Various results of number theory

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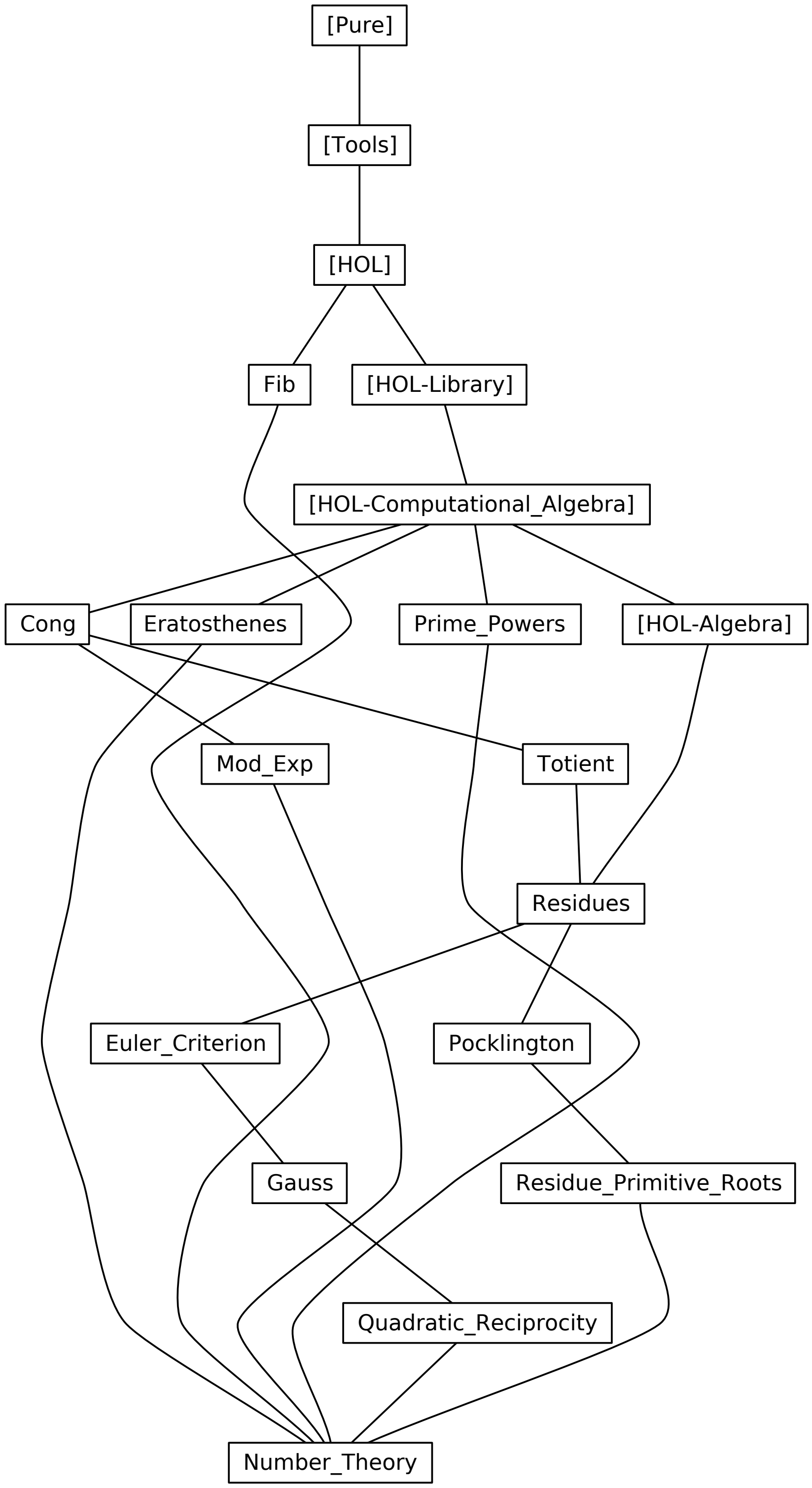
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1 The fibonacci function

theory *Fib*
  imports *Complex-Main*
begin

1.1 Fibonacci numbers

fun *fib* :: *nat* ⇒ *nat*
  where
    fib0: *fib 0* = *0*
  | *fib1*: *fib* (*Suc 0*) = *1*
  | *fib2*: *fib* (*Suc* (*Suc n*)) = *fib* (*Suc n*) + *fib n*

1.2 Basic Properties

lemma *fib-1* [*simp*]: *fib 1* = *1*
  by (*metis One-nat-def fib1*)

lemma *fib-2* [*simp*]: *fib 2* = *1*
  using *fib.simps*(*3*) [*of 0*] **by** (*simp add*: *numeral-2-eq-2*)

lemma *fib-plus-2*: *fib* (*n* + *2*) = *fib* (*n* + *1*) + *fib n*
  by (*metis Suc-eq-plus1 add-2-eq-Suc′ fib.simps*(*3*))

lemma *fib-add*: *fib* (*Suc* (*n* + *k*)) = *fib* (*Suc k*) * *fib* (*Suc n*) + *fib k* * *fib n*
  by (*induct n rule*: *fib.induct*) (*auto simp add*: *field-simps*)

lemma *fib-neq-0-nat*: *n* > *0* ⟹ *fib n* > *0*
  by (*induct n rule*: *fib.induct*) *auto*

lemma *fib-Suc-mono*: *fib m* ≤ *fib* (*Suc m*)
by(*induction m*) *auto*

lemma *fib-mono*: *m* ≤ *n* ⟹ *fib m* ≤ *fib n*
by (*simp add*: *fib-Suc-mono lift-Suc-mono-le*)

1.3 More efficient code

The naive approach is very inefficient since the branching recursion leads to many values of *fib* being computed multiple times. We can avoid this by "remembering" the last two values in the sequence, yielding a tail-recursive version. This is far from optimal (it takes roughly $O(n \cdot M(n))$ time where $M(n)$ is the time required to multiply two n-bit integers), but much better than the naive version, which is exponential.

fun *gen-fib* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat*
  where
    gen-fib a b 0 = *a*
  | *gen-fib a b* (*Suc 0*) = *b*

| *gen-fib a b* (*Suc* (*Suc n*)) = *gen-fib b* (*a* + *b*) (*Suc n*)

lemma *gen-fib-recurrence*: *gen-fib a b* (*Suc* (*Suc n*)) = *gen-fib a b n* + *gen-fib a b* (*Suc n*)
by (*induct a b n rule*: *gen-fib.induct*) *simp-all*

lemma *gen-fib-fib*: *gen-fib* (*fib n*) (*fib* (*Suc n*)) *m* = *fib* (*n* + *m*)
by (*induct m rule*: *fib.induct*) (*simp-all del*: *gen-fib.simps*(*3*) *add*: *gen-fib-recurrence*)

lemma *fib-conv-gen-fib*: *fib n* = *gen-fib 0 1 n*
using *gen-fib-fib*[*of 0 n*] **by** *simp*

declare *fib-conv-gen-fib* [*code*]

1.4 A Few Elementary Results

Concrete Mathematics, page 278: Cassini's identity. The proof is much easier using integers, not natural numbers!

lemma *fib-Cassini-int*: *int* (*fib* (*Suc* (*Suc n*)) ∗ *fib n*) − *int*((*fib* (*Suc n*))^2) = −((−*1*)^*n*)
by (*induct n rule*: *fib.induct*) (*auto simp add*: *field-simps power2-eq-square power-add*)

lemma *fib-Cassini-nat*:
fib (*Suc* (*Suc n*)) ∗ *fib n* =
(*if even n then* (*fib* (*Suc n*))^2 − *1 else* (*fib* (*Suc n*))^2 + *1*)
using *fib-Cassini-int* [*of n*] **by** (*auto simp del*: *of-nat-mult of-nat-power*)

1.5 Law 6.111 of Concrete Mathematics

lemma *coprime-fib-Suc-nat*: *coprime* (*fib n*) (*fib* (*Suc n*))
apply (*induct n rule*: *fib.induct*)
apply (*simp-all add*: *coprime-iff-gcd-eq-1 algebra-simps*)
apply (*simp add*: *add.assoc* [*symmetric*])
done

lemma *gcd-fib-add*:
gcd (*fib m*) (*fib* (*n* + *m*)) = *gcd* (*fib m*) (*fib n*)
proof (*cases m*)
case *0*
then show *?thesis*
by *simp*
next
case (*Suc q*)
from *coprime-fib-Suc-nat* [*of q*]
have *coprime* (*fib* (*Suc q*)) (*fib q*)
by (*simp add*: *ac-simps*)
have *gcd* (*fib q*) (*fib* (*Suc q*)) = *Suc 0*
using *coprime-fib-Suc-nat* [*of q*] **by** *simp*

```
  then have ∗: gcd (fib n ∗ fib q) (fib n ∗ fib (Suc q)) = fib n
    by (simp add: gcd-mult-distrib-nat [symmetric])
  moreover have gcd (fib (Suc q)) (fib n ∗ fib q + fib (Suc n) ∗ fib (Suc q)) =
    gcd (fib (Suc q)) (fib n ∗ fib q)
    using gcd-add-mult [of fib (Suc q)] by (simp add: ac-simps)
  moreover have gcd (fib (Suc q)) (fib n ∗ fib (Suc q)) = fib (Suc q)
    by simp
  ultimately show ?thesis
    using Suc ‹coprime (fib (Suc q)) (fib q)›
    by (auto simp add: fib-add algebra-simps gcd-mult-right-right-cancel)
qed

lemma gcd-fib-diff: m ≤ n =⇒ gcd (fib m) (fib (n − m)) = gcd (fib m) (fib n)
  by (simp add: gcd-fib-add [symmetric, of - n−m])

lemma gcd-fib-mod: 0 < m =⇒ gcd (fib m) (fib (n mod m)) = gcd (fib m) (fib n)
proof (induct n rule: less-induct)
  case (less n)
  show gcd (fib m) (fib (n mod m)) = gcd (fib m) (fib n)
  proof (cases m < n)
    case True
    then have m ≤ n by auto
    with ‹0 < m› have 0 < n by auto
    with ‹0 < m› ‹m < n› have ∗: n − m < n by auto
    have gcd (fib m) (fib (n mod m)) = gcd (fib m) (fib ((n − m) mod m))
      by (simp add: mod-if [of n]) (use ‹m < n› in auto)
    also have . . . = gcd (fib m) (fib (n − m))
      by (simp add: less.hyps ∗ ‹0 < m›)
    also have . . . = gcd (fib m) (fib n)
      by (simp add: gcd-fib-diff ‹m ≤ n›)
    finally show gcd (fib m) (fib (n mod m)) = gcd (fib m) (fib n) .
  next
    case False
    then show gcd (fib m) (fib (n mod m)) = gcd (fib m) (fib n)
      by (cases m = n) auto
  qed
qed

lemma fib-gcd: fib (gcd m n) = gcd (fib m) (fib n)  — Law 6.111
  by (induct m n rule: gcd-nat-induct) (simp-all add: gcd-non-0-nat gcd.commute
gcd-fib-mod)

theorem fib-mult-eq-sum-nat: fib (Suc n) ∗ fib n = (∑ k ∈ {..n}. fib k ∗ fib k)
  by (induct n rule: nat.induct) (auto simp add: field-simps)
```

1.6 Closed form

```
lemma fib-closed-form:
  fixes φ ψ :: real
```

```
  defines φ ≡ (1 + sqrt 5) / 2
    and ψ ≡ (1 − sqrt 5) / 2
  shows of-nat (fib n) = (φ ^ n − ψ ^ n) / sqrt 5
proof (induct n rule: fib.induct)
  fix n :: nat
  assume IH1: of-nat (fib n) = (φ ^ n − ψ ^ n) / sqrt 5
  assume IH2: of-nat (fib (Suc n)) = (φ ^ Suc n − ψ ^ Suc n) / sqrt 5
  have of-nat (fib (Suc (Suc n))) = of-nat (fib (Suc n)) + of-nat (fib n) by simp
  also have ... = (φ^n ∗ (φ + 1) − ψ^n ∗ (ψ + 1)) / sqrt 5
    by (simp add: IH1 IH2 field-simps)
  also have φ + 1 = φ² by (simp add: φ-def field-simps power2-eq-square)
  also have ψ + 1 = ψ² by (simp add: ψ-def field-simps power2-eq-square)
  also have φ^n ∗ φ² − ψ^n ∗ ψ² = φ ^ Suc (Suc n) − ψ ^ Suc (Suc n)
    by (simp add: power2-eq-square)
  finally show of-nat (fib (Suc (Suc n))) = (φ ^ Suc (Suc n) − ψ ^ Suc (Suc n))
/ sqrt 5 .
qed (simp-all add: φ-def ψ-def field-simps)

lemma fib-closed-form':
  fixes φ ψ :: real
  defines φ ≡ (1 + sqrt 5) / 2
    and ψ ≡ (1 − sqrt 5) / 2
  assumes n > 0
  shows fib n = round (φ ^ n / sqrt 5)
proof (rule sym, rule round-unique')
  have |φ ^ n / sqrt 5 − of-int (int (fib n))| = |ψ| ^ n / sqrt 5
    by (simp add: fib-closed-form[folded φ-def ψ-def] field-simps power-abs)
  also {
    from assms have |ψ|^n ≤ |ψ|^1
      by (intro power-decreasing) (simp-all add: algebra-simps real-le-lsqrt)
    also have ... < sqrt 5 / 2 by (simp add: ψ-def field-simps)
    finally have |ψ|^n / sqrt 5 < 1/2 by (simp add: field-simps)
  }
  finally show |φ ^ n / sqrt 5 − of-int (int (fib n))| < 1/2 .
qed

lemma fib-asymptotics:
  fixes φ :: real
  defines φ ≡ (1 + sqrt 5) / 2
  shows (λn. real (fib n) / (φ ^ n / sqrt 5)) ⟶ 1
proof −
  define ψ :: real where ψ ≡ (1 − sqrt 5) / 2
  have φ > 1 by (simp add: φ-def)
  then have ∗: φ ≠ 0 by auto
  have (λn. (ψ / φ) ^ n) ⟶ 0
   by (rule LIMSEQ-power-zero) (simp-all add: φ-def ψ-def field-simps add-pos-pos)
  then have (λn. 1 − (ψ / φ) ^ n) ⟶ 1 − 0
    by (intro tendsto-diff tendsto-const)
  with ∗ have (λn. (φ ^ n − ψ ^ n) / φ ^ n) ⟶ 1
```

by (*simp add*: *field-simps*)
then show *?thesis*
by (*simp add*: *fib-closed-form* φ*-def* ψ*-def*)
qed

1.7 Divide-and-Conquer recurrence

The following divide-and-conquer recurrence allows for a more efficient computation of Fibonacci numbers; however, it requires memoisation of values to be reasonably efficient, cutting the number of values to be computed to logarithmically many instead of linearly many. The vast majority of the computation time is then actually spent on the multiplication, since the output number is exponential in the input number.

lemma *fib-rec-odd*:
fixes φ ψ :: *real*
defines $\varphi \equiv (1 + sqrt\ 5) / 2$
and $\psi \equiv (1 - sqrt\ 5) / 2$
shows $fib\ (Suc\ (2 * n)) = fib\ n\hat{}2 + fib\ (Suc\ n)\hat{}2$
proof −
have $of\text{-}nat\ (fib\ n\hat{}2 + fib\ (Suc\ n)\hat{}2) = ((\varphi\ \hat{}\ n - \psi\ \hat{}\ n)^2 + (\varphi * \varphi\ \hat{}\ n - \psi * \psi\ \hat{}\ n)^2)/5$
by (*simp add*: *fib-closed-form*[*folded* φ*-def* ψ*-def*] *field-simps power2-eq-square*)
also
let $?A = \varphi\hat{}(2 * n) + \psi\hat{}(2 * n) - 2*(\varphi * \psi)\hat{}n + \varphi\hat{}(2 * n + 2) + \psi\hat{}(2 * n + 2) - 2*(\varphi * \psi)\hat{}(n + 1)$
have $(\varphi\ \hat{}\ n - \psi\ \hat{}\ n)^2 + (\varphi * \varphi\ \hat{}\ n - \psi * \psi\ \hat{}\ n)^2 = ?A$
by (*simp add*: *power2-eq-square algebra-simps power-mult power-mult-distrib*)
also have $\varphi * \psi = -1$
by (*simp add*: φ*-def* ψ*-def field-simps*)
then have $?A = \varphi\hat{}(2 * n + 1) * (\varphi + inverse\ \varphi) + \psi\hat{}(2 * n + 1) * (\psi + inverse\ \psi)$
by (*auto simp*: *field-simps power2-eq-square*)
also have $1 + sqrt\ 5 > 0$
by (*auto intro*: *add-pos-pos*)
then have $\varphi + inverse\ \varphi = sqrt\ 5$
by (*simp add*: φ*-def field-simps*)
also have $\psi + inverse\ \psi = -sqrt\ 5$
by (*simp add*: ψ*-def field-simps*)
also have $(\varphi\ \hat{}(2 * n + 1) * sqrt\ 5 + \psi\ \hat{}(2 * n + 1) * - sqrt\ 5) / 5 = (\varphi\ \hat{}(2 * n + 1) - \psi\ \hat{}(2 * n + 1)) * (sqrt\ 5 / 5)$
by (*simp add*: *field-simps*)
also have $sqrt\ 5 / 5 = inverse\ (sqrt\ 5)$
by (*simp add*: *field-simps*)
also have $(\varphi\ \hat{}(2 * n + 1) - \psi\ \hat{}(2 * n + 1)) * \ldots = of\text{-}nat\ (fib\ (Suc\ (2 * n)))$
by (*simp add*: *fib-closed-form*[*folded* φ*-def* ψ*-def*] *divide-inverse*)
finally show *?thesis*
by (*simp only*: *of-nat-eq-iff*)

qed

lemma *fib-rec-even*: $fib\ (2 * n) = (fib\ (n - 1) + fib\ (n + 1)) * fib\ n$
proof (*induct n*)
  case *0*
  then show *?case* **by** *simp*
next
  case (*Suc n*)
  let $?rfib = \lambda x.\ real\ (fib\ x)$
  have $2 * (Suc\ n) = Suc\ (Suc\ (2 * n))$ **by** *simp*
  also have $real\ (fib\ \ldots) = ?rfib\ n\hat{}2 + ?rfib\ (Suc\ n)\hat{}2 + (?rfib\ (n - 1) + ?rfib\ (n + 1)) * ?rfib\ n$
    by (*simp add*: *fib-rec-odd Suc*)
  also have $(?rfib\ (n - 1) + ?rfib\ (n + 1)) * ?rfib\ n = (2 * ?rfib\ (n + 1) - ?rfib\ n) * ?rfib\ n$
    by (*cases n*) *simp-all*
  also have $?rfib\ n\hat{}2 + ?rfib\ (Suc\ n)\hat{}2 + \ldots = (?rfib\ (Suc\ n) + 2 * ?rfib\ n) * ?rfib\ (Suc\ n)$
    by (*simp add*: *algebra-simps power2-eq-square*)
  also have $\ldots = real\ ((fib\ (Suc\ n - 1) + fib\ (Suc\ n + 1)) * fib\ (Suc\ n))$ **by** *simp*
  finally show *?case* **by** (*simp only*: *of-nat-eq-iff*)
qed

lemma *fib-rec-even'*: $fib\ (2 * n) = (2 * fib\ (n - 1) + fib\ n) * fib\ n$
  by (*subst fib-rec-even*, *cases n*) *simp-all*

lemma *fib-rec*:
  $fib\ n =$
    $(if\ n = 0\ then\ 0\ else\ if\ n = 1\ then\ 1$
     $else\ if\ even\ n\ then\ let\ n' = n\ div\ 2;\ fn = fib\ n'\ in\ (2 * fib\ (n' - 1) + fn) * fn$
     $else\ let\ n' = n\ div\ 2\ in\ fib\ n'\ \hat{}\ 2 + fib\ (Suc\ n')\ \hat{}\ 2)$
  by (*auto elim*: *evenE oddE simp*: *fib-rec-odd fib-rec-even' Let-def*)

1.8 Fibonacci and Binomial Coefficients

lemma *sum-drop-zero*: $(\sum k = 0..Suc\ n.\ if\ 0{<}k\ then\ (f\ (k - 1))\ else\ 0) = (\sum j = 0..n.\ f\ j)$
  by (*induct n*) *auto*

lemma *sum-choose-drop-zero*:
  $(\sum k = 0..Suc\ n.\ if\ k = 0\ then\ 0\ else\ (Suc\ n - k)\ choose\ (k - 1)) =$
    $(\sum j = 0..n.\ (n{-}j)\ choose\ j)$
  by (*rule trans* [*OF sum.cong sum-drop-zero*]) *auto*

lemma *ne-diagonal-fib*: $(\sum k = 0..n.\ (n{-}k)\ choose\ k) = fib\ (Suc\ n)$
proof (*induct n rule*: *fib.induct*)
  case *1*
  show *?case* **by** *simp*

next
  case *2*
  show *?case* **by** *simp*
next
  case (*3 n*)
  have ($\sum k = 0..Suc\ n.\ Suc\ (Suc\ n) - k\ choose\ k$) =
    ($\sum k = 0..Suc\ n.\ (Suc\ n - k\ choose\ k) + (if\ k = 0\ then\ 0\ else\ (Suc\ n - k\ choose\ (k - 1)))$)
    by (*rule sum.cong*) (*simp-all add*: *choose-reduce-nat*)
  also have ... =
    ($\sum k = 0..Suc\ n.\ Suc\ n - k\ choose\ k$) +
    ($\sum k = 0..Suc\ n.\ if\ k{=}0\ then\ 0\ else\ (Suc\ n - k\ choose\ (k - 1))$)
    by (*simp add*: *sum.distrib*)
  also have ... = ($\sum k = 0..Suc\ n.\ Suc\ n - k\ choose\ k$) + ($\sum j = 0..n.\ n - j\ choose\ j$)
    by (*metis sum-choose-drop-zero*)
  finally show *?case* **using** *3*
    by *simp*
qed

end

2 Congruence

theory *Cong*
  imports *HOL−Computational-Algebra.Primes*
begin

2.1 Generic congruences

context *unique-euclidean-semiring*
begin

definition *cong* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ (‹(*1*[- = -] ′(′ *mod* -′))›)
  where *cong b c a* $\longleftrightarrow$ *b mod a* = *c mod a*

abbreviation *notcong* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ (‹(*1*[- ≠ -] ′(′ *mod* -′))›)
  where *notcong b c a* ≡ ¬ *cong b c a*

lemma *cong-refl* [*simp*]:
  [*b* = *b*] (*mod a*)
  by (*simp add*: *cong-def*)

lemma *cong-sym*:
  [*b* = *c*] (*mod a*) $\Longrightarrow$ [*c* = *b*] (*mod a*)
  by (*simp add*: *cong-def*)

lemma *cong-sym-eq*:
  [*b* = *c*] (*mod a*) $\longleftrightarrow$ [*c* = *b*] (*mod a*)

by (*auto simp add*: *cong-def*)

lemma *cong-trans* [*trans*]:
$[b = c] \ (mod \ a) \Longrightarrow [c = d] \ (mod \ a) \Longrightarrow [b = d] \ (mod \ a)$
by (*simp add*: *cong-def*)

lemma *cong-mult-self-right*:
$[b * a = 0] \ (mod \ a)$
by (*simp add*: *cong-def*)

lemma *cong-mult-self-left*:
$[a * b = 0] \ (mod \ a)$
by (*simp add*: *cong-def*)

lemma *cong-mod-left* [*simp*]:
$[b \ mod \ a = c] \ (mod \ a) \longleftrightarrow [b = c] \ (mod \ a)$
by (*simp add*: *cong-def*)

lemma *cong-mod-right* [*simp*]:
$[b = c \ mod \ a] \ (mod \ a) \longleftrightarrow [b = c] \ (mod \ a)$
by (*simp add*: *cong-def*)

lemma *cong-0* [*simp*, *presburger*]:
$[b = c] \ (mod \ 0) \longleftrightarrow b = c$
by (*simp add*: *cong-def*)

lemma *cong-1* [*simp*, *presburger*]:
$[b = c] \ (mod \ 1)$
by (*simp add*: *cong-def*)

lemma *cong-dvd-iff*:
$a \ dvd \ b \longleftrightarrow a \ dvd \ c$ **if** $[b = c] \ (mod \ a)$
using *that* **by** (*auto simp*: *cong-def dvd-eq-mod-eq-0*)

lemma *cong-0-iff*: $[b = 0] \ (mod \ a) \longleftrightarrow a \ dvd \ b$
by (*simp add*: *cong-def dvd-eq-mod-eq-0*)

lemma *cong-add*:
$[b = c] \ (mod \ a) \Longrightarrow [d = e] \ (mod \ a) \Longrightarrow [b + d = c + e] \ (mod \ a)$
by (*auto simp add*: *cong-def intro*: *mod-add-cong*)

lemma *cong-mult*:
$[b = c] \ (mod \ a) \Longrightarrow [d = e] \ (mod \ a) \Longrightarrow [b * d = c * e] \ (mod \ a)$
by (*auto simp add*: *cong-def intro*: *mod-mult-cong*)

lemma *cong-scalar-right*:
$[b = c] \ (mod \ a) \Longrightarrow [b * d = c * d] \ (mod \ a)$
by (*simp add*: *cong-mult*)

lemma *cong-scalar-left*:
  $[b = c] \ (mod\ a) \Longrightarrow [d * b = d * c] \ (mod\ a)$
  by (*simp add*: *cong-mult*)

lemma *cong-pow*:
  $[b = c] \ (mod\ a) \Longrightarrow [b \ \hat{}\ n = c \ \hat{}\ n] \ (mod\ a)$
  by (*simp add*: *cong-def power-mod* [*symmetric*, *of b n a*] *power-mod* [*symmetric*, *of c n a*])

lemma *cong-sum*:
  $[sum\ f\ A = sum\ g\ A] \ (mod\ a)$ **if** $\bigwedge x.\ x \in A \Longrightarrow [f\ x = g\ x] \ (mod\ a)$
  using *that* **by** (*induct A rule*: *infinite-finite-induct*) (*auto intro*: *cong-add*)

lemma *cong-prod*:
  $[prod\ f\ A = prod\ g\ A] \ (mod\ a)$ **if** $(\bigwedge x.\ x \in A \Longrightarrow [f\ x = g\ x] \ (mod\ a))$
  using *that* **by** (*induct A rule*: *infinite-finite-induct*) (*auto intro*: *cong-mult*)

lemma *mod-mult-cong-right*:
  $[c\ mod\ (a * b) = d] \ (mod\ a) \longleftrightarrow [c = d] \ (mod\ a)$
  by (*simp add*: *cong-def mod-mod-cancel mod-add-left-eq*)

lemma *mod-mult-cong-left*:
  $[c\ mod\ (b * a) = d] \ (mod\ a) \longleftrightarrow [c = d] \ (mod\ a)$
  using *mod-mult-cong-right* [*of c a b d*] **by** (*simp add*: *ac-simps*)

end

context *unique-euclidean-ring*
begin

lemma *cong-diff*:
  $[b = c] \ (mod\ a) \Longrightarrow [d = e] \ (mod\ a) \Longrightarrow [b - d = c - e] \ (mod\ a)$
  by (*auto simp add*: *cong-def intro*: *mod-diff-cong*)

lemma *cong-diff-iff-cong-0*:
  $[b - c = 0] \ (mod\ a) \longleftrightarrow [b = c] \ (mod\ a)$ (**is** $?P \longleftrightarrow ?Q$)
proof
  assume $?P$
  then have $[b - c + c = 0 + c] \ (mod\ a)$
    by (*rule cong-add*) *simp*
  then show $?Q$
    by *simp*
next
  assume $?Q$
  with *cong-diff* [*of b c a c c*] **show** $?P$
    by *simp*
qed

lemma *cong-minus-minus-iff*:

$[- \ b = - \ c] \ (mod \ a) \longleftrightarrow [b = c] \ (mod \ a)$
using *cong-diff-iff-cong-0* [*of b c a*] *cong-diff-iff-cong-0* [*of* $-$ *b* $-$ *c a*]
by (*simp add*: *cong-0-iff dvd-diff-commute*)

lemma *cong-modulus-minus-iff* [*iff*]:
$[b = c] \ (mod \ - \ a) \longleftrightarrow [b = c] \ (mod \ a)$
using *cong-diff-iff-cong-0* [*of b c a*] *cong-diff-iff-cong-0* [*of b c* $-a$]
by (*simp add*: *cong-0-iff*)

lemma *cong-iff-dvd-diff*:
$[a = b] \ (mod \ m) \longleftrightarrow m \ dvd \ (a - b)$
by (*simp add*: *cong-0-iff* [*symmetric*] *cong-diff-iff-cong-0*)

lemma *cong-iff-lin*:
$[a = b] \ (mod \ m) \longleftrightarrow (\exists \, k. \ b = a + m * k)$ (**is** $?P \longleftrightarrow ?Q$)
proof $-$
have $?P \longleftrightarrow m \ dvd \ b - a$
by (*simp add*: *cong-iff-dvd-diff dvd-diff-commute*)
also have $\ldots \longleftrightarrow ?Q$
by (*auto simp add*: *algebra-simps elim*!: *dvdE*)
finally show *?thesis*
by *simp*
qed

lemma *cong-add-lcancel*:
$[a + x = a + y] \ (mod \ n) \longleftrightarrow [x = y] \ (mod \ n)$
by (*simp add*: *cong-iff-lin algebra-simps*)

lemma *cong-add-rcancel*:
$[x + a = y + a] \ (mod \ n) \longleftrightarrow [x = y] \ (mod \ n)$
by (*simp add*: *cong-iff-lin algebra-simps*)

lemma *cong-add-lcancel-0*:
$[a + x = a] \ (mod \ n) \longleftrightarrow [x = 0] \ (mod \ n)$
using *cong-add-lcancel* [*of a x 0 n*] **by** *simp*

lemma *cong-add-rcancel-0*:
$[x + a = a] \ (mod \ n) \longleftrightarrow [x = 0] \ (mod \ n)$
using *cong-add-rcancel* [*of x a 0 n*] **by** *simp*

lemma *cong-dvd-modulus*:
$[x = y] \ (mod \ n)$ **if** $[x = y] \ (mod \ m)$ **and** *n dvd m*
using *that* **by** (*auto intro*: *dvd-trans simp add*: *cong-iff-dvd-diff*)

lemma *cong-modulus-mult*:
$[x = y] \ (mod \ m)$ **if** $[x = y] \ (mod \ m * n)$
using *that* **by** (*simp add*: *cong-iff-dvd-diff*) (*rule dvd-mult-left*)

end

lemma *cong-abs* [*simp*]:
  $[x = y]$ (*mod* $|m|$) $\longleftrightarrow$ $[x = y]$ (*mod* *m*)
  for *x y* :: *'a* :: {*unique-euclidean-ring*, *linordered-idom*}
  by (*simp add*: *cong-iff-dvd-diff*)

lemma *cong-square*:
  prime p $\Longrightarrow$ *0* $<$ *a* $\Longrightarrow$ $[a * a = 1]$ (*mod p*) $\Longrightarrow$ $[a = 1]$ (*mod p*) $\vee$ $[a = - 1]$ (*mod p*)
  for *a p* :: *'a* :: {*normalization-semidom*, *linordered-idom*, *unique-euclidean-ring*}
  by (*auto simp add*: *cong-iff-dvd-diff square-diff-one-factored dest*: *prime-dvd-multD*)

lemma *cong-mult-rcancel*:
  $[a * k = b * k]$ (*mod m*) $\longleftrightarrow$ $[a = b]$ (*mod m*)
  if *coprime k m* **for** *a k m* :: *'a*::{*unique-euclidean-ring*, *ring-gcd*}
  using *that* **by** (*auto simp add*: *cong-iff-dvd-diff left-diff-distrib* [*symmetric*] *ac-simps coprime-dvd-mult-right-iff*)

lemma *cong-mult-lcancel*:
  $[k * a = k * b]$ (*mod m*) = $[a = b]$ (*mod m*)
  if *coprime k m* **for** *a k m* :: *'a*::{*unique-euclidean-ring*, *ring-gcd*}
  using *that cong-mult-rcancel* [*of k m a b*] **by** (*simp add*: *ac-simps*)

lemma *coprime-cong-mult*:
  $[a = b]$ (*mod m*) $\Longrightarrow$ $[a = b]$ (*mod n*) $\Longrightarrow$ *coprime m n* $\Longrightarrow$ $[a = b]$ (*mod m* $*$ *n*)
  for *a b* :: *'a* :: {*unique-euclidean-ring*, *semiring-gcd*}
  by (*simp add*: *cong-iff-dvd-diff divides-mult*)

lemma *cong-gcd-eq*:
  gcd a m = *gcd b m* **if** $[a = b]$ (*mod m*)
  for *a b* :: *'a* :: {*unique-euclidean-semiring*, *euclidean-semiring-gcd*}
proof (*cases m* = *0*)
  case *True*
  with *that* **show** *?thesis*
    by *simp*
next
  case *False*
  moreover have *gcd* (*a mod m*) *m* = *gcd* (*b mod m*) *m*
    using *that* **by** (*simp add*: *cong-def*)
  ultimately show *?thesis*
    by *simp*
qed

lemma *cong-imp-coprime*:
  $[a = b]$ (*mod m*) $\Longrightarrow$ *coprime a m* $\Longrightarrow$ *coprime b m*
  for *a b* :: *'a* :: {*unique-euclidean-semiring*, *euclidean-semiring-gcd*}
  by (*auto simp add*: *coprime-iff-gcd-eq-1 dest*: *cong-gcd-eq*)

lemma *cong-cong-prod-coprime*:

$[x = y]$ $(mod\ (\prod i{\in}A.\ m\ i))$ **if**
  $(\forall\, i{\in}A.\ [x = y]\ (mod\ m\ i))$
  $(\forall\, i{\in}A.\ (\forall\, j{\in}A.\ i \neq j \longrightarrow coprime\ (m\ i)\ (m\ j)))$
for $x\ y :: 'a :: \{unique\text{-}euclidean\text{-}ring,\ semiring\text{-}gcd\}$
using *that* **by** (*induct A rule*: *infinite-finite-induct*)
  (*auto intro*!: *coprime-cong-mult prod-coprime-right*)

2.2 Congruences on *nat* and *int*

lemma *cong-int-iff*:
  $[int\ m = int\ q]\ (mod\ int\ n) \longleftrightarrow [m = q]\ (mod\ n)$
  by (*simp add*: *cong-def of-nat-mod* [*symmetric*])

lemma *cong-Suc-0* [*simp*, *presburger*]:
  $[m = n]\ (mod\ Suc\ 0)$
  using *cong-1* [*of m n*] **by** *simp*

lemma *cong-diff-nat*:
  $[a - c = b - d]\ (mod\ m)$ **if** $[a = b]\ (mod\ m)$ $[c = d]\ (mod\ m)$
    and $a \geq c$ $b \geq d$ **for** $a\ b\ c\ d\ m :: nat$
proof −
  have $[c + (a - c) = d + (b - d)]\ (mod\ m)$
    using *that* **by** *simp*
  with ‹$[c = d]\ (mod\ m)$› **have** $[c + (a - c) = c + (b - d)]\ (mod\ m)$
    using *mod-add-cong* **by** (*auto simp add*: *cong-def*) *fastforce*
  then show *?thesis*
    by (*simp add*: *cong-def nat-mod-eq-iff*)
qed

lemma *cong-diff-iff-cong-0-nat*:
  $[a - b = 0]\ (mod\ m) \longleftrightarrow [a = b]\ (mod\ m)$ **if** $a \geq b$ **for** $a\ b :: nat$
  using *that* **by** (*auto simp add*: *cong-def le-imp-diff-is-add dest*: *nat-mod-eq-lemma*)

lemma *cong-diff-iff-cong-0-nat'*:
  $[nat\ |int\ a - int\ b| = 0]\ (mod\ m) \longleftrightarrow [a = b]\ (mod\ m)$
proof (*cases* $b \leq a$)
  case *True*
  then show *?thesis*
    by (*simp add*: *nat-diff-distrib' cong-diff-iff-cong-0-nat* [*of b a m*])
next
  case *False*
  then have $a \leq b$
    by *simp*
  then show *?thesis*
    by (*simp add*: *nat-diff-distrib' cong-diff-iff-cong-0-nat* [*of a b m*])
      (*auto simp add*: *cong-def*)
qed

lemma *cong-altdef-nat*:

$a \geq b \Longrightarrow [a = b]\ (mod\ m) \longleftrightarrow m\ dvd\ (a - b)$
for $a\ b :: nat$
by (*simp add*: *cong-0-iff* [*symmetric*] *cong-diff-iff-cong-0-nat*)

lemma *cong-altdef-nat'*:
$[a = b]\ (mod\ m) \longleftrightarrow m\ dvd\ nat\ |int\ a - int\ b|$
using *cong-diff-iff-cong-0-nat'* [*of a b m*]
by (*simp only*: *cong-0-iff* [*symmetric*])

lemma *cong-mult-rcancel-nat*:
$[a * k = b * k]\ (mod\ m) \longleftrightarrow [a = b]\ (mod\ m)$
if *coprime k m* **for** $a\ k\ m :: nat$
proof −
have $[a * k = b * k]\ (mod\ m) \longleftrightarrow m\ dvd\ nat\ |int\ (a * k) - int\ (b * k)|$
by (*simp add*: *cong-altdef-nat'*)
also have $\ldots \longleftrightarrow m\ dvd\ nat\ |(int\ a - int\ b) * int\ k|$
by (*simp add*: *algebra-simps*)
also have $\ldots \longleftrightarrow m\ dvd\ nat\ |int\ a - int\ b| * k$
by (*simp add*: *abs-mult nat-times-as-int*)
also have $\ldots \longleftrightarrow m\ dvd\ nat\ |int\ a - int\ b|$
by (*rule coprime-dvd-mult-left-iff*) (*use* ‹*coprime k m*› **in** ‹*simp add*: *ac-simps*›)
also have $\ldots \longleftrightarrow [a = b]\ (mod\ m)$
by (*simp add*: *cong-altdef-nat'*)
finally show *?thesis* **.**
qed

lemma *cong-mult-lcancel-nat*:
$[k * a = k * b]\ (mod\ m) = [a = b]\ (mod\ m)$
if *coprime k m* **for** $a\ k\ m :: nat$
using *that* **by** (*simp add*: *cong-mult-rcancel-nat ac-simps*)

lemma *coprime-cong-mult-nat*:
$[a = b]\ (mod\ m) \Longrightarrow [a = b]\ (mod\ n) \Longrightarrow coprime\ m\ n \Longrightarrow [a = b]\ (mod\ m * n)$
for $a\ b :: nat$
by (*simp add*: *cong-altdef-nat' divides-mult*)

lemma *cong-less-imp-eq-nat*: $0 \leq a \Longrightarrow a < m \Longrightarrow 0 \leq b \Longrightarrow b < m \Longrightarrow [a = b]\ (mod\ m) \Longrightarrow a = b$
for $a\ b :: nat$
by (*auto simp add*: *cong-def*)

lemma *cong-less-imp-eq-int*: $0 \leq a \Longrightarrow a < m \Longrightarrow 0 \leq b \Longrightarrow b < m \Longrightarrow [a = b]\ (mod\ m) \Longrightarrow a = b$
for $a\ b :: int$
by (*auto simp add*: *cong-def*)

lemma *cong-less-unique-nat*: $0 < m \Longrightarrow (\exists!b.\ 0 \leq b \land b < m \land [a = b]\ (mod\ m))$
for $a\ m :: nat$

by (*auto simp*: *cong-def*) (*metis mod-less-divisor mod-mod-trivial*)

lemma *cong-less-unique-int*: $0 < m \Longrightarrow (\exists!b.\ 0 \le b \wedge b < m \wedge [a = b]\ (mod\ m))$
for $a\ m$ :: *int*
by (*auto simp*: *cong-def*) (*metis mod-mod-trivial pos-mod-conj*)

lemma *cong-iff-lin-nat*: $([a = b]\ (mod\ m)) \longleftrightarrow (\exists\, k1\ k2.\ b + k1 * m = a + k2 * m)$
(**is** *?lhs* = *?rhs*)
for $a\ b$ :: *nat*
proof
assume *?lhs*
show *?rhs*
proof (*cases* $b \le a$)
case *True*
with ‹*?lhs*› **show** *?rhs*
by (*metis cong-altdef-nat dvd-def le-add-diff-inverse add-0-right mult-0 mult.commute*)
next
case *False*
with ‹*?lhs*› **show** *?rhs*
by (*metis cong-def mult.commute nat-le-linear nat-mod-eq-lemma*)
qed
next
assume *?rhs*
then show *?lhs*
by (*metis cong-def mult.commute nat-mod-eq-iff*)
qed

lemma *cong-cong-mod-nat*: $[a = b]\ (mod\ m) \longleftrightarrow [a\ mod\ m = b\ mod\ m]\ (mod\ m)$
for $a\ b$ :: *nat*
by *simp*

lemma *cong-cong-mod-int*: $[a = b]\ (mod\ m) \longleftrightarrow [a\ mod\ m = b\ mod\ m]\ (mod\ m)$
for $a\ b$ :: *int*
by *simp*

lemma *cong-add-lcancel-nat*: $[a + x = a + y]\ (mod\ n) \longleftrightarrow [x = y]\ (mod\ n)$
for $a\ x\ y$ :: *nat*
by (*simp add*: *cong-iff-lin-nat*)

lemma *cong-add-rcancel-nat*: $[x + a = y + a]\ (mod\ n) \longleftrightarrow [x = y]\ (mod\ n)$
for $a\ x\ y$ :: *nat*
by (*simp add*: *cong-iff-lin-nat*)

lemma *cong-add-lcancel-0-nat*: $[a + x = a]\ (mod\ n) \longleftrightarrow [x = 0]\ (mod\ n)$
for $a\ x$ :: *nat*
using *cong-add-lcancel-nat* [*of a x 0 n*] **by** *simp*

lemma *cong-add-rcancel-0-nat*: $[x + a = a]\ (mod\ n) \longleftrightarrow [x = 0]\ (mod\ n)$

for *a x* :: *nat*
using *cong-add-rcancel-nat* [*of x a 0 n*] **by** *simp*

lemma *cong-dvd-modulus-nat*: $[x = y]$ (*mod m*) $\Longrightarrow$ *n dvd m* $\Longrightarrow$ $[x = y]$ (*mod n*)
for *x y* :: *nat*
unfolding *cong-iff-lin-nat dvd-def*
by (*metis mult.commute mult.left-commute*)

lemma *cong-to-1-nat*:
fixes *a* :: *nat*
assumes $[a = 1]$ (*mod n*)
shows *n dvd* $(a - 1)$
proof (*cases* $a = 0$)
case *True*
then show *?thesis* **by** *force*
next
case *False*
with *assms* **show** *?thesis* **by** (*metis cong-altdef-nat leI less-one*)
qed

lemma *cong-0-1-nat'*: $[0 = Suc\ 0]$ (*mod n*) $\longleftrightarrow$ $n = Suc\ 0$
by (*auto simp*: *cong-def*)

lemma *cong-0-1-nat*: $[0 = 1]$ (*mod n*) $\longleftrightarrow$ $n = 1$
for *n* :: *nat*
by (*auto simp*: *cong-def*)

lemma *cong-0-1-int*: $[0 = 1]$ (*mod n*) $\longleftrightarrow$ $n = 1 \lor n = -1$
for *n* :: *int*
by (*auto simp*: *cong-def zmult-eq-1-iff*)

lemma *cong-to-1'-nat*: $[a = 1]$ (*mod n*) $\longleftrightarrow$ $a = 0 \land n = 1 \lor (\exists m.\ a = 1 + m * n)$
for *a* :: *nat*
by (*metis add.right-neutral cong-0-1-nat cong-iff-lin-nat cong-to-1-nat dvd-div-mult-self leI le-add-diff-inverse less-one mult-eq-if*)

lemma *cong-le-nat*: $y \le x \Longrightarrow [x = y]$ (*mod n*) $\longleftrightarrow (\exists q.\ x = q * n + y)$
for *x y* :: *nat*
by (*auto simp add*: *cong-altdef-nat le-imp-diff-is-add elim*!: *dvdE*)

lemma *cong-solve-nat*:
fixes *a* :: *nat*
shows $\exists x.\ [a * x = gcd\ a\ n]$ (*mod n*)
proof (*cases* $a = 0 \lor n = 0$)
case *True*
then show *?thesis*
by (*force simp add*: *cong-0-iff cong-sym*)

```
next
  case False
  then show ?thesis
    using bezout-nat [of a n]
    by auto (metis cong-add-rcancel-0-nat cong-mult-self-left)
qed

lemma cong-solve-int:
  fixes a :: int
  shows ∃x. [a ∗ x = gcd a n] (mod n)
    by (metis bezout-int cong-iff-lin mult.commute)

lemma cong-solve-dvd-nat:
  fixes a :: nat
  assumes gcd a n dvd d
  shows ∃x. [a ∗ x = d] (mod n)
proof −
  from cong-solve-nat [of a] obtain x where [a ∗ x = gcd a n](mod n)
    by auto
  then have [(d div gcd a n) ∗ (a ∗ x) = (d div gcd a n) ∗ gcd a n] (mod n)
    using cong-scalar-left by blast
  also from assms have (d div gcd a n) ∗ gcd a n = d
    by (rule dvd-div-mult-self)
  also have (d div gcd a n) ∗ (a ∗ x) = a ∗ (d div gcd a n ∗ x)
    by auto
  finally show ?thesis
    by auto
qed

lemma cong-solve-dvd-int:
  fixes a::int
  assumes b: gcd a n dvd d
  shows ∃x. [a ∗ x = d] (mod n)
proof −
  from cong-solve-int [of a] obtain x where [a ∗ x = gcd a n](mod n)
    by auto
  then have [(d div gcd a n) ∗ (a ∗ x) = (d div gcd a n) ∗ gcd a n] (mod n)
    using cong-scalar-left by blast
  also from b have (d div gcd a n) ∗ gcd a n = d
    by (rule dvd-div-mult-self)
  also have (d div gcd a n) ∗ (a ∗ x) = a ∗ (d div gcd a n ∗ x)
    by auto
  finally show ?thesis
    by auto
qed

lemma cong-solve-coprime-nat:
  ∃x. [a ∗ x = Suc 0] (mod n) if coprime a n
  using that cong-solve-nat [of a n] by auto
```

