Measure and Probability Theory

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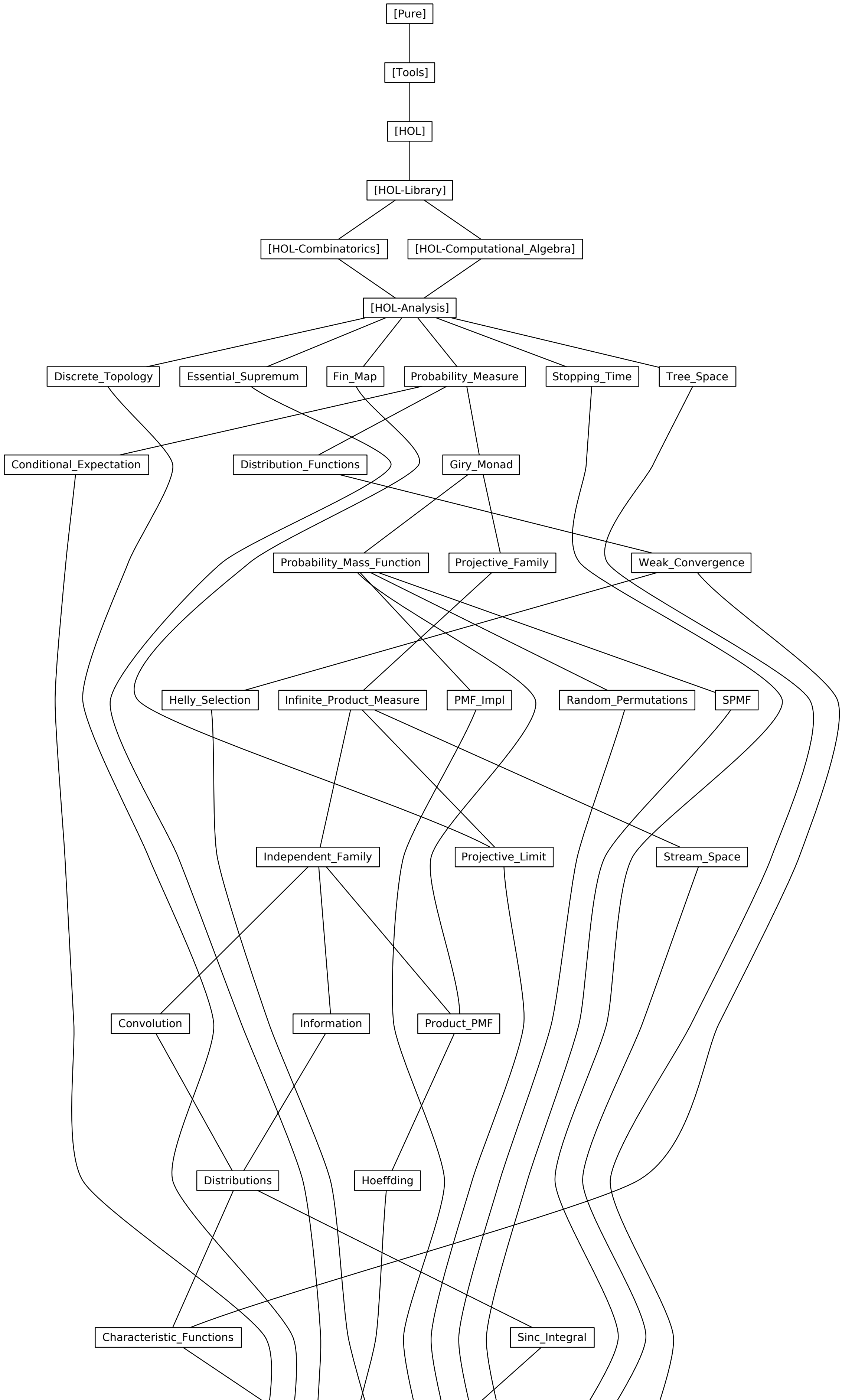
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1 Probability measure

theory *Probability-Measure*
  imports *HOL−Analysis.Analysis*
begin

locale *prob-space* = *finite-measure* +
  assumes *emeasure-space-1*: *emeasure M* (*space M*) = *1*

lemma *prob-spaceI*[*Pure.intro*!]:
  assumes ∗: *emeasure M* (*space M*) = *1*
  shows *prob-space M*
⟨*proof*⟩

lemma *prob-space-imp-sigma-finite*: *prob-space M* ⟹ *sigma-finite-measure M*
  ⟨*proof*⟩

abbreviation (**in** *prob-space*) *events* ≡ *sets M*
abbreviation (**in** *prob-space*) *prob* ≡ *measure M*
abbreviation (**in** *prob-space*) *random-variable* M' *X* ≡ *X* ∈ *measurable M* M'
abbreviation (**in** *prob-space*) *expectation* ≡ $integral^L$ *M*
abbreviation (**in** *prob-space*) *variance X* ≡ $integral^L$ *M* (λ*x*. (*X x* − *expectation* *X*)^2)

lemma (**in** *prob-space*) *finite-measure* [*simp*]: *finite-measure M*
  ⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-distr*:
  assumes *f*: *f* ∈ *measurable M* M' **shows** *prob-space* (*distr M* M' *f*)
⟨*proof*⟩

lemma *prob-space-distrD*:
  assumes *f*: *f* ∈ *measurable M N* **and** *M*: *prob-space* (*distr M N f*) **shows** *prob-space M*
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space*: *prob* (*space M*) = *1*
  ⟨*proof*⟩

lemma (**in** *prob-space*) *prob-le-1*[*simp*, *intro*]: *prob A* ≤ *1*
  ⟨*proof*⟩

lemma (**in** *prob-space*) *not-empty*: *space M* ≠ {}
  ⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-eq-1-AE*:
  S ∈ *sets M* ⟹ *AE x in M*. *x* ∈ *S* ⟹ *emeasure M S* = *1*
  ⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-le-1*: *emeasure M S* $\leq$ *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-ge-1-iff*: *emeasure M A* $\geq$ *1* $\longleftrightarrow$ *emeasure M A* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-iff-emeasure-eq-1*:
assumes [*measurable*]: *Measurable.pred M P*
shows (*AE x in M. P x*) $\longleftrightarrow$ *emeasure M* {*x*∈*space M. P x*} = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-le-1*: *emeasure M X* $\leq$ *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-ge-1-iff*: *measure M A* $\geq$ *1* $\longleftrightarrow$ *measure M A* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-I-eq-1*:
assumes *emeasure M* {*x*∈*space M. P x*} = *1* {*x*∈*space M. P x*} ∈ *sets M*
shows *AE x in M. P x*
⟨*proof*⟩

lemma *prob-space-restrict-space*:
S ∈ *sets M* $\Longrightarrow$ *emeasure M S* = *1* $\Longrightarrow$ *prob-space* (*restrict-space M S*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-compl*:
assumes *A*: *A* ∈ *events*
shows *prob* (*space M* − *A*) = *1* − *prob A*
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-in-set-eq-1*:
assumes *A*[*measurable*]: *A* ∈ *events* **shows** (*AE x in M. x* ∈ *A*) $\longleftrightarrow$ *prob A* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-False*: (*AE x in M. False*) $\longleftrightarrow$ *False*
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-prob-1*:
assumes *prob A* = *1* **shows** *AE x in M. x* ∈ *A*
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-const*[*simp*]: (*AE x in M. P*) $\longleftrightarrow$ *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *ae-filter-bot*: *ae-filter M* $\neq$ *bot*

⟨*proof*⟩

lemma (**in** *prob-space*) *AE-contr*:
assumes *ae*: $AE\ \omega\ in\ M.\ P\ \omega$ $AE\ \omega\ in\ M.\ \neg\ P\ \omega$
shows *False*
⟨*proof*⟩

lemma (**in** *prob-space*) *integral-ge-const*:
fixes c :: *real*
shows $integrable\ M\ f \Longrightarrow (AE\ x\ in\ M.\ c \leq f\ x) \Longrightarrow c \leq (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *integral-le-const*:
fixes c :: *real*
shows $integrable\ M\ f \Longrightarrow (AE\ x\ in\ M.\ f\ x \leq c) \Longrightarrow (\int x.\ f\ x\ \partial M) \leq c$
⟨*proof*⟩

lemma (**in** *prob-space*) *nn-integral-ge-const*:
$(AE\ x\ in\ M.\ c \leq f\ x) \Longrightarrow c \leq (\int^{+} x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *expectation-less*:
fixes X :: - ⇒ *real*
assumes [*simp*]: *integrable M X*
assumes *gt*: $AE\ x\ in\ M.\ X\ x < b$
shows $expectation\ X < b$
⟨*proof*⟩

lemma (**in** *prob-space*) *expectation-greater*:
fixes X :: - ⇒ *real*
assumes [*simp*]: *integrable M X*
assumes *gt*: $AE\ x\ in\ M.\ a < X\ x$
shows $a < expectation\ X$
⟨*proof*⟩

lemma (**in** *prob-space*) *jensens-inequality*:
fixes q :: *real* ⇒ *real*
assumes *X*: $integrable\ M\ X$ $AE\ x\ in\ M.\ X\ x \in I$
assumes *I*: $I = \{a\ {<}..{<}\ b\} \vee I = \{a\ {<}..\} \vee I = \{..{<}\ b\} \vee I = UNIV$
assumes *q*: $integrable\ M\ (\lambda x.\ q\ (X\ x))$ $convex\text{-}on\ I\ q$
shows $q\ (expectation\ X) \leq expectation\ (\lambda x.\ q\ (X\ x))$
⟨*proof*⟩

1.1 Introduce binder for probability

syntax
-prob :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹('𝒫'((/- in -./ -)'))›)

translations

$\mathcal{P}(x \; in \; M. \; P)$ => $CONST \; measure \; M \; \{x \in CONST \; space \; M. \; P\}$

⟨ML⟩

definition
cond-prob M P Q $= \mathcal{P}(\omega \; in \; M. \; P \; \omega \wedge Q \; \omega) \; / \; \mathcal{P}(\omega \; in \; M. \; Q \; \omega)$

syntax
-conditional-prob :: *pttrn* ⇒ *logic* ⇒ *logic* ⇒ *logic* ⇒ *logic* (‹('$\mathcal{P}$'(- in -. - |/ -'))›)

translations
$\mathcal{P}(x \; in \; M. \; P \mid Q)$ => $CONST \; cond\text{-}prob \; M \; (\lambda x. \; P) \; (\lambda x. \; Q)$

lemma (**in** *prob-space*) *AE-E-prob*:
assumes *ae*: *AE x in M. P x*
obtains *S* **where** $S \subseteq \{x \in space \; M. \; P \; x\}$ $S \in events$ $prob \; S = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-neg*: $\{x \in space \; M. \; P \; x\} \in events \Longrightarrow \mathcal{P}(x \; in \; M. \; \neg \; P \; x) = 1 - \mathcal{P}(x \; in \; M. \; P \; x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-AE*:
$(AE \; x \; in \; M. \; P \; x \longleftrightarrow Q \; x) \Longrightarrow \{x \in space \; M. \; P \; x\} \in events \Longrightarrow \{x \in space \; M. \; Q \; x\} \in events \Longrightarrow \mathcal{P}(x \; in \; M. \; P \; x) = \mathcal{P}(x \; in \; M. \; Q \; x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-0-AE*:
assumes *not*: *AE x in M.* ¬ *P x* **shows** $\mathcal{P}(x \; in \; M. \; P \; x) = 0$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-Collect-eq-0*:
$\{x \in space \; M. \; P \; x\} \in sets \; M \Longrightarrow \mathcal{P}(x \; in \; M. \; P \; x) = 0 \longleftrightarrow (AE \; x \; in \; M. \; \neg \; P \; x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-Collect-eq-1*:
$\{x \in space \; M. \; P \; x\} \in sets \; M \Longrightarrow \mathcal{P}(x \; in \; M. \; P \; x) = 1 \longleftrightarrow (AE \; x \; in \; M. \; P \; x)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-0*:
$A \in sets \; M \Longrightarrow prob \; A = 0 \longleftrightarrow (AE \; x \; in \; M. \; x \notin A)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-eq-1*:
$A \in sets \; M \Longrightarrow prob \; A = 1 \longleftrightarrow (AE \; x \; in \; M. \; x \in A)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-sums*:
assumes *P*: $\bigwedge n. \; \{x \in space \; M. \; P \; n \; x\} \in events$

assumes *Q*: $\{x{\in}space\ M.\ Q\ x\} \in events$
assumes *ae*: $AE\ x\ in\ M.\ (\forall n.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists!n.\ P\ n\ x))$
shows $(\lambda n.\ \mathcal{P}(x\ in\ M.\ P\ n\ x))\ sums\ \mathcal{P}(x\ in\ M.\ Q\ x)$
$\langle proof \rangle$

lemma (**in** *prob-space*) *prob-sum*:
assumes [*simp*, *intro*]: *finite I*
assumes *P*: $\bigwedge n.\ n \in I \Longrightarrow \{x{\in}space\ M.\ P\ n\ x\} \in events$
assumes *Q*: $\{x{\in}space\ M.\ Q\ x\} \in events$
assumes *ae*: $AE\ x\ in\ M.\ (\forall n{\in}I.\ P\ n\ x \longrightarrow Q\ x) \wedge (Q\ x \longrightarrow (\exists!n{\in}I.\ P\ n\ x))$
shows $\mathcal{P}(x\ in\ M.\ Q\ x) = (\sum n{\in}I.\ \mathcal{P}(x\ in\ M.\ P\ n\ x))$
$\langle proof \rangle$

lemma (**in** *prob-space*) *prob-EX-countable*:
assumes *sets*: $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}space\ M.\ P\ i\ x\} \in sets\ M$ **and** *I*: *countable I*
assumes *disj*: $AE\ x\ in\ M.\ \forall i{\in}I.\ \forall j{\in}I.\ P\ i\ x \longrightarrow P\ j\ x \longrightarrow i = j$
shows $\mathcal{P}(x\ in\ M.\ \exists i{\in}I.\ P\ i\ x) = (\int^{+} i.\ \mathcal{P}(x\ in\ M.\ P\ i\ x)\ \partial count\text{-}space\ I)$
$\langle proof \rangle$

lemma (**in** *prob-space*) *cond-prob-eq-AE*:
assumes *P*: $AE\ x\ in\ M.\ Q\ x \longrightarrow P\ x \longleftrightarrow P'\ x\ \{x{\in}space\ M.\ P\ x\} \in events$ $\{x{\in}space\ M.\ P'\ x\} \in events$
assumes *Q*: $AE\ x\ in\ M.\ Q\ x \longleftrightarrow Q'\ x\ \{x{\in}space\ M.\ Q\ x\} \in events\ \{x{\in}space$ $M.\ Q'\ x\} \in events$
shows $cond\text{-}prob\ M\ P\ Q = cond\text{-}prob\ M\ P'\ Q'$
$\langle proof \rangle$

lemma (**in** *prob-space*) *joint-distribution-Times-le-fst*:
$random\text{-}variable\ MX\ X \Longrightarrow random\text{-}variable\ MY\ Y \Longrightarrow A \in sets\ MX \Longrightarrow B \in sets\ MY$
$\Longrightarrow emeasure\ (distr\ M\ (MX \bigotimes_M MY)\ (\lambda x.\ (X\ x,\ Y\ x)))\ (A \times B) \leq emeasure\ (distr\ M\ MX\ X)\ A$
$\langle proof \rangle$

lemma (**in** *prob-space*) *joint-distribution-Times-le-snd*:
$random\text{-}variable\ MX\ X \Longrightarrow random\text{-}variable\ MY\ Y \Longrightarrow A \in sets\ MX \Longrightarrow B \in sets\ MY$
$\Longrightarrow emeasure\ (distr\ M\ (MX \bigotimes_M MY)\ (\lambda x.\ (X\ x,\ Y\ x)))\ (A \times B) \leq emeasure\ (distr\ M\ MY\ Y)\ B$
$\langle proof \rangle$

lemma (**in** *prob-space*) *variance-eq*:
fixes $X :: 'a \Rightarrow real$
assumes [*simp*]: *integrable M X*
assumes [*simp*]: $integrable\ M\ (\lambda x.\ (X\ x)^2)$
shows $variance\ X = expectation\ (\lambda x.\ (X\ x)^2) - (expectation\ X)^2$
$\langle proof \rangle$

lemma (**in** *prob-space*) *variance-positive*: $0 \leq variance\ (X::'a \Rightarrow real)$
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-mean-zero*:
$expectation\ X = 0 \Longrightarrow variance\ X = expectation\ (\lambda x.\ (X\ x)\hat{}2)$
⟨*proof*⟩

theorem (**in** *prob-space*) *Chebyshev-inequality*:
assumes [*measurable*]: *random-variable borel f*
assumes *integrable M* $(\lambda x.\ f\ x\ \hat{}\ 2)$
defines $\mu \equiv expectation\ f$
assumes $a > 0$
shows $prob\ \{x{\in}space\ M.\ |f\ x - \mu| \geq a\} \leq variance\ f\ /\ a^2$
unfolding *μ-def*
proof (*rule second-moment-method*)
have *integrable*: *integrable M f*
using *assms* **by** (*blast dest*: *square-integrable-imp-integrable*)
show *integrable M* $(\lambda x.\ (f\ x - expectation\ f)^2)$
using *assms integrable* **unfolding** *power2-eq-square ring-distribs*
by (*intro Bochner-Integration.integrable-diff*) *auto*
qed (*use assms* **in** *auto*)

locale *pair-prob-space* = *pair-sigma-finite M1 M2* + *M1*: *prob-space M1* + *M2*: *prob-space M2* **for** *M1 M2*

sublocale *pair-prob-space* ⊆ *P?*: *prob-space* $M1 \bigotimes_M M2$
⟨*proof*⟩

locale *product-prob-space* = *product-sigma-finite M* **for** $M :: 'i \Rightarrow 'a\ measure$ +
fixes $I :: 'i\ set$
assumes *prob-space*: $\bigwedge i.\ prob\text{-}space\ (M\ i)$

sublocale *product-prob-space* ⊆ *M?*: *prob-space M i* **for** *i*
⟨*proof*⟩

locale *finite-product-prob-space* = *finite-product-sigma-finite M I* + *product-prob-space M I* **for** *M I*

sublocale *finite-product-prob-space* ⊆ *prob-space* $\Pi_M\ i{\in}I.\ M\ i$
⟨*proof*⟩

lemma (**in** *finite-product-prob-space*) *prob-times*:
assumes *X*: $\bigwedge i.\ i \in I \Longrightarrow X\ i \in sets\ (M\ i)$
shows $prob\ (\Pi_E\ i{\in}I.\ X\ i) = (\prod i{\in}I.\ M.prob\ i\ (X\ i))$
⟨*proof*⟩

lemma *product-prob-spaceI*:
assumes $\bigwedge i.\ prob\text{-}space\ (M\ i)$
shows *product-prob-space M*

⟨*proof*⟩

1.2 Distributions

definition *distributed* :: $'a$ *measure* $\Rightarrow$ $'b$ *measure* $\Rightarrow$ $('a \Rightarrow 'b) \Rightarrow ('b \Rightarrow$ *ennreal*$)$ $\Rightarrow$ *bool*
where
distributed M N X f $\longleftrightarrow$
distr M N X = *density N f* $\wedge$ *f* $\in$ *borel-measurable N* $\wedge$ *X* $\in$ *measurable M N*

lemma
assumes *distributed M N X f*
shows *distributed-distr-eq-density*: *distr M N X* = *density N f*
and *distributed-measurable*: *X* $\in$ *measurable M N*
and *distributed-borel-measurable*: *f* $\in$ *borel-measurable N*
⟨*proof*⟩

lemma
assumes *D*: *distributed M N X f*
shows *distributed-measurable'*[*measurable-dest*]:
g $\in$ *measurable L M* $\Longrightarrow$ $(\lambda x.\ X\ (g\ x))$ $\in$ *measurable L N*
and *distributed-borel-measurable'*[*measurable-dest*]:
h $\in$ *measurable L N* $\Longrightarrow$ $(\lambda x.\ f\ (h\ x))$ $\in$ *borel-measurable L*
⟨*proof*⟩

lemma *distributed-real-measurable*:
$(\bigwedge x.\ x \in$ *space N* $\Longrightarrow 0 \leq f\ x) \Longrightarrow$ *distributed M N X* $(\lambda x.$ *ennreal* $(f\ x)) \Longrightarrow f$ $\in$ *borel-measurable N*
⟨*proof*⟩

lemma *distributed-real-measurable'*:
$(\bigwedge x.\ x \in$ *space N* $\Longrightarrow 0 \leq f\ x) \Longrightarrow$ *distributed M N X* $(\lambda x.$ *ennreal* $(f\ x)) \Longrightarrow$
h $\in$ *measurable L N* $\Longrightarrow (\lambda x.\ f\ (h\ x)) \in$ *borel-measurable L*
⟨*proof*⟩

lemma *joint-distributed-measurable1*:
distributed M $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ $f \Longrightarrow$ *h1* $\in$ *measurable N M* $\Longrightarrow (\lambda x.$ X $(h1\ x)) \in$ *measurable N S*
⟨*proof*⟩

lemma *joint-distributed-measurable2*:
distributed M $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ $f \Longrightarrow$ *h2* $\in$ *measurable N M* $\Longrightarrow (\lambda x.$ Y $(h2\ x)) \in$ *measurable N T*
⟨*proof*⟩

lemma *distributed-count-space*:
assumes *X*: *distributed M* (*count-space A*) *X P* **and** *a*: $a \in A$ **and** *A*: *finite A*
shows $P\ a$ = *emeasure M* $(X -\text{`}\ \{a\} \cap$ *space M*$)$
⟨*proof*⟩

lemma *distributed-cong-density*:
$(AE\ x\ in\ N.\ f\ x = g\ x) \Longrightarrow g \in borel\text{-}measurable\ N \Longrightarrow f \in borel\text{-}measurable\ N \Longrightarrow$
$distributed\ M\ N\ X\ f \longleftrightarrow distributed\ M\ N\ X\ g$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-imp-emeasure-nonzero*:
assumes *X*: $distributed\ M\ MX\ X\ Px$
shows $emeasure\ MX\ \{x \in space\ MX.\ Px\ x \neq 0\} \neq 0$
⟨*proof*⟩

lemma *subdensity*:
assumes *T*: $T \in measurable\ P\ Q$
assumes *f*: $distributed\ M\ P\ X\ f$
assumes *g*: $distributed\ M\ Q\ Y\ g$
assumes *Y*: $Y = T \circ X$
shows $AE\ x\ in\ P.\ g\ (T\ x) = 0 \longrightarrow f\ x = 0$
⟨*proof*⟩

lemma *subdensity-real*:
fixes $g :: 'a \Rightarrow real$ **and** $f :: 'b \Rightarrow real$
assumes *T*: $T \in measurable\ P\ Q$
assumes *f*: $distributed\ M\ P\ X\ f$
assumes *g*: $distributed\ M\ Q\ Y\ g$
assumes *Y*: $Y = T \circ X$
shows $(AE\ x\ in\ P.\ 0 \leq g\ (T\ x)) \Longrightarrow (AE\ x\ in\ P.\ 0 \leq f\ x) \Longrightarrow AE\ x\ in\ P.\ g\ (T\ x) = 0 \longrightarrow f\ x = 0$
⟨*proof*⟩

lemma *distributed-emeasure*:
$distributed\ M\ N\ X\ f \Longrightarrow A \in sets\ N \Longrightarrow emeasure\ M\ (X -` A \cap space\ M) = (\int^{+}x.\ f\ x * indicator\ A\ x\ \partial N)$
⟨*proof*⟩

lemma *distributed-nn-integral*:
$distributed\ M\ N\ X\ f \Longrightarrow g \in borel\text{-}measurable\ N \Longrightarrow (\int^{+}x.\ f\ x * g\ x\ \partial N) = (\int^{+}x.\ g\ (X\ x)\ \partial M)$
⟨*proof*⟩

lemma *distributed-integral*:
$distributed\ M\ N\ X\ f \Longrightarrow g \in borel\text{-}measurable\ N \Longrightarrow (\bigwedge x.\ x \in space\ N \Longrightarrow 0 \leq f\ x) \Longrightarrow$
$(\int x.\ f\ x * g\ x\ \partial N) = (\int x.\ g\ (X\ x)\ \partial M)$
⟨*proof*⟩

lemma *distributed-transform-integral*:
assumes *Px*: $distributed\ M\ N\ X\ Px\ \bigwedge x.\ x \in space\ N \Longrightarrow 0 \leq Px\ x$
assumes $distributed\ M\ P\ Y\ Py\ \bigwedge x.\ x \in space\ P \Longrightarrow 0 \leq Py\ x$

