ f)\ ((\ {*f*}\ g)\ (star\text{-}of\ x + y)) -$
$star\text{-}of\ (f\ (g\ x)))\ /\ ((\ {*f*}\ g)\ (star\text{-}of\ x + y) - star\text{-}of\ (g\ x)) \approx$
$star\text{-}of\ Da$
$\langle proof \rangle$

From other version of differentiability

$f\ (x + h) - f\ x$ −−−−−−−−−−−−−−−−−− $\approx Db\ h$

lemma *NSDERIVD2*: $[|\ NSDERIV\ g\ x :> Db;\ y \in Infinitesimal;\ y \neq 0\ |]$
$==> ((\ {*f*}\ g)\ (star\text{-}of(x) + y) - star\text{-}of(g\ x))\ /\ y$
$\approx star\text{-}of(Db)$
$\langle proof \rangle$

This proof uses both definitions of differentiability.

lemma *NSDERIV-chain*:
$NSDERIV\ f\ (g\ x)\ :>\ Da \Longrightarrow NSDERIV\ g\ x\ :>\ Db \Longrightarrow NSDERIV\ (f \circ g)\ x\ :>\ Da * Db$
⟨*proof*⟩

Differentiation of natural number powers.

lemma *NSDERIV-Id* [*simp*]: $NSDERIV\ (\lambda x.\ x)\ x\ :>\ 1$
⟨*proof*⟩

lemma *NSDERIV-cmult-Id* [*simp*]: $NSDERIV\ ((*)\ c)\ x\ :>\ c$
⟨*proof*⟩

lemma *NSDERIV-inverse*:
fixes $x :: 'a::real\text{-}normed\text{-}field$
assumes $x \neq 0$ — can't get rid of $x \neq (0::'a)$ because it isn't continuous at zero
shows $NSDERIV\ (\lambda x.\ inverse\ x)\ x\ :>\ -\ (inverse\ x\ \widehat{}\ Suc\ (Suc\ 0))$
⟨*proof*⟩

13.2.1 Equivalence of NS and Standard definitions

lemma *divideR-eq-divide*: $x\ /_R\ y = x\ /\ y$
⟨*proof*⟩

Now equivalence between *NSDERIV* and *DERIV*.

lemma *NSDERIV-DERIV-iff*: $NSDERIV\ f\ x\ :>\ D \longleftrightarrow DERIV\ f\ x\ :>\ D$
⟨*proof*⟩

NS version.

lemma *NSDERIV-pow*: $NSDERIV\ (\lambda x.\ x\ \widehat{}\ n)\ x\ :>\ real\ n * (x\ \widehat{}\ (n - Suc\ 0))$
⟨*proof*⟩

Derivative of inverse.

lemma *NSDERIV-inverse-fun*:
$NSDERIV\ f\ x\ :>\ d \Longrightarrow f\ x \neq 0 \Longrightarrow$
$NSDERIV\ (\lambda x.\ inverse\ (f\ x))\ x\ :>\ (-\ (d * inverse\ (f\ x\ \widehat{}\ Suc\ (Suc\ 0))))$
for $x :: 'a::\{real\text{-}normed\text{-}field\}$
⟨*proof*⟩

Derivative of quotient.

lemma *NSDERIV-quotient*:
fixes $x :: 'a::real\text{-}normed\text{-}field$
shows $NSDERIV\ f\ x\ :>\ d \Longrightarrow NSDERIV\ g\ x\ :>\ e \Longrightarrow g\ x \neq 0 \Longrightarrow$
$NSDERIV\ (\lambda y.\ f\ y\ /\ g\ y)\ x\ :>\ (d * g\ x - (e * f\ x))\ /\ (g\ x\ \widehat{}\ Suc\ (Suc\ 0))$
⟨*proof*⟩

lemma *CARAT-NSDERIV*:
$NSDERIV\ f\ x\ :>\ l \Longrightarrow \exists\, g.\ (\forall\, z.\ f\ z - f\ x = g\ z * (z - x)) \land isNSCont\ g\ x \land g\ x = l$

⟨*proof*⟩

lemma *hypreal-eq-minus-iff3*: $x = y + z \longleftrightarrow x + - z = y$
for $x\ y\ z$ :: *hypreal*
⟨*proof*⟩

lemma *CARAT-DERIVD*:
assumes *all*: $\forall z.\ f\ z - f\ x = g\ z * (z - x)$
and *nsc*: *isNSCont* $g\ x$
shows *NSDERIV* $f\ x$:> $g\ x$
⟨*proof*⟩

13.2.2 Differentiability predicate

lemma *NSdifferentiableD*: f *NSdifferentiable* $x \Longrightarrow \exists D.$ *NSDERIV* $f\ x$:> D
⟨*proof*⟩

lemma *NSdifferentiableI*: *NSDERIV* $f\ x$:> $D \Longrightarrow f$ *NSdifferentiable* x
⟨*proof*⟩

13.3 (NS) Increment

lemma *incrementI*:
f *NSdifferentiable* $x \Longrightarrow$
increment $f\ x\ h = (\ {*f*}\ f)$ (*hypreal-of-real* $x + h$) $-$ *hypreal-of-real* ($f\ x$)
⟨*proof*⟩

lemma *incrementI2*:
NSDERIV $f\ x$:> $D \Longrightarrow$
increment $f\ x\ h = (\ {*f*}\ f)$ (*hypreal-of-real* $x + h$) $-$ *hypreal-of-real* ($f\ x$)
⟨*proof*⟩

The Increment theorem – Keisler p. 65.

lemma *increment-thm*:
assumes *NSDERIV* $f\ x$:> $D\ h \in$ *Infinitesimal* $h \neq 0$
shows $\exists e \in$ *Infinitesimal*. *increment* $f\ x\ h$ = *hypreal-of-real* $D * h + e * h$
⟨*proof*⟩

lemma *increment-approx-zero*: *NSDERIV* $f\ x$:> $D \Longrightarrow h \approx 0 \Longrightarrow h \neq 0 \Longrightarrow$ *increment* $f\ x\ h \approx 0$
⟨*proof*⟩

end

14 Nonstandard Extensions of Transcendental Functions

theory *HTranscendental*

imports *Complex-Main HSeries HDeriv*
begin

definition
exphr :: *real* ⇒ *hypreal* **where**
— define exponential function using standard part
$exphr\ x \equiv\ st(sumhr\ (0,\ whn,\ \lambda n.\ inverse\ (fact\ n) * (x\ \hat{}\ n)))$

definition
sinhr :: *real* ⇒ *hypreal* **where**
$sinhr\ x \equiv st(sumhr\ (0,\ whn,\ \lambda n.\ sin\text{-}coeff\ n * x\ \hat{}\ n))$

definition
coshr :: *real* ⇒ *hypreal* **where**
$coshr\ x \equiv st(sumhr\ (0,\ whn,\ \lambda n.\ cos\text{-}coeff\ n * x\ \hat{}\ n))$