lemma *cong-solve-coprime-int*:
  $\exists x. [a * x = 1] \ (mod\ n)$ **if** *coprime a n* **for** *a n x :: int*
  using *that cong-solve-int* [*of a n*] **by** (*auto simp add*: *zabs-def split*: *if-splits*)

lemma *coprime-iff-invertible-nat*:
  coprime a m $\longleftrightarrow$ ($\exists x. [a * x = Suc\ 0] \ (mod\ m)$) (**is** *?P* $\longleftrightarrow$ *?Q*)
proof
  assume *?P* **then show** *?Q*
    by (*auto dest*!: *cong-solve-coprime-nat*)
next
  assume *?Q*
  then obtain *b* **where** $[a * b = Suc\ 0] \ (mod\ m)$
    by *blast*
  with *coprime-mod-left-iff* [*of m a * b*] **show** *?P*
    by (*cases* $m = 0 \lor m = 1$)
      (*unfold cong-def*, *auto simp add*: *cong-def*)
qed

lemma *coprime-iff-invertible-int*:
  coprime a m $\longleftrightarrow$ ($\exists x. [a * x = 1] \ (mod\ m)$) (**is** *?P* $\longleftrightarrow$ *?Q*) **for** *m :: int*
proof
  assume *?P* **then show** *?Q*
    by (*auto dest*: *cong-solve-coprime-int*)
next
  assume *?Q*
  then obtain *b* **where** $[a * b = 1] \ (mod\ m)$
    by *blast*
  with *coprime-mod-left-iff* [*of m a * b*] **show** *?P*
    by (*cases* $m = 0 \lor m = 1$)
      (*unfold cong-def*, *auto simp add*: *zmult-eq-1-iff*)
qed

lemma *coprime-iff-invertible'-nat*:
  assumes $m > 0$
  shows *coprime a m* $\longleftrightarrow$ ($\exists x. 0 \le x \land x < m \land [a * x = Suc\ 0] \ (mod\ m)$)
proof −
  have $\bigwedge b. [\![0 < m; [a * b = Suc\ 0] \ (mod\ m)]\!] \Longrightarrow \exists b'<m. [a * b' = Suc\ 0] \ (mod$
$m)$
    by (*metis cong-def mod-less-divisor* [*OF assms*] *mod-mult-right-eq*)
  then show *?thesis*
    using *assms coprime-iff-invertible-nat* **by** *auto*
qed

lemma *coprime-iff-invertible'-int*:
  fixes *m :: int*
  assumes $m > 0$
  shows *coprime a m* $\longleftrightarrow$ ($\exists x. 0 \le x \land x < m \land [a * x = 1] \ (mod\ m)$)
proof −

have $\bigwedge b.\ [\![0 < m; [a * b = 1]\ (mod\ m)]\!] \Longrightarrow \exists\, b'{<}m.\ [a * b' = 1]\ (mod\ m)$
by (*meson cong-less-unique-int cong-scalar-left cong-sym cong-trans*)
then show *?thesis*
by (*metis assms coprime-iff-invertible-int cong-def cong-mult-lcancel mod-pos-pos-trivial pos-mod-conj*)
qed

lemma *cong-cong-lcm-nat*: $[x = y]\ (mod\ a) \Longrightarrow [x = y]\ (mod\ b) \Longrightarrow [x = y]\ (mod\ lcm\ a\ b)$
for $x\ y$ :: *nat*
by (*meson cong-altdef-nat′ lcm-least*)

lemma *cong-cong-lcm-int*: $[x = y]\ (mod\ a) \Longrightarrow [x = y]\ (mod\ b) \Longrightarrow [x = y]\ (mod\ lcm\ a\ b)$
for $x\ y$ :: *int*
by (*auto simp add*: *cong-iff-dvd-diff lcm-least*)

lemma *cong-cong-prod-coprime-nat*:
$[x = y]\ (mod\ (\prod i{\in}A.\ m\ i))$ **if**
$(\forall\, i{\in}A.\ [x = y]\ (mod\ m\ i))$
$(\forall\, i{\in}A.\ (\forall\, j{\in}A.\ i \neq j \longrightarrow coprime\ (m\ i)\ (m\ j)))$
for $x\ y$ :: *nat*
using *that* **by** (*induct A rule*: *infinite-finite-induct*)
(*auto intro*!: *coprime-cong-mult-nat prod-coprime-right*)

lemma *binary-chinese-remainder-nat*:
fixes *m1 m2* :: *nat*
assumes *a*: *coprime m1 m2*
shows $\exists\, x.\ [x = u1]\ (mod\ m1) \wedge [x = u2]\ (mod\ m2)$
proof −
have $\exists\, b1\ b2.\ [b1 = 1]\ (mod\ m1) \wedge [b1 = 0]\ (mod\ m2) \wedge [b2 = 0]\ (mod\ m1) \wedge [b2 = 1]\ (mod\ m2)$
proof −
from *cong-solve-coprime-nat* [*OF a*] **obtain** *x1* **where** *1*: $[m1 * x1 = 1]\ (mod\ m2)$
by *auto*
from *a* **have** *b*: *coprime m2 m1*
by (*simp add*: *ac-simps*)
from *cong-solve-coprime-nat* [*OF b*] **obtain** *x2* **where** *2*: $[m2 * x2 = 1]\ (mod\ m1)$
by *auto*
have $[m1 * x1 = 0]\ (mod\ m1)$
by (*simp add*: *cong-mult-self-left*)
moreover have $[m2 * x2 = 0]\ (mod\ m2)$
by (*simp add*: *cong-mult-self-left*)
ultimately show *?thesis*
using *1 2* **by** *blast*
qed
then obtain *b1 b2*

where [*b1* = *1*] (*mod m1*) **and** [*b1* = *0*] (*mod m2*)
and [*b2* = *0*] (*mod m1*) **and** [*b2* = *1*] (*mod m2*)
by *blast*
let *?x* = *u1* ∗ *b1* + *u2* ∗ *b2*
have [*?x* = *u1* ∗ *1* + *u2* ∗ *0*] (*mod m1*)
using ‹[*b1* = *1*] (*mod m1*)› ‹[*b2* = *0*] (*mod m1*)› *cong-add cong-scalar-left* **by** *blast*
then have [*?x* = *u1*] (*mod m1*) **by** *simp*
have [*?x* = *u1* ∗ *0* + *u2* ∗ *1*] (*mod m2*)
using ‹[*b1* = *0*] (*mod m2*)› ‹[*b2* = *1*] (*mod m2*)› *cong-add cong-scalar-left* **by** *blast*
then have [*?x* = *u2*] (*mod m2*)
by *simp*
with ‹[*?x* = *u1*] (*mod m1*)› **show** *?thesis*
by *blast*
qed

lemma *binary-chinese-remainder-int*:
fixes *m1 m2* :: *int*
assumes *a*: *coprime m1 m2*
shows ∃ *x*. [*x* = *u1*] (*mod m1*) ∧ [*x* = *u2*] (*mod m2*)
proof −
have ∃ *b1 b2*. [*b1* = *1*] (*mod m1*) ∧ [*b1* = *0*] (*mod m2*) ∧ [*b2* = *0*] (*mod m1*) ∧ [*b2* = *1*] (*mod m2*)
proof −
from *cong-solve-coprime-int* [*OF a*] **obtain** *x1* **where** *1*: [*m1* ∗ *x1* = *1*] (*mod m2*)
by *auto*
from *a* **have** *b*: *coprime m2 m1*
by (*simp add*: *ac-simps*)
from *cong-solve-coprime-int* [*OF b*] **obtain** *x2* **where** *2*: [*m2* ∗ *x2* = *1*] (*mod m1*)
by *auto*
have [*m1* ∗ *x1* = *0*] (*mod m1*)
by (*simp add*: *cong-mult-self-left*)
moreover have [*m2* ∗ *x2* = *0*] (*mod m2*)
by (*simp add*: *cong-mult-self-left*)
ultimately show *?thesis*
using *1 2* **by** *blast*
qed
then obtain *b1 b2*
where [*b1* = *1*] (*mod m1*) **and** [*b1* = *0*] (*mod m2*)
and [*b2* = *0*] (*mod m1*) **and** [*b2* = *1*] (*mod m2*)
by *blast*
let *?x* = *u1* ∗ *b1* + *u2* ∗ *b2*
have [*?x* = *u1* ∗ *1* + *u2* ∗ *0*] (*mod m1*)
using ‹[*b1* = *1*] (*mod m1*)› ‹[*b2* = *0*] (*mod m1*)› *cong-add cong-scalar-left* **by** *blast*
then have [*?x* = *u1*] (*mod m1*) **by** *simp*

have $[?x = u1 * 0 + u2 * 1]$ $(mod\ m2)$
using ‹$[b1 = 0]$ $(mod\ m2)$› ‹$[b2 = 1]$ $(mod\ m2)$› *cong-add cong-scalar-left* **by** *blast*
then have $[?x = u2]$ $(mod\ m2)$ **by** *simp*
with ‹$[?x = u1]$ $(mod\ m1)$› **show** *?thesis*
by *blast*
qed

lemma *cong-modulus-mult-nat*: $[x = y]$ $(mod\ m * n) \Longrightarrow [x = y]$ $(mod\ m)$
for $x\ y :: nat$
by (*metis cong-def mod-mult-cong-right*)

lemma *cong-less-modulus-unique-nat*: $[x = y]$ $(mod\ m) \Longrightarrow x < m \Longrightarrow y < m \Longrightarrow x = y$
for $x\ y :: nat$
by (*simp add*: *cong-def*)

lemma *binary-chinese-remainder-unique-nat*:
fixes $m1\ m2 :: nat$
assumes *a*: *coprime m1 m2*
and *nz*: $m1 \neq 0$ $m2 \neq 0$
shows $\exists!x.\ x < m1 * m2 \wedge [x = u1]\ (mod\ m1) \wedge [x = u2]\ (mod\ m2)$
proof −
obtain *y* **where** *y1*: $[y = u1]$ $(mod\ m1)$ **and** *y2*: $[y = u2]$ $(mod\ m2)$
using *binary-chinese-remainder-nat* [*OF a*] **by** *blast*
let $?x = y\ mod\ (m1 * m2)$
from *nz* **have** *less*: $?x < m1 * m2$
by *auto*
have *1*: $[?x = u1]$ $(mod\ m1)$
using *y1 mod-mult-cong-right* **by** *blast*
have *2*: $[?x = u2]$ $(mod\ m2)$
using *y2 mod-mult-cong-left* **by** *blast*
have $z = ?x$ **if** $z < m1 * m2$ $[z = u1]$ $(mod\ m1)$ $[z = u2]$ $(mod\ m2)$ **for** *z*
proof −
have $[?x = z]$ $(mod\ m1)$
by (*metis 1 cong-def that*(*2*))
moreover have $[?x = z]$ $(mod\ m2)$
by (*metis 2 cong-def that*(*3*))
ultimately have $[?x = z]$ $(mod\ m1 * m2)$
using *a* **by** (*auto intro*: *coprime-cong-mult-nat simp add*: *mod-mult-cong-left mod-mult-cong-right*)
with ‹$z < m1 * m2$› ‹$?x < m1 * m2$› **show** $z = ?x$
by (*auto simp add*: *cong-def*)
qed
with *less 1 2* **show** *?thesis*
by *blast*
qed

lemma *chinese-remainder-nat*:

fixes A :: $'a$ *set*
  and m :: $'a \Rightarrow$ *nat*
  and u :: $'a \Rightarrow$ *nat*
assumes *fin*: *finite* A
  and *cop*: $\forall\, i \in A.\ \forall\, j \in A.\ i \neq j \longrightarrow$ *coprime* $(m\ i)\ (m\ j)$
shows $\exists\, x.\ \forall\, i \in A.\ [x = u\ i]\ (mod\ m\ i)$
proof $-$
  have $\exists\, b.\ (\forall\, i \in A.\ [b\ i = 1]\ (mod\ m\ i) \wedge [b\ i = 0]\ (mod\ (\prod j \in A - \{i\}.\ m\ j)))$
  proof (*rule finite-set-choice*, *rule fin*, *rule ballI*)
    fix i
    assume $i \in A$
    with *cop* **have** *coprime* $(\prod j \in A - \{i\}.\ m\ j)\ (m\ i)$
      by (*intro prod-coprime-left*) *auto*
    then have $\exists\, x.\ [(\prod j \in A - \{i\}.\ m\ j) * x = Suc\ 0]\ (mod\ m\ i)$
      by (*elim cong-solve-coprime-nat*)
    then obtain x **where** $[(\prod j \in A - \{i\}.\ m\ j) * x = 1]\ (mod\ m\ i)$
      by *auto*
    moreover have $[(\prod j \in A - \{i\}.\ m\ j) * x = 0]\ (mod\ (\prod j \in A - \{i\}.\ m\ j))$
      by (*simp add*: *cong-0-iff*)
    ultimately show $\exists\, a.\ [a = 1]\ (mod\ m\ i) \wedge [a = 0]\ (mod\ prod\ m\ (A - \{i\}))$
      by *blast*
  qed
  then obtain b **where** b: $\bigwedge i.\ i \in A \Longrightarrow [b\ i = 1]\ (mod\ m\ i) \wedge [b\ i = 0]\ (mod$
$(\prod j \in A - \{i\}.\ m\ j))$
    by *blast*
  let $?x = \sum i{\in}A.\ (u\ i) * (b\ i)$
  show *?thesis*
  proof (*rule exI*, *clarify*)
    fix i
    assume a: $i \in A$
    show $[?x = u\ i]\ (mod\ m\ i)$
    proof $-$
      from *fin* a **have** $?x = (\sum j \in \{i\}.\ u\ j * b\ j) + (\sum j \in A - \{i\}.\ u\ j * b\ j)$
        by (*subst sum.union-disjoint* [*symmetric*]) (*auto intro*: *sum.cong*)
      then have $[?x = u\ i * b\ i + (\sum j \in A - \{i\}.\ u\ j * b\ j)]\ (mod\ m\ i)$
        by *auto*
      also have $[u\ i * b\ i + (\sum j \in A - \{i\}.\ u\ j * b\ j) =$
          $u\ i * 1 + (\sum j \in A - \{i\}.\ u\ j * 0)]\ (mod\ m\ i)$
      proof (*intro cong-add cong-scalar-left cong-sum*)
        show $[b\ i = 1]\ (mod\ m\ i)$
          using a b **by** *blast*
        show $[b\ x = 0]\ (mod\ m\ i)$ **if** $x \in A - \{i\}$ **for** x
        proof $-$
          have $x \in A$ $x \neq i$
            using *that* **by** *auto*
          then show *?thesis*
            using a b [*OF* ‹$x \in A$›] *cong-dvd-modulus-nat fin* **by** *blast*
        qed
      qed

```
      finally show ?thesis
        by simp
    qed
  qed
qed

lemma coprime-cong-prod-nat: [x = y] (mod (∏ i∈A. m i))
  if ⋀i j. ⟦i ∈ A; j ∈ A; i ≠ j⟧ ⟹ coprime (m i) (m j)
    and ⋀i. i ∈ A ⟹ [x = y] (mod m i) for x y :: nat
  using that
proof (induct A rule: infinite-finite-induct)
  case (insert x A)
  then show ?case
    by simp (metis coprime-cong-mult-nat prod-coprime-right)
qed auto

lemma chinese-remainder-unique-nat:
  fixes A :: 'a set
    and m :: 'a ⇒ nat
    and u :: 'a ⇒ nat
  assumes fin: finite A
    and nz: ∀ i∈A. m i ≠ 0
    and cop: ∀ i∈A. ∀ j∈A. i ≠ j ⟶ coprime (m i) (m j)
  shows ∃!x. x < (∏ i∈A. m i) ∧ (∀ i∈A. [x = u i] (mod m i))
proof −
  from chinese-remainder-nat [OF fin cop]
  obtain y where one: (∀ i∈A. [y = u i] (mod m i))
    by blast
  let ?x = y mod (∏ i∈A. m i)
  from fin nz have prodnz: (∏ i∈A. m i) ≠ 0
    by auto
  then have less: ?x < (∏ i∈A. m i)
    by auto
  have cong: ∀ i∈A. [?x = u i] (mod m i)
    using fin one
    by (auto simp add: cong-def dvd-prod-eqI mod-mod-cancel)
  have unique: ∀ z. z < (∏ i∈A. m i) ∧ (∀ i∈A. [z = u i] (mod m i)) ⟶ z = ?x
  proof clarify
    fix z
    assume zless: z < (∏ i∈A. m i)
    assume zcong: (∀ i∈A. [z = u i] (mod m i))
    have ∀ i∈A. [?x = z] (mod m i)
      using cong zcong by (auto simp add: cong-def)
    with fin cop have [?x = z] (mod (∏ i∈A. m i))
      by (intro coprime-cong-prod-nat) auto
    with zless less show z = ?x
      by (auto simp add: cong-def)
  qed
  from less cong unique show ?thesis
```

by *blast*
qed

end

3 Fundamental facts about Euler's totient function

theory *Totient*
imports
Complex-Main
HOL−Computational-Algebra.Primes
Cong
begin

definition *totatives* :: *nat* ⇒ *nat set* **where**
totatives n = {*k* ∈ {*0*<..*n*}. *coprime k n*}

lemma *in-totatives-iff*: *k* ∈ *totatives n* ⟷ *k* > *0* ∧ *k* ≤ *n* ∧ *coprime k n*
by (*simp add*: *totatives-def*)

lemma *totatives-code* [*code*]: *totatives n* = *Set.filter* (λ*k*. *coprime k n*) {*0*<..*n*}
by (*simp add*: *totatives-def Set.filter-def*)

lemma *finite-totatives* [*simp*]: *finite* (*totatives n*)
by (*simp add*: *totatives-def*)

lemma *totatives-subset*: *totatives n* ⊆ {*0*<..*n*}
by (*auto simp*: *totatives-def*)

lemma *zero-not-in-totatives* [*simp*]: *0* ∉ *totatives n*
by (*auto simp*: *totatives-def*)

lemma *totatives-le*: *x* ∈ *totatives n* ⟹ *x* ≤ *n*
by (*auto simp*: *totatives-def*)

lemma *totatives-less*:
assumes *x* ∈ *totatives n n* > *1*
shows *x* < *n*
proof −
from *assms* **have** *x* ≠ *n* **by** (*auto simp*: *totatives-def*)
with *totatives-le*[*OF assms*(*1*)] **show** *?thesis* **by** *simp*
qed

lemma *totatives-0* [*simp*]: *totatives 0* = {}
by (*auto simp*: *totatives-def*)

lemma *totatives-1* [*simp*]: *totatives 1* = {*Suc 0*}

```
  by (auto simp: totatives-def)

lemma totatives-Suc-0 [simp]: totatives (Suc 0) = {Suc 0}
  by (auto simp: totatives-def)

lemma one-in-totatives [simp]: n > 0 ⟹ Suc 0 ∈ totatives n
  by (auto simp: totatives-def)

lemma totatives-eq-empty-iff [simp]: totatives n = {} ⟷ n = 0
  using one-in-totatives[of n] by (auto simp del: one-in-totatives)

lemma minus-one-in-totatives:
  assumes n ≥ 2
  shows n − 1 ∈ totatives n
  using assms coprime-diff-one-left-nat [of n] by (simp add: in-totatives-iff)

lemma power-in-totatives:
  assumes m > 1 coprime m g
  shows   g ^ i mod m ∈ totatives m
proof −
  have ¬m dvd g ^ i
  proof
    assume m dvd g ^ i
    hence ¬coprime m (g ^ i)
      using ‹m > 1› by (subst coprime-absorb-left) auto
    with ‹coprime m g› show False by simp
  qed
  with assms show ?thesis
    by (auto simp: totatives-def coprime-commute intro!: Nat.gr0I)
qed

lemma totatives-prime-power-Suc:
  assumes prime p
  shows   totatives (p ^ Suc n) = {0<..p^Suc n} − (λm. p ∗ m) ‘ {0<..p^n}
proof safe
  fix m assume m: p ∗ m ∈ totatives (p ^ Suc n) and m: m ∈ {0<..p^n}
  thus False using assms by (auto simp: totatives-def gcd-mult-left)
next
  fix k assume k: k ∈ {0<..p^Suc n} k ∉ (λm. p ∗ m) ‘ {0<..p^n}
  from k have ¬(p dvd k) by (auto elim!: dvdE)
  hence coprime k (p ^ Suc n)
    using prime-imp-coprime [OF assms, of k]
    by (cases n > 0) (auto simp add: ac-simps)
  with k show k ∈ totatives (p ^ Suc n) by (simp add: totatives-def)
qed (auto simp: totatives-def)

lemma totatives-prime: prime p ⟹ totatives p = {0<..<p}
  using totatives-prime-power-Suc [of p 0] by auto
```

lemma *bij-betw-totatives*:
  assumes $m1 > 1$ $m2 > 1$ *coprime m1 m2*
  shows *bij-betw* (λ*x*. (*x mod m1*, *x mod m2*)) (*totatives* (*m1* ∗ *m2*))
      (*totatives m1* × *totatives m2*)
  unfolding *bij-betw-def*
proof
  show *inj-on* (λ*x*. (*x mod m1*, *x mod m2*)) (*totatives* (*m1* ∗ *m2*))
  proof (*intro inj-onI*, *clarify*)
    fix *x y* **assume** *xy*: *x* ∈ *totatives* (*m1* ∗ *m2*) *y* ∈ *totatives* (*m1* ∗ *m2*)
          x mod m1 = *y mod m1* *x mod m2* = *y mod m2*
    have *ex*: ∃!*z*. *z* < *m1* ∗ *m2* ∧ [*z* = *x*] (*mod m1*) ∧ [*z* = *x*] (*mod m2*)
      by (*rule binary-chinese-remainder-unique-nat*) (*insert assms*, *simp-all*)
    have *x* < *m1* ∗ *m2* ∧ [*x* = *x*] (*mod m1*) ∧ [*x* = *x*] (*mod m2*)
        y < *m1* ∗ *m2* ∧ [*y* = *x*] (*mod m1*) ∧ [*y* = *x*] (*mod m2*)
      using *xy assms* **by** (*simp-all add*: *totatives-less one-less-mult cong-def*)
    from *this*[*THEN the1-equality*[*OF ex*]] **show** *x* = *y* **by** *simp*
  qed
next
  show (λ*x*. (*x mod m1*, *x mod m2*)) ' *totatives* (*m1* ∗ *m2*) = *totatives m1* × *totatives m2*
  proof *safe*
    fix *x* **assume** *x* ∈ *totatives* (*m1* ∗ *m2*)
    with *assms* **show** *x mod m1* ∈ *totatives m1* *x mod m2* ∈ *totatives m2*
      using *coprime-common-divisor* [*of x m1 m1*] *coprime-common-divisor* [*of x m2 m2*]
      by (*auto simp add*: *in-totatives-iff mod-greater-zero-iff-not-dvd*)
  next
    fix *a b* **assume** *ab*: *a* ∈ *totatives m1* *b* ∈ *totatives m2*
    with *assms* **have** *ab'*: *a* < *m1* *b* < *m2* **by** (*auto simp*: *totatives-less*)
    with *binary-chinese-remainder-unique-nat*[*OF assms*(*3*), *of a b*] **obtain** *x*
      where *x*: *x* < *m1* ∗ *m2* *x mod m1* = *a* *x mod m2* = *b* **by** (*auto simp*: *cong-def*)
    from *x ab assms*(*3*) **have** *x* ∈ *totatives* (*m1* ∗ *m2*)
      by (*auto intro*: *ccontr simp add*: *in-totatives-iff*)
    with *x* **show** (*a*, *b*) ∈ (λ*x*. (*x mod m1*, *x mod m2*)) ' *totatives* (*m1*∗*m2*) **by** *blast*
  qed
qed

lemma *bij-betw-totatives-gcd-eq*:
  fixes *n d* :: *nat*
  assumes *d dvd n* *n* > *0*
  shows *bij-betw* (λ*k*. *k* ∗ *d*) (*totatives* (*n div d*)) {*k*∈{*0*<..*n*}. *gcd k n* = *d*}
  unfolding *bij-betw-def*
proof
  show *inj-on* (λ*k*. *k* ∗ *d*) (*totatives* (*n div d*))
    by (*auto simp*: *inj-on-def*)
next
  show (λ*k*. *k* ∗ *d*) ' *totatives* (*n div d*) = {*k*∈{*0*<..*n*}. *gcd k n* = *d*}
  proof (*intro equalityI subsetI*, *goal-cases*)

case (*1 k*)
then show *?case* **using** *assms*
by (*auto elim*: *dvdE simp add*: *in-totatives-iff ac-simps gcd-mult-right*)
next
case (*2 k*)
hence *d dvd k* **by** *auto*
then obtain *l* **where** *k*: *k* = *l* ∗ *d* **by** (*elim dvdE*) *auto*
from *2 assms* **show** *?case*
using *gcd-mult-right* [*of* - *d l*]
by (*auto intro*: *gcd-eq-1-imp-coprime elim*!: *dvdE simp add*: *k image-iff in-totatives-iff ac-simps*)
qed
qed

definition *totient* :: *nat* ⇒ *nat* **where**
totient n = *card* (*totatives n*)

primrec *totient-naive* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *nat* **where**
totient-naive 0 acc n = *acc*
| *totient-naive* (*Suc k*) *acc n* =
(*if coprime* (*Suc k*) *n then totient-naive k* (*acc* + *1*) *n else totient-naive k acc n*)

lemma *totient-naive*:
totient-naive k acc n = *card* {*x* ∈ {*0*<..*k*}. *coprime x n*} + *acc*
proof (*induction k arbitrary*: *acc*)
case (*Suc k acc*)
have *totient-naive* (*Suc k*) *acc n* =
(*if coprime* (*Suc k*) *n then 1 else 0*) + *card* {*x* ∈ {*0*<..*k*}. *coprime x n*} + *acc*
using *Suc* **by** *simp*
also have (*if coprime* (*Suc k*) *n then 1 else 0*) =
card (*if coprime* (*Suc k*) *n then* {*Suc k*} *else* {}) **by** *auto*
also have . . . + *card* {*x* ∈ {*0*<..*k*}. *coprime x n*} =
card ((*if coprime* (*Suc k*) *n then* {*Suc k*} *else* {}) ∪ {*x* ∈ {*0*<..*k*}. *coprime x n*})
by (*intro card-Un-disjoint* [*symmetric*]) *auto*
also have ((*if coprime* (*Suc k*) *n then* {*Suc k*} *else* {}) ∪ {*x* ∈ {*0*<..*k*}. *coprime x n*}) =
{*x* ∈ {*0*<..*Suc k*}. *coprime x n*} **by** (*auto elim*: *le-SucE*)
finally show *?case* **.**
qed *simp-all*

lemma *totient-code-naive* [*code*]: *totient n* = *totient-naive n 0 n*
by (*subst totient-naive*) (*simp add*: *totient-def totatives-def*)

lemma *totient-le*: *totient n* ≤ *n*
proof −
have *card* (*totatives n*) ≤ *card* {*0*<..*n*}

by (*intro card-mono*) (*auto simp*: *totatives-def*)
thus *?thesis* **by** (*simp add*: *totient-def*)
qed

lemma *totient-less*:
assumes $n > 1$
shows *totient* $n < n$
proof −
from *assms* **have** *card* (*totatives n*) $\leq$ *card* $\{0<..<n\}$
using *totatives-less*[*of - n*] *totatives-subset*[*of n*] **by** (*intro card-mono*) *auto*
with *assms* **show** *?thesis* **by** (*simp add*: *totient-def*)
qed

lemma *totient-0* [*simp*]: *totient* $0 = 0$
by (*simp add*: *totient-def*)

lemma *totient-Suc-0* [*simp*]: *totient* (*Suc 0*) = *Suc 0*
by (*simp add*: *totient-def*)

lemma *totient-1* [*simp*]: *totient 1* = *Suc 0*
by *simp*

lemma *totient-0-iff* [*simp*]: *totient* $n = 0 \longleftrightarrow n = 0$
by (*auto simp*: *totient-def*)

lemma *totient-gt-0-iff* [*simp*]: *totient* $n > 0 \longleftrightarrow n > 0$
by (*auto intro*: *Nat.gr0I*)

lemma *totient-gt-1*:
assumes $n > 2$
shows *totient* $n > 1$
proof −
have $\{1, n - 1\} \subseteq$ *totatives n*
using *assms coprime-diff-one-left-nat*[*of n*] **by** (*auto simp*: *totatives-def*)
hence *card* $\{1, n - 1\} \leq$ *card* (*totatives n*)
by (*intro card-mono*) *auto*
thus *?thesis* **using** *assms*
by (*simp add*: *totient-def*)
qed

lemma *card-gcd-eq-totient*:
$n > 0 \Longrightarrow d$ *dvd* $n \Longrightarrow$ *card* $\{k \in \{0<..n\}.\ gcd\ k\ n = d\}$ = *totient* (*n div d*)
unfolding *totient-def* **by** (*rule sym*, *rule bij-betw-same-card*[*OF bij-betw-totatives-gcd-eq*])

lemma *totient-divisor-sum*: $(\sum d \mid d\ dvd\ n.\ totient\ d) = n$
proof (*cases* $n = 0$)
case *False*
hence $n > 0$ **by** *simp*
define *A* **where** $A = (\lambda d.\ \{k \in \{0<..n\}.\ gcd\ k\ n = d\})$

have ∗: *card* (*A d*) = *totient* (*n div d*) **if** *d*: *d dvd n* **for** *d*
  using ‹*n* > *0*› **and** *d* **unfolding** *A-def* **by** (*rule card-gcd-eq-totient*)
have *n* = *card* {*1*..*n*} **by** *simp*
also have {*1*..*n*} = (⋃*d*∈{*d*. *d dvd n*}. *A d*) **by** *safe* (*auto simp*: *A-def*)
also have *card* ... = (∑ *d* | *d dvd n*. *card* (*A d*))
  using ‹*n* > *0*› **by** (*intro card-UN-disjoint*) (*auto simp*: *A-def*)
also have ... = (∑ *d* | *d dvd n*. *totient* (*n div d*)) **by** (*intro sum.cong refl* ∗)
auto
also have ... = (∑ *d* | *d dvd n*. *totient d*) **using** ‹*n* > *0*›
  by (*intro sum.reindex-bij-witness*[*of* - (*div*) *n* (*div*) *n*]) (*auto elim*: *dvdE*)
finally show *?thesis* **..**
qed *auto*

lemma *totient-mult-coprime*:
  assumes *coprime m n*
  shows *totient* (*m* ∗ *n*) = *totient m* ∗ *totient n*
proof (*cases m* > *1* ∧ *n* > *1*)
  case *True*
  hence *mn*: *m* > *1 n* > *1* **by** *simp-all*
  have *totient* (*m* ∗ *n*) = *card* (*totatives* (*m* ∗ *n*)) **by** (*simp add*: *totient-def*)
  also have ... = *card* (*totatives m* × *totatives n*)
    using *bij-betw-totatives* [*OF mn* ‹*coprime m n*›] **by** (*rule bij-betw-same-card*)
  also have ... = *totient m* ∗ *totient n* **by** (*simp add*: *totient-def*)
  finally show *?thesis* **.**
next
  case *False*
  with *assms* **show** *?thesis* **by** (*cases m*; *cases n*) *auto*
qed

lemma *totient-prime-power-Suc*:
  assumes *prime p*
  shows *totient* (*p* ^ *Suc n*) = *p* ^ *n* ∗ (*p* − *1*)
proof −
  from *assms* **have** *totient* (*p* ^ *Suc n*) = *card* ({*0*<..*p* ^ *Suc n*} − (∗) *p* ‘ {*0*<..*p*
^ *n*})
    unfolding *totient-def* **by** (*subst totatives-prime-power-Suc*) *simp-all*
  also from *assms* **have** ... = *p* ^ *Suc n* − *card* ((∗) *p* ‘ {*0*<..*p*^*n*})
    by (*subst card-Diff-subset*) (*auto intro*: *prime-gt-0-nat*)
  also from *assms* **have** *card* ((∗) *p* ‘ {*0*<..*p*^*n*}) = *p* ^ *n*
    by (*subst card-image*) (*auto simp*: *inj-on-def*)
  also have *p* ^ *Suc n* − *p* ^ *n* = *p* ^ *n* ∗ (*p* − *1*) **by** (*simp add*: *algebra-simps*)
  finally show *?thesis* **.**
qed

lemma *totient-prime-power*:
  assumes *prime p n* > *0*
  shows *totient* (*p* ^ *n*) = *p* ^ (*n* − *1*) ∗ (*p* − *1*)
  using *totient-prime-power-Suc*[*of p n* − *1*] *assms* **by** *simp*

```
lemma totient-imp-prime:
  assumes totient p = p − 1 p > 0
  shows   prime p
proof (cases p = 1)
  case True
  with assms show ?thesis by auto
next
  case False
  with assms have p: p > 1 by simp
  have x ∈ {0<..<p} if x ∈ totatives p for x
    using that and p by (cases x = p) (auto simp: totatives-def)
  with assms have ∗: totatives p = {0<..<p}
    by (intro card-subset-eq) (auto simp: totient-def)
  have ∗∗: False if x ≠ 1 x ≠ p x dvd p for x
  proof −
    from that have nz: x ≠ 0 by (auto intro!: Nat.gr0I)
    from that and p have le: x ≤ p by (intro dvd-imp-le) auto
    from that and nz have ¬coprime x p
      by (auto elim: dvdE)
    hence x ∉ totatives p by (simp add: totatives-def)
    also note ∗
    finally show False using that and le by auto
  qed
  hence (∀ m. m dvd p ⟶ m = 1 ∨ m = p) by blast
  with p show ?thesis by (subst prime-nat-iff) (auto dest: ∗∗)
qed

lemma totient-prime:
  assumes prime p
  shows   totient p = p − 1
  using totient-prime-power-Suc[of p 0] assms by simp

lemma totient-2 [simp]: totient 2 = 1
  and totient-3 [simp]: totient 3 = 2
  and totient-5 [simp]: totient 5 = 4
  and totient-7 [simp]: totient 7 = 6
  by (subst totient-prime; simp)+

lemma totient-4 [simp]: totient 4 = 2
  and totient-8 [simp]: totient 8 = 4
  and totient-9 [simp]: totient 9 = 6
 using totient-prime-power[of 2 2] totient-prime-power[of 2 3] totient-prime-power[of
3 2]
  by simp-all

lemma totient-6 [simp]: totient 6 = 2
  using totient-mult-coprime [of 2 3] coprime-add-one-right [of 2]
  by simp
```

lemma *totient-even*:
  assumes $n > 2$
  shows *even* (*totient* n)
proof (*cases* $\exists p.$ *prime* $p \wedge p \neq 2 \wedge p$ *dvd* n)
  case *True*
  then obtain p **where** p: *prime* p $p \neq 2$ p *dvd* n **by** *auto*
  from ‹$p \neq 2$› **have** $p = 0 \vee p = 1 \vee p > 2$ **by** *auto*
  with $p(1)$ **have** *odd* p **using** *prime-odd-nat*[*of* p] **by** *auto*
  define k **where** $k =$ *multiplicity* p n
  from p *assms* **have** *k-pos*: $k > 0$ **unfolding** *k-def* **by** (*subst multiplicity-gt-zero-iff*) *auto*
  have p ^ k *dvd* n **unfolding** *k-def* **by** (*simp add*: *multiplicity-dvd*)
  then obtain m **where** m: $n = p$ ^ $k * m$ **by** (*elim dvdE*)
  with *assms* **have** *m-pos*: $m > 0$ **by** (*auto intro*!: *Nat.gr0I*)
  from *k-def m-pos* p **have** $\neg$ p *dvd* m
   by (*subst* (*asm*) m) (*auto intro*!: *Nat.gr0I simp*: *prime-elem-multiplicity-mult-distrib*

prime-elem-multiplicity-eq-zero-iff)
  with ‹*prime* p› **have** *coprime* p m
    by (*rule prime-imp-coprime*)
  with ‹$k > 0$› **have** *coprime* (p ^ k) m
    by *simp*
  then show *?thesis* **using** p *k-pos* ‹*odd* p›
    by (*auto simp add*: m *totient-mult-coprime totient-prime-power*)
next
  case *False*
  from *assms* **have** $n = (\prod p\in$*prime-factors* $n.$ p ^ *multiplicity* p n)
    by (*intro Primes.prime-factorization-nat*) *auto*
  also from *False* **have** $\ldots = (\prod p\in$*prime-factors* $n.$ *if* $p = 2$ *then* 2 ^ *multiplicity* 2 n *else* 1)
    by (*intro prod.cong refl*) *auto*
  also have $\ldots = 2$ ^ *multiplicity* 2 n
   by (*subst prod.delta*[*OF finite-set-mset*]) (*auto simp*: *prime-factors-multiplicity*)
  finally have n: $n = 2$ ^ *multiplicity* 2 n **.**
  have *multiplicity* 2 $n = 0 \vee$ *multiplicity* 2 $n = 1 \vee$ *multiplicity* 2 $n > 1$ **by** *force*
  with n *assms* **have** *multiplicity* 2 $n > 1$ **by** *auto*
  thus *?thesis* **by** (*subst* n) (*simp add*: *totient-prime-power*)
qed

lemma *totient-prod-coprime*:
  assumes *pairwise coprime* (f ‘ A) *inj-on* f A
  shows *totient* (*prod* f A) $= (\prod a\in A.$ *totient* (f a))
  using *assms*
proof (*induction* A *rule*: *infinite-finite-induct*)
  case (*insert* x A)
  have $*$: *coprime* (*prod* f A) (f x)
  proof (*rule prod-coprime-left*)
    fix y
    assume $y \in A$

```
    with ‹x ∉ A› have y ≠ x
      by auto
    with ‹x ∉ A› ‹y ∈ A› ‹inj-on f (insert x A)› have f y ≠ f x
      using inj-onD [of f insert x A y x]
      by auto
    with ‹y ∈ A› show coprime (f y) (f x)
      using pairwiseD [OF ‹pairwise coprime (f ‘ insert x A)›]
      by auto
  qed
  from insert.hyps have prod f (insert x A) = prod f A * f x by simp
  also have totient ... = totient (prod f A) * totient (f x)
    using insert.hyps insert.prems by (intro totient-mult-coprime *)
  also have totient (prod f A) = (∏ a∈A. totient (f a))
    using insert.prems by (intro insert.IH) (auto dest: pairwise-subset)
  also from insert.hyps have ... * totient (f x) = (∏ a∈insert x A. totient (f a))
by simp
  finally show ?case .
qed simp-all


lemma prime-power-eq-imp-eq:
  fixes p q :: 'a :: factorial-semiring
  assumes prime p prime q m > 0
  assumes p ^ m = q ^ n
  shows   p = q
proof (rule ccontr)
  assume pq: p ≠ q
  from assms have m = multiplicity p (p ^ m)
    by (subst multiplicity-prime-power) auto
  also note ‹p ^ m = q ^ n›
  also from assms pq have multiplicity p (q ^ n) = 0
    by (subst multiplicity-distinct-prime-power) auto
  finally show False using ‹m > 0› by simp
qed

lemma totient-formula1:
  assumes n > 0
  shows   totient n = (∏ p∈prime-factors n. p ^ (multiplicity p n − 1) * (p − 1))
proof −
  from assms have n = (∏ p∈prime-factors n. p ^ multiplicity p n)
    by (rule prime-factorization-nat)
  also have totient ... = (∏ x∈prime-factors n. totient (x ^ multiplicity x n))
  proof (rule totient-prod-coprime)
    show pairwise coprime ((λp. p ^ multiplicity p n) ‘ prime-factors n)
    proof (rule pairwiseI, clarify)
      fix p q assume *: p ∈# prime-factorization n q ∈# prime-factorization n
                 p ^ multiplicity p n ≠ q ^ multiplicity q n
      then have multiplicity p n > 0 multiplicity q n > 0
        by (simp-all add: prime-factors-multiplicity)
```

```
      with * primes-coprime [of p q] show coprime (p ^ multiplicity p n) (q ^
multiplicity q n)
        by auto
    qed
  next
    show inj-on (λp. p ^ multiplicity p n) (prime-factors n)
    proof
      fix p q assume pq: p ∈# prime-factorization n q ∈# prime-factorization n
                         p ^ multiplicity p n = q ^ multiplicity q n
      from assms and pq have prime p prime q multiplicity p n > 0
        by (simp-all add: prime-factors-multiplicity)
      from prime-power-eq-imp-eq[OF this pq(3)] show p = q .
    qed
  qed
  also have ... = (∏ p∈prime-factors n. p ^(multiplicity p n − 1) * (p − 1))
   by (intro prod.cong refl totient-prime-power) (auto simp: prime-factors-multiplicity)
  finally show ?thesis .
qed

lemma totient-dvd:
  assumes m dvd n
  shows   totient m dvd totient n
proof (cases m = 0 ∨ n = 0)
  case False
  let ?M = λp m :: nat. multiplicity p m − 1
  have (∏ p∈prime-factors m. p ^ ?M p m * (p − 1)) dvd
          (∏ p∈prime-factors n. p ^ ?M p n * (p − 1)) using assms False
   by (intro prod-dvd-prod-subset2 mult-dvd-mono dvd-refl le-imp-power-dvd diff-le-mono
             dvd-prime-factors dvd-imp-multiplicity-le) auto
  with False show ?thesis by (simp add: totient-formula1)
qed (insert assms, auto)

lemma totient-dvd-mono:
  assumes m dvd n n > 0
  shows   totient m ≤ totient n
  by (cases m = 0) (insert assms, auto intro: dvd-imp-le totient-dvd)


lemma prime-factors-power: n > 0 ⟹ prime-factors (x ^ n) = prime-factors x
  by (cases x = 0; cases n = 0)
       (auto simp: prime-factors-multiplicity prime-elem-multiplicity-power-distrib
zero-power)

lemma totient-formula2:
  real (totient n) = real n * (∏ p∈prime-factors n. 1 − 1 / real p)
proof (cases n = 0)
  case False
  have real (totient n) = (∏ p∈prime-factors n. real
          (p ^(multiplicity p n − 1) * (p − 1)))
```

using *False* **by** (*subst totient-formula1*) *simp-all*
also have ... = ($\prod$ *p*∈*prime-factors n. real* (*p* ^ *multiplicity p n*) ∗ (*1* − *1* / *real p*))
by (*intro prod.cong refl*) (*auto simp add*: *field-simps prime-factors-multiplicity prime-ge-Suc-0-nat of-nat-diff power-Suc* [*symmetric*] *simp del*: *power-Suc*)
also have ... = *real* ($\prod$ *p*∈*prime-factors n. p* ^ *multiplicity p n*) ∗
($\prod$ *p*∈*prime-factors n. 1* − *1* / *real p*) **by** (*subst prod.distrib*) *auto*
also have ($\prod$ *p*∈*prime-factors n. p* ^ *multiplicity p n*) = *n*
using *False* **by** (*intro Primes.prime-factorization-nat* [*symmetric*]) *auto*
finally show *?thesis* **.**
qed *auto*

lemma *totient-gcd*: *totient* (*a* ∗ *b*) ∗ *totient* (*gcd a b*) = *totient a* ∗ *totient b* ∗ *gcd a b*
proof (*cases a* = *0* ∨ *b* = *0*)
case *False*
let *?P* = *prime-factors* :: *nat* ⇒ *nat set*
have *real* (*totient a* ∗ *totient b* ∗ *gcd a b*) = *real* (*a* ∗ *b* ∗ *gcd a b*) ∗
(($\prod$ *p*∈*?P a. 1* − *1* / *real p*) ∗ ($\prod$ *p*∈*?P b. 1* − *1* / *real p*))
by (*simp add*: *totient-formula2*)
also have *?P a* = (*?P a* − *?P b*) ∪ (*?P a* ∩ *?P b*) **by** *auto*
also have ($\prod$ *p*∈.... *1* − *1* / *real p*) =
($\prod$ *p*∈*?P a* − *?P b. 1* − *1* / *real p*) ∗ ($\prod$ *p*∈*?P a* ∩ *?P b. 1* − *1* / *real p*)
by (*rule prod.union-disjoint*) *blast+*
also have ... ∗ ($\prod$ *p*∈*?P b. 1* − *1* / *real p*) = ($\prod$ *p*∈*?P a* − *?P b. 1* − *1* / *real p*) ∗
($\prod$ *p*∈*?P b. 1* − *1* / *real p*) ∗ ($\prod$ *p*∈*?P a* ∩ *?P b. 1* − *1* / *real p*) (**is** - = *?A* ∗ -)
by (*simp only*: *mult-ac*)
also have *?A* = ($\prod$ *p*∈*?P a* − *?P b* ∪ *?P b. 1* − *1* / *real p*)
by (*rule prod.union-disjoint* [*symmetric*]) *blast+*
also have *?P a* − *?P b* ∪ *?P b* = *?P a* ∪ *?P b* **by** *blast*
also have *real* (*a* ∗ *b* ∗ *gcd a b*) ∗ (($\prod$ *p*∈.... *1* − *1* / *real p*) ∗
($\prod$ *p*∈*?P a* ∩ *?P b. 1* − *1* / *real p*)) = *real* (*totient* (*a* ∗ *b*) ∗ *totient* (*gcd a b*))
using *False* **by** (*simp add*: *totient-formula2 prime-factors-product prime-factorization-gcd*)
finally show *?thesis* **by** (*simp only*: *of-nat-eq-iff*)
qed *auto*

lemma *totient-mult*: *totient* (*a* ∗ *b*) = *totient a* ∗ *totient b* ∗ *gcd a b div totient* (*gcd a b*)
by (*subst totient-gcd* [*symmetric*]) *simp*

lemma *of-nat-eq-1-iff*: *of-nat x* = (*1* :: *'a* :: {*semiring-1*, *semiring-char-0*}) ⟷ *x* = *1*
by (*fact of-nat-eq-1-iff*)

lemma *odd-imp-coprime-nat*:
  assumes *odd* (*n*::*nat*)
  shows *coprime n 2*
proof −
  from *assms* **obtain** *k* **where** *n*: *n* = *Suc* (*2* ∗ *k*) **by** (*auto elim*!: *oddE*)
  have *coprime* (*Suc* (*2* ∗ *k*)) (*2* ∗ *k*)
    by (*fact coprime-Suc-left-nat*)
  then show *?thesis* **using** *n*
    by *simp*
qed

lemma *totient-double*: *totient* (*2* ∗ *n*) = (*if even n then 2* ∗ *totient n else totient n*)
  by (*simp add*: *totient-mult ac-simps odd-imp-coprime-nat*)

lemma *totient-power-Suc*: *totient* (*n* ^ *Suc m*) = *n* ^ *m* ∗ *totient n*
proof (*induction m arbitrary*: *n*)
  case (*Suc m n*)
  have *totient* (*n* ^ *Suc* (*Suc m*)) = *totient* (*n* ∗ *n* ^ *Suc m*) **by** *simp*
  also have . . . = *n* ^ *Suc m* ∗ *totient n*
    using *Suc.IH* **by** (*subst totient-mult*) *simp*
  finally show *?case* **.**
qed *simp-all*

lemma *totient-power*: *m* > *0* ⟹ *totient* (*n* ^ *m*) = *n* ^ (*m* − *1*) ∗ *totient n*
  using *totient-power-Suc*[*of n m* − *1*] **by** (*cases m*) *simp-all*

lemma *totient-gcd-lcm*: *totient* (*gcd a b*) ∗ *totient* (*lcm a b*) = *totient a* ∗ *totient b*
proof (*cases a* = *0* ∨ *b* = *0*)
  case *False*
  let *?P* = *prime-factors* :: *nat* ⇒ *nat set* **and** *?f* = λ*p*::*nat*. *1* − *1* / *real p*
  have *real* (*totient* (*gcd a b*) ∗ *totient* (*lcm a b*)) = *real* (*gcd a b* ∗ *lcm a b*) ∗
        (*prod ?f* (*?P a* ∩ *?P b*) ∗ *prod ?f* (*?P a* ∪ *?P b*))
    using *False* **unfolding** *of-nat-mult*
    by (*simp add*: *totient-formula2 prime-factorization-gcd prime-factorization-lcm*)
  also have *gcd a b* ∗ *lcm a b* = *a* ∗ *b* **by** *simp*
  also have *?P a* ∪ *?P b* = (*?P a* − *?P a* ∩ *?P b*) ∪ *?P b* **by** *blast*
  also have *prod ?f* . . . = *prod ?f* (*?P a* − *?P a* ∩ *?P b*) ∗ *prod ?f* (*?P b*)
    by (*rule prod.union-disjoint*) *blast*+
  also have *prod ?f* (*?P a* ∩ *?P b*) ∗ . . . =
            prod ?f (*?P a* ∩ *?P b* ∪ (*?P a* − *?P a* ∩ *?P b*)) ∗ *prod ?f* (*?P b*)
    by (*subst prod.union-disjoint*) *auto*
  also have *?P a* ∩ *?P b* ∪ (*?P a* − *?P a* ∩ *?P b*) = *?P a* **by** *blast*
  also have *real* (*a* ∗ *b*) ∗ (*prod ?f* (*?P a*) ∗ *prod ?f* (*?P b*)) = *real* (*totient a* ∗ *totient b*)
    using *False* **by** (*simp add*: *totient-formula2*)
  finally show *?thesis* **by** (*simp only*: *of-nat-eq-iff*)
qed *auto*

end

4 Residue rings

theory *Residues*
imports
  Cong
  HOL−Algebra.Multiplicative-Group
  Totient
begin

definition *QuadRes* :: *int* ⇒ *int* ⇒ *bool*
  where *QuadRes p a* = (∃ *y*. ([*y*^*2* = *a*] (*mod p*)))

definition *Legendre* :: *int* ⇒ *int* ⇒ *int*
  where *Legendre a p* =
    (*if* ([*a* = *0*] (*mod p*)) *then 0*
     else if QuadRes p a then 1
     else −*1*)