assumes Y: $Y = T \circ X$ **and** T: $T \in$ *measurable* N P **and** f: $f \in$ *borel-measurable* P
shows $(\int x.\ Py\ x * f\ x\ \partial P) = (\int x.\ Px\ x * f\ (T\ x)\ \partial N)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-unique*:
assumes Px: *distributed* $M\ S\ X\ Px$
assumes Py: *distributed* $M\ S\ X\ Py$
shows *AE* x *in* S. $Px\ x = Py\ x$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-jointI*:
assumes *sigma-finite-measure* S *sigma-finite-measure* T
assumes X[*measurable*]: $X \in$ *measurable* $M\ S$ **and** Y[*measurable*]: $Y \in$ *measurable* $M\ T$
assumes [*measurable*]: $f \in$ *borel-measurable* $(S \bigotimes_M T)$ **and** f: *AE* x *in* $S \bigotimes_M T$. $0 \le f\ x$
assumes *eq*: $\bigwedge A\ B.\ A \in$ *sets* $S \Longrightarrow B \in$ *sets* $T \Longrightarrow$
emeasure M $\{x \in$ *space* $M.\ X\ x \in A \wedge Y\ x \in B\} = (\int^+ x.\ (\int^+ y.\ f\ (x, y) *$ *indicator* $B\ y\ \partial T) *$ *indicator* $A\ x\ \partial S)$
shows *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ f$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-swap*:
assumes *sigma-finite-measure* S *sigma-finite-measure* T
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *distributed* $M\ (T \bigotimes_M S)\ (\lambda x.\ (Y\ x,\ X\ x))\ (\lambda(x, y).\ Pxy\ (y, x))$
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-marginal1*:
assumes *sigma-finite-measure* S *sigma-finite-measure* T
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
defines $Px \equiv \lambda x.\ (\int^+ z.\ Pxy\ (x, z)\ \partial T)$
shows *distributed* $M\ S\ X\ Px$
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-marginal2*:
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *distributed* $M\ T\ Y\ (\lambda y.\ (\int^+ x.\ Pxy\ (x, y)\ \partial S))$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint1*:
assumes T: *sigma-finite-measure* T
assumes S: *sigma-finite-measure* S
assumes Px: *distributed* $M\ S\ X\ Px$
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *AE* x *in* S. $Px\ x = (\int^+ y.\ Pxy\ (x, y)\ \partial T)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint2*:
  assumes *T*: *sigma-finite-measure T*
  assumes *S*: *sigma-finite-measure S*
  assumes *Py*: *distributed M T Y Py*
  assumes *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
  shows $AE\ y\ in\ T.\ Py\ y = (\int^+ x.\ Pxy\ (x,\ y)\ \partial S)$
  ⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-joint-indep'*:
  assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
  assumes *X*[*measurable*]: *distributed M S X Px* **and** *Y*[*measurable*]: *distributed M T Y Py*
  assumes *indep*: $distr\ M\ S\ X \bigotimes_M distr\ M\ T\ Y = distr\ M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$
  shows $distributed\ M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ (\lambda(x,\ y).\ Px\ x * Py\ y)$
  ⟨*proof*⟩

lemma *distributed-integrable*:
  $distributed\ M\ N\ X\ f \Longrightarrow g \in borel\text{-}measurable\ N \Longrightarrow (\bigwedge x.\ x \in space\ N \Longrightarrow 0 \le f\ x) \Longrightarrow$
    $integrable\ N\ (\lambda x.\ f\ x * g\ x) \longleftrightarrow integrable\ M\ (\lambda x.\ g\ (X\ x))$
  ⟨*proof*⟩

lemma *distributed-transform-integrable*:
  assumes *Px*: $distributed\ M\ N\ X\ Px\ \bigwedge x.\ x \in space\ N \Longrightarrow 0 \le Px\ x$
  assumes $distributed\ M\ P\ Y\ Py\ \bigwedge x.\ x \in space\ P \Longrightarrow 0 \le Py\ x$
  assumes *Y*: $Y = (\lambda x.\ T\ (X\ x))$ **and** *T*: $T \in measurable\ N\ P$ **and** *f*: $f \in borel\text{-}measurable\ P$
  shows $integrable\ P\ (\lambda x.\ Py\ x * f\ x) \longleftrightarrow integrable\ N\ (\lambda x.\ Px\ x * f\ (T\ x))$
⟨*proof*⟩

lemma *distributed-integrable-var*:
  fixes $X :: 'a \Rightarrow real$
  shows $distributed\ M\ lborel\ X\ (\lambda x.\ ennreal\ (f\ x)) \Longrightarrow (\bigwedge x.\ 0 \le f\ x) \Longrightarrow$
    $integrable\ lborel\ (\lambda x.\ f\ x * x) \Longrightarrow integrable\ M\ X$
  ⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-variance*:
  fixes $f::real \Rightarrow real$
  assumes *D*: *distributed M lborel X f* **and** [*simp*]: $\bigwedge x.\ 0 \le f\ x$
  shows $variance\ X = (\int x.\ x^2 * f\ (x + expectation\ X)\ \partial lborel)$
⟨*proof*⟩

lemma (**in** *prob-space*) *variance-affine*:
  fixes $f::real \Rightarrow real$
  assumes [*arith*]: $b \neq 0$
  assumes *D*[*intro*]: *distributed M lborel X f*
  assumes [*simp*]: *prob-space* (*density lborel f*)

assumes $I[simp]$: *integrable* $M\ X$
assumes $I2[simp]$: *integrable* $M\ (\lambda x.\ (X\ x)^2)$
shows *variance* $(\lambda x.\ a + b * X\ x) = b^2 * variance\ X$
⟨*proof*⟩

definition
simple-distributed $M\ X\ f \longleftrightarrow$
$(\forall x.\ 0 \leq f\ x) \land$
distributed M (*count-space* $(X`space\ M))\ X\ (\lambda x.\ ennreal\ (f\ x)) \land$
finite $(X`space\ M)$

lemma *simple-distributed-nonneg*[*dest*]: *simple-distributed* $M\ X\ f \Longrightarrow 0 \leq f\ x$
⟨*proof*⟩

lemma *simple-distributed*:
simple-distributed $M\ X\ Px \Longrightarrow$ *distributed* M (*count-space* $(X`space\ M))\ X\ Px$
⟨*proof*⟩

lemma *simple-distributed-finite*[*dest*]: *simple-distributed* $M\ X\ P \Longrightarrow$ *finite* $(X`space\ M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-simple-function-superset*:
assumes X: *simple-function* $M\ X\ \bigwedge x.\ x \in X\ `\ space\ M \Longrightarrow P\ x = measure\ M\ (X\ -`\ \{x\} \cap space\ M)$
assumes A: $X`space\ M \subseteq A$ *finite* A
defines $S \equiv$ *count-space* A **and** $P' \equiv (\lambda x.\ if\ x \in X`space\ M\ then\ P\ x\ else\ 0)$
shows *distributed* $M\ S\ X\ P'$
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributedI*:
assumes X: *simple-function* $M\ X$
$\bigwedge x.\ 0 \leq P\ x$
$\bigwedge x.\ x \in X\ `\ space\ M \Longrightarrow P\ x = measure\ M\ (X\ -`\ \{x\} \cap space\ M)$
shows *simple-distributed* $M\ X\ P$
⟨*proof*⟩

lemma *simple-distributed-joint-finite*:
assumes X: *simple-distributed* $M\ (\lambda x.\ (X\ x,\ Y\ x))\ Px$
shows *finite* $(X\ `\ space\ M)$ *finite* $(Y\ `\ space\ M)$
⟨*proof*⟩

lemma *simple-distributed-joint2-finite*:
assumes X: *simple-distributed* $M\ (\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))\ Px$
shows *finite* $(X\ `\ space\ M)$ *finite* $(Y\ `\ space\ M)$ *finite* $(Z\ `\ space\ M)$
⟨*proof*⟩

lemma *simple-distributed-simple-function*:
simple-distributed $M\ X\ Px \Longrightarrow$ *simple-function* $M\ X$

⟨*proof*⟩

lemma *simple-distributed-measure*:
simple-distributed M X P $\Longrightarrow$ *a* $\in$ *X‘space M* $\Longrightarrow$ *P a* = *measure M* (*X* −‘ {*a*} $\cap$ *space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-joint*:
assumes *X*: *simple-distributed M* (λx. (*X x*, *Y x*)) *Px*
defines *S* $\equiv$ *count-space* (*X‘space M*) $\bigotimes_M$ *count-space* (*Y‘space M*)
defines *P* $\equiv$ (λx. *if x* $\in$ (λx. (*X x*, *Y x*))‘*space M then Px x else 0*)
shows *distributed M S* (λx. (*X x*, *Y x*)) *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-joint2*:
assumes *X*: *simple-distributed M* (λx. (*X x*, *Y x*, *Z x*)) *Px*
defines *S* $\equiv$ *count-space* (*X‘space M*) $\bigotimes_M$ *count-space* (*Y‘space M*) $\bigotimes_M$ *count-space* (*Z‘space M*)
defines *P* $\equiv$ (λx. *if x* $\in$ (λx. (*X x*, *Y x*, *Z x*))‘*space M then Px x else 0*)
shows *distributed M S* (λx. (*X x*, *Y x*, *Z x*)) *P*
⟨*proof*⟩

lemma (**in** *prob-space*) *simple-distributed-sum-space*:
assumes *X*: *simple-distributed M X f*
shows *sum f* (*X‘space M*) = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-marginal-eq-joint-simple*:
assumes *Px*: *simple-function M X*
assumes *Py*: *simple-distributed M Y Py*
assumes *Pxy*: *simple-distributed M* (λx. (*X x*, *Y x*)) *Pxy*
assumes *y*: *y* $\in$ *Y‘space M*
shows *Py y* = ($\sum$ *x*$\in$*X‘space M*. *if* (*x*, *y*) $\in$ (λx. (*X x*, *Y x*)) ‘ *space M then Pxy* (*x*, *y*) *else 0*)
⟨*proof*⟩

lemma *distributedI-real*:
fixes *f* :: *'a* $\Rightarrow$ *real*
assumes *gen*: *sets M1* = *sigma-sets* (*space M1*) *E* **and** *Int-stable E*
and *A*: *range A* $\subseteq$ *E* ($\bigcup$*i*::*nat*. *A i*) = *space M1* $\bigwedge$*i*. *emeasure* (*distr M M1 X*) (*A i*) $\neq \infty$
and *X*: *X* $\in$ *measurable M M1*
and *f*: *f* $\in$ *borel-measurable M1 AE x in M1*. *0* $\leq$ *f x*
and *eq*: $\bigwedge$*A*. *A* $\in$ *E* $\Longrightarrow$ *emeasure M* (*X* −‘ *A* $\cap$ *space M*) = ($\int^+$ *x*. *f x* * *indicator A x* ∂*M1*)
shows *distributed M M1 X f*
⟨*proof*⟩

lemma *distributedI-borel-atMost*:

fixes f :: *real* $\Rightarrow$ *real*
assumes [*measurable*]: $X \in$ *borel-measurable M*
and [*measurable*]: $f \in$ *borel-measurable borel* **and** f[*simp*]: *AE x in lborel*. $0 \leq f\ x$
and *g-eq*: $\bigwedge a.\ (\int^{+} x.\ f\ x * \mathit{indicator}\ \{..a\}\ x\ \partial lborel) = \mathit{ennreal}\ (g\ a)$
and *M-eq*: $\bigwedge a.\ \mathit{emeasure}\ M\ \{x{\in}\mathit{space}\ M.\ X\ x \leq a\} = \mathit{ennreal}\ (g\ a)$
shows *distributed M lborel X f*
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-params*:
assumes X: *distributed M MX X* ($\lambda x.$ *indicator A x* / *measure MX A*)
shows $A \in$ *sets MX measure MX A* $\neq 0$
⟨*proof*⟩

lemma *prob-space-uniform-measure*:
assumes A: *emeasure M A* $\neq 0$ *emeasure M A* $\neq \infty$
shows *prob-space* (*uniform-measure M A*)
⟨*proof*⟩

lemma *prob-space-uniform-count-measure*: *finite A* $\Longrightarrow A \neq \{\} \Longrightarrow$ *prob-space* (*uniform-count-measure A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *measure-uniform-measure-eq-cond-prob*:
assumes [*measurable*]: *Measurable.pred M P Measurable.pred M Q*
shows $\mathcal{P}$(*x in uniform-measure M* $\{x{\in}\mathit{space}\ M.\ Q\ x\}$. $P\ x$) = $\mathcal{P}$(*x in M*. $P\ x \mid Q\ x$)
⟨*proof*⟩

lemma *prob-space-point-measure*:
finite S $\Longrightarrow (\bigwedge s.\ s \in S \Longrightarrow 0 \leq p\ s) \Longrightarrow (\sum s{\in}S.\ p\ s) = 1 \Longrightarrow$ *prob-space* (*point-measure S p*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distr-pair-fst*: *distr* ($N \bigotimes_M M$) *N fst* = N
⟨*proof*⟩

lemma (**in** *product-prob-space*) *distr-reorder*:
assumes *inj-on t J* $t \in J \rightarrow K$ *finite K*
shows *distr* (*PiM K M*) (Pi_M J ($\lambda x.\ M\ (t\ x)$)) ($\lambda\omega.\ \lambda n{\in}J.\ \omega\ (t\ n)$) = *PiM J* ($\lambda x.\ M\ (t\ x)$)
⟨*proof*⟩

lemma (**in** *product-prob-space*) *distr-restrict*:
$J \subseteq K \Longrightarrow$ *finite K* $\Longrightarrow (\Pi_M\ i{\in}J.\ M\ i) = \mathit{distr}\ (\Pi_M\ i{\in}K.\ M\ i)\ (\Pi_M\ i{\in}J.\ M\ i)\ (\lambda f.\ \mathit{restrict}\ f\ J)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-prod-emb*[*simp*]:

assumes L: $J \subseteq L$ *finite* L **and** X: $X \in$ *sets* $(Pi_M\ J\ M)$
shows *emeasure* $(Pi_M\ L\ M)$ (*prod-emb* $L\ M\ J\ X$) = *emeasure* $(Pi_M\ J\ M)$ X
⟨*proof*⟩

lemma *emeasure-distr-restrict*:
assumes $I \subseteq K$ **and** Q[*measurable-cong*]: *sets* Q = *sets* (*PiM* K M) **and** A[*measurable*]: $A \in$ *sets* (*PiM* I M)
shows *emeasure* (*distr* Q (*PiM* I M) ($\lambda\omega$. *restrict* ω I)) A = *emeasure* Q (*prod-emb* K M I A)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-completion*: *prob-space* (*completion* M)
⟨*proof*⟩

lemma *distr-PiM-finite-prob-space*:
assumes *fin*: *finite* I
assumes *product-prob-space* M
assumes *product-prob-space* M'
assumes [*measurable*]: $\bigwedge i.\ i \in I \Longrightarrow f \in$ *measurable* $(M\ i)$ $(M'\ i)$
shows *distr* (*PiM* I M) (*PiM* I M') (*compose* I f) = *PiM* I (λi. *distr* $(M\ i)$ $(M'\ i)$ f)
⟨*proof*⟩

end

2 Distribution Functions

Shows that the cumulative distribution function (cdf) of a distribution (a measure on the reals) is nondecreasing and right continuous, which tends to 0 and 1 in either direction.

Conversely, every such function is the cdf of a unique distribution. This direction defines the measure in the obvious way on half-open intervals, and then applies the Caratheodory extension theorem.

theory *Distribution-Functions*
imports *Probability-Measure*
begin

lemma *UN-Ioc-eq-UNIV*: $(\bigcup n.\ \{ -real\ n <..\ real\ n\}) = UNIV$
⟨*proof*⟩

2.1 Properties of cdf's

definition
cdf :: *real measure* ⇒ *real* ⇒ *real*
where
cdf $M \equiv \lambda x.$ *measure* M $\{..x\}$

lemma *cdf-def2*: $cdf\ M\ x = measure\ M\ \{..x\}$
⟨*proof*⟩

locale *finite-borel-measure* = *finite-measure M* **for** $M :: real\ measure$ +
assumes *M-is-borel*: $sets\ M = sets\ borel$
begin

lemma *sets-M*[*intro*]: $a \in sets\ borel \Longrightarrow a \in sets\ M$
⟨*proof*⟩

lemma *cdf-diff-eq*:
assumes $x < y$
shows $cdf\ M\ y - cdf\ M\ x = measure\ M\ \{x<..y\}$
⟨*proof*⟩

lemma *cdf-nondecreasing*: $x \le y \Longrightarrow cdf\ M\ x \le cdf\ M\ y$
⟨*proof*⟩

lemma *borel-UNIV*: $space\ M = UNIV$
⟨*proof*⟩

lemma *cdf-nonneg*: $cdf\ M\ x \ge 0$
⟨*proof*⟩

lemma *cdf-bounded*: $cdf\ M\ x \le measure\ M\ (space\ M)$
⟨*proof*⟩

lemma *cdf-lim-infty*:
$((\lambda i.\ cdf\ M\ (real\ i)) \longlongrightarrow measure\ M\ (space\ M))$
⟨*proof*⟩

lemma *cdf-lim-at-top*: $(cdf\ M \longlongrightarrow measure\ M\ (space\ M))\ at\text{-}top$
⟨*proof*⟩

lemma *cdf-lim-neg-infty*: $((\lambda i.\ cdf\ M\ (-\ real\ i)) \longlongrightarrow 0)$
⟨*proof*⟩

lemma *cdf-lim-at-bot*: $(cdf\ M \longlongrightarrow 0)\ at\text{-}bot$
⟨*proof*⟩

lemma *cdf-is-right-cont*: $continuous\ (at\text{-}right\ a)\ (cdf\ M)$
⟨*proof*⟩

lemma *cdf-at-left*: $(cdf\ M \longlongrightarrow measure\ M\ \{..<a\})\ (at\text{-}left\ a)$
⟨*proof*⟩

lemma *isCont-cdf*: $isCont\ (cdf\ M)\ x \longleftrightarrow measure\ M\ \{x\} = 0$
⟨*proof*⟩

lemma *countable-atoms*: *countable* $\{x.\ measure\ M\ \{x\} > 0\}$
⟨*proof*⟩

end

locale *real-distribution* = *prob-space M* **for** *M* :: *real measure* +
assumes *events-eq-borel* [*simp*, *measurable-cong*]: *sets M* = *sets borel*
begin

lemma *finite-borel-measure-M*: *finite-borel-measure M*
⟨*proof*⟩

sublocale *finite-borel-measure M*
⟨*proof*⟩

lemma *space-eq-univ* [*simp*]: *space M* = *UNIV*
⟨*proof*⟩

lemma *cdf-bounded-prob*: $\bigwedge x.\ cdf\ M\ x \le 1$
⟨*proof*⟩

lemma *cdf-lim-infty-prob*: $(\lambda i.\ cdf\ M\ (real\ i)) \longrightarrow 1$
⟨*proof*⟩

lemma *cdf-lim-at-top-prob*: $(cdf\ M \longrightarrow 1)\ at\text{-}top$
⟨*proof*⟩

lemma *measurable-finite-borel* [*simp*]:
$f \in borel\text{-}measurable\ borel \Longrightarrow f \in borel\text{-}measurable\ M$
⟨*proof*⟩

end

lemma (**in** *prob-space*) *real-distribution-distr* [*intro*, *simp*]:
random-variable borel X $\Longrightarrow$ *real-distribution* (*distr M borel X*)
⟨*proof*⟩

2.2 Uniqueness

lemma (**in** *finite-borel-measure*) *emeasure-Ioc*:
assumes $a \le b$ **shows** $emeasure\ M\ \{a <..\ b\} = cdf\ M\ b - cdf\ M\ a$
⟨*proof*⟩

lemma *cdf-unique'*:
fixes *M1 M2*
assumes *finite-borel-measure M1* **and** *finite-borel-measure M2*
assumes *cdf M1* = *cdf M2*
shows *M1* = *M2*
⟨*proof*⟩

lemma *cdf-unique*:
$real\text{-}distribution\ M1 \Longrightarrow real\text{-}distribution\ M2 \Longrightarrow cdf\ M1 = cdf\ M2 \Longrightarrow M1 = M2$
⟨*proof*⟩

lemma
fixes $F :: real \Rightarrow real$
assumes *nondecF* : $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$
and *right-cont-F* : $\bigwedge a.\ continuous\ (at\text{-}right\ a)\ F$
and *lim-F-at-bot* : $(F \longrightarrow 0)\ at\text{-}bot$
and *lim-F-at-top* : $(F \longrightarrow m)\ at\text{-}top$
and *m*: $0 \leq m$
shows *interval-measure-UNIV*: $emeasure\ (interval\text{-}measure\ F)\ UNIV = m$
and *finite-borel-measure-interval-measure*: $finite\text{-}borel\text{-}measure\ (interval\text{-}measure\ F)$
⟨*proof*⟩

lemma *real-distribution-interval-measure*:
fixes $F :: real \Rightarrow real$
assumes *nondecF* : $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$ **and**
right-cont-F : $\bigwedge a.\ continuous\ (at\text{-}right\ a)\ F$ **and**
lim-F-at-bot : $(F \longrightarrow 0)\ at\text{-}bot$ **and**
lim-F-at-top : $(F \longrightarrow 1)\ at\text{-}top$
shows $real\text{-}distribution\ (interval\text{-}measure\ F)$
⟨*proof*⟩

lemma
fixes $F :: real \Rightarrow real$
assumes *nondecF* : $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$ **and**
right-cont-F : $\bigwedge a.\ continuous\ (at\text{-}right\ a)\ F$ **and**
lim-F-at-bot : $(F \longrightarrow 0)\ at\text{-}bot$
shows *emeasure-interval-measure-Iic*: $emeasure\ (interval\text{-}measure\ F)\ \{..\ x\} = F\ x$
and *measure-interval-measure-Iic*: $measure\ (interval\text{-}measure\ F)\ \{..\ x\} = F\ x$
⟨*proof*⟩

lemma *cdf-interval-measure*:
$(\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y) \Longrightarrow (\bigwedge a.\ continuous\ (at\text{-}right\ a)\ F) \Longrightarrow (F \longrightarrow 0)\ at\text{-}bot \Longrightarrow cdf\ (interval\text{-}measure\ F) = F$
⟨*proof*⟩

end

3 Weak Convergence of Functions and Distributions

Properties of weak convergence of functions and measures, including the portmanteau theorem.

theory *Weak-Convergence*
  imports *Distribution-Functions*
begin

4 Weak Convergence of Functions

definition
  $\mathit{weak\text{-}conv} :: (\mathit{nat} \Rightarrow (\mathit{real} \Rightarrow \mathit{real})) \Rightarrow (\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{bool}$
where
  $\mathit{weak\text{-}conv}\ \mathit{F\text{-}seq}\ F \equiv \forall x.\ \mathit{isCont}\ F\ x \longrightarrow (\lambda n.\ \mathit{F\text{-}seq}\ n\ x) \longlonglongrightarrow F\ x$

5 Weak Convergence of Distributions

definition
  $\mathit{weak\text{-}conv\text{-}m} :: (\mathit{nat} \Rightarrow \mathit{real\ measure}) \Rightarrow \mathit{real\ measure} \Rightarrow \mathit{bool}$
where
  $\mathit{weak\text{-}conv\text{-}m}\ \mathit{M\text{-}seq}\ M \equiv \mathit{weak\text{-}conv}\ (\lambda n.\ \mathit{cdf}\ (\mathit{M\text{-}seq}\ n))\ (\mathit{cdf}\ M)$

6 Skorohod's theorem

locale *right-continuous-mono* =
  fixes $f :: \mathit{real} \Rightarrow \mathit{real}$ **and** $a\ b :: \mathit{real}$
  assumes *cont*: $\bigwedge x.\ \mathit{continuous}\ (\mathit{at\text{-}right}\ x)\ f$
  assumes *mono*: *mono f*
  assumes *bot*: $(f \longlonglongrightarrow a)\ \mathit{at\text{-}bot}$
  assumes *top*: $(f \longlonglongrightarrow b)\ \mathit{at\text{-}top}$
begin

abbreviation $I :: \mathit{real} \Rightarrow \mathit{real}$ **where**
  $I\ \omega \equiv \mathit{Inf}\ \{x.\ \omega \le f\ x\}$

lemma *pseudoinverse*: **assumes** $a < \omega$ $\omega < b$ **shows** $\omega \le f\ x \longleftrightarrow I\ \omega \le x$
⟨*proof*⟩

lemma *pseudoinverse'*: $\forall \omega \in \{a<..<b\}.\ \forall x.\ \omega \le f\ x \longleftrightarrow I\ \omega \le x$
  ⟨*proof*⟩

lemma *mono-I*: $\mathit{mono\text{-}on}\ I\ \{a <..< b\}$
  ⟨*proof*⟩

end

locale *cdf-distribution* = *real-distribution*
begin

abbreviation $C \equiv$ *cdf M*

sublocale *right-continuous-mono C 0 1*
⟨*proof*⟩

lemma *measurable-C*[*measurable*]: $C \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *measurable-CI*[*measurable*]: $I \in$ *borel-measurable* (*restrict-space borel* {*0*<..<*1*})
⟨*proof*⟩

lemma *emeasure-distr-I*: *emeasure* (*distr* (*restrict-space lborel* {*0*<..<*1*::*real*}) *borel* *I*) *UNIV* = *1*
⟨*proof*⟩

lemma *distr-I-eq-M*: *distr* (*restrict-space lborel* {*0*<..<*1*::*real*}) *borel I* = *M* (**is** *?I* = -)
⟨*proof*⟩

end

context
fixes μ :: *nat* ⇒ *real measure*
and *M* :: *real measure*
assumes μ: ⋀*n*. *real-distribution* (μ *n*)
assumes *M*: *real-distribution M*
assumes μ*-to-M*: *weak-conv-m* μ *M*
begin

theorem *Skorohod*:
∃ (Ω :: *real measure*) (*Y-seq* :: *nat* ⇒ *real* ⇒ *real*) (*Y* :: *real* ⇒ *real*).
prob-space Ω ∧
(∀ *n*. *Y-seq n* ∈ *measurable* Ω *borel*) ∧
(∀ *n*. *distr* Ω *borel* (*Y-seq n*) = μ *n*) ∧
Y ∈ *measurable* Ω *lborel* ∧
distr Ω *borel Y* = *M* ∧
(∀ *x* ∈ *space* Ω. (λ*n*. *Y-seq n x*) ⟶ *Y x*)
⟨*proof*⟩

The Portmanteau theorem, that is, the equivalence of various definitions of weak convergence.