14.1 Nonstandard Extension of Square Root Function

lemma *STAR-sqrt-zero* [*simp*]: $(*f* \ sqrt)\ 0 = 0$
⟨*proof*⟩

lemma *STAR-sqrt-one* [*simp*]: $(*f* \ sqrt)\ 1 = 1$
⟨*proof*⟩

lemma *hypreal-sqrt-pow2-iff*: $((*f* \ sqrt)(x)\ \hat{}\ 2 = x) = (0 \le x)$
⟨*proof*⟩

lemma *hypreal-sqrt-gt-zero-pow2*: $\bigwedge x.\ 0 < x \Longrightarrow (*f* \ sqrt)\ (x)\ \hat{}\ 2 = x$
⟨*proof*⟩

lemma *hypreal-sqrt-pow2-gt-zero*: $0 < x \Longrightarrow 0 < (*f* \ sqrt)\ (x)\ \hat{}\ 2$
⟨*proof*⟩

lemma *hypreal-sqrt-not-zero*: $0 < x \Longrightarrow (*f* \ sqrt)\ (x) \neq 0$
⟨*proof*⟩

lemma *hypreal-inverse-sqrt-pow2*:
$0 < x \Longrightarrow inverse\ ((*f* \ sqrt)(x))\ \hat{}\ 2 = inverse\ x$
⟨*proof*⟩

lemma *hypreal-sqrt-mult-distrib*:
$\bigwedge x\ y.\ [\![0 < x;\ 0 < y]\!] \Longrightarrow$
$(*f* \ sqrt)(x*y) = (*f* \ sqrt)(x) * (*f* \ sqrt)(y)$
⟨*proof*⟩

lemma *hypreal-sqrt-mult-distrib2*:
$[\![0 \le x;\ 0 \le y]\!] \Longrightarrow\ (*f* \ sqrt)(x*y) = (*f* \ sqrt)(x) * (*f* \ sqrt)(y)$
⟨*proof*⟩

lemma *hypreal-sqrt-approx-zero* [*simp*]:
assumes $0 < x$
shows $((*f* \ sqrt)\ x \approx 0) \longleftrightarrow (x \approx 0)$
⟨*proof*⟩

lemma *hypreal-sqrt-approx-zero2* [*simp*]:
$0 \leq x \Longrightarrow ((*f* \ sqrt)(x) \approx 0) = (x \approx 0)$
⟨*proof*⟩

lemma *hypreal-sqrt-gt-zero*: $\bigwedge x.\ 0 < x \Longrightarrow 0 < (*f* \ sqrt)(x)$
⟨*proof*⟩

lemma *hypreal-sqrt-ge-zero*: $0 \leq x \Longrightarrow 0 \leq (*f* \ sqrt)(x)$
⟨*proof*⟩

lemma *hypreal-sqrt-lessI*:
$\bigwedge x\ u.\ [\![0 < u;\ x < u^2]\!] \Longrightarrow (*f* \ sqrt)\ x < u$
⟨*proof*⟩

lemma *hypreal-sqrt-hrabs* [*simp*]: $\bigwedge x.\ (*f* \ sqrt)(x^2) = |x|$
⟨*proof*⟩

lemma *hypreal-sqrt-hrabs2* [*simp*]: $\bigwedge x.\ (*f* \ sqrt)(x*x) = |x|$
⟨*proof*⟩

lemma *hypreal-sqrt-hyperpow-hrabs* [*simp*]:
$\bigwedge x.\ (*f* \ sqrt)(x\ pow\ (hypnat\text{-}of\text{-}nat\ 2)) = |x|$
⟨*proof*⟩

lemma *star-sqrt-HFinite*: $[\![x \in HFinite;\ 0 \leq x]\!] \Longrightarrow (*f* \ sqrt)\ x \in HFinite$
⟨*proof*⟩

lemma *st-hypreal-sqrt*:
assumes $x \in HFinite\ 0 \leq x$
shows $st((*f* \ sqrt)\ x) = (*f* \ sqrt)(st\ x)$
⟨*proof*⟩

lemma *hypreal-sqrt-sum-squares-ge1* [*simp*]: $\bigwedge x\ y.\ x \leq (*f* \ sqrt)(x^2 + y^2)$
⟨*proof*⟩

lemma *HFinite-hypreal-sqrt-imp-HFinite*:
$[\![0 \leq x;\ (*f* \ sqrt)\ x \in HFinite]\!] \Longrightarrow x \in HFinite$
⟨*proof*⟩

lemma *HFinite-hypreal-sqrt-iff* [*simp*]:
$0 \leq x \Longrightarrow ((*f* \ sqrt)\ x \in HFinite) = (x \in HFinite)$
⟨*proof*⟩

lemma *Infinitesimal-hypreal-sqrt*:

$[\![0 \leq x;\ x \in \mathit{Infinitesimal}]\!] \Longrightarrow (\ {*f*}\ \mathit{sqrt})\ x \in \mathit{Infinitesimal}$
⟨proof⟩

lemma *Infinitesimal-hypreal-sqrt-imp-Infinitesimal*:
$[\![0 \leq x;\ (\ {*f*}\ \mathit{sqrt})\ x \in \mathit{Infinitesimal}]\!] \Longrightarrow x \in \mathit{Infinitesimal}$
⟨proof⟩

lemma *Infinitesimal-hypreal-sqrt-iff* [*simp*]:
$0 \leq x \Longrightarrow ((\ {*f*}\ \mathit{sqrt})\ x \in \mathit{Infinitesimal}) = (x \in \mathit{Infinitesimal})$
⟨proof⟩

lemma *HInfinite-hypreal-sqrt*:
$[\![0 \leq x;\ x \in \mathit{HInfinite}]\!] \Longrightarrow (\ {*f*}\ \mathit{sqrt})\ x \in \mathit{HInfinite}$
⟨proof⟩

lemma *HInfinite-hypreal-sqrt-imp-HInfinite*:
$[\![0 \leq x;\ (\ {*f*}\ \mathit{sqrt})\ x \in \mathit{HInfinite}]\!] \Longrightarrow x \in \mathit{HInfinite}$
⟨proof⟩

lemma *HInfinite-hypreal-sqrt-iff* [*simp*]:
$0 \leq x \Longrightarrow ((\ {*f*}\ \mathit{sqrt})\ x \in \mathit{HInfinite}) = (x \in \mathit{HInfinite})$
⟨proof⟩

lemma *HFinite-exp* [*simp*]:
$\mathit{sumhr}\ (0,\ \mathit{whn},\ \lambda n.\ \mathit{inverse}\ (\mathit{fact}\ n) * x \mathbin{\hat{}} n) \in \mathit{HFinite}$
⟨proof⟩

lemma *exphr-zero* [*simp*]: $\mathit{exphr}\ 0 = 1$
⟨proof⟩

lemma *coshr-zero* [*simp*]: $\mathit{coshr}\ 0 = 1$
⟨proof⟩

lemma *STAR-exp-zero-approx-one* [*simp*]: $(\ {*f*}\ \mathit{exp})\ (0{::}\mathit{hypreal}) \approx 1$
⟨proof⟩

lemma *STAR-exp-Infinitesimal*:
assumes $x \in \mathit{Infinitesimal}$ **shows** $(\ {*f*}\ \mathit{exp})\ (x{::}\mathit{hypreal}) \approx 1$
⟨proof⟩

lemma *STAR-exp-epsilon* [*simp*]: $(\ {*f*}\ \mathit{exp})\ \varepsilon \approx 1$
⟨proof⟩

lemma *STAR-exp-add*:
$\bigwedge(x{::}'a{::}\ \{\mathit{banach},\mathit{real\text{-}normed\text{-}field}\}\ \mathit{star})\ y.\ (\ {*f*}\ \mathit{exp})(x + y) = (\ {*f*}\ \mathit{exp})\ x * (\ {*f*}\ \mathit{exp})\ y$
⟨proof⟩

lemma *exphr-hypreal-of-real-exp-eq*: $\mathit{exphr}\ x = \mathit{hypreal\text{-}of\text{-}real}\ (\mathit{exp}\ x)$