4.1 A locale for residue rings

definition *residue-ring* :: *int* ⇒ *int ring*
  where
    residue-ring m =
      (|*carrier* = {*0*..*m* − *1*},
       monoid.mult = λ*x y*. (*x* ∗ *y*) *mod m*,
       one = *1*,
       zero = *0*,
       add = λ*x y*. (*x* + *y*) *mod m*|)

locale *residues* =
  fixes *m* :: *int* **and** *R* (**structure**)
  assumes *m-gt-one*: *m* > *1*
  defines *R* ≡ *residue-ring m*
begin

lemma *abelian-group*: *abelian-group R*
proof −
  have ∃ *y*∈{*0*..*m* − *1*}. (*x* + *y*) *mod m* = *0* **if** *0* ≤ *x x* < *m* **for** *x*
  proof (*cases x* = *0*)
    case *True*
    with *m-gt-one* **show** *?thesis* **by** *simp*
  next
    case *False*
    then have (*x* + (*m* − *x*)) *mod m* = *0*
      by *simp*
    with *m-gt-one that* **show** *?thesis*
     by (*metis False atLeastAtMost-iff diff-ge-0-iff-ge diff-left-mono int-one-le-iff-zero-less*

less-le)
  qed
  with *m-gt-one* **show** *?thesis*
    by (*fastforce simp add*: *R-def residue-ring-def mod-add-right-eq ac-simps intro*!: *abelian-groupI*)
qed

lemma *comm-monoid*: *comm-monoid R*
  unfolding *R-def residue-ring-def*
  apply (*rule comm-monoidI*)
    using *m-gt-one* **apply** *auto*
  apply (*metis mod-mult-right-eq mult.assoc mult.commute*)
  apply (*metis mult.commute*)
  done

lemma *cring*: *cring R*
  apply (*intro cringI abelian-group comm-monoid*)
  unfolding *R-def residue-ring-def*
  apply (*auto simp add*: *comm-semiring-class.distrib mod-add-eq mod-mult-left-eq*)
  done

end

sublocale *residues* < *cring*
  by (*rule cring*)

context *residues*
begin

These lemmas translate back and forth between internal and external concepts.

lemma *res-carrier-eq*: *carrier R* = $\{0..m - 1\}$
  by (*auto simp*: *R-def residue-ring-def*)

lemma *res-add-eq*: $x \oplus y = (x + y)$ *mod m*
  by (*auto simp*: *R-def residue-ring-def*)

lemma *res-mult-eq*: $x \otimes y = (x * y)$ *mod m*
  by (*auto simp*: *R-def residue-ring-def*)

lemma *res-zero-eq*: $\mathbf{0} = 0$
  by (*auto simp*: *R-def residue-ring-def*)

lemma *res-one-eq*: $\mathbf{1} = 1$
  by (*auto simp*: *R-def residue-ring-def units-of-def*)

lemma *res-units-eq*: *Units R* = $\{x.\ 0 < x \wedge x < m \wedge$ *coprime x m*$\}$
  using *m-gt-one*

apply (*auto simp add*: *Units-def R-def residue-ring-def ac-simps invertible-coprime intro*: *ccontr*)
apply (*subst* (*asm*) *coprime-iff-invertible′-int*)
apply (*auto simp add*: *cong-def*)
done

lemma *res-neg-eq*: $\ominus$ *x* = (− *x*) *mod m*
using *m-gt-one* **unfolding** *R-def a-inv-def m-inv-def residue-ring-def*
apply *simp*
apply (*rule the-equality*)
apply (*simp add*: *mod-add-right-eq*)
apply (*simp add*: *add.commute mod-add-right-eq*)
apply (*metis add.right-neutral minus-add-cancel mod-add-right-eq mod-pos-pos-trivial*)
done

lemma *finite* [*iff*]: *finite* (*carrier R*)
by (*simp add*: *res-carrier-eq*)

lemma *finite-Units* [*iff*]: *finite* (*Units R*)
by (*simp add*: *finite-ring-finite-units*)

The function $a \mapsto a\ mod\ m$ maps the integers to the residue classes. The following lemmas show that this mapping respects addition and multiplication on the integers.

lemma *mod-in-carrier* [*iff*]: *a mod m* ∈ *carrier R*
unfolding *res-carrier-eq*
using *insert m-gt-one* **by** *auto*

lemma *add-cong*: (*x mod m*) $\oplus$ (*y mod m*) = (*x* + *y*) *mod m*
by (*auto simp*: *R-def residue-ring-def mod-simps*)

lemma *mult-cong*: (*x mod m*) $\otimes$ (*y mod m*) = (*x* ∗ *y*) *mod m*
by (*auto simp*: *R-def residue-ring-def mod-simps*)

lemma *zero-cong*: **0** = *0*
by (*auto simp*: *R-def residue-ring-def*)

lemma *one-cong*: **1** = *1 mod m*
using *m-gt-one* **by** (*auto simp*: *R-def residue-ring-def*)

lemma *pow-cong*: (*x mod m*) [^] *n* = *x*^*n mod m*
using *m-gt-one*
apply (*induct n*)
apply (*auto simp add*: *nat-pow-def one-cong*)
apply (*metis mult.commute mult-cong*)
done

lemma *neg-cong*: $\ominus$ (*x mod m*) = (− *x*) *mod m*

by (*metis mod-minus-eq res-neg-eq*)

lemma (**in** *residues*) *prod-cong*: *finite A* $\Longrightarrow$ ($\bigotimes i{\in}A.$ (*f i*) *mod m*) = ($\prod i{\in}A.$ *f i*) *mod m*
by (*induct set*: *finite*) (*auto simp*: *one-cong mult-cong*)

lemma (**in** *residues*) *sum-cong*: *finite A* $\Longrightarrow$ ($\bigoplus i{\in}A.$ (*f i*) *mod m*) = ($\sum i{\in}A.$ *f i*) *mod m*
by (*induct set*: *finite*) (*auto simp*: *zero-cong add-cong*)

lemma *mod-in-res-units* [*simp*]:
assumes *1* < *m* **and** *coprime a m*
shows *a mod m* ∈ *Units R*
proof (*cases a mod m = 0*)
case *True*
with *assms* **show** *?thesis*
by (*auto simp add*: *res-units-eq gcd-red-int* [*symmetric*])
next
case *False*
from *assms* **have** *0* < *m* **by** *simp*
then have *0* ≤ *a mod m* **by** (*rule pos-mod-sign* [*of m a*])
with *False* **have** *0* < *a mod m* **by** *simp*
with *assms* **show** *?thesis*
by (*auto simp add*: *res-units-eq gcd-red-int* [*symmetric*] *ac-simps*)
qed

lemma *res-eq-to-cong*: (*a mod m*) = (*b mod m*) ⟷ [*a* = *b*] (*mod m*)
by (*auto simp*: *cong-def*)

Simplifying with these will translate a ring equation in R to a congruence.

lemmas *res-to-cong-simps* =
add-cong mult-cong pow-cong one-cong
prod-cong sum-cong neg-cong res-eq-to-cong

Other useful facts about the residue ring.

lemma *one-eq-neg-one*: **1** = ⊖ **1** $\Longrightarrow$ *m* = *2*
apply (*simp add*: *res-one-eq res-neg-eq*)
apply (*metis add.commute add-diff-cancel mod-mod-trivial one-add-one uminus-add-conv-diff*
zero-neq-one zmod-zminus1-eq-if)
done

end

4.2 Prime residues

locale *residues-prime* =
fixes *p* :: *nat* **and** *R* (**structure**)
assumes *p-prime* [*intro*]: *prime p*

```
  defines R ≡ residue-ring (int p)

sublocale residues-prime < residues p
  unfolding R-def residues-def
  using p-prime apply auto
  apply (metis (full-types) of-nat-1 of-nat-less-iff prime-gt-1-nat)
  done

context residues-prime
begin

lemma p-coprime-left:
  coprime p a ⟷ ¬ p dvd a
 using p-prime by (auto intro: prime-imp-coprime dest: coprime-common-divisor)

lemma p-coprime-right:
  coprime a p  ⟷ ¬ p dvd a
  using p-coprime-left [of a] by (simp add: ac-simps)

lemma p-coprime-left-int:
  coprime (int p) a ⟷ ¬ int p dvd a
 using p-prime by (auto intro: prime-imp-coprime dest: coprime-common-divisor)

lemma p-coprime-right-int:
  coprime a (int p) ⟷ ¬ int p dvd a
  using p-coprime-left-int [of a] by (simp add: ac-simps)

lemma is-field: field R
proof −
  have 0 < x ⟹ x < int p ⟹ coprime (int p) x for x
    by (rule prime-imp-coprime) (auto simp add: zdvd-not-zless)
  then show ?thesis
    by (intro cring.field-intro2 cring)
      (auto simp add: res-carrier-eq res-one-eq res-zero-eq res-units-eq ac-simps)
qed

lemma res-prime-units-eq: Units R = {1..p − 1}
  apply (subst res-units-eq)
  apply (auto simp add: p-coprime-right-int zdvd-not-zless)
  done

end

sublocale residues-prime < field
  by (rule is-field)
```

5 Test cases: Euler's theorem and Wilson's theorem

5.1 Euler's theorem

lemma (**in** *residues*) *totatives-eq*:
 totatives (*nat m*) = *nat* ‘ *Units R*
proof −
 from *m-gt-one* **have** $|m| > 1$
  by *simp*
 then have *totatives* (*nat* $|m|$) = *nat* ‘ *abs* ‘ *Units R*
  by (*auto simp add*: *totatives-def res-units-eq image-iff le-less*)
   (*use m-gt-one zless-nat-eq-int-zless* **in** *force*)
 moreover have $|m| = m$ *abs* ‘ *Units R* = *Units R*
  using *m-gt-one* **by** (*auto simp add*: *res-units-eq image-iff*)
 ultimately show *?thesis*
  by *simp*
qed

lemma (**in** *residues*) *totient-eq*:
 totient (*nat m*) = *card* (*Units R*)
proof −
 have ∗: *inj-on nat* (*Units R*)
  by (*rule inj-onI*) (*auto simp add*: *res-units-eq*)
 then show *?thesis*
  by (*simp add*: *totient-def totatives-eq card-image*)
qed

lemma (**in** *residues-prime*) *totient-eq*: *totient p* = *p* − *1*
 using *totient-eq* **by** (*simp add*: *res-prime-units-eq*)

lemma (**in** *residues*) *euler-theorem*:
 assumes *coprime a m*
 shows [*a* ^ *totient* (*nat m*) = *1*] (*mod m*)
proof −
 have *a* ^ *totient* (*nat m*) *mod m* = *1 mod m*
   by (*metis assms finite-Units m-gt-one mod-in-res-units one-cong totient-eq pow-cong units-power-order-eq-one*)
 then show *?thesis*
  using *res-eq-to-cong* **by** *blast*
qed

lemma *euler-theorem*:
 fixes *a m* :: *nat*
 assumes *coprime a m*
 shows [*a* ^ *totient m* = *1*] (*mod m*)
proof (*cases m* = *0* ∨ *m* = *1*)
 case *True*
 then show *?thesis* **by** *auto*

```
next
  case False
  with assms show ?thesis
    using residues.euler-theorem [of int m int a] cong-int-iff
    by (auto simp add: residues-def gcd-int-def) fastforce
qed

lemma fermat-theorem:
  fixes p a :: nat
  assumes prime p and ¬ p dvd a
  shows [a ^ (p − 1) = 1] (mod p)
proof −
  from assms prime-imp-coprime [of p a] have coprime a p
    by (auto simp add: ac-simps)
  then have [a ^ totient p = 1] (mod p)
     by (rule euler-theorem)
  also have totient p = p − 1
    by (rule totient-prime) (rule assms)
  finally show ?thesis .
qed
```

5.2 Wilson's theorem

```
lemma (in field) inv-pair-lemma: x ∈ Units R =⇒ y ∈ Units R =⇒
    {x, inv x} ≠ {y, inv y} =⇒ {x, inv x} ∩ {y, inv y} = {}
  apply auto
  apply (metis Units-inv-inv)+
  done

lemma (in residues-prime) wilson-theorem1:
  assumes a: p > 2
  shows [fact (p − 1) = (−1::int)] (mod p)
proof −
  let ?Inverse-Pairs = {{x, inv x}| x. x ∈ Units R − {1, ⊖ 1}}
  have UR: Units R = {1, ⊖ 1} ∪ ⋃ ?Inverse-Pairs
    by auto
  have (⨂ i∈Units R. i) = (⨂ i∈{1, ⊖ 1}. i) ⊗ (⨂ i∈⋃ ?Inverse-Pairs. i)
    apply (subst UR)
    apply (subst finprod-Un-disjoint)
         apply (auto intro: funcsetI)
    using inv-one apply auto[1]
    using inv-eq-neg-one-eq apply auto
    done
  also have (⨂ i∈{1, ⊖ 1}. i) = ⊖ 1
    apply (subst finprod-insert)
        apply auto
    apply (frule one-eq-neg-one)
    using a apply force
    done
```

```
  also have (⨂ i∈(⋃ ?Inverse-Pairs). i) = (⨂ A∈?Inverse-Pairs. (⨂ y∈A. y))
    apply (subst finprod-Union-disjoint)
       apply (auto simp: pairwise-def disjnt-def)
     apply (metis Units-inv-inv)+
    done
  also have ... = 1
    apply (rule finprod-one-eqI)
     apply auto
    apply (subst finprod-insert)
        apply auto
    apply (metis inv-eq-self)
    done
  finally have (⨂ i∈Units R. i) = ⊖ 1
    by simp
  also have (⨂ i∈Units R. i) = (⨂ i∈Units R. i mod p)
    by (rule finprod-cong') (auto simp: res-units-eq)
  also have ... = (∏ i∈Units R. i) mod p
    by (rule prod-cong) auto
  also have ... = fact (p − 1) mod p
    apply (simp add: fact-prod)
    using assms
    apply (subst res-prime-units-eq)
    apply (simp add: int-prod zmod-int prod-int-eq)
    done
  finally have fact (p − 1) mod p = ⊖ 1 .
  then show ?thesis
    by (simp add: cong-def res-neg-eq res-one-eq zmod-int)
qed

lemma wilson-theorem:
  assumes prime p
  shows [fact (p − 1) = − 1] (mod p)
proof (cases p = 2)
  case True
  then show ?thesis
    by (simp add: cong-def fact-prod)
next
  case False
  then show ?thesis
    using assms prime-ge-2-nat
    by (metis residues-prime.wilson-theorem1 residues-prime.intro le-eq-less-or-eq)
qed
```

This result can be transferred to the multiplicative group of $\mathbb{Z}/p\mathbb{Z}$ for p prime.

```
lemma mod-nat-int-pow-eq:
  fixes n :: nat and p a :: int
  shows a ≥ 0 ⟹ p ≥ 0 ⟹ (nat a ^ n) mod (nat p) = nat ((a ^ n) mod p)
 by (simp add: int-one-le-iff-zero-less nat-mod-distrib order-less-imp-le nat-power-eq[symmetric])
```

theorem *residue-prime-mult-group-has-gen*:
 fixes *p* :: *nat*
 assumes *prime-p* : *prime p*
 shows $\exists\, a \in \{1\ ..\ p - 1\}.\ \{1\ ..\ p - 1\} = \{a\widehat{\ }i\ mod\ p|i\ .\ i \in UNIV\}$
proof −
  have $p \geq 2$
    using *prime-gt-1-nat*[*OF prime-p*] **by** *simp*
  interpret *R*: *residues-prime p residue-ring p*
    by (*simp add*: *residues-prime-def prime-p*)
  have *car*: *carrier* (*residue-ring* (*int p*)) − $\{\mathbf{0}_{residue\text{-}ring\ (int\ p)}\} = \{1\ ..\ int\ p - 1\}$
    by (*auto simp add*: *R.zero-cong R.res-carrier-eq*)

  have $x\ [\widehat{\ }]_{residue\text{-}ring\ (int\ p)}\ i = x\ \widehat{\ }\ i\ mod\ (int\ p)$
    if $x \in \{1\ ..\ int\ p - 1\}$ **for** *x* **and** *i* :: *nat*
    using *that R.pow-cong*[*of x i*] **by** *auto*
  moreover
  obtain *a* **where** *a*: $a \in \{1\ ..\ int\ p - 1\}$
    and *a-gen*: $\{1\ ..\ int\ p - 1\} = \{a[\widehat{\ }]_{residue\text{-}ring\ (int\ p)}i|i{::}nat\ .\ i \in UNIV\}$
    using *field.finite-field-mult-group-has-gen*[*OF R.is-field*]
    by (*auto simp add*: *car*[*symmetric*] *carrier-mult-of*)
  moreover
  have *nat* ‘ $\{1\ ..\ int\ p - 1\} = \{1\ ..\ p - 1\}$ (**is** *?L* = *?R*)
  proof
    have $n \in ?R$ **if** $n \in ?L$ **for** *n*
      using *that* ‹$p{\geq}2$› **by** *force*
    then show $?L \subseteq ?R$ **by** *blast*
    have $n \in ?L$ **if** $n \in ?R$ **for** *n*
      using *that* ‹$p{\geq}2$› **by** (*auto intro*: *rev-image-eqI* [*of int n*])
    then show $?R \subseteq ?L$ **by** *blast*
  qed
  moreover
  have *nat* ‘ $\{a\widehat{\ }i\ mod\ (int\ p)\ |\ i{::}nat.\ i \in UNIV\} = \{nat\ a\widehat{\ }i\ mod\ p\ |\ i\ .\ i \in UNIV\}$ (**is** *?L* = *?R*)
  proof
    have $x \in ?R$ **if** $x \in ?L$ **for** *x*
    proof −
      from *that* **obtain** *i* **where** *i*: $x = nat\ (a\widehat{\ }i\ mod\ (int\ p))$
        by *blast*
      then have $x = nat\ a\ \widehat{\ }\ i\ mod\ p$
        using *mod-nat-int-pow-eq*[*of a int p i*] *a* ‹$p{\geq}2$› **by** *auto*
      with *i* **show** *?thesis* **by** *blast*
    qed
    then show $?L \subseteq ?R$ **by** *blast*
    have $x \in ?L$ **if** $x \in ?R$ **for** *x*
    proof −
      from *that* **obtain** *i* **where** *i*: $x = nat\ a\widehat{\ }i\ mod\ p$
        by *blast*

```
      with mod-nat-int-pow-eq[of a int p i] a ‹p≥2› show ?thesis
        by auto
    qed
    then show ?R ⊆ ?L by blast
  qed
  ultimately have {1 .. p − 1} = {nat a^i mod p | i. i ∈ UNIV}
    by presburger
  moreover from a have nat a ∈ {1 .. p − 1} by force
  ultimately show ?thesis ..
qed
```

5.3 Upper bound for the number of n-th roots

```
lemma roots-mod-prime-bound:
  fixes n c p :: nat
  assumes prime p n > 0
  defines A ≡ {x∈{..<p}. [x ^ n = c] (mod p)}
  shows   card A ≤ n
proof −
  define R where R = residue-ring (int p)
  from assms(1) interpret residues-prime p R
    by unfold-locales (simp-all add: R-def)
  interpret R: UP-domain R UP R by (unfold-locales)

  let ?f = UnivPoly.monom (UP R) 1_R n ⊖_(UP R) UnivPoly.monom (UP R) (int
(c mod p)) 0
  have in-carrier: int (c mod p) ∈ carrier R
    using prime-gt-1-nat[OF assms(1)] by (simp add: R-def residue-ring-def)

  have deg R ?f = n
    using assms in-carrier by (simp add: R.deg-minus-eq)
 hence f-not-zero: ?f ≠ 0_UP R using assms by (auto simp add : R.deg-nzero-nzero)
  have roots-bound: finite {a ∈ carrier R. UnivPoly.eval R R id a ?f = 0_R} ∧
                  card {a ∈ carrier R. UnivPoly.eval R R id a ?f = 0_R} ≤ deg R ?f
                   using finite in-carrier by (intro R.roots-bound[OF - f-not-zero])
simp
  have subs: {x ∈ carrier R. x [^]_R n = int (c mod p)} ⊆
              {a ∈ carrier R. UnivPoly.eval R R id a ?f = 0_R}
    using in-carrier by (auto simp: R.evalRR-simps)
  then have card {x ∈ carrier R. x [^]_R n = int (c mod p)} ≤
             card {a ∈ carrier R. UnivPoly.eval R R id a ?f = 0_R}
    using finite by (intro card-mono) auto
  also have ... ≤ n
    using ‹deg R ?f = n› roots-bound by linarith
  also {
    fix x assume x ∈ carrier R
    hence x [^]_R n = (x ^ n) mod (int p)
      by (subst pow-cong [symmetric]) (auto simp: R-def residue-ring-def)
  }
```

hence $\{x \in \mathit{carrier}\ R.\ x\ [\widehat{\ }]_R\ n = \mathit{int}\ (c\ \mathit{mod}\ p)\} = \{x \in \mathit{carrier}\ R.\ [x\ \widehat{\ }\ n = \mathit{int}\ c]\ (\mathit{mod}\ p)\}$
by (*fastforce simp*: *cong-def zmod-int*)
also have *bij-betw int A* $\{x \in \mathit{carrier}\ R.\ [x\ \widehat{\ }\ n = \mathit{int}\ c]\ (\mathit{mod}\ p)\}$
by (*rule bij-betwI*[*of int - - nat*])
(*use cong-int-iff* **in** ‹*force simp*: *R-def residue-ring-def A-def*›)+
from *bij-betw-same-card*[*OF this*] **have** *card* $\{x \in \mathit{carrier}\ R.\ [x\ \widehat{\ }\ n = \mathit{int}\ c]\ (\mathit{mod}\ p)\} = \mathit{card}\ A$ **..**
finally show *?thesis* **.**
qed

end

6 The sieve of Eratosthenes

theory *Eratosthenes*
imports *Main HOL−Computational-Algebra.Primes*
begin

6.1 Preliminary: strict divisibility

context *dvd*
begin

abbreviation *dvd-strict* :: $'a \Rightarrow 'a \Rightarrow \mathit{bool}$ (**infixl** *dvd′-strict 50*)
where
$b\ \mathit{dvd\text{-}strict}\ a \equiv b\ \mathit{dvd}\ a \wedge \neg\ a\ \mathit{dvd}\ b$

end

6.2 Main corpus

The sieve is modelled as a list of booleans, where *False* means *marked out*.

type-synonym *marks* = *bool list*

definition *numbers-of-marks* :: $\mathit{nat} \Rightarrow \mathit{marks} \Rightarrow \mathit{nat\ set}$
where
numbers-of-marks n bs = *fst* ‘ $\{x \in \mathit{set}\ (\mathit{enumerate}\ n\ \mathit{bs}).\ \mathit{snd}\ x\}$

lemma *numbers-of-marks-simps* [*simp*, *code*]:
numbers-of-marks n [] = {}
numbers-of-marks n (*True* # *bs*) = *insert n* (*numbers-of-marks* (*Suc n*) *bs*)
numbers-of-marks n (*False* # *bs*) = *numbers-of-marks* (*Suc n*) *bs*
by (*auto simp add*: *numbers-of-marks-def intro*!: *image-eqI*)

lemma *numbers-of-marks-Suc*:
numbers-of-marks (*Suc n*) *bs* = *Suc* ‘ *numbers-of-marks n bs*

by (*auto simp add*: *numbers-of-marks-def enumerate-Suc-eq image-iff Bex-def*)

lemma *numbers-of-marks-replicate-False* [*simp*]:
numbers-of-marks n (*replicate m False*) = {}
by (*auto simp add*: *numbers-of-marks-def enumerate-replicate-eq*)

lemma *numbers-of-marks-replicate-True* [*simp*]:
numbers-of-marks n (*replicate m True*) = {*n*..<*n*+*m*}
by (*auto simp add*: *numbers-of-marks-def enumerate-replicate-eq image-def*)

lemma *in-numbers-of-marks-eq*:
m ∈ *numbers-of-marks n bs* ⟷ *m* ∈ {*n*..<*n* + *length bs*} ∧ *bs* ! (*m* − *n*)
by (*simp add*: *numbers-of-marks-def in-set-enumerate-eq image-iff add.commute*)

lemma *sorted-list-of-set-numbers-of-marks*:
sorted-list-of-set (*numbers-of-marks n bs*) = *map fst* (*filter snd* (*enumerate n bs*))
by (*auto simp add*: *numbers-of-marks-def distinct-map*
intro!: *sorted-filter distinct-filter inj-onI sorted-distinct-set-unique*)

Marking out multiples in a sieve

definition *mark-out* :: *nat* ⇒ *marks* ⇒ *marks*
where
mark-out n bs = *map* (λ(*q*, *b*). *b* ∧ ¬ *Suc n dvd Suc* (*Suc q*)) (*enumerate n bs*)

lemma *mark-out-Nil* [*simp*]: *mark-out n* [] = []
by (*simp add*: *mark-out-def*)

lemma *length-mark-out* [*simp*]: *length* (*mark-out n bs*) = *length bs*
by (*simp add*: *mark-out-def*)

lemma *numbers-of-marks-mark-out*:
numbers-of-marks n (*mark-out m bs*) = {*q* ∈ *numbers-of-marks n bs*. ¬ *Suc m dvd Suc q* − *n*}
by (*auto simp add*: *numbers-of-marks-def mark-out-def in-set-enumerate-eq image-iff*
nth-enumerate-eq less-eq-dvd-minus)

Auxiliary operation for efficient implementation

definition *mark-out-aux* :: *nat* ⇒ *nat* ⇒ *marks* ⇒ *marks*
where
mark-out-aux n m bs =
map (λ(*q*, *b*). *b* ∧ (*q* < *m* + *n* ∨ ¬ *Suc n dvd Suc* (*Suc q*) + (*n* − *m mod Suc n*))) (*enumerate n bs*)

lemma *mark-out-code* [*code*]: *mark-out n bs* = *mark-out-aux n n bs*
proof −
have *aux*: *False*
if *A*: *Suc n dvd Suc* (*Suc a*)
and *B*: *a* < *n* + *n*

```
    and C: n ≤ a
    for a
  proof (cases n = 0)
    case True
    with A B C show ?thesis by simp
  next
    case False
    define m where m = Suc n
    then have m > 0 by simp
    from False have n > 0 by simp
    from A obtain q where q: Suc (Suc a) = Suc n ∗ q by (rule dvdE)
    have q > 0
    proof (rule ccontr)
      assume ¬ q > 0
      with q show False by simp
    qed
    with ‹n > 0› have Suc n ∗ q ≥ 2 by (auto simp add: gr0-conv-Suc)
    with q have a: a = Suc n ∗ q − 2 by simp
    with B have q + n ∗ q < n + n + 2 by auto
    then have m ∗ q < m ∗ 2 by (simp add: m-def)
    with ‹m > 0› have q < 2 by simp
    with ‹q > 0› have q = 1 by simp
    with a have a = n − 1 by simp
    with ‹n > 0› C show False by simp
  qed
  show ?thesis
    by (auto simp add: mark-out-def mark-out-aux-def in-set-enumerate-eq intro:
aux)
qed

lemma mark-out-aux-simps [simp, code]:
  mark-out-aux n m [] = []
  mark-out-aux n 0 (b # bs) = False # mark-out-aux n n bs
  mark-out-aux n (Suc m) (b # bs) = b # mark-out-aux n m bs
proof goal-cases
  case 1
  show ?case
    by (simp add: mark-out-aux-def)
next
  case 2
  show ?case
    by (auto simp add: mark-out-code [symmetric] mark-out-aux-def mark-out-def
      enumerate-Suc-eq in-set-enumerate-eq less-eq-dvd-minus)
next
  case 3
  { define v where v = Suc m
    define w where w = Suc n
    fix q
    assume m + n ≤ q
```

then obtain *r* **where** *q*: $q = m + n + r$ **by** (*auto simp add*: *le-iff-add*)
{ **fix** *u*
from *w-def* **have** $u\ mod\ w < w$ **by** *simp*
then have $u + (w - u\ mod\ w) = w + (u - u\ mod\ w)$
by *simp*
then have $u + (w - u\ mod\ w) = w + u\ div\ w * w$
by (*simp add*: *minus-mod-eq-div-mult*)
}
then have $w\ dvd\ v + w + r + (w - v\ mod\ w) \longleftrightarrow w\ dvd\ m + w + r + (w - m\ mod\ w)$
by (*simp add*: *add.assoc add.left-commute* [*of m*] *add.left-commute* [*of v*] *dvd-add-left-iff dvd-add-right-iff*)
moreover from *q* **have** $Suc\ q = m + w + r$ **by** (*simp add*: *w-def*)
moreover from *q* **have** $Suc\ (Suc\ q) = v + w + r$ **by** (*simp add*: *v-def w-def*)
ultimately have $w\ dvd\ Suc\ (Suc\ (q + (w - v\ mod\ w))) \longleftrightarrow w\ dvd\ Suc\ (q + (w - m\ mod\ w))$
by (*simp only*: *add-Suc* [*symmetric*])
then have $Suc\ n\ dvd\ Suc\ (Suc\ (Suc\ (q + n) - Suc\ m\ mod\ Suc\ n)) \longleftrightarrow Suc\ n\ dvd\ Suc\ (Suc\ (q + n - m\ mod\ Suc\ n))$
by (*simp add*: *v-def w-def Suc-diff-le trans-le-add2*)
}
then show *?case*
by (*auto simp add*: *mark-out-aux-def enumerate-Suc-eq in-set-enumerate-eq not-less*)
qed

Main entry point to sieve

fun *sieve* :: $nat \Rightarrow marks \Rightarrow marks$
where
sieve n [] = []
| *sieve n* (*False* # *bs*) = *False* # *sieve* (*Suc n*) *bs*
| *sieve n* (*True* # *bs*) = *True* # *sieve* (*Suc n*) (*mark-out n bs*)

There are the following possible optimisations here:

- *sieve* can abort as soon as *n* is too big to let *mark-out* have any effect.
- Search for further primes can be given up as soon as the search position exceeds the square root of the maximum candidate.

This is left as an constructive exercise to the reader.

lemma *numbers-of-marks-sieve*:
numbers-of-marks (*Suc n*) (*sieve n bs*) =
$\{q \in numbers\text{-}of\text{-}marks\ (Suc\ n)\ bs.\ \forall m \in numbers\text{-}of\text{-}marks\ (Suc\ n)\ bs.\ \neg\ m\ dvd\text{-}strict\ q\}$
proof (*induct n bs rule*: *sieve.induct*)
case *1*
show *?case* **by** *simp*

```
next
  case 2
  then show ?case by simp
next
  case (3 n bs)
  have aux: n ∈ Suc ‘ M ⟷ n > 0 ∧ n − 1 ∈ M (is ?lhs ⟷ ?rhs) for M n
  proof
    show ?rhs if ?lhs using that by auto
    show ?lhs if ?rhs
    proof −
      from that have n > 0 and n − 1 ∈ M by auto
      then have Suc (n − 1) ∈ Suc ‘ M by blast
      with ‹n > 0› show n ∈ Suc ‘ M by simp
    qed
  qed
  have aux1: False if Suc (Suc n) ≤ m and m dvd Suc n for m :: nat
  proof −
    from ‹m dvd Suc n› obtain q where Suc n = m ∗ q ..
    with ‹Suc (Suc n) ≤ m› have Suc (m ∗ q) ≤ m by simp
    then have m ∗ q < m by arith
    then have q = 0 by simp
    with ‹Suc n = m ∗ q› show ?thesis by simp
  qed
  have aux2: m dvd q
    if 1: ∀ q>0. 1 < q ⟶ Suc n < q ⟶ q ≤ Suc (n + length bs) ⟶
      bs ! (q − Suc (Suc n)) ⟶ ¬ Suc n dvd q ⟶ q dvd m ⟶ m dvd q
    and 2: ¬ Suc n dvd m q dvd m
    and 3: Suc n < q q ≤ Suc (n + length bs) bs ! (q − Suc (Suc n))
    for m q :: nat
  proof −
    from 1 have ∗: ⋀q. Suc n < q ⟹ q ≤ Suc (n + length bs) ⟹
      bs ! (q − Suc (Suc n)) ⟹ ¬ Suc n dvd q ⟹ q dvd m ⟹ m dvd q
      by auto
    from 2 have ¬ Suc n dvd q by (auto elim: dvdE)
    moreover note 3
    moreover note ‹q dvd m›
    ultimately show ?thesis by (auto intro: ∗)
  qed
  from 3 show ?case
   apply (simp-all add: numbers-of-marks-mark-out numbers-of-marks-Suc Compr-image-eq
      inj-image-eq-iff in-numbers-of-marks-eq Ball-def imp-conjL aux)
    apply safe
    apply (simp-all add: less-diff-conv2 le-diff-conv2 dvd-minus-self not-less)
    apply (clarsimp dest!: aux1)
    apply (simp add: Suc-le-eq less-Suc-eq-le)
    apply (rule aux2)
    apply (clarsimp dest!: aux1)+
    done
qed
```

Relation of the sieve algorithm to actual primes

definition *primes-upto* :: *nat* ⇒ *nat list*
where
  primes-upto n = *sorted-list-of-set* {*m*. *m* ≤ *n* ∧ *prime m*}

lemma *set-primes-upto*: *set* (*primes-upto n*) = {*m*. *m* ≤ *n* ∧ *prime m*}
  by (*simp add*: *primes-upto-def*)

lemma *sorted-primes-upto* [*iff*]: *sorted* (*primes-upto n*)
  by (*simp add*: *primes-upto-def*)

lemma *distinct-primes-upto* [*iff*]: *distinct* (*primes-upto n*)
  by (*simp add*: *primes-upto-def*)

lemma *set-primes-upto-sieve*:
  set (*primes-upto n*) = *numbers-of-marks 2* (*sieve 1* (*replicate* (*n* − *1*) *True*))
proof −
  consider *n* = *0* ∨ *n* = *1* | *n* > *1* **by** *arith*
  then show *?thesis*
  proof *cases*
    case *1*
    then show *?thesis*
      by (*auto simp add*: *numbers-of-marks-sieve numeral-2-eq-2 set-primes-upto*
        dest: *prime-gt-Suc-0-nat*)
  next
    case *2*
    {
      fix *m q*
      assume *Suc* (*Suc 0*) ≤ *q*
        and *q* < *Suc n*
        and *m dvd q*
      then have *m* < *Suc n* **by** (*auto dest*: *dvd-imp-le*)
      assume ∗: ∀ *m*∈{*Suc* (*Suc 0*)..<*Suc n*}. *m dvd q* ⟶ *q dvd m*
        and *m dvd q* **and** *m* ≠ *1*
      have *m* = *q*
      proof (*cases m* = *0*)
        case *True* **with** ‹*m dvd q*› **show** *?thesis* **by** *simp*
      next
        case *False* **with** ‹*m* ≠ *1*› **have** *Suc* (*Suc 0*) ≤ *m* **by** *arith*
        with ‹*m* < *Suc n*› ∗ ‹*m dvd q*› **have** *q dvd m* **by** *simp*
        with ‹*m dvd q*› **show** *?thesis* **by** (*simp add*: *dvd-antisym*)
      qed
    }
    then have *aux*: ⋀*m q*. *Suc* (*Suc 0*) ≤ *q* ⟹
      q < *Suc n* ⟹
      m dvd q ⟹
      ∀ *m*∈{*Suc* (*Suc 0*)..<*Suc n*}. *m dvd q* ⟶ *q dvd m* ⟹
      m dvd q ⟹ *m* ≠ *q* ⟹ *m* = *1* **by** *auto*
    from *2* **show** *?thesis*

apply (*auto simp add*: *numbers-of-marks-sieve numeral-2-eq-2 set-primes-upto dest*: *prime-gt-Suc-0-nat*)
apply (*metis One-nat-def Suc-le-eq less-not-refl prime-nat-iff*)
apply (*metis One-nat-def Suc-le-eq aux prime-nat-iff*)
done
qed
qed

lemma *primes-upto-sieve* [*code*]:
primes-upto n = *map fst* (*filter snd* (*enumerate 2* (*sieve 1* (*replicate* (*n* − *1*) *True*))))
proof −
have *primes-upto n* = *sorted-list-of-set* (*numbers-of-marks 2* (*sieve 1* (*replicate* (*n* − *1*) *True*)))
apply (*rule sorted-distinct-set-unique*)
apply (*simp-all only*: *set-primes-upto-sieve numbers-of-marks-def*)
apply *auto*
done
then show *?thesis*
by (*simp add*: *sorted-list-of-set-numbers-of-marks*)
qed

lemma *prime-in-primes-upto*: *prime n* ⟷ *n* ∈ *set* (*primes-upto n*)
by (*simp add*: *set-primes-upto*)

6.3 Application: smallest prime beyond a certain number

definition *smallest-prime-beyond* :: *nat* ⇒ *nat*
where
smallest-prime-beyond n = (*LEAST p*. *prime p* ∧ *p* ≥ *n*)

lemma *prime-smallest-prime-beyond* [*iff*]: *prime* (*smallest-prime-beyond n*) (**is** *?P*)
and *smallest-prime-beyond-le* [*iff*]: *smallest-prime-beyond n* ≥ *n* (**is** *?Q*)
proof −
let *?least* = *LEAST p*. *prime p* ∧ *p* ≥ *n*
from *primes-infinite* **obtain** *q* **where** *prime q* ∧ *q* ≥ *n*
by (*metis finite-nat-set-iff-bounded-le mem-Collect-eq nat-le-linear*)
then have *prime ?least* ∧ *?least* ≥ *n*
by (*rule LeastI*)
then show *?P* **and** *?Q*
by (*simp-all add*: *smallest-prime-beyond-def*)
qed

lemma *smallest-prime-beyond-smallest*: *prime p* ⟹ *p* ≥ *n* ⟹ *smallest-prime-beyond n* ≤ *p*
by (*simp only*: *smallest-prime-beyond-def*) (*auto intro*: *Least-le*)

lemma *smallest-prime-beyond-eq*:

prime p $\Longrightarrow$ *p* $\geq$ *n* $\Longrightarrow$ ($\bigwedge$*q. prime q* $\Longrightarrow$ *q* $\geq$ *n* $\Longrightarrow$ *q* $\geq$ *p*) $\Longrightarrow$ *smallest-prime-beyond n* = *p*
by (*simp only*: *smallest-prime-beyond-def*) (*auto intro*: *Least-equality*)

definition *smallest-prime-between* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat option*
where
smallest-prime-between m n =
(*if* ($\exists$ *p. prime p* $\wedge$ *m* $\leq$ *p* $\wedge$ *p* $\leq$ *n*) *then Some* (*smallest-prime-beyond m*) *else None*)

lemma *smallest-prime-between-None*:
smallest-prime-between m n = *None* $\longleftrightarrow$ ($\forall$ *q. m* $\leq$ *q* $\wedge$ *q* $\leq$ *n* $\longrightarrow$ $\neg$ *prime q*)
by (*auto simp add*: *smallest-prime-between-def*)

lemma *smallest-prime-betwen-Some*:
smallest-prime-between m n = *Some p* $\longleftrightarrow$ *smallest-prime-beyond m* = *p* $\wedge$ *p* $\leq$ *n*
by (*auto simp add*: *smallest-prime-between-def dest*: *smallest-prime-beyond-smallest* [*of - m*])

lemma [*code*]: *smallest-prime-between m n* = *List.find* (λ*p. p* $\geq$ *m*) (*primes-upto n*)
proof −
have *List.find* (λ*p. p* $\geq$ *m*) (*primes-upto n*) = *Some* (*smallest-prime-beyond m*)
if *assms*: *m* $\leq$ *p prime p p* $\leq$ *n* **for** *p*
proof −
define *A* **where** *A* = {*p. p* $\leq$ *n* $\wedge$ *prime p* $\wedge$ *m* $\leq$ *p*}
from *assms* **have** *smallest-prime-beyond m* $\leq$ *p*
by (*auto intro*: *smallest-prime-beyond-smallest*)
from *this* ‹*p* $\leq$ *n*› **have** $*$: *smallest-prime-beyond m* $\leq$ *n*
by (*rule order-trans*)
from *assms* **have** *ex*: $\exists$ *p*$\leq$*n. prime p* $\wedge$ *m* $\leq$ *p*
by *auto*
then have *finite A*
by (*auto simp add*: *A-def*)
with $*$ **have** *Min A* = *smallest-prime-beyond m*
by (*auto simp add*: *A-def intro*: *Min-eqI smallest-prime-beyond-smallest*)
with *ex sorted-primes-upto* **show** *?thesis*
by (*auto simp add*: *set-primes-upto sorted-find-Min A-def*)
qed
then show *?thesis*
by (*auto simp add*: *smallest-prime-between-def find-None-iff set-primes-upto intro*!: *sym* [*of - None*])
qed

definition *smallest-prime-beyond-aux* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*
where
smallest-prime-beyond-aux k n = *smallest-prime-beyond n*

lemma [*code*]:
  smallest-prime-beyond-aux k n =
    (*case smallest-prime-between n* (*k* ∗ *n*) *of*
      Some p ⇒ *p*
    | *None* ⇒ *smallest-prime-beyond-aux* (*Suc k*) *n*)
  by (*simp add*: *smallest-prime-beyond-aux-def smallest-prime-betwen-Some split*: *option.split*)

lemma [*code*]: *smallest-prime-beyond n* = *smallest-prime-beyond-aux 2 n*
  by (*simp add*: *smallest-prime-beyond-aux-def*)

end

7 Fast modular exponentiation

theory *Mod-Exp*
  imports *Cong HOL−Library.Power-By-Squaring*
begin

context *euclidean-semiring-cancel*
begin

definition *mod-exp-aux* :: *'a* ⇒ *'a* ⇒ *'a* ⇒ *nat* ⇒ *'a*
  where *mod-exp-aux m* = *efficient-funpow* (λ*x y*. *x* ∗ *y mod m*)

lemma *mod-exp-aux-code* [*code*]:
  mod-exp-aux m y x n =
     (*if n* = *0 then y*
      else if n = *1 then* (*x* ∗ *y*) *mod m*
      else if even n then mod-exp-aux m y ((*x* ∗ *x*) *mod m*) (*n div 2*)
      else mod-exp-aux m ((*x* ∗ *y*) *mod m*) ((*x* ∗ *x*) *mod m*) (*n div 2*))
  unfolding *mod-exp-aux-def* **by** (*rule efficient-funpow-code*)

lemma *mod-exp-aux-correct*:
  mod-exp-aux m y x n mod m = (*x* ^ *n* ∗ *y*) *mod m*
proof −
  have *mod-exp-aux m y x n* = *efficient-funpow* (λ*x y*. *x* ∗ *y mod m*) *y x n*
    by (*simp add*: *mod-exp-aux-def*)
  also have ... = ((λ*y*. *x* ∗ *y mod m*) ^^ *n*) *y*
   by (*rule efficient-funpow-correct*) (*simp add*: *mod-mult-left-eq mod-mult-right-eq mult-ac*)
  also have ((λ*y*. *x* ∗ *y mod m*) ^^ *n*) *y mod m* = (*x* ^ *n* ∗ *y*) *mod m*
  proof (*induction n*)
    case (*Suc n*)
    hence *x* ∗ ((λ*y*. *x* ∗ *y mod m*) ^^ *n*) *y mod m* = *x* ∗ *x* ^ *n* ∗ *y mod m*
      by (*metis mod-mult-right-eq mult.assoc*)
    thus *?case* **by** *auto*
  qed *auto*
  finally show *?thesis* **.**

qed

definition *mod-exp* :: *'a* ⇒ *nat* ⇒ *'a* ⇒ *'a*
  where *mod-exp b e m* = (*b* ^ *e*) *mod m*

lemma *mod-exp-code* [*code*]: *mod-exp b e m* = *mod-exp-aux m 1 b e mod m*
  by (*simp add*: *mod-exp-def mod-exp-aux-correct*)

end

lemmas [*code-abbrev*] = *mod-exp-def*[**where** *?'a* = *nat*] *mod-exp-def*[**where** *?'a* = *int*]

lemma *cong-power-nat-code* [*code-unfold*]:
  [*b* ^ *e* = (*x* ::*nat*)] (*mod m*) ⟷ *mod-exp b e m* = *x mod m*
  by (*simp add*: *mod-exp-def cong-def*)

lemma *cong-power-int-code* [*code-unfold*]:
  [*b* ^ *e* = (*x* ::*int*)] (*mod m*) ⟷ *mod-exp b e m* = *x mod m*
  by (*simp add*: *mod-exp-def cong-def*)