theorem *weak-conv-imp-bdd-ae-continuous-conv*:
fixes
f :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*}

assumes
discont-null: M ($\{x.\ \neg$ *isCont f x*$\}$) $= 0$ **and**
f-bdd: $\bigwedge x.$ *norm* $(f\ x) \leq B$ **and**
[*measurable*]: $f \in$ *borel-measurable borel*
shows
$(\lambda\ n.$ *integral*^{L} $(\mu\ n)\ f) \longrightarrow$ *integral*^{L} $M\ f$
⟨*proof*⟩

theorem *weak-conv-imp-integral-bdd-continuous-conv*:
fixes f :: *real* $\Rightarrow$ *'a*::{*banach*, *second-countable-topology*}
assumes
$\bigwedge x.$ *isCont f x* **and**
$\bigwedge x.$ *norm* $(f\ x) \leq B$
shows
$(\lambda\ n.$ *integral*^{L} $(\mu\ n)\ f) \longrightarrow$ *integral*^{L} $M\ f$
⟨*proof*⟩

theorem *weak-conv-imp-continuity-set-conv*:
fixes f :: *real* $\Rightarrow$ *real*
assumes [*measurable*]: $A \in$ *sets borel* **and** M (*frontier* A) $= 0$
shows $(\lambda n.$ *measure* $(\mu\ n)\ A) \longrightarrow$ *measure* $M\ A$
⟨*proof*⟩

end

definition
cts-step :: *real* $\Rightarrow$ *real* $\Rightarrow$ *real* $\Rightarrow$ *real*
where
cts-step $a\ b\ x \equiv$ *if* $x \leq a$ *then* 1 *else if* $x \geq b$ *then* 0 *else* $(b - x) / (b - a)$

lemma *cts-step-uniformly-continuous*:
assumes [*arith*]: $a < b$
shows *uniformly-continuous-on UNIV* (*cts-step* $a\ b$)
⟨*proof*⟩

lemma (**in** *real-distribution*) *integrable-cts-step*: $a < b \Longrightarrow$ *integrable* M (*cts-step* $a\ b$)
⟨*proof*⟩

lemma (**in** *real-distribution*) *cdf-cts-step*:
assumes [*arith*]: $x < y$
shows *cdf* $M\ x \leq$ *integral*^{L} M (*cts-step* $x\ y$) **and** *integral*^{L} M (*cts-step* $x\ y$) $\leq$ *cdf* $M\ y$
⟨*proof*⟩

context
fixes *M-seq* :: *nat* $\Rightarrow$ *real measure*
and M :: *real measure*
assumes *distr-M-seq* [*simp*]: $\bigwedge n.$ *real-distribution* (*M-seq* n)

assumes *distr-M* [*simp*]: *real-distribution M*
begin

theorem *continuity-set-conv-imp-weak-conv*:
fixes $f :: real \Rightarrow real$
assumes $*$: $\bigwedge A.\ A \in sets\ borel \Longrightarrow M\ (frontier\ A) = 0 \Longrightarrow (\lambda\ n.\ (measure\ (M\text{-}seq\ n)\ A)) \longrightarrow measure\ M\ A$
shows *weak-conv-m M-seq M*
⟨*proof*⟩

theorem *integral-cts-step-conv-imp-weak-conv*:
assumes *integral-conv*: $\bigwedge x\ y.\ x < y \Longrightarrow (\lambda n.\ integral^L\ (M\text{-}seq\ n)\ (cts\text{-}step\ x\ y)) \longrightarrow integral^L\ M\ (cts\text{-}step\ x\ y)$
shows *weak-conv-m M-seq M*
⟨*proof*⟩

theorem *integral-bdd-continuous-conv-imp-weak-conv*:
assumes
$\bigwedge f.\ (\bigwedge x.\ isCont\ f\ x) \Longrightarrow (\bigwedge x.\ abs\ (f\ x) \le 1) \Longrightarrow (\lambda n.\ integral^L\ (M\text{-}seq\ n)\ f{::}real) \longrightarrow integral^L\ M\ f$
shows
weak-conv-m M-seq M
⟨*proof*⟩

end

end

7 The Giry monad

theory *Giry-Monad*
imports *Probability-Measure HOL−Library.Monad-Syntax*
begin

7.1 Sub-probability spaces

locale *subprob-space* = *finite-measure* +
assumes *emeasure-space-le-1*: $emeasure\ M\ (space\ M) \le 1$
assumes *subprob-not-empty*: $space\ M \neq \{\}$

lemma *subprob-spaceI*[*Pure.intro*!]:
assumes $*$: $emeasure\ M\ (space\ M) \le 1$
assumes $space\ M \neq \{\}$
shows *subprob-space M*
⟨*proof*⟩

lemma (**in** *subprob-space*) *emeasure-subprob-space-less-top*: $emeasure\ M\ A \neq top$
⟨*proof*⟩

lemma *prob-space-imp-subprob-space*:
  prob-space $M \Longrightarrow$ *subprob-space* M
  ⟨*proof*⟩

lemma *subprob-space-imp-sigma-finite*: *subprob-space* $M \Longrightarrow$ *sigma-finite-measure* M
  ⟨*proof*⟩

sublocale *prob-space* $\subseteq$ *subprob-space*
  ⟨*proof*⟩

lemma *subprob-space-sigma* [*simp*]: $\Omega \neq \{\} \Longrightarrow$ *subprob-space* (*sigma* Ω X)
⟨*proof*⟩

lemma *subprob-space-null-measure*: *space* $M \neq \{\} \Longrightarrow$ *subprob-space* (*null-measure* M)
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-space-distr*:
  assumes *f*: $f \in$ *measurable* M M' **and** *space* $M' \neq \{\}$ **shows** *subprob-space* (*distr* M M' f)
⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-emeasure-le-1*: *emeasure* M $X \leq 1$
  ⟨*proof*⟩

lemma (**in** *subprob-space*) *subprob-measure-le-1*: *measure* M $X \leq 1$
  ⟨*proof*⟩

lemma (**in** *subprob-space*) *nn-integral-le-const*:
  assumes $0 \leq c$ *AE* x *in* M. $f\ x \leq c$
  shows $(\int^+ x.\ f\ x\ \partial M) \leq c$
⟨*proof*⟩

lemma *emeasure-density-distr-interval*:
  fixes h :: *real* $\Rightarrow$ *real* **and** g :: *real* $\Rightarrow$ *real* **and** g' :: *real* $\Rightarrow$ *real*
  assumes [*simp*]: $a \leq b$
  assumes *Mf*[*measurable*]: $f \in$ *borel-measurable borel*
  assumes *Mg*[*measurable*]: $g \in$ *borel-measurable borel*
  assumes *Mg'*[*measurable*]: $g' \in$ *borel-measurable borel*
  assumes *Mh*[*measurable*]: $h \in$ *borel-measurable borel*
  assumes *prob*: *subprob-space* (*density lborel f*)
  assumes *nonnegf*: $\bigwedge x.\ f\ x \geq 0$
  assumes *derivg*: $\bigwedge x.\ x \in \{a..b\} \Longrightarrow$ (g *has-real-derivative* g' x) (*at* x)
  assumes *contg'*: *continuous-on* $\{a..b\}$ g'
  assumes *mono*: *strict-mono-on* g $\{a..b\}$ **and** *inv*: $\bigwedge x.\ h\ x \in \{a..b\} \Longrightarrow g\ (h\ x) = x$
  assumes *range*: $\{a..b\} \subseteq$ *range* h
  shows *emeasure* (*distr* (*density lborel f*) *lborel h*) $\{a..b\} =$

emeasure (*density lborel* ($\lambda x.$ *f* (*g x*) $*$ *g′ x*)) {*a..b*}
⟨*proof*⟩

locale *pair-subprob-space* =
pair-sigma-finite M1 M2 + *M1*: *subprob-space M1* + *M2*: *subprob-space M2* **for** *M1 M2*

sublocale *pair-subprob-space* $\subseteq$ *P?*: *subprob-space M1* $\bigotimes_M$ *M2*
⟨*proof*⟩

lemma *subprob-space-null-measure-iff*:
subprob-space (*null-measure M*) $\longleftrightarrow$ *space M* $\neq$ {}
⟨*proof*⟩

lemma *subprob-space-restrict-space*:
assumes *M*: *subprob-space M*
and *A*: *A* $\cap$ *space M* $\in$ *sets M A* $\cap$ *space M* $\neq$ {}
shows *subprob-space* (*restrict-space M A*)
⟨*proof*⟩

definition *subprob-algebra* :: *′a measure* $\Rightarrow$ *′a measure measure* **where**
subprob-algebra K =
(*SUP A* $\in$ *sets K*. *vimage-algebra* {*M*. *subprob-space M* $\wedge$ *sets M* = *sets K*} ($\lambda M.$ *emeasure M A*) *borel*)

lemma *space-subprob-algebra*: *space* (*subprob-algebra A*) = {*M*. *subprob-space M* $\wedge$ *sets M* = *sets A*}
⟨*proof*⟩

lemma *subprob-algebra-cong*: *sets M* = *sets N* $\Longrightarrow$ *subprob-algebra M* = *subprob-algebra N*
⟨*proof*⟩

lemma *measurable-emeasure-subprob-algebra*[*measurable*]:
a $\in$ *sets A* $\Longrightarrow$ ($\lambda M.$ *emeasure M a*) $\in$ *borel-measurable* (*subprob-algebra A*)
⟨*proof*⟩

lemma *measurable-measure-subprob-algebra*[*measurable*]:
a $\in$ *sets A* $\Longrightarrow$ ($\lambda M.$ *measure M a*) $\in$ *borel-measurable* (*subprob-algebra A*)
⟨*proof*⟩

lemma *subprob-measurableD*:
assumes *N*: *N* $\in$ *measurable M* (*subprob-algebra S*) **and** *x*: *x* $\in$ *space M*
shows *space* (*N x*) = *space S*
and *sets* (*N x*) = *sets S*
and *measurable* (*N x*) *K* = *measurable S K*
and *measurable K* (*N x*) = *measurable K S*
⟨*proof*⟩

⟨*ML*⟩

context
  fixes K M N **assumes** K: $K \in$ *measurable* M (*subprob-algebra* N)
begin

lemma *subprob-space-kernel*: $a \in$ *space* $M \Longrightarrow$ *subprob-space* $(K\ a)$
  ⟨*proof*⟩

lemma *sets-kernel*: $a \in$ *space* $M \Longrightarrow$ *sets* $(K\ a) =$ *sets* N
  ⟨*proof*⟩

lemma *measurable-emeasure-kernel*[*measurable*]:
    $A \in$ *sets* $N \Longrightarrow (\lambda a.$ *emeasure* $(K\ a)\ A) \in$ *borel-measurable* M
  ⟨*proof*⟩

end

lemma *measurable-subprob-algebra*:
  $(\bigwedge a.\ a \in$ *space* $M \Longrightarrow$ *subprob-space* $(K\ a)) \Longrightarrow$
  $(\bigwedge a.\ a \in$ *space* $M \Longrightarrow$ *sets* $(K\ a) =$ *sets* $N) \Longrightarrow$
  $(\bigwedge A.\ A \in$ *sets* $N \Longrightarrow (\lambda a.$ *emeasure* $(K\ a)\ A) \in$ *borel-measurable* $M) \Longrightarrow$
  $K \in$ *measurable* M (*subprob-algebra* N)
  ⟨*proof*⟩

lemma *measurable-submarkov*:
  $K \in$ *measurable* M (*subprob-algebra* M) $\longleftrightarrow$
    $(\forall x \in$ *space* M. *subprob-space* $(K\ x) \wedge$ *sets* $(K\ x) =$ *sets* $M) \wedge$
    $(\forall A \in$ *sets* M. $(\lambda x.$ *emeasure* $(K\ x)\ A) \in$ *measurable* M *borel*$)$
⟨*proof*⟩

lemma *measurable-subprob-algebra-generated*:
  assumes *eq*: *sets* $N =$ *sigma-sets* Ω G **and** *Int-stable* G $G \subseteq$ *Pow* Ω
  assumes *subsp*: $\bigwedge a.\ a \in$ *space* $M \Longrightarrow$ *subprob-space* $(K\ a)$
  assumes *sets*: $\bigwedge a.\ a \in$ *space* $M \Longrightarrow$ *sets* $(K\ a) =$ *sets* N
  assumes $\bigwedge A.\ A \in G \Longrightarrow (\lambda a.$ *emeasure* $(K\ a)\ A) \in$ *borel-measurable* M
  assumes Ω: $(\lambda a.$ *emeasure* $(K\ a)\ \Omega) \in$ *borel-measurable* M
  shows $K \in$ *measurable* M (*subprob-algebra* N)
⟨*proof*⟩

lemma *space-subprob-algebra-empty-iff*:
  space (*subprob-algebra* N) $= \{\} \longleftrightarrow$ *space* $N = \{\}$
⟨*proof*⟩

lemma *nn-integral-measurable-subprob-algebra*[*measurable*]:
  assumes f: $f \in$ *borel-measurable* N
  shows $(\lambda M.\ integral^N\ M\ f) \in$ *borel-measurable* (*subprob-algebra* N) (**is** - $\in$ *?B*)
  ⟨*proof*⟩

lemma *measurable-distr*:
assumes [*measurable*]: $f \in$ *measurable* M N
shows ($\lambda M'$. *distr* M' N f) $\in$ *measurable* (*subprob-algebra* M) (*subprob-algebra* N)
⟨*proof*⟩

lemma *emeasure-space-subprob-algebra*[*measurable*]:
(λa. *emeasure* a (*space* a)) $\in$ *borel-measurable* (*subprob-algebra* N)
⟨*proof*⟩

lemma *integrable-measurable-subprob-algebra*[*measurable*]:
fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes [*measurable*]: $f \in$ *borel-measurable* N
shows *Measurable.pred* (*subprob-algebra* N) (λM. *integrable* M f)
⟨*proof*⟩

lemma *integral-measurable-subprob-algebra*[*measurable*]:
fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes f [*measurable*]: $f \in$ *borel-measurable* N
shows (λM. *integral*^{L} M f) $\in$ *subprob-algebra* $N \rightarrow_M$ *borel*
⟨*proof*⟩

lemma *measurable-pair-measure*:
assumes f: $f \in$ *measurable* M (*subprob-algebra* N)
assumes g: $g \in$ *measurable* M (*subprob-algebra* L)
shows (λx. f $x \bigotimes_M g$ x) $\in$ *measurable* M (*subprob-algebra* ($N \bigotimes_M L$))
⟨*proof*⟩

lemma *restrict-space-measurable*:
assumes X: $X \neq \{\}$ $X \in$ *sets* K
assumes N: $N \in$ *measurable* M (*subprob-algebra* K)
shows (λx. *restrict-space* (N x) X) $\in$ *measurable* M (*subprob-algebra* (*restrict-space* K X))
⟨*proof*⟩

7.2 Properties of "return"

definition *return* :: $'a$ *measure* $\Rightarrow 'a \Rightarrow 'a$ *measure* **where**
return R x = *measure-of* (*space* R) (*sets* R) (λA. *indicator* A x)

lemma *space-return*[*simp*]: *space* (*return* M x) = *space* M
⟨*proof*⟩

lemma *sets-return*[*simp*]: *sets* (*return* M x) = *sets* M
⟨*proof*⟩

lemma *measurable-return1*[*simp*]: *measurable* (*return* N x) L = *measurable* N L
⟨*proof*⟩

lemma *measurable-return2*[*simp*]: *measurable L* (*return N x*) = *measurable L N*
⟨*proof*⟩

lemma *return-sets-cong*: *sets M* = *sets N* ⟹ *return M* = *return N*
⟨*proof*⟩

lemma *return-cong*: *sets A* = *sets B* ⟹ *return A x* = *return B x*
⟨*proof*⟩

lemma *emeasure-return*[*simp*]:
assumes $A \in$ *sets M*
shows *emeasure* (*return M x*) *A* = *indicator A x*
⟨*proof*⟩

lemma *prob-space-return*: $x \in$ *space M* ⟹ *prob-space* (*return M x*)
⟨*proof*⟩

lemma *subprob-space-return*: $x \in$ *space M* ⟹ *subprob-space* (*return M x*)
⟨*proof*⟩

lemma *subprob-space-return-ne*:
assumes *space M* ≠ {} **shows** *subprob-space* (*return M x*)
⟨*proof*⟩

lemma *measure-return*: **assumes** *X*: $X \in$ *sets M* **shows** *measure* (*return M x*) *X* = *indicator X x*
⟨*proof*⟩

lemma *AE-return*:
assumes [*simp*]: $x \in$ *space M* **and** [*measurable*]: *Measurable.pred M P*
shows (*AE y in return M x. P y*) ⟷ *P x*
⟨*proof*⟩

lemma *nn-integral-return*:
assumes $x \in$ *space M* $g \in$ *borel-measurable M*
shows ($\int^+$ *a. g a* ∂*return M x*) = *g x*
⟨*proof*⟩

lemma *integral-return*:
fixes *g* :: - ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes $x \in$ *space M* $g \in$ *borel-measurable M*
shows ($\int$ *a. g a* ∂*return M x*) = *g x*
⟨*proof*⟩

lemma *return-measurable*[*measurable*]: *return N* ∈ *measurable N* (*subprob-algebra N*)
⟨*proof*⟩

lemma *distr-return*:
assumes $f \in$ *measurable* M N **and** $x \in$ *space* M
shows *distr* (*return* M x) N f = *return* N (f x)
⟨*proof*⟩

lemma *return-restrict-space*:
$\Omega \in$ *sets* $M \Longrightarrow$ *return* (*restrict-space* M Ω) x = *restrict-space* (*return* M x) Ω
⟨*proof*⟩

lemma *measurable-distr2*:
assumes f[*measurable*]: *case-prod* $f \in$ *measurable* ($L \bigotimes_M M$) N
assumes g[*measurable*]: $g \in$ *measurable* L (*subprob-algebra* M)
shows (λx. *distr* (g x) N (f x)) $\in$ *measurable* L (*subprob-algebra* N)
⟨*proof*⟩

lemma *nn-integral-measurable-subprob-algebra2*:
assumes f[*measurable*]: ($\lambda(x, y)$. f x y) $\in$ *borel-measurable* ($M \bigotimes_M N$)
assumes N[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *integral*^{N} (L x) (f x)) $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *emeasure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *emeasure* (L x) (A x)) $\in$ *borel-measurable* M
⟨*proof*⟩

lemma *measure-measurable-subprob-algebra2*:
assumes A[*measurable*]: (*SIGMA* x:*space* M. A x) $\in$ *sets* ($M \bigotimes_M N$)
assumes L[*measurable*]: $L \in$ *measurable* M (*subprob-algebra* N)
shows (λx. *measure* (L x) (A x)) $\in$ *borel-measurable* M
⟨*proof*⟩

definition *select-sets* M = (*SOME* N. *sets* M = *sets* (*subprob-algebra* N))

lemma *select-sets1*:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *sets* M = *sets* (*subprob-algebra* (*select-sets* M))
⟨*proof*⟩

lemma *sets-select-sets*[*simp*]:
assumes *sets*: *sets* M = *sets* (*subprob-algebra* N)
shows *sets* (*select-sets* M) = *sets* N
⟨*proof*⟩

lemma *space-select-sets*[*simp*]:
sets M = *sets* (*subprob-algebra* N) $\Longrightarrow$ *space* (*select-sets* M) = *space* N
⟨*proof*⟩

7.3 Join

definition *join* :: $'a$ *measure measure* $\Rightarrow$ $'a$ *measure* **where**
join M = *measure-of* (*space* (*select-sets M*)) (*sets* (*select-sets M*)) (λB. $\int^{+}$ M'. *emeasure* M' *B* ∂M)

lemma
shows *space-join*[*simp*]: *space* (*join M*) = *space* (*select-sets M*)
and *sets-join*[*simp*]: *sets* (*join M*) = *sets* (*select-sets M*)
⟨*proof*⟩

lemma *emeasure-join*:
assumes *M*[*simp*, *measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*) **and** *A*: $A \in$ *sets N*
shows *emeasure* (*join M*) *A* = ($\int^{+}$ M'. *emeasure* M' *A* ∂M)
⟨*proof*⟩

lemma *measurable-join*:
join $\in$ *measurable* (*subprob-algebra* (*subprob-algebra N*)) (*subprob-algebra N*)
⟨*proof*⟩

lemma *nn-integral-join*:
assumes *f*: $f \in$ *borel-measurable N*
and *M*[*measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*)
shows ($\int^{+}x$. *f x* ∂*join M*) = ($\int^{+}M'$. $\int^{+}x$. *f x* $\partial M'$ ∂M)
⟨*proof*⟩

lemma *measurable-join1*:
⟦ $f \in$ *measurable N K*; *sets M* = *sets* (*subprob-algebra N*) ⟧
$\Longrightarrow$ $f \in$ *measurable* (*join M*) *K*
⟨*proof*⟩

lemma
fixes *f* :: - $\Rightarrow$ *real*
assumes *f-measurable* [*measurable*]: $f \in$ *borel-measurable N*
and *f-bounded*: $\bigwedge x.\ x \in$ *space N* $\Longrightarrow$ $|f\ x| \le B$
and *M* [*measurable-cong*]: *sets M* = *sets* (*subprob-algebra N*)
and *fin*: *finite-measure M*
and *M-bounded*: *AE* M' *in M*. *emeasure* M' (*space* M') $\le$ *ennreal* B'
shows *integrable-join*: *integrable* (*join M*) *f* (**is** *?integrable*)
and *integral-join*: *integral*^{L} (*join M*) *f* = $\int$ M'. *integral*^{L} M' *f* ∂M (**is** *?integral*)
⟨*proof*⟩

lemma *join-assoc*:
assumes *M*[*measurable-cong*]: *sets M* = *sets* (*subprob-algebra* (*subprob-algebra N*))
shows *join* (*distr M* (*subprob-algebra N*) *join*) = *join* (*join M*)
⟨*proof*⟩

lemma *join-return*:

assumes *sets M = sets N* **and** *subprob-space M*
shows *join (return (subprob-algebra N) M) = M*
⟨*proof*⟩

lemma *join-return'*:
assumes *sets N = sets M*
shows *join (distr M (subprob-algebra N) (return N)) = M*
⟨*proof*⟩

lemma *join-distr-distr*:
fixes *f :: 'a ⇒ 'b* **and** *M :: 'a measure measure* **and** *N :: 'b measure*
assumes *sets M = sets (subprob-algebra R)* **and** *f ∈ measurable R N*
shows *join (distr M (subprob-algebra N) (λM. distr M N f)) = distr (join M) N f* (**is** *?r = ?l*)
⟨*proof*⟩

definition *bind :: 'a measure ⇒ ('a ⇒ 'b measure) ⇒ 'b measure* **where**
bind M f = (if space M = {} then count-space {} else
join (distr M (subprob-algebra (f (SOME x. x ∈ space M))) f))

adhoc-overloading *Monad-Syntax.bind bind*

lemma *bind-empty*:
space M = {} ⟹ bind M f = count-space {}
⟨*proof*⟩

lemma *bind-nonempty*:
space M ≠ {} ⟹ bind M f = join (distr M (subprob-algebra (f (SOME x. x ∈ space M))) f)
⟨*proof*⟩

lemma *sets-bind-empty*: *sets M = {} ⟹ sets (bind M f) = {{}}*
⟨*proof*⟩

lemma *space-bind-empty*: *space M = {} ⟹ space (bind M f) = {}*
⟨*proof*⟩

lemma *sets-bind[simp, measurable-cong]*:
assumes *f*: *⋀x. x ∈ space M ⟹ sets (f x) = sets N* **and** *M*: *space M ≠ {}*
shows *sets (bind M f) = sets N*
⟨*proof*⟩

lemma *space-bind[simp]*:
assumes *⋀x. x ∈ space M ⟹ sets (f x) = sets N* **and** *space M ≠ {}*
shows *space (bind M f) = space N*
⟨*proof*⟩

lemma *bind-cong-All*:
assumes *∀x ∈ space M. f x = g x*

shows $bind\ M\ f = bind\ M\ g$
$\langle proof \rangle$

lemma *bind-cong*:
$M = N \Longrightarrow (\bigwedge x.\ x \in space\ M \Longrightarrow f\ x = g\ x) \Longrightarrow bind\ M\ f = bind\ N\ g$
$\langle proof \rangle$

lemma *bind-nonempty'*:
assumes $f \in measurable\ M\ (subprob\text{-}algebra\ N)\ x \in space\ M$
shows $bind\ M\ f = join\ (distr\ M\ (subprob\text{-}algebra\ N)\ f)$
$\langle proof \rangle$

lemma *bind-nonempty''*:
assumes $f \in measurable\ M\ (subprob\text{-}algebra\ N)\ space\ M \neq \{\}$
shows $bind\ M\ f = join\ (distr\ M\ (subprob\text{-}algebra\ N)\ f)$
$\langle proof \rangle$

lemma *emeasure-bind*:
$[\![space\ M \neq \{\}; f \in measurable\ M\ (subprob\text{-}algebra\ N); X \in sets\ N]\!]$
$\Longrightarrow emeasure\ (M \gg= f)\ X = \int^{+} x.\ emeasure\ (f\ x)\ X\ \partial M$
$\langle proof \rangle$

lemma *nn-integral-bind*:
assumes f: $f \in borel\text{-}measurable\ B$
assumes N: $N \in measurable\ M\ (subprob\text{-}algebra\ B)$
shows $(\int^{+} x.\ f\ x\ \partial(M \gg= N)) = (\int^{+} x.\ \int^{+} y.\ f\ y\ \partial N\ x\ \partial M)$
$\langle proof \rangle$

lemma *AE-bind*:
assumes $N[measurable]$: $N \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes $P[measurable]$: $Measurable.pred\ B\ P$
shows $(AE\ x\ in\ M \gg= N.\ P\ x) \longleftrightarrow (AE\ x\ in\ M.\ AE\ y\ in\ N\ x.\ P\ y)$
$\langle proof \rangle$

lemma *measurable-bind'*:
assumes $M1$: $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and**
$M2$: $case\text{-}prod\ g \in measurable\ (M \bigotimes_M N)\ (subprob\text{-}algebra\ R)$
shows $(\lambda x.\ bind\ (f\ x)\ (g\ x)) \in measurable\ M\ (subprob\text{-}algebra\ R)$
$\langle proof \rangle$

lemma *measurable-bind*[*measurable* (*raw*)]:
assumes $M1$: $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and**
$M2$: $(\lambda x.\ g\ (fst\ x)\ (snd\ x)) \in measurable\ (M \bigotimes_M N)\ (subprob\text{-}algebra\ R)$
shows $(\lambda x.\ bind\ (f\ x)\ (g\ x)) \in measurable\ M\ (subprob\text{-}algebra\ R)$
$\langle proof \rangle$

lemma *measurable-bind2*:
assumes $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and** $g \in measurable\ N\ (subprob\text{-}algebra\ R)$

shows $(\lambda x.\ bind\ (f\ x)\ g) \in measurable\ M\ (subprob\text{-}algebra\ R)$
⟨*proof*⟩

lemma *subprob-space-bind*:
assumes *subprob-space M* $f \in measurable\ M\ (subprob\text{-}algebra\ N)$
shows *subprob-space* $(M \gg= f)$
⟨*proof*⟩

lemma
fixes $f :: - \Rightarrow real$
assumes *f-measurable* [*measurable*]: $f \in borel\text{-}measurable\ K$
and *f-bounded*: $\bigwedge x.\ x \in space\ K \Longrightarrow |f\ x| \leq B$
and *N* [*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ K)$
and *fin*: *finite-measure M*
and *M-bounded*: $AE\ x\ in\ M.\ emeasure\ (N\ x)\ (space\ (N\ x)) \leq ennreal\ B'$
shows *integrable-bind*: *integrable* (*bind M N*) *f* (**is** *?integrable*)
and *integral-bind*: $integral^L\ (bind\ M\ N)\ f = \int\ x.\ integral^L\ (N\ x)\ f\ \partial M$ (**is** *?integral*)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-bind*:
assumes *ae*: $AE\ x\ in\ M.\ prob\text{-}space\ (N\ x)$
and *N*[*measurable*]: $N \in measurable\ M\ (subprob\text{-}algebra\ S)$
shows *prob-space* $(M \gg= N)$
⟨*proof*⟩

lemma (**in** *subprob-space*) *bind-in-space*:
$A \in measurable\ M\ (subprob\text{-}algebra\ N) \Longrightarrow (M \gg= A) \in space\ (subprob\text{-}algebra\ N)$
⟨*proof*⟩

lemma (**in** *subprob-space*) *measure-bind*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and** *X*: $X \in sets\ N$
shows $measure\ (M \gg= f)\ X = \int x.\ measure\ (f\ x)\ X\ \partial M$
⟨*proof*⟩

lemma *emeasure-bind-const*:
$space\ M \neq \{\} \Longrightarrow X \in sets\ N \Longrightarrow subprob\text{-}space\ N \Longrightarrow$
$emeasure\ (M \gg= (\lambda x.\ N))\ X = emeasure\ N\ X * emeasure\ M\ (space\ M)$
⟨*proof*⟩

lemma *emeasure-bind-const'*:
assumes *subprob-space M subprob-space N*
shows $emeasure\ (M \gg= (\lambda x.\ N))\ X = emeasure\ N\ X * emeasure\ M\ (space\ M)$
⟨*proof*⟩

lemma *emeasure-bind-const-prob-space*:
assumes *prob-space M subprob-space N*
shows $emeasure\ (M \gg= (\lambda x.\ N))\ X = emeasure\ N\ X$