⟨*proof*⟩

lemma *starfun-exp-ge-add-one-self* [*simp*]: $\bigwedge x{::}hypreal.\ 0 \leq x \Longrightarrow (1 + x) \leq (*f* exp)\ x$
⟨*proof*⟩

exp maps infinities to infinities

lemma *starfun-exp-HInfinite*:
fixes $x :: hypreal$
assumes $x \in HInfinite\ 0 \leq x$
shows $(*f* exp)\ x \in HInfinite$
⟨*proof*⟩

lemma *starfun-exp-minus*:
$\bigwedge x{::}'a{::}\ \{banach,real\text{-}normed\text{-}field\}\ star.\ (*f* exp)\ (-x) = inverse((*f* exp)\ x)$
⟨*proof*⟩

exp maps infinitesimals to infinitesimals

lemma *starfun-exp-Infinitesimal*:
fixes $x :: hypreal$
assumes $x \in HInfinite\ x \leq 0$
shows $(*f* exp)\ x \in Infinitesimal$
⟨*proof*⟩

lemma *starfun-exp-gt-one* [*simp*]: $\bigwedge x{::}hypreal.\ 0 < x \Longrightarrow 1 < (*f* exp)\ x$
⟨*proof*⟩

abbreviation *real-ln* :: $real \Rightarrow real$ **where**
$real\text{-}ln \equiv ln$

lemma *starfun-ln-exp* [*simp*]: $\bigwedge x.\ (*f* real\text{-}ln)\ ((*f* exp)\ x) = x$
⟨*proof*⟩

lemma *starfun-exp-ln-iff* [*simp*]: $\bigwedge x.\ ((*f* exp)((*f* real\text{-}ln)\ x) = x) = (0 < x)$
⟨*proof*⟩

lemma *starfun-exp-ln-eq*: $\bigwedge u\ x.\ (*f* exp)\ u = x \Longrightarrow (*f* real\text{-}ln)\ x = u$
⟨*proof*⟩

lemma *starfun-ln-less-self* [*simp*]: $\bigwedge x.\ 0 < x \Longrightarrow (*f* real\text{-}ln)\ x < x$
⟨*proof*⟩

lemma *starfun-ln-ge-zero* [*simp*]: $\bigwedge x.\ 1 \leq x \Longrightarrow 0 \leq (*f* real\text{-}ln)\ x$
⟨*proof*⟩

lemma *starfun-ln-gt-zero* [*simp*]: $\bigwedge x\ .1 < x \Longrightarrow 0 < (*f* real\text{-}ln)\ x$
⟨*proof*⟩

lemma *starfun-ln-not-eq-zero* [*simp*]: $\bigwedge x.\ [\![0 < x;\ x \neq 1]\!] \Longrightarrow (*f* real\text{-}ln)\ x \neq 0$

⟨proof⟩

lemma *starfun-ln-HFinite*: $[\![x \in \mathit{HFinite};\ 1 \le x]\!] \Longrightarrow (\ {*f*}\ \mathit{real\text{-}ln})\ x \in \mathit{HFinite}$
⟨proof⟩

lemma *starfun-ln-inverse*: $\bigwedge x.\ 0 < x \Longrightarrow (\ {*f*}\ \mathit{real\text{-}ln})\ (\mathit{inverse}\ x) = -(\ {*f*}\ \mathit{ln})\ x$
⟨proof⟩

lemma *starfun-abs-exp-cancel*: $\bigwedge x.\ |(\ {*f*}\ \mathit{exp})\ (x{::}\mathit{hypreal})| = (\ {*f*}\ \mathit{exp})\ x$
⟨proof⟩

lemma *starfun-exp-less-mono*: $\bigwedge x\ y{::}\mathit{hypreal}.\ x < y \Longrightarrow (\ {*f*}\ \mathit{exp})\ x < (\ {*f*}\ \mathit{exp})\ y$
⟨proof⟩

lemma *starfun-exp-HFinite*:
fixes $x :: \mathit{hypreal}$
assumes $x \in \mathit{HFinite}$
shows $(\ {*f*}\ \mathit{exp})\ x \in \mathit{HFinite}$
⟨proof⟩

lemma *starfun-exp-add-HFinite-Infinitesimal-approx*:
fixes $x :: \mathit{hypreal}$
shows $[\![x \in \mathit{Infinitesimal};\ z \in \mathit{HFinite}]\!] \Longrightarrow (\ {*f*}\ \mathit{exp})\ (z + x{::}\mathit{hypreal}) \approx (\ {*f*}\ \mathit{exp})\ z$
⟨proof⟩

lemma *starfun-ln-HInfinite*:
$[\![x \in \mathit{HInfinite};\ 0 < x]\!] \Longrightarrow (\ {*f*}\ \mathit{real\text{-}ln})\ x \in \mathit{HInfinite}$
⟨proof⟩

lemma *starfun-exp-HInfinite-Infinitesimal-disj*:
fixes $x :: \mathit{hypreal}$
shows $x \in \mathit{HInfinite} \Longrightarrow (\ {*f*}\ \mathit{exp})\ x \in \mathit{HInfinite} \vee (\ {*f*}\ \mathit{exp})\ (x{::}\mathit{hypreal}) \in \mathit{Infinitesimal}$
⟨proof⟩

lemma *starfun-ln-HFinite-not-Infinitesimal*:
$[\![x \in \mathit{HFinite} - \mathit{Infinitesimal};\ 0 < x]\!] \Longrightarrow (\ {*f*}\ \mathit{real\text{-}ln})\ x \in \mathit{HFinite}$
⟨proof⟩

lemma *starfun-ln-Infinitesimal-HInfinite*:
assumes $x \in \mathit{Infinitesimal}\ 0 < x$
shows $(\ {*f*}\ \mathit{real\text{-}ln})\ x \in \mathit{HInfinite}$
⟨proof⟩

lemma *starfun-ln-less-zero*: $\bigwedge x.\ [\![0 < x;\ x < 1]\!] \Longrightarrow (\ {*f*}\ \mathit{real\text{-}ln})\ x < 0$

⟨*proof*⟩

lemma *starfun-ln-Infinitesimal-less-zero*:
$[\![x \in \mathit{Infinitesimal};\ 0 < x]\!] \Longrightarrow (\ {*}f{*}\ \mathit{real\text{-}ln})\ x < 0$
⟨*proof*⟩

lemma *starfun-ln-HInfinite-gt-zero*:
$[\![x \in \mathit{HInfinite};\ 0 < x]\!] \Longrightarrow 0 < (\ {*}f{*}\ \mathit{real\text{-}ln})\ x$
⟨*proof*⟩

lemma *HFinite-sin* [*simp*]: $\mathit{sumhr}\ (0,\ \mathit{whn},\ \lambda n.\ \mathit{sin\text{-}coeff}\ n * x \,\widehat{}\, n) \in \mathit{HFinite}$
⟨*proof*⟩

lemma *STAR-sin-zero* [*simp*]: $(\ {*}f{*}\ \mathit{sin})\ 0 = 0$
⟨*proof*⟩

lemma *STAR-sin-Infinitesimal* [*simp*]:
fixes $x :: {'a}::\{\mathit{real\text{-}normed\text{-}field},\mathit{banach}\}\ \mathit{star}$
assumes $x \in \mathit{Infinitesimal}$
shows $(\ {*}f{*}\ \mathit{sin})\ x \approx x$
⟨*proof*⟩

lemma *HFinite-cos* [*simp*]: $\mathit{sumhr}\ (0,\ \mathit{whn},\ \lambda n.\ \mathit{cos\text{-}coeff}\ n * x \,\widehat{}\, n) \in \mathit{HFinite}$
⟨*proof*⟩

lemma *STAR-cos-zero* [*simp*]: $(\ {*}f{*}\ \mathit{cos})\ 0 = 1$
⟨*proof*⟩

lemma *STAR-cos-Infinitesimal* [*simp*]:
fixes $x :: {'a}::\{\mathit{real\text{-}normed\text{-}field},\mathit{banach}\}\ \mathit{star}$
assumes $x \in \mathit{Infinitesimal}$
shows $(\ {*}f{*}\ \mathit{cos})\ x \approx 1$
⟨*proof*⟩

lemma *STAR-tan-zero* [*simp*]: $(\ {*}f{*}\ \mathit{tan})\ 0 = 0$
⟨*proof*⟩

lemma *STAR-tan-Infinitesimal* [*simp*]:
assumes $x \in \mathit{Infinitesimal}$
shows $(\ {*}f{*}\ \mathit{tan})\ x \approx x$
⟨*proof*⟩

lemma *STAR-sin-cos-Infinitesimal-mult*:
fixes $x :: {'a}::\{\mathit{real\text{-}normed\text{-}field},\mathit{banach}\}\ \mathit{star}$
shows $x \in \mathit{Infinitesimal} \Longrightarrow (\ {*}f{*}\ \mathit{sin})\ x * (\ {*}f{*}\ \mathit{cos})\ x \approx x$
⟨*proof*⟩

lemma *HFinite-pi*: $\mathit{hypreal\text{-}of\text{-}real}\ \mathit{pi} \in \mathit{HFinite}$