The following rules allow the simplifier to evaluate *mod-exp* efficiently.

lemma *eval-mod-exp-aux* [*simp*]:
  mod-exp-aux m y x 0 = *y*
  mod-exp-aux m y x (*Suc 0*) = (*x* ∗ *y*) *mod m*
  mod-exp-aux m y x (*numeral* (*num.Bit0 n*)) =
    mod-exp-aux m y (x^2 *mod m*) (*numeral n*)
  mod-exp-aux m y x (*numeral* (*num.Bit1 n*)) =
    mod-exp-aux m ((*x* ∗ *y*) *mod m*) (x^2 *mod m*) (*numeral n*)
proof −
  define *n'* **where** *n'* = (*numeral n* :: *nat*)
  have [*simp*]: *n'* ≠ *0* **by** (*auto simp*: *n'-def*)

  show *mod-exp-aux m y x 0* = *y* **and** *mod-exp-aux m y x* (*Suc 0*) = (*x* ∗ *y*) *mod m*
    by (*simp-all add*: *mod-exp-aux-def*)

  have *numeral* (*num.Bit0 n*) = (*2* ∗ *n'*)
    by (*subst numeral.numeral-Bit0*) (*simp del*: *arith-simps add*: *n'-def*)
  also have *mod-exp-aux m y x* ... = *mod-exp-aux m y* (*x*^*2 mod m*) *n'*
    by (*subst mod-exp-aux-code*) (*simp-all add*: *power2-eq-square*)
  finally show *mod-exp-aux m y x* (*numeral* (*num.Bit0 n*)) =
          mod-exp-aux m y (x^2 *mod m*) (*numeral n*)
    by (*simp add*: *n'-def*)

  have *numeral* (*num.Bit1 n*) = *Suc* (*2* ∗ *n'*)
    by (*subst numeral.numeral-Bit1*) (*simp del*: *arith-simps add*: *n'-def*)
  also have *mod-exp-aux m y x* ... = *mod-exp-aux m* ((*x* ∗ *y*) *mod m*) (*x*^*2 mod*

m) *n'*
  by (*subst mod-exp-aux-code*) (*simp-all add*: *power2-eq-square*)
 finally show *mod-exp-aux m y x* (*numeral* (*num.Bit1 n*)) =
          mod-exp-aux m ((*x* ∗ *y*) *mod m*) (x^2 *mod m*) (*numeral n*)
  by (*simp add*: *n'-def*)
qed

lemma *eval-mod-exp* [*simp*]:
 mod-exp b' 0 m' = *1 mod m'*
 mod-exp b' 1 m' = *b' mod m'*
 mod-exp b' (*Suc 0*) *m'* = *b' mod m'*
 mod-exp b' e' 0 = *b'* ^ *e'*
 mod-exp b' e' 1 = *0*
 mod-exp b' e' (*Suc 0*) = *0*
 mod-exp 0 1 m' = *0*
 mod-exp 0 (*Suc 0*) *m'* = *0*
 mod-exp 0 (*numeral e*) *m'* = *0*
 mod-exp 1 e' m' = *1 mod m'*
 mod-exp (*Suc 0*) *e' m'* = *1 mod m'*
 mod-exp (*numeral b*) (*numeral e*) (*numeral m*) =
   mod-exp-aux (*numeral m*) *1* (*numeral b*) (*numeral e*) *mod numeral m*
 by (*simp-all add*: *mod-exp-def mod-exp-aux-correct*)

end

theory *Euler-Criterion*
imports *Residues*
begin

context
 fixes *p* :: *nat*
 fixes *a* :: *int*

 assumes *p-prime*: *prime p*
 assumes *p-ge-2*: *2* < *p*
 assumes *p-a-relprime*: [*a* ≠ *0*](*mod p*)
begin

private lemma *odd-p*: *odd p*
 using *p-ge-2 p-prime prime-odd-nat* **by** *blast*

private lemma *p-minus-1-int*:
 int (*p* − *1*) = *int p* − *1*
 by (*metis of-nat-1 of-nat-diff p-prime prime-ge-1-nat*)

private lemma *p-not-eq-Suc-0* [*simp*]:
 p ≠ *Suc 0*
 using *p-ge-2* **by** *simp*

```
private lemma one-mod-int-p-eq [simp]:
  1 mod int p = 1
proof −
  from p-ge-2 have int 1 mod int p = int 1
    by simp
  then show ?thesis
    by simp
qed

private lemma E-1:
  assumes QuadRes (int p) a
  shows [a ^ ((p − 1) div 2) = 1] (mod int p)
proof −
  from assms obtain b where b: [b ^ 2 = a] (mod int p)
    unfolding QuadRes-def by blast
  then have [a ^ ((p − 1) div 2) = b ^ (2 ∗ ((p − 1) div 2))] (mod int p)
    by (simp add: cong-pow cong-sym power-mult)
  then have [a ^ ((p − 1) div 2) = b ^ (p − 1)] (mod int p)
    using odd-p by force
  moreover have [b ^ (p − 1) = 1] (mod int p)
  proof −
    have [nat |b| ^ (p − 1) = 1] (mod p)
    using p-prime proof (rule fermat-theorem)
      from b p-a-relprime show ¬ p dvd nat |b|
        by (auto simp add: cong-iff-dvd-diff power2-eq-square)
          (metis cong-iff-dvd-diff cong-dvd-iff dvd-mult2)
    qed
    then have nat |b| ^ (p − 1) mod p = 1 mod p
      by (simp add: cong-def)
    then have int (nat |b| ^ (p − 1) mod p) = int (1 mod p)
      by simp
    moreover from odd-p have |b| ^ (p − Suc 0) = b ^ (p − Suc 0)
      by (simp add: power-even-abs)
    ultimately show ?thesis
      by (auto simp add: cong-def zmod-int)
  qed
  ultimately show ?thesis
    by (auto intro: cong-trans)
qed

private definition S1 :: int set where S1 = {0 <.. int p − 1}

private definition P :: int ⇒ int ⇒ bool where
  P x y ⟷ [x ∗ y = a] (mod p) ∧ y ∈ S1

private definition f-1 :: int ⇒ int where
  f-1 x = (THE y. P x y)
```

private definition *f* :: *int* ⇒ *int set* **where**
f x = {*x*, *f-1 x*}

private definition *S2* :: *int set set* **where** *S2* = *f* ' *S1*

private lemma *P-lemma*: **assumes** *x* ∈ *S1*
shows ∃! *y*. *P x y*
proof −
have ¬ *p dvd x* **using** *assms zdvd-not-zless S1-def* **by** *auto*
hence *co-xp*: *coprime x p* **using** *p-prime prime-imp-coprime-int*[*of p x*]
by (*simp add*: *ac-simps*)
then obtain *y'* **where** *y'*: [*x* ∗ *y'* = *1*] (*mod p*) **using** *cong-solve-coprime-int* **by** *blast*
moreover define *y* **where** *y* = *y'* ∗ *a mod p*
ultimately have [*x* ∗ *y* = *a*] (*mod p*)
using *mod-mult-right-eq* [*of x y'* ∗ *a p*]
cong-scalar-right [*of x* ∗ *y' 1 int p a*]
by (*auto simp add*: *cong-def ac-simps*)
moreover have *y* ∈ {*0* .. *int p* − *1*} **unfolding** *y-def* **using** *p-ge-2* **by** *auto*
hence *y* ∈ *S1* **using** *calculation cong-iff-dvd-diff p-a-relprime S1-def cong-dvd-iff* **by** *fastforce*
ultimately have *P x y* **unfolding** *P-def* **by** *blast*
moreover {
fix *y1 y2*
assume *P x y1 P x y2*
moreover hence [*y1* = *y2*] (*mod p*) **unfolding** *P-def*
using *co-xp cong-mult-lcancel*[*of x p y1 y2*] *cong-sym cong-trans* **by** *blast*
ultimately have *y1* = *y2* **unfolding** *P-def S1-def* **using** *cong-less-imp-eq-int* **by** *auto*
}
ultimately show *?thesis* **by** *blast*
qed

private lemma *f-1-lemma-1*: **assumes** *x* ∈ *S1*
shows *P x* (*f-1 x*) **using** *assms P-lemma theI'*[*of P x*] *f-1-def* **by** *presburger*

private lemma *f-1-lemma-2*: **assumes** *x* ∈ *S1*
shows *f-1* (*f-1 x*) = *x*
using *assms f-1-lemma-1*[*of x*] *f-1-def P-lemma*[*of f-1 x*] *P-def* **by** (*auto simp*: *mult.commute*)

private lemma *f-lemma-1*: **assumes** *x* ∈ *S1*
shows *f x* = *f* (*f-1 x*) **using** *f-def f-1-lemma-2*[*of x*] *assms* **by** *auto*

private lemma *l1*: **assumes** ¬ *QuadRes p a x* ∈ *S1*
shows *x* ≠ *f-1 x*
using *f-1-lemma-1*[*of x*] *assms* **unfolding** *P-def QuadRes-def power2-eq-square* **by** *fastforce*

private lemma *l2*: **assumes** $\neg$ *QuadRes p a x* $\in$ *S1*
  shows $[\prod\ (f\ x) = a]\ (mod\ p)$
  using *assms l1 f-1-lemma-1 P-def f-def* **by** *auto*

private lemma *l3*: **assumes** $x \in S2$
  shows *finite x* **using** *assms f-def S2-def* **by** *auto*

private lemma *l4*: $S1 = \bigcup\ S2$ **using** *f-1-lemma-1 P-def f-def S2-def* **by** *auto*

private lemma *l5*: **assumes** $x \in S2\ y \in S2\ x \neq y$
  shows $x \cap y = \{\}$
proof −
  obtain *a b* **where** *ab*: $x = f\ a\ a \in S1\ y = f\ b\ b \in S1$ **using** *assms S2-def* **by** *auto*
  hence $a \neq b\ a \neq f\text{-}1\ b\ f\text{-}1\ a \neq b$ **using** *assms*(*3*) *f-lemma-1* **by** *blast+*
  moreover hence $f\text{-}1\ a \neq f\text{-}1\ b$ **using** *f-1-lemma-2*[*of a*] *f-1-lemma-2*[*of b*] *ab* **by** *force*
  ultimately show *?thesis* **using** *f-def ab* **by** *fastforce*
qed

private lemma *l6*: *prod Prod S2* $= \prod\ S1$
  using *prod.Union-disjoint*[*of S2* $\lambda x.\ x$] *l3 l4 l5* **unfolding** *comp-def* **by** *auto*

private lemma *l7*: *fact n* $= \prod\ \{0 <..\ int\ n\}$
proof (*induction n*)
case (*Suc n*)
  have *int* (*Suc n*) $=$ *int n* $+ 1$ **by** *simp*
  hence *insert* (*int* (*Suc n*)) $\{0<..int\ n\} = \{0<..int\ (Suc\ n)\}$ **by** *auto*
  thus *?case* **using** *prod.insert*[*of* $\{0<..int\ n\}$ *int* (*Suc n*) $\lambda x.\ x$] *Suc fact-Suc* **by** *auto*
qed *simp*

private lemma *l8*: *fact* $(p - 1) = \prod\ S1$ **using** *l7*[*of* $p - 1$] *S1-def p-minus-1-int* **by** *presburger*

private lemma *l9*: [*prod Prod S2* $= -1$] (*mod p*)
  using *l6 l8 wilson-theorem*[*of p*] *p-prime* **by** *presburger*

private lemma *l10*: **assumes** *card S* $= n\ \bigwedge x.\ x \in S \Longrightarrow [g\ x = a]\ (mod\ p)$
  shows [*prod g S* $= a$ ^ n] (*mod p*) **using** *assms*
proof (*induction n arbitrary*: *S*)
case *0*
  thus *?case* **using** *card-0-eq*[*of S*] *prod.empty prod.infinite* **by** *fastforce*
next
case (*Suc n*)
  then obtain *x* **where** *x*: $x \in S$ **by** *force*
  define S' **where** $S' = S - \{x\}$
  hence [*prod g* $S' = a$ ^ n] (*mod int p*)
    using *x Suc*(*1*)[*of S'*] *Suc*(*2*) *Suc*(*3*) **by** (*simp add*: *card-ge-0-finite*)

moreover have *prod g S = g x ∗ prod g S′*
  using *x S′-def Suc(2) prod.remove[of S x g]* **by** *fastforce*
ultimately show *?case* **using** *x Suc(3) cong-mult*
  by *simp blast*
qed

private lemma *l11*: **assumes** ¬ *QuadRes p a*
  shows *card S2 = (p − 1) div 2*
proof −
  have *sum card S2 = 2 ∗ card S2*
    using *sum.cong[of S2 S2 card λx. 2] l1 f-def S2-def assms* **by** *fastforce*
  moreover have *p − 1 = sum card S2*
    using *l4 card-UN-disjoint[of S2 λx. x] l3 l5 S1-def S2-def* **by** *auto*
  ultimately show *?thesis* **by** *linarith*
qed

private lemma *l12*: **assumes** ¬ *QuadRes p a*
  shows *[prod Prod S2 = a ^ ((p − 1) div 2)] (mod p)*
  using *assms l2 l10 l11* **unfolding** *S2-def* **by** *blast*

private lemma *E-2*: **assumes** ¬ *QuadRes p a*
  shows *[a ^ ((p − 1) div 2) = −1] (mod p)* **using** *l9 l12 cong-trans cong-sym assms* **by** *blast*

lemma *euler-criterion-aux*: *[(Legendre a p) = a ^ ((p − 1) div 2)] (mod p)*
  using *p-a-relprime* **by** *(auto simp add: Legendre-def*
    intro!: cong-sym [of - 1] cong-sym [of - − 1]
    dest: E-1 E-2)

end

theorem *euler-criterion*: **assumes** *prime p 2 < p*
  shows *[(Legendre a p) = a ^ ((p − 1) div 2)] (mod p)*
proof *(cases [a = 0] (mod p))*
  case *True*
  then have *[a ^ ((p − 1) div 2) = 0 ^ ((p − 1) div 2)] (mod p)*
    using *cong-pow* **by** *blast*
  moreover have *(0::int) ^ ((p − 1) div 2) = 0*
    using *zero-power [of (p − 1) div 2] assms(2)* **by** *simp*
  ultimately have *[a ^ ((p − 1) div 2) = 0] (mod p)*
    using *True assms(1) prime-dvd-power-int-iff*
    by *(simp add: cong-iff-dvd-diff)*
  then show *?thesis* **unfolding** *Legendre-def* **using** *True cong-sym*
    by *auto*
next
  case *False*
  then show *?thesis*
    using *euler-criterion-aux assms* **by** *presburger*
qed

hide-fact *euler-criterion-aux*

end

8 Gauss' Lemma

theory *Gauss*
imports *Euler-Criterion*
begin

lemma *cong-prime-prod-zero-nat*:
$[a * b = 0]$ $(mod\ p) \Longrightarrow prime\ p \Longrightarrow [a = 0]$ $(mod\ p) \vee [b = 0]$ $(mod\ p)$
for a :: *nat*
by (*auto simp add*: *cong-altdef-nat prime-dvd-mult-iff*)

lemma *cong-prime-prod-zero-int*:
$[a * b = 0]$ $(mod\ p) \Longrightarrow prime\ p \Longrightarrow [a = 0]$ $(mod\ p) \vee [b = 0]$ $(mod\ p)$
for a :: *int*
by (*simp add*: *cong-0-iff prime-dvd-mult-iff*)

locale *GAUSS* =
fixes p :: *nat*
fixes a :: *int*
assumes *p-prime*: *prime p*
assumes *p-ge-2*: $2 < p$
assumes *p-a-relprime*: $[a \neq 0](mod\ p)$
assumes *a-nonzero*: $0 < a$
begin

definition $A = \{0::int <.. ((int\ p - 1)\ div\ 2)\}$
definition $B = (\lambda x.\ x * a)$ ‘ A
definition $C = (\lambda x.\ x\ mod\ p)$ ‘ B
definition $D = C \cap \{..\ (int\ p - 1)\ div\ 2\}$
definition $E = C \cap \{(int\ p - 1)\ div\ 2 <..\}$
definition $F = (\lambda x.\ (int\ p - x))$ ‘ E

8.1 Basic properties of p

lemma *odd-p*: *odd p*
by (*metis p-prime p-ge-2 prime-odd-nat*)

lemma *p-minus-one-l*: $(int\ p - 1)\ div\ 2 < p$
proof −
have $(p - 1)\ div\ 2 \leq (p - 1)\ div\ 1$
by (*metis div-by-1 div-le-dividend*)
also have $\ldots = p - 1$ **by** *simp*
finally show *?thesis*

using *p-ge-2* **by** *arith*
qed

lemma *p-eq2*: $int\ p = (2 * ((int\ p - 1)\ div\ 2)) + 1$
using *odd-p p-ge-2 nonzero-mult-div-cancel-left* [*of* $2\ p - 1$] **by** *simp*

lemma *p-odd-int*: **obtains** $z :: int$ **where** $int\ p = 2 * z + 1$ $0 < z$
proof
let $?z = (int\ p - 1)\ div\ 2$
show $int\ p = 2 * ?z + 1$ **by** (*rule p-eq2*)
show $0 < ?z$
using *p-ge-2* **by** *linarith*
qed

8.2 Basic Properties of the Gauss Sets

lemma *finite-A*: *finite A*
by (*auto simp add*: *A-def*)

lemma *finite-B*: *finite B*
by (*auto simp add*: *B-def finite-A*)

lemma *finite-C*: *finite C*
by (*auto simp add*: *C-def finite-B*)

lemma *finite-D*: *finite D*
by (*auto simp add*: *D-def finite-C*)

lemma *finite-E*: *finite E*
by (*auto simp add*: *E-def finite-C*)

lemma *finite-F*: *finite F*
by (*auto simp add*: *F-def finite-E*)

lemma *C-eq*: $C = D \cup E$
by (*auto simp add*: *C-def D-def E-def*)

lemma *A-card-eq*: $card\ A = nat\ ((int\ p - 1)\ div\ 2)$
by (*auto simp add*: *A-def*)

lemma *inj-on-xa-A*: $inj\text{-}on\ (\lambda x.\ x * a)\ A$
using *a-nonzero* **by** (*simp add*: *A-def inj-on-def*)

definition $ResSet :: int \Rightarrow int\ set \Rightarrow bool$
where $ResSet\ m\ X \longleftrightarrow (\forall y1\ y2.\ y1 \in X \wedge y2 \in X \wedge [y1 = y2]\ (mod\ m) \longrightarrow y1 = y2)$

lemma *ResSet-image*:
$0 < m \Longrightarrow ResSet\ m\ A \Longrightarrow \forall x \in A.\ \forall y \in A.\ ([f\ x = f\ y](mod\ m) \longrightarrow x = y)$

$\Longrightarrow$ *ResSet m* $(f$ ‘ $A)$
by (*auto simp add*: *ResSet-def*)

lemma *A-res*: *ResSet p A*
using *p-ge-2* **by** (*auto simp add*: *A-def ResSet-def intro*!: *cong-less-imp-eq-int*)

lemma *B-res*: *ResSet p B*
proof −
have ∗: $x = y$
if *a*: $[x * a = y * a]$ (*mod p*)
and *b*: $0 < x$
and *c*: $x \le (int\ p - 1)\ div\ 2$
and *d*: $0 < y$
and *e*: $y \le (int\ p - 1)\ div\ 2$
for *x y*
proof −
from *p-a-relprime* **have** ¬ *p dvd a*
by (*simp add*: *cong-0-iff*)
with *p-prime prime-imp-coprime* [*of - nat* $|a|$]
have *coprime a* (*int p*)
by (*simp-all add*: *ac-simps*)
with *a cong-mult-rcancel* [*of a int p x y*] **have** $[x = y]$ (*mod p*)
by *simp*
with *cong-less-imp-eq-int* [*of x y p*] *p-minus-one-l*
order-le-less-trans [*of x* (*int p* − *1*) *div 2 p*]
order-le-less-trans [*of y* (*int p* − *1*) *div 2 p*]
show *?thesis*
by (*metis b c cong-less-imp-eq-int d e zero-less-imp-eq-int of-nat-0-le-iff*)
qed
show *?thesis*
using *p-ge-2 p-a-relprime p-minus-one-l*
by (*metis* ∗ *A-def A-res B-def GAUSS.ResSet-image GAUSS-axioms greaterThanAt-Most-iff odd-p odd-pos of-nat-0-less-iff*)
qed

lemma *SR-B-inj*: *inj-on* ($\lambda x.\ x\ mod\ p$) *B*
proof −
have *False*
if *a*: $x * a\ mod\ p = y * a\ mod\ p$
and *b*: $0 < x$
and *c*: $x \le (int\ p - 1)\ div\ 2$
and *d*: $0 < y$
and *e*: $y \le (int\ p - 1)\ div\ 2$
and *f*: $x \ne y$
for *x y*
proof −
from *a* **have** *a′*: $[x * a = y * a]$(*mod p*)
using *cong-def* **by** *blast*
from *p-a-relprime* **have** ¬*p dvd a*

by (*simp add*: *cong-0-iff*)
with *p-prime prime-imp-coprime* [*of - nat* |*a*|]
have *coprime a* (*int p*)
by (*simp-all add*: *ac-simps*)
with *a′ cong-mult-rcancel* [*of a int p x y*]
have [$x = y$] (*mod p*) **by** *simp*
with *cong-less-imp-eq-int* [*of x y p*] *p-minus-one-l*
order-le-less-trans [*of x* (*int p* − *1*) *div 2 p*]
order-le-less-trans [*of y* (*int p* − *1*) *div 2 p*]
have $x = y$
by (*metis b c cong-less-imp-eq-int d e zero-less-imp-eq-int of-nat-0-le-iff*)
then show *?thesis*
by (*simp add*: *f*)
qed
then show *?thesis*
by (*auto simp add*: *B-def inj-on-def A-def*) *metis*
qed

lemma *inj-on-pminusx-E*: *inj-on* ($\lambda x.\ p - x$) *E*
apply (*auto simp add*: *E-def C-def B-def A-def*)
apply (*rule inj-on-inverseI* [**where** g = (−) (*int p*)])
apply *auto*
done

lemma *nonzero-mod-p*: $0 < x \Longrightarrow x <$ *int p* $\Longrightarrow [x \neq 0]$(*mod p*)
for *x* :: *int*
by (*simp add*: *cong-def*)

lemma *A-ncong-p*: $x \in A \Longrightarrow [x \neq 0]$(*mod p*)
by (*rule nonzero-mod-p*) (*auto simp add*: *A-def*)

lemma *A-greater-zero*: $x \in A \Longrightarrow 0 < x$
by (*auto simp add*: *A-def*)

lemma *B-ncong-p*: $x \in B \Longrightarrow [x \neq 0]$(*mod p*)
by (*auto simp*: *B-def p-prime p-a-relprime A-ncong-p dest*: *cong-prime-prod-zero-int*)

lemma *B-greater-zero*: $x \in B \Longrightarrow 0 < x$
using *a-nonzero* **by** (*auto simp add*: *B-def A-greater-zero*)

lemma *B-mod-greater-zero*:
$0 < x$ *mod int p* **if** $x \in B$
proof −
from *that* **have** *x mod int p* $\neq 0$
using *B-ncong-p cong-def cong-mult-self-left* **by** *blast*
moreover from *that* **have** $0 < x$
by (*rule B-greater-zero*)
then have $0 \leq$ *x mod int p*
by (*auto simp add*: *mod-int-pos-iff intro*: *neq-le-trans*)

ultimately show *?thesis*
  by *simp*
qed

lemma *C-greater-zero*: $y \in C \Longrightarrow 0 < y$
  by (*auto simp add*: *C-def B-mod-greater-zero*)

lemma *F-subset*: $F \subseteq \{x.\ 0 < x \wedge x \leq ((int\ p - 1)\ div\ 2)\}$
  apply (*auto simp add*: *F-def E-def C-def*)
  apply (*metis p-ge-2 Divides.pos-mod-bound nat-int zless-nat-conj*)
  apply (*auto intro*: *p-odd-int*)
  done

lemma *D-subset*: $D \subseteq \{x.\ 0 < x \wedge x \leq ((p - 1)\ div\ 2)\}$
  by (*auto simp add*: *D-def C-greater-zero*)

lemma *F-eq*: $F = \{x.\ \exists\, y \in A.\ (x = p - ((y * a)\ mod\ p) \wedge (int\ p - 1)\ div\ 2 < (y * a)\ mod\ p)\}$
  by (*auto simp add*: *F-def E-def D-def C-def B-def A-def*)

lemma *D-eq*: $D = \{x.\ \exists\, y \in A.\ (x = (y * a)\ mod\ p \wedge (y * a)\ mod\ p \leq (int\ p - 1)\ div\ 2)\}$
  by (*auto simp add*: *D-def C-def B-def A-def*)

lemma *all-A-relprime*:
  coprime x p **if** $x \in A$
proof −
  from *A-ncong-p* [*OF that*] **have** $\neg$ *int p dvd x*
    by (*simp add*: *cong-0-iff*)
  with *p-prime* **show** *?thesis*
    by (*metis* (*no-types*) *coprime-commute prime-imp-coprime prime-nat-int-transfer*)
qed

lemma *A-prod-relprime*: *coprime* (*prod id A*) *p*
  by (*auto intro*: *prod-coprime-left all-A-relprime*)

8.3 Relationships Between Gauss Sets

lemma *StandardRes-inj-on-ResSet*: *ResSet m X* $\Longrightarrow$ *inj-on* ($\lambda b.\ b\ mod\ m$) *X*
  by (*auto simp add*: *ResSet-def inj-on-def cong-def*)

lemma *B-card-eq-A*: *card B* = *card A*
  using *finite-A* **by** (*simp add*: *finite-A B-def inj-on-xa-A card-image*)

lemma *B-card-eq*: *card B* = *nat* ((*int p* − *1*) *div 2*)
  by (*simp add*: *B-card-eq-A A-card-eq*)

lemma *F-card-eq-E*: *card F* = *card E*
  using *finite-E* **by** (*simp add*: *F-def inj-on-pminusx-E card-image*)

lemma *C-card-eq-B*: *card C* = *card B*
proof −
  have *inj-on* (λ*x*. *x mod p*) *B*
    by (*metis SR-B-inj*)
  then show *?thesis*
    by (*metis C-def card-image*)
qed

lemma *D-E-disj*: *D* ∩ *E* = {}
  by (*auto simp add*: *D-def E-def*)

lemma *C-card-eq-D-plus-E*: *card C* = *card D* + *card E*
  by (*auto simp add*: *C-eq card-Un-disjoint D-E-disj finite-D finite-E*)

lemma *C-prod-eq-D-times-E*: *prod id E* ∗ *prod id D* = *prod id C*
  by (*metis C-eq D-E-disj finite-D finite-E inf-commute prod.union-disjoint sup-commute*)

lemma *C-B-zcong-prod*: [*prod id C* = *prod id B*] (*mod p*)
  apply (*auto simp add*: *C-def*)
  apply (*insert finite-B SR-B-inj*)
  apply (*drule prod.reindex* [*of* λ*x*. *x mod int p B id*])
  apply *auto*
  apply (*rule cong-prod*)
  apply (*auto simp add*: *cong-def*)
  done

lemma *F-Un-D-subset*: (*F* ∪ *D*) ⊆ *A*
  by (*intro Un-least subset-trans* [*OF F-subset*] *subset-trans* [*OF D-subset*]) (*auto simp*: *A-def*)

lemma *F-D-disj*: (*F* ∩ *D*) = {}
proof (*auto simp add*: *F-eq D-eq*)
  fix *y z* :: *int*
  assume *p* − (*y* ∗ *a*) *mod p* = (*z* ∗ *a*) *mod p*
  then have [(*y* ∗ *a*) *mod p* + (*z* ∗ *a*) *mod p* = *0*] (*mod p*)
    by (*metis add.commute diff-eq-eq dvd-refl cong-def dvd-eq-mod-eq-0 mod-0*)
  moreover have [*y* ∗ *a* = (*y* ∗ *a*) *mod p*] (*mod p*)
    by (*metis cong-def mod-mod-trivial*)
  ultimately have [*a* ∗ (*y* + *z*) = *0*] (*mod p*)
    by (*metis cong-def mod-add-left-eq mod-add-right-eq mult.commute ring-class.ring-distribs*(*1*))
  with *p-prime a-nonzero p-a-relprime* **have** *a*: [*y* + *z* = *0*] (*mod p*)
    by (*auto dest*!: *cong-prime-prod-zero-int*)
  assume *b*: *y* ∈ *A* **and** *c*: *z* ∈ *A*
  then have *0* < *y* + *z*
    by (*auto simp*: *A-def*)
  moreover from *b c p-eq2* **have** *y* + *z* < *p*
    by (*auto simp*: *A-def*)
  ultimately show *False*

by (*metis a nonzero-mod-p*)
qed

lemma *F-Un-D-card*: *card* $(F \cup D) =$ *nat* $((p - 1)$ *div* $2)$
proof −
have *card* $(F \cup D) =$ *card* $E +$ *card* D
by (*auto simp add*: *finite-F finite-D F-D-disj card-Un-disjoint F-card-eq-E*)
then have *card* $(F \cup D) =$ *card* C
by (*simp add*: *C-card-eq-D-plus-E*)
then show *card* $(F \cup D) =$ *nat* $((p - 1)$ *div* $2)$
by (*simp add*: *C-card-eq-B B-card-eq*)
qed

lemma *F-Un-D-eq-A*: $F \cup D = A$
using *finite-A F-Un-D-subset A-card-eq F-Un-D-card* **by** (*auto simp add*: *card-seteq*)

lemma *prod-D-F-eq-prod-A*: *prod id* $D\ *$ *prod id* $F =$ *prod id* A
by (*metis F-D-disj F-Un-D-eq-A Int-commute Un-commute finite-D finite-F prod.union-disjoint*)

lemma *prod-F-zcong*: [*prod id* $F = ((-1)$ ^ (*card* E)) $*$ *prod id* E] (*mod* p)
proof −
have *FE*: *prod id* $F =$ *prod* $((-)\ p)\ E$
apply (*auto simp add*: *F-def*)
apply (*insert finite-E inj-on-pminusx-E*)
apply (*drule prod.reindex*)
apply *auto*
done
then have $\forall x \in E.$ [$(p-x)$ *mod* $p = -\ x$](*mod* p)
by (*metis cong-def minus-mod-self1 mod-mod-trivial*)
then have [*prod* $((\lambda x.\ x$ *mod* $p) \circ ((-)\ p))\ E =$ *prod* (*uminus*) E](*mod* p)
using *finite-E p-ge-2 cong-prod* [*of* E $(\lambda x.\ x$ *mod* $p) \circ ((-)\ p)$ *uminus* p]
by *auto*
then have *two*: [*prod id* $F =$ *prod* (*uminus*) E](*mod* p)
by (*metis FE cong-cong-mod-int cong-refl cong-prod minus-mod-self1*)
have *prod uminus* $E = (-1)$ ^ *card* $E\ *$ *prod id* E
using *finite-E* **by** (*induct set*: *finite*) *auto*
with *two* **show** *?thesis*
by *simp*
qed

8.4 Gauss' Lemma

lemma *aux*: *prod id* $A * (-\ 1)$ ^ *card* $E * a$ ^ *card* $A * (-\ 1)$ ^ *card* $E =$ *prod id* $A * a$ ^ *card* A
by *auto*

theorem *pre-gauss-lemma*: [a ^ *nat*((*int* $p - 1)$ *div* $2) = (-1)$ ^ (*card* E)] (*mod* p)
proof −

have [*prod id A* = *prod id F* ∗ *prod id D*](*mod p*)
  by (*auto simp*: *prod-D-F-eq-prod-A mult.commute cong del*: *prod.cong-simp*)
then have [*prod id A* = ((−*1*) ^ (*card E*) ∗ *prod id E*) ∗ *prod id D*] (*mod p*)
  by (*rule cong-trans*) (*metis cong-scalar-right prod-F-zcong*)
then have [*prod id A* = ((−*1*) ^ (*card E*) ∗ *prod id C*)] (*mod p*)
  using *finite-D finite-E* **by** (*auto simp add*: *ac-simps C-prod-eq-D-times-E C-eq D-E-disj prod.union-disjoint*)
then have [*prod id A* = ((−*1*) ^ (*card E*) ∗ *prod id B*)] (*mod p*)
  by (*rule cong-trans*) (*metis C-B-zcong-prod cong-scalar-left*)
then have [*prod id A* = ((−*1*) ^ (*card E*) ∗ *prod id* ((λ*x*. *x* ∗ *a*) ‘ *A*))] (*mod p*)
  by (*simp add*: *B-def*)
then have [*prod id A* = ((−*1*) ^ (*card E*) ∗ *prod* (λ*x*. *x* ∗ *a*) *A*)] (*mod p*)
  by (*simp add*: *inj-on-xa-A prod.reindex*)
moreover have *prod* (λ*x*. *x* ∗ *a*) *A* = *prod* (λ*x*. *a*) *A* ∗ *prod id A*
  using *finite-A* **by** (*induct set*: *finite*) *auto*
ultimately have [*prod id A* = ((−*1*) ^ (*card E*) ∗ (*prod* (λ*x*. *a*) *A* ∗ *prod id A*))] (*mod p*)
  by *simp*
then have [*prod id A* = ((−*1*) ^ (*card E*) ∗ *a* ^ (*card A*) ∗ *prod id A*)](*mod p*)
  by (*rule cong-trans*)
    (*simp add*: *cong-scalar-left cong-scalar-right finite-A prod-constant ac-simps*)
then have *a*: [*prod id A* ∗ (−*1*) ^ (*card E*) =
    ((−*1*) ^ (*card E*) ∗ *a* ^ (*card A*) ∗ *prod id A* ∗ (−*1*) ^ (*card E*))](*mod p*)
  by (*rule cong-scalar-right*)
then have [*prod id A* ∗ (−*1*) ^ (*card E*) = *prod id A* ∗
    (−*1*) ^ (*card E*) ∗ *a* ^ (*card A*) ∗ (−*1*) ^ (*card E*)](*mod p*)
  by (*rule cong-trans*) (*simp add*: *a ac-simps*)
then have [*prod id A* ∗ (−*1*) ^ (*card E*) = *prod id A* ∗ *a* ^ (*card A*)](*mod p*)
  by (*rule cong-trans*) (*simp add*: *aux cong del*: *prod.cong-simp*)
with *A-prod-relprime* **have** [(− *1*) ^ *card E* = *a* ^ *card A*](*mod p*)
  by (*metis cong-mult-lcancel*)
then show *?thesis*
  by (*simp add*: *A-card-eq cong-sym*)
qed

theorem *gauss-lemma*: *Legendre a p* = (−*1*) ^ (*card E*)
proof −
from *euler-criterion p-prime p-ge-2* **have** [*Legendre a p* = *a* ^ (*nat* (((*p*) − *1*) *div 2*))] (*mod p*)
  by *auto*
moreover have *int* ((*p* − *1*) *div 2*) = (*int p* − *1*) *div 2*
  using *p-eq2* **by** *linarith*
then have [*a* ^ *nat* (*int* ((*p* − *1*) *div 2*)) = *a* ^ *nat* ((*int p* − *1*) *div 2*)] (*mod int p*)
  by *force*
ultimately have [*Legendre a p* = (−*1*) ^ (*card E*)] (*mod p*)
  using *pre-gauss-lemma cong-trans* **by** *blast*
moreover from *p-a-relprime* **have** *Legendre a p* = *1* ∨ *Legendre a p* = −*1*
  by (*auto simp add*: *Legendre-def*)

moreover have $(-1::int)$ ^ $(card\ E) = 1 \vee (-1::int)$ ^ $(card\ E) = -1$
  using *neg-one-even-power neg-one-odd-power* **by** *blast*
moreover have $[1 \neq -1]$ $(mod\ int\ p)$
  using *cong-iff-dvd-diff* [**where** $'a=int$] *nonzero-mod-p*[*of 2*] *p-odd-int*
  by *fastforce*
ultimately show *?thesis*
  by (*auto simp add*: *cong-sym*)
qed

end

end

theory *Quadratic-Reciprocity*
imports *Gauss*
begin

The proof is based on Gauss's fifth proof, which can be found at https://www.lehigh.edu/~shw2/q-recip/gauss5.pdf.

locale *QR* =
  fixes $p :: nat$
  fixes $q :: nat$
  assumes *p-prime*: *prime p*
  assumes *p-ge-2*: $2 < p$
  assumes *q-prime*: *prime q*
  assumes *q-ge-2*: $2 < q$
  assumes *pq-neq*: $p \neq q$
begin

lemma *odd-p*: *odd p*
  using *p-ge-2 p-prime prime-odd-nat* **by** *blast*

lemma *p-ge-0*: $0 < int\ p$
  by (*simp add*: *p-prime prime-gt-0-nat*)

lemma *p-eq2*: $int\ p = (2 * ((int\ p - 1)\ div\ 2)) + 1$
  using *odd-p* **by** *simp*

lemma *odd-q*: *odd q*
  using *q-ge-2 q-prime prime-odd-nat* **by** *blast*

lemma *q-ge-0*: $0 < int\ q$
  by (*simp add*: *q-prime prime-gt-0-nat*)

lemma *q-eq2*: $int\ q = (2 * ((int\ q - 1)\ div\ 2)) + 1$
  using *odd-q* **by** *simp*

lemma *pq-eq2*: $int\ p * int\ q = (2 * ((int\ p * int\ q - 1)\ div\ 2)) + 1$

using *odd-p odd-q* **by** *simp*

lemma *pq-coprime*: *coprime p q*
using *pq-neq p-prime primes-coprime-nat q-prime* **by** *blast*

lemma *pq-coprime-int*: *coprime* (*int p*) (*int q*)
by (*simp add*: *gcd-int-def pq-coprime*)

lemma *qp-ineq*: $int\ p * k \leq (int\ p * int\ q - 1)\ div\ 2 \longleftrightarrow k \leq (int\ q - 1)\ div\ 2$
proof −
have $2 * int\ p * k \leq int\ p * int\ q - 1 \longleftrightarrow 2 * k \leq int\ q - 1$
using *p-ge-0* **by** *auto*
then show *?thesis* **by** *auto*
qed

lemma *QRqp*: *QR q p*
using *QR-def QR-axioms* **by** *simp*

lemma *pq-commute*: $int\ p * int\ q = int\ q * int\ p$
by *simp*

lemma *pq-ge-0*: $int\ p * int\ q > 0$
using *p-ge-0 q-ge-0 mult-pos-pos* **by** *blast*

definition $r = ((p - 1)\ div\ 2) * ((q - 1)\ div\ 2)$
definition $m = card\ (GAUSS.E\ p\ q)$
definition $n = card\ (GAUSS.E\ q\ p)$

abbreviation $Res\ k \equiv \{0\ ..\ k - 1\}$ **for** $k :: int$
abbreviation $Res\text{-}ge\text{-}0\ k \equiv \{0 <..\ k - 1\}$ **for** $k :: int$
abbreviation $Res\text{-}0\ k \equiv \{0::int\}$ **for** $k :: int$
abbreviation $Res\text{-}l\ k \equiv \{0 <..\ (k - 1)\ div\ 2\}$ **for** $k :: int$
abbreviation $Res\text{-}h\ k \equiv \{(k - 1)\ div\ 2 <..\ k - 1\}$ **for** $k :: int$

abbreviation *Sets-pq r0 r1 r2* $\equiv$
$\{(x::int).\ x \in r0\ (int\ p * int\ q) \land x\ mod\ p \in r1\ (int\ p) \land x\ mod\ q \in r2\ (int\ q)\}$

definition $A =$ *Sets-pq Res-l Res-l Res-h*
definition $B =$ *Sets-pq Res-l Res-h Res-l*
definition $C =$ *Sets-pq Res-h Res-h Res-l*
definition $D =$ *Sets-pq Res-l Res-h Res-h*
definition $E =$ *Sets-pq Res-l Res-0 Res-h*
definition $F =$ *Sets-pq Res-l Res-h Res-0*

definition $a = card\ A$
definition $b = card\ B$
definition $c = card\ C$
definition $d = card\ D$
definition $e = card\ E$

```
definition f = card F

lemma Gpq: GAUSS p q
  using p-prime pq-neq p-ge-2 q-prime
  by (auto simp: GAUSS-def cong-iff-dvd-diff dest: primes-dvd-imp-eq)

lemma Gqp: GAUSS q p
  by (simp add: QRqp QR.Gpq)

lemma QR-lemma-01: (λx. x mod q) ‘ E = GAUSS.E q p
proof
  have x ∈ E ⟶ x mod int q ∈ GAUSS.E q p if x ∈ E for x
  proof −
    from that obtain k where k: x = int p ∗ k
      unfolding E-def by blast
    from that E-def have x ∈ Res-l (int p ∗ int q)
      by blast
    then have k ∈ GAUSS.A q
      using Gqp GAUSS.A-def k qp-ineq by (simp add: zero-less-mult-iff)
    then have x mod q ∈ GAUSS.E q p
     using GAUSS.C-def[of q p] Gqp k GAUSS.B-def[of q p] that GAUSS.E-def[of
q p]
      by (force simp: E-def)
    then show ?thesis by auto
  qed
  then show (λx. x mod int q) ‘ E ⊆ GAUSS.E q p
    by auto
  show GAUSS.E q p ⊆ (λx. x mod q) ‘ E
  proof
    fix x
    assume x: x ∈ GAUSS.E q p
    then obtain ka where ka: ka ∈ GAUSS.A q x = (ka ∗ p) mod q
      by (auto simp: Gqp GAUSS.B-def GAUSS.C-def GAUSS.E-def)
    then have ka ∗ p ∈ Res-l (int p ∗ int q)
      using Gqp p-ge-0 qp-ineq by (simp add: GAUSS.A-def Groups.mult-ac(2))
    then show x ∈ (λx. x mod q) ‘ E
      using ka x Gqp q-ge-0 by (force simp: E-def GAUSS.E-def)
  qed
qed

lemma QR-lemma-02: e = n
proof −
  have x = y if x: x ∈ E and y: y ∈ E and mod: x mod q = y mod q for x y
  proof −
    obtain p-inv where p-inv: [int p ∗ p-inv = 1] (mod int q)
      using pq-coprime-int cong-solve-coprime-int by blast
    from x y E-def obtain kx ky where k: x = int p ∗ kx y = int p ∗ ky
      using dvd-def[of p x] by blast
    with x y E-def have 0 < x int p ∗ kx ≤ (int p ∗ int q − 1) div 2
```