⟨*proof*⟩

lemma *bind-return*:
assumes $f \in$ *measurable* M (*subprob-algebra* N) **and** $x \in$ *space* M
shows *bind* (*return* M x) $f = f\ x$
⟨*proof*⟩

lemma *bind-return'*:
shows *bind* M (*return* M) $= M$
⟨*proof*⟩

lemma *distr-bind*:
assumes N: $N \in$ *measurable* M (*subprob-algebra* K) *space* $M \neq \{\}$
assumes f: $f \in$ *measurable* K R
shows *distr* ($M \gg= N$) R f = ($M \gg=$ ($\lambda x.$ *distr* ($N\ x$) R f))
⟨*proof*⟩

lemma *bind-distr*:
assumes f[*measurable*]: $f \in$ *measurable* M X
assumes N[*measurable*]: $N \in$ *measurable* X (*subprob-algebra* K) **and** *space* $M \neq$ $\{\}$
shows (*distr* M X $f \gg= N$) = ($M \gg=$ ($\lambda x.$ N ($f\ x$)))
⟨*proof*⟩

lemma *bind-count-space-singleton*:
assumes *subprob-space* ($f\ x$)
shows *count-space* $\{x\} \gg= f = f\ x$
⟨*proof*⟩

lemma *restrict-space-bind*:
assumes N: $N \in$ *measurable* M (*subprob-algebra* K)
assumes *space* $M \neq \{\}$
assumes X[*simp*]: $X \in$ *sets* K $X \neq \{\}$
shows *restrict-space* (*bind* M N) X = *bind* M ($\lambda x.$ *restrict-space* ($N\ x$) X)
⟨*proof*⟩

lemma *bind-restrict-space*:
assumes A: $A \cap$ *space* $M \neq \{\}$ $A \cap$ *space* $M \in$ *sets* M
and f: $f \in$ *measurable* (*restrict-space* M A) (*subprob-algebra* N)
shows *restrict-space* M $A \gg= f = M \gg=$ ($\lambda x.$ *if* $x \in A$ *then* $f\ x$ *else null-measure* (f (*SOME* $x.$ $x \in A \wedge x \in$ *space* M)))
(**is** *?lhs* = *?rhs* **is** - = $M \gg=$ *?f*)
⟨*proof*⟩

lemma *bind-const'*: $[\![$*prob-space* M; *subprob-space* $N]\!] \Longrightarrow M \gg= (\lambda x.\ N) = N$
⟨*proof*⟩

lemma *bind-return-distr*:
space $M \neq \{\} \Longrightarrow f \in$ *measurable* M $N \Longrightarrow$ *bind* M (*return* $N \circ f$) = *distr* M

$N\ f$
$\langle proof \rangle$

lemma *bind-return-distr'*:
$space\ M \neq \{\} \Longrightarrow f \in measurable\ M\ N \Longrightarrow bind\ M\ (\lambda x.\ return\ N\ (f\ x)) = distr\ M\ N\ f$
$\langle proof \rangle$

lemma *bind-assoc*:
fixes $f :: 'a \Rightarrow 'b\ measure$ **and** $g :: 'b \Rightarrow 'c\ measure$
assumes $M1$: $f \in measurable\ M\ (subprob\text{-}algebra\ N)$ **and** $M2$: $g \in measurable\ N\ (subprob\text{-}algebra\ R)$
shows $bind\ (bind\ M\ f)\ g = bind\ M\ (\lambda x.\ bind\ (f\ x)\ g)$
$\langle proof \rangle$

lemma *double-bind-assoc*:
assumes Mg: $g \in measurable\ N\ (subprob\text{-}algebra\ N')$
assumes Mf: $f \in measurable\ M\ (subprob\text{-}algebra\ M')$
assumes Mh: $case\text{-}prod\ h \in measurable\ (M \bigotimes_M M')\ N$
shows $do\ \{x \leftarrow M;\ y \leftarrow f\ x;\ g\ (h\ x\ y)\} = do\ \{x \leftarrow M;\ y \leftarrow f\ x;\ return\ N\ (h\ x\ y)\} \gg= g$
$\langle proof \rangle$

lemma (**in** *prob-space*) *M-in-subprob*[*measurable* (*raw*)]: $M \in space\ (subprob\text{-}algebra\ M)$
$\langle proof \rangle$

lemma (**in** *pair-prob-space*) *pair-measure-eq-bind*:
$(M1 \bigotimes_M M2) = (M1 \gg= (\lambda x.\ M2 \gg= (\lambda y.\ return\ (M1 \bigotimes_M M2)\ (x,\ y))))$
$\langle proof \rangle$

lemma (**in** *pair-prob-space*) *bind-rotate*:
assumes C[*measurable*]: $(\lambda(x,\ y).\ C\ x\ y) \in measurable\ (M1 \bigotimes_M M2)\ (subprob\text{-}algebra\ N)$
shows $(M1 \gg= (\lambda x.\ M2 \gg= (\lambda y.\ C\ x\ y))) = (M2 \gg= (\lambda y.\ M1 \gg= (\lambda x.\ C\ x\ y)))$
$\langle proof \rangle$

lemma *bind-return''*: $sets\ M = sets\ N \Longrightarrow M \gg= return\ N = M$
$\langle proof \rangle$

lemma (**in** *prob-space*) *distr-const*[*simp*]:
$c \in space\ N \Longrightarrow distr\ M\ N\ (\lambda x.\ c) = return\ N\ c$
$\langle proof \rangle$

lemma *return-count-space-eq-density*:
$return\ (count\text{-}space\ M)\ x = density\ (count\text{-}space\ M)\ (indicator\ \{x\})$
$\langle proof \rangle$

lemma *null-measure-in-space-subprob-algebra* [*simp*]:
$\mathit{null\text{-}measure}\ M \in \mathit{space}\ (\mathit{subprob\text{-}algebra}\ M) \longleftrightarrow \mathit{space}\ M \neq \{\}$
⟨*proof*⟩

7.4 Giry monad on probability spaces

definition *prob-algebra* :: $'a\ \mathit{measure} \Rightarrow 'a\ \mathit{measure}\ \mathit{measure}$ **where**
$\mathit{prob\text{-}algebra}\ K = \mathit{restrict\text{-}space}\ (\mathit{subprob\text{-}algebra}\ K)\ \{M.\ \mathit{prob\text{-}space}\ M\}$

lemma *space-prob-algebra*: $\mathit{space}\ (\mathit{prob\text{-}algebra}\ M) = \{N.\ \mathit{sets}\ N = \mathit{sets}\ M \land \mathit{prob\text{-}space}\ N\}$
⟨*proof*⟩

lemma *measurable-measure-prob-algebra*[*measurable*]:
$a \in \mathit{sets}\ A \Longrightarrow (\lambda M.\ \mathit{Sigma\text{-}Algebra.measure}\ M\ a) \in \mathit{prob\text{-}algebra}\ A \rightarrow_M \mathit{borel}$
⟨*proof*⟩

lemma *measurable-prob-algebraD*:
$f \in N \rightarrow_M \mathit{prob\text{-}algebra}\ M \Longrightarrow f \in N \rightarrow_M \mathit{subprob\text{-}algebra}\ M$
⟨*proof*⟩

lemma *measure-measurable-prob-algebra2*:
$\mathit{Sigma}\ (\mathit{space}\ M)\ A \in \mathit{sets}\ (M \bigotimes_M N) \Longrightarrow L \in M \rightarrow_M \mathit{prob\text{-}algebra}\ N \Longrightarrow$
$(\lambda x.\ \mathit{Sigma\text{-}Algebra.measure}\ (L\ x)\ (A\ x)) \in \mathit{borel\text{-}measurable}\ M$
⟨*proof*⟩

lemma *measurable-prob-algebraI*:
$(\bigwedge x.\ x \in \mathit{space}\ N \Longrightarrow \mathit{prob\text{-}space}\ (f\ x)) \Longrightarrow f \in N \rightarrow_M \mathit{subprob\text{-}algebra}\ M \Longrightarrow f \in N \rightarrow_M \mathit{prob\text{-}algebra}\ M$
⟨*proof*⟩

lemma *measurable-distr-prob-space*:
assumes f: $f \in M \rightarrow_M N$
shows $(\lambda M'.\ \mathit{distr}\ M'\ N\ f) \in \mathit{prob\text{-}algebra}\ M \rightarrow_M \mathit{prob\text{-}algebra}\ N$
⟨*proof*⟩

lemma *measurable-return-prob-space*[*measurable*]: $\mathit{return}\ N \in N \rightarrow_M \mathit{prob\text{-}algebra}\ N$
⟨*proof*⟩

lemma *measurable-distr-prob-space2*[*measurable* (*raw*)]:
assumes f: $g \in L \rightarrow_M \mathit{prob\text{-}algebra}\ M$ $(\lambda(x, y).\ f\ x\ y) \in L \bigotimes_M M \rightarrow_M N$
shows $(\lambda x.\ \mathit{distr}\ (g\ x)\ N\ (f\ x)) \in L \rightarrow_M \mathit{prob\text{-}algebra}\ N$
⟨*proof*⟩

lemma *measurable-bind-prob-space*:
assumes f: $f \in M \rightarrow_M \mathit{prob\text{-}algebra}\ N$ **and** g: $g \in N \rightarrow_M \mathit{prob\text{-}algebra}\ R$
shows $(\lambda x.\ \mathit{bind}\ (f\ x)\ g) \in M \rightarrow_M \mathit{prob\text{-}algebra}\ R$
⟨*proof*⟩

lemma *measurable-bind-prob-space2*[*measurable* (*raw*)]:
assumes *f*: $f \in M \rightarrow_M$ *prob-algebra N* **and** *g*: $(\lambda(x, y).\ g\ x\ y) \in (M \bigotimes_M N) \rightarrow_M$ *prob-algebra R*
shows $(\lambda x.\ bind\ (f\ x)\ (g\ x)) \in M \rightarrow_M$ *prob-algebra R*
⟨*proof*⟩

lemma *measurable-prob-algebra-generated*:
assumes *eq*: *sets N* = *sigma-sets* Ω *G* **and** *Int-stable G* $G \subseteq$ *Pow* Ω
assumes *subsp*: $\bigwedge a.\ a \in$ *space M* $\Longrightarrow$ *prob-space* (*K a*)
assumes *sets*: $\bigwedge a.\ a \in$ *space M* $\Longrightarrow$ *sets* (*K a*) = *sets N*
assumes $\bigwedge A.\ A \in G \Longrightarrow (\lambda a.\ emeasure\ (K\ a)\ A) \in$ *borel-measurable M*
shows $K \in$ *measurable M* (*prob-algebra N*)
⟨*proof*⟩

lemma *in-space-prob-algebra*:
$x \in$ *space* (*prob-algebra M*) $\Longrightarrow$ *emeasure x* (*space M*) = *1*
⟨*proof*⟩

lemma *prob-space-pair*:
assumes *prob-space M prob-space N* **shows** *prob-space* $(M \bigotimes_M N)$
⟨*proof*⟩

lemma *measurable-pair-prob*[*measurable*]:
$f \in M \rightarrow_M$ *prob-algebra N* $\Longrightarrow g \in M \rightarrow_M$ *prob-algebra L* $\Longrightarrow (\lambda x.\ f\ x \bigotimes_M g\ x) \in M \rightarrow_M$ *prob-algebra* $(N \bigotimes_M L)$
⟨*proof*⟩

lemma *emeasure-bind-prob-algebra*:
assumes *A*: $A \in$ *space* (*prob-algebra N*)
assumes *B*: $B \in N \rightarrow_M$ *prob-algebra L*
assumes *X*: $X \in$ *sets L*
shows *emeasure* (*bind A B*) *X* = $(\int^+ x.\ emeasure\ (B\ x)\ X\ \partial A)$
⟨*proof*⟩

lemma *prob-space-bind'*:
assumes *A*: $A \in$ *space* (*prob-algebra M*) **and** *B*: $B \in M \rightarrow_M$ *prob-algebra N*
shows *prob-space* $(A \gg= B)$
⟨*proof*⟩

lemma *sets-bind'*:
assumes *A*: $A \in$ *space* (*prob-algebra M*) **and** *B*: $B \in M \rightarrow_M$ *prob-algebra N*
shows *sets* $(A \gg= B)$ = *sets N*
⟨*proof*⟩

lemma *bind-cong-AE'*:
assumes *M*: $M \in$ *space* (*prob-algebra L*)
and *f*: $f \in L \rightarrow_M$ *prob-algebra N* **and** *g*: $g \in L \rightarrow_M$ *prob-algebra N*

and *ae*: $AE\ x\ in\ M.\ f\ x = g\ x$
shows $bind\ M\ f = bind\ M\ g$
$\langle proof \rangle$

lemma *density-discrete*:
$countable\ A \Longrightarrow sets\ N = Set.Pow\ A \Longrightarrow (\bigwedge x.\ f\ x \geq 0) \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x = emeasure\ N\ \{x\}) \Longrightarrow$
$density\ (count\text{-}space\ A)\ f = N$
$\langle proof \rangle$

lemma *distr-density-discrete*:
fixes f'
assumes $countable\ A$
assumes $f' \in borel\text{-}measurable\ M$
assumes $g \in measurable\ M\ (count\text{-}space\ A)$
defines $f \equiv \lambda x.\ \int^{+} t.\ (if\ g\ t = x\ then\ 1\ else\ 0) * f'\ t\ \partial M$
assumes $\bigwedge x.\ \ x \in space\ M \Longrightarrow g\ x \in A$
shows $density\ (count\text{-}space\ A)\ (\lambda x.\ f\ x) = distr\ (density\ M\ f')\ (count\text{-}space\ A)\ g$
$\langle proof \rangle$

lemma *bind-cong-AE*:
assumes $M = N$
assumes *f*: $f \in measurable\ N\ (subprob\text{-}algebra\ B)$
assumes *g*: $g \in measurable\ N\ (subprob\text{-}algebra\ B)$
assumes *ae*: $AE\ x\ in\ N.\ f\ x = g\ x$
shows $bind\ M\ f = bind\ N\ g$
$\langle proof \rangle$

lemma *bind-cong-simp*: $M = N \Longrightarrow (\bigwedge x.\ x{\in}space\ M =simp=> f\ x = g\ x) \Longrightarrow bind\ M\ f = bind\ N\ g$
$\langle proof \rangle$

lemma *sets-bind-measurable*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes *M*: $space\ M \neq \{\}$
shows $sets\ (M \gg= f) = sets\ B$
$\langle proof \rangle$

lemma *space-bind-measurable*:
assumes *f*: $f \in measurable\ M\ (subprob\text{-}algebra\ B)$
assumes *M*: $space\ M \neq \{\}$
shows $space\ (M \gg= f) = space\ B$
$\langle proof \rangle$

lemma *bind-distr-return*:
$f \in M \rightarrow_M N \Longrightarrow g \in N \rightarrow_M L \Longrightarrow space\ M \neq \{\} \Longrightarrow$
$distr\ M\ N\ f \gg= (\lambda x.\ return\ L\ (g\ x)) = distr\ M\ L\ (\lambda x.\ g\ (f\ x))$
$\langle proof \rangle$

lemma (**in** *prob-space*) *AE-eq-constD*:
assumes $AE\ x\ in\ M.\ x = y$
shows $M = return\ M\ y\ y \in space\ M$
⟨*proof*⟩

end

8 Projective Family

theory *Projective-Family*
imports *Giry-Monad*
begin

lemma *vimage-restrict-preseve-mono*:
assumes J: $J \subseteq I$
and *sets*: $A \subseteq (\Pi_E\ i{\in}J.\ S\ i)$ $B \subseteq (\Pi_E\ i{\in}J.\ S\ i)$ **and** *ne*: $(\Pi_E\ i{\in}I.\ S\ i) \neq \{\}$
and *eq*: $(\lambda x.\ restrict\ x\ J) -` A \cap (\Pi_E\ i{\in}I.\ S\ i) \subseteq (\lambda x.\ restrict\ x\ J) -` B \cap (\Pi_E\ i{\in}I.\ S\ i)$
shows $A \subseteq B$
⟨*proof*⟩

locale *projective-family* =
fixes $I :: 'i\ set$ **and** $P :: 'i\ set \Rightarrow ('i \Rightarrow 'a)\ measure$ **and** $M :: 'i \Rightarrow 'a\ measure$
assumes P: $\bigwedge J\ H.\ J \subseteq H \Longrightarrow finite\ H \Longrightarrow H \subseteq I \Longrightarrow P\ J = distr\ (P\ H)\ (PiM\ J\ M)\ (\lambda f.\ restrict\ f\ J)$
assumes *prob-space-P*: $\bigwedge J.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow prob\text{-}space\ (P\ J)$
begin

lemma *sets-P*: $finite\ J \Longrightarrow J \subseteq I \Longrightarrow sets\ (P\ J) = sets\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *space-P*: $finite\ J \Longrightarrow J \subseteq I \Longrightarrow space\ (P\ J) = space\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *not-empty-M*: $i \in I \Longrightarrow space\ (M\ i) \neq \{\}$
⟨*proof*⟩

lemma *not-empty*: $space\ (PiM\ I\ M) \neq \{\}$
⟨*proof*⟩

abbreviation
$emb\ L\ K \equiv prod\text{-}emb\ L\ M\ K$

lemma *emb-preserve-mono*:
assumes $J \subseteq L\ L \subseteq I$ **and** *sets*: $X \in sets\ (Pi_M\ J\ M)$ $Y \in sets\ (Pi_M\ J\ M)$
assumes $emb\ L\ J\ X \subseteq emb\ L\ J\ Y$
shows $X \subseteq Y$
⟨*proof*⟩

lemma *emb-injective*:
assumes L: $J \subseteq L$ $L \subseteq I$ **and** X: $X \in sets\ (Pi_M\ J\ M)$ **and** Y: $Y \in sets\ (Pi_M\ J\ M)$
shows $emb\ L\ J\ X = emb\ L\ J\ Y \Longrightarrow X = Y$
⟨*proof*⟩

lemma *emeasure-P*: $J \subseteq K \Longrightarrow finite\ K \Longrightarrow K \subseteq I \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow P\ K\ (emb\ K\ J\ X) = P\ J\ X$
⟨*proof*⟩

inductive-set *generator* :: $('i \Rightarrow 'a)\ set\ set$ **where**
$finite\ J \Longrightarrow J \subseteq I \Longrightarrow X \in sets\ (Pi_M\ J\ M) \Longrightarrow emb\ I\ J\ X \in generator$

lemma *algebra-generator*: $algebra\ (space\ (PiM\ I\ M))\ generator$
⟨*proof*⟩

interpretation *generator*: $algebra\ space\ (PiM\ I\ M)\ generator$
⟨*proof*⟩

lemma *sets-PiM-generator*: $sets\ (PiM\ I\ M) = sigma\text{-}sets\ (space\ (PiM\ I\ M))\ generator$
⟨*proof*⟩

definition *mu-G* (μG) **where**
$\mu G\ A = (THE\ x.\ \forall J \subseteq I.\ finite\ J \longrightarrow (\forall X \in sets\ (Pi_M\ J\ M).\ A = emb\ I\ J\ X \longrightarrow x = emeasure\ (P\ J)\ X))$

definition *lim* :: $('i \Rightarrow 'a)\ measure$ **where**
$lim = extend\text{-}measure\ (space\ (PiM\ I\ M))\ generator\ (\lambda x.\ x)\ \mu G$

lemma *space-lim*[*simp*]: $space\ lim = space\ (PiM\ I\ M)$
⟨*proof*⟩

lemma *sets-lim*[*simp*, *measurable*]: $sets\ lim = sets\ (PiM\ I\ M)$
⟨*proof*⟩

lemma *mu-G-spec*:
assumes J: $finite\ J$ $J \subseteq I$ $X \in sets\ (Pi_M\ J\ M)$
shows $\mu G\ (emb\ I\ J\ X) = emeasure\ (P\ J)\ X$
⟨*proof*⟩

lemma *positive-mu-G*: $positive\ generator\ \mu G$
⟨*proof*⟩

lemma *additive-mu-G*: $additive\ generator\ \mu G$
⟨*proof*⟩

lemma *emeasure-lim*:
assumes JX: $finite\ J$ $J \subseteq I$ $X \in sets\ (PiM\ J\ M)$

assumes *cont*: $\bigwedge J\ X.\ (\bigwedge i.\ J\ i \subseteq I) \Longrightarrow incseq\ J \Longrightarrow (\bigwedge i.\ finite\ (J\ i)) \Longrightarrow (\bigwedge i.\ X\ i \in sets\ (PiM\ (J\ i)\ M)) \Longrightarrow$
$decseq\ (\lambda i.\ emb\ I\ (J\ i)\ (X\ i)) \Longrightarrow 0 < (INF\ i.\ P\ (J\ i)\ (X\ i)) \Longrightarrow (\bigcap i.\ emb\ I\ (J\ i)\ (X\ i)) \neq \{\}$
shows $emeasure\ lim\ (emb\ I\ J\ X) = P\ J\ X$
⟨*proof*⟩

end

sublocale $product\text{-}prob\text{-}space \subseteq projective\text{-}family\ I\ \lambda J.\ PiM\ J\ M\ M$
⟨*proof*⟩

Proof due to Ionescu Tulcea.