⟨*proof*⟩

lemma *STAR-sin-Infinitesimal-divide*:
fixes $x :: 'a::\{real\text{-}normed\text{-}field,banach\}$ *star*
shows $[\![x \in Infinitesimal;\ x \neq 0]\!] \Longrightarrow (\ {*f*}\ sin)\ x/x \approx 1$
⟨*proof*⟩

14.2 Proving $\sin *(1/n) \times 1/(1/n) \approx 1$ for $n = \infty$

lemma *lemma-sin-pi*:
$n \in HNatInfinite$
$\Longrightarrow (\ {*f*}\ sin)\ (inverse\ (hypreal\text{-}of\text{-}hypnat\ n))/(inverse\ (hypreal\text{-}of\text{-}hypnat\ n))$
≈ 1
⟨*proof*⟩

lemma *STAR-sin-inverse-HNatInfinite*:
$n \in HNatInfinite$
$\Longrightarrow (\ {*f*}\ sin)\ (inverse\ (hypreal\text{-}of\text{-}hypnat\ n)) * hypreal\text{-}of\text{-}hypnat\ n \approx 1$
⟨*proof*⟩

lemma *Infinitesimal-pi-divide-HNatInfinite*:
$N \in HNatInfinite$
$\Longrightarrow hypreal\text{-}of\text{-}real\ pi/(hypreal\text{-}of\text{-}hypnat\ N) \in Infinitesimal$
⟨*proof*⟩

lemma *pi-divide-HNatInfinite-not-zero* [*simp*]:
$N \in HNatInfinite \Longrightarrow hypreal\text{-}of\text{-}real\ pi/(hypreal\text{-}of\text{-}hypnat\ N) \neq 0$
⟨*proof*⟩

lemma *STAR-sin-pi-divide-HNatInfinite-approx-pi*:
assumes $n \in HNatInfinite$
shows $({*f*}\ sin)\ (hypreal\text{-}of\text{-}real\ pi\ /\ hypreal\text{-}of\text{-}hypnat\ n) * hypreal\text{-}of\text{-}hypnat\ n$
$\approx$
$hypreal\text{-}of\text{-}real\ pi$
⟨*proof*⟩

lemma *STAR-sin-pi-divide-HNatInfinite-approx-pi2*:
$n \in HNatInfinite$
$\Longrightarrow hypreal\text{-}of\text{-}hypnat\ n * (\ {*f*}\ sin)\ (hypreal\text{-}of\text{-}real\ pi/(hypreal\text{-}of\text{-}hypnat\ n))$
$\approx hypreal\text{-}of\text{-}real\ pi$
⟨*proof*⟩

lemma *starfunNat-pi-divide-n-Infinitesimal*:
$N \in HNatInfinite \Longrightarrow (\ {*f*}\ (\lambda x.\ pi\ /\ real\ x))\ N \in Infinitesimal$
⟨*proof*⟩

lemma *STAR-sin-pi-divide-n-approx*:
assumes $N \in HNatInfinite$

shows $(*f* \; sin) \; ((*f* \; (\lambda x. \; pi \; / \; real \; x)) \; N) \approx hypreal\text{-}of\text{-}real \; pi/(hypreal\text{-}of\text{-}hypnat \; N)$
⟨*proof*⟩

lemma *NSLIMSEQ-sin-pi*: $(\lambda n. \; real \; n * sin \; (pi \; / \; real \; n)) \longrightarrow_{NS} pi$
⟨*proof*⟩

lemma *NSLIMSEQ-cos-one*: $(\lambda n. \; cos \; (pi \; / \; real \; n)) \longrightarrow_{NS} 1$
⟨*proof*⟩

lemma *NSLIMSEQ-sin-cos-pi*:
$(\lambda n. \; real \; n * sin \; (pi \; / \; real \; n) * cos \; (pi \; / \; real \; n)) \longrightarrow_{NS} pi$
⟨*proof*⟩

A familiar approximation to *cos x* when *x* is small

lemma *STAR-cos-Infinitesimal-approx*:
fixes $x :: 'a::\{real\text{-}normed\text{-}field,banach\} \; star$
shows $x \in Infinitesimal \Longrightarrow (*f* \; cos) \; x \approx 1 - x^2$
⟨*proof*⟩

lemma *STAR-cos-Infinitesimal-approx2*:
fixes $x :: hypreal$
assumes $x \in Infinitesimal$
shows $(*f* \; cos) \; x \approx 1 - (x^2)/2$
⟨*proof*⟩

end

15 Non-Standard Complex Analysis

theory *NSCA*
imports *NSComplex HTranscendental*
begin

abbreviation

SComplex :: *hcomplex set* **where**
SComplex ≡ *Standard*

definition — standard part map
stc :: *hcomplex* => *hcomplex* **where**
$stc \; x = (SOME \; r. \; x \in HFinite \wedge r \in SComplex \wedge r \approx x)$

15.1 Closure Laws for SComplex, the Standard Complex Numbers

lemma *SComplex-minus-iff* [*simp*]: $(-x \in SComplex) = (x \in SComplex)$
⟨*proof*⟩

lemma *SComplex-add-cancel*:
$[\![x + y \in SComplex;\ y \in SComplex]\!] \Longrightarrow x \in SComplex$
⟨*proof*⟩

lemma *SReal-hcmod-hcomplex-of-complex* [*simp*]:
$hcmod\ (hcomplex\text{-}of\text{-}complex\ r) \in \mathbb{R}$
⟨*proof*⟩

lemma *SReal-hcmod-numeral*: $hcmod\ (numeral\ w\ ::hcomplex) \in \mathbb{R}$
⟨*proof*⟩

lemma *SReal-hcmod-SComplex*: $x \in SComplex \Longrightarrow hcmod\ x \in \mathbb{R}$
⟨*proof*⟩

lemma *SComplex-divide-numeral*:
$r \in SComplex \Longrightarrow r/(numeral\ w::hcomplex) \in SComplex$
⟨*proof*⟩

lemma *SComplex-UNIV-complex*:
$\{x.\ hcomplex\text{-}of\text{-}complex\ x \in SComplex\} = (UNIV::complex\ set)$
⟨*proof*⟩

lemma *SComplex-iff*: $(x \in SComplex) = (\exists\, y.\ x = hcomplex\text{-}of\text{-}complex\ y)$
⟨*proof*⟩

lemma *hcomplex-of-complex-image*:
$range\ hcomplex\text{-}of\text{-}complex = SComplex$
⟨*proof*⟩

lemma *inv-hcomplex-of-complex-image*: $inv\ hcomplex\text{-}of\text{-}complex\ `SComplex = UNIV$
⟨*proof*⟩

lemma *SComplex-hcomplex-of-complex-image*:
$[\![\exists\, x.\ x \in P;\ P \le SComplex]\!] \Longrightarrow \exists\, Q.\ P = hcomplex\text{-}of\text{-}complex\ `\ Q$
⟨*proof*⟩

lemma *SComplex-SReal-dense*:
$[\![x \in SComplex;\ y \in SComplex;\ hcmod\ x < hcmod\ y$
$]\!] \Longrightarrow \exists\, r \in Reals.\ hcmod\ x< r \wedge r < hcmod\ y$
⟨*proof*⟩

15.2 The Finite Elements form a Subring

lemma *HFinite-hcmod-hcomplex-of-complex* [*simp*]:
$hcmod\ (hcomplex\text{-}of\text{-}complex\ r) \in HFinite$
⟨*proof*⟩

lemma *HFinite-hcmod-iff* [*simp*]: $hcmod\ x \in HFinite \longleftrightarrow x \in HFinite$
⟨*proof*⟩

lemma *HFinite-bounded-hcmod*:
$[\![x \in HFinite;\ y \le hcmod\ x;\ 0 \le y]\!] \Longrightarrow y \in HFinite$
⟨*proof*⟩