$0 < y$ *int p* $*$ *ky* $\leq$ (*int p* $*$ *int q* $-$ *1*) *div 2*
using *greaterThanAtMost-iff mem-Collect-eq* **by** *blast+*
with *k* **have** $0 \leq kx$ $kx < q$ $0 \leq ky$ $ky < q$
using *qp-ineq* **by** (*simp-all add*: *zero-less-mult-iff*)
moreover from *mod k* **have** (*p-inv* $*$ (*p* $*$ *kx*)) *mod q* = (*p-inv* $*$ (*p* $*$ *ky*)) *mod q*
using *mod-mult-cong* **by** *blast*
then have (*p* $*$ *p-inv* $*$ *kx*) *mod q* = (*p* $*$ *p-inv* $*$ *ky*) *mod q*
by (*simp add*: *algebra-simps*)
then have *kx mod q* = *ky mod q*
using *p-inv mod-mult-cong*[*of p* $*$ *p-inv q 1*]
by (*auto simp*: *cong-def*)
then have [*kx* = *ky*] (*mod q*)
unfolding *cong-def* **by** *blast*
ultimately show *?thesis*
using *cong-less-imp-eq-int k* **by** *blast*
qed
then have *inj-on* ($\lambda x.$ *x mod q*) *E*
by (*auto simp*: *inj-on-def*)
then show *?thesis*
using *QR-lemma-01 card-image e-def n-def* **by** *fastforce*
qed

lemma *QR-lemma-03*: *f* = *m*
proof $-$
have *F* = *QR.E q p*
unfolding *F-def pq-commute* **using** *QRqp QR.E-def*[*of q p*] **by** *fastforce*
then have *f* = *QR.e q p*
unfolding *f-def* **using** *QRqp QR.e-def*[*of q p*] **by** *presburger*
then show *?thesis*
using *QRqp QR.QR-lemma-02 m-def QRqp QR.n-def* **by** *presburger*
qed

definition *f-1* :: *int* $\Rightarrow$ *int* $\times$ *int*
where *f-1 x* = ((*x mod p*), (*x mod q*))

definition *P-1* :: *int* $\times$ *int* $\Rightarrow$ *int* $\Rightarrow$ *bool*
where *P-1 res x* $\longleftrightarrow$ *x mod p* = *fst res* $\wedge$ *x mod q* = *snd res* $\wedge$ *x* $\in$ *Res* (*int p* $*$ *int q*)

definition *g-1* :: *int* $\times$ *int* $\Rightarrow$ *int*
where *g-1 res* = (*THE x. P-1 res x*)

lemma *P-1-lemma*:
fixes *res* :: *int* $\times$ *int*
assumes $0 \leq$ *fst res* *fst res* $< p$ $0 \leq$ *snd res* *snd res* $< q$
shows $\exists!x.$ *P-1 res x*
proof $-$
obtain *y k1 k2* **where** *yk*: *y* = *nat* (*fst res*) + *k1* $*$ *p* *y* = *nat* (*snd res*) + *k2* $*$

q
    using *chinese-remainder*[*of p q*] *pq-coprime p-ge-0 q-ge-0* **by** *fastforce*
  have *fst res* = *int* (*y* − *k1* ∗ *p*)
    using ‹*0* ≤ *fst res*› *yk*(*1*) **by** *simp*
  moreover have *snd res* = *int* (*y* − *k2* ∗ *q*)
    using ‹*0* ≤ *snd res*› *yk*(*2*) **by** *simp*
  ultimately have *res*: *res* = (*int* (*y* − *k1* ∗ *p*), *int* (*y* − *k2* ∗ *q*))
    by (*simp add*: *prod-eq-iff*)
  have *y*: *k1* ∗ *p* ≤ *y k2* ∗ *q* ≤ *y*
    using *yk* **by** *simp-all*
  from *y* **have** ∗: [*y* = *nat* (*fst res*)] (*mod p*) [*y* = *nat* (*snd res*)] (*mod q*)
    by (*auto simp add*: *res cong-le-nat intro*: *exI* [*of* - *k1*] *exI* [*of* - *k2*])
  from ∗ **have** (*y mod* (*int p* ∗ *int q*)) *mod int p* = *fst res* (*y mod* (*int p* ∗ *int q*)) *mod int q* = *snd res*
    using *y* **apply** (*auto simp add*: *res of-nat-mult* [*symmetric*] *of-nat-mod* [*symmetric*] *mod-mod-cancel simp del*: *of-nat-mult*)
      apply (*metis* ‹*fst res* = *int* (*y* − *k1* ∗ *p*)› *assms*(*1*) *assms*(*2*) *cong-def mod-pos-pos-trivial nat-int of-nat-mod*)
     apply (*metis* ‹*snd res* = *int* (*y* − *k2* ∗ *q*)› *assms*(*3*) *assms*(*4*) *cong-def mod-pos-pos-trivial nat-int of-nat-mod*)
    done
  then obtain *x* **where** *P-1 res x*
    unfolding *P-1-def*
    using *Divides.pos-mod-bound Divides.pos-mod-sign pq-ge-0* **by** *fastforce*
  moreover have *a* = *b* **if** *P-1 res a P-1 res b* **for** *a b*
  proof −
    from *that* **have** *int p* ∗ *int q dvd a* − *b*
      using *divides-mult*[*of int p a* − *b int q*] *pq-coprime-int mod-eq-dvd-iff* [*of a* - *b*]
      unfolding *P-1-def* **by** *force*
    with *that* **show** *?thesis*
      using *dvd-imp-le-int*[*of a* − *b*] **unfolding** *P-1-def* **by** *fastforce*
  qed
  ultimately show *?thesis* **by** *auto*
qed

lemma *g-1-lemma*:
  fixes *res* :: *int* × *int*
  assumes *0* ≤ *fst res fst res* < *p 0* ≤ *snd res snd res* < *q*
  shows *P-1 res* (*g-1 res*)
  using *assms P-1-lemma* [*of res*] *theI′* [*of P-1 res*] *g-1-def*
  by *auto*

definition *BuC* = *Sets-pq Res-ge-0 Res-h Res-l*

lemma *finite-BuC* [*simp*]:
  finite BuC
proof −
  {

```
    fix p q :: nat
    have finite {x. 0 < x ∧ x < int p ∗ int q}
      by simp
    then have finite {x.
      0 < x ∧
      x < int p ∗ int q ∧
      (int p − 1) div 2
      < x mod int p ∧
      x mod int p < int p ∧
      0 < x mod int q ∧
      x mod int q ≤ (int q − 1) div 2}
      by (auto intro: rev-finite-subset)
  }
  then show ?thesis
    by (simp add: BuC-def)
qed

lemma QR-lemma-04: card BuC = card (Res-h p × Res-l q)
  using card-bij-eq[of f-1 BuC Res-h p × Res-l q g-1]
proof
  show inj-on f-1 BuC
  proof
    fix x y
    assume ∗: x ∈ BuC y ∈ BuC f-1 x = f-1 y
    then have int p ∗ int q dvd x − y
      using f-1-def pq-coprime-int divides-mult[of int p x − y int q]
        mod-eq-dvd-iff[of x - y]
      by auto
    with ∗ show x = y
      using dvd-imp-le-int[of x − y int p ∗ int q] unfolding BuC-def by force
  qed
  show inj-on g-1 (Res-h p × Res-l q)
  proof
    fix x y
    assume ∗: x ∈ Res-h p × Res-l q y ∈ Res-h p × Res-l q g-1 x = g-1 y
    then have 0 ≤ fst x fst x < p 0 ≤ snd x snd x < q
        0 ≤ fst y fst y < p 0 ≤ snd y snd y < q
      using mem-Sigma-iff prod.collapse by fastforce+
    with ∗ show x = y
      using g-1-lemma[of x] g-1-lemma[of y] P-1-def by fastforce
  qed
  show g-1 ‘ (Res-h p × Res-l q) ⊆ BuC
  proof
    fix y
    assume y ∈ g-1 ‘ (Res-h p × Res-l q)
    then obtain x where x: y = g-1 x x ∈ Res-h p × Res-l q
      by blast
    then have P-1 x y
      using g-1-lemma by fastforce
```

with x **show** $y \in BuC$
unfolding *P-1-def BuC-def mem-Collect-eq* **using** *SigmaE prod.sel* **by** *fastforce*
qed
qed (*auto simp*: *finite-subset f-1-def*, *simp-all add*: *BuC-def*)

lemma *QR-lemma-05*: *card* (*Res-h p* × *Res-l q*) = *r*
proof −
have *card* (*Res-l q*) = (*q* − *1*) *div 2 card* (*Res-h p*) = (*p* − *1*) *div 2*
using *p-eq2* **by** *force+*
then show *?thesis*
unfolding *r-def* **using** *card-cartesian-product*[*of Res-h p Res-l q*] **by** *presburger*
qed

lemma *QR-lemma-06*: *b* + *c* = *r*
proof −
have *B* ∩ *C* = {} *finite B finite C B* ∪ *C* = *BuC*
unfolding *B-def C-def BuC-def* **by** *fastforce+*
then show *?thesis*
unfolding *b-def c-def* **using** *card.empty card-Un-Int QR-lemma-04 QR-lemma-05* **by** *fastforce*
qed

definition *f-2*:: *int* ⇒ *int*
where *f-2 x* = (*int p* ∗ *int q*) − *x*

lemma *f-2-lemma-1*: *f-2* (*f-2 x*) = *x*
by (*simp add*: *f-2-def*)

lemma *f-2-lemma-2*: [*f-2 x* = *int p* − *x*] (*mod p*)
by (*simp add*: *f-2-def cong-iff-dvd-diff*)

lemma *f-2-lemma-3*: *f-2 x* ∈ *S* ⟹ *x* ∈ *f-2* ‘ *S*
using *f-2-lemma-1*[*of x*] *image-eqI*[*of x f-2 f-2 x S*] **by** *presburger*

lemma *QR-lemma-07*:
f-2 ‘ *Res-l* (*int p* ∗ *int q*) = *Res-h* (*int p* ∗ *int q*)
f-2 ‘ *Res-h* (*int p* ∗ *int q*) = *Res-l* (*int p* ∗ *int q*)
proof −
have *1*: *f-2* ‘ *Res-l* (*int p* ∗ *int q*) ⊆ *Res-h* (*int p* ∗ *int q*)
by (*force simp*: *f-2-def*)
have *2*: *f-2* ‘ *Res-h* (*int p* ∗ *int q*) ⊆ *Res-l* (*int p* ∗ *int q*)
using *pq-eq2* **by** (*fastforce simp*: *f-2-def*)
from *2* **have** *3*: *Res-h* (*int p* ∗ *int q*) ⊆ *f-2* ‘ *Res-l* (*int p* ∗ *int q*)
using *f-2-lemma-3* **by** *blast*
from *1* **have** *4*: *Res-l* (*int p* ∗ *int q*) ⊆ *f-2* ‘ *Res-h* (*int p* ∗ *int q*)
using *f-2-lemma-3* **by** *blast*
from *1 3* **show** *f-2* ‘ *Res-l* (*int p* ∗ *int q*) = *Res-h* (*int p* ∗ *int q*)
by *blast*

  from *2 4* **show** *f-2 ‘ Res-h* (*int p* ∗ *int q*) = *Res-l* (*int p* ∗ *int q*)
    by *blast*
qed

lemma *QR-lemma-08*:
  f-2 x mod p ∈ *Res-l p* ⟷ *x mod p* ∈ *Res-h p*
  f-2 x mod p ∈ *Res-h p* ⟷ *x mod p* ∈ *Res-l p*
  using *f-2-lemma-2*[*of x*] *cong-def*[*of f-2 x p* − *x p*] *minus-mod-self2*[*of x p*]
    zmod-zminus1-eq-if[*of x p*] *p-eq2*
  by *auto*

lemma *QR-lemma-09*:
  f-2 x mod q ∈ *Res-l q* ⟷ *x mod q* ∈ *Res-h q*
  f-2 x mod q ∈ *Res-h q* ⟷ *x mod q* ∈ *Res-l q*
  using *QRqp QR.QR-lemma-08 f-2-def QR.f-2-def pq-commute* **by** *auto*

lemma *QR-lemma-10*: *a* = *c*
  unfolding *a-def c-def*
  apply (*rule card-bij-eq*[*of f-2 A C f-2*])
  unfolding *A-def C-def*
  using *QR-lemma-07 QR-lemma-08 QR-lemma-09* **apply** ((*simp add*: *inj-on-def*
f-2-def), *blast*)+
  apply *fastforce*+
  done

definition *BuD* = *Sets-pq Res-l Res-h Res-ge-0*
definition *BuDuF* = *Sets-pq Res-l Res-h Res*

definition *f-3* :: *int* ⇒ *int* × *int*
  where *f-3 x* = (*x mod p*, *x div p* + *1*)

definition *g-3* :: *int* × *int* ⇒ *int*
  where *g-3 x* = *fst x* + (*snd x* − *1*) ∗ *p*

lemma *QR-lemma-11*: *card BuDuF* = *card* (*Res-h p* × *Res-l q*)
  using *card-bij-eq*[*of f-3 BuDuF Res-h p* × *Res-l q g-3*]
proof
  show *f-3 ‘ BuDuF* ⊆ *Res-h p* × *Res-l q*
  proof
    fix *y*
    assume *y* ∈ *f-3 ‘ BuDuF*
    then obtain *x* **where** *x*: *y* = *f-3 x x* ∈ *BuDuF*
      by *blast*
    then have *x* ≤ *int p* ∗ (*int q* − *1*) *div 2* + (*int p* − *1*) *div 2*
      unfolding *BuDuF-def* **using** *p-eq2 int-distrib*(*4*) **by** *auto*
    moreover from *x* **have** (*int p* − *1*) *div 2* ≤ − *1* + *x mod p*
      by (*auto simp*: *BuDuF-def*)
    moreover have *int p* ∗ (*int q* − *1*) *div 2* = *int p* ∗ ((*int q* − *1*) *div 2*)
      by (*subst div-mult1-eq*) (*simp add*: *odd-q*)

then have $p * (int\ q - 1)\ div\ 2 = p * ((int\ q + 1)\ div\ 2 - 1)$
  by *fastforce*
ultimately have $x \leq p * ((int\ q + 1)\ div\ 2 - 1) - 1 + x\ mod\ p$
  by *linarith*
then have $x\ div\ p < (int\ q + 1)\ div\ 2 - 1$
  using *mult.commute*[*of int p x div p*] *p-ge-0 div-mult-mod-eq*[*of x p*]
    and *mult-less-cancel-left-pos*[*of p x div p* (*int q* + *1*) *div 2* − *1*]
  by *linarith*
moreover from x **have** $0 < x\ div\ p + 1$
  using *pos-imp-zdiv-neg-iff*[*of p x*] *p-ge-0* **by** (*auto simp*: *BuDuF-def*)
ultimately show $y \in Res\text{-}h\ p \times Res\text{-}l\ q$
  using x **by** (*auto simp*: *BuDuF-def f-3-def*)
qed
show *inj-on g-3* (*Res-h p* × *Res-l q*)
proof
  have ∗: $f\text{-}3\ (g\text{-}3\ x) = x$ **if** $x \in Res\text{-}h\ p \times Res\text{-}l\ q$ **for** x
  proof −
    from *that* **have** ∗: $(fst\ x + (snd\ x - 1) * int\ p)\ mod\ int\ p = fst\ x$
      by *force*
    from *that* **have** $(fst\ x + (snd\ x - 1) * int\ p)\ div\ int\ p + 1 = snd\ x$
      by *auto*
    with ∗ **show** $f\text{-}3\ (g\text{-}3\ x) = x$
      by (*simp add*: *f-3-def g-3-def*)
  qed
  fix $x\ y$
  assume $x \in Res\text{-}h\ p \times Res\text{-}l\ q\ \ y \in Res\text{-}h\ p \times Res\text{-}l\ q\ \ g\text{-}3\ x = g\text{-}3\ y$
  from *this* ∗[*of x*] ∗[*of y*] **show** $x = y$
    by *presburger*
qed
show *g-3* ‘ (*Res-h p* × *Res-l q*) ⊆ *BuDuF*
proof
  fix y
  assume $y \in g\text{-}3$ ‘ $(Res\text{-}h\ p \times Res\text{-}l\ q)$
  then obtain x **where** x: $x \in Res\text{-}h\ p \times Res\text{-}l\ q$ **and** y: $y = g\text{-}3\ x$
    by *blast*
  then have $snd\ x \leq (int\ q - 1)\ div\ 2$
    by *force*
  moreover have $int\ p * ((int\ q - 1)\ div\ 2) = (int\ p * int\ q - int\ p)\ div\ 2$
    using *int-distrib*(*4*) *div-mult1-eq*[*of int p int q* − *1 2*] *odd-q* **by** *fastforce*
  ultimately have $(snd\ x) * int\ p \leq (int\ q * int\ p - int\ p)\ div\ 2$
    using *mult-right-mono*[*of snd x* (*int q* − *1*) *div 2 p*] *mult.commute*[*of* (*int q*
− *1*) *div 2 p*]
      pq-commute
    by *presburger*
  then have $(snd\ x - 1) * int\ p \leq (int\ q * int\ p - 1)\ div\ 2 - int\ p$
    using *p-ge-0 int-distrib*(*3*) **by** *auto*
  moreover from x **have** $fst\ x \leq int\ p - 1$ **by** *force*
  ultimately have $fst\ x + (snd\ x - 1) * int\ p \leq (int\ p * int\ q - 1)\ div\ 2$
    using *pq-commute* **by** *linarith*

moreover from x **have** $0 < fst\ x\ 0 \leq (snd\ x - 1) * p$
by *fastforce+*
ultimately show $y \in BuDuF$
unfolding *BuDuF-def* **using** *q-ge-0 x g-3-def y* **by** *auto*
qed
show *finite BuDuF* **unfolding** *BuDuF-def* **by** *fastforce*
qed (*simp add*: *inj-on-inverseI*[*of BuDuF g-3*] *f-3-def g-3-def QR-lemma-05*)+

lemma *QR-lemma-12*: $b + d + m = r$
proof −
have $B \cap D = \{\}$ *finite B finite D* $B \cup D = BuD$
unfolding *B-def D-def BuD-def* **by** *fastforce+*
then have $b + d = card\ BuD$
unfolding *b-def d-def* **using** *card-Un-Int* **by** *fastforce*
moreover have $BuD \cap F = \{\}$ *finite BuD finite F*
unfolding *BuD-def F-def* **by** *fastforce+*
moreover have $BuD \cup F = BuDuF$
unfolding *BuD-def F-def BuDuF-def*
using *q-ge-0 ivl-disj-un-singleton*(5)[*of 0 int q* − *1*] **by** *auto*
ultimately show *?thesis*
using *QR-lemma-03 QR-lemma-05 QR-lemma-11 card-Un-disjoint*[*of BuD F*]
unfolding *b-def d-def f-def*
by *presburger*
qed

lemma *QR-lemma-13*: $a + d + n = r$
proof −
have $A = QR.B\ q\ p$
unfolding *A-def pq-commute* **using** *QRqp QR.B-def*[*of q p*] **by** *blast*
then have $a = QR.b\ q\ p$
using *a-def QRqp QR.b-def*[*of q p*] **by** *presburger*
moreover have $D = QR.D\ q\ p$
unfolding *D-def pq-commute* **using** *QRqp QR.D-def*[*of q p*] **by** *blast*
then have $d = QR.d\ q\ p$
using *d-def QRqp QR.d-def*[*of q p*] **by** *presburger*
moreover have $n = QR.m\ q\ p$
using *n-def QRqp QR.m-def*[*of q p*] **by** *presburger*
moreover have $r = QR.r\ q\ p$
unfolding *r-def* **using** *QRqp QR.r-def*[*of q p*] **by** *auto*
ultimately show *?thesis*
using *QRqp QR.QR-lemma-12* **by** *presburger*
qed

lemma *QR-lemma-14*: $(-1::int)\ \hat{}\ (m + n) = (-1)\ \hat{}\ r$
proof −
have $m + n + 2 * d = r$
using *QR-lemma-06 QR-lemma-10 QR-lemma-12 QR-lemma-13* **by** *auto*
then show *?thesis*
using *power-add*[*of* −*1::int m + n 2 * d*] **by** *fastforce*

qed

lemma *Quadratic-Reciprocity*:
  Legendre p q $*$ *Legendre q p* = $(-1::int)$ ^ $((p - 1)$ *div* $2 * ((q - 1)$ *div* $2))$
  using *Gpq Gqp GAUSS.gauss-lemma power-add*[*of* $-1::int$ *m n*] *QR-lemma-14*
  unfolding *r-def m-def n-def* **by** *auto*

end

theorem *Quadratic-Reciprocity*:
  assumes *prime p* $2 < p$ *prime q* $2 < q$ $p \neq q$
  shows *Legendre p q* $*$ *Legendre q p* = $(-1::int)$ ^ $((p - 1)$ *div* $2 * ((q - 1)$ *div* $2))$
  using *QR.Quadratic-Reciprocity QR-def assms* **by** *blast*

theorem *Quadratic-Reciprocity-int*:
  assumes *prime* (*nat p*) $2 < p$ *prime* (*nat q*) $2 < q$ $p \neq q$
  shows *Legendre p q* $*$ *Legendre q p* = $(-1::int)$ ^ (*nat* $((p - 1)$ *div* $2 * ((q - 1)$ *div* $2)))$
proof $-$
  from *assms* **have** $0 \leq (p - 1)$ *div* 2 **by** *simp*
  moreover have (*nat p* $- 1)$ *div* 2 = *nat* $((p - 1)$ *div* $2)$ (*nat q* $- 1)$ *div* 2 = *nat* $((q - 1)$ *div* $2)$
    by *fastforce+*
  ultimately have (*nat p* $- 1)$ *div* $2 * (($*nat q* $- 1)$ *div* $2)$ = *nat* $((p - 1)$ *div* $2 * ((q - 1)$ *div* $2))$
    using *nat-mult-distrib* **by** *presburger*
  moreover have $2 <$ *nat p* $2 <$ *nat q* *nat p* $\neq$ *nat q* *int* (*nat p*) = *p* *int* (*nat q*) = *q*
    using *assms* **by** *linarith+*
  ultimately show *?thesis*
    using *Quadratic-Reciprocity*[*of nat p nat q*] *assms* **by** *presburger*
qed

end

9 Pocklington's Theorem for Primes

theory *Pocklington*
imports *Residues*
begin

9.1 Lemmas about previously defined terms

lemma *prime-nat-iff''*: *prime* $(p::nat) \longleftrightarrow p \neq 0 \wedge p \neq 1 \wedge (\forall m.\ 0 < m \wedge m < p \longrightarrow$ *coprime p m*)
  apply (*auto simp add*: *prime-nat-iff*)
   apply (*rule coprimeI*)
   apply (*auto dest*: *nat-dvd-not-less simp add*: *ac-simps*)

apply (*metis One-nat-def dvd-1-iff-1 dvd-pos-nat gcd-nat.order-iff is-unit-gcd linorder-neqE-nat nat-dvd-not-less*)
done

lemma *finite-number-segment*: *card* { *m*. *0* < *m* ∧ *m* < *n* } = *n* − *1*
proof −
have { *m*. *0* < *m* ∧ *m* < *n* } = {*1*..<*n*} **by** *auto*
then show *?thesis* **by** *simp*
qed

9.2 Some basic theorems about solving congruences

lemma *cong-solve*:
fixes *n* :: *nat*
assumes *an*: *coprime a n*
shows ∃ *x*. [*a* ∗ *x* = *b*] (*mod n*)
proof (*cases a* = *0*)
case *True*
with *an* **show** *?thesis*
by (*simp add*: *cong-def*)
next
case *False*
from *bezout-add-strong-nat* [*OF this*]
obtain *d x y* **where** *dxy*: *d dvd a d dvd n a* ∗ *x* = *n* ∗ *y* + *d* **by** *blast*
from *dxy*(*1*,*2*) **have** *d1*: *d* = *1*
using *assms coprime-common-divisor* [*of a n d*] **by** *simp*
with *dxy*(*3*) **have** *a* ∗ *x* ∗ *b* = (*n* ∗ *y* + *1*) ∗ *b*
by *simp*
then have *a* ∗ (*x* ∗ *b*) = *n* ∗ (*y* ∗ *b*) + *b*
by (*auto simp*: *algebra-simps*)
then have *a* ∗ (*x* ∗ *b*) *mod n* = (*n* ∗ (*y* ∗ *b*) + *b*) *mod n*
by *simp*
then have *a* ∗ (*x* ∗ *b*) *mod n* = *b mod n*
by (*simp add*: *mod-add-left-eq*)
then have [*a* ∗ (*x* ∗ *b*) = *b*] (*mod n*)
by (*simp only*: *cong-def*)
then show *?thesis* **by** *blast*
qed

lemma *cong-solve-unique*:
fixes *n* :: *nat*
assumes *an*: *coprime a n* **and** *nz*: *n* ≠ *0*
shows ∃!*x*. *x* < *n* ∧ [*a* ∗ *x* = *b*] (*mod n*)
proof −
from *cong-solve*[*OF an*] **obtain** *x* **where** *x*: [*a* ∗ *x* = *b*] (*mod n*)
by *blast*
let *?P* = λ*x*. *x* < *n* ∧ [*a* ∗ *x* = *b*] (*mod n*)
let *?x* = *x mod n*
from *x* **have** ∗: [*a* ∗ *?x* = *b*] (*mod n*)

by (*simp add*: *cong-def mod-mult-right-eq*[*of a x n*])
from *mod-less-divisor*[*of n x*] *nz* $*$ **have** *Px*: *?P ?x* **by** *simp*
have $y = ?x$ **if** *Py*: $y < n$ $[a * y = b]$ (*mod n*) **for** *y*
proof −
from *Py*(*2*) $*$ **have** $[a * y = a * ?x]$ (*mod n*)
by (*simp add*: *cong-def*)
then have $[y = ?x]$ (*mod n*)
by (*metis an cong-mult-lcancel-nat*)
with *mod-less*[*OF Py*(*1*)] *mod-less-divisor*[*of n x*] *nz*
show *?thesis*
by (*simp add*: *cong-def*)
qed
with *Px* **show** *?thesis* **by** *blast*
qed

lemma *cong-solve-unique-nontrivial*:
fixes *p* :: *nat*
assumes *p*: *prime p*
and *pa*: *coprime p a*
and *x0*: $0 < x$
and *xp*: $x < p$
shows $\exists!y.\ 0 < y \wedge y < p \wedge [x * y = a]$ (*mod p*)
proof −
from *pa* **have** *ap*: *coprime a p*
by (*simp add*: *ac-simps*)
from *x0 xp p* **have** *px*: *coprime x p*
by (*auto simp add*: *prime-nat-iff''* *ac-simps*)
obtain *y* **where** *y*: $y < p$ $[x * y = a]$ (*mod p*) $\forall z.\ z < p \wedge [x * z = a]$ (*mod p*) $\longrightarrow z = y$
by (*metis cong-solve-unique neq0-conv p prime-gt-0-nat px*)
have $y \neq 0$
proof
assume $y = 0$
with *y*(*2*) **have** *p dvd a*
using *cong-dvd-iff* **by** *auto*
with *not-prime-1 p pa* **show** *False*
by (*auto simp add*: *gcd-nat.order-iff*)
qed
with *y* **show** *?thesis*
by *blast*
qed

lemma *cong-unique-inverse-prime*:
fixes *p* :: *nat*
assumes *prime p* **and** $0 < x$ **and** $x < p$
shows $\exists!y.\ 0 < y \wedge y < p \wedge [x * y = 1]$ (*mod p*)
by (*rule cong-solve-unique-nontrivial*) (*use assms* **in** *simp-all*)

lemma *chinese-remainder-coprime-unique*:

fixes *a* :: *nat*
assumes *ab*: *coprime a b* **and** *az*: $a \neq 0$ **and** *bz*: $b \neq 0$
and *ma*: *coprime m a* **and** *nb*: *coprime n b*
shows $\exists!x.$ *coprime* $x\ (a * b) \wedge x < a * b \wedge [x = m]\ (mod\ a) \wedge [x = n]\ (mod\ b)$
proof −
let $?P = \lambda x.\ x < a * b \wedge [x = m]\ (mod\ a) \wedge [x = n]\ (mod\ b)$
from *binary-chinese-remainder-unique-nat*[*OF ab az bz*]
obtain *x* **where** *x*: $x < a * b\ [x = m]\ (mod\ a)\ [x = n]\ (mod\ b)\ \forall y.\ ?P\ y \longrightarrow y = x$
by *blast*
from *ma nb x* **have** *coprime x a coprime x b*
using *cong-imp-coprime cong-sym* **by** *blast+*
then have *coprime* $x\ (a*b)$
by *simp*
with *x* **show** *?thesis*
by *blast*
qed

9.3 Lucas's theorem

lemma *lucas-coprime-lemma*:
fixes *n* :: *nat*
assumes *m*: $m \neq 0$ **and** *am*: $[a\hat{\ }m = 1]\ (mod\ n)$
shows *coprime a n*
proof −
consider $n = 1 \mid n = 0 \mid n > 1$ **by** *arith*
then show *?thesis*
proof *cases*
case *1*
then show *?thesis* **by** *simp*
next
case *2*
with *am m* **show** *?thesis*
by *simp*
next
case *3*
from *m* **obtain** m' **where** m': $m = Suc\ m'$ **by** (*cases m*) *blast+*
have $d = 1$ **if** *d*: *d dvd a d dvd n* **for** *d*
proof −
from *am mod-less*[*OF* ‹$n > 1$›] **have** *am1*: $a\hat{\ }m\ mod\ n = 1$
by (*simp add*: *cong-def*)
from *dvd-mult2*[*OF d*(*1*), *of* $a\hat{\ }m'$] **have** *dam*: *d dvd* $a\hat{\ }m$
by (*simp add*: m')
from *dvd-mod-iff*[*OF d*(*2*), *of* $a\hat{\ }m$] *dam am1* **show** *?thesis*
by *simp*
qed
then show *?thesis*
by (*auto intro*: *coprimeI*)
qed

```
qed

lemma lucas-weak:
  fixes n :: nat
  assumes n: n ≥ 2
    and an: [a ^ (n − 1) = 1] (mod n)
    and nm: ∀ m. 0 < m ∧ m < n − 1 ⟶ ¬ [a ^ m = 1] (mod n)
  shows prime n
proof (rule totient-imp-prime)
  show totient n = n − 1
  proof (rule ccontr)
    have [a ^ totient n = 1] (mod n)
      by (rule euler-theorem, rule lucas-coprime-lemma [of n − 1]) (use n an in
auto)
    moreover assume totient n ≠ n − 1
    then have totient n > 0 totient n < n − 1
      using ‹n ≥ 2› and totient-less[of n] by simp-all
    ultimately show False
      using nm by auto
  qed
qed (use n in auto)

lemma nat-exists-least-iff: (∃ (n::nat). P n) ⟷ (∃ n. P n ∧ (∀ m < n. ¬ P m))
  by (metis ex-least-nat-le not-less0)

lemma nat-exists-least-iff': (∃ (n::nat). P n) ⟷ P (Least P) ∧ (∀ m < (Least P).
¬ P m)
  (is ?lhs ⟷ ?rhs)
proof
  show ?lhs if ?rhs
    using that by blast
  show ?rhs if ?lhs
  proof −
    from ‹?lhs› obtain n where n: P n by blast
    let ?x = Least P
    have ¬ P m if m < ?x for m
      by (rule not-less-Least[OF that])
    with LeastI-ex[OF ‹?lhs›] show ?thesis
      by blast
  qed
qed

theorem lucas:
  assumes n2: n ≥ 2 and an1: [a^(n − 1) = 1] (mod n)
    and pn: ∀ p. prime p ∧ p dvd n − 1 ⟶ [a^((n − 1) div p) ≠ 1] (mod n)
  shows prime n
proof−
  from n2 have n01: n ≠ 0 n ≠ 1 n − 1 ≠ 0
    by arith+
```

from *mod-less-divisor*[*of n 1*] *n01* **have** *onen*: *1 mod n = 1*
  by *simp*
from *lucas-coprime-lemma*[*OF n01*(*3*) *an1*] *cong-imp-coprime an1*
have *an*: *coprime a n coprime* (*a* ^ (*n* − *1*)) *n*
  using ‹*n* ≥ *2*› **by** *simp-all*
have *False* **if** *H0*: ∃ *m*. *0* < *m* ∧ *m* < *n* − *1* ∧ [*a* ^ *m* = *1*] (*mod n*) (**is** ∃ *m*. *?P m*)
proof −
  from *H0*[*unfolded nat-exists-least-iff*[*of ?P*]] **obtain** *m* **where**
    m: *0* < *m m* < *n* − *1* [*a* ^ *m* = *1*] (*mod n*) ∀ *k* <*m*. ¬*?P k*
    by *blast*
  have *False* **if** *nm1*: (*n* − *1*) *mod m* > *0*
  proof −
    from *mod-less-divisor*[*OF m*(*1*)] **have** *th0*:(*n* − *1*) *mod m* < *m* **by** *blast*
    let *?y* = *a*^((*n* − *1*) *div m* ∗ *m*)
    note *mdeq* = *div-mult-mod-eq*[*of* (*n* − *1*) *m*]
    have *yn*: *coprime ?y n*
      using *an*(*1*) **by** (*cases* (*n* − *Suc 0*) *div m* ∗ *m* = *0*) *auto*
    have *?y mod n* = (*a*^*m*)^((*n* − *1*) *div m*) *mod n*
      by (*simp add*: *algebra-simps power-mult*)
    also have . . . = (*a*^*m mod n*)^((*n* − *1*) *div m*) *mod n*
      using *power-mod*[*of a*^*m n* (*n* − *1*) *div m*] **by** *simp*
    also have . . . = *1* **using** *m*(*3*)[*unfolded cong-def onen*] *onen*
      by (*metis power-one*)
    finally have ∗: *?y mod n* = *1* **.**
    have ∗∗: [*?y* ∗ *a* ^ ((*n* − *1*) *mod m*) = *?y*∗ *1*] (*mod n*)
      using *an1*[*unfolded cong-def onen*] *onen*
        div-mult-mod-eq[*of* (*n* − *1*) *m*, *symmetric*]
      by (*simp add*:*power-add*[*symmetric*] *cong-def* ∗ *del*: *One-nat-def*)
    have [*a* ^ ((*n* − *1*) *mod m*) = *1*] (*mod n*)
      by (*metis cong-mult-rcancel-nat mult.commute* ∗∗ *yn*)
    with *m*(*4*)[*rule-format*, *OF th0*] *nm1*
      less-trans[*OF mod-less-divisor*[*OF m*(*1*), *of n* − *1*] *m*(*2*)] **show** *?thesis*
      by *blast*
  qed
  then have (*n* − *1*) *mod m* = *0* **by** *auto*
  then have *mn*: *m dvd n* − *1* **by** *presburger*
  then obtain *r* **where** *r*: *n* − *1* = *m* ∗ *r*
    unfolding *dvd-def* **by** *blast*
  from *n01 r m*(*2*) **have** *r01*: *r* ≠ *0 r* ≠ *1* **by** *auto*
  obtain *p* **where** *p*: *prime p p dvd r*
    by (*metis prime-factor-nat r01*(*2*))
  then have *th*: *prime p* ∧ *p dvd n* − *1*
    unfolding *r* **by** (*auto intro*: *dvd-mult*)
  from *r* **have** (*a* ^ ((*n* − *1*) *div p*)) *mod n* = (*a*^(*m*∗*r div p*)) *mod n*
    by (*simp add*: *power-mult*)
  also have . . . = (*a*^(*m*∗(*r div p*))) *mod n*
    using *div-mult1-eq*[*of m r p*] *p*(*2*)[*unfolded dvd-eq-mod-eq-0*] **by** *simp*
  also have . . . = ((*a*^*m*)^(*r div p*)) *mod n*

by (*simp add*: *power-mult*)
also have ... = ((a^m *mod* n)^(r *div* p)) *mod* n
using *power-mod* ..
also from *m*(*3*) *onen* **have** ... = *1*
by (*simp add*: *cong-def*)
finally have [(a ^((n − 1) *div* p))= *1*] (*mod* n)
using *onen* **by** (*simp add*: *cong-def*)
with *pn th* **show** *?thesis* **by** *blast*
qed
then have ∀ m. *0* < m ∧ m < n − *1* ⟶ ¬ [a ^ m = *1*] (*mod* n)
by *blast*
then show *?thesis* **by** (*rule lucas-weak*[*OF n2 an1*])
qed

9.4 Definition of the order of a number mod n

definition *ord n a* = (*if coprime n a then Least* (λd. d > *0* ∧ [a ^d = *1*] (*mod* n)) *else 0*)

This has the expected properties.

lemma *coprime-ord*:
fixes *n*::*nat*
assumes *coprime n a*
shows *ord n a* > *0* ∧ [a ^(*ord n a*) = *1*] (*mod* n) ∧ (∀ m. *0* < m ∧ m < *ord n a* ⟶ [a^ m ≠ *1*] (*mod* n))
proof−
let *?P* = λd. *0* < d ∧ [a ^ d = *1*] (*mod* n)
from *bigger-prime*[*of a*] **obtain** *p* **where** *p*: *prime p a* < *p*
by *blast*
from *assms* **have** *o*: *ord n a* = *Least ?P*
by (*simp add*: *ord-def*)
have *ex*: ∃ m>*0*. *?P m*
proof (*cases n* ≥ *2*)
case *True*
moreover from *assms* **have** *coprime a n*
by (*simp add*: *ac-simps*)
then have [a ^ *totient n* = *1*] (*mod* n)
by (*rule euler-theorem*)
ultimately show *?thesis*
by (*auto intro*: *exI* [**where** x = *totient n*])
next
case *False*
then have n = *0* ∨ n = *1*
by *auto*
with *assms* **show** *?thesis*
by *auto*
qed
from *nat-exists-least-iff′*[*of ?P*] *ex assms* **show** *?thesis*
unfolding *o*[*symmetric*] **by** *auto*

qed

With the special value *0* for non-coprime case, it's more convenient.

lemma *ord-works*: $[a \mathbin{\hat{}} (ord\ n\ a) = 1]\ (mod\ n) \wedge (\forall m.\ 0 < m \wedge m < ord\ n\ a \longrightarrow \neg\ [a \mathbin{\hat{}} m = 1]\ (mod\ n))$
  for $n :: nat$
   by (*cases coprime n a*) (*use coprime-ord*[*of n a*] **in** ‹*auto simp add*: *ord-def cong-def*›)

lemma *ord*: $[a \mathbin{\hat{}} (ord\ n\ a) = 1]\ (mod\ n)$
  for $n :: nat$
  using *ord-works* **by** *blast*

lemma *ord-minimal*: $0 < m \Longrightarrow m < ord\ n\ a \Longrightarrow \neg\ [a \mathbin{\hat{}} m = 1]\ (mod\ n)$
  for $n :: nat$
  using *ord-works* **by** *blast*

lemma *ord-eq-0*: $ord\ n\ a = 0 \longleftrightarrow \neg\ coprime\ n\ a$
  for $n :: nat$
  by (*cases coprime n a*) (*simp add*: *coprime-ord*, *simp add*: *ord-def*)

lemma *divides-rexp*: $x\ dvd\ y \Longrightarrow x\ dvd\ (y \mathbin{\hat{}} Suc\ n)$
  for $x\ y :: nat$
  by (*simp add*: *dvd-mult2*[*of x y*])

lemma *ord-divides*: $[a \mathbin{\hat{}} d = 1]\ (mod\ n) \longleftrightarrow ord\ n\ a\ dvd\ d$
  (**is** *?lhs* ⟷ *?rhs*)
  for $n :: nat$
proof
  assume *?rhs*
  then obtain *k* **where** $d = ord\ n\ a * k$
    unfolding *dvd-def* **by** *blast*
  then have $[a \mathbin{\hat{}} d = (a \mathbin{\hat{}} (ord\ n\ a)\ mod\ n) \mathbin{\hat{}} k]\ (mod\ n)$
    by (*simp add* : *cong-def power-mult power-mod*)
  also have $[(a \mathbin{\hat{}} (ord\ n\ a)\ mod\ n) \mathbin{\hat{}} k = 1]\ (mod\ n)$
    using *ord*[*of a n*, *unfolded cong-def*]
    by (*simp add*: *cong-def power-mod*)
  finally show *?lhs* **.**
next
  assume *?lhs*
  show *?rhs*
  proof (*cases coprime n a*)
    case *prem*: *False*
    then have *o*: $ord\ n\ a = 0$ **by** (*simp add*: *ord-def*)
    show *?thesis*
    proof (*cases d*)
      case *0*
      with *o prem* **show** *?thesis* **by** (*simp add*: *cong-def*)
    next

```
      case (Suc d')
      then have d0: d ≠ 0 by simp
      from prem obtain p where p: p dvd n p dvd a p ≠ 1
        by (auto elim: not-coprimeE)
      from ‹?lhs› obtain q1 q2 where q12: a ^ d + n * q1 = 1 + n * q2
        using prem d0 lucas-coprime-lemma
        by (auto elim: not-coprimeE simp add: ac-simps)
      then have a ^ d + n * q1 − n * q2 = 1 by simp
       with dvd-diff-nat [OF dvd-add [OF divides-rexp]]  dvd-mult2 Suc p have p
dvd 1
        by metis
      with p(3) have False by simp
      then show ?thesis ..
    qed
  next
    case H: True
    let ?o = ord n a
    let ?q = d div ord n a
    let ?r = d mod ord n a
    have eqo: [(a^?o)^?q = 1] (mod n)
      using cong-pow ord-works by fastforce
    from H have onz: ?o ≠ 0 by (simp add: ord-eq-0)
    then have opos: ?o > 0 by simp
    from div-mult-mod-eq[of d ord n a] ‹?lhs›
    have [a^(?o*?q + ?r) = 1] (mod n)
      by (simp add: cong-def mult.commute)
    then have [(a^?o)^?q * (a^?r) = 1] (mod n)
      by (simp add: cong-def power-mult[symmetric] power-add[symmetric])
    then have th: [a^?r = 1] (mod n)
      using eqo mod-mult-left-eq[of (a^?o)^?q a^?r n]
     by (simp add: cong-def del: One-nat-def) (metis mod-mult-left-eq nat-mult-1)
    show ?thesis
    proof (cases ?r = 0)
      case True
      then show ?thesis by (simp add: dvd-eq-mod-eq-0)
    next
      case False
      with mod-less-divisor[OF opos, of d] have r0o:?r >0 ∧ ?r < ?o by simp
      from conjunct2[OF ord-works[of a n], rule-format, OF r0o] th
      show ?thesis by blast
    qed
  qed
qed

lemma order-divides-totient:
  ord n a dvd totient n if coprime n a
  using that euler-theorem [of a n]
  by (simp add: ord-divides [symmetric] ac-simps)
```

```
lemma order-divides-expdiff:
  fixes n::nat and a::nat assumes na: coprime n a
  shows [a^d = a^e] (mod n) ⟷ [d = e] (mod (ord n a))
proof -
  have th: [a^d = a^e] (mod n) ⟷ [d = e] (mod (ord n a))
    if na: coprime n a and ed: (e::nat) ≤ d
    for n a d e :: nat
  proof -
    from na ed have ∃ c. d = e + c by presburger
    then obtain c where c: d = e + c ..
    from na have an: coprime a n
      by (simp add: ac-simps)
    then have aen: coprime (a ^ e) n
      by (cases e > 0) simp-all
    from an have acn: coprime (a ^ c) n
      by (cases c > 0) simp-all
    from c have [a^d = a^e] (mod n) ⟷ [a^(e + c) = a^(e + 0)] (mod n)
      by simp
    also have ... ⟷ [a^e* a^c = a^e *a^0] (mod n) by (simp add: power-add)
    also have  ... ⟷ [a ^ c = 1] (mod n)
      using cong-mult-lcancel-nat [OF aen, of a^c a^0] by simp
    also have ... ⟷ ord n a dvd c
      by (simp only: ord-divides)
    also have ... ⟷ [e + c = e + 0] (mod ord n a)
      by (auto simp add: cong-altdef-nat)
    finally show ?thesis
      using c by simp
  qed
  consider e ≤ d | d ≤ e by arith
  then show ?thesis
  proof cases
    case 1
    with na show ?thesis by (rule th)
  next
    case 2
    from th[OF na this] show ?thesis
      by (metis cong-sym)
  qed
qed

lemma ord-not-coprime [simp]: ¬coprime n a ⟹ ord n a = 0
  by (simp add: ord-def)

lemma ord-1 [simp]: ord 1 n = 1
proof -
  have (LEAST k. k > 0) = (1 :: nat)
    by (rule Least-equality) auto
  thus ?thesis by (simp add: ord-def)
qed
```

```
lemma ord-1-right [simp]: ord (n::nat) 1 = 1
  using ord-divides[of 1 1 n] by simp

lemma ord-Suc-0-right [simp]: ord (n::nat) (Suc 0) = 1
  using ord-divides[of 1 1 n] by simp

lemma ord-0-nat [simp]: ord 0 (n :: nat) = (if n = 1 then 1 else 0)
proof -
  have (LEAST k. k > 0) = (1 :: nat)
    by (rule Least-equality) auto
  thus ?thesis by (auto simp: ord-def)
qed

lemma ord-0-right-nat [simp]: ord (n :: nat) 0 = (if n = 1 then 1 else 0)
proof -
  have (LEAST k. k > 0) = (1 :: nat)
    by (rule Least-equality) auto
  thus ?thesis by (auto simp: ord-def)
qed

lemma ord-divides': [a ^ d = Suc 0] (mod n) = (ord n a dvd d)
  using ord-divides[of a d n] by simp

lemma ord-Suc-0 [simp]: ord (Suc 0) n = 1
  using ord-1[where 'a = nat] by (simp del: ord-1)

lemma ord-mod [simp]: ord n (k mod n) = ord n k
  by (cases n = 0) (auto simp add: ord-def cong-def power-mod)

lemma ord-gt-0-iff [simp]: ord (n::nat) x > 0 ⟷ coprime n x
  using ord-eq-0[of n x] by auto

lemma ord-eq-Suc-0-iff: ord n (x::nat) = Suc 0 ⟷ [x = 1] (mod n)
  using ord-divides[of x 1 n] by (auto simp: ord-divides')

lemma ord-cong:
  assumes [k1 = k2] (mod n)
  shows   ord n k1 = ord n k2
proof -
  have ord n (k1 mod n) = ord n (k2 mod n)
    by (simp only: assms[unfolded cong-def])
  thus ?thesis by simp
qed

lemma ord-nat-code [code-unfold]:
  ord n a =
    (if n = 0 then if a = 1 then 1 else 0 else
      if coprime n a then Min (Set.filter (λk. [a ^ k = 1] (mod n)) {0<..n}) else
```

```
0)
proof (cases coprime n a ∧ n > 0)
  case True
  define A where A = {k∈{0<..n}. [a ^ k = 1] (mod n)}
  define k where k = (LEAST k. k > 0 ∧ [a ^ k = 1] (mod n))
  have totient: totient n ∈ A
    using euler-theorem[of a n] True
    by (auto simp: A-def coprime-commute intro!: Nat.gr0I totient-le)
  moreover have finite A by (auto simp: A-def)
  ultimately have *: Min A ∈ A and ∀ y. y ∈ A ⟶ Min A ≤ y
    by (auto intro: Min-in)

  have k > 0 ∧ [a ^ k = 1] (mod n)
     unfolding k-def by (rule LeastI[of - totient n]) (use totient in ‹auto simp:
A-def›)
  moreover have k ≤ totient n
    unfolding k-def by (intro Least-le) (use totient in ‹auto simp: A-def›)
  ultimately have k ∈ A using totient-le[of n] by (auto simp: A-def)
  hence Min A ≤ k by (intro Min-le) (auto simp: ‹finite A›)
  moreover from * have k ≤ Min A
    unfolding k-def by (intro Least-le) (auto simp: A-def)
 ultimately show ?thesis using True by (simp add: ord-def k-def A-def Set.filter-def)
qed auto

theorem ord-modulus-mult-coprime:
  fixes x :: nat
  assumes coprime m n
  shows   ord (m * n) x = lcm (ord m x) (ord n x)
proof (intro dvd-antisym)
  have [x ^ lcm (ord m x) (ord n x) = 1] (mod (m * n))
    using assms by (intro coprime-cong-mult-nat assms) (auto simp: ord-divides')
  thus ord (m * n) x dvd lcm (ord m x) (ord n x)
    by (simp add: ord-divides')
next
  show lcm (ord m x) (ord n x) dvd ord (m * n) x
  proof (intro lcm-least)
    show ord m x dvd ord (m * n) x
      using cong-modulus-mult-nat[of x ^ ord (m * n) x 1 m n] assms
      by (simp add: ord-divides')
    show ord n x dvd ord (m * n) x
      using cong-modulus-mult-nat[of x ^ ord (m * n) x 1 n m] assms
      by (simp add: ord-divides' mult.commute)
  qed
qed

corollary ord-modulus-prod-coprime:
  assumes finite A ⋀i j. i ∈ A ⟹ j ∈ A ⟹ i ≠ j ⟹ coprime (f i) (f j)
  shows   ord (∏ i∈A. f i :: nat) x = (LCM i∈A. ord (f i) x)
  using assms by (induction A rule: finite-induct)
```

```
    (simp, simp, subst ord-modulus-mult-coprime, auto intro!: prod-coprime-right)

lemma ord-power-aux:
  fixes m x k a :: nat
  defines l ≡ ord m a
  shows   ord m (a ^ k) * gcd k l = l
proof (rule dvd-antisym)
  have [a ^ lcm k l = 1] (mod m)
    unfolding ord-divides by (simp add: l-def)
  also have lcm k l = k * (l div gcd k l)
    by (simp add: lcm-nat-def div-mult-swap)
  finally have ord m (a ^ k) dvd l div gcd k l
    unfolding ord-divides [symmetric] by (simp add: power-mult [symmetric])
  thus ord m (a ^ k) * gcd k l dvd l
    by (cases l = 0) (auto simp: dvd-div-iff-mult)

  have [(a ^ k) ^ ord m (a ^ k) = 1] (mod m)
    by (rule ord)
  also have (a ^ k) ^ ord m (a ^ k) = a ^(k * ord m (a ^ k))
    by (simp add: power-mult)
  finally have ord m a dvd k * ord m (a ^ k)
    by (simp add: ord-divides')
  hence l dvd gcd (k * ord m (a ^ k)) (l * ord m (a ^ k))
    by (intro gcd-greatest dvd-triv-left) (auto simp: l-def ord-divides')
  also have gcd (k * ord m (a ^ k)) (l * ord m (a ^ k)) = ord m (a ^ k) * gcd k l
    by (subst gcd-mult-distrib-nat) (auto simp: mult-ac)
  finally show l dvd ord m (a ^ k) * gcd k l .
qed

theorem ord-power: coprime m a ⟹ ord m (a ^ k :: nat) = ord m a div gcd k
(ord m a)
  using ord-power-aux[of m a k] by (metis div-mult-self-is-m gcd-pos-nat ord-eq-0)

lemma inj-power-mod:
  assumes coprime n (a :: nat)
  shows   inj-on (λk. a ^ k mod n) {..<ord n a}
proof
  fix k l assume *: k ∈ {..<ord n a} l ∈ {..<ord n a} a ^ k mod n = a ^ l mod n
  have k = l if k < l l < ord n a [a ^ k = a ^ l] (mod n) for k l
  proof −
    have l = k + (l − k) using that by simp
    also have a ^ ... = a ^ k * a ^(l − k)
      by (simp add: power-add)
    also have [... = a ^ l * a ^(l − k)] (mod n)
      using that by (intro cong-mult) auto
    finally have [a ^ l * a ^(l − k) = a ^ l * 1] (mod n)
      by (simp add: cong-sym-eq)
    with assms have [a ^(l − k) = 1] (mod n)
      by (subst (asm) cong-mult-lcancel-nat) (auto simp: coprime-commute)
```