locale *Ionescu-Tulcea* =
fixes $P :: nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow 'a\ measure$ **and** $M :: nat \Rightarrow 'a\ measure$
assumes $P[measurable]$: $\bigwedge i.\ P\ i \in measurable\ (PiM\ \{0..<i\}\ M)\ (subprob\text{-}algebra\ (M\ i))$
assumes *prob-space-P*: $\bigwedge i\ x.\ x \in space\ (PiM\ \{0..<i\}\ M) \Longrightarrow prob\text{-}space\ (P\ i\ x)$
begin

lemma *non-empty*[*simp*]: $space\ (M\ i) \neq \{\}$
⟨*proof*⟩

lemma *space-PiM-not-empty*[*simp*]: $space\ (PiM\ UNIV\ M) \neq \{\}$
⟨*proof*⟩

lemma *space-P*: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (P\ n\ x) = space\ (M\ n)$
⟨*proof*⟩

lemma *sets-P*[*measurable-cong*]: $x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (P\ n\ x) = sets\ (M\ n)$
⟨*proof*⟩

definition $eP :: nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)\ measure$ **where**
$eP\ n\ \omega = distr\ (P\ n\ \omega)\ (PiM\ \{0..<Suc\ n\}\ M)\ (fun\text{-}upd\ \omega\ n)$

lemma *measurable-eP*[*measurable*]:
$eP\ n \in measurable\ (PiM\ \{0..<\ n\}\ M)\ (subprob\text{-}algebra\ (PiM\ \{0..<Suc\ n\}\ M))$
⟨*proof*⟩

lemma *space-eP*:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow space\ (eP\ n\ x) = space\ (PiM\ \{0..<Suc\ n\}\ M)$
⟨*proof*⟩

lemma *sets-eP*[*measurable*]:
$x \in space\ (PiM\ \{0..<n\}\ M) \Longrightarrow sets\ (eP\ n\ x) = sets\ (PiM\ \{0..<Suc\ n\}\ M)$
⟨*proof*⟩

lemma *prob-space-eP*: $x \in$ *space* (*PiM* $\{0..<n\}$ M) $\Longrightarrow$ *prob-space* (*eP* n x)
⟨*proof*⟩

lemma *nn-integral-eP*:
$\omega \in$ *space* (*PiM* $\{0..<n\}$ M) $\Longrightarrow f \in$ *borel-measurable* (*PiM* $\{0..<Suc\ n\}$ M) $\Longrightarrow$
$(\int^{+} x.\ f\ x\ \partial eP\ n\ \omega) = (\int^{+} x.\ f\ (\omega(n := x))\ \partial P\ n\ \omega)$
⟨*proof*⟩

lemma *emeasure-eP*:
assumes ω[*simp*]: $\omega \in$ *space* (*PiM* $\{0..<n\}$ M) **and** A[*measurable*]: $A \in$ *sets* (*PiM* $\{0..<Suc\ n\}$ M)
shows *eP* n ω A = P n ω $((\lambda x.\ \omega(n := x))$ −‘ $A \cap$ *space* $(M\ n))$
⟨*proof*⟩

primrec C :: *nat* ⇒ *nat* ⇒ (*nat* ⇒ $'a$) ⇒ (*nat* ⇒ $'a$) *measure* **where**
C n 0 ω = *return* (*PiM* $\{0..<n\}$ M) ω
| C n (*Suc* m) ω = C n m ω $\gg=$ *eP* $(n + m)$

lemma *measurable-C*[*measurable*]:
C n $m \in$ *measurable* (*PiM* $\{0..<n\}$ M) (*subprob-algebra* (*PiM* $\{0..<n + m\}$ M))
⟨*proof*⟩

lemma *space-C*:
$x \in$ *space* (*PiM* $\{0..<n\}$ M) $\Longrightarrow$ *space* (C n m x) = *space* (*PiM* $\{0..<n + m\}$ M)
⟨*proof*⟩

lemma *sets-C*[*measurable-cong*]:
$x \in$ *space* (*PiM* $\{0..<n\}$ M) $\Longrightarrow$ *sets* (C n m x) = *sets* (*PiM* $\{0..<n + m\}$ M)
⟨*proof*⟩

lemma *prob-space-C*: $x \in$ *space* (*PiM* $\{0..<n\}$ M) $\Longrightarrow$ *prob-space* (C n m x)
⟨*proof*⟩

lemma *split-C*:
assumes ω: $\omega \in$ *space* (*PiM* $\{0..<n\}$ M) **shows** (C n m ω $\gg=$ C $(n + m)$ l) = C n $(m + l)$ ω
⟨*proof*⟩

lemma *nn-integral-C*:
assumes $m \le m'$ **and** f[*measurable*]: $f \in$ *borel-measurable* (*PiM* $\{0..<n+m\}$ M)
and *nonneg*: $\bigwedge x.\ x \in$ *space* (*PiM* $\{0..<n+m\}$ M) $\Longrightarrow 0 \le f\ x$
and x: $x \in$ *space* (*PiM* $\{0..<n\}$ M)
shows $(\int^{+} x.\ f\ x\ \partial C\ n\ m\ x) = (\int^{+} x.\ f$ (*restrict* x $\{0..<n+m\}$) $\partial C\ n\ m'\ x)$
⟨*proof*⟩

lemma *emeasure-C*:
assumes $m \leq m'$ **and** $A[measurable]$: $A \in sets\ (PiM\ \{0..<n+m\}\ M)$ **and** $[simp]$: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $emeasure\ (C\ n\ m'\ x)\ (prod\text{-}emb\ \{0..<n + m'\}\ M\ \{0..<n+m\}\ A) = emeasure\ (C\ n\ m\ x)\ A$
$\langle proof \rangle$

lemma *distr-C*:
assumes $m \leq m'$ **and** $[simp]$: $x \in space\ (PiM\ \{0..<n\}\ M)$
shows $C\ n\ m\ x = distr\ (C\ n\ m'\ x)\ (PiM\ \{0..<n+m\}\ M)\ (\lambda x.\ restrict\ x\ \{0..<n+m\})$
$\langle proof \rangle$

definition *up-to* :: $nat\ set \Rightarrow nat$ **where**
$up\text{-}to\ J = (LEAST\ n.\ \forall\, i \geq n.\ i \notin J)$

lemma *up-to-less*: $finite\ J \Longrightarrow i \in J \Longrightarrow i < up\text{-}to\ J$
$\langle proof \rangle$

lemma *up-to-iff*: $finite\ J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow (\forall\, i \in J.\ i < n)$
$\langle proof \rangle$

lemma *up-to-iff-Ico*: $finite\ J \Longrightarrow up\text{-}to\ J \leq n \longleftrightarrow J \subseteq \{0..<n\}$
$\langle proof \rangle$

lemma *up-to*: $finite\ J \Longrightarrow J \subseteq \{0..< up\text{-}to\ J\}$
$\langle proof \rangle$

lemma *up-to-mono*: $J \subseteq H \Longrightarrow finite\ H \Longrightarrow up\text{-}to\ J \leq up\text{-}to\ H$
$\langle proof \rangle$

definition *CI* :: $nat\ set \Rightarrow (nat \Rightarrow 'a)\ measure$ **where**
$CI\ J = distr\ (C\ 0\ (up\text{-}to\ J)\ (\lambda x.\ undefined))\ (PiM\ J\ M)\ (\lambda f.\ restrict\ f\ J)$

sublocale *PF*: $projective\text{-}family\ UNIV\ CI$
$\langle proof \rangle$

lemma *emeasure-CI'*:
$finite\ J \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow CI\ J\ X = C\ 0\ (up\text{-}to\ J)\ (\lambda\text{-}.\ undefined)\ (PF.emb\ \{0..<up\text{-}to\ J\}\ J\ X)$
$\langle proof \rangle$

lemma *emeasure-CI*:
$J \subseteq \{0..<n\} \Longrightarrow X \in sets\ (PiM\ J\ M) \Longrightarrow CI\ J\ X = C\ 0\ n\ (\lambda\text{-}.\ undefined)\ (PF.emb\ \{0..<n\}\ J\ X)$
$\langle proof \rangle$

lemma *lim*:
assumes J: $finite\ J$ **and** X: $X \in sets\ (PiM\ J\ M)$

shows *emeasure PF.lim* (*PF.emb UNIV J X*) = *emeasure* (*CI J*) *X*
⟨*proof*⟩

lemma *distr-lim*: **assumes** *J*[*simp*]: *finite J* **shows** *distr PF.lim* (*PiM J M*) ($\lambda x.$ *restrict x J*) = *CI J*
⟨*proof*⟩

end

lemma (**in** *product-prob-space*) *emeasure-lim-emb*:
assumes ∗: *finite J J* ⊆ *I X* ∈ *sets* (*PiM J M*)
shows *emeasure lim* (*emb I J X*) = *emeasure* (Pi_M *J M*) *X*
⟨*proof*⟩

end

9 Infinite Product Measure

theory *Infinite-Product-Measure*
imports *Probability-Measure Projective-Family*
begin

lemma (**in** *product-prob-space*) *distr-PiM-restrict-finite*:
assumes *finite J J* ⊆ *I*
shows *distr* (*PiM I M*) (*PiM J M*) ($\lambda x.$ *restrict x J*) = *PiM J M*
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-emb′*:
$J \subseteq I \Longrightarrow$ *finite* $J \Longrightarrow X \in$ *sets* (*PiM J M*) $\Longrightarrow$ *emeasure* (Pi_M *I M*) (*emb I J X*) = *PiM J M X*
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-emb*:
$J \subseteq I \Longrightarrow$ *finite* $J \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow X\ i \in$ *sets* $(M\ i)) \Longrightarrow$
emeasure (Pi_M *I M*) (*emb I J* (Pi_E *J X*)) = ($\prod$ *i*∈*J*. *emeasure* (*M i*) (*X i*))
⟨*proof*⟩

sublocale *product-prob-space* ⊆ *P?*: *prob-space* Pi_M *I M*
⟨*proof*⟩

lemma *prob-space-PiM*:
assumes *M*: $\bigwedge i.\ i \in I \Longrightarrow$ *prob-space* (*M i*) **shows** *prob-space* (*PiM I M*)
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect*:
assumes *X*: $J \subseteq I$ *finite J* $\bigwedge i.\ i \in J \Longrightarrow X\ i \in$ *sets* (*M i*)
shows *emeasure* (Pi_M *I M*) {*x*∈*space* (Pi_M *I M*). ∀ *i*∈*J*. *x i* ∈ *X i*} = ($\prod$ *i*∈*J*. *emeasure* (*M i*) (*X i*))
⟨*proof*⟩

lemma (**in** *product-prob-space*) *emeasure-PiM-Collect-single*:
assumes X: $i \in I$ $A \in$ *sets* $(M\ i)$
shows *emeasure* $(Pi_M\ I\ M)$ $\{x{\in}space\ (Pi_M\ I\ M).\ x\ i \in A\}$ = *emeasure* $(M\ i)\ A$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *measure-PiM-emb*:
assumes $J \subseteq I$ *finite* J $\bigwedge i.\ i \in J \Longrightarrow X\ i \in sets\ (M\ i)$
shows *measure* $(PiM\ I\ M)$ $(emb\ I\ J\ (Pi_E\ J\ X)) = (\prod\ i{\in}J.\ measure\ (M\ i)\ (X\ i))$
⟨*proof*⟩

lemma *sets-Collect-single'*:
$i \in I \Longrightarrow \{x{\in}space\ (M\ i).\ P\ x\} \in sets\ (M\ i) \Longrightarrow \{x{\in}space\ (PiM\ I\ M).\ P\ (x\ i)\} \in sets\ (PiM\ I\ M)$
⟨*proof*⟩

lemma (**in** *finite-product-prob-space*) *finite-measure-PiM-emb*:
$(\bigwedge i.\ i \in I \Longrightarrow A\ i \in sets\ (M\ i)) \Longrightarrow measure\ (PiM\ I\ M)\ (Pi_E\ I\ A) = (\prod i{\in}I.\ measure\ (M\ i)\ (A\ i))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *PiM-component*:
assumes $i \in I$
shows *distr* $(PiM\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *PiM-eq*:
assumes M': *sets* M' = *sets* $(PiM\ I\ M)$
assumes *eq*: $\bigwedge J\ F.\ finite\ J \Longrightarrow J \subseteq I \Longrightarrow (\bigwedge j.\ j \in J \Longrightarrow F\ j \in sets\ (M\ j)) \Longrightarrow$
emeasure M' $(prod\text{-}emb\ I\ M\ J\ (\Pi_E\ j{\in}J.\ F\ j)) = (\prod j{\in}J.\ emeasure\ (M\ j)\ (F\ j))$
shows $M' = (PiM\ I\ M)$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *AE-component*: $i \in I \Longrightarrow AE\ x\ in\ M\ i.\ P\ x \Longrightarrow AE\ x\ in\ PiM\ I\ M.\ P\ (x\ i)$
⟨*proof*⟩

lemma *emeasure-PiM-emb*:
assumes M: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$
assumes J: $J \subseteq I$ *finite* J **and** A: $\bigwedge i.\ i \in J \Longrightarrow A\ i \in sets\ (M\ i)$
shows *emeasure* $(Pi_M\ I\ M)$ $(prod\text{-}emb\ I\ M\ J\ (Pi_E\ J\ A)) = (\prod i{\in}J.\ emeasure\ (M\ i)\ (A\ i))$
⟨*proof*⟩

lemma *distr-pair-PiM-eq-PiM*:
fixes i' :: $'i$ **and** I :: $'i$ *set* **and** M :: $'i \Rightarrow 'a$ *measure*
assumes M: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$ *prob-space* $(M\ i')$

shows $distr\ (M\ i' \bigotimes_M\ (\Pi_M\ i{\in}I.\ M\ i))\ (\Pi_M\ i{\in}insert\ i'\ I.\ M\ i)\ (\lambda(x,\ X).\ X(i' := x)) =$
$(\Pi_M\ i{\in}insert\ i'\ I.\ M\ i)$ (**is** $?L = -$)
$\langle proof \rangle$

lemma *distr-PiM-reindex*:
assumes M: $\bigwedge i.\ i \in K \Longrightarrow prob\text{-}space\ (M\ i)$
assumes f: $inj\text{-}on\ f\ I\ f \in I \to K$
shows $distr\ (Pi_M\ K\ M)\ (\Pi_M\ i{\in}I.\ M\ (f\ i))\ (\lambda\omega.\ \lambda n{\in}I.\ \omega\ (f\ n)) = (\Pi_M\ i{\in}I.\ M\ (f\ i))$
(**is** $distr\ ?K\ ?I\ ?t = ?I$)
$\langle proof \rangle$

lemma *distr-PiM-component*:
assumes M: $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$
assumes $i \in I$
shows $distr\ (Pi_M\ I\ M)\ (M\ i)\ (\lambda\omega.\ \omega\ i) = M\ i$
$\langle proof \rangle$

lemma *AE-PiM-component*:
$(\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)) \Longrightarrow i \in I \Longrightarrow AE\ x\ in\ M\ i.\ P\ x \Longrightarrow AE\ x\ in\ PiM\ I\ M.\ P\ (x\ i)$
$\langle proof \rangle$

lemma *decseq-emb-PiE*:
$incseq\ J \Longrightarrow decseq\ (\lambda i.\ prod\text{-}emb\ I\ M\ (J\ i)\ (\Pi_E\ j{\in}J\ i.\ X\ j))$
$\langle proof \rangle$

9.1 Sequence space

definition $comb\text{-}seq :: nat \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a)$ **where**
$comb\text{-}seq\ i\ \omega\ \omega'\ j = (if\ j < i\ then\ \omega\ j\ else\ \omega'\ (j - i))$

lemma *split-comb-seq*: $P\ (comb\text{-}seq\ i\ \omega\ \omega'\ j) \longleftrightarrow (j < i \longrightarrow P\ (\omega\ j)) \wedge (\forall k.\ j = i + k \longrightarrow P\ (\omega'\ k))$
$\langle proof \rangle$

lemma *split-comb-seq-asm*: $P\ (comb\text{-}seq\ i\ \omega\ \omega'\ j) \longleftrightarrow \neg\ ((j < i \wedge \neg\ P\ (\omega\ j)) \vee (\exists k.\ j = i + k \wedge \neg\ P\ (\omega'\ k)))$
$\langle proof \rangle$

lemma *measurable-comb-seq*:
$(\lambda(\omega,\ \omega').\ comb\text{-}seq\ i\ \omega\ \omega') \in measurable\ ((\Pi_M\ i{\in}UNIV.\ M) \bigotimes_M (\Pi_M\ i{\in}UNIV.\ M))\ (\Pi_M\ i{\in}UNIV.\ M)$
$\langle proof \rangle$

lemma *measurable-comb-seq'*[*measurable (raw)*]:
assumes f: $f \in measurable\ N\ (\Pi_M\ i{\in}UNIV.\ M)$ **and** g: $g \in measurable\ N\ (\Pi_M\ i{\in}UNIV.\ M)$

shows $(\lambda x.\ comb\text{-}seq\ i\ (f\ x)\ (g\ x)) \in measurable\ N\ (\Pi_M\ i{\in}UNIV.\ M)$
⟨*proof*⟩

lemma *comb-seq-0*: $comb\text{-}seq\ 0\ \omega\ \omega' = \omega'$
⟨*proof*⟩

lemma *comb-seq-Suc*: $comb\text{-}seq\ (Suc\ n)\ \omega\ \omega' = comb\text{-}seq\ n\ \omega\ (case\text{-}nat\ (\omega\ n)\ \omega')$
⟨*proof*⟩

lemma *comb-seq-Suc-0*[*simp*]: $comb\text{-}seq\ (Suc\ 0)\ \omega = case\text{-}nat\ (\omega\ 0)$
⟨*proof*⟩

lemma *comb-seq-less*: $i < n \Longrightarrow comb\text{-}seq\ n\ \omega\ \omega'\ i = \omega\ i$
⟨*proof*⟩

lemma *comb-seq-add*: $comb\text{-}seq\ n\ \omega\ \omega'\ (i + n) = \omega'\ i$
⟨*proof*⟩

lemma *case-nat-comb-seq*: $case\text{-}nat\ s'\ (comb\text{-}seq\ n\ \omega\ \omega')\ (i + n) = case\text{-}nat\ (case\text{-}nat\ s'\ \omega\ n)\ \omega'\ i$
⟨*proof*⟩

lemma *case-nat-comb-seq'*:
$case\text{-}nat\ s\ (comb\text{-}seq\ i\ \omega\ \omega') = comb\text{-}seq\ (Suc\ i)\ (case\text{-}nat\ s\ \omega)\ \omega'$
⟨*proof*⟩

locale *sequence-space* = *product-prob-space* $\lambda i.\ M\ UNIV :: nat\ set$ **for** M
begin

abbreviation $S \equiv \Pi_M\ i{\in}UNIV{::}nat\ set.\ M$

lemma *infprod-in-sets*[*intro*]:
fixes $E :: nat \Rightarrow 'a\ set$ **assumes** E: $\bigwedge i.\ E\ i \in sets\ M$
shows $Pi\ UNIV\ E \in sets\ S$
⟨*proof*⟩

lemma *measure-PiM-countable*:
fixes $E :: nat \Rightarrow 'a\ set$ **assumes** E: $\bigwedge i.\ E\ i \in sets\ M$
shows $(\lambda n.\ \prod i{\le}n.\ measure\ M\ (E\ i)) \longrightarrow measure\ S\ (Pi\ UNIV\ E)$
⟨*proof*⟩

lemma *nat-eq-diff-eq*:
fixes $a\ b\ c :: nat$
shows $c \le b \Longrightarrow a = b - c \longleftrightarrow a + c = b$
⟨*proof*⟩

lemma *PiM-comb-seq*:
$distr\ (S \bigotimes_M S)\ S\ (\lambda(\omega, \omega').\ comb\text{-}seq\ i\ \omega\ \omega') = S$ (**is** $?D = -$)
⟨*proof*⟩

lemma *PiM-iter*:
  $distr\ (M \bigotimes_M S)\ S\ (\lambda(s, \omega).\ case\text{-}nat\ s\ \omega) = S$ (**is** $?D = \text{-}$)
$\langle proof \rangle$

end

lemma *PiM-return*:
  assumes *finite I*
  assumes $[measurable]$: $\bigwedge i.\ i \in I \Longrightarrow \{a\ i\} \in sets\ (M\ i)$
  shows $PiM\ I\ (\lambda i.\ return\ (M\ i)\ (a\ i)) = return\ (PiM\ I\ M)\ (restrict\ a\ I)$
$\langle proof \rangle$

lemma *distr-PiM-finite-prob-space'*:
  assumes *fin*: *finite I*
  assumes $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M\ i)$
  assumes $\bigwedge i.\ i \in I \Longrightarrow prob\text{-}space\ (M'\ i)$
  assumes $[measurable]$: $\bigwedge i.\ i \in I \Longrightarrow f \in measurable\ (M\ i)\ (M'\ i)$
  shows $distr\ (PiM\ I\ M)\ (PiM\ I\ M')\ (compose\ I\ f) = PiM\ I\ (\lambda i.\ distr\ (M\ i)\ (M'\ i)\ f)$
$\langle proof \rangle$

end

10 Independent families of events, event sets, and random variables

theory *Independent-Family*
  imports *Infinite-Product-Measure*
begin

definition (**in** *prob-space*)
  $indep\text{-}sets\ F\ I \longleftrightarrow (\forall i{\in}I.\ F\ i \subseteq events) \wedge$
    $(\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow finite\ J \longrightarrow (\forall A{\in}Pi\ J\ F.\ prob\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ prob\ (A\ j))))$

definition (**in** *prob-space*)
  $indep\text{-}set\ A\ B \longleftrightarrow indep\text{-}sets\ (case\text{-}bool\ A\ B)\ UNIV$

definition (**in** *prob-space*)
  indep-events-def-alt: $indep\text{-}events\ A\ I \longleftrightarrow indep\text{-}sets\ (\lambda i.\ \{A\ i\})\ I$

lemma (**in** *prob-space*) *indep-events-def*:
  $indep\text{-}events\ A\ I \longleftrightarrow (A`I \subseteq events) \wedge$
    $(\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow finite\ J \longrightarrow prob\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ prob\ (A\ j)))$
  $\langle proof \rangle$

lemma (**in** *prob-space*) *indep-eventsI*:

$(\bigwedge i.\ i \in I \Longrightarrow F\ i \in sets\ M) \Longrightarrow (\bigwedge J.\ J \subseteq I \Longrightarrow finite\ J \Longrightarrow J \neq \{\} \Longrightarrow prob\ (\bigcap i{\in}J.\ F\ i) = (\prod i{\in}J.\ prob\ (F\ i))) \Longrightarrow indep\text{-}events\ F\ I$
⟨*proof*⟩

definition (**in** *prob-space*)
$indep\text{-}event\ A\ B \longleftrightarrow indep\text{-}events\ (case\text{-}bool\ A\ B)\ UNIV$

lemma (**in** *prob-space*) *indep-sets-cong*:
$I = J \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow F\ i = G\ i) \Longrightarrow indep\text{-}sets\ F\ I \longleftrightarrow indep\text{-}sets\ G\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-events-finite-index-events*:
$indep\text{-}events\ F\ I \longleftrightarrow (\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow finite\ J \longrightarrow indep\text{-}events\ F\ J)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-finite-index-sets*:
$indep\text{-}sets\ F\ I \longleftrightarrow (\forall J{\subseteq}I.\ J \neq \{\} \longrightarrow finite\ J \longrightarrow indep\text{-}sets\ F\ J)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono-index*:
$J \subseteq I \Longrightarrow indep\text{-}sets\ F\ I \Longrightarrow indep\text{-}sets\ F\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono-sets*:
assumes *indep*: $indep\text{-}sets\ F\ I$
assumes *mono*: $\bigwedge i.\ i{\in}I \Longrightarrow G\ i \subseteq F\ i$
shows $indep\text{-}sets\ G\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-mono*:
assumes *indep*: $indep\text{-}sets\ F\ I$
assumes *mono*: $J \subseteq I\ \bigwedge i.\ i{\in}J \Longrightarrow G\ i \subseteq F\ i$
shows $indep\text{-}sets\ G\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setsI*:
assumes $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq events$
and $\bigwedge A\ J.\ J \neq \{\} \Longrightarrow J \subseteq I \Longrightarrow finite\ J \Longrightarrow (\forall j{\in}J.\ A\ j \in F\ j) \Longrightarrow prob\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ prob\ (A\ j))$
shows $indep\text{-}sets\ F\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setsD*:
assumes $indep\text{-}sets\ F\ I$ **and** $J \subseteq I\ J \neq \{\}\ finite\ J\ \forall j{\in}J.\ A\ j \in F\ j$
shows $prob\ (\bigcap j{\in}J.\ A\ j) = (\prod j{\in}J.\ prob\ (A\ j))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setI*:
assumes *ev*: $A \subseteq events\ B \subseteq events$

and *indep*: $\bigwedge a\ b.\ a \in A \Longrightarrow b \in B \Longrightarrow prob\ (a \cap b) = prob\ a * prob\ b$
shows *indep-set A B*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-setD*:
assumes *indep*: *indep-set A B* **and** *ev*: $a \in A\ b \in B$
shows $prob\ (a \cap b) = prob\ a * prob\ b$
⟨*proof*⟩

lemma (**in** *prob-space*)
assumes *indep*: *indep-set A B*
shows *indep-setD-ev1*: $A \subseteq events$
and *indep-setD-ev2*: $B \subseteq events$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-Dynkin*:
assumes *indep*: *indep-sets F I*
shows *indep-sets* ($\lambda i.$ *Dynkin* (*space M*) (*F i*)) *I*
(**is** *indep-sets ?F I*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-sigma*:
assumes *indep*: *indep-sets F I*
assumes *stable*: $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* (*F i*)
shows *indep-sets* ($\lambda i.$ *sigma-sets* (*space M*) (*F i*)) *I*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-sigma-sets-iff*:
assumes $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* (*F i*)
shows *indep-sets* ($\lambda i.$ *sigma-sets* (*space M*) (*F i*)) *I* $\longleftrightarrow$ *indep-sets F I*
⟨*proof*⟩

definition (**in** *prob-space*)
indep-vars-def2: *indep-vars* M' *X I* $\longleftrightarrow$
($\forall i{\in}I.$ *random-variable* (M' *i*) (*X i*)) $\wedge$
indep-sets ($\lambda i.$ { *X i* −‘ *A* $\cap$ *space M* | *A*. $A \in$ *sets* (M' *i*)}) *I*

definition (**in** *prob-space*)
indep-var Ma A Mb B $\longleftrightarrow$ *indep-vars* (*case-bool Ma Mb*) (*case-bool A B*) *UNIV*

lemma (**in** *prob-space*) *indep-vars-def*:
indep-vars M' *X I* $\longleftrightarrow$
($\forall i{\in}I.$ *random-variable* (M' *i*) (*X i*)) $\wedge$
indep-sets ($\lambda i.$ *sigma-sets* (*space M*) { *X i* −‘ *A* $\cap$ *space M* | *A*. $A \in$ *sets* (M'
i)}) *I*
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-eq*:
indep-var S X T Y $\longleftrightarrow$