15.3 The Complex Infinitesimals form a Subring

lemma *Infinitesimal-hcmod-iff*:
$(z \in Infinitesimal) = (hcmod\ z \in Infinitesimal)$
⟨*proof*⟩

lemma *HInfinite-hcmod-iff*: $(z \in HInfinite) = (hcmod\ z \in HInfinite)$
⟨*proof*⟩

lemma *HFinite-diff-Infinitesimal-hcmod*:
$x \in HFinite - Infinitesimal \Longrightarrow hcmod\ x \in HFinite - Infinitesimal$
⟨*proof*⟩

lemma *hcmod-less-Infinitesimal*:
$[\![e \in Infinitesimal;\ hcmod\ x < hcmod\ e]\!] \Longrightarrow x \in Infinitesimal$
⟨*proof*⟩

lemma *hcmod-le-Infinitesimal*:
$[\![e \in Infinitesimal;\ hcmod\ x \le hcmod\ e]\!] \Longrightarrow x \in Infinitesimal$
⟨*proof*⟩

15.4 The "Infinitely Close" Relation

lemma *approx-SComplex-mult-cancel-zero*:
$[\![a \in SComplex;\ a \neq 0;\ a*x \approx 0]\!] \Longrightarrow x \approx 0$
⟨*proof*⟩

lemma *approx-mult-SComplex1*: $[\![a \in SComplex;\ x \approx 0]\!] \Longrightarrow x*a \approx 0$
⟨*proof*⟩

lemma *approx-mult-SComplex2*: $[\![a \in SComplex;\ x \approx 0]\!] \Longrightarrow a*x \approx 0$
⟨*proof*⟩

lemma *approx-mult-SComplex-zero-cancel-iff* [*simp*]:
$[\![a \in SComplex;\ a \neq 0]\!] \Longrightarrow (a*x \approx 0) = (x \approx 0)$
⟨*proof*⟩

lemma *approx-SComplex-mult-cancel*:
$[\![a \in SComplex;\ a \neq 0;\ a*w \approx a*z]\!] \Longrightarrow w \approx z$
⟨*proof*⟩

lemma *approx-SComplex-mult-cancel-iff1* [*simp*]:
$[\![a \in SComplex;\ a \neq 0]\!] \Longrightarrow (a*w \approx a*z) = (w \approx z)$
⟨*proof*⟩

lemma *approx-hcmod-approx-zero*: $(x \approx y) = (hcmod\ (y - x) \approx 0)$
⟨*proof*⟩

lemma *approx-approx-zero-iff*: $(x \approx 0) = (hcmod\ x \approx 0)$
⟨*proof*⟩

lemma *approx-minus-zero-cancel-iff* [*simp*]: $(-x \approx 0) = (x \approx 0)$
⟨*proof*⟩

lemma *Infinitesimal-hcmod-add-diff*:
$u \approx 0 \Longrightarrow hcmod(x + u) - hcmod\ x \in Infinitesimal$
⟨*proof*⟩

lemma *approx-hcmod-add-hcmod*: $u \approx 0 \Longrightarrow hcmod(x + u) \approx hcmod\ x$
⟨*proof*⟩

15.5 Zero is the Only Infinitesimal Complex Number

lemma *Infinitesimal-less-SComplex*:
$[\![x \in SComplex;\ y \in Infinitesimal;\ 0 < hcmod\ x]\!] \Longrightarrow hcmod\ y < hcmod\ x$
⟨*proof*⟩

lemma *SComplex-Int-Infinitesimal-zero*: $SComplex\ Int\ Infinitesimal = \{0\}$
⟨*proof*⟩

lemma *SComplex-Infinitesimal-zero*:
$[\![x \in SComplex;\ x \in Infinitesimal]\!] \Longrightarrow x = 0$
⟨*proof*⟩

lemma *SComplex-HFinite-diff-Infinitesimal*:
$[\![x \in SComplex;\ x \neq 0]\!] \Longrightarrow x \in HFinite - Infinitesimal$
⟨*proof*⟩

lemma *numeral-not-Infinitesimal* [*simp*]:
$numeral\ w \neq (0::hcomplex) \Longrightarrow (numeral\ w::hcomplex) \notin Infinitesimal$
⟨*proof*⟩

lemma *approx-SComplex-not-zero*:
$[\![y \in SComplex;\ x \approx y;\ y\neq 0]\!] \Longrightarrow x \neq 0$
⟨*proof*⟩

lemma *SComplex-approx-iff*:
$[\![x \in SComplex;\ y \in SComplex]\!] \Longrightarrow (x \approx y) = (x = y)$
⟨*proof*⟩

lemma *approx-unique-complex*:
$[\![r \in SComplex;\ s \in SComplex;\ r \approx x;\ s \approx x]\!] \Longrightarrow r = s$

$\langle proof \rangle$

15.6 Properties of *hRe*, *hIm* and *HComplex*

lemma *abs-hRe-le-hcmod*: $\bigwedge x.\ |hRe\ x| \leq hcmod\ x$
$\langle proof \rangle$

lemma *abs-hIm-le-hcmod*: $\bigwedge x.\ |hIm\ x| \leq hcmod\ x$
$\langle proof \rangle$

lemma *Infinitesimal-hRe*: $x \in Infinitesimal \Longrightarrow hRe\ x \in Infinitesimal$
$\langle proof \rangle$

lemma *Infinitesimal-hIm*: $x \in Infinitesimal \Longrightarrow hIm\ x \in Infinitesimal$
$\langle proof \rangle$

lemma *Infinitesimal-HComplex*:
assumes x: $x \in Infinitesimal$ **and** y: $y \in Infinitesimal$
shows $HComplex\ x\ y \in Infinitesimal$
$\langle proof \rangle$

lemma *hcomplex-Infinitesimal-iff*:
$(x \in Infinitesimal) \longleftrightarrow (hRe\ x \in Infinitesimal \wedge hIm\ x \in Infinitesimal)$
$\langle proof \rangle$

lemma *hRe-diff* [*simp*]: $\bigwedge x\ y.\ hRe\ (x - y) = hRe\ x - hRe\ y$
$\langle proof \rangle$

lemma *hIm-diff* [*simp*]: $\bigwedge x\ y.\ hIm\ (x - y) = hIm\ x - hIm\ y$
$\langle proof \rangle$

lemma *approx-hRe*: $x \approx y \Longrightarrow hRe\ x \approx hRe\ y$
$\langle proof \rangle$

lemma *approx-hIm*: $x \approx y \Longrightarrow hIm\ x \approx hIm\ y$
$\langle proof \rangle$

lemma *approx-HComplex*:
$[\![a \approx b;\ c \approx d]\!] \Longrightarrow HComplex\ a\ c \approx HComplex\ b\ d$
$\langle proof \rangle$

lemma *hcomplex-approx-iff*:
$(x \approx y) = (hRe\ x \approx hRe\ y \wedge hIm\ x \approx hIm\ y)$
$\langle proof \rangle$

lemma *HFinite-hRe*: $x \in HFinite \Longrightarrow hRe\ x \in HFinite$
$\langle proof \rangle$

lemma *HFinite-hIm*: $x \in HFinite \Longrightarrow hIm\ x \in HFinite$

⟨proof⟩

lemma *HFinite-HComplex*:
assumes $x \in \mathit{HFinite}$ $y \in \mathit{HFinite}$
shows $\mathit{HComplex}\ x\ y \in \mathit{HFinite}$
⟨proof⟩

lemma *hcomplex-HFinite-iff*:
$(x \in \mathit{HFinite}) = (\mathit{hRe}\ x \in \mathit{HFinite} \wedge \mathit{hIm}\ x \in \mathit{HFinite})$
⟨proof⟩

lemma *hcomplex-HInfinite-iff*:
$(x \in \mathit{HInfinite}) = (\mathit{hRe}\ x \in \mathit{HInfinite} \vee \mathit{hIm}\ x \in \mathit{HInfinite})$
⟨proof⟩

lemma *hcomplex-of-hypreal-approx-iff* [*simp*]:
$(\mathit{hcomplex\text{-}of\text{-}hypreal}\ x \approx \mathit{hcomplex\text{-}of\text{-}hypreal}\ z) = (x \approx z)$
⟨proof⟩

lemma *stc-part-Ex*:
assumes $x \in \mathit{HFinite}$
shows $\exists\, t \in \mathit{SComplex}.\ x \approx t$
⟨proof⟩

lemma *stc-part-Ex1*: $x \in \mathit{HFinite} \Longrightarrow \exists!t.\ t \in \mathit{SComplex} \wedge x \approx t$
⟨proof⟩