hence *ord n a dvd l − k*
by (*simp add*: *ord-divides′*)
from *dvd-imp-le*[*OF this*] **and** ‹*l* < *ord n a*› **have** *l − k = 0*
by (*cases l − k = 0*) *auto*
with ‹*k* < *l*› **show** *k = l* **by** *simp*
qed
from *this*[*of k l*] **and** *this*[*of l k*] **and** ∗ **show** *k = l*
by (*cases k l rule*: *linorder-cases*) (*auto simp*: *cong-def*)
qed

lemma *ord-eq-2-iff*: *ord n* (*x* :: *nat*) = *2* ⟷ [*x* ≠ *1*] (*mod n*) ∧ [x^2 = *1*] (*mod n*)
proof
assume *x*: [*x* ≠ *1*] (*mod n*) ∧ [x^2 = *1*] (*mod n*)
hence *coprime n x*
by (*metis coprime-commute lucas-coprime-lemma zero-neq-numeral*)
with *x* **have** *ord n x dvd 2 ord n x* ≠ *1 ord n x* > *0*
by (*auto simp*: *ord-divides′ ord-eq-Suc-0-iff*)
thus *ord n x = 2* **by** (*auto dest*!: *dvd-imp-le simp del*: *ord-gt-0-iff*)
qed (*use ord-divides*[*of - 2*] *ord-divides*[*of - 1*] **in** *auto*)

lemma *square-mod-8-eq-1-iff*: [x^2 = *1*] (*mod 8*) ⟷ *odd* (*x* :: *nat*)
proof −
have [x^2 = *1*] (*mod 8*) ⟷ ((*x mod 8*)^2 *mod 8* = *1*)
by (*simp add*: *power-mod cong-def*)
also have ... ⟷ *x mod 8* ∈ {*1*, *3*, *5*, *7*}
proof
assume *x*: (*x mod 8*)^2 *mod 8* = *1*
have *x mod 8* ∈ {..<*8*} **by** *simp*
also have {..<*8*} = {*0*, *1*, *2*, *3*, *4*, *5*, *6*, *7*::*nat*}
by (*simp add*: *lessThan-nat-numeral lessThan-Suc insert-commute*)
finally have *x-cases*: *x mod 8* ∈ {*0*, *1*, *2*, *3*, *4*, *5*, *6*, *7*} **.**
from *x* **have** *x mod 8* ∉ {*0*, *2*, *4*, *6*}
using *x* **by** (*auto intro*: *Nat.gr0I*)
with *x-cases* **show** *x mod 8* ∈ {*1*, *3*, *5*, *7*} **by** *simp*
qed *auto*
also have ... ⟷ *odd* (*x mod 8*)
by (*auto elim*!: *oddE*)
also have ... ⟷ *odd x*
by *presburger*
finally show *?thesis* **.**
qed

lemma *ord-twopow-aux*:
assumes *k* ≥ *3* **and** *odd* (*x* :: *nat*)
shows [*x* ^ (*2* ^ (*k* − *2*)) = *1*] (*mod* (*2* ^ *k*))
using *assms*(*1*)
proof (*induction k rule*: *dec-induct*)
case *base*

from *assms* **have** $[x^2 = 1]$ (*mod 8*)
  by (*subst square-mod-8-eq-1-iff*) *auto*
  thus *?case* **by** *simp*
next
  case (*step k*)
  define k' **where** $k' = k - 2$
  have *k*: $k = Suc\ (Suc\ k')$
    using ‹$k \geq 3$› **by** (*simp add*: *k′-def*)
  from ‹$k \geq 3$› **have** $2 * k \geq Suc\ k$ **by** *presburger*

  from ‹*odd x*› **have** $x > 0$ **by** (*intro Nat.gr0I*) *auto*
  from *step.IH* **have** $2 \mathbin{\hat{}} k\ dvd\ (x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 2)) - 1)$
    by (*rule cong-to-1-nat*)
  then obtain *t* **where** $x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 2)) - 1 = t * 2 \mathbin{\hat{}} k$
    by *auto*
  hence $x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 2)) = t * 2 \mathbin{\hat{}} k + 1$
   by (*metis* ‹$0 < x$› *add.commute add-diff-inverse-nat less-one neq0-conv power-eq-0-iff*)
  hence $(x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 2))) \mathbin{\hat{}} 2 = (t * 2 \mathbin{\hat{}} k + 1) \mathbin{\hat{}} 2$
    by (*rule arg-cong*)
  hence $[(x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 2))) \mathbin{\hat{}} 2 = (t * 2 \mathbin{\hat{}} k + 1) \mathbin{\hat{}} 2]$ (*mod* $(2 \mathbin{\hat{}} Suc\ k)$)
    by *simp*
  also have $(x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 2))) \mathbin{\hat{}} 2 = x \mathbin{\hat{}} (2 \mathbin{\hat{}} (k - 1))$
    by (*simp-all add*: *power-even-eq*[*symmetric*] *power-mult k*)
  also have $(t * 2 \mathbin{\hat{}} k + 1) \mathbin{\hat{}} 2 = t^2 * 2 \mathbin{\hat{}} (2 * k) + t * 2 \mathbin{\hat{}} Suc\ k + 1$
    by (*subst power2-eq-square*)
      (*auto simp*: *algebra-simps k power2-eq-square*[*of t*]
            power-even-eq[*symmetric*] *power-add* [*symmetric*])
  also have $[\ldots = 0 + 0 + 1]$ (*mod* $2 \mathbin{\hat{}} Suc\ k$)
    using ‹$2 * k \geq Suc\ k$›
    by (*intro cong-add*)
     (*auto simp*: *cong-0-iff intro*: *dvd-mult*[*OF le-imp-power-dvd*] *simp del*: *power-Suc*)
  finally show *?case* **by** *simp*
qed

lemma *ord-twopow-3-5*:
  assumes $k \geq 3$ $x\ mod\ 8 \in \{3, 5 :: nat\}$
  shows $ord\ (2 \mathbin{\hat{}} k)\ x = 2 \mathbin{\hat{}} (k - 2)$
  using *assms*(*1*)
proof (*induction k rule*: *less-induct*)
  have $x\ mod\ 8 = 3 \lor x\ mod\ 8 = 5$ **using** *assms* **by** *auto*
  hence *odd x* **by** *presburger*
  case (*less k*)
  from ‹$k \geq 3$› **consider** $k = 3 \mid k = 4 \mid k \geq 5$ **by** *force*
  thus *?case*
  proof *cases*
    case *1*
    thus *?thesis* **using** *assms*
      by (*auto simp*: *ord-eq-2-iff cong-def simp flip*: *power-mod*[*of x*])
  next

```
    case 2
    from assms have x mod 8 = 3 ∨ x mod 8 = 5 by auto
    hence x': x mod 16 = 3 ∨ x mod 16 = 5 ∨ x mod 16 = 11 ∨ x mod 16 = 13
      using mod-double-modulus[of 8 x] by auto
    hence [x ^ 4 = 1] (mod 16) using assms
      by (auto simp: cong-def simp flip: power-mod[of x])
    hence ord 16 x dvd 2^2 by (simp add: ord-divides')
    then obtain l where l: ord 16 x = 2 ^ l l ≤ 2
      by (subst (asm) divides-primepow-nat) auto

    have [x ^ 2 ≠ 1] (mod 16)
      using x' by (auto simp: cong-def simp flip: power-mod[of x])
    hence ¬ord 16 x dvd 2 by (simp add: ord-divides')
    with l have l = 2
      using le-imp-power-dvd[of l 1 2] by (cases l ≤ 1) auto
    with l show ?thesis by (simp add: ‹k = 4›)
  next
    case 3
    define k' where k' = k − 2
    have k': k' ≥ 2 and [simp]: k = Suc (Suc k')
      using 3 by (simp-all add: k'-def)
    have IH: ord (2 ^ k') x = 2 ^ (k' − 2) ord (2 ^ Suc k') x = 2 ^ (k' − 1)
      using less.IH[of k'] less.IH[of Suc k'] 3 by simp-all
    from IH have cong: [x ^ (2 ^ (k' − 2)) = 1] (mod (2 ^ k'))
      by (simp-all add: ord-divides')
    have notcong: [x ^ (2 ^ (k' − 2)) ≠ 1] (mod (2 ^ Suc k'))
    proof
      assume [x ^ (2 ^ (k' − 2)) = 1] (mod (2 ^ Suc k'))
      hence ord (2 ^ Suc k') x dvd 2 ^ (k' − 2)
        by (simp add: ord-divides')
      also have ord (2 ^ Suc k') x = 2 ^ (k' − 1)
        using IH by simp
      finally have k' − 1 ≤ k' − 2
        by (rule power-dvd-imp-le) auto
      with ‹k' ≥ 2› show False by simp
    qed

    have 2 ^ k' + 1 < 2 ^ k' + (2 ^ k' :: nat)
      using one-less-power[of 2::nat k'] k' by (intro add-strict-left-mono) auto
   with cong notcong have cong': x ^ (2 ^ (k' − 2)) mod 2 ^ Suc k' = 1 + 2 ^ k'
       using mod-double-modulus[of 2 ^ k' x ^ 2 ^ (k' − 2)] k' by (auto simp:
cong-def)

    hence x ^ (2 ^ (k' − 2)) mod 2 ^ k = 1 + 2 ^ k' ∨
           x ^ (2 ^ (k' − 2)) mod 2 ^ k = 1 + 2 ^ k' + 2 ^ Suc k'
      using mod-double-modulus[of 2 ^ Suc k' x ^ 2 ^ (k' − 2)] by auto
    hence eq: [x ^ 2 ^ (k' − 1) = 1 + 2 ^ (k − 1)] (mod 2 ^ k)
    proof
      assume *: x ^ (2 ^ (k' − 2)) mod (2 ^ k) = 1 + 2 ^ k'
```

have $[x \hat{~} (2 \hat{~} (k' - 2)) = x \hat{~} (2 \hat{~} (k' - 2))\ mod\ 2 \hat{~} k]\ (mod\ 2 \hat{~} k)$
by *simp*
also have $[x \hat{~} (2 \hat{~} (k' - 2))\ mod\ (2 \hat{~} k) = 1 + 2 \hat{~} k']\ (mod\ 2 \hat{~} k)$
by (*subst* $*$) *auto*
finally have $[(x \hat{~} 2 \hat{~} (k' - 2)) \hat{~} 2 = (1 + 2 \hat{~} k') \hat{~} 2]\ (mod\ 2 \hat{~} k)$
by (*rule cong-pow*)
hence $[x \hat{~} 2 \hat{~} Suc\ (k' - 2) = (1 + 2 \hat{~} k') \hat{~} 2]\ (mod\ 2 \hat{~} k)$
by (*simp add*: *power-mult* [*symmetric*] *power-Suc2* [*symmetric*] *del*: *power-Suc*)
also have $Suc\ (k' - 2) = k' - 1$
using k' **by** *simp*
also have $(1 + 2 \hat{~} k' :: nat)^2 = 1 + 2 \hat{~} (k - 1) + 2 \hat{~} (2 * k')$
by (*subst power2-eq-square*) (*simp add*: *algebra-simps flip*: *power-add*)
also have $(2 \hat{~} k :: nat)\ dvd\ 2 \hat{~} (2 * k')$
using k' **by** (*intro le-imp-power-dvd*) *auto*
hence $[1 + 2 \hat{~} (k - 1) + 2 \hat{~} (2 * k') = 1 + 2 \hat{~} (k - 1) + (0 :: nat)]\ (mod$
$2 \hat{~} k)$
by (*intro cong-add*) (*auto simp*: *cong-0-iff*)
finally show $[x \hat{~} 2 \hat{~} (k' - 1) = 1 + 2 \hat{~} (k - 1)]\ (mod\ 2 \hat{~} k)$
by *simp*
next
assume $*$: $x \hat{~} (2 \hat{~} (k' - 2))\ mod\ 2 \hat{~} k = 1 + 2 \hat{~} k' + 2 \hat{~} Suc\ k'$
have $[x \hat{~} (2 \hat{~} (k' - 2)) = x \hat{~} (2 \hat{~} (k' - 2))\ mod\ 2 \hat{~} k]\ (mod\ 2 \hat{~} k)$
by *simp*
also have $[x \hat{~} (2 \hat{~} (k' - 2))\ mod\ (2 \hat{~} k) = 1 + 3 * 2 \hat{~} k']\ (mod\ 2 \hat{~} k)$
by (*subst* $*$) *auto*
finally have $[(x \hat{~} 2 \hat{~} (k' - 2)) \hat{~} 2 = (1 + 3 * 2 \hat{~} k') \hat{~} 2]\ (mod\ 2 \hat{~} k)$
by (*rule cong-pow*)
hence $[x \hat{~} 2 \hat{~} Suc\ (k' - 2) = (1 + 3 * 2 \hat{~} k') \hat{~} 2]\ (mod\ 2 \hat{~} k)$
by (*simp add*: *power-mult* [*symmetric*] *power-Suc2* [*symmetric*] *del*: *power-Suc*)
also have $Suc\ (k' - 2) = k' - 1$
using k' **by** *simp*
also have $(1 + 3 * 2 \hat{~} k' :: nat)^2 = 1 + 2 \hat{~} (k - 1) + 2 \hat{~} k + 9 * 2 \hat{~} (2$
$* k')$
by (*subst power2-eq-square*) (*simp add*: *algebra-simps flip*: *power-add*)
also have $(2 \hat{~} k :: nat)\ dvd\ 9 * 2 \hat{~} (2 * k')$
using k' **by** (*intro dvd-mult le-imp-power-dvd*) *auto*
hence $[1 + 2 \hat{~} (k - 1) + 2 \hat{~} k + 9 * 2 \hat{~} (2 * k') = 1 + 2 \hat{~} (k - 1) + 0$
$+ (0 :: nat)]$
$(mod\ 2 \hat{~} k)$
by (*intro cong-add*) (*auto simp*: *cong-0-iff*)
finally show $[x \hat{~} 2 \hat{~} (k' - 1) = 1 + 2 \hat{~} (k - 1)]\ (mod\ 2 \hat{~} k)$
by *simp*
qed

have *notcong'*: $[x \hat{~} 2 \hat{~} (k - 3) \neq 1]\ (mod\ 2 \hat{~} k)$
proof
assume $[x \hat{~} 2 \hat{~} (k - 3) = 1]\ (mod\ 2 \hat{~} k)$
hence $[x \hat{~} 2 \hat{~} (k' - 1) - x \hat{~} 2 \hat{~} (k' - 1) = 1 + 2 \hat{~} (k - 1) - 1]\ (mod\ 2$
$\hat{~} k)$

```
        by (intro cong-diff-nat eq) auto
      hence [2 ^ (k − 1) = (0 :: nat)] (mod 2 ^ k)
        by (simp add: cong-sym-eq)
      hence 2 ^ k dvd 2 ^ (k − 1)
        by (simp add: cong-0-iff)
      hence k ≤ k − 1
        by (rule power-dvd-imp-le) auto
      thus False by simp
    qed

    have [x ^ 2 ^ (k − 2) = 1] (mod 2 ^ k)
      using ord-twopow-aux[of k x] ‹odd x› ‹k ≥ 3› by simp
    hence ord (2 ^ k) x dvd 2 ^ (k − 2)
      by (simp add: ord-divides')
    then obtain l where l: l ≤ k − 2 ord (2 ^ k) x = 2 ^ l
      using divides-primepow-nat[of 2 ord (2 ^ k) x k − 2] by auto

    from notcong' have ¬ord (2 ^ k) x dvd 2 ^ (k − 3)
      by (simp add: ord-divides')
    with l have l = k − 2
      using le-imp-power-dvd[of l k − 3 2] by (cases l ≤ k − 3) auto
    with l show ?thesis by simp
  qed
qed

lemma ord-4-3 [simp]: ord 4 (3::nat) = 2
proof −
  have [3 ^ 2 = (1 :: nat)] (mod 4)
    by (simp add: cong-def)
  hence ord 4 (3::nat) dvd 2
    by (subst (asm) ord-divides) auto
  hence ord 4 (3::nat) ≤ 2
    by (intro dvd-imp-le) auto
  moreover have ord 4 (3::nat) ≠ 1
    by (auto simp: ord-eq-Suc-0-iff cong-def)
  moreover have ord 4 (3::nat) ≠ 0
    by (auto simp: gcd-non-0-nat coprime-iff-gcd-eq-1)
  ultimately show ord 4 (3 :: nat) = 2
    by linarith
qed

lemma elements-with-ord-1: n > 0 ⟹ {x∈totatives n. ord n x = Suc 0} = {1}
  by (auto simp: ord-eq-Suc-0-iff cong-def totatives-less)

lemma residue-prime-has-primroot:
  fixes p :: nat
  assumes prime p
  shows ∃ a∈totatives p. ord p a = p − 1
proof −
```

```
  from residue-prime-mult-group-has-gen[OF assms]
    obtain a where a: a ∈ {1..p−1} {1..p−1} = {a ^ i mod p |i. i ∈ UNIV} by
blast
  from a have coprime p a
    using a assms by (intro prime-imp-coprime) (auto dest: dvd-imp-le)
  with a(1) have a ∈ totatives p by (auto simp: totatives-def coprime-commute)

  have p − 1 = card {1..p−1} by simp
  also have {1..p−1} = {a ^ i mod p |i. i ∈ UNIV} by fact
  also have {a ^ i mod p |i. i ∈ UNIV} = (λi. a ^ i mod p) ` {..<ord p a}
  proof (intro equalityI subsetI)
    fix x assume x ∈ {a ^ i mod p |i. i ∈ UNIV}
    then obtain i where [simp]: x = a ^ i mod p by auto

    have [a ^ i = a ^(i mod ord p a)] (mod p)
      using ‹coprime p a› by (subst order-divides-expdiff) auto
    hence ∃j. a ^ i mod p = a ^ j mod p ∧ j < ord p a
      using ‹coprime p a› by (intro exI[of - i mod ord p a]) (auto simp: cong-def)
    thus x ∈ (λi. a ^ i mod p) ` {..<ord p a}
      by auto
  qed auto
  also have card ... = ord p a
    using inj-power-mod[OF ‹coprime p a›] by (subst card-image) auto
  finally show ?thesis using ‹a ∈ totatives p›
    by auto
qed
```

9.5 Another trivial primality characterization

```
lemma prime-prime-factor: prime n ⟷ n ≠ 1 ∧ (∀p. prime p ∧ p dvd n ⟶ p
= n)
  (is ?lhs ⟷ ?rhs)
  for n :: nat
proof (cases n = 0 ∨ n = 1)
  case True
  then show ?thesis
    by (metis bigger-prime dvd-0-right not-prime-1 not-prime-0)
next
  case False
  show ?thesis
  proof
    assume prime n
    then show ?rhs
      by (metis not-prime-1 prime-nat-iff)
  next
    assume ?rhs
    with False show prime n
      by (auto simp: prime-nat-iff) (metis One-nat-def prime-factor-nat prime-nat-iff)
  qed
```

qed

lemma *prime-divisor-sqrt*: *prime n* $\longleftrightarrow$ $n \neq 1 \wedge (\forall d.\ d\ dvd\ n \wedge d^2 \leq n \longrightarrow d = 1)$
  for $n :: nat$
proof −
  consider $n = 0 \mid n = 1 \mid n \neq 0\ n \neq 1$ **by** *blast*
  then show *?thesis*
  proof *cases*
    case *1*
    then show *?thesis* **by** *simp*
  next
    case *2*
    then show *?thesis* **by** *simp*
  next
    case *n*: *3*
    then have *np*: $n > 1$ **by** *arith*
    {
      fix *d*
      assume *d*: $d\ dvd\ n\ d^2 \leq n$
        and *H*: $\forall m.\ m\ dvd\ n \longrightarrow m = 1 \vee m = n$
      from *H d* **have** *d1n*: $d = 1 \vee d = n$ **by** *blast*
      then have $d = 1$
      proof
        assume *dn*: $d = n$
        from *n* **have** $n^2 > n * 1$
          by (*simp add*: *power2-eq-square*)
        with *dn d*(*2*) **show** *?thesis* **by** *simp*
      qed
    }
    moreover
    {
      fix *d* **assume** *d*: *d dvd n* **and** *H*: $\forall d'.\ d'\ dvd\ n \wedge d'^2 \leq n \longrightarrow d' = 1$
      from *d n* **have** $d \neq 0$
        by (*metis dvd-0-left-iff*)
      then have *dp*: $d > 0$ **by** *simp*
      from *d*[*unfolded dvd-def*] **obtain** *e* **where** *e*: $n= d*e$ **by** *blast*
      from *n dp e* **have** *ep*:$e > 0$ **by** *simp*
      from *dp ep* **have** $d^2 \leq n \vee e^2 \leq n$
        by (*auto simp add*: *e power2-eq-square mult-le-cancel-left*)
      then have $d = 1 \vee d = n$
      proof
        assume $d^2 \leq n$
        with *H*[*rule-format*, *of d*] *d* **have** $d = 1$ **by** *blast*
        then show *?thesis* **..**
      next
        assume *h*: $e^2 \leq n$
        from *e* **have** *e dvd n* **by** (*simp add*: *dvd-def mult.commute*)
        with *H*[*rule-format*, *of e*] *h* **have** $e = 1$ **by** *simp*

```
      with e have d = n by simp
      then show ?thesis ..
    qed
  }
  ultimately show ?thesis
    unfolding prime-nat-iff using np n(2) by blast
 qed
qed

lemma prime-prime-factor-sqrt:
 prime (n::nat) ⟷ n ≠ 0 ∧ n ≠ 1 ∧ (∄p. prime p ∧ p dvd n ∧ p² ≤ n)
 (is ?lhs ⟷ ?rhs)
proof −
 consider n = 0 | n = 1 | n ≠ 0 n ≠ 1
   by blast
 then show ?thesis
 proof cases
   case 1
   then show ?thesis by (metis not-prime-0)
 next
   case 2
   then show ?thesis by (metis not-prime-1)
 next
   case n: 3
   show ?thesis
   proof
     assume ?lhs
     from this[unfolded prime-divisor-sqrt] n show ?rhs
       by (metis prime-prime-factor)
   next
     assume ?rhs
     {
       fix d
       assume d: d dvd n d² ≤ n d ≠ 1
       then obtain p where p: prime p p dvd d
         by (metis prime-factor-nat)
       from d(1) n have dp: d > 0
         by (metis dvd-0-left neq0-conv)
       from mult-mono[OF dvd-imp-le[OF p(2) dp] dvd-imp-le[OF p(2) dp]] d(2)
       have p² ≤ n unfolding power2-eq-square by arith
       with ‹?rhs› n p(1) dvd-trans[OF p(2) d(1)] have False
         by blast
     }
     with n prime-divisor-sqrt show ?lhs by auto
   qed
 qed
qed
```

9.6 Pocklington theorem

```
lemma pocklington-lemma:
  fixes p :: nat
  assumes n: n ≥ 2 and nqr: n − 1 = q ∗ r
    and an: [a^(n − 1) = 1] (mod n)
    and aq: ∀ p. prime p ∧ p dvd q ⟶ coprime (a ^((n − 1) div p) − 1) n
    and pp: prime p and pn: p dvd n
  shows [p = 1] (mod q)
proof −
  have p01: p ≠ 0 p ≠ 1
    using pp by (auto intro: prime-gt-0-nat)
  obtain k where k: a ^(q ∗ r) − 1 = n ∗ k
    by (metis an cong-to-1-nat dvd-def nqr)
  from pn[unfolded dvd-def] obtain l where l: n = p ∗ l
    by blast
  have a0: a ≠ 0
  proof
    assume a = 0
    with n have a^(n − 1) = 0
      by (simp add: power-0-left)
    with n an mod-less[of 1 n] show False
      by (simp add: power-0-left cong-def)
  qed
  with n nqr have aqr0: a ^(q ∗ r) ≠ 0
    by simp
  then have (a ^(q ∗ r) − 1) + 1  = a ^(q ∗ r)
    by simp
  with k l have a ^(q ∗ r) = p ∗ l ∗ k + 1
    by simp
  then have a ^(r ∗ q) + p ∗ 0 = 1 + p ∗ (l ∗ k)
    by (simp add: ac-simps)
  then have odq: ord p (a^r) dvd q
    unfolding ord-divides[symmetric] power-mult[symmetric]
    by (metis an cong-dvd-modulus-nat mult.commute nqr pn)
  from odq[unfolded dvd-def] obtain d where d: q = ord p (a^r) ∗ d
    by blast
  have d1: d = 1
  proof (rule ccontr)
    assume d1: d ≠ 1
    obtain P where P: prime P P dvd d
      by (metis d1 prime-factor-nat)
    from d dvd-mult[OF P(2), of ord p (a^r)] have Pq: P dvd q by simp
    from aq P(1) Pq have caP:coprime (a^((n − 1) div P) − 1) n by blast
    from Pq obtain s where s: q = P∗s unfolding dvd-def by blast
    from P(1) have P0: P ≠ 0
      by (metis not-prime-0)
    from P(2) obtain t where t: d = P∗t unfolding dvd-def by blast
    from d s t P0  have s': ord p (a^r) ∗ t = s
      by (metis mult.commute mult-cancel1 mult.assoc)
```

```
    have ord p (a^r) * t*r = r * ord p (a^r) * t
      by (metis mult.assoc mult.commute)
    then have exps: a^(ord p (a^r) * t*r) = ((a ^ r) ^ ord p (a^r)) ^ t
      by (simp only: power-mult)
    then have [((a ^ r) ^ ord p (a^r)) ^ t= 1] (mod p)
      by (metis cong-pow ord power-one)
    then have pd0: p dvd a^(ord p (a^r) * t*r) - 1
      by (metis cong-to-1-nat exps)
    from nqr s s' have (n - 1) div P = ord p (a^r) * t*r
      using P0 by simp
    with caP have coprime (a ^(ord p (a ^ r) * t * r) - 1) n
      by simp
    with p01 pn pd0 coprime-common-divisor [of - n p] show False
      by auto
  qed
  with d have o: ord p (a^r) = q by simp
  from pp totient-prime [of p] have totient-eq: totient p = p - 1
    by simp
  {
    fix d
    assume d: d dvd p d dvd a d ≠ 1
    from pp[unfolded prime-nat-iff] d have dp: d = p by blast
    from n have n ≠ 0 by simp
    then have False using d dp pn an
      by auto (metis One-nat-def Suc-lessI
          ‹1 < p ∧ (∀m. m dvd p ⟶ m = 1 ∨ m = p)› ‹a ^ (q * r) = p *
l * k + 1› add-diff-cancel-left' dvd-diff-nat dvd-power dvd-triv-left gcd-nat.trans
nat-dvd-not-less nqr zero-less-diff zero-less-one)
  }
  then have cpa: coprime p a
    by (auto intro: coprimeI)
  then have arp: coprime (a ^ r) p
    by (cases r > 0) (simp-all add: ac-simps)
  from euler-theorem [OF arp, simplified ord-divides] o totient-eq have q dvd (p
- 1)
    by simp
  then obtain d where d:p - 1 = q * d
    unfolding dvd-def by blast
  have p ≠ 0
    by (metis p01(1))
  with d have p + q * 0 = 1 + q * d by simp
  then show ?thesis
    by (metis cong-iff-lin-nat mult.commute)
qed

theorem pocklington:
  assumes n: n ≥ 2 and nqr: n - 1 = q * r and sqr: n ≤ q^2
    and an: [a^(n - 1) = 1] (mod n)
    and aq: ∀p. prime p ∧ p dvd q ⟶ coprime (a^((n - 1) div p) - 1) n
```

shows *prime n*
unfolding *prime-prime-factor-sqrt*[*of n*]
proof −
let *?ths* = $n \neq 0 \wedge n \neq 1 \wedge (\nexists p.\ prime\ p \wedge p\ dvd\ n \wedge p^2 \leq n)$
from *n* **have** *n01*: $n \neq 0$ $n \neq 1$ **by** *arith+*
{
fix *p*
assume *p*: *prime p p dvd n* $p^2 \leq n$
from *p*(*3*) *sqr* **have** $p\hat{~}(Suc\ 1) \leq q\hat{~}(Suc\ 1)$
by (*simp add*: *power2-eq-square*)
then have *pq*: $p \leq q$
by (*metis le0 power-le-imp-le-base*)
from *pocklington-lemma*[*OF n nqr an aq p*(*1*,*2*)] **have** ∗: $q\ dvd\ p - 1$
by (*metis cong-to-1-nat*)
have $p - 1 \neq 0$
using *prime-ge-2-nat* [*OF p*(*1*)] **by** *arith*
with *pq* ∗ **have** *False*
by (*simp add*: *nat-dvd-not-less*)
}
with *n01* **show** *?ths* **by** *blast*
qed

Variant for application, to separate the exponentiation.

lemma *pocklington-alt*:
assumes *n*: $n \geq 2$ **and** *nqr*: $n - 1 = q * r$ **and** *sqr*: $n \leq q^2$
and *an*: $[a\hat{~}(n - 1) = 1]\ (mod\ n)$
and *aq*: $\forall p.\ prime\ p \wedge p\ dvd\ q \longrightarrow (\exists b.\ [a\hat{~}((n - 1)\ div\ p) = b]\ (mod\ n) \wedge$
$coprime\ (b - 1)\ n)$
shows *prime n*
proof −
{
fix *p*
assume *p*: *prime p p dvd q*
from *aq*[*rule-format*] *p* **obtain** *b* **where** *b*: $[a\hat{~}((n - 1)\ div\ p) = b]\ (mod\ n)$
$coprime\ (b - 1)\ n$
by *blast*
have *a0*: $a \neq 0$
proof
assume *a0*: $a = 0$
from *n an* **have** $[0 = 1]\ (mod\ n)$
unfolding *a0 power-0-left* **by** *auto*
then show *False*
using *n* **by** (*simp add*: *cong-def dvd-eq-mod-eq-0*[*symmetric*])
qed
then have *a1*: $a \geq 1$ **by** *arith*
from *one-le-power*[*OF a1*] **have** *ath*: $1 \leq a\ \hat{~}((n - 1)\ div\ p)$ **.**
have *b0*: $b \neq 0$
proof
assume *b0*: $b = 0$

from *p(2) nqr* **have** $(n - 1)$ *mod p* $= 0$
by (*metis mod-0 mod-mod-cancel mod-mult-self1-is-0*)
with *div-mult-mod-eq*[*of n* $- 1$ *p*]
have $(n - 1)$ *div p* $*$ *p*$= n - 1$ **by** *auto*
then have *eq*: $(a\widehat{\ }((n - 1)\ div\ p))\widehat{\ }p = a\widehat{\ }(n - 1)$
by (*simp only*: *power-mult*[*symmetric*])
have $p - 1 \neq 0$
using *prime-ge-2-nat* [*OF p*(*1*)] **by** *arith*
then have *pS*: *Suc* $(p - 1) = p$ **by** *arith*
from *b* **have** *d*: *n dvd* $a\widehat{\ }((n - 1)\ div\ p)$
unfolding *b0* **by** *auto*
from *divides-rexp*[*OF d*, *of p* $- 1$] *pS eq cong-dvd-iff* [*OF an*] *n* **show** *False*
by *simp*
qed
then have *b1*: $b \geq 1$ **by** *arith*
from *cong-imp-coprime*[*OF Cong.cong-diff-nat*[*OF cong-sym* [*OF b*(*1*)] *cong-refl*
[*of 1*] *b1*]]
ath b1 b nqr
have *coprime* $(a\ \widehat{\ }((n - 1)\ div\ p) - 1)\ n$
by *simp*
}
then have $\forall p.\ prime\ p \wedge p\ dvd\ q \longrightarrow coprime\ (a\ \widehat{\ }((n - 1)\ div\ p) - 1)\ n$
by *blast*
then show *?thesis* **by** (*rule pocklington*[*OF n nqr sqr an*])
qed

9.7 Prime factorizations

definition *primefact ps n* $\longleftrightarrow$ *foldr* $(*)$ *ps* $1 = n \wedge (\forall p \in$ *set ps. prime p*)

lemma *primefact*:
fixes *n* :: *nat*
assumes *n*: $n \neq 0$
shows $\exists ps.$ *primefact ps n*
proof $-$
obtain *xs* **where** *xs*: *mset xs* = *prime-factorization n*
using *ex-mset* [*of prime-factorization n*] **by** *blast*
from *assms* **have** *n* = *prod-mset* (*prime-factorization n*)
by (*simp add*: *prod-mset-prime-factorization*)
also have ... = *prod-mset* (*mset xs*) **by** (*simp add*: *xs*)
also have ... = *foldr* $(*)$ *xs 1* **by** (*induct xs*) *simp-all*
finally have *foldr* $(*)$ *xs* $1 = n$ **..**
moreover from *xs* **have** $\forall p \in \#$*mset xs. prime p* **by** *auto*
ultimately have *primefact xs n* **by** (*auto simp*: *primefact-def*)
then show *?thesis* **..**
qed

lemma *primefact-contains*:
fixes *p* :: *nat*

```
  assumes pf: primefact ps n
    and p: prime p
    and pn: p dvd n
  shows p ∈ set ps
  using pf p pn
proof (induct ps arbitrary: p n)
  case Nil
  then show ?case by (auto simp: primefact-def)
next
  case (Cons q qs)
  from Cons.prems[unfolded primefact-def]
  have q: prime q q * foldr (*) qs 1 = n ∀ p ∈set qs. prime p
    and p: prime p p dvd q * foldr (*) qs 1
    by simp-all
  consider p dvd q | p dvd foldr (*) qs 1
    by (metis p prime-dvd-mult-eq-nat)
  then show ?case
  proof cases
    case 1
    with p(1) q(1) have p = q
      unfolding prime-nat-iff by auto
    then show ?thesis by simp
  next
    case prem: 2
    from q(3) have pqs: primefact qs (foldr (*) qs 1)
      by (simp add: primefact-def)
    from Cons.hyps[OF pqs p(1) prem] show ?thesis by simp
  qed
qed

lemma primefact-variant: primefact ps n ⟷ foldr (*) ps 1 = n ∧ list-all prime
ps
  by (auto simp add: primefact-def list-all-iff)
```

Variant of Lucas theorem.

```
lemma lucas-primefact:
  assumes n: n ≥ 2 and an: [a^(n − 1) = 1] (mod n)
    and psn: foldr (*) ps 1 = n − 1
    and psp: list-all (λp. prime p ∧ ¬ [a^((n − 1) div p) = 1] (mod n)) ps
  shows prime n
proof −
  {
    fix p
    assume p: prime p p dvd n − 1 [a ^ ((n − 1) div p) = 1] (mod n)
    from psn psp have psn1: primefact ps (n − 1)
      by (auto simp add: list-all-iff primefact-variant)
    from p(3) primefact-contains[OF psn1 p(1,2)] psp
    have False by (induct ps) auto
  }
```

with *lucas*[*OF n an*] **show** *?thesis* **by** *blast*
qed

Variant of Pocklington theorem.

lemma *pocklington-primefact*:
  assumes *n*: $n \geq 2$ **and** *qrn*: $q*r = n - 1$ **and** *nq2*: $n \leq q^2$
    and *arnb*: $(a\hat{\ }r)$ *mod* $n = b$ **and** *psq*: *foldr* $(*)$ *ps* $1 = q$
    and *bqn*: $(b\hat{\ }q)$ *mod* $n = 1$
    and *psp*: *list-all* $(\lambda p.$ *prime* $p \wedge$ *coprime* $((b\hat{\ }(q$ *div* $p))$ *mod* $n - 1)$ $n)$ *ps*
  shows *prime n*
proof −
  from *bqn psp qrn*
  have *bqn*: $a\ \hat{\ }(n - 1)$ *mod* $n = 1$
    and *psp*: *list-all* $(\lambda p.$ *prime* $p \wedge$ *coprime* $(a\hat{\ }(r *(q$ *div* $p))$ *mod* $n - 1)$ $n)$ *ps*
    unfolding *arnb*[*symmetric*] *power-mod*
    by (*simp-all add*: *power-mult*[*symmetric*] *algebra-simps*)
  from *n* **have** *n0*: $n > 0$ **by** *arith*
  from *div-mult-mod-eq*[*of* $a\hat{\ }(n - 1)$ *n*]
    mod-less-divisor[*OF n0*, *of* $a\hat{\ }(n - 1)$]
  have *an1*: $[a\ \hat{\ }(n - 1) = 1]$ (*mod n*)
    by (*metis bqn cong-def mod-mod-trivial*)
  have *coprime* $(a\ \hat{\ }((n - 1)$ *div* $p) - 1)$ *n* **if** *p*: *prime p p dvd q* **for** *p*
  proof −
    from *psp psq* **have** *pfpsq*: *primefact ps q*
      by (*auto simp add*: *primefact-variant list-all-iff*)
    from *psp primefact-contains*[*OF pfpsq p*]
    have *p'*: *coprime* $(a\ \hat{\ }(r * (q$ *div* $p))$ *mod* $n - 1)$ *n*
      by (*simp add*: *list-all-iff*)
    from *p prime-nat-iff* **have** *p01*: $p \neq 0$ $p \neq 1$ $p =$ *Suc* $(p - 1)$
      by *auto*
    from *div-mult1-eq*[*of r q p*] *p*(*2*)
    have *eq1*: $r*$ $(q$ *div* $p) = (n - 1)$ *div* p
      unfolding *qrn*[*symmetric*] *dvd-eq-mod-eq-0* **by** (*simp add*: *mult.commute*)
    have *ath*: $a \leq b \Longrightarrow a \neq 0 \Longrightarrow 1 \leq a \wedge 1 \leq b$ **for** *a b* :: *nat*
      by *arith*
    {
      assume $a\ \hat{\ }((n - 1)$ *div* $p)$ *mod* $n = 0$
      then obtain *s* **where** *s*: $a\ \hat{\ }((n - 1)$ *div* $p) = n * s$
        by *blast*
      then have *eq0*: $(a\hat{\ }((n - 1)$ *div* $p))\hat{\ }p = (n*s)\hat{\ }p$ **by** *simp*
      from *qrn*[*symmetric*] **have** *qn1*: *q dvd* $n - 1$
        by (*auto simp*: *dvd-def*)
      from *dvd-trans*[*OF p*(*2*) *qn1*] **have** *npp*: $(n - 1)$ *div* $p * p = n - 1$
        by *simp*
      with *eq0* **have** $a\ \hat{\ }(n - 1) = (n * s)\ \hat{\ }p$
        by (*simp add*: *power-mult*[*symmetric*])
      with *bqn p01* **have** $1 = (n * s)\hat{\ }($*Suc* $(p - 1))$ *mod n*
        by *simp*
      also have $\ldots = 0$ **by** (*simp add*: *mult.assoc*)

```
    finally have False by simp
  }
  then have *: a ^ ((n − 1) div p) mod n ≠ 0 by auto
  have [a ^ ((n − 1) div p) mod n = a ^ ((n − 1) div p)] (mod n)
    by (simp add: cong-def)
  with ath[OF mod-less-eq-dividend *]
  have [a ^ ((n − 1) div p) mod n − 1 = a ^ ((n − 1) div p) − 1] (mod n)
    by (simp add: cong-diff-nat)
  then show ?thesis
    by (metis cong-imp-coprime eq1 p′)
 qed
 with pocklington[OF n qrn[symmetric] nq2 an1] show ?thesis
  by blast
qed

end
```