$(random\text{-}variable\ S\ X \wedge random\text{-}variable\ T\ Y) \wedge$
indep-set
$(sigma\text{-}sets\ (space\ M)\ \{\ X\ -`\ A \cap space\ M\ |\ A.\ A \in sets\ S\})$
$(sigma\text{-}sets\ (space\ M)\ \{\ Y\ -`\ A \cap space\ M\ |\ A.\ A \in sets\ T\})$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets2-eq*:
$indep\text{-}set\ A\ B \longleftrightarrow A \subseteq events \wedge B \subseteq events \wedge (\forall a{\in}A.\ \forall b{\in}B.\ prob\ (a \cap b) = prob\ a * prob\ b)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-set-sigma-sets*:
assumes *indep-set A B*
assumes *A*: *Int-stable A* **and** *B*: *Int-stable B*
shows *indep-set* (*sigma-sets* (*space M*) *A*) (*sigma-sets* (*space M*) *B*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-eventsI-indep-vars*:
assumes *indep*: *indep-vars N X I*
assumes *P*: $\bigwedge i.\ i \in I \Longrightarrow \{x{\in}space\ (N\ i).\ P\ i\ x\} \in sets\ (N\ i)$
shows $indep\text{-}events\ (\lambda i.\ \{x{\in}space\ M.\ P\ i\ (X\ i\ x)\})\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-collect-sigma*:
fixes $I :: 'j \Rightarrow 'i\ set$ **and** $J :: 'j\ set$ **and** $E :: 'i \Rightarrow 'a\ set\ set$
assumes *indep*: $indep\text{-}sets\ E\ (\bigcup j{\in}J.\ I\ j)$
assumes *Int-stable*: $\bigwedge i\ j.\ j \in J \Longrightarrow i \in I\ j \Longrightarrow Int\text{-}stable\ (E\ i)$
assumes *disjoint*: *disjoint-family-on I J*
shows $indep\text{-}sets\ (\lambda j.\ sigma\text{-}sets\ (space\ M)\ (\bigcup i{\in}I\ j.\ E\ i))\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-restrict*:
assumes *ind*: $indep\text{-}vars\ M'\ X\ I$ **and** *K*: $\bigwedge j.\ j \in L \Longrightarrow K\ j \subseteq I$ **and** *J*: *disjoint-family-on K L*
shows $indep\text{-}vars\ (\lambda j.\ PiM\ (K\ j)\ M')\ (\lambda j\ \omega.\ restrict\ (\lambda i.\ X\ i\ \omega)\ (K\ j))\ L$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-restrict*:
assumes *ind*: $indep\text{-}vars\ M'\ X\ I$ **and** *AB*: $A \cap B = \{\}\ A \subseteq I\ B \subseteq I$
shows $indep\text{-}var\ (PiM\ A\ M')\ (\lambda\omega.\ restrict\ (\lambda i.\ X\ i\ \omega)\ A)\ (PiM\ B\ M')\ (\lambda\omega.\ restrict\ (\lambda i.\ X\ i\ \omega)\ B)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-subset*:
assumes $indep\text{-}vars\ M'\ X\ I\ J \subseteq I$
shows $indep\text{-}vars\ M'\ X\ J$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-cong*:

$I = J \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow X\ i = Y\ i) \Longrightarrow (\bigwedge i.\ i \in I \Longrightarrow M'\ i = N'\ i) \Longrightarrow$ *indep-vars* $M'\ X\ I \longleftrightarrow$ *indep-vars* $N'\ Y\ J$
⟨*proof*⟩

definition (**in** *prob-space*) *tail-events* **where**
tail-events $A = (\bigcap n.$ *sigma-sets* (*space* M) $(\bigcup\ (A\ `\ \{n..\})))$

lemma (**in** *prob-space*) *tail-events-sets*:
assumes A: $\bigwedge i$::*nat*. $A\ i \subseteq$ *events*
shows *tail-events* $A \subseteq$ *events*
⟨*proof*⟩

lemma (**in** *prob-space*) *sigma-algebra-tail-events*:
assumes $\bigwedge i$::*nat*. *sigma-algebra* (*space* M) ($A\ i$)
shows *sigma-algebra* (*space* M) (*tail-events* A)
⟨*proof*⟩

lemma (**in** *prob-space*) *kolmogorov-0-1-law*:
fixes A :: *nat* $\Rightarrow$ *'a set set*
assumes $\bigwedge i$::*nat*. *sigma-algebra* (*space* M) ($A\ i$)
assumes *indep*: *indep-sets* A *UNIV*
and X: $X \in$ *tail-events* A
shows *prob* $X = 0 \lor$ *prob* $X = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *borel-0-1-law*:
fixes F :: *nat* $\Rightarrow$ *'a set*
assumes *F2*: *indep-events* F *UNIV*
shows *prob* $(\bigcap n.\ \bigcup m \in \{n..\}.\ F\ m) = 0 \lor$ *prob* $(\bigcap n.\ \bigcup m \in \{n..\}.\ F\ m) = 1$
⟨*proof*⟩

lemma (**in** *prob-space*) *borel-0-1-law-AE*:
fixes P :: *nat* $\Rightarrow$ *'a* $\Rightarrow$ *bool*
assumes *indep-events* ($\lambda m.\ \{x \in$ *space* $M.\ P\ m\ x\}$) *UNIV* (**is** *indep-events ?P -*)
shows (*AE x in M*. *infinite* $\{m.\ P\ m\ x\}$) $\lor$ (*AE x in M*. *finite* $\{m.\ P\ m\ x\}$)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-finite*:
assumes I: $I \neq \{\}$ *finite* I
and F: $\bigwedge i.\ i \in I \Longrightarrow F\ i \subseteq$ *events* $\bigwedge i.\ i \in I \Longrightarrow$ *space* $M \in F\ i$
shows *indep-sets* $F\ I \longleftrightarrow (\forall A \in Pi\ I\ F.$ *prob* $(\bigcap j \in I.\ A\ j) = (\prod j \in I.$ *prob* $(A\ j)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-finite*:
fixes I :: *'i set*
assumes I: $I \neq \{\}$ *finite* I
and M': $\bigwedge i.\ i \in I \Longrightarrow$ *sets* $(M'\ i) =$ *sigma-sets* (*space* $(M'\ i)$) $(E\ i)$
and *rv*: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable* $(M'\ i)$ $(X\ i)$

and *Int-stable*: $\bigwedge i.\ i \in I \Longrightarrow$ *Int-stable* $(E\ i)$
and *space*: $\bigwedge i.\ i \in I \Longrightarrow$ *space* $(M'\ i) \in E\ i$ **and** *closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq$ *Pow* (*space* $(M'\ i)$)
shows *indep-vars* $M'\ X\ I \longleftrightarrow$
$(\forall A\in(\Pi\ i\in I.\ E\ i).\ prob\ (\bigcap j\in I.\ X\ j -\text{`}\ A\ j \cap space\ M) = (\prod j\in I.\ prob\ (X\ j -\text{`}\ A\ j \cap space\ M)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-compose*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i.\ Y\ i \circ X\ i)\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-compose2*:
assumes *indep-vars* $M'\ X\ I$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow Y\ i \in$ *measurable* $(M'\ i)\ (N\ i)$
shows *indep-vars* $N\ (\lambda i\ x.\ Y\ i\ (X\ i\ x))\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-compose*:
assumes *indep-var* $M1\ X1\ M2\ X2\ Y1 \in$ *measurable* $M1\ N1\ Y2 \in$ *measurable* $M2\ N2$
shows *indep-var* $N1\ (Y1 \circ X1)\ N2\ (Y2 \circ X2)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-Min*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow real$
assumes I: *finite* $I\ i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X$ (*insert* $i\ I$)
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ Min\ ((\lambda i.\ X\ i\ \omega)\text{`}I))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-sum*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow real$
assumes I: *finite* $I\ i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X$ (*insert* $i\ I$)
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ \sum i\in I.\ X\ i\ \omega)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-prod*:
fixes $X :: 'i \Rightarrow 'a \Rightarrow real$
assumes I: *finite* $I\ i \notin I$ **and** *indep*: *indep-vars* $(\lambda\text{-}.\ borel)\ X$ (*insert* $i\ I$)
shows *indep-var borel* $(X\ i)$ *borel* $(\lambda\omega.\ \prod i\in I.\ X\ i\ \omega)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varsD-finite*:
assumes X: *indep-vars* $M'\ X\ I$
assumes I: $I \neq \{\}$ *finite* $I\ \bigwedge i.\ i \in I \Longrightarrow A\ i \in$ *sets* $(M'\ i)$
shows *prob* $(\bigcap i\in I.\ X\ i -\text{`}\ A\ i \cap space\ M) = (\prod i\in I.\ prob\ (X\ i -\text{`}\ A\ i \cap space\ M))$

⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varsD*:
assumes X: *indep-vars* M' X I
assumes I: $J \neq \{\}$ *finite* J $J \subseteq I$ $\bigwedge i.\ i \in J \Longrightarrow A\ i \in$ *sets* $(M'\ i)$
shows *prob* $(\bigcap i{\in}J.\ X\ i\ -`\ A\ i \cap$ *space* $M) = (\prod i{\in}J.$ *prob* $(X\ i\ -`\ A\ i \cap$ *space* $M))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM*:
fixes I :: *'i set* **and** X :: $'i \Rightarrow 'a \Rightarrow 'b$
assumes $I \neq \{\}$
assumes *rv*: $\bigwedge i.$ *random-variable* $(M'\ i)\ (X\ i)$
shows *indep-vars* M' X $I \longleftrightarrow$
distr M $(\Pi_M\ i{\in}I.\ M'\ i)$ $(\lambda x.\ \lambda i{\in}I.\ X\ i\ x) = (\Pi_M\ i{\in}I.$ *distr* M $(M'\ i)$ $(X\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-iff-distr-eq-PiM'*:
fixes I :: *'i set* **and** X :: $'i \Rightarrow 'a \Rightarrow 'b$
assumes $I \neq \{\}$
assumes *rv*: $\bigwedge i.\ i \in I \Longrightarrow$ *random-variable* $(M'\ i)\ (X\ i)$
shows *indep-vars* M' X $I \longleftrightarrow$
distr M $(\Pi_M\ i{\in}I.\ M'\ i)$ $(\lambda x.\ \lambda i{\in}I.\ X\ i\ x) = (\Pi_M\ i{\in}I.$ *distr* M $(M'\ i)$ $(X\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-varD*:
assumes *indep*: *indep-var* *Ma* A *Mb* B
assumes *sets*: $Xa \in$ *sets* *Ma* $Xb \in$ *sets* *Mb*
shows *prob* $((\lambda x.\ (A\ x,\ B\ x))\ -`\ (Xa \times Xb) \cap$ *space* $M) =$
prob $(A\ -`\ Xa \cap$ *space* $M) *$ *prob* $(B\ -`\ Xb \cap$ *space* $M)$
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-indep-random-variable*:
assumes *ind*[*simp*]: *indep-var* N X N Y
assumes [*simp*]: $A \in$ *sets* N $B \in$ *sets* N
shows $\mathcal{P}(x\ in\ M.\ X\ x \in A \wedge Y\ x \in B) = \mathcal{P}(x\ in\ M.\ X\ x \in A) * \mathcal{P}(x\ in\ M.\ Y\ x \in B)$
⟨*proof*⟩

lemma (**in** *prob-space*)
assumes *indep-var* S X T Y
shows *indep-var-rv1*: *random-variable* S X
and *indep-var-rv2*: *random-variable* T Y
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-var-distribution-eq*:
indep-var S X T $Y \longleftrightarrow$ *random-variable* S $X \wedge$ *random-variable* T $Y \wedge$
distr M S $X \bigotimes_M$ *distr* M T $Y =$ *distr* M $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ (**is** -

$\longleftrightarrow$ - $\wedge$ - $\wedge$ *?S* $\bigotimes_M$ *?T* = *?J*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-joint-indep*:
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *X*: *distributed M S X Px* **and** *Y*: *distributed M T Y Py*
assumes *indep*: *indep-var S X T Y*
shows *distributed M* (*S* $\bigotimes_M$ *T*) ($\lambda x.$ (*X x*, *Y x*)) ($\lambda(x, y).$ *Px x* $*$ *Py y*)
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-nn-integral*:
assumes *I*: *finite I indep-vars* (λ-. *borel*) *X I* $\bigwedge i\ \omega.\ i \in I \Longrightarrow 0 \le X\ i\ \omega$
shows ($\int^+\omega.$ ($\prod i{\in}I.\ X\ i\ \omega$) ∂M) = ($\prod i{\in}I.\ \int^+\omega.\ X\ i\ \omega\ \partial M$)
⟨*proof*⟩

lemma (**in** *prob-space*)
fixes *X* :: *'i* ⇒ *'a* ⇒ *'b*::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *I*: *finite I indep-vars* (λ-. *borel*) *X I* $\bigwedge i.\ i \in I \Longrightarrow$ *integrable M* (*X i*)
shows *indep-vars-lebesgue-integral*: ($\int\omega.$ ($\prod i{\in}I.\ X\ i\ \omega$) ∂M) = ($\prod i{\in}I.\ \int\omega.\ X\ i\ \omega\ \partial M$) (**is** *?eq*)
and *indep-vars-integrable*: *integrable M* ($\lambda\omega.$ ($\prod i{\in}I.\ X\ i\ \omega$)) (**is** *?int*)
⟨*proof*⟩

lemma (**in** *prob-space*)
fixes *X1 X2* :: *'a* ⇒ *'b*::{*real-normed-field*, *banach*, *second-countable-topology*}
assumes *indep-var borel X1 borel X2 integrable M X1 integrable M X2*
shows *indep-var-lebesgue-integral*: ($\int\omega.\ X1\ \omega * X2\ \omega\ \partial M$) = ($\int\omega.\ X1\ \omega\ \partial M$) $*$ ($\int\omega.\ X2\ \omega\ \partial M$) (**is** *?eq*)
and *indep-var-integrable*: *integrable M* ($\lambda\omega.\ X1\ \omega * X2\ \omega$) (**is** *?int*)
⟨*proof*⟩

end

11 Convolution Measure

theory *Convolution*
imports *Independent-Family*
begin

lemma (**in** *finite-measure*) *sigma-finite-measure*: *sigma-finite-measure M*
⟨*proof*⟩

definition *convolution* :: (*'a* :: *ordered-euclidean-space*) *measure* ⇒ *'a measure* ⇒ *'a measure* (**infix** $\star$ *50*) **where**
convolution M N = *distr* (*M* $\bigotimes_M$ *N*) *borel* ($\lambda(x, y).\ x + y$)

lemma
shows *space-convolution*[*simp*]: *space* (*convolution M N*) = *space borel*
and *sets-convolution*[*simp*]: *sets* (*convolution M N*) = *sets borel*

and *measurable-convolution1* [*simp*]: *measurable A* (*convolution M N*) = *measurable A borel*
and *measurable-convolution2* [*simp*]: *measurable* (*convolution M N*) *B* = *measurable borel B*
⟨*proof*⟩

lemma *nn-integral-convolution*:
assumes *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*measurable*]: $f \in$ *borel-measurable borel*
shows $(\int^+ x.\ f\ x\ \partial convolution\ M\ N) = (\int^+ x.\ \int^+ y.\ f\ (x + y)\ \partial N\ \partial M)$
⟨*proof*⟩

lemma *convolution-emeasure*:
assumes $A \in$ *sets borel finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
assumes [*simp*]: *space M* = *space N space N* = *space borel*
shows *emeasure* $(M \star N)\ A = \int^+ x.\ (emeasure\ N\ \{a.\ a + x \in A\})\ \partial M$
⟨*proof*⟩

lemma *convolution-emeasure'*:
assumes [*simp*]:$A \in$ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel*
shows *emeasure* $(M \star N)\ A = \int^+ x.\ \int^+ y.\ (indicator\ A\ (x + y))\ \partial N\ \partial M$
⟨*proof*⟩

lemma *convolution-finite*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*]: *sets N* = *sets borel sets M* = *sets borel*
shows *finite-measure* $(M \star N)$
⟨*proof*⟩

lemma *convolution-emeasure-3*:
assumes [*simp*, *measurable*]: $A \in$ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* $(L \star (M \star N))\ A = \int^+ x.\ \int^+ y.\ \int^+ z.\ indicator\ A\ (x + y + z)\ \partial N\ \partial M\ \partial L$
⟨*proof*⟩

lemma *convolution-emeasure-3'*:
assumes [*simp*, *measurable*]:$A \in$ *sets borel*
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*measurable-cong*, *simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows *emeasure* $((L \star M) \star N)\ A = \int^+ x.\ \int^+ y.\ \int^+ z.\ indicator\ A\ (x + y + z)\ \partial N\ \partial M\ \partial L$
⟨*proof*⟩

lemma *convolution-commutative*:
assumes [*simp*]: *finite-measure M finite-measure N*
assumes [*measurable-cong*, *simp*]: *sets N* = *sets borel sets M* = *sets borel*
shows $(M \star N) = (N \star M)$
⟨*proof*⟩

lemma *convolution-associative*:
assumes [*simp*]: *finite-measure M finite-measure N finite-measure L*
assumes [*simp*]: *sets N* = *sets borel sets M* = *sets borel sets L* = *sets borel*
shows $(L \star (M \star N)) = ((L \star M) \star N)$
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-random-variable*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable borel X*
assumes [*simp*, *measurable*]: *random-variable borel Y*
shows *distr M borel* ($\lambda x.\ X\ x + Y\ x$) = *convolution* (*distr M borel X*) (*distr M borel Y*)
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-random-variable-lborel*:
assumes *ind*: *indep-var borel X borel Y*
assumes [*simp*, *measurable*]: *random-variable lborel X*
assumes [*simp*, *measurable*]:*random-variable lborel Y*
shows *distr M lborel* ($\lambda x.\ X\ x + Y\ x$) = *convolution* (*distr M lborel X*) (*distr M lborel Y*)
⟨*proof*⟩

lemma *convolution-density*:
fixes *f g* :: *real* ⇒ *ennreal*
assumes [*measurable*]: *f* ∈ *borel-measurable borel g* ∈ *borel-measurable borel*
assumes [*simp*]:*finite-measure* (*density lborel f*) *finite-measure* (*density lborel g*)
shows *density lborel f* ⋆ *density lborel g* = *density lborel* ($\lambda x.\ \int^+ y.\ f\ (x - y) * g\ y\ \partial lborel$)
(**is** *?l* = *?r*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-finite-measure-density*:
distributed M N X f ⟹ *finite-measure* (*density N f*)
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-convolution*:
fixes *f* :: *real* ⇒ -
fixes *g* :: *real* ⇒ -
assumes *indep*: *indep-var borel X borel Y*
assumes *X*: *distributed M lborel X f*
assumes *Y*: *distributed M lborel Y g*

shows *distributed M lborel* ($\lambda x.\ X\ x + Y\ x$) ($\lambda x.\ \int^{+} y.\ f\ (x - y) * g\ y\ \partial lborel$)
⟨*proof*⟩

lemma *prob-space-convolution-density*:
fixes *f*:: *real* ⇒ -
fixes *g*:: *real* ⇒ -
assumes [*measurable*]: *f*∈ *borel-measurable borel*
assumes [*measurable*]: *g*∈ *borel-measurable borel*
assumes *gt-0*[*simp*]: $\bigwedge x.\ 0 \le f\ x\ \bigwedge x.\ 0 \le g\ x$
assumes *prob-space* (*density lborel f*) (**is** *prob-space ?F*)
assumes *prob-space* (*density lborel g*) (**is** *prob-space ?G*)
shows *prob-space* (*density lborel* ($\lambda x. \int^{+} y.\ f\ (x - y) * g\ y\ \partial lborel$)) (**is** *prob-space ?D*)
⟨*proof*⟩

end

12 Information theory

theory *Information*
imports
Independent-Family
begin

lemma *log-le*: $1 < a \Longrightarrow 0 < x \Longrightarrow x \le y \Longrightarrow log\ a\ x \le log\ a\ y$
⟨*proof*⟩

lemma *log-less*: $1 < a \Longrightarrow 0 < x \Longrightarrow x < y \Longrightarrow log\ a\ x < log\ a\ y$
⟨*proof*⟩

lemma *sum-cartesian-product'*:
($\sum x{\in}A \times B.\ f\ x$) = ($\sum x{\in}A.\ sum\ (\lambda y.\ f\ (x, y))\ B$)
⟨*proof*⟩

lemma *split-pairs*:
$((A, B) = X) \longleftrightarrow (fst\ X = A \wedge snd\ X = B)$ **and**
$(X = (A, B)) \longleftrightarrow (fst\ X = A \wedge snd\ X = B)$ ⟨*proof*⟩

12.1 Information theory

locale *information-space* = *prob-space* +
fixes *b* :: *real* **assumes** *b-gt-1*: $1 < b$

context *information-space*
begin

Introduce some simplification rules for logarithm of base b.

lemma *log-neg-const*:
assumes $x \le 0$

shows *log b x = log b 0*
⟨*proof*⟩

lemma *log-mult-eq*:
log b (A ∗ B) = (if 0 < A ∗ B then log b |A| + log b |B| else log b 0)
⟨*proof*⟩

lemma *log-inverse-eq*:
log b (inverse B) = (if 0 < B then − log b B else log b 0)
⟨*proof*⟩

lemma *log-divide-eq*:
log b (A / B) = (if 0 < A ∗ B then log b |A| − log b |B| else log b 0)
⟨*proof*⟩

lemmas *log-simps = log-mult-eq log-inverse-eq log-divide-eq*

end

12.2 Kullback−Leibler divergence

The Kullback−Leibler divergence is also known as relative entropy or Kullback−Leibler distance.

definition
entropy-density b M N = log b ∘ enn2real ∘ RN-deriv M N

definition
KL-divergence b M N = integral^L N (entropy-density b M N)

lemma *measurable-entropy-density*[*measurable*]: *entropy-density b M N ∈ borel-measurable M*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density*:
fixes *f :: 'a ⇒ real*
assumes *1 < b*
assumes *f*[*measurable*]: *f ∈ borel-measurable M* **and** *nn*: *AE x in M. 0 ≤ f x*
shows *KL-divergence b M (density M f) = (∫ x. f x ∗ log b (f x) ∂M)*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density-density*:
fixes *f g :: 'a ⇒ real*
assumes *1 < b*
assumes *f*: *f ∈ borel-measurable M AE x in M. 0 ≤ f x*
assumes *g*: *g ∈ borel-measurable M AE x in M. 0 ≤ g x*
assumes *ac*: *AE x in M. f x = 0 ⟶ g x = 0*
shows *KL-divergence b (density M f) (density M g) = (∫ x. g x ∗ log b (g x / f x) ∂M)*
⟨*proof*⟩

lemma (**in** *information-space*) *KL-gt-0*:
  fixes $D :: 'a \Rightarrow real$
  assumes *prob-space* (*density M D*)
  assumes *D*: $D \in$ *borel-measurable M AE x in M*. $0 \leq D\ x$
  assumes *int*: *integrable M* ($\lambda x.\ D\ x * log\ b\ (D\ x)$)
  assumes *A*: *density M D* $\neq M$
  shows $0 <$ *KL-divergence b M* (*density M D*)
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-same-eq-0*: *KL-divergence b M M* $= 0$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-eq-0-iff-eq*:
  fixes $D :: 'a \Rightarrow real$
  assumes *prob-space* (*density M D*)
  assumes *D*: $D \in$ *borel-measurable M AE x in M*. $0 \leq D\ x$
  assumes *int*: *integrable M* ($\lambda x.\ D\ x * log\ b\ (D\ x)$)
  shows *KL-divergence b M* (*density M D*) $= 0 \longleftrightarrow$ *density M D* $= M$
  ⟨*proof*⟩

lemma (**in** *information-space*) *KL-eq-0-iff-eq-ac*:
  fixes $D :: 'a \Rightarrow real$
  assumes *prob-space N*
  assumes *ac*: *absolutely-continuous M N sets N* $=$ *sets M*
  assumes *int*: *integrable N* (*entropy-density b M N*)
  shows *KL-divergence b M N* $= 0 \longleftrightarrow N = M$
⟨*proof*⟩

lemma (**in** *information-space*) *KL-nonneg*:
  assumes *prob-space* (*density M D*)
  assumes *D*: $D \in$ *borel-measurable M AE x in M*. $0 \leq D\ x$
  assumes *int*: *integrable M* ($\lambda x.\ D\ x * log\ b\ (D\ x)$)
  shows $0 \leq$ *KL-divergence b M* (*density M D*)
  ⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *KL-density-density-nonneg*:
  fixes $f\ g :: 'a \Rightarrow real$
  assumes $1 < b$
  assumes *f*: $f \in$ *borel-measurable M AE x in M*. $0 \leq f\ x$ *prob-space* (*density M f*)
  assumes *g*: $g \in$ *borel-measurable M AE x in M*. $0 \leq g\ x$ *prob-space* (*density M g*)
  assumes *ac*: *AE x in M*. $f\ x = 0 \longrightarrow g\ x = 0$
  assumes *int*: *integrable M* ($\lambda x.\ g\ x * log\ b\ (g\ x\ /\ f\ x)$)
  shows $0 \leq$ *KL-divergence b* (*density M f*) (*density M g*)
⟨*proof*⟩

12.3 Finite Entropy

definition (**in** *information-space*) *finite-entropy* :: $'b$ *measure* $\Rightarrow$ ($'a \Rightarrow$ $'b$) $\Rightarrow$ ($'b$ $\Rightarrow$ *real*) $\Rightarrow$ *bool*
where
  finite-entropy S X f $\longleftrightarrow$
    distributed M S X f $\wedge$
    integrable S (λx. *f x* $*$ *log b* (*f x*)) $\wedge$
    ($\forall x \in$*space S*. $0 \leq f\ x$)

lemma (**in** *information-space*) *finite-entropy-simple-function*:
  assumes *X*: *simple-function M X*
  shows *finite-entropy* (*count-space* (*X'space M*)) *X* (λa. *measure M* $\{x \in$ *space M*. *X x* $= a\}$)
  ⟨*proof*⟩

lemma *ac-fst*:
  assumes *sigma-finite-measure T*
  shows *absolutely-continuous S* (*distr* ($S \bigotimes_M T$) *S fst*)
⟨*proof*⟩

lemma *ac-snd*:
  assumes *sigma-finite-measure T*
  shows *absolutely-continuous T* (*distr* ($S \bigotimes_M T$) *T snd*)
⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-integrable*:
  finite-entropy S X Px $\Longrightarrow$ *integrable S* (λx. *Px x* $*$ *log b* (*Px x*))
  ⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-distributed*:
  finite-entropy S X Px $\Longrightarrow$ *distributed M S X Px*
  ⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-nn*:
  finite-entropy S X Px $\Longrightarrow x \in$ *space S* $\Longrightarrow 0 \leq$ *Px x*
  ⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-measurable*:
  finite-entropy S X Px $\Longrightarrow$ *Px* $\in S \rightarrow_M$ *borel*
  ⟨*proof*⟩

lemma (**in** *information-space*) *subdensity-finite-entropy*:
  fixes *g* :: $'b \Rightarrow$ *real* **and** *f* :: $'c \Rightarrow$ *real*
  assumes *T*: $T \in$ *measurable P Q*
  assumes *f*: *finite-entropy P X f*
  assumes *g*: *finite-entropy Q Y g*
  assumes *Y*: $Y = T \circ X$
  shows *AE x in P*. *g* (*T x*) $= 0 \longrightarrow f\ x = 0$
  ⟨*proof*⟩

lemma (**in** *information-space*) *finite-entropy-integrable-transform*:
finite-entropy S X Px $\Longrightarrow$ *distributed M T Y Py* $\Longrightarrow$ $(\bigwedge x.\ x \in$ *space T* $\Longrightarrow 0 \leq$ *Py x*$)$ $\Longrightarrow$
$X = (\lambda x.\ f\ (Y\ x)) \Longrightarrow f \in$ *measurable T S* $\Longrightarrow$ *integrable T* $(\lambda x.$ *Py x* $*$ *log b* $($*Px* $(f\ x)))$
$\langle$*proof*$\rangle$