15.7 Theorems About Monads

lemma *monad-zero-hcmod-iff*: $(x \in \mathit{monad}\ 0) = (\mathit{hcmod}\ x \in \mathit{monad}\ 0)$
⟨proof⟩

15.8 Theorems About Standard Part

lemma *stc-approx-self*: $x \in \mathit{HFinite} \Longrightarrow \mathit{stc}\ x \approx x$
⟨proof⟩

lemma *stc-SComplex*: $x \in \mathit{HFinite} \Longrightarrow \mathit{stc}\ x \in \mathit{SComplex}$
⟨proof⟩

lemma *stc-HFinite*: $x \in \mathit{HFinite} \Longrightarrow \mathit{stc}\ x \in \mathit{HFinite}$
⟨proof⟩

lemma *stc-unique*: $[\![y \in \mathit{SComplex};\ y \approx x]\!] \Longrightarrow \mathit{stc}\ x = y$
⟨proof⟩

lemma *stc-SComplex-eq* [*simp*]: $x \in \mathit{SComplex} \Longrightarrow \mathit{stc}\ x = x$
⟨proof⟩

lemma *stc-eq-approx*:
$[\![x \in HFinite;\ y \in HFinite;\ stc\ x = stc\ y]\!] \Longrightarrow x \approx y$
$\langle proof \rangle$

lemma *approx-stc-eq*:
$[\![x \in HFinite;\ y \in HFinite;\ x \approx y]\!] \Longrightarrow stc\ x = stc\ y$
$\langle proof \rangle$

lemma *stc-eq-approx-iff*:
$[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow (x \approx y) = (stc\ x = stc\ y)$
$\langle proof \rangle$

lemma *stc-Infinitesimal-add-SComplex*:
$[\![x \in SComplex;\ e \in Infinitesimal]\!] \Longrightarrow stc(x + e) = x$
$\langle proof \rangle$

lemma *stc-Infinitesimal-add-SComplex2*:
$[\![x \in SComplex;\ e \in Infinitesimal]\!] \Longrightarrow stc(e + x) = x$
$\langle proof \rangle$

lemma *HFinite-stc-Infinitesimal-add*:
$x \in HFinite \Longrightarrow \exists\, e \in Infinitesimal.\ x = stc(x) + e$
$\langle proof \rangle$

lemma *stc-add*:
$[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow stc\ (x + y) = stc(x) + stc(y)$
$\langle proof \rangle$

lemma *stc-zero*: $stc\ 0 = 0$
$\langle proof \rangle$

lemma *stc-one*: $stc\ 1 = 1$
$\langle proof \rangle$

lemma *stc-minus*: $y \in HFinite \Longrightarrow stc(-y) = -stc(y)$
$\langle proof \rangle$

lemma *stc-diff*:
$[\![x \in HFinite;\ y \in HFinite]\!] \Longrightarrow stc\ (x-y) = stc(x) - stc(y)$
$\langle proof \rangle$

lemma *stc-mult*:
$[\![x \in HFinite;\ y \in HFinite]\!]$
$\Longrightarrow stc\ (x * y) = stc(x) * stc(y)$
$\langle proof \rangle$

lemma *stc-Infinitesimal*: $x \in Infinitesimal \Longrightarrow stc\ x = 0$
$\langle proof \rangle$

lemma *stc-not-Infinitesimal*: $stc(x) \neq 0 \Longrightarrow x \notin Infinitesimal$
⟨*proof*⟩

lemma *stc-inverse*:
$[\![x \in HFinite;\ stc\ x \neq 0]\!] \Longrightarrow stc(inverse\ x) = inverse\ (stc\ x)$
⟨*proof*⟩

lemma *stc-divide* [*simp*]:
$[\![x \in HFinite;\ y \in HFinite;\ stc\ y \neq 0]\!]$
$\Longrightarrow stc(x/y) = (stc\ x)\ /\ (stc\ y)$
⟨*proof*⟩

lemma *stc-idempotent* [*simp*]: $x \in HFinite \Longrightarrow stc(stc(x)) = stc(x)$
⟨*proof*⟩

lemma *HFinite-HFinite-hcomplex-of-hypreal*:
$z \in HFinite \Longrightarrow hcomplex\text{-}of\text{-}hypreal\ z \in HFinite$
⟨*proof*⟩

lemma *SComplex-SReal-hcomplex-of-hypreal*:
$x \in \mathbb{R} \Longrightarrow hcomplex\text{-}of\text{-}hypreal\ x \in SComplex$
⟨*proof*⟩

lemma *stc-hcomplex-of-hypreal*:
$z \in HFinite \Longrightarrow stc(hcomplex\text{-}of\text{-}hypreal\ z) = hcomplex\text{-}of\text{-}hypreal\ (st\ z)$
⟨*proof*⟩

lemma *hmod-stc-eq*:
assumes $x \in HFinite$
shows $hcmod(stc\ x) = st(hcmod\ x)$
⟨*proof*⟩

lemma *Infinitesimal-hcnj-iff* [*simp*]:
$(hcnj\ z \in Infinitesimal) \longleftrightarrow (z \in Infinitesimal)$
⟨*proof*⟩

end

16 Star-transforms in NSA, Extending Sets of Complex Numbers and Complex Functions

theory *CStar*
imports *NSCA*
begin

16.1 Properties of the $*$-Transform Applied to Sets of Reals

lemma *STARC-hcomplex-of-complex-Int*: $*s*$ $X \cap$ *SComplex* = *hcomplex-of-complex* ‘ X
⟨*proof*⟩

lemma *lemma-not-hcomplexA*: $x \notin$ *hcomplex-of-complex* ‘ $A \Longrightarrow \forall y \in A.\ x \neq$ *hcomplex-of-complex* y
⟨*proof*⟩

16.2 Theorems about Nonstandard Extensions of Functions

lemma *starfunC-hcpow*: $\bigwedge Z.$ ($*f*$ $(\lambda z.\ z \mathbin{\hat{}} n))\ Z = Z$ *pow hypnat-of-nat* n
⟨*proof*⟩

lemma *starfunCR-cmod*: $*f*$ *cmod* = *hcmod*
⟨*proof*⟩

16.3 Internal Functions - Some Redundancy With $*f*$ Now

lemma *starfun-Re*: ($*f*$ $(\lambda x.\ Re\ (f\ x))) = (\lambda x.\ hRe\ ((\ *f*\ f)\ x))$
⟨*proof*⟩

lemma *starfun-Im*: ($*f*$ $(\lambda x.\ Im\ (f\ x))) = (\lambda x.\ hIm\ ((\ *f*\ f)\ x))$
⟨*proof*⟩

lemma *starfunC-eq-Re-Im-iff*:
($*f*$ $f)\ x = z \longleftrightarrow$ ($*f*$ $(\lambda x.\ Re\ (f\ x)))\ x = hRe\ z \wedge$ ($*f*$ $(\lambda x.\ Im\ (f\ x)))\ x = hIm\ z$
⟨*proof*⟩

lemma *starfunC-approx-Re-Im-iff*:
($*f*$ $f)\ x \approx z \longleftrightarrow$ ($*f*$ $(\lambda x.\ Re\ (f\ x)))\ x \approx hRe\ z \wedge$ ($*f*$ $(\lambda x.\ Im\ (f\ x)))\ x \approx hIm\ z$
⟨*proof*⟩

end

17 Limits, Continuity and Differentiation for Complex Functions

theory *CLim*
imports *CStar*
begin

declare *epsilon-not-zero* [*simp*]

lemma *lemma-complex-mult-inverse-squared* [*simp*]: $x \neq 0 \Longrightarrow x * (inverse\ x)^2 = inverse\ x$
for x :: *complex*
⟨*proof*⟩