10 Prime powers

```
theory Prime-Powers
 imports Complex-Main HOL−Computational-Algebra.Primes HOL−Library.FuncSet
begin

definition aprimedivisor :: ′a :: normalization-semidom ⇒ ′a where
 aprimedivisor q = (SOME p. prime p ∧ p dvd q)

definition primepow :: ′a :: normalization-semidom ⇒ bool where
 primepow n ⟷ (∃ p k. prime p ∧ k > 0 ∧ n = p ^ k)

definition primepow-factors :: ′a :: normalization-semidom ⇒ ′a set where
 primepow-factors n = {x. primepow x ∧ x dvd n}

lemma primepow-gt-Suc-0: primepow n ⟹ n > Suc 0
 using one-less-power[of p::nat for p] by (auto simp: primepow-def prime-nat-iff)

lemma
 assumes prime p p dvd n
 shows prime-aprimedivisor: prime (aprimedivisor n)
  and aprimedivisor-dvd:   aprimedivisor n dvd n
proof −
 from assms have ∃ p. prime p ∧ p dvd n by auto
 from someI-ex[OF this] show prime (aprimedivisor n) aprimedivisor n dvd n
   unfolding aprimedivisor-def by (simp-all add: conj-commute)
qed

lemma
 assumes n ≠ 0 ¬is-unit (n :: ′a :: factorial-semiring)
 shows prime-aprimedivisor′: prime (aprimedivisor n)
  and aprimedivisor-dvd′:   aprimedivisor n dvd n
```

proof −
  from *someI-ex*[*OF prime-divisor-exists*[*OF assms*]]
    show *prime* (*aprimedivisor n*) *aprimedivisor n dvd n*
      unfolding *aprimedivisor-def* **by** (*simp-all add*: *conj-commute*)
qed

lemma *aprimedivisor-of-prime* [*simp*]:
  assumes *prime p*
  shows   *aprimedivisor p* = *p*
proof −
  from *assms* **have** ∃ *q*. *prime q* ∧ *q dvd p* **by** *auto*
  from *someI-ex*[*OF this*, *folded aprimedivisor-def*] *assms* **show** *?thesis*
    by (*auto intro*: *primes-dvd-imp-eq*)
qed

lemma *aprimedivisor-pos-nat*: (*n*::*nat*) > *1* ⟹ *aprimedivisor n* > *0*
  using *aprimedivisor-dvd'*[*of n*] **by** (*auto elim*: *dvdE intro*!: *Nat.gr0I*)

lemma *aprimedivisor-primepow-power*:
  assumes *primepow n k* > *0*
  shows   *aprimedivisor* (*n* ^ *k*) = *aprimedivisor n*
proof −
  from *assms* **obtain** *p l* **where** *l*: *prime p l* > *0 n* = *p* ^ *l*
    by (*auto simp*: *primepow-def*)
  from *l assms* **have** ∗: *prime* (*aprimedivisor* (*n* ^ *k*)) *aprimedivisor* (*n* ^ *k*) *dvd n* ^ *k*
    by (*intro prime-aprimedivisor*[*of p*] *aprimedivisor-dvd*[*of p*] *dvd-power*;
      simp add: *power-mult* [*symmetric*])+
  from ∗ *l* **have** *aprimedivisor* (*n* ^ *k*) *dvd p* ^ (*l* ∗ *k*) **by** (*simp add*: *power-mult*)
  with *assms* ∗ *l* **have** *aprimedivisor* (*n* ^ *k*) *dvd p*
    by (*subst* (*asm*) *prime-dvd-power-iff*) *simp-all*
  with *l assms* **have** *aprimedivisor* (*n* ^ *k*) = *p*
    by (*intro primes-dvd-imp-eq prime-aprimedivisor l*) (*auto simp*: *power-mult* [*symmetric*])
  moreover from *l* **have** *aprimedivisor n dvd p* ^ *l*
    by (*auto intro*: *aprimedivisor-dvd simp*: *prime-gt-0-nat*)
  with *assms l* **have** *aprimedivisor n dvd p*
    by (*subst* (*asm*) *prime-dvd-power-iff*) (*auto intro*!: *prime-aprimedivisor simp*: *prime-gt-0-nat*)
  with *l assms* **have** *aprimedivisor n* = *p*
    by (*intro primes-dvd-imp-eq prime-aprimedivisor l*) *auto*
  ultimately show *?thesis* **by** *simp*
qed

lemma *aprimedivisor-prime-power*:
  assumes *prime p k* > *0*
  shows   *aprimedivisor* (*p* ^ *k*) = *p*
proof −
  from *assms* **have** ∗: *prime* (*aprimedivisor* (*p* ^ *k*)) *aprimedivisor* (*p* ^ *k*) *dvd p*

```
^ k
   by (intro prime-aprimedivisor[of p] aprimedivisor-dvd[of p]; simp add: prime-nat-iff)+
  from assms * have aprimedivisor (p ^ k) dvd p
    by (subst (asm) prime-dvd-power-iff) simp-all
  with assms * show aprimedivisor (p ^ k) = p by (intro primes-dvd-imp-eq)
qed

lemma prime-factorization-primepow:
  assumes primepow n
  shows   prime-factorization n =
            replicate-mset (multiplicity (aprimedivisor n) n) (aprimedivisor n)
  using assms
 by (auto simp: primepow-def aprimedivisor-prime-power prime-factorization-prime-power)

lemma primepow-decompose:
  fixes n :: 'a :: factorial-semiring-multiplicative
  assumes primepow n
  shows   aprimedivisor n ^ multiplicity (aprimedivisor n) n = n
proof -
  from assms have n ≠ 0 by (intro notI) (auto simp: primepow-def)
  hence n = unit-factor n * prod-mset (prime-factorization n)
    by (subst prod-mset-prime-factorization) simp-all
 also from assms have unit-factor n = 1 by (auto simp: primepow-def unit-factor-power)
  also have prime-factorization n =
              replicate-mset (multiplicity (aprimedivisor n) n) (aprimedivisor n)
    by (intro prime-factorization-primepow assms)
  also have prod-mset ... = aprimedivisor n ^ multiplicity (aprimedivisor n) n
by simp
  finally show ?thesis by simp
qed

lemma prime-power-not-one:
  assumes prime p k > 0
  shows   p ^ k ≠ 1
proof
  assume p ^ k = 1
  hence is-unit (p ^ k) by simp
  thus False using assms by (simp add: is-unit-power-iff)
qed

lemma zero-not-primepow [simp]: ¬primepow 0
  by (auto simp: primepow-def)

lemma one-not-primepow [simp]: ¬primepow 1
  by (auto simp: primepow-def prime-power-not-one)

lemma primepow-not-unit [simp]: primepow p ⟹ ¬is-unit p
  by (auto simp: primepow-def is-unit-power-iff)
```

lemma *not-primepow-Suc-0-nat* [*simp*]: ¬*primepow* (*Suc 0*)
  using *primepow-gt-Suc-0*[*of Suc 0*] **by** *auto*

lemma *primepow-gt-0-nat*: *primepow n* ⟹ *n* > (*0*::*nat*)
  using *primepow-gt-Suc-0*[*of n*] **by** *simp*

lemma *unit-factor-primepow*:
  fixes *p* :: ′*a* :: *factorial-semiring-multiplicative*
  shows *primepow p* ⟹ *unit-factor p* = *1*
  by (*auto simp*: *primepow-def unit-factor-power*)

lemma *aprimedivisor-primepow*:
  assumes *prime p p dvd n primepow* (*n* :: ′*a* :: *factorial-semiring-multiplicative*)
  shows *aprimedivisor* (*p* ∗ *n*) = *p aprimedivisor n* = *p*
proof −
  from *assms* **have** [*simp*]: *n* ≠ *0* **by** *auto*
  define *q* **where** *q* = *aprimedivisor n*
  with *assms* **have** *q*: *prime q* **by** (*auto simp*: *q-def intro*!: *prime-aprimedivisor*)
  from ‹*primepow n*› **have** *n*: *n* = *q* ^ *multiplicity q n*
    by (*simp add*: *primepow-decompose q-def*)
  have *nz*: *multiplicity q n* ≠ *0*
  proof
    assume *multiplicity q n* = *0*
    with *n* **have** *n*′: *n* = *unit-factor n* **by** *simp*
    have *is-unit n* **by** (*subst n*′, *rule unit-factor-is-unit*) (*insert assms*, *auto*)
    with *assms* **show** *False* **by** *auto*
  qed
  with ‹*prime p*› ‹*p dvd n*› *q* **have** *p dvd q*
    by (*subst* (*asm*) *n*) (*auto intro*: *prime-dvd-power*)
  with ‹*prime p*› *q* **have** *p* = *q* **by** (*intro primes-dvd-imp-eq*)
  thus *aprimedivisor n* = *p* **by** (*simp add*: *q-def*)

  define *r* **where** *r* = *aprimedivisor* (*p* ∗ *n*)
  with *assms* **have** *r*: *r dvd* (*p* ∗ *n*) *prime r* **unfolding** *r-def*
    by (*intro aprimedivisor-dvd*[*of p*] *prime-aprimedivisor*[*of p*]; *simp*)+
  hence *r dvd q* ^ *Suc* (*multiplicity q n*)
    by (*subst* (*asm*) *n*) (*auto simp*: ‹*p* = *q*› *dest*: *dvd-unit-imp-unit*)
  with *r* **have** *r dvd q*
   by (*auto intro*: *prime-dvd-power-nat simp*: *prime-dvd-mult-iff dest*: *prime-dvd-power*)
  with *r q* **have** *r* = *q* **by** (*intro primes-dvd-imp-eq*)
  thus *aprimedivisor* (*p* ∗ *n*) = *p* **by** (*simp add*: *r-def* ‹*p* = *q*›)
qed

lemma *power-eq-prime-powerD*:
  fixes *p* :: ′*a* :: *factorial-semiring*
  assumes *prime p n* > *0 x* ^ *n* = *p* ^ *k*
  shows ∃ *i*. *normalize x* = *normalize* (*p* ^ *i*)
proof −
  have *normalize x* = *normalize* (*p* ^ *multiplicity p x*)

proof (*rule multiplicity-eq-imp-eq*)
fix *q* :: *'a* **assume** *prime q*
from *assms* **have** *multiplicity q* (*x* ^ *n*) = *multiplicity q* (*p* ^ *k*) **by** *simp*
with ‹*prime q*› **and** *assms* **have** *n* ∗ *multiplicity q x* = *k* ∗ *multiplicity q p*
by (*subst* (*asm*) (*1 2*) *prime-elem-multiplicity-power-distrib*) (*auto simp*: *power-0-left*)
with *assms* **and** ‹*prime q*› **show** *multiplicity q x* = *multiplicity q* (*p* ^ *multiplicity p x*)
by (*cases p* = *q*) (*auto simp*: *multiplicity-distinct-prime-power prime-multiplicity-other*)
qed (*insert assms*, *auto simp*: *power-0-left*)
thus *?thesis* **by** *auto*
qed

lemma *primepow-power-iff*:
fixes *p* :: *'a* :: *factorial-semiring-multiplicative*
assumes *unit-factor p* = *1*
shows *primepow* (*p* ^ *n*) ⟷ *primepow p* ∧ *n* > *0*
proof *safe*
assume *primepow* (*p* ^ *n*)
hence *n*: *n* ≠ *0* **by** (*auto intro*!: *Nat.gr0I*)
thus *n* > *0* **by** *simp*
from *assms* **have** [*simp*]: *normalize p* = *p*
using *normalize-mult-unit-factor*[*of p*] **by** (*simp only*: *mult.right-neutral*)
from ‹*primepow* (*p* ^ *n*)› **obtain** *q k* **where** ∗: *k* > *0 prime q p* ^ *n* = *q* ^ *k*
by (*auto simp*: *primepow-def*)
with *power-eq-prime-powerD*[*of q n p k*] *n*
obtain *i* **where** *eq*: *normalize p* = *normalize* (*q* ^ *i*) **by** *auto*
with *primepow-not-unit*[*OF* ‹*primepow* (*p* ^ *n*)›] **have** *i* ≠ *0*
by (*intro notI*) (*simp add*: *normalize-1-iff is-unit-power-iff del*: *primepow-not-unit*)
with ‹*normalize p* = *normalize* (*q* ^ *i*)› ‹*prime q*› **show** *primepow p*
by (*auto simp*: *normalize-power primepow-def intro*!: *exI*[*of - q*] *exI*[*of - i*])
next
assume *primepow p n* > *0*
then obtain *q k* **where** ∗: *k* > *0 prime q p* = *q* ^ *k* **by** (*auto simp*: *primepow-def*)
with ‹*n* > *0*› **show** *primepow* (*p* ^ *n*)
by (*auto simp*: *primepow-def power-mult intro*!: *exI*[*of - q*] *exI*[*of - k* ∗ *n*])
qed

lemma *primepow-power-iff-nat*:
p > *0* ⟹ *primepow* (*p* ^ *n*) ⟷ *primepow* (*p* :: *nat*) ∧ *n* > *0*
by (*rule primepow-power-iff*) (*simp-all add*: *unit-factor-nat-def*)

lemma *primepow-prime* [*simp*]: *prime n* ⟹ *primepow n*
by (*auto simp*: *primepow-def intro*!: *exI*[*of - n*] *exI*[*of - 1*::*nat*])

lemma *primepow-prime-power* [*simp*]:
prime (*p* :: *'a* :: *factorial-semiring-multiplicative*) ⟹ *primepow* (*p* ^ *n*) ⟷ *n* > *0*

by (*subst primepow-power-iff*) *auto*

lemma *aprimedivisor-vimage*:
assumes *prime* (*p* :: *'a* :: *factorial-semiring-multiplicative*)
shows *aprimedivisor* −' {*p*} ∩ *primepow-factors n* = {*p* ^ *k* |*k*. *k* > *0* ∧ *p* ^ *k* *dvd n*}
proof *safe*
fix *q* **assume** *q*: *q* ∈ *primepow-factors n*
hence *q'*: *q* ≠ *0* *q* ≠ *1* **by** (*auto simp*: *primepow-def primepow-factors-def prime-power-not-one*)
let *?n* = *multiplicity* (*aprimedivisor q*) *q*
from *q q'* **have** *q* = *aprimedivisor q* ^ *?n* ∧ *?n* > *0* ∧ *aprimedivisor q* ^ *?n dvd n*
by (*auto simp*: *primepow-decompose primepow-factors-def prime-multiplicity-gt-zero-iff prime-aprimedivisor' prime-imp-prime-elem aprimedivisor-dvd'*)
thus ∃ *k*. *q* = *aprimedivisor q* ^ *k* ∧ *k* > *0* ∧ *aprimedivisor q* ^ *k dvd n* **..**
next
fix *k* :: *nat* **assume** *k*: *p* ^ *k dvd n k* > *0*
with *assms* **show** *p* ^ *k* ∈ *aprimedivisor* −' {*p*}
by (*auto simp*: *aprimedivisor-prime-power*)
with *assms k* **show** *p* ^ *k* ∈ *primepow-factors n*
by (*auto simp*: *primepow-factors-def primepow-def aprimedivisor-prime-power intro*: *Suc-leI*)
qed

lemma *aprimedivisor-nat*:
assumes *n* ≠ (*Suc 0*::*nat*)
shows *prime* (*aprimedivisor n*) *aprimedivisor n dvd n*
proof −
from *assms* **have** ∃ *p*. *prime p* ∧ *p dvd n* **by** (*intro prime-factor-nat*) *auto*
from *someI-ex*[*OF this*, *folded aprimedivisor-def*]
show *prime* (*aprimedivisor n*) *aprimedivisor n dvd n* **by** *blast*+
qed

lemma *aprimedivisor-gt-Suc-0*:
assumes *n* ≠ *Suc 0*
shows *aprimedivisor n* > *Suc 0*
proof −
from *assms* **have** *prime* (*aprimedivisor n*) **by** (*rule aprimedivisor-nat*)
thus *aprimedivisor n* > *Suc 0* **by** (*simp add*: *prime-nat-iff*)
qed

lemma *aprimedivisor-le-nat*:
assumes *n* > *Suc 0*
shows *aprimedivisor n* ≤ *n*
proof −
from *assms* **have** *aprimedivisor n dvd n* **by** (*intro aprimedivisor-nat*) *simp-all*
with *assms* **show** *aprimedivisor n* ≤ *n*
by (*intro dvd-imp-le*) *simp-all*

qed

lemma *bij-betw-primepows*:
  bij-betw (λ(*p*,*k*). *p* ^ *Suc k* :: *'a* :: *factorial-semiring-multiplicative*)
    (*Collect prime* × *UNIV*) (*Collect primepow*)
proof (*rule bij-betwI* [**where** *?g* = (λ*n*. (*aprimedivisor n*, *multiplicity* (*aprimedivisor n*) *n* − *1*))],
    goal-cases)
  case *1*
  show (λ(*p*, *k*). *p* ^ *Suc k* :: *'a*) ∈ *Collect prime* × *UNIV* → *Collect primepow*
    by (*auto intro*!: *primepow-prime-power simp del*: *power-Suc*)
next
  case *2*
  show *?case*
    by (*auto simp*: *primepow-def prime-aprimedivisor*)
next
  case (*3 n*)
  thus *?case*
    by (*auto simp*: *aprimedivisor-prime-power simp del*: *power-Suc*)
next
  case (*4 n*)
  hence ∗: *0* < *multiplicity* (*aprimedivisor n*) *n*
    by (*subst prime-multiplicity-gt-zero-iff*)
      (*auto intro*!: *prime-imp-prime-elem aprimedivisor-dvd simp*: *primepow-def prime-aprimedivisor*)
  have *aprimedivisor n* ∗ *aprimedivisor n* ^ (*multiplicity* (*aprimedivisor n*) *n* − *Suc 0*) =
      aprimedivisor n ^ *Suc* (*multiplicity* (*aprimedivisor n*) *n* − *Suc 0*) **by** *simp*
  also from ∗ **have** *Suc* (*multiplicity* (*aprimedivisor n*) *n* − *Suc 0*) =
      multiplicity (*aprimedivisor n*) *n*
    by (*subst Suc-diff-Suc*) (*auto simp*: *prime-multiplicity-gt-zero-iff*)
  also have *aprimedivisor n* ^ . . . = *n*
    using *4* **by** (*subst primepow-decompose*) *auto*
  finally show *?case* **by** *auto*
qed

lemma *primepow-multD*:
  assumes *primepow* (*a* ∗ *b* :: *nat*)
  shows *a* = *1* ∨ *primepow a b* = *1* ∨ *primepow b*
proof −
  from *assms* **obtain** *p k* **where** *k*: *k* > *0 a* ∗ *b* = *p* ^ *k prime p*
    unfolding *primepow-def* **by** *auto*
  then obtain *i j* **where** *a* = *p* ^ *i b* = *p* ^ *j*
    using *prime-power-mult-nat*[*of p a b*] **by** *blast*
  with ‹*prime p*› **show** *a* = *1* ∨ *primepow a b* = *1* ∨ *primepow b* **by** *auto*
qed

lemma *primepow-mult-aprimedivisorI*:

```
  assumes primepow (n :: 'a :: factorial-semiring-multiplicative)
  shows   primepow (aprimedivisor n * n)
 by (subst (2) primepow-decompose[OF assms, symmetric], subst power-Suc [symmetric],
     subst primepow-prime-power)
    (insert assms, auto intro!: prime-aprimedivisor' dest: primepow-gt-Suc-0)

lemma primepow-factors-altdef:
  fixes x :: 'a :: factorial-semiring-multiplicative
  assumes x ≠ 0
   shows primepow-factors x = {p ^ k |p k. p ∈ prime-factors x ∧ k ∈ {0<..
multiplicity p x}}
proof (intro equalityI subsetI)
  fix q assume q ∈ primepow-factors x
  then obtain p k where pk: prime p k > 0 q = p ^ k q dvd x
    unfolding primepow-factors-def primepow-def by blast
  moreover have k ≤ multiplicity p x using pk assms by (intro multiplicity-geI)
auto
  ultimately show q ∈ {p ^ k |p k. p ∈ prime-factors x ∧ k ∈ {0<.. multiplicity
p x}}
    by (auto simp: prime-factors-multiplicity intro!: exI[of - p] exI[of - k])
qed (auto simp: primepow-factors-def prime-factors-multiplicity multiplicity-dvd')

lemma finite-primepow-factors:
  assumes x ≠ (0 :: 'a :: factorial-semiring-multiplicative)
  shows   finite (primepow-factors x)
proof -
  have finite (SIGMA p:prime-factors x. {0<..multiplicity p x})
    by (intro finite-SigmaI) simp-all
  hence finite ((λ(p,k). p ^ k) ` ...) (is finite ?A) by (rule finite-imageI)
  also have ?A = primepow-factors x
    using assms by (subst primepow-factors-altdef) fast+
  finally show ?thesis .
qed

lemma aprimedivisor-primepow-factors-conv-prime-factorization:
  assumes [simp]: n ≠ (0 :: 'a :: factorial-semiring-multiplicative)
 shows   image-mset aprimedivisor (mset-set (primepow-factors n)) = prime-factorization
n
         (is ?lhs = ?rhs)
proof (intro multiset-eqI)
  fix p :: 'a
  show count ?lhs p = count ?rhs p
  proof (cases prime p)
    case False
    have p ∉# image-mset aprimedivisor (mset-set (primepow-factors n))
    proof
      assume p ∈# image-mset aprimedivisor (mset-set (primepow-factors n))
      then obtain q where p = aprimedivisor q q ∈ primepow-factors n
        by (auto simp: finite-primepow-factors)
```

with *False prime-aprimedivisor'*[*of q*] **have** $q = 0 \vee$ *is-unit q* **by** *auto*
with ‹$q \in$ *primepow-factors n*› **show** *False* **by** (*auto simp*: *primepow-factors-def primepow-def*)
qed
hence *count ?lhs p* = *0* **by** (*simp only*: *Multiset.not-in-iff*)
with *False* **show** *?thesis* **by** (*simp add*: *count-prime-factorization*)
next
case *True*
hence *p*: $p \neq 0$ $\neg$*is-unit p* **by** *auto*
have *count ?lhs p* = *card* (*aprimedivisor* −' {*p*} ∩ *primepow-factors n*)
by (*simp add*: *count-image-mset finite-primepow-factors*)
also have *aprimedivisor* −' {*p*} ∩ *primepow-factors n* = {*p*^*k* |*k*. *k* > *0* ∧ *p* ^ *k dvd n*}
using *True* **by** (*rule aprimedivisor-vimage*)
also from *True* **have** ... = (λ*k*. *p* ^ *k*) ' {*0*<..*multiplicity p n*}
by (*subst power-dvd-iff-le-multiplicity*) *auto*
also from *p True* **have** *card* ... = *multiplicity p n*
by (*subst card-image*) (*auto intro*!: *inj-onI dest*: *prime-power-inj*)
also from *True* **have** ... = *count* (*prime-factorization n*) *p*
by (*simp add*: *count-prime-factorization*)
finally show *?thesis* **.**
qed
qed

lemma *prime-elem-aprimedivisor-nat*: *d* > *Suc 0* ⟹ *prime-elem* (*aprimedivisor d*)
using *prime-aprimedivisor'*[*of d*] **by** *simp*

lemma *aprimedivisor-gt-0-nat* [*simp*]: *d* > *Suc 0* ⟹ *aprimedivisor d* > *0*
using *prime-aprimedivisor'*[*of d*] **by** (*simp add*: *prime-gt-0-nat*)

lemma *aprimedivisor-gt-Suc-0-nat* [*simp*]: *d* > *Suc 0* ⟹ *aprimedivisor d* > *Suc 0*
using *prime-aprimedivisor'*[*of d*] **by** (*simp add*: *prime-gt-Suc-0-nat*)

lemma *aprimedivisor-not-Suc-0-nat* [*simp*]: *d* > *Suc 0* ⟹ *aprimedivisor d* ≠ *Suc 0*
using *aprimedivisor-gt-Suc-0*[*of d*] **by** (*intro notI*) *auto*

lemma *multiplicity-aprimedivisor-gt-0-nat* [*simp*]:
d > *Suc 0* ⟹ *multiplicity* (*aprimedivisor d*) *d* > *0*
by (*subst multiplicity-gt-zero-iff*) (*auto intro*: *aprimedivisor-dvd'*)

lemma *primepowI*:
prime p ⟹ *k* > *0* ⟹ *p* ^ *k* = *n* ⟹ *primepow n* ∧ *aprimedivisor n* = *p*
unfolding *primepow-def* **by** (*auto simp*: *aprimedivisor-prime-power*)

lemma *not-primepowI*:
assumes *prime p prime q p* ≠ *q p dvd n q dvd n*

shows $\neg$*primepow n*
using *assms* **by** (*auto simp*: *primepow-def dest*!: *prime-dvd-power*[*rotated*] *dest*: *primes-dvd-imp-eq*)

lemma *sum-prime-factorization-conv-sum-primepow-factors*:
fixes *n* :: *'a* :: *factorial-semiring-multiplicative*
assumes $n \neq 0$
shows ($\sum$ *q*$\in$*primepow-factors n. f* (*aprimedivisor q*)) = ($\sum$ *p*$\in$#*prime-factorization n. f p*)
proof −
from *assms* **have** *prime-factorization n* = *image-mset aprimedivisor* (*mset-set* (*primepow-factors n*))
by (*rule aprimedivisor-primepow-factors-conv-prime-factorization* [*symmetric*])
also have ($\sum$ *p*$\in$#.... *f p*) = ($\sum$ *q*$\in$*primepow-factors n. f* (*aprimedivisor q*))
by (*simp add*: *image-mset.compositionality sum-unfold-sum-mset o-def*)
finally show *?thesis* **..**
qed

lemma *multiplicity-aprimedivisor-Suc-0-iff*:
assumes *primepow* (*n* :: *'a* :: *factorial-semiring-multiplicative*)
shows *multiplicity* (*aprimedivisor n*) *n* = *Suc 0* $\longleftrightarrow$ *prime n*
by (*subst* (*3*) *primepow-decompose* [*OF assms*, *symmetric*])
(*insert assms*, *auto simp add*: *prime-power-iff intro*!: *prime-aprimedivisor'*)

definition *mangoldt* :: *nat* $\Rightarrow$ *'a* :: *real-algebra-1* **where**
mangoldt n = (*if primepow n then of-real* (*ln* (*real* (*aprimedivisor n*))) *else 0*)

lemma *mangoldt-0* [*simp*]: *mangoldt 0* = *0*
by (*simp add*: *mangoldt-def*)

lemma *mangoldt-Suc-0* [*simp*]: *mangoldt* (*Suc 0*) = *0*
by (*simp add*: *mangoldt-def*)

lemma *of-real-mangoldt* [*simp*]: *of-real* (*mangoldt n*) = *mangoldt n*
by (*simp add*: *mangoldt-def*)

lemma *mangoldt-sum*:
assumes $n \neq 0$
shows ($\sum$ *d* | *d dvd n. mangoldt d* :: *'a* :: *real-algebra-1*) = *of-real* (*ln* (*real n*))
proof −
have ($\sum$ *d* | *d dvd n. mangoldt d* :: *'a*) = *of-real* ($\sum$ *d* | *d dvd n. mangoldt d*) **by** *simp*
also have ($\sum$ *d* | *d dvd n. mangoldt d*) = ($\sum$ *d* $\in$ *primepow-factors n. ln* (*real* (*aprimedivisor d*)))
using *assms* **by** (*intro sum.mono-neutral-cong-right*) (*auto simp*: *primepow-factors-def mangoldt-def*)
also have ... = *ln* (*real* ($\prod$ *d* $\in$ *primepow-factors n. aprimedivisor d*))
using *assms finite-primepow-factors*[*of n*]

by (*subst ln-prod* [*symmetric*])
(*auto simp*: *primepow-factors-def intro*!: *aprimedivisor-pos-nat*
intro: *Nat.gr0I primepow-gt-Suc-0*)
also have *primepow-factors n* =
(λ(*p*,*k*). *p* ^ *k*) ' (*SIGMA p*:*prime-factors n*. {*0*<..*multiplicity p n*})
(**is** - = - ' *?A*) **by** (*subst primepow-factors-altdef*[*OF assms*]) *fast*+
also have *prod aprimedivisor* ... = ($\prod$(*p*,*k*)$\in$*?A*. *aprimedivisor* (*p* ^ *k*))
by (*subst prod.reindex*)
(*auto simp*: *inj-on-def prime-power-inj″ prime-factors-multiplicity*
prod.Sigma [*symmetric*] *case-prod-unfold*)
also have ... = ($\prod$(*p*,*k*)$\in$*?A*. *p*)
by (*intro prod.cong refl*) (*auto simp*: *aprimedivisor-prime-power prime-factors-multiplicity*)
also have ... = ($\prod$*x*$\in$*prime-factors n*. $\prod$*k*$\in${*0*<..*multiplicity x n*}. *x*)
by (*rule prod.Sigma* [*symmetric*]) *auto*
also have ... = ($\prod$*x*$\in$*prime-factors n*. *x* ^ *multiplicity x n*)
by (*intro prod.cong refl*) (*simp add*: *prod-constant*)
also have ... = *n* **using** *assms* **by** (*intro prime-factorization-nat* [*symmetric*])
simp
finally show *?thesis* **.**
qed

lemma *mangoldt-primepow*:
prime p $\Longrightarrow$ *mangoldt* (*p* ^ *k*) = (*if k* > *0 then of-real* (*ln* (*real p*)) *else 0*)
by (*simp add*: *mangoldt-def aprimedivisor-prime-power*)

lemma *mangoldt-primepow′* [*simp*]: *prime p* $\Longrightarrow$ *k* > *0* $\Longrightarrow$ *mangoldt* (*p* ^ *k*) =
of-real (*ln* (*real p*))
by (*subst mangoldt-primepow*) *auto*

lemma *mangoldt-prime* [*simp*]: *prime p* $\Longrightarrow$ *mangoldt p* = *of-real* (*ln* (*real p*))
using *mangoldt-primepow*[*of p 1*] **by** *simp*

lemma *mangoldt-nonneg*: *0* $\leq$ (*mangoldt d* :: *real*)
using *aprimedivisor-gt-Suc-0-nat*[*of d*]
by (*auto simp*: *mangoldt-def of-nat-le-iff*[*of 1 x* **for** *x*, *unfolded of-nat-1*] *Suc-le-eq*
intro!: *ln-ge-zero dest*: *primepow-gt-Suc-0*)

lemma *norm-mangoldt* [*simp*]:
norm (*mangoldt n* :: ′*a* :: *real-normed-algebra-1*) = *mangoldt n*
proof (*cases primepow n*)
case *True*
hence *prime* (*aprimedivisor n*)
by (*intro prime-aprimedivisor′*)
(*auto simp*: *primepow-def prime-gt-0-nat*)
hence *aprimedivisor n* > *1* **by** (*simp add*: *prime-gt-Suc-0-nat*)
with *True* **show** *?thesis* **by** (*auto simp*: *mangoldt-def abs-if*)
qed (*auto simp*: *mangoldt-def*)

lemma *Re-mangoldt* [*simp*]: *Re* (*mangoldt n*) = *mangoldt n*

and *Im-mangoldt* [*simp*]: *Im* (*mangoldt n*) = *0*
by (*simp-all add*: *mangoldt-def*)

lemma *abs-mangoldt* [*simp*]: *abs* (*mangoldt n* :: *real*) = *mangoldt n*
using *norm-mangoldt*[*of n*, **where** *?'a* = *real*, *unfolded real-norm-def*] **.**

lemma *mangoldt-le*:
assumes $n > 0$
shows *mangoldt n* ≤ *ln n*
proof (*cases primepow n*)
case *True*
from *True* **have** *prime* (*aprimedivisor n*)
by (*intro prime-aprimedivisor'*)
(*auto simp*: *primepow-def prime-gt-0-nat*)
hence *gt-1*: *aprimedivisor n* > *1* **by** (*simp add*: *prime-gt-Suc-0-nat*)
from *True* **have** *mangoldt n* = *ln* (*aprimedivisor n*)
by (*simp add*: *mangoldt-def*)
also have ... ≤ *ln n* **using** *True gt-1*
by (*subst ln-le-cancel-iff*) (*auto intro*!: *Nat.gr0I dvd-imp-le aprimedivisor-dvd'*)
finally show *?thesis* **.**
qed (*insert assms*, *auto simp*: *mangoldt-def*)

end

11 Primitive roots in residue rings and Carmichael's function

theory *Residue-Primitive-Roots*
imports *Pocklington*
begin

This theory develops the notions of primitive roots (generators) in residue rings. It also provides a definition and all the basic properties of Carmichael's function $\lambda(n)$, which is strongly related to this. The proofs mostly follow Apostol's presentation

11.1 Primitive roots in residue rings

A primitive root of a residue ring modulo n is an element g that *generates* the ring, i. e. such that for each x coprime to n there exists an i such that $x = g^i$. A simpler definition is that g must have the same order as the cardinality of the multiplicative group, which is $\varphi(n)$.

Note that for convenience, this definition does *not* demand $g < n$.

inductive *residue-primroot* :: *nat* ⇒ *nat* ⇒ *bool* **where**
$n > 0$ ⟹ *coprime n g* ⟹ *ord n g* = *totient n* ⟹ *residue-primroot n g*

lemma *residue-primroot-def* [*code*]:
  residue-primroot n x $\longleftrightarrow$ *n* > *0* $\wedge$ *coprime n x* $\wedge$ *ord n x* = *totient n*
  by (*simp add*: *residue-primroot.simps*)

lemma *not-residue-primroot-0* [*simp*]: ~*residue-primroot 0 x*
  by (*auto simp*: *residue-primroot-def*)

lemma *residue-primroot-mod* [*simp*]: *residue-primroot n* (*x mod n*) = *residue-primroot n x*
  by (*cases n* = *0*) (*simp-all add*: *residue-primroot-def*)

lemma *residue-primroot-cong*:
  assumes [*x* = *x'*] (*mod n*)
  shows *residue-primroot n x* = *residue-primroot n x'*
proof −
  have *residue-primroot n x* = *residue-primroot n* (*x mod n*)
    by *simp*
  also have *x mod n* = *x' mod n*
    using *assms* **by** (*simp add*: *cong-def*)
  also have *residue-primroot n* (*x' mod n*) = *residue-primroot n x'*
    by *simp*
  finally show *?thesis* **.**
qed

lemma *not-residue-primroot-0-right* [*simp*]: *residue-primroot n 0* $\longleftrightarrow$ *n* = *1*
  by (*auto simp*: *residue-primroot-def*)

lemma *residue-primroot-1-iff*: *residue-primroot n* (*Suc 0*) $\longleftrightarrow$ *n* ∈ {*1*, *2*}
proof
  assume ∗: *residue-primroot n* (*Suc 0*)
  with *totient-gt-1*[*of n*] **have** *n* ≤ *2* **by** (*cases n* ≤ *2*) (*auto simp*: *residue-primroot-def*)
  hence *n* ∈ {*0*, *1*, *2*} **by** *auto*
  thus *n* ∈ {*1*, *2*} **using** ∗ **by** (*auto simp*: *residue-primroot-def*)
qed (*auto simp*: *residue-primroot-def*)

11.2 Primitive roots modulo a prime

For prime p, we now analyse the number of elements in the ring $\mathbb{Z}/p\mathbb{Z}$ whose order is precisely d for each d.

context
  fixes *n* :: *nat* **and** ψ
  assumes *n*: *n* > *1*
  defines ψ ≡ (λ*d*. *card* {*x*∈*totatives n*. *ord n x* = *d*})
begin

lemma *elements-with-ord-restrict-totatives*:
  d > *0* $\Longrightarrow$ {*x*∈{..<*n*}. *ord n x* = *d*} = {*x*∈*totatives n*. *ord n x* = *d*}
  using *n* **by** (*auto simp*: *totatives-def coprime-commute intro*!: *Nat.gr0I le-neq-trans*)

```
lemma prime-elements-with-ord:
  assumes ψ d ≠ 0 and prime n
      and a: a ∈ totatives n ord n a = d a ≠ 1
  shows   inj-on (λk. a ^ k mod n) {..<d}
    and   {x∈{..<n}. [x ^ d = 1] (mod n)} = (λk. a ^ k mod n) ‘ {..<d}
    and   bij-betw (λk. a ^ k mod n) (totatives d) {x∈{..<n}. ord n x = d}
proof −
  show inj: inj-on (λk. a ^ k mod n) {..<d}
    using inj-power-mod[of n a] a by (auto simp: totatives-def coprime-commute)
  from a have d > 0 by (auto simp: totatives-def coprime-commute)
  moreover have d ≠ 1 using a n
    by (auto simp: ord-eq-Suc-0-iff totatives-less cong-def)
  ultimately have d: d > 1 by simp

  have ∗: (λk. a ^ k mod n) ‘ {..<d} = {x∈{..<n}. [x ^ d = 1] (mod n)}
  proof (rule card-seteq)
    have card {x∈{..<n}. [x ^ d = 1] (mod n)} ≤ d
       using assms a by (intro roots-mod-prime-bound) (auto simp: totatives-def
coprime-commute)
    also have ... = card ((λk. a ^ k mod n) ‘ {..<d})
      using inj by (subst card-image) auto
    finally show card {x ∈ {..<n}. [x ^ d = 1] (mod n)} ≤ ... .
  next
    show (λk. a ^ k mod n) ‘ {..<d} ⊆ {x ∈ {..<n}. [x ^ d = 1] (mod n)}
    proof safe
      fix k assume k < d
      have [(a ^ d) ^ k = 1 ^ k] (mod n)
        by (intro cong-pow) (use a in ‹auto simp: ord-divides'›)
      thus [(a ^ k mod n) ^ d = 1] (mod n)
        by (simp add: power-mult [symmetric] cong-def power-mod mult.commute)
    qed (use ‹prime n› in ‹auto dest: prime-gt-1-nat›)
  qed auto
  thus {x∈{..<n}. [x ^ d = 1] (mod n)} = (λk. a ^ k mod n) ‘ {..<d} ..

  show bij-betw (λk. a ^ k mod n) (totatives d) {x∈{..<n}. ord n x = d}
    unfolding bij-betw-def
  proof (intro conjI inj-on-subset[OF inj] equalityI subsetI)
    fix b assume b ∈ (λk. a ^ k mod n) ‘ totatives d
    then obtain k where b = a ^ k mod n k ∈ totatives d by auto
    thus b ∈ {b ∈ {..<n}. ord n b = d}
      using n a by (simp add: ord-power totatives-def coprime-commute)
  next
    fix b assume b ∈ {x ∈ {..<n}. ord n x = d}
    hence b: ord n b = d b < n by auto
    with d have coprime n b using ord-eq-0[of n b] by auto
    from b have b ∈ {x∈{..<n}. [x ^ d = 1] (mod n)}
      by (auto simp: ord-divides')
    with ∗ obtain k where k: k < d b = a ^ k mod n
      by blast
```

with $b(2)$ n a d **have** d *div* *gcd* k d = *ord* n b
using ‹*coprime* n b› **by** (*auto simp*: *ord-power*)
also have *ord* n b = d **by** (*simp add*: b)
finally have *coprime* k d
unfolding *coprime-iff-gcd-eq-1* **using** d a **by** (*subst* (*asm*) *div-eq-dividend-iff*) *auto*
with k b d **have** $k \in$ *totatives* d **by** (*auto simp*: *totatives-def intro*!: *Nat.gr0I*)
with k **show** $b \in (\lambda k.\ a$ ^ k *mod* n) ‘ *totatives* d **by** *blast*
qed (*use* d n **in** ‹*auto simp*: *totatives-less*›)
qed

lemma *prime-card-elements-with-ord*:
assumes ψ $d \neq 0$ **and** *prime* n
shows ψ d = *totient* d
proof (*cases* $d = 1$)
case *True*
have ψ $1 = 1$
using *elements-with-ord-1*[*of* n] n **by** (*simp add*: ψ-*def*)
thus *?thesis* **using** *True* **by** *simp*
next
case *False*
from *assms* **obtain** a **where** a: $a \in$ *totatives* n *ord* n $a = d$
by (*auto simp*: ψ-*def*)
from a **have** $d > 0$ **by** (*auto intro*!: *Nat.gr0I simp*: *ord-eq-0 totatives-def coprime-commute*)
from a **and** *False* **have** $a \neq 1$ **by** *auto*
from *bij-betw-same-card*[*OF prime-elements-with-ord*(*3*)[*OF assms a this*]] **show** ψ d = *totient* d
using *elements-with-ord-restrict-totatives*[*of* d] *False* a ‹$d > 0$›
by (*simp add*: ψ-*def totient-def*)
qed

lemma *prime-sum-card-elements-with-ord-eq-totient*:
($\sum d$ | d *dvd* *totient* n. ψ d) = *totient* n
proof −
have *totient* n = *card* (*totatives* n)
by (*simp add*: *totient-def*)
also have *totatives* n = ($\bigcup d \in \{d.$ d *dvd* *totient* $n\}$. $\{x \in$ *totatives* n. *ord* n $x = d\}$)
by (*force simp*: *order-divides-totient totatives-def coprime-commute*)
also have *card* ... = ($\sum d$ | d *dvd* *totient* n. ψ d)
unfolding ψ-*def* **using** n **by** (*subst card-UN-disjoint*) (*auto intro*!: *finite-divisors-nat*)
finally show *?thesis* **..**
qed

We can now show that the number of elements of order d is $\varphi(d)$ if $d \mid p-1$ and 0 otherwise.

theorem *prime-card-elements-with-ord-eq-totient*:
assumes *prime* n

```
  shows   ψ d = (if d dvd n − 1 then totient d else 0)
proof (cases d dvd totient n)
  case False
  thus ?thesis using order-divides-totient[of n] assms
    by (auto simp: ψ-def totient-prime totatives-def coprime-commute[of n])
next
  case True
  have ψ d = totient d
  proof (rule ccontr)
    assume neq: ψ d ≠ totient d
    have le: ψ d ≤ totient d if d dvd totient n for d
      using prime-card-elements-with-ord[of d] assms by (cases ψ d = 0) auto
    from neq and le[of d] and True have less: ψ d < totient d by auto

    have totient n = (∑ d | d dvd totient n. ψ d)
      using prime-sum-card-elements-with-ord-eq-totient ..
    also have ... < (∑ d | d dvd totient n. totient d)
      by (rule sum-strict-mono-ex1)
         (use n le less assms True in ‹auto intro!: finite-divisors-nat›)
    also have ... = totient n
      using totient-divisor-sum .
    finally show False by simp
  qed
  with True show ?thesis using assms by (simp add: totient-prime)
qed
```

As a corollary, we get that the number of primitive roots modulo a prime p is $\varphi(p-1)$. Since this number is positive, we also get that there *is* at least one primitive root modulo p.

```
lemma
  assumes prime n
  shows   prime-card-primitive-roots: card {x∈totatives n. ord n x = n − 1} =
totient (n − 1)
                                     card {x∈{..<n}. ord n x = n − 1} = totient (n − 1)
  and     prime-primitive-root-exists: ∃ x. residue-primroot n x
proof −
  show *: card {x∈totatives n. ord n x = n − 1} = totient (n − 1)
    using prime-card-elements-with-ord-eq-totient[of n − 1] assms
    by (auto simp: totient-prime ψ-def)
  thus card {x∈{..<n}. ord n x = n − 1} = totient (n − 1)
    using assms n elements-with-ord-restrict-totatives[of n − 1] by simp

  note *
  also have totient (n − 1) > 0 using n by auto
  finally show ∃ x. residue-primroot n x using assms prime-gt-1-nat[of n]
    by (subst (asm) card-gt-0-iff)
       (auto simp: residue-primroot-def totient-prime totatives-def coprime-commute)
qed
```

end

11.3 Primitive roots modulo powers of an odd prime

Any primitive root g modulo an odd prime p is also a primitive root modulo p^k for all $k > 0$ if $[g^{p-1} \neq 1] \pmod{p^2}$. To show this, we first need the following lemma.

```
lemma residue-primroot-power-prime-power-neq-1:
  assumes k ≥ 2
  assumes p: prime p odd p and residue-primroot p g and [g ^(p − 1) ≠ 1] (mod
p^2)
  shows   [g ^ totient (p ^(k − 1)) ≠ 1] (mod (p ^ k))
  using assms(1)
proof (induction k rule: dec-induct)
  case base
  thus ?case using assms by (simp add: totient-prime)
next
  case (step k)
  from p have p > 2
    using prime-gt-1-nat[of p] by (cases p = 2) auto
  from assms have g: g > 0 by (auto intro!: Nat.gr0I)
  have [g ^ totient (p ^(k − 1)) = 1] (mod p ^(k − 1))
    using assms by (intro euler-theorem)
                (auto simp: residue-primroot-def totatives-def coprime-commute)
  from cong-to-1-nat[OF this]
    obtain t where ∗: g ^ totient (p ^(k − 1)) − 1 = p ^(k − 1) ∗ t by auto
  have t: g ^ totient (p ^(k − 1)) = p ^(k − 1) ∗ t + 1
    using g by (subst ∗ [symmetric]) auto

  have ¬p dvd t
  proof
    assume p dvd t
    then obtain q where [simp]: t = p ∗ q by auto
    from t have g ^ totient (p ^(k − 1)) = p ^ k ∗ q + 1
      using ‹k ≥ 2› by (cases k) auto
    hence [g ^ totient (p ^(k − 1)) = p ^ k ∗ q + 1] (mod p ^ k)
      by simp
    also have [p ^ k ∗ q + 1 = 0 ∗ q + 1] (mod p ^ k)
      by (intro cong-add cong-mult) (auto simp: cong-0-iff)
    finally have [g ^ totient (p ^(k − 1)) = 1] (mod p ^ k)
      by simp
    with step.IH show False by contradiction
  qed

  from t have (g ^ totient (p ^(k − 1))) ^ p = (p ^(k − 1) ∗ t + 1) ^ p
    by (rule arg-cong)
  also have (g ^ totient (p ^(k − 1))) ^ p = g ^(p ∗ totient (p ^(k − 1)))
    by (simp add: power-mult [symmetric] mult.commute)
  also have p ∗ totient (p ^(k − 1)) = totient (p ^ k)
```

using *p* ‹$k \geq 2$› **by** (*simp add*: *totient-prime-power Suc-diff-Suc flip*: *power-Suc*)
also have $(p \mathbin{\hat{}} (k - 1) * t + 1) \mathbin{\hat{}} p = (\sum i{\le}p.\ (p\ choose\ i) * t \mathbin{\hat{}} i * p \mathbin{\hat{}} (i * (k - 1)))$
by (*subst binomial*) (*simp-all add*: *mult-ac power-mult-distrib power-mult* [*symmetric*])
finally have $[g \mathbin{\hat{}} totient\ (p \mathbin{\hat{}} k) = (\sum i{\le}p.\ (p\ choose\ i) * t \mathbin{\hat{}} i * p \mathbin{\hat{}} (i * (k - 1)))]$
$(mod\ (p \mathbin{\hat{}} Suc\ k))$ (**is** [- = *?rhs*] (*mod* -)) **by** *simp*

also have $[?rhs = (\sum i{\le}p.\ if\ i \le 1\ then\ (p\ choose\ i) * t \mathbin{\hat{}} i * p \mathbin{\hat{}} (i * (k - 1))\ else\ 0)]$
$(mod\ (p \mathbin{\hat{}} Suc\ k))$ (**is** [*sum ?f* - = *sum ?g* -] (*mod* -))
proof (*intro cong-sum*)
fix *i* **assume** *i*: $i \in \{..p\}$
consider $i \le 1 \mid i = 2 \mid i > 2$ **by** *force*
thus $[?f\ i = ?g\ i]\ (mod\ (p \mathbin{\hat{}} Suc\ k))$
proof *cases*
assume *i*: $i > 2$
have $Suc\ k \le 3 * (k - 1)$
using ‹$k \geq 2$› **by** (*simp add*: *algebra-simps*)
also have $3 * (k - 1) \le i * (k - 1)$
using *i* **by** (*intro mult-right-mono*) *auto*
finally have $p \mathbin{\hat{}} Suc\ k\ dvd\ ?f\ i$
by (*intro dvd-mult le-imp-power-dvd*)
thus $[?f\ i = ?g\ i]\ (mod\ (p \mathbin{\hat{}} Suc\ k))$
by (*simp add*: *cong-0-iff*)
next
assume [*simp*]: $i = 2$
have $?f\ i = p * (p - 1)\ div\ 2 * t^2 * p \mathbin{\hat{}} (2 * (k - 1))$
using *choose-two*[*of p*] **by** *simp*
also have $p * (p - 1)\ div\ 2 = (p - 1)\ div\ 2 * p$
using ‹*odd p*› **by** (*auto elim*!: *oddE*)
also have $\ldots * t^2 * p \mathbin{\hat{}} (2 * (k - 1)) = (p - 1)\ div\ 2 * t^2 * (p * p \mathbin{\hat{}} (2 * (k - 1)))$
by (*simp add*: *algebra-simps*)
also have $p * p \mathbin{\hat{}} (2 * (k - 1)) = p \mathbin{\hat{}} (2 * k - 1)$
using ‹$k \geq 2$› **by** (*cases k*) *auto*
also have $p \mathbin{\hat{}} Suc\ k\ dvd\ (p - 1)\ div\ 2 * t^2 * p \mathbin{\hat{}} (2 * k - 1)$
using ‹$k \geq 2$› **by** (*intro dvd-mult le-imp-power-dvd*) *auto*
finally show $[?f\ i = ?g\ i]\ (mod\ (p \mathbin{\hat{}} Suc\ k))$
by (*simp add*: *cong-0-iff*)
qed *auto*
qed
also have $(\sum i{\le}p.\ ?g\ i) = (\sum i{\le}1.\ ?f\ i)$
using *p prime-gt-1-nat*[*of p*] **by** (*intro sum.mono-neutral-cong-right*) *auto*
also have $\ldots = 1 + t * p \mathbin{\hat{}} k$
using *choose-two*[*of p*] ‹$k \geq 2$› **by** (*cases k*) *simp-all*
finally have *eq*: $[g \mathbin{\hat{}} totient\ (p \mathbin{\hat{}} k) = 1 + t * p \mathbin{\hat{}} k]\ (mod\ p \mathbin{\hat{}} Suc\ k)$ **.**

have $[g \mathbin{\hat{}} totient\ (p \mathbin{\hat{}} k) \neq 1]\ (mod\ p \mathbin{\hat{}} Suc\ k)$

proof
  assume $[g \,\hat{}\, \mathit{totient}\ (p \,\hat{}\, k) = 1]\ (\mathit{mod}\ p \,\hat{}\, \mathit{Suc}\ k)$
  hence $[g \,\hat{}\, \mathit{totient}\ (p \,\hat{}\, k) - g \,\hat{}\, \mathit{totient}\ (p \,\hat{}\, k) = 1 + t * p \,\hat{}\, k - 1]\ (\mathit{mod}\ p \,\hat{}\, \mathit{Suc}\ k)$
    by (*intro cong-diff-nat eq*) *auto*
  hence $[t * p \,\hat{}\, k = 0]\ (\mathit{mod}\ p \,\hat{}\, \mathit{Suc}\ k)$
    by (*simp add*: *cong-sym-eq*)
  hence $p * p \,\hat{}\, k\ \mathit{dvd}\ t * p \,\hat{}\, k$
    by (*simp add*: *cong-0-iff*)
  hence $p\ \mathit{dvd}\ t$ **using** ‹$p > 2$› **by** *simp*
  with ‹$\neg p\ \mathit{dvd}\ t$› **show** *False* **by** *contradiction*
qed
thus *?case* **by** *simp*
qed