12.4 Mutual Information

definition (**in** *prob-space*)
mutual-information b S T X Y $=$
KL-divergence b $($*distr M S X* $\bigotimes_M$ *distr M T Y*$)$ $($*distr M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x)))$

lemma (**in** *information-space*) *mutual-information-indep-vars*:
fixes *S T X Y*
defines $P \equiv$ *distr M S X* $\bigotimes_M$ *distr M T Y*
defines $Q \equiv$ *distr M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$
shows *indep-var S X T Y* $\longleftrightarrow$
$($*random-variable S X* $\wedge$ *random-variable T Y* $\wedge$
absolutely-continuous P Q $\wedge$ *integrable Q* $($*entropy-density b P Q*$)$ $\wedge$
mutual-information b S T X Y $= 0)$
$\langle$*proof*$\rangle$

abbreviation (**in** *information-space*)
mutual-information-Pow $(\mathcal{I}'(\text{-}\ ;\ \text{-}'))$ **where**
$\mathcal{I}(X\ ;\ Y) \equiv$ *mutual-information b* $($*count-space* $(X$‘*space M*$))$ $($*count-space* $(Y$‘*space M*$))$ *X Y*

lemma (**in** *information-space*)
fixes *Pxy* $::$ $'b \times 'c \Rightarrow$ *real* **and** *Px* $::$ $'b \Rightarrow$ *real* **and** *Py* $::$ $'c \Rightarrow$ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Fx*: *finite-entropy S X Px* **and** *Fy*: *finite-entropy T Y Py*
assumes *Fxy*: *finite-entropy* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
defines $f \equiv \lambda x.$ *Pxy x* $*$ *log b* $($*Pxy x* $/$ $($*Px* $($*fst x*$)$ $*$ *Py* $($*snd x*$)))$
shows *mutual-information-distr'*: *mutual-information b S T X Y* $=$ *integral*^{L} $(S \bigotimes_M T)$ *f* (**is** *?M* $=$ *?R*)
and *mutual-information-nonneg'*: $0 \leq$ *mutual-information b S T X Y*
$\langle$*proof*$\rangle$

lemma (**in** *information-space*)
fixes *Pxy* $::$ $'b \times 'c \Rightarrow$ *real* **and** *Px* $::$ $'b \Rightarrow$ *real* **and** *Py* $::$ $'c \Rightarrow$ *real*
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes *Px*: *distributed M S X Px* **and** *Px-nn*: $\bigwedge x.\ x \in$ *space S* $\Longrightarrow 0 \leq$ *Px x*
and *Py*: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge y.\ y \in$ *space T* $\Longrightarrow 0 \leq$ *Py y*
and *Pxy*: *distributed M* $(S \bigotimes_M T)$ $(\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*: $\bigwedge x\ y.\ x \in$ *space S* $\Longrightarrow y \in$ *space T* $\Longrightarrow 0 \leq$ *Pxy* $(x,\ y)$
defines $f \equiv \lambda x.$ *Pxy x* $*$ *log b* $($*Pxy x* $/$ $($*Px* $($*fst x*$)$ $*$ *Py* $($*snd x*$)))$

shows *mutual-information-distr*: $mutual\text{-}information\ b\ S\ T\ X\ Y = integral^L\ (S \bigotimes_M T)\ f$ (**is** $?M = ?R$)
and *mutual-information-nonneg*: $integrable\ (S \bigotimes_M T)\ f \Longrightarrow 0 \le mutual\text{-}information\ b\ S\ T\ X\ Y$
⟨*proof*⟩

lemma (**in** *information-space*)
fixes $Pxy :: 'b \times 'c \Rightarrow real$ **and** $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$
assumes *sigma-finite-measure S sigma-finite-measure T*
assumes $Px[measurable]$: *distributed M S X Px* **and** *Px-nn*: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \le Px\ x$
and $Py[measurable]$: *distributed M T Y Py* **and** *Py-nn*: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \le Py\ x$
and $Pxy[measurable]$: $distributed\ M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
and *Pxy-nn*: $\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \le Pxy\ x$
assumes *ae*: $AE\ x\ in\ S.\ AE\ y\ in\ T.\ Pxy\ (x,\ y) = Px\ x * Py\ y$
shows *mutual-information-eq-0*: $mutual\text{-}information\ b\ S\ T\ X\ Y = 0$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-simple-distributed*:
assumes *X*: *simple-distributed M X Px* **and** *Y*: *simple-distributed M Y Py*
assumes *XY*: $simple\text{-}distributed\ M\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows $\mathcal{I}(X\ ;\ Y) = (\sum (x,\ y) \in (\lambda x.\ (X\ x,\ Y\ x))\ `space\ M.\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ (Px\ x * Py\ y)))$
⟨*proof*⟩

lemma (**in** *information-space*)
fixes $Pxy :: 'b \times 'c \Rightarrow real$ **and** $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$
assumes *Px*: *simple-distributed M X Px* **and** *Py*: *simple-distributed M Y Py*
assumes *Pxy*: $simple\text{-}distributed\ M\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
assumes *ae*: $\forall x \in space\ M.\ Pxy\ (X\ x,\ Y\ x) = Px\ (X\ x) * Py\ (Y\ x)$
shows *mutual-information-eq-0-simple*: $\mathcal{I}(X\ ;\ Y) = 0$
⟨*proof*⟩

12.5 Entropy

definition (**in** *prob-space*) $entropy :: real \Rightarrow 'b\ measure \Rightarrow ('a \Rightarrow 'b) \Rightarrow real$ **where**
$entropy\ b\ S\ X = -\ KL\text{-}divergence\ b\ S\ (distr\ M\ S\ X)$

abbreviation (**in** *information-space*)
entropy-Pow ($\mathcal{H}'(\text{-}')$) **where**
$\mathcal{H}(X) \equiv entropy\ b\ (count\text{-}space\ (X`space\ M))\ X$

lemma (**in** *prob-space*) *distributed-RN-deriv*:
assumes *X*: *distributed M S X Px*
shows $AE\ x\ in\ S.\ RN\text{-}deriv\ S\ (density\ S\ Px)\ x = Px\ x$
⟨*proof*⟩

lemma (**in** *information-space*)

fixes $X :: 'a \Rightarrow 'b$
assumes $X[measurable]$: *distributed M MX X f* **and** *nn*: $\bigwedge x.\ x \in space\ MX \Longrightarrow 0 \leq f\ x$
shows *entropy-distr*: $entropy\ b\ MX\ X = - (\int x.\ f\ x * log\ b\ (f\ x)\ \partial MX)$ (**is** *?eq*)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le*:
fixes $Px :: 'b \Rightarrow real$ **and** $MX :: 'b\ measure$
assumes $X[measurable]$: *distributed M MX X Px* **and** $Px\text{-}nn[simp]$: $\bigwedge x.\ x \in space\ MX \Longrightarrow 0 \leq Px\ x$
and *fin*: $emeasure\ MX\ \{x \in space\ MX.\ Px\ x \neq 0\} \neq top$
and *int*: $integrable\ MX\ (\lambda x.\ - Px\ x * log\ b\ (Px\ x))$
shows $entropy\ b\ MX\ X \leq log\ b\ (measure\ MX\ \{x \in space\ MX.\ Px\ x \neq 0\})$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-space*:
fixes $Px :: 'b \Rightarrow real$ **and** $MX :: 'b\ measure$
assumes X: *distributed M MX X Px* **and** $Px\text{-}nn[simp]$: $\bigwedge x.\ x \in space\ MX \Longrightarrow 0 \leq Px\ x$
and *fin*: *finite-measure MX*
and *int*: $integrable\ MX\ (\lambda x.\ - Px\ x * log\ b\ (Px\ x))$
shows $entropy\ b\ MX\ X \leq log\ b\ (measure\ MX\ (space\ MX))$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-uniform*:
assumes X: $distributed\ M\ MX\ X\ (\lambda x.\ indicator\ A\ x\ /\ measure\ MX\ A)$ (**is** *distributed - - - ?f*)
shows $entropy\ b\ MX\ X = log\ b\ (measure\ MX\ A)$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-simple-distributed*:
$simple\text{-}distributed\ M\ X\ f \Longrightarrow \mathcal{H}(X) = - (\sum x{\in}X\text{`}space\ M.\ f\ x * log\ b\ (f\ x))$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-card-not-0*:
assumes X: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq log\ b\ (card\ (X\ \text{`}\ space\ M \cap \{x.\ f\ x \neq 0\}))$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-le-card*:
assumes X: *simple-distributed M X f*
shows $\mathcal{H}(X) \leq log\ b\ (real\ (card\ (X\ \text{`}\ space\ M)))$
⟨*proof*⟩

12.6 Conditional Mutual Information

definition (**in** *prob-space*)
$conditional\text{-}mutual\text{-}information\ b\ MX\ MY\ MZ\ X\ Y\ Z \equiv$
$mutual\text{-}information\ b\ MX\ (MY \bigotimes_M MZ)\ X\ (\lambda x.\ (Y\ x,\ Z\ x))\ -$

mutual-information b MX MZ X Z

abbreviation (**in** *information-space*)
conditional-mutual-information-Pow ($\mathcal{I}'$(- ; - | - ')) **where**
$\mathcal{I}(X\ ;\ Y\ |\ Z) \equiv$ *conditional-mutual-information b*
(*count-space* (*X* ' *space M*)) (*count-space* (*Y* ' *space M*)) (*count-space* (*Z* ' *space M*)) *X Y Z*

lemma (**in** *information-space*)
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T* **and** *P*: *sigma-finite-measure P*
assumes *Px*[*measurable*]: *distributed M S X Px*
and *Px-nn*[*simp*]: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
assumes *Pz*[*measurable*]: *distributed M P Z Pz*
and *Pz-nn*[*simp*]: $\bigwedge z.\ z \in space\ P \Longrightarrow 0 \leq Pz\ z$
assumes *Pyz*[*measurable*]: *distributed M* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
and *Pyz-nn*[*simp*]: $\bigwedge y\ z.\ y \in space\ T \Longrightarrow z \in space\ P \Longrightarrow 0 \leq Pyz\ (y,\ z)$
assumes *Pxz*[*measurable*]: *distributed M* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
and *Pxz-nn*[*simp*]: $\bigwedge x\ z.\ x \in space\ S \Longrightarrow z \in space\ P \Longrightarrow 0 \leq Pxz\ (x,\ z)$
assumes *Pxyz*[*measurable*]: *distributed M* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
and *Pxyz-nn*[*simp*]: $\bigwedge x\ y\ z.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow z \in space\ P \Longrightarrow 0 \leq Pxyz\ (x,\ y,\ z)$
assumes *I1*: *integrable* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Px\ x * Pyz\ (y,\ z))))$
assumes *I2*: *integrable* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda(x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxz\ (x,\ z)\ /\ (Px\ x * Pz\ z)))$
shows *conditional-mutual-information-generic-eq*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg*: $0 \leq$ *conditional-mutual-information b S T P X Y Z* (**is** *?nonneg*)
⟨*proof*⟩

lemma (**in** *information-space*)
fixes *Px* :: - ⇒ *real*
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T* **and** *P*: *sigma-finite-measure P*
assumes *Fx*: *finite-entropy S X Px*
assumes *Fz*: *finite-entropy P Z Pz*
assumes *Fyz*: *finite-entropy* $(T \bigotimes_M P)$ $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
assumes *Fxz*: *finite-entropy* $(S \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
assumes *Fxyz*: *finite-entropy* $(S \bigotimes_M T \bigotimes_M P)$ $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
shows *conditional-mutual-information-generic-eq'*: *conditional-mutual-information b S T P X Y Z*
$= (\int (x,\ y,\ z).\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z)))\ \partial(S \bigotimes_M T \bigotimes_M P))$ (**is** *?eq*)
and *conditional-mutual-information-generic-nonneg'*: $0 \leq$ *conditional-mutual-information*

$b\ S\ T\ P\ X\ Y\ Z$ (**is** *?nonneg*)
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-eq*:
assumes *Pz*: *simple-distributed M Z Pz*
assumes *Pyz*: *simple-distributed M* $(\lambda x.\ (Y\ x,\ Z\ x))$ *Pyz*
assumes *Pxz*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Z\ x))$ *Pxz*
assumes *Pxyz*: *simple-distributed M* $(\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))$ *Pxyz*
shows $\mathcal{I}(X\ ;\ Y \mid Z) =$
$(\sum (x,\ y,\ z) \in (\lambda x.\ (X\ x,\ Y\ x,\ Z\ x))`space\ M.\ Pxyz\ (x,\ y,\ z) * log\ b\ (Pxyz\ (x,\ y,\ z)\ /\ (Pxz\ (x,\ z) * (Pyz\ (y,z)\ /\ Pz\ z))))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-nonneg*:
assumes *X*: *simple-function M X* **and** *Y*: *simple-function M Y* **and** *Z*: *simple-function M Z*
shows $0 \leq \mathcal{I}(X\ ;\ Y \mid Z)$
⟨*proof*⟩

12.7 Conditional Entropy

definition (**in** *prob-space*)
$conditional\text{-}entropy\ b\ S\ T\ X\ Y = -\ (\int (x,\ y).\ log\ b\ (enn2real\ (RN\text{-}deriv\ (S \bigotimes_M T)\ (distr\ M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x)))\ (x,\ y))\ /$
$enn2real\ (RN\text{-}deriv\ T\ (distr\ M\ T\ Y)\ y))\ \partial distr\ M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x)))$

abbreviation (**in** *information-space*)
conditional-entropy-Pow $(\mathcal{H}'(\text{-} \mid \text{-}'))$ **where**
$\mathcal{H}(X \mid Y) \equiv conditional\text{-}entropy\ b\ (count\text{-}space\ (X`space\ M))\ (count\text{-}space\ (Y`space\ M))\ X\ Y$

lemma (**in** *information-space*) *conditional-entropy-generic-eq*:
fixes $Pxy :: \text{-} \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Py*[*measurable*]: *distributed M T Y Py* **and** *Py-nn*[*simp*]: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*
and *Pxy-nn*[*simp*]: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \leq Pxy\ (x,\ y)$
shows $conditional\text{-}entropy\ b\ S\ T\ X\ Y = -\ (\int (x,\ y).\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ Py\ y)\ \partial(S \bigotimes_M T))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq-entropy*:
fixes $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$
assumes *S*: *sigma-finite-measure S* **and** *T*: *sigma-finite-measure T*
assumes *Py*[*measurable*]: *distributed M T Y Py*
and *Py-nn*[*simp*]: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes *Pxy*[*measurable*]: *distributed M* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$ *Pxy*

and $Pxy\text{-}nn[simp]$: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \le Pxy\ (x,\ y)$
assumes $I1$: $integrable\ (S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
assumes $I2$: $integrable\ (S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
shows $conditional\text{-}entropy\ b\ S\ T\ X\ Y = entropy\ b\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x)) - entropy\ b\ T\ Y$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq-entropy-simple*:
assumes X: *simple-function* $M\ X$ **and** Y: *simple-function* $M\ Y$
shows $\mathcal{H}(X \mid Y) = entropy\ b\ (count\text{-}space\ (X`space\ M) \bigotimes_M count\text{-}space\ (Y`space\ M))\ (\lambda x.\ (X\ x,\ Y\ x)) - \mathcal{H}(Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-eq*:
assumes Y: *simple-distributed* $M\ Y\ Py$
assumes XY: *simple-distributed* $M\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows $\mathcal{H}(X \mid Y) = -\ (\sum (x,\ y)\in(\lambda x.\ (X\ x,\ Y\ x))\ `\ space\ M.\ Pxy\ (x,\ y) * log\ b\ (Pxy\ (x,\ y)\ /\ Py\ y))$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-mutual-information-eq-conditional-entropy*:
assumes X: *simple-function* $M\ X$ **and** Y: *simple-function* $M\ Y$
shows $\mathcal{I}(X\ ;\ X \mid Y) = \mathcal{H}(X \mid Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-nonneg*:
assumes X: *simple-function* $M\ X$ **and** Y: *simple-function* $M\ Y$ **shows** $0 \le \mathcal{H}(X \mid Y)$
⟨*proof*⟩

12.8 Equalities

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy-distr*:
fixes $Px :: 'b \Rightarrow real$ **and** $Py :: 'c \Rightarrow real$ **and** $Pxy :: ('b \times 'c) \Rightarrow real$
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes $Px[measurable]$: *distributed* $M\ S\ X\ Px$
and $Px\text{-}nn[simp]$: $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \le Px\ x$
and $Py[measurable]$: *distributed* $M\ T\ Y\ Py$
and $Py\text{-}nn[simp]$: $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \le Py\ x$
and $Pxy[measurable]$: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
and $Pxy\text{-}nn[simp]$: $\bigwedge x\ y.\ x \in space\ S \Longrightarrow y \in space\ T \Longrightarrow 0 \le Pxy\ (x,\ y)$
assumes Ix: $integrable(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes Iy: $integrable(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes Ixy: $integrable(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows $mutual\text{-}information\ b\ S\ T\ X\ Y = entropy\ b\ S\ X + entropy\ b\ T\ Y - entropy\ b\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy'*:

fixes Px :: $'b \Rightarrow real$ **and** Py :: $'c \Rightarrow real$ **and** Pxy :: $('b \times 'c) \Rightarrow real$
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes Px: *distributed* $M\ S\ X\ Px$ $\bigwedge x.\ x \in space\ S \Longrightarrow 0 \leq Px\ x$
and Py: *distributed* $M\ T\ Y\ Py$ $\bigwedge x.\ x \in space\ T \Longrightarrow 0 \leq Py\ x$
assumes Pxy: *distributed* $M\ (S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
$\bigwedge x.\ x \in space\ (S \bigotimes_M T) \Longrightarrow 0 \leq Pxy\ x$
assumes Ix: $integrable(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Px\ (fst\ x)))$
assumes Iy: $integrable(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Py\ (snd\ x)))$
assumes Ixy: $integrable(S \bigotimes_M T)\ (\lambda x.\ Pxy\ x * log\ b\ (Pxy\ x))$
shows *mutual-information* $b\ S\ T\ X\ Y$ = *entropy* $b\ S\ X$ − *conditional-entropy* b $S\ T\ X\ Y$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-eq-entropy-conditional-entropy*:
assumes *sf-X*: *simple-function* $M\ X$ **and** *sf-Y*: *simple-function* $M\ Y$
shows $\mathcal{I}(X\ ;\ Y) = \mathcal{H}(X) - \mathcal{H}(X\ |\ Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *mutual-information-nonneg-simple*:
assumes *sf-X*: *simple-function* $M\ X$ **and** *sf-Y*: *simple-function* $M\ Y$
shows $0 \leq \mathcal{I}(X\ ;\ Y)$
⟨*proof*⟩

lemma (**in** *information-space*) *conditional-entropy-less-eq-entropy*:
assumes X: *simple-function* $M\ X$ **and** Z: *simple-function* $M\ Z$
shows $\mathcal{H}(X\ |\ Z) \leq \mathcal{H}(X)$
⟨*proof*⟩

lemma (**in** *information-space*)
fixes Px :: $'b \Rightarrow real$ **and** Py :: $'c \Rightarrow real$ **and** Pxy :: $('b \times 'c) \Rightarrow real$
assumes S: *sigma-finite-measure* S **and** T: *sigma-finite-measure* T
assumes Px: *finite-entropy* $S\ X\ Px$ **and** Py: *finite-entropy* $T\ Y\ Py$
assumes Pxy: *finite-entropy* $(S \bigotimes_M T)\ (\lambda x.\ (X\ x,\ Y\ x))\ Pxy$
shows *conditional-entropy* $b\ S\ T\ X\ Y \leq$ *entropy* $b\ S\ X$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-chain-rule*:
assumes X: *simple-function* $M\ X$ **and** Y: *simple-function* $M\ Y$
shows $\mathcal{H}(\lambda x.\ (X\ x,\ Y\ x)) = \mathcal{H}(X) + \mathcal{H}(Y|X)$
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-partition*:
assumes X: *simple-function* $M\ X$
shows $\mathcal{H}(X) = \mathcal{H}(f \circ X) + \mathcal{H}(X|f \circ X)$
⟨*proof*⟩

corollary (**in** *information-space*) *entropy-data-processing*:
assumes X: *simple-function* $M\ X$ **shows** $\mathcal{H}(f \circ X) \leq \mathcal{H}(X)$
⟨*proof*⟩

corollary (**in** *information-space*) *entropy-of-inj*:
  assumes *X*: *simple-function M X* **and** *inj*: *inj-on f* (*X'space M*)
  shows $\mathcal{H}(f \circ X) = \mathcal{H}(X)$
⟨*proof*⟩

end

13 Properties of Various Distributions

theory *Distributions*
  imports *Convolution Information*
begin

lemma (**in** *prob-space*) *distributed-affine*:
  fixes *f* :: *real* ⇒ *ennreal*
  assumes *f*: *distributed M lborel X f*
  assumes *c*: $c \neq 0$
  shows *distributed M lborel* ($\lambda x.\ t + c * X\ x$) ($\lambda x.\ f\ ((x - t) / c) / |c|$)
  ⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-affineI*:
  fixes *f* :: *real* ⇒ *ennreal* **and** *c* :: *real*
  assumes *f*: *distributed M lborel* ($\lambda x.\ (X\ x - t) / c$) ($\lambda x.\ |c| * f\ (x * c + t)$)
  assumes *c*: $c \neq 0$
  shows *distributed M lborel X f*
⟨*proof*⟩

lemma (**in** *prob-space*) *distributed-AE2*:
  assumes [*measurable*]: *distributed M N X f Measurable.pred N P*
  shows (*AE x in M. P* (*X x*)) ⟷ (*AE x in N.* $0 < f\ x$ ⟶ *P x*)
⟨*proof*⟩

13.1 Erlang

lemma *nn-intergal-power-times-exp-Icc*:
  assumes [*arith*]: $0 \leq a$
  shows ($\int^{+} x.$ *ennreal* ($x\hat{\ }k * exp\ (-x)$) $*$ *indicator* $\{0\ ..\ a\}$ *x* ∂*lborel*) =
    ($1 - (\sum n{\leq}k.\ (a\hat{\ }n * exp\ (-a)) / fact\ n)) * fact\ k$ (**is** *?I* = -)
⟨*proof*⟩

lemma *nn-intergal-power-times-exp-Ici*:
  shows ($\int^{+} x.$ *ennreal* ($x\hat{\ }k * exp\ (-x)$) $*$ *indicator* $\{0\ ..\}$ *x* ∂*lborel*) = *real-of-nat* (*fact k*)
⟨*proof*⟩

definition *erlang-density* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real* **where**
  erlang-density k l x = (*if* $x < 0$ *then 0 else* ($l\hat{\ }(Suc\ k) * x\hat{\ }k * exp\ (- l * x)) / fact\ k$)

definition *erlang-CDF* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real* **where**
erlang-CDF k l x = (*if x* < *0 then 0 else 1* − ($\sum$ *n*≤*k*. ((*l* ∗ *x*)^*n* ∗ *exp* (− *l* ∗ *x*) / *fact n*)))

lemma *erlang-density-nonneg*[*simp*]: *0* ≤ *l* ⟹ *0* ≤ *erlang-density k l x*
⟨*proof*⟩

lemma *borel-measurable-erlang-density*[*measurable*]: *erlang-density k l* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *erlang-CDF-transform*: *0* < *l* ⟹ *erlang-CDF k l a* = *erlang-CDF k 1* (*l* ∗ *a*)
⟨*proof*⟩

lemma *erlang-CDF-nonneg*[*simp*]: **assumes** *0* < *l* **shows** *0* ≤ *erlang-CDF k l x*
⟨*proof*⟩

lemma *nn-integral-erlang-density*:
assumes [*arith*]: *0* < *l*
shows ($\int^+$ *x*. *ennreal* (*erlang-density k l x*) ∗ *indicator* {.. *a*} *x* ∂*lborel*) = *erlang-CDF k l a*
⟨*proof*⟩

lemma *emeasure-erlang-density*:
0 < *l* ⟹ *emeasure* (*density lborel* (*erlang-density k l*)) {.. *a*} = *erlang-CDF k l a*
⟨*proof*⟩

lemma *nn-integral-erlang-ith-moment*:
fixes *k i* :: *nat* **and** *l* :: *real*
assumes [*arith*]: *0* < *l*
shows ($\int^+$ *x*. *ennreal* (*erlang-density k l x* ∗ *x* ^ *i*) ∂*lborel*) = *fact* (*k* + *i*) / (*fact k* ∗ *l* ^ *i*)
⟨*proof*⟩

lemma *prob-space-erlang-density*:
assumes *l*[*arith*]: *0* < *l*
shows *prob-space* (*density lborel* (*erlang-density k l*)) (**is** *prob-space ?D*)
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-le*:
assumes *D*: *distributed M lborel X* (*erlang-density k l*)
assumes [*simp*, *arith*]: *0* < *l* *0* ≤ *a*
shows $\mathcal{P}$(*x in M*. *X x* ≤ *a*) = *erlang-CDF k l a*
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-gt*:

assumes $D[simp]$: *distributed M lborel X* $($*erlang-density k l*$)$
assumes $[arith]$: $0 < l\ 0 \le a$
shows $\mathcal{P}(x\ in\ M.\ a < X\ x\ ) = 1 - (erlang\text{-}CDF\ k\ l\ a)$
⟨*proof*⟩

lemma *erlang-CDF-at0*: $erlang\text{-}CDF\ k\ l\ 0 = 0$
⟨*proof*⟩

lemma *erlang-distributedI*:
assumes $X[measurable]$: $X \in borel\text{-}measurable\ M$ **and** $[arith]$: $0 < l$
and *X-distr*: $\bigwedge a.\ 0 \le a \Longrightarrow emeasure\ M\ \{x{\in}space\ M.\ X\ x \le a\} = erlang\text{-}CDF\ k\ l\ a$
shows *distributed M lborel X* $($*erlang-density k l*$)$
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-iff*:
assumes $[arith]$: $0{<}l$
shows *distributed M lborel X* $($*erlang-density k l*$) \longleftrightarrow$
$(X \in borel\text{-}measurable\ M \wedge 0 < l \wedge (\forall\, a{\ge}0.\ \mathcal{P}(x\ in\ M.\ X\ x \le a) = erlang\text{-}CDF\ k\ l\ a\ ))$
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-mult-const*:
assumes *erlX*: *distributed M lborel X* $($*erlang-density k l*$)$
assumes *a-pos*$[arith]$: $0 < \alpha\ \ 0 < l$
shows *distributed M lborel* $(\lambda x.\ \alpha * X\ x)$ $($*erlang-density k* $(l\ /\ \alpha))$
⟨*proof*⟩

lemma (**in** *prob-space*) *has-bochner-integral-erlang-ith-moment*:
fixes $k\ i$ :: *nat* **and** l :: *real*
assumes $[arith]$: $0 < l$ **and** D: *distributed M lborel X* $($*erlang-density k l*$)$
shows *has-bochner-integral M* $(\lambda x.\ X\ x\ \hat{}\ i)$ $(fact\ (k + i)\ /\ (fact\ k * l\ \hat{}\ i))$
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-ith-moment-integrable*:
$0 < l \Longrightarrow$ *distributed M lborel X* $($*erlang-density k l*$) \Longrightarrow$ *integrable M* $(\lambda x.\ X\ x\ \hat{}\ i)$
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-ith-moment*:
$0 < l \Longrightarrow$ *distributed M lborel X* $($*erlang-density k l*$) \Longrightarrow$
expectation $(\lambda x.\ X\ x\ \hat{}\ i) = fact\ (k + i)\ /\ (fact\ k * l\ \hat{}\ i)$
⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-variance*:
assumes $[arith]$: $0 < l$ **and** *distributed M lborel X* $($*erlang-density k l*$)$
shows *variance* $X = (k + 1)\ /\ l^2$
⟨*proof*⟩