Changing the quantified variable. Install earlier?

lemma *all-shift*: $(\forall x::'a::comm\text{-}ring\text{-}1.\ P\ x) \longleftrightarrow (\forall x.\ P\ (x - a))$
⟨*proof*⟩

lemma *complex-add-minus-iff* [*simp*]: $x + - a = 0 \longleftrightarrow x = a$
for $x\ a$ :: *complex*
⟨*proof*⟩

lemma *complex-add-eq-0-iff* [*iff*]: $x + y = 0 \longleftrightarrow y = - x$
for $x\ y$ :: *complex*
⟨*proof*⟩

17.1 Limit of Complex to Complex Function

lemma *NSLIM-Re*: $f\ -a\rightarrow_{NS}\ L \Longrightarrow (\lambda x.\ Re\ (f\ x))\ -a\rightarrow_{NS}\ Re\ L$
⟨*proof*⟩

lemma *NSLIM-Im*: $f\ -a\rightarrow_{NS}\ L \Longrightarrow (\lambda x.\ Im\ (f\ x))\ -a\rightarrow_{NS}\ Im\ L$
⟨*proof*⟩

lemma *LIM-Re*: $f\ -a\rightarrow\ L \Longrightarrow (\lambda x.\ Re\ (f\ x))\ -a\rightarrow\ Re\ L$
for f :: $'a::real\text{-}normed\text{-}vector \Rightarrow complex$
⟨*proof*⟩

lemma *LIM-Im*: $f\ -a\rightarrow\ L \Longrightarrow (\lambda x.\ Im\ (f\ x))\ -a\rightarrow\ Im\ L$
for f :: $'a::real\text{-}normed\text{-}vector \Rightarrow complex$
⟨*proof*⟩

lemma *LIM-cnj*: $f\ -a\rightarrow\ L \Longrightarrow (\lambda x.\ cnj\ (f\ x))\ -a\rightarrow\ cnj\ L$
for f :: $'a::real\text{-}normed\text{-}vector \Rightarrow complex$
⟨*proof*⟩

lemma *LIM-cnj-iff*: $((\lambda x.\ cnj\ (f\ x))\ -a\rightarrow\ cnj\ L) \longleftrightarrow f\ -a\rightarrow\ L$
for f :: $'a::real\text{-}normed\text{-}vector \Rightarrow complex$
⟨*proof*⟩

lemma *starfun-norm*: $(\ {*f*}\ (\lambda x.\ norm\ (f\ x))) = (\lambda x.\ hnorm\ ((\ {*f*}\ f)\ x))$
⟨*proof*⟩

lemma *star-of-Re* [*simp*]: $star\text{-}of\ (Re\ x) = hRe\ (star\text{-}of\ x)$
⟨*proof*⟩

lemma *star-of-Im* [*simp*]: $star\text{-}of\ (Im\ x) = hIm\ (star\text{-}of\ x)$
⟨*proof*⟩

Another equivalence result.

lemma *NSCLIM-NSCRLIM-iff*: $f\ -x\rightarrow_{NS}\ L \longleftrightarrow (\lambda y.\ cmod\ (f\ y - L))\ -x\rightarrow_{NS}\ 0$
⟨*proof*⟩

Much, much easier standard proof.

lemma *CLIM-CRLIM-iff*: $f\ -x\rightarrow\ L \longleftrightarrow (\lambda y.\ cmod\ (f\ y - L))\ -x\rightarrow\ 0$
for f :: $'a$::*real-normed-vector* $\Rightarrow$ *complex*
⟨*proof*⟩

So this is nicer nonstandard proof.

lemma *NSCLIM-NSCRLIM-iff2*: $f\ -x\rightarrow_{NS}\ L \longleftrightarrow (\lambda y.\ cmod\ (f\ y - L))\ -x\rightarrow_{NS}\ 0$
⟨*proof*⟩

lemma *NSLIM-NSCRLIM-Re-Im-iff*:
$f\ -a\rightarrow_{NS}\ L \longleftrightarrow (\lambda x.\ Re\ (f\ x))\ -a\rightarrow_{NS}\ Re\ L \wedge (\lambda x.\ Im\ (f\ x))\ -a\rightarrow_{NS}\ Im\ L$
⟨*proof*⟩

lemma *LIM-CRLIM-Re-Im-iff*: $f\ -a\rightarrow\ L \longleftrightarrow (\lambda x.\ Re\ (f\ x))\ -a\rightarrow\ Re\ L \wedge (\lambda x.\ Im\ (f\ x))\ -a\rightarrow\ Im\ L$
for f :: $'a$::*real-normed-vector* $\Rightarrow$ *complex*
⟨*proof*⟩

17.2 Continuity

lemma *NSLIM-isContc-iff*: $f\ -a\rightarrow_{NS}\ f\ a \longleftrightarrow (\lambda h.\ f\ (a + h))\ -0\rightarrow_{NS}\ f\ a$
⟨*proof*⟩

17.3 Functions from Complex to Reals

lemma *isNSContCR-cmod* [*simp*]: *isNSCont cmod a*
⟨*proof*⟩

lemma *isContCR-cmod* [*simp*]: *isCont cmod a*
⟨*proof*⟩

lemma *isCont-Re*: $isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ Re\ (f\ x))\ a$
for f :: $'a$::*real-normed-vector* $\Rightarrow$ *complex*
⟨*proof*⟩

lemma *isCont-Im*: $isCont\ f\ a \Longrightarrow isCont\ (\lambda x.\ Im\ (f\ x))\ a$
for f :: $'a$::*real-normed-vector* $\Rightarrow$ *complex*
⟨*proof*⟩

17.4 Differentiation of Natural Number Powers

lemma *CDERIV-pow* [*simp*]: $DERIV\ (\lambda x.\ x\ \hat{}\ n)\ x :> complex\text{-}of\text{-}real\ (real\ n) * (x\ \hat{}\ (n - Suc\ 0))$

⟨*proof*⟩

Nonstandard version.

lemma *NSCDERIV-pow*: $NSDERIV\ (\lambda x.\ x\ \hat{}\ n)\ x :> complex\text{-}of\text{-}real\ (real\ n) * (x\ \hat{}\ (n - 1))$
⟨*proof*⟩

Can't relax the premise $x \neq (0::'a)$: it isn't continuous at zero.

lemma *NSCDERIV-inverse*: $x \neq 0 \Longrightarrow NSDERIV\ (\lambda x.\ inverse\ x)\ x :> - (inverse\ x)^2$
for $x :: complex$
⟨*proof*⟩

lemma *CDERIV-inverse*: $x \neq 0 \Longrightarrow DERIV\ (\lambda x.\ inverse\ x)\ x :> - (inverse\ x)^2$
for $x :: complex$
⟨*proof*⟩

17.5 Derivative of Reciprocals (Function *inverse*)

lemma *CDERIV-inverse-fun*:
$DERIV\ f\ x :> d \Longrightarrow f\ x \neq 0 \Longrightarrow DERIV\ (\lambda x.\ inverse\ (f\ x))\ x :> - (d * inverse\ ((f\ x)^2))$
for $x :: complex$
⟨*proof*⟩

lemma *NSCDERIV-inverse-fun*:
$NSDERIV\ f\ x :> d \Longrightarrow f\ x \neq 0 \Longrightarrow NSDERIV\ (\lambda x.\ inverse\ (f\ x))\ x :> - (d * inverse\ ((f\ x)^2))$
for $x :: complex$
⟨*proof*⟩

17.6 Derivative of Quotient

lemma *CDERIV-quotient*:
$DERIV\ f\ x :> d \Longrightarrow DERIV\ g\ x :> e \Longrightarrow g(x) \neq 0 \Longrightarrow$
$DERIV\ (\lambda y.\ f\ y\ /\ g\ y)\ x :> (d * g\ x - (e * f\ x))\ /\ (g\ x)^2$
for $x :: complex$
⟨*proof*⟩

lemma *NSCDERIV-quotient*:
$NSDERIV\ f\ x :> d \Longrightarrow NSDERIV\ g\ x :> e \Longrightarrow g\ x \neq (0::complex) \Longrightarrow$
$NSDERIV\ (\lambda y.\ f\ y\ /\ g\ y)\ x :> (d * g\ x - (e * f\ x))\ /\ (g\ x)^2$
⟨*proof*⟩