We can now show that primitive roots modulo p with the above condition are indeed also primitive roots modulo p^k.

proposition *residue-primroot-prime-lift-iff*:
  assumes p: *prime* p *odd* p **and** *residue-primroot* $p\ g$
  shows $(\forall k>0.\ \mathit{residue\text{-}primroot}\ (p \,\hat{}\, k)\ g) \longleftrightarrow [g \,\hat{}\, (p - 1) \neq 1]\ (\mathit{mod}\ p^2)$
proof −
  from *assms* **have** g: *coprime* $p\ g$ *ord* $p\ g = p - 1$
    by (*auto simp*: *residue-primroot-def totient-prime*)
  show *?thesis*
  proof
    assume $\forall k>0.\ \mathit{residue\text{-}primroot}\ (p \,\hat{}\, k)\ g$
    hence *residue-primroot* $(p^2)\ g$ **by** *auto*
    hence $\mathit{ord}\ (p^2)\ g = \mathit{totient}\ (p^2)$
      by (*simp-all add*: *residue-primroot-def*)
    thus $[g \,\hat{}\, (p - 1) \neq 1]\ (\mathit{mod}\ p^2)$
      using g *assms prime-gt-1-nat*[*of* p]
      by (*auto simp*: *ord-divides′ totient-prime-power*)
  next
    assume g': $[g \,\hat{}\, (p - 1) \neq 1]\ (\mathit{mod}\ p^2)$

    have *residue-primroot* $(p \,\hat{}\, k)\ g$ **if** $k > 0$ **for** k
    proof (*cases* $k = 1$)
      case *False*
      with *that* **have** k: $k > 1$ **by** *simp*
      from g **have** *coprime*: *coprime* $(p \,\hat{}\, k)\ g$
        by (*auto simp*: *totatives-def coprime-commute*)

      define t **where** $t = \mathit{ord}\ (p \,\hat{}\, k)\ g$
      have $[g \,\hat{}\, t = 1]\ (\mathit{mod}\ (p \,\hat{}\, k))$
        by (*simp add*: *t-def ord-divides′*)
      also have $p \,\hat{}\, k = p * p \,\hat{}\, (k - 1)$
        using k **by** (*cases* k) *auto*
      finally have $[g \,\hat{}\, t = 1]\ (\mathit{mod}\ p)$
        by (*rule cong-modulus-mult-nat*)

```
      hence totient p dvd t
        using g p by (simp add: ord-divides' totient-prime)
      then obtain q where t: t = totient p * q by auto

      have t dvd totient (p ^ k)
        using coprime by (simp add: t-def order-divides-totient)
      with t p k have q dvd p ^(k − 1) using prime-gt-1-nat[of p]
        by (auto simp: totient-prime totient-prime-power)
      then obtain b where b: b ≤ k − 1 q = p ^ b
        using divides-primepow-nat[of p q k − 1] p by auto

      have b = k − 1
      proof (rule ccontr)
        assume b ≠ k − 1
        with b have b < k − 1 by simp
        have t = p ^ b * (p − 1)
          using b p by (simp add: t totient-prime)
        also have ... dvd p ^(k − 2) * (p − 1)
          using ‹b < k − 1› by (intro mult-dvd-mono le-imp-power-dvd) auto
        also have ... = totient (p ^(k − 1))
          using k p by (simp add: totient-prime-power numeral-2-eq-2)
        finally have [g ^ totient (p ^(k − 1)) = 1] (mod (p ^ k))
          by (simp add: ord-divides' t-def)
       with residue-primroot-power-prime-power-neq-1[of k p g] p k assms g' show
False
          by auto
      qed
      hence t = totient (p ^ k)
        using p k by (simp add: t b totient-prime totient-prime-power)
      thus residue-primroot (p ^ k) g
        using g one-less-power[of p k] prime-gt-1-nat[of p] p k
        by (simp add: residue-primroot-def t-def)
    qed (use assms in auto)
    thus ∀k>0. residue-primroot (p ^ k) g by blast
  qed
qed
```

If p is an odd prime, there is always a primitive root g modulo p, and if g does not fulfil the above assumption required for it to be liftable to p^k, we can use $g + p$, which is also a primitive root modulo p and *does* fulfil the assumption.

This shows that any modulus that is a power of an odd prime has a primitive root.

```
theorem residue-primroot-odd-prime-power-exists:
  assumes p: prime p odd p
  obtains g where ∀k>0. residue-primroot (p ^ k) g
proof −
  obtain g where g: residue-primroot p g
```

using *prime-primitive-root-exists*[*of p*] *assms prime-gt-1-nat*[*of p*] **by** *auto*

have $\exists\, g.\ residue\text{-}primroot\ p\ g \wedge [g\ \widehat{}\ (p - 1) \neq 1]\ (mod\ p^2)$
proof $(cases\ [g\ \widehat{}\ (p - 1) = 1]\ (mod\ p^2))$
case *True*
define g' **where** $g' = p + g$
note *g*
also have $residue\text{-}primroot\ p\ g \longleftrightarrow residue\text{-}primroot\ p\ g'$
unfolding g'*-def* **by** (*rule residue-primroot-cong*) (*auto simp*: *cong-def*)
finally have g': $residue\text{-}primroot\ p\ g'$ **.**

have $[g'\ \widehat{}\ (p - 1) = (\sum k{\le}p{-}1.\ ((p{-}1)\ choose\ k) * g\ \widehat{}\ (p - Suc\ k) * p\ \widehat{}\ k)]\ (mod\ p^2)$
(**is** $[- = ?rhs]\ (mod\ -)$) **by** (*simp add*: g'*-def binomial mult-ac*)
also have $[?rhs = (\sum k{\le}p{-}1.\ if\ k \le 1\ then\ ((p{-}1)\ choose\ k) * g\ \widehat{}\ (p - Suc\ k) * p\ \widehat{}\ k\ else\ 0)]\ (mod\ p^2)$
(**is** $[sum\ ?f\ - = sum\ ?g\ -]\ (mod\ -)$)
proof (*intro cong-sum*)
fix *k* **assume** $k \in \{..p{-}1\}$
show $[?f\ k = ?g\ k]\ (mod\ p^2)$
proof $(cases\ k \le 1)$
case *False*
have $p^2\ dvd\ ?f\ k$
using *False* **by** (*intro dvd-mult le-imp-power-dvd*) *auto*
thus *?thesis* **using** *False* **by** (*simp add*: *cong-0-iff*)
qed *auto*
qed
also have $sum\ ?g\ \{..p{-}1\} = sum\ ?f\ \{0,\ 1\}$
using *prime-gt-1-nat*[*of p*] *p* **by** (*intro sum.mono-neutral-cong-right*) *auto*
also have $\ldots = g\ \widehat{}\ (p - 1) + p * (p - 1) * g\ \widehat{}\ (p - 2)$
using *p* **by** (*simp add*: *numeral-2-eq-2*)
also have $[g\ \widehat{}\ (p - 1) + p * (p - 1) * g\ \widehat{}\ (p - 2) = 1 + p * (p - 1) * g\ \widehat{}\ (p - 2)]\ (mod\ p^2)$
by (*intro cong-add True*) *auto*
finally have $[g'\ \widehat{}\ (p - 1) = 1 + p * (p - 1) * g\ \widehat{}\ (p - 2)]\ (mod\ p^2)$ **.**

moreover have $[1 + p * (p - 1) * g\ \widehat{}\ (p - 2) \neq 1]\ (mod\ p^2)$
proof
assume $[1 + p * (p - 1) * g\ \widehat{}\ (p - 2) = 1]\ (mod\ p^2)$
hence $[1 + p * (p - 1) * g\ \widehat{}\ (p - 2) - 1 = 1 - 1]\ (mod\ p^2)$
by (*intro cong-diff-nat*) *auto*
hence $p * p\ dvd\ p * ((p - 1) * g\ \widehat{}\ (p - 2))$
by (*auto simp*: *cong-0-iff power2-eq-square*)
hence $p\ dvd\ (p - 1) * g\ \widehat{}\ (p - 2)$
using *p* **by** *simp*
hence $p\ dvd\ g\ \widehat{}\ (p - 2)$
using *p dvd-imp-le*[*of p p* $-$ *Suc 0*] *prime-gt-1-nat*[*of p*]
by (*auto simp*: *prime-dvd-mult-iff*)
also have $\ldots\ dvd\ g\ \widehat{}\ (p - 1)$

by (*intro le-imp-power-dvd*) *auto*
finally have $[g \mathbin{\hat{}} (p - 1) = 0]$ (*mod p*)
by (*simp add*: *cong-0-iff*)
hence $[0 = g \mathbin{\hat{}} (p - 1)]$ (*mod p*)
by (*simp add*: *cong-sym-eq*)

also from ‹*residue-primroot p g*› **have** $[g \mathbin{\hat{}} (p - 1) = 1]$ (*mod p*)
using *p* **by** (*auto simp*: *residue-primroot-def ord-divides′ totient-prime*)
finally have $[0 = 1]$ (*mod p*) **.**
thus *False* **using** *prime-gt-1-nat*[*of p*] *p* **by** (*simp add*: *cong-def*)
qed

ultimately have $[g' \mathbin{\hat{}} (p - 1) \neq 1]$ (*mod* p^2)
by (*simp add*: *cong-def*)
thus $\exists$ *g. residue-primroot p g* $\wedge$ $[g \mathbin{\hat{}} (p - 1) \neq 1]$ (*mod* p^2)
using g' **by** *blast*
qed (*use g* **in** *auto*)
thus *?thesis*
using *residue-primroot-prime-lift-iff*[*OF assms*] *that* **by** *blast*
qed

11.4 Carmichael's function

Carmichael's function $\lambda(n)$ gives the LCM of the orders of all elements in the residue ring modulo n – or, equivalently, the maximum order, as we will show later. Algebraically speaking, it is the exponent of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$.

It is not to be confused with Liouville's function, which is also denoted by $\lambda(n)$.

definition *Carmichael* **where**
Carmichael n = (*LCM a* $\in$ *totatives n. ord n a*)

lemma *Carmichael-0* [*simp*]: *Carmichael 0* = *1*
by (*simp add*: *Carmichael-def*)

lemma *Carmichael-1* [*simp*]: *Carmichael 1* = *1*
by (*simp add*: *Carmichael-def*)

lemma *Carmichael-Suc-0* [*simp*]: *Carmichael* (*Suc 0*) = *1*
by (*simp add*: *Carmichael-def*)

lemma *ord-dvd-Carmichael*:
assumes *n* > *1 coprime n k*
shows *ord n k dvd Carmichael n*
proof −
have *k mod n* $\in$ *totatives n*
using *assms* **by** (*auto simp*: *totatives-def coprime-commute intro*!: *Nat.gr0I*)
hence *ord n* (*k mod n*) *dvd Carmichael n*

```
    by (simp add: Carmichael-def del: ord-mod)
  thus ?thesis by simp
qed

lemma Carmichael-divides:
  assumes Carmichael n dvd k coprime n a
  shows   [a ^ k = 1] (mod n)
proof (cases n < 2 ∨ a = 1)
  case False
  hence ord n a dvd Carmichael n
    using False assms by (intro ord-dvd-Carmichael) auto
  also have ... dvd k by fact
  finally have ord n a dvd k .
  thus ?thesis using ord-divides by auto
next
  case True
  then consider a = 1 | n = 0 | n = 1 by force
  thus ?thesis using assms by cases auto
qed

lemma Carmichael-dvd-totient: Carmichael n dvd totient n
  unfolding Carmichael-def
proof (intro Lcm-least, safe)
  fix a assume a ∈ totatives n
  hence [a ^ totient n = 1] (mod n)
    by (intro euler-theorem) (auto simp: totatives-def)
  thus ord n a dvd totient n
    using ord-divides by blast
qed

lemma Carmichael-dvd-mono-coprime:
  assumes coprime m n m > 1 n > 1
  shows   Carmichael m dvd Carmichael (m ∗ n)
  unfolding Carmichael-def[of m]
proof (intro Lcm-least, safe)
  fix x assume x: x ∈ totatives m
  from assms have totatives n ≠ {} by simp
  then obtain y where y: y ∈ totatives n by blast

  from binary-chinese-remainder-nat[OF assms(1), of x y]
  obtain z where z: [z = x] (mod m) [z = y] (mod n) by blast
  have z': coprime z n coprime z m
   by (rule cong-imp-coprime; use x y z in ‹force simp: totatives-def cong-sym-eq›)+

  from z have ord m x = ord m z
    by (intro ord-cong) (auto simp: cong-sym-eq)
  also have ord m z dvd ord (m ∗ n) z
    using assms by (auto simp: ord-modulus-mult-coprime)
  also from z' assms have ... dvd Carmichael (m ∗ n)
```

by (*intro ord-dvd-Carmichael*) (*auto simp*: *coprime-commute intro*!:*one-less-mult*)
finally show *ord m x dvd Carmichael* (*m* ∗ *n*) **.**
qed

λ distributes over the product of coprime numbers similarly to φ, but with LCM instead of multiplication:

lemma *Carmichael-mult-coprime*:
assumes *coprime m n*
shows *Carmichael* (*m* ∗ *n*) = *lcm* (*Carmichael m*) (*Carmichael n*)
proof (*cases m* ≤ *1* ∨ *n* ≤ *1*)
case *True*
hence *m* = *0* ∨ *n* = *0* ∨ *m* = *1* ∨ *n* = *1* **by** *force*
thus *?thesis* **using** *assms* **by** *auto*
next
case *False*
show *?thesis*
proof (*rule dvd-antisym*)
show *Carmichael* (*m* ∗ *n*) *dvd lcm* (*Carmichael m*) (*Carmichael n*)
unfolding *Carmichael-def*[*of m* ∗ *n*]
proof (*intro Lcm-least*, *safe*)
fix *x* **assume** *x*: *x* ∈ *totatives* (*m* ∗ *n*)
have *ord* (*m* ∗ *n*) *x* = *lcm* (*ord m x*) (*ord n x*)
using *assms x* **by** (*subst ord-modulus-mult-coprime*) (*auto simp*: *coprime-commute totatives-def*)
also have ... *dvd lcm* (*Carmichael m*) (*Carmichael n*)
using *False x*
by (*intro lcm-mono ord-dvd-Carmichael*) (*auto simp*: *totatives-def coprime-commute*)
finally show *ord* (*m* ∗ *n*) *x dvd* ... **.**
qed
next
show *lcm* (*Carmichael m*) (*Carmichael n*) *dvd Carmichael* (*m* ∗ *n*)
using *Carmichael-dvd-mono-coprime*[*of m n*]
Carmichael-dvd-mono-coprime[*of n m*] *assms False*
by (*auto intro*!: *lcm-least simp*: *coprime-commute mult.commute*)
qed
qed

lemma *Carmichael-pos* [*simp*, *intro*]: *Carmichael n* > *0*
by (*auto simp*: *Carmichael-def ord-eq-0 totatives-def coprime-commute intro*!: *Nat.gr0I*)

lemma *Carmichael-nonzero* [*simp*]: *Carmichael n* ≠ *0*
by *simp*

lemma *power-Carmichael-eq-1*:
assumes *n* > *1 coprime n x*
shows [*x* ^ *Carmichael n* = *1*] (*mod n*)
using *ord-dvd-Carmichael*[*of n x*] *assms*

by (*auto simp*: *ord-divides'*)

lemma *Carmichael-2* [*simp*]: *Carmichael 2 = 1*
  using *Carmichael-dvd-totient*[*of 2*] **by** *simp*

lemma *Carmichael-4* [*simp*]: *Carmichael 4 = 2*
proof −
  have *Carmichael 4 dvd 2*
    using *Carmichael-dvd-totient*[*of 4*] **by** *simp*
  hence *Carmichael 4 ≤ 2* **by** (*rule dvd-imp-le*) *auto*
  moreover have *Carmichael 4 ≠ 1*
    using *power-Carmichael-eq-1*[*of 4::nat 3*]
    unfolding *coprime-iff-gcd-eq-1* **by** (*auto simp*: *gcd-non-0-nat cong-def*)
  ultimately show *?thesis*
    using *Carmichael-pos*[*of 4*] **by** *linarith*
qed

lemma *residue-primroot-Carmichael*:
  assumes *residue-primroot n g*
  shows *Carmichael n = totient n*
proof (*cases n = 1*)
  case *False*
  show *?thesis*
  proof (*intro dvd-antisym Carmichael-dvd-totient*)
    have *ord n g dvd Carmichael n*
    using *assms False* **by** (*intro ord-dvd-Carmichael*) (*auto simp*: *residue-primroot-def*)
    thus *totient n dvd Carmichael n*
      using *assms* **by** (*auto simp*: *residue-primroot-def*)
  qed
qed *auto*

lemma *Carmichael-odd-prime-power*:
  assumes *prime p odd p k > 0*
  shows *Carmichael (p ^ k) = p ^ (k − 1) ∗ (p − 1)*
proof −
  from *assms* **obtain** *g* **where** *residue-primroot (p ^ k) g*
    using *residue-primroot-odd-prime-power-exists*[*of p*] *assms* **by** *metis*
  hence *Carmichael (p ^ k) = totient (p ^ k)*
    by (*intro residue-primroot-Carmichael*[*of p ^ k g*]) *auto*
  with *assms* **show** *?thesis* **by** (*simp add*: *totient-prime-power*)
qed

lemma *Carmichael-prime*:
  assumes *prime p*
  shows *Carmichael p = p − 1*
proof (*cases even p*)
  case *True*
  with *assms* **have** *p = 2*
    using *primes-dvd-imp-eq two-is-prime-nat* **by** *blast*

```
  thus ?thesis by simp
next
  case False
  with Carmichael-odd-prime-power[of p 1] assms show ?thesis by simp
qed

lemma Carmichael-twopow-ge-8:
  assumes k ≥ 3
  shows  Carmichael (2 ^ k) = 2 ^ (k − 2)
proof (intro dvd-antisym)
  have 2 ^ (k − 2) = ord (2 ^ k) (3 :: nat)
    using ord-twopow-3-5[of k 3] assms by simp
  also have ... dvd Carmichael (2 ^ k)
    using assms one-less-power[of 2::nat k] by (intro ord-dvd-Carmichael) auto
  finally show 2 ^ (k − 2) dvd ... .
next
  show Carmichael (2 ^ k) dvd 2 ^ (k − 2)
    unfolding Carmichael-def
  proof (intro Lcm-least, safe)
    fix x assume x ∈ totatives (2 ^ k)
    hence odd x by (auto simp: totatives-def)
    hence [x ^ 2 ^ (k − 2) = 1] (mod 2 ^ k)
      using assms ord-twopow-aux[of k x] by auto
    thus ord (2 ^ k) x dvd 2 ^ (k − 2)
      by (simp add: ord-divides')
  qed
qed

lemma Carmichael-twopow:
  Carmichael (2 ^ k) = (if k ≤ 2 then 2 ^ (k − 1) else 2 ^ (k − 2))
proof −
  have k = 0 ∨ k = 1 ∨ k = 2 ∨ k ≥ 3 by linarith
  thus ?thesis by (auto simp: Carmichael-twopow-ge-8)
qed

lemma Carmichael-prime-power:
  assumes prime p k > 0
  shows  Carmichael (p ^ k) =
           (if p = 2 ∧ k > 2 then 2 ^ (k − 2) else p ^ (k − 1) ∗ (p − 1))
proof (cases p = 2)
  case True
  thus ?thesis by (simp add: Carmichael-twopow)
next
  case False
  with assms have odd p p > 2
    using prime-odd-nat[of p] prime-gt-1-nat[of p] by auto
  thus ?thesis
    using assms Carmichael-odd-prime-power[of p k] by simp
qed
```

```
lemma Carmichael-prod-coprime:
  assumes finite A ⋀i j. i ∈ A ⟹ j ∈ A ⟹ i ≠ j ⟹ coprime (f i) (f j)
  shows   Carmichael (∏i∈A. f i) = (LCM i∈A. Carmichael (f i))
  using assms by (induction A rule: finite-induct)
               (simp, simp, subst Carmichael-mult-coprime[OF prod-coprime-right],
auto)
```

Since λ distributes over coprime factors and we know the value of $\lambda(p^k)$ for prime p, we can now give a closed formula for $\lambda(n)$ in terms of the prime factorisation of n:

```
theorem Carmichael-closed-formula:
  Carmichael n =
    (LCM p∈prime-factors n. let k = multiplicity p n
                             in  if p = 2 ∧ k > 2 then 2 ^ (k − 2) else p ^ (k − 1) ∗
(p − 1))
  (is - = Lcm ?A)
proof (cases n = 0)
  case False
  hence n = (∏p∈prime-factors n. p ^ multiplicity p n)
    using prime-factorization-nat by blast
  also have Carmichael … =
              (LCM p∈prime-factors n. Carmichael (p ^ multiplicity p n))
   by (subst Carmichael-prod-coprime) (auto simp: in-prime-factors-iff primes-coprime)
  also have (λp. Carmichael (p ^ multiplicity p n)) ‘ prime-factors n = ?A
    by (intro image-cong)
       (auto simp: Let-def Carmichael-prime-power prime-factors-multiplicity)
  finally show ?thesis .
qed auto

corollary even-Carmichael:
  assumes n > 2
  shows   even (Carmichael n)
proof (cases ∃k. n = 2 ^ k)
  case True
  then obtain k where [simp]: n = 2 ^ k by auto
  from assms have k ≠ 0 k ≠ 1 by (auto intro!: Nat.gr0I)
  hence k ≥ 2 by auto
  thus ?thesis by (auto simp: Carmichael-twopow)
next
  case False
  from assms have n ≠ 0 by auto
  from False have ∃p∈prime-factors n. p ≠ 2
    using assms Ex-other-prime-factor[of n 2] by auto
  from divide-out-primepow-ex[OF ‹n ≠ 0› this]
  obtain p k n'
    where p:
      p ≠ 2
      prime p
```

```
    p dvd n
    ¬ p dvd n'
    0 < k
    n = p ^ k * n' .
  from p have coprime: coprime (p ^ k) n'
    using p prime-imp-coprime by auto
  have odd p
    using p primes-dvd-imp-eq[of 2 p] by auto
  have even (Carmichael (p ^ k))
    using p ‹odd p› by (auto simp: Carmichael-prime-power)
  with p coprime show ?thesis
    by (auto simp: Carmichael-mult-coprime intro!: dvd-lcmI1)
qed

lemma eval-Carmichael:
  assumes prime-factorization n = A
  shows    Carmichael n = (LCM p ∈ set-mset A.
             let k = count A p in if p = 2 ∧ k > 2 then 2 ^(k − 2) else p ^(k −
1) * (p − 1))
  unfolding assms [symmetric] Carmichael-closed-formula
 by (intro arg-cong[where f = Lcm] image-cong) (auto simp: Let-def count-prime-factorization)
```

Any residue ring always contains a λ-root, i. e. an element whose order is $\lambda(n)$.

```
theorem Carmichael-root-exists:
  assumes n > (0::nat)
  obtains g where g ∈ totatives n and ord n g = Carmichael n
proof (cases n = 1)
  case True
  thus ?thesis by (intro that[of 1]) (auto simp: totatives-def)
next
  case False
  have primepow: ∃ g. coprime (p ^ k) g ∧ ord (p ^ k) g = Carmichael (p ^ k)
    if pk: prime p k > 0 for p k
  proof (cases p = 2)
    case [simp]: True
    from ‹k > 0› consider k = 1 | k = 2 | k ≥ 3 by force
    thus ?thesis
    proof cases
      assume k = 1
      thus ?thesis by (intro exI[of - 1]) auto
    next
      assume [simp]: k = 2
      have coprime 4 (3::nat)
        by (auto simp: coprime-iff-gcd-eq-1 gcd-non-0-nat)
      thus ?thesis by (intro exI[of - 3]) auto
    next
      assume k: k ≥ 3
      have coprime (2 ^ k :: nat) 3 by auto
```

thus *?thesis* **using** *k*
by (*intro exI*[*of - 3*]) (*auto simp*: *ord-twopow-3-5 Carmichael-twopow*)
qed
next
case *False*
hence *odd p* **using** ‹*prime p*›
using *primes-dvd-imp-eq two-is-prime-nat* **by** *blast*
then obtain *g* **where** *residue-primroot* (p ^ k) *g*
using *residue-primroot-odd-prime-power-exists*[*of p*] *pk* **by** *metis*
thus *?thesis* **using** *False pk*
by (*intro exI*[*of - g*])
(*auto simp*: *Carmichael-prime-power residue-primroot-def totient-prime-power*)
qed

define *P* **where** P = *prime-factors n*
define *k* **where** k = (λp. *multiplicity p n*)
have $\forall p \in P$. $\exists g$. *coprime* (p ^ k p) g $\wedge$ *ord* (p ^ k p) g = *Carmichael* (p ^ k p)
using *primepow* **by** (*auto simp*: *P-def k-def prime-factors-multiplicity*)
hence $\exists g$. $\forall p \in P$. *coprime* (p ^ k p) (g p) $\wedge$ *ord* (p ^ k p) (g p) = *Carmichael* (p ^ k p)
by (*subst* (*asm*) *bchoice-iff*)
then obtain *g* **where** *g*: $\bigwedge p$. $p \in P \Longrightarrow$ *coprime* (p ^ k p) (g p)
$\bigwedge p$. $p \in P \Longrightarrow$ *ord* (p ^ k p) (g p) = *Carmichael* (p ^ k p) **by** *metis*
have $\exists x$. $\forall i \in P$. [x = g i] (*mod* i ^ k i)
by (*intro chinese-remainder-nat*)
(*auto simp*: *P-def k-def in-prime-factors-iff primes-coprime*)
then obtain *x* **where** *x*: $\bigwedge p$. $p \in P \Longrightarrow$ [x = g p] (*mod* p ^ k p) **by** *metis*

have n = ($\prod p \in P$. p ^ k p)
using *assms* **unfolding** *P-def k-def* **by** (*rule prime-factorization-nat*)
also have *ord* ... x = (*LCM* $p \in P$. *ord* (p ^ k p) x)
by (*intro ord-modulus-prod-coprime*) (*auto simp*: *P-def in-prime-factors-iff primes-coprime*)
also have (λp. *ord* (p ^ k p) x) ‘ P = (λp. *ord* (p ^ k p) (g p)) ‘ P
by (*intro image-cong ord-cong x*) *auto*
also have ... = (λp. *Carmichael* (p ^ k p)) ‘ P
by (*intro image-cong g*) *auto*
also have *Lcm* ... = *Carmichael* ($\prod p \in P$. p ^ k p)
by (*intro Carmichael-prod-coprime* [*symmetric*])
(*auto simp*: *P-def in-prime-factors-iff primes-coprime*)
also have ($\prod p \in P$. p ^ k p) = n
using *assms* **unfolding** *P-def k-def* **by** (*rule prime-factorization-nat* [*symmetric*])
finally have *ord n x* = *Carmichael n* **.**
moreover from *this* **have** *coprime n x*
by (*cases coprime n x*) (*auto split*: *if-splits*)
ultimately show *?thesis* **using** *assms False*
by (*intro that*[*of x mod n*])
(*auto simp*: *totatives-def coprime-commute coprime-absorb-left intro*!: *Nat.gr0I*)

qed

This also means that the Carmichael number is not only the LCM of the orders of the elements of the residue ring, but indeed the maximum of the orders.

lemma *Carmichael-altdef*:
  Carmichael n = (*if n* = *0 then 1 else Max* (*ord n* ' *totatives n*))
proof (*cases n* = *0*)
  case *False*
  have *Carmichael n* = *Max* (*ord n* ' *totatives n*)
  proof (*intro antisym Max.boundedI Max.coboundedI*)
    fix *k* **assume** *k*: *k* ∈ *ord n* ' *totatives n*
    thus *k* ≤ *Carmichael n*
    proof (*cases n* = *1*)
      case *False*
      with ‹*n* ≠ *0*› **have** *n* > *1* **by** *linarith*
      thus *?thesis* **using** *k* ‹*n* ≠ *0*›
        by (*intro dvd-imp-le*)
          (*auto intro*!: *ord-dvd-Carmichael simp*: *totatives-def coprime-commute*)
    qed *auto*
  next
    obtain *g* **where** *g* ∈ *totatives n* **and** *ord n g* = *Carmichael n*
      using *Carmichael-root-exists*[*of n*] ‹*n* ≠ *0*› **by** *auto*
    thus *Carmichael n* ∈ *ord n* ' *totatives n* **by** *force*
  qed (*use* ‹*n* ≠ *0*› **in** *auto*)
  thus *?thesis* **using** *False* **by** *simp*
qed *auto*

11.5 Existence of primitive roots for general moduli

We now related Carmichael's function to the existence of primitive roots and, in the end, use this to show precisely which moduli have primitive roots and which do not.

The first criterion for the existence of a primitive root is this: A primitive root modulo n exists iff $\lambda(n) = \varphi(n)$.

lemma *Carmichael-eq-totient-imp-primroot*:
  assumes *n* > *0* **and** *Carmichael n* = *totient n*
  shows ∃ *g. residue-primroot n g*
proof −
  from ‹*n* > *0*› **obtain** *g* **where** *g* ∈ *totatives n* **and** *ord n g* = *Carmichael n*
    using *Carmichael-root-exists*[*of n*] **by** *metis*
  with *assms* **show** *?thesis* **by** (*auto simp*: *residue-primroot-def totatives-def coprime-commute*)
qed

theorem *residue-primroot-iff-Carmichael*:
  (∃ *g. residue-primroot n g*) ⟷ *Carmichael n* = *totient n* ∧ *n* > *0*
proof *safe*

```
  fix g assume g: residue-primroot n g
  thus n > 0 by (auto simp: residue-primroot-def)
  have coprime n g by (rule ccontr) (use g in ‹auto simp: residue-primroot-def›)

  show Carmichael n = totient n
  proof (cases n = 1)
    case False
    with ‹n > 0› have n > 1 by auto
    with ‹coprime n g› have ord n g dvd Carmichael n
      by (intro ord-dvd-Carmichael) auto
    thus ?thesis using g by (intro dvd-antisym Carmichael-dvd-totient)
                         (auto simp: residue-primroot-def)
  qed auto
qed (use Carmichael-eq-totient-imp-primroot[of n] in auto)
```

Any primitive root modulo mn for coprime m, n is also a primitive root modulo m and n. The converse does not hold in general.

```
lemma residue-primroot-modulus-mult-coprimeD:
  assumes coprime m n and residue-primroot (m * n) g
  shows   residue-primroot m g residue-primroot n g
proof -
  have *: m > 0 n > 0 coprime m g coprime n g
         lcm (ord m g) (ord n g) = totient m * totient n
    using assms
   by (auto simp: residue-primroot-def ord-modulus-mult-coprime totient-mult-coprime)

  define a b where a = totient m div ord m g and b = totient n div ord n g
  have ab: totient m = ord m g * a totient n = ord n g * b
    using * by (auto simp: a-def b-def order-divides-totient)

  have a = 1 b = 1 coprime (ord m g) (ord n g)
   unfolding coprime-iff-gcd-eq-1 using * by (auto simp: ab lcm-nat-def dvd-div-eq-mult
ord-eq-0)
  with ab and * show residue-primroot m g residue-primroot n g
    by (auto simp: residue-primroot-def)
qed
```

If a primitive root modulo mn exists for coprime m, n, then $\lambda(m)$ and $\lambda(n)$ must also be coprime. This is helpful in establishing that there are no primitive roots modulo mn by showing e. g. that $\lambda(m)$ and $\lambda(n)$ are both even.

```
lemma residue-primroot-modulus-mult-coprime-imp-Carmichael-coprime:
  assumes coprime m n and residue-primroot (m * n) g
  shows   coprime (Carmichael m) (Carmichael n)
proof -
  from residue-primroot-modulus-mult-coprimeD[OF assms]
    have *: residue-primroot m g residue-primroot n g by auto
  hence [simp]: Carmichael m = totient m Carmichael n = totient n
    by (simp-all add: residue-primroot-Carmichael)
```

from $*$ **have** *mn*: $m > 0$ $n > 0$ **by** (*auto simp*: *residue-primroot-def*)

from *assms* **have** *Carmichael* $(m * n) =$ *totient* $(m * n) \wedge n > 0$
  using *residue-primroot-iff-Carmichael*[*of* $m * n$] **by** *auto*
with *assms* **have** *lcm* (*totient* m) (*totient* n) $=$ *totient* $m *$ *totient* n
  by (*simp add*: *Carmichael-mult-coprime totient-mult-coprime*)
thus *?thesis* **unfolding** *coprime-iff-gcd-eq-1* **using** *mn*
  by (*simp add*: *lcm-nat-def dvd-div-eq-mult*)
qed

The following moduli are precisely those that have primitive roots.

definition *cyclic-moduli* :: *nat set* **where**
  cyclic-moduli $= \{1, 2, 4\} \cup \{p \mathbin{\hat{}} k \mid p\ k.\ \mathit{prime}\ p \wedge \mathit{odd}\ p \wedge k > 0\} \cup$
    $\{2 * p \mathbin{\hat{}} k \mid p\ k.\ \mathit{prime}\ p \wedge \mathit{odd}\ p \wedge k > 0\}$

theorem *residue-primroot-iff-in-cyclic-moduli*:
  $(\exists g.$ *residue-primroot* $m\ g) \longleftrightarrow m \in$ *cyclic-moduli*
proof $-$
  have $(\exists g.$ *residue-primroot* $m\ g)$ **if** $m \in$ *cyclic-moduli*
  using *that* **unfolding** *cyclic-moduli-def*
  by (*intro Carmichael-eq-totient-imp-primroot*)
    (*auto dest*: *prime-gt-0-nat simp*: *Carmichael-prime-power totient-prime-power*
      Carmichael-mult-coprime totient-mult-coprime)

  moreover have $\neg(\exists g.$ *residue-primroot* $m\ g)$ **if** $m \notin$ *cyclic-moduli*
  proof (*cases* $m = 0$)
    case *False*
    with *that* **have** [*simp*]: $m > 0$ $m \neq 1$ **by** (*auto simp*: *cyclic-moduli-def*)
    show *?thesis*
    proof (*cases* $\exists k.\ m = 2 \mathbin{\hat{}} k$)
      case *True*
      then obtain k **where** [*simp*]: $m = 2 \mathbin{\hat{}} k$ **by** *auto*
      with *that* **have** $k \neq 0 \wedge k \neq 1 \wedge k \neq 2$ **by** (*auto simp*: *cyclic-moduli-def*)
      hence $k \geq 3$ **by** *auto*
      thus *?thesis* **by** (*subst residue-primroot-iff-Carmichael*)
        (*auto simp*: *Carmichael-twopow totient-prime-power*)
    next
      case *False*
      hence $\exists p \in$ *prime-factors* $m.\ p \neq 2$
        using *Ex-other-prime-factor*[*of* m 2] **by** *auto*
      from *divide-out-primepow-ex*[*OF* ‹$m \neq 0$› *this*]
      obtain $p\ k\ m'$ **where** p: $p \neq 2$ *prime* p p *dvd* m $\neg p$ *dvd* m' $0 < k$ $m = p \mathbin{\hat{}} k$
$* m'$
        by *metis*
      have *odd* p
        using p *primes-dvd-imp-eq*[*of* 2 p] **by** *auto*
      from p **have** *coprime*: *coprime* $(p \mathbin{\hat{}} k)$ m'
        using p *prime-imp-coprime* **by** *auto*
      have $m \in$ *cyclic-moduli* **if** $m' = 1$

```
      using that p ‹odd p› by (auto simp: cyclic-moduli-def)
    moreover have m ∈ cyclic-moduli if m′ = 2
    proof −
      have m = 2 ∗ p ^ k using p that by (simp add: mult.commute)
      thus m ∈ cyclic-moduli using p ‹odd p›
        unfolding cyclic-moduli-def by blast
    qed
    moreover have m′ ≠ 0 using p by (intro notI) auto
    ultimately have m′ ≠ 0 ∧m′ ≠ 1 ∧ m′ ≠ 2 using that by auto
    hence m′ > 2 by linarith

    show ?thesis
    proof
      assume ∃ g. residue-primroot m g
      with coprime p have coprime′: coprime (p − 1) (Carmichael m′)
       using residue-primroot-modulus-mult-coprime-imp-Carmichael-coprime[OF
coprime]
        by (auto simp: Carmichael-prime-power)
      moreover have even (p − 1) even (Carmichael m′)
        using ‹m′ > 2› ‹odd p› by (auto intro!: even-Carmichael)
      ultimately show False by force
    qed
  qed
qed auto

ultimately show ?thesis by metis
qed

lemma residue-primroot-is-generator:
  assumes m > 1 and residue-primroot m g
  shows   bij-betw (λi. g ^ i mod m) {..<totient m} (totatives m)
  unfolding bij-betw-def
proof
  from assms have [simp]: ord m g = totient m
    by (simp add: residue-primroot-def)
  from assms have coprime m g by (simp add: residue-primroot-def)
  hence inj-on (λi. g ^ i mod m) {..<ord m g}
    by (intro inj-power-mod)
  thus inj: inj-on (λi. g ^ i mod m) {..<totient m}
    by simp

  show (λi. g ^ i mod m) ‘ {..<totient m} = totatives m
  proof (rule card-subset-eq)
    show card ((λi. g ^ i mod m) ‘ {..<totient m}) = card (totatives m)
      using inj by (subst card-image) (auto simp: totient-def)
    show (λi. g ^ i mod m) ‘ {..<totient m} ⊆ totatives m
      using ‹m > 1› ‹coprime m g› power-in-totatives[of m g] by blast
  qed auto
qed
```

Given one primitive root g, all the primitive roots are powers g^i for $1 \leq i \leq \varphi(n)$ with $\gcd(i, \varphi(n)) = 1$.

```
theorem residue-primroot-bij-betw-primroots:
  assumes m > 1 and residue-primroot m g
  shows   bij-betw (λi. g ^ i mod m) (totatives (totient m))
                                {g∈totatives m. residue-primroot m g}
proof (cases m = 2)
  case [simp]: True
  have [simp]: totatives 2 = {1}
    by (auto simp: totatives-def elim!: oddE)
  from assms have odd g
    by (auto simp: residue-primroot-def)
  hence pow-eq: (λi. g ^ i mod m) = (λ-. 1)
    by (auto simp: fun-eq-iff mod-2-eq-odd)
  have {g ∈ totatives m. residue-primroot m g} = {1}
    by (auto simp: residue-primroot-def)
  thus ?thesis using pow-eq by (auto simp: bij-betw-def)
next
  case False
  thus ?thesis unfolding bij-betw-def
  proof safe
    from assms False have m > 2 by auto
    from assms ‹m > 2› have totient m > 1 by (intro totient-gt-1) auto
    from assms have [simp]: ord m g = totient m
      by (simp add: residue-primroot-def)
    from assms have coprime m g by (simp add: residue-primroot-def)
    hence inj-on (λi. g ^ i mod m) {..<ord m g}
      by (intro inj-power-mod)
    thus inj-on (λi. g ^ i mod m) (totatives (totient m))
      by (rule inj-on-subset)
      (use assms ‹totient m > 1› in ‹auto simp: totatives-less residue-primroot-def›)

    {
      fix i assume i: i ∈ totatives (totient m)
      from ‹coprime m g› and ‹m > 2› show g ^ i mod m ∈ totatives m
        by (intro power-in-totatives) auto
      show residue-primroot m (g ^ i mod m)
        using i ‹m > 2› ‹coprime m g›
      by (auto simp: residue-primroot-def coprime-commute ord-power totatives-def)
    }
    {
      fix x assume x: x ∈ totatives m residue-primroot m x
      then obtain i where i: i < totient m x = (g ^ i mod m)
      using assms residue-primroot-is-generator[of m g] by (auto simp: bij-betw-def)
     from i x ‹m > 2› have i > 0 by (intro Nat.gr0I) (auto simp: residue-primroot-1-iff)
      have totient m div gcd i (totient m) = totient m
       using x i ‹coprime m g› by (auto simp add: residue-primroot-def ord-power)
      hence coprime i (totient m)
      unfolding coprime-iff-gcd-eq-1 using assms by (subst (asm) dvd-div-eq-mult)
```

auto
with *i* ‹*i* > *0*› **have** *i* ∈ *totatives* (*totient m*) **by** (*auto simp*: *totatives-def*)
thus *x* ∈ (λ*i*. *g* ^ *i mod m*) ‘ *totatives* (*totient m*) **using** *i* **by** *auto*
}
qed
qed

It follows from the above statement that any residue ring modulo n that *has* primitive roots has exactly $\varphi(\varphi(n))$ of them.

corollary *card-residue-primroots*:
assumes ∃ *g*. *residue-primroot m g*
shows *card* {*g*∈*totatives m*. *residue-primroot m g*} = *totient* (*totient m*)
proof (*cases m* = *1*)
case [*simp*]: *True*
have {*g* ∈ *totatives m*. *residue-primroot m g*} = {*1*}
by (*auto simp*: *residue-primroot-def*)
thus *?thesis* **by** *simp*
next
case *False*
from *assms* **obtain** *g* **where** *g*: *residue-primroot m g* **by** *auto*
hence *m* ≠ *0* **by** (*intro notI*) *auto*
with ‹*m* ≠ *1*› **have** *m* > *1* **by** *linarith*
from *this g* **have** *bij-betw* (λ*i*. *g* ^ *i mod m*) (*totatives* (*totient m*))
{*g*∈*totatives m*. *residue-primroot m g*}
by (*rule residue-primroot-bij-betw-primroots*)
hence *card* (*totatives* (*totient m*)) = *card* {*g*∈*totatives m*. *residue-primroot m g*}
by (*rule bij-betw-same-card*)
thus *?thesis* **by** (*simp add*: *totient-def*)
qed

corollary *card-residue-primroots′*:
card {*g*∈*totatives m*. *residue-primroot m g*} =
(*if m* ∈ *cyclic-moduli then totient* (*totient m*) *else 0*)
by (*simp add*: *residue-primroot-iff-in-cyclic-moduli* [*symmetric*] *card-residue-primroots*)

As an example, we evaluate λ(122200) using the prime factorisation.

lemma *Carmichael 122200* = *1380*
proof −
have *prime-factorization* (*2*^*3* ∗ *5*^*2* ∗ *13* ∗ *47*) = {#*2*, *2*, *2*, *5*, *5*, *13*, *47*::*nat*#}
by (*intro prime-factorization-eqI*) *auto*
from *eval-Carmichael*[*OF this*] **show** *Carmichael 122200* = *1380*
by (*simp add*: *lcm-nat-def gcd-non-0-nat*)
qed

end

12 Comprehensive number theory

theory *Number-Theory*
imports
Fib
Residues
Eratosthenes
Mod-Exp
Quadratic-Reciprocity
Pocklington
Prime-Powers
Residue-Primitive-Roots
begin

end