17.7 Caratheodory Formulation of Derivative at a Point: Standard Proof

lemma *CARAT-CDERIVD*:
$(\forall z.\ f\ z - f\ x = g\ z * (z - x)) \wedge isNSCont\ g\ x \wedge g\ x = l \Longrightarrow NSDERIV\ f\ x :> l$

⟨*proof*⟩

end

18 Logarithms: Non-Standard Version

theory *HLog*
imports *HTranscendental*
begin

definition *powhr* :: *hypreal* ⇒ *hypreal* ⇒ *hypreal* (**infixr** *powhr 80*)
where [*transfer-unfold*]: *x powhr a* = *starfun2* (*powr*) *x a*

definition *hlog* :: *hypreal* ⇒ *hypreal* ⇒ *hypreal*
where [*transfer-unfold*]: *hlog a x* = *starfun2 log a x*

lemma *powhr*: (*star-n X*) *powhr* (*star-n Y*) = *star-n* (λ*n*. (*X n*) *powr* (*Y n*))
⟨*proof*⟩

lemma *powhr-one-eq-one* [*simp*]: ⋀*a*. *1 powhr a* = *1*
⟨*proof*⟩

lemma *powhr-mult*: ⋀*a x y*. *0* < *x* ⟹ *0* < *y* ⟹ (*x* ∗ *y*) *powhr a* = (*x powhr a*) ∗ (*y powhr a*)
⟨*proof*⟩

lemma *powhr-gt-zero* [*simp*]: ⋀*a x*. *0* < *x powhr a* ⟷ *x* ≠ *0*
⟨*proof*⟩

lemma *powhr-not-zero* [*simp*]: ⋀*a x*. *x powhr a* ≠ *0* ⟷ *x* ≠ *0*
⟨*proof*⟩

lemma *powhr-divide*: ⋀*a x y*. *0* ≤ *x* ⟹ *0* ≤ *y* ⟹ (*x* / *y*) *powhr a* = (*x powhr a*) / (*y powhr a*)
⟨*proof*⟩

lemma *powhr-add*: ⋀*a b x*. *x powhr* (*a* + *b*) = (*x powhr a*) ∗ (*x powhr b*)
⟨*proof*⟩

lemma *powhr-powhr*: ⋀*a b x*. (*x powhr a*) *powhr b* = *x powhr* (*a* ∗ *b*)
⟨*proof*⟩

lemma *powhr-powhr-swap*: ⋀*a b x*. (*x powhr a*) *powhr b* = (*x powhr b*) *powhr a*
⟨*proof*⟩

lemma *powhr-minus*: ⋀*a x*. *x powhr* (− *a*) = *inverse* (*x powhr a*)
⟨*proof*⟩

lemma *powhr-minus-divide*: *x powhr* (− *a*) = *1* / (*x powhr a*)

⟨*proof*⟩

lemma *powhr-less-mono*: $\bigwedge a\ b\ x.\ a < b \Longrightarrow 1 < x \Longrightarrow x\ powhr\ a < x\ powhr\ b$
⟨*proof*⟩

lemma *powhr-less-cancel*: $\bigwedge a\ b\ x.\ x\ powhr\ a < x\ powhr\ b \Longrightarrow 1 < x \Longrightarrow a < b$
⟨*proof*⟩

lemma *powhr-less-cancel-iff* [*simp*]: $1 < x \Longrightarrow x\ powhr\ a < x\ powhr\ b \longleftrightarrow a < b$
⟨*proof*⟩

lemma *powhr-le-cancel-iff* [*simp*]: $1 < x \Longrightarrow x\ powhr\ a \leq x\ powhr\ b \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *hlog*: $hlog\ (star\text{-}n\ X)\ (star\text{-}n\ Y) = star\text{-}n\ (\lambda n.\ log\ (X\ n)\ (Y\ n))$
⟨*proof*⟩

lemma *hlog-starfun-ln*: $\bigwedge x.\ (\ *f*\ ln)\ x = hlog\ ((\ *f*\ exp)\ 1)\ x$
⟨*proof*⟩

lemma *powhr-hlog-cancel* [*simp*]: $\bigwedge a\ x.\ 0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow a\ powhr\ (hlog\ a\ x) = x$
⟨*proof*⟩

lemma *hlog-powhr-cancel* [*simp*]: $\bigwedge a\ y.\ 0 < a \Longrightarrow a \neq 1 \Longrightarrow hlog\ a\ (a\ powhr\ y) = y$
⟨*proof*⟩

lemma *hlog-mult*:
$\bigwedge a\ x\ y.\ 0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow hlog\ a\ (x * y) = hlog\ a\ x + hlog\ a\ y$
⟨*proof*⟩

lemma *hlog-as-starfun*: $\bigwedge a\ x.\ 0 < a \Longrightarrow a \neq 1 \Longrightarrow hlog\ a\ x = (\ *f*\ ln)\ x\ /\ (\ *f*\ ln)\ a$
⟨*proof*⟩

lemma *hlog-eq-div-starfun-ln-mult-hlog*:
$\bigwedge a\ b\ x.\ 0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < b \Longrightarrow b \neq 1 \Longrightarrow 0 < x \Longrightarrow$
$hlog\ a\ x = ((\ *f*\ ln)\ b\ /\ (\ *f*\ ln)\ a) * hlog\ b\ x$
⟨*proof*⟩

lemma *powhr-as-starfun*: $\bigwedge a\ x.\ x\ powhr\ a = (if\ x = 0\ then\ 0\ else\ (\ *f*\ exp)\ (a * (\ *f*\ real\text{-}ln)\ x))$
⟨*proof*⟩

lemma *HInfinite-powhr*:
$x \in HInfinite \Longrightarrow 0 < x \Longrightarrow a \in HFinite - Infinitesimal \Longrightarrow 0 < a \Longrightarrow x\ powhr\ a \in HInfinite$

⟨*proof*⟩

lemma *hlog-hrabs-HInfinite-Infinitesimal*:
$x \in HFinite - Infinitesimal \Longrightarrow a \in HInfinite \Longrightarrow 0 < a \Longrightarrow hlog\ a\ |x| \in Infinitesimal$
⟨*proof*⟩

lemma *hlog-HInfinite-as-starfun*: $a \in HInfinite \Longrightarrow 0 < a \Longrightarrow hlog\ a\ x = (\ {*}f{*}\ ln)\ x\ /\ (\ {*}f{*}\ ln)\ a$
⟨*proof*⟩

lemma *hlog-one* [*simp*]: $\bigwedge a.\ hlog\ a\ 1 = 0$
⟨*proof*⟩

lemma *hlog-eq-one* [*simp*]: $\bigwedge a.\ 0 < a \Longrightarrow a \neq 1 \Longrightarrow hlog\ a\ a = 1$
⟨*proof*⟩

lemma *hlog-inverse*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow hlog\ a\ (inverse\ x) = -\ hlog\ a\ x$
⟨*proof*⟩

lemma *hlog-divide*: $0 < a \Longrightarrow a \neq 1 \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow hlog\ a\ (x\ /\ y) = hlog\ a\ x - hlog\ a\ y$
⟨*proof*⟩

lemma *hlog-less-cancel-iff* [*simp*]:
$\bigwedge a\ x\ y.\ 1 < a \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow hlog\ a\ x < hlog\ a\ y \longleftrightarrow x < y$
⟨*proof*⟩

lemma *hlog-le-cancel-iff* [*simp*]: $1 < a \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow hlog\ a\ x \leq hlog\ a\ y \longleftrightarrow x \leq y$
⟨*proof*⟩

end

theory *Hyperreal*
imports *HLog*
begin

end
theory *Hypercomplex*
imports *CLim Hyperreal*
begin

end

theory *Nonstandard-Analysis*

imports *Hypercomplex*
begin

end