13.2 Exponential distribution

abbreviation *exponential-density* :: *real* ⇒ *real* ⇒ *real* **where**
exponential-density ≡ *erlang-density 0*

lemma *exponential-density-def*:
exponential-density l x = (*if x* < *0 then 0 else l* ∗ *exp* (− *x* ∗ *l*))
⟨*proof*⟩

lemma *erlang-CDF-0*: *erlang-CDF 0 l a* = (*if 0* ≤ *a then 1* − *exp* (− *l* ∗ *a*) *else 0*)
⟨*proof*⟩

lemma *prob-space-exponential-density*: *0* < *l* ⟹ *prob-space* (*density lborel* (*exponential-density l*))
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributedD-le*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0* ≤ *a* **and** *l*: *0* < *l*
shows 𝒫(*x in M*. *X x* ≤ *a*) = *1* − *exp* (− *a* ∗ *l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributedD-gt*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0* ≤ *a* **and** *l*: *0* < *l*
shows 𝒫(*x in M*. *a* < *X x*) = *exp* (− *a* ∗ *l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-memoryless*:
assumes *D*: *distributed M lborel X* (*exponential-density l*) **and** *a*: *0* ≤ *a* **and** *l*: *0* < *l* **and** *t*: *0* ≤ *t*
shows 𝒫(*x in M*. *a* + *t* < *X x* | *a* < *X x*) = 𝒫(*x in M*. *t* < *X x*)
⟨*proof*⟩

lemma *exponential-distributedI*:
assumes *X*[*measurable*]: *X* ∈ *borel-measurable M* **and** [*arith*]: *0* < *l*
and *X-distr*: ⋀*a*. *0* ≤ *a* ⟹ *emeasure M* {*x*∈*space M*. *X x* ≤ *a*} = *1* − *exp* (− *a* ∗ *l*)
shows *distributed M lborel X* (*exponential-density l*)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-iff*:
assumes *0* < *l*
shows *distributed M lborel X* (*exponential-density l*) ⟷
(*X* ∈ *borel-measurable M* ∧ (∀ *a*≥*0*. 𝒫(*x in M*. *X x* ≤ *a*) = *1* − *exp* (− *a* ∗ *l*)))
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-expectation*:

$0 < l \Longrightarrow$ *distributed M lborel X* (*exponential-density l*) $\Longrightarrow$ *expectation X* $= 1$ / l
⟨*proof*⟩

lemma *exponential-density-nonneg*: $0 < l \Longrightarrow 0 \leq$ *exponential-density l x*
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-min*:
assumes $0 < l$ $0 < u$
assumes *expX*: *distributed M lborel X* (*exponential-density l*)
assumes *expY*: *distributed M lborel Y* (*exponential-density u*)
assumes *ind*: *indep-var borel X borel Y*
shows *distributed M lborel* ($\lambda x.$ *min* (*X x*) (*Y x*)) (*exponential-density* ($l + u$))
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-Min*:
assumes *finI*: *finite I*
assumes *A*: $I \neq \{\}$
assumes *l*: $\bigwedge i.\ i \in I \Longrightarrow 0 < l\ i$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*exponential-density* (*l i*))
assumes *ind*: *indep-vars* ($\lambda i.$ *borel*) *X I*
shows *distributed M lborel* ($\lambda x.$ *Min* (($\lambda i.$ *X i x*)'*I*)) (*exponential-density* ($\sum i{\in}I.$ *l i*))
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-variance*:
$0 < l \Longrightarrow$ *distributed M lborel X* (*exponential-density l*) $\Longrightarrow$ *variance X* $= 1 / l^2$
⟨*proof*⟩

lemma *nn-integral-zero'*: *AE x in M*. $f\ x = 0 \Longrightarrow (\int^+ x.\ f\ x\ \partial M) = 0$
⟨*proof*⟩

lemma *convolution-erlang-density*:
fixes k_1 k_2 :: *nat*
assumes [*simp*, *arith*]: $0 < l$
shows ($\lambda x.\ \int^+ y.$ *ennreal* (*erlang-density* k_1 l $(x - y)$) $*$ *ennreal* (*erlang-density* k_2 l y) ∂*lborel*) =
(*erlang-density* (*Suc* k_1 + *Suc* k_2 − *1*) *l*)
(**is** *?LHS = ?RHS*)
⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-erlang*:
assumes *indep*: *indep-var borel X borel Y*
assumes [*simp*, *arith*]: $0 < l$
assumes *erlX*: *distributed M lborel X* (*erlang-density* k_1 *l*)
assumes *erlY*: *distributed M lborel Y* (*erlang-density* k_2 *l*)
shows *distributed M lborel* ($\lambda x.$ *X x* + *Y x*) (*erlang-density* (*Suc* k_1 + *Suc* k_2 − *1*) *l*)

⟨*proof*⟩

lemma (**in** *prob-space*) *erlang-distributed-sum*:
assumes *finI* : *finite I*
assumes *A*: $I \neq \{\}$
assumes [*simp*, *arith*]: $0 < l$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* $(X\ i)$ (*erlang-density* $(k\ i)\ l$)
assumes *ind*: *indep-vars* ($\lambda i.$ *borel*) *X I*
shows *distributed M lborel* ($\lambda x.\ \sum i{\in}I.\ X\ i\ x$) (*erlang-density* $((\sum i{\in}I.\ Suc\ (k\ i)) - 1)\ l$)
⟨*proof*⟩

lemma (**in** *prob-space*) *exponential-distributed-sum*:
assumes *finI*: *finite I*
assumes *A*: $I \neq \{\}$
assumes *l*: $0 < l$
assumes *expX*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* $(X\ i)$ (*exponential-density l*)
assumes *ind*: *indep-vars* ($\lambda i.$ *borel*) *X I*
shows *distributed M lborel* ($\lambda x.\ \sum i{\in}I.\ X\ i\ x$) (*erlang-density* ((*card I*) $-$ *1*) *l*)
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-exponential*:
assumes *l*[*simp*, *arith*]: $0 < l$
assumes *D*: *distributed M lborel X* (*exponential-density l*)
shows *entropy b lborel X* = *log b* (*exp 1* / *l*)
⟨*proof*⟩

13.3 Uniform distribution

lemma *uniform-distrI*:
assumes *X*: $X \in$ *measurable M M'*
and *A*: $A \in$ *sets M' emeasure M' A* $\neq \infty$ *emeasure M' A* $\neq 0$
assumes *distr*: $\bigwedge B.\ B \in$ *sets M'* $\Longrightarrow$ *emeasure M* (X −‘ $B \cap$ *space M*) = *emeasure M'* $(A \cap B)$ / *emeasure M' A*
shows *distr M M' X* = *uniform-measure M' A*
⟨*proof*⟩

lemma *uniform-distrI-borel*:
fixes *A* :: *real set*
assumes *X*[*measurable*]: $X \in$ *borel-measurable M* **and** *A*: *emeasure lborel A* = *ennreal r* $0 < r$
and [*measurable*]: $A \in$ *sets borel*
assumes *distr*: $\bigwedge a.$ *emeasure M* $\{x{\in}$*space M*$.\ X\ x \le a\}$ = *emeasure lborel* $(A \cap \{..\ a\})$ / *r*
shows *distributed M lborel X* ($\lambda x.$ *indicator A x* / *measure lborel A*)
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distrI-borel-atLeastAtMost*:
fixes *a b* :: *real*

assumes X: $X \in$ *borel-measurable* M **and** $a < b$
assumes *distr*: $\bigwedge t.\ a \leq t \Longrightarrow t \leq b \Longrightarrow \mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)$
shows *distributed M lborel X* $(\lambda x.\ indicator\ \{a..b\}\ x\ /\ measure\ lborel\ \{a..b\})$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-measure*:
fixes $a\ b$ :: *real*
assumes D: *distributed M lborel X* $(\lambda x.\ indicator\ \{a\ ..\ b\}\ x\ /\ measure\ lborel\ \{a\ ..\ b\})$
assumes t: $a \leq t\ t \leq b$
shows $\mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-bounds*:
fixes $a\ b$ :: *real*
assumes D: *distributed M lborel X* $(\lambda x.\ indicator\ \{a\ ..\ b\}\ x\ /\ measure\ lborel\ \{a\ ..\ b\})$
shows $a < b$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-iff*:
fixes $a\ b$ :: *real*
shows *distributed M lborel X* $(\lambda x.\ indicator\ \{a..b\}\ x\ /\ measure\ lborel\ \{a..b\}) \longleftrightarrow$
$(X \in$ *borel-measurable* $M \wedge a < b \wedge (\forall t \in \{a\ ..\ b\}.\ \mathcal{P}(x\ in\ M.\ X\ x \leq t) = (t - a)\ /\ (b - a)))$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-expectation*:
fixes $a\ b$ :: *real*
assumes D: *distributed M lborel X* $(\lambda x.\ indicator\ \{a\ ..\ b\}\ x\ /\ measure\ lborel\ \{a\ ..\ b\})$
shows *expectation* $X = (a + b)\ /\ 2$
⟨*proof*⟩

lemma (**in** *prob-space*) *uniform-distributed-variance*:
fixes $a\ b$ :: *real*
assumes D: *distributed M lborel X* $(\lambda x.\ indicator\ \{a\ ..\ b\}\ x\ /\ measure\ lborel\ \{a\ ..\ b\})$
shows *variance* $X = (b - a)^2\ /\ 12$
⟨*proof*⟩

13.4 Normal distribution

definition *normal-density* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* **where**
normal-density $\mu\ \sigma\ x = 1\ /\ sqrt\ (2 * pi * \sigma^2) * exp\ (-(x - \mu)^2 /\ (2 * \sigma^2))$

abbreviation *std-normal-density* :: *real* ⇒ *real* **where**
std-normal-density ≡ *normal-density 0 1*

lemma *std-normal-density-def*: *std-normal-density* $x = (1\ /\ sqrt\ (2 * pi)) * exp\ (- x^2\ /\ 2)$
⟨*proof*⟩

lemma *normal-density-nonneg*[*simp*]: $0 \leq$ *normal-density* $\mu\ \sigma\ x$
⟨*proof*⟩

lemma *normal-density-pos*: $0 < \sigma \Longrightarrow 0 <$ *normal-density* $\mu\ \sigma\ x$
⟨*proof*⟩

lemma *borel-measurable-normal-density*[*measurable*]: *normal-density* $\mu\ \sigma \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *gaussian-moment-0*:
has-bochner-integral lborel $(\lambda x.\ indicator\ \{0..\}\ x *_R exp\ (- x^2))\ (sqrt\ pi\ /\ 2)$
⟨*proof*⟩

lemma *gaussian-moment-1*:
has-bochner-integral lborel $(\lambda x{::}real.\ indicator\ \{0..\}\ x *_R (exp\ (- x^2) * x))\ (1\ /\ 2)$
⟨*proof*⟩

lemma
fixes k :: *nat*
shows *gaussian-moment-even-pos*:
has-bochner-integral lborel $(\lambda x{::}real.\ indicator\ \{0..\}\ x *_R (exp\ (-x^2)*x\widehat{\ }(2 * k)))$
$((sqrt\ pi\ /\ 2) * (fact\ (2 * k)\ /\ (2\ \widehat{\ }(2 * k) * fact\ k)))$
(**is** *?even*)
and *gaussian-moment-odd-pos*:
has-bochner-integral lborel $(\lambda x{::}real.\ indicator\ \{0..\}\ x *_R (exp\ (-x^2)*x\widehat{\ }(2 * k + 1)))\ (fact\ k\ /\ 2)$
(**is** *?odd*)
⟨*proof*⟩

context
fixes k :: *nat* **and** $\mu\ \sigma$ :: *real* **assumes** [*arith*]: $0 < \sigma$
begin

lemma *normal-moment-even*:
has-bochner-integral lborel $(\lambda x.\ normal\text{-}density\ \mu\ \sigma\ x * (x - \mu)\ \widehat{\ }(2 * k))\ (fact\ (2 * k)\ /\ ((2\ /\ \sigma^2)\widehat{\ }k * fact\ k))$
⟨*proof*⟩

lemma *normal-moment-abs-odd*:
has-bochner-integral lborel $(\lambda x.\ normal\text{-}density\ \mu\ \sigma\ x * |x - \mu|\widehat{\ }(2 * k + 1))\ (2\widehat{\ }k * \sigma\widehat{\ }(2 * k + 1) * fact\ k * sqrt\ (2\ /\ pi))$

⟨*proof*⟩

lemma *normal-moment-odd*:
has-bochner-integral lborel ($\lambda x.$ *normal-density* μ σ $x * (x - \mu)$^$(2 * k + 1)$) *0*
⟨*proof*⟩

lemma *integral-normal-moment-even*:
$integral^L$ *lborel* ($\lambda x.$ *normal-density* μ σ $x * (x - \mu)$^$(2 * k)$) = *fact* $(2 * k)$ / $((2 / \sigma^2)$^$k *$ *fact* $k)$
⟨*proof*⟩

lemma *integral-normal-moment-abs-odd*:
$integral^L$ *lborel* ($\lambda x.$ *normal-density* μ σ $x * |x - \mu|$^$(2 * k + 1)$) = 2 ^ $k * \sigma$ ^ $(2 * k + 1) *$ *fact* $k *$ *sqrt* $(2 / pi)$
⟨*proof*⟩

lemma *integral-normal-moment-odd*:
$integral^L$ *lborel* ($\lambda x.$ *normal-density* μ σ $x * (x - \mu)$^$(2 * k + 1)$) = *0*
⟨*proof*⟩

end

context
fixes σ :: *real*
assumes σ*-pos*[*arith*]: $0 < \sigma$
begin

lemma *normal-moment-nz-1*: *has-bochner-integral lborel* ($\lambda x.$ *normal-density* μ σ $x * x$) μ
⟨*proof*⟩

lemma *integral-normal-moment-nz-1*:
$integral^L$ *lborel* ($\lambda x.$ *normal-density* μ σ $x * x$) = μ
⟨*proof*⟩

lemma *integrable-normal-moment-nz-1*: *integrable lborel* ($\lambda x.$ *normal-density* μ σ $x * x$)
⟨*proof*⟩

lemma *integrable-normal-moment*: *integrable lborel* ($\lambda x.$ *normal-density* μ σ $x * (x - \mu)$^k)
⟨*proof*⟩

lemma *integrable-normal-moment-abs*: *integrable lborel* ($\lambda x.$ *normal-density* μ σ $x * |x - \mu|$^k)
⟨*proof*⟩

lemma *integrable-normal-density*[*simp*, *intro*]: *integrable lborel* (*normal-density* μ

σ)
⟨proof⟩

lemma *integral-normal-density*[*simp*]: ($\int x.$ *normal-density* μ σ x ∂*lborel*) = *1*
⟨proof⟩

lemma *prob-space-normal-density*:
prob-space (*density lborel* (*normal-density* μ σ))
⟨proof⟩

end

context
fixes k :: *nat*
begin

lemma *std-normal-moment-even*:
has-bochner-integral lborel ($\lambda x.$ *std-normal-density* $x * x$ ^ ($2 * k$)) (*fact* ($2 * k$) / (2^k * *fact* k))
⟨proof⟩

lemma *std-normal-moment-abs-odd*:
has-bochner-integral lborel ($\lambda x.$ *std-normal-density* $x * |x|$^($2 * k + 1$)) (*sqrt* ($2/pi$) * 2^k * *fact* k)
⟨proof⟩

lemma *std-normal-moment-odd*:
has-bochner-integral lborel ($\lambda x.$ *std-normal-density* $x * x$^($2 * k + 1$)) *0*
⟨proof⟩

lemma *integral-std-normal-moment-even*:
integral^L *lborel* ($\lambda x.$ *std-normal-density* $x * x$^($2*k$)) = *fact* ($2 * k$) / (2^k * *fact* k)
⟨proof⟩

lemma *integral-std-normal-moment-abs-odd*:
integral^L *lborel* ($\lambda x.$ *std-normal-density* $x * |x|$^($2 * k + 1$)) = *sqrt* (2 / *pi*) * 2^k * *fact* k
⟨proof⟩

lemma *integral-std-normal-moment-odd*:
integral^L *lborel* ($\lambda x.$ *std-normal-density* $x * x$^($2 * k + 1$)) = *0*
⟨proof⟩

lemma *integrable-std-normal-moment-abs*: *integrable lborel* ($\lambda x.$ *std-normal-density* $x * |x|$^k)
⟨proof⟩

lemma *integrable-std-normal-moment*: *integrable lborel* (λx. *std-normal-density* $x * x$^k)
  ⟨*proof*⟩

end

lemma (**in** *prob-space*) *normal-density-affine*:
  assumes X: *distributed M lborel X* (*normal-density* μ σ)
  assumes [*simp*, *arith*]: $0 < \sigma$ $\alpha \neq 0$
  shows *distributed M lborel* (λx. $\beta + \alpha * X x$) (*normal-density* ($\beta + \alpha * \mu$) ($|\alpha| * \sigma$))
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-standard-normal-convert*:
  assumes *pos-var*[*simp*, *arith*]: $0 < \sigma$
  shows *distributed M lborel X* (*normal-density* μ σ) = *distributed M lborel* (λx. ($X x - \mu$) / σ) *std-normal-density*
⟨*proof*⟩

lemma *conv-normal-density-zero-mean*:
  assumes [*simp*, *arith*]: $0 < \sigma$ $0 < \tau$
  shows (λx. $\int^+ y$. *ennreal* (*normal-density* 0 σ ($x - y$) $*$ *normal-density* 0 τ y) ∂*lborel*) =
    normal-density 0 (*sqrt* ($\sigma^2 + \tau^2$)) (**is** *?LHS* = *?RHS*)
⟨*proof*⟩

lemma *conv-std-normal-density*:
  (λx. $\int^+ y$. *ennreal* (*std-normal-density* ($x - y$) $*$ *std-normal-density* y) ∂*lborel*) =
  (*normal-density* 0 (*sqrt* 2))
  ⟨*proof*⟩

lemma (**in** *prob-space*) *add-indep-normal*:
  assumes *indep*: *indep-var borel X borel Y*
  assumes *pos-var*[*arith*]: $0 < \sigma$ $0 < \tau$
  assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density* μ σ)
  assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density* ν τ)
  shows *distributed M lborel* (λx. $X x + Y x$) (*normal-density* ($\mu + \nu$) (*sqrt* ($\sigma^2 + \tau^2$)))
⟨*proof*⟩

lemma (**in** *prob-space*) *diff-indep-normal*:
  assumes *indep*[*simp*]: *indep-var borel X borel Y*
  assumes [*simp*, *arith*]: $0 < \sigma$ $0 < \tau$
  assumes *normalX*[*simp*]: *distributed M lborel X* (*normal-density* μ σ)
  assumes *normalY*[*simp*]: *distributed M lborel Y* (*normal-density* ν τ)
  shows *distributed M lborel* (λx. $X x - Y x$) (*normal-density* ($\mu - \nu$) (*sqrt* ($\sigma^2 + \tau^2$)))

⟨*proof*⟩

lemma (**in** *prob-space*) *sum-indep-normal*:
assumes *finite I I* $\neq$ {} *indep-vars* (λ*i. borel*) *X I*
assumes $\bigwedge i.\ i \in I \Longrightarrow 0 < \sigma\ i$
assumes *normal*: $\bigwedge i.\ i \in I \Longrightarrow$ *distributed M lborel* (*X i*) (*normal-density* (μ *i*) (σ *i*))
shows *distributed M lborel* ($\lambda x.\ \sum i{\in}I.\ X\ i\ x$) (*normal-density* ($\sum i{\in}I.\ \mu\ i$) (*sqrt* ($\sum i{\in}I.\ (\sigma\ i)^2$)))
⟨*proof*⟩

lemma (**in** *prob-space*) *standard-normal-distributed-expectation*:
assumes *D*: *distributed M lborel X std-normal-density*
shows *expectation X* = *0*
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-distributed-expectation*:
assumes σ[*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *expectation X* = μ
⟨*proof*⟩

lemma (**in** *prob-space*) *normal-distributed-variance*:
fixes *a b* :: *real*
assumes [*simp*, *arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *variance X* = σ^2
⟨*proof*⟩

lemma (**in** *prob-space*) *standard-normal-distributed-variance*:
distributed M lborel X std-normal-density $\Longrightarrow$ *variance X* = *1*
⟨*proof*⟩

lemma (**in** *information-space*) *entropy-normal-density*:
assumes [*arith*]: $0 < \sigma$
assumes *D*: *distributed M lborel X* (*normal-density* μ σ)
shows *entropy b lborel X* = *log b* ($2 * pi * exp\ 1 * \sigma^2$) / *2*
⟨*proof*⟩

end

14 Characteristic Functions

theory *Characteristic-Functions*
imports *Weak-Convergence Independent-Family Distributions*
begin

lemma *mult-min-right*: $a \geq 0 \Longrightarrow (a :: real) * min\ b\ c = min\ (a * b)\ (a * c)$
⟨*proof*⟩

lemma *sequentially-even-odd*:
assumes *E*: *eventually* ($\lambda n.$ *P* ($2 * n$)) *sequentially* **and** *O*: *eventually* ($\lambda n.$ *P* ($2 * n + 1$)) *sequentially*
shows *eventually P sequentially*
⟨*proof*⟩

lemma *limseq-even-odd*:
assumes ($\lambda n.$ f ($2 * n$)) $\longrightarrow$ (l :: $'a$ :: *topological-space*)
and ($\lambda n.$ f ($2 * n + 1$)) $\longrightarrow$ l
shows $f \longrightarrow l$
⟨*proof*⟩

14.1 Application of the FTC: integrating $e^i x$

abbreviation *iexp* :: *real* ⇒ *complex* **where**
iexp ≡ ($\lambda x.$ *exp* (i ∗ *complex-of-real* x))

lemma *isCont-iexp* [*simp*]: *isCont iexp x*
⟨*proof*⟩

lemma *has-vector-derivative-iexp*[*derivative-intros*]:
(*iexp has-vector-derivative* i ∗ *iexp x*) (*at x within s*)
⟨*proof*⟩

lemma *interval-integral-iexp*:
fixes *a b* :: *real*
shows (*CLBINT x=a..b. iexp x*) = i ∗ *iexp a* − i ∗ *iexp b*
⟨*proof*⟩

14.2 The Characteristic Function of a Real Measure.

definition
char :: *real measure* ⇒ *real* ⇒ *complex*
where
char M t = *CLINT x*|*M*. *iexp* ($t * x$)

lemma (**in** *real-distribution*) *char-zero*: *char M 0* = *1*
⟨*proof*⟩

lemma (**in** *prob-space*) *integrable-iexp*:
assumes *f*: *f* ∈ *borel-measurable M* $\bigwedge x.$ *Im* (*f x*) = *0*
shows *integrable M* ($\lambda x.$ *exp* (i ∗ (*f x*)))
⟨*proof*⟩

lemma (**in** *real-distribution*) *cmod-char-le-1*: *norm* (*char M t*) ≤ *1*
⟨*proof*⟩

lemma (**in** *real-distribution*) *isCont-char*: *isCont* (*char M*) *t*
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-measurable* [*measurable*]: *char M* $\in$ *borel-measurable borel*
⟨*proof*⟩

14.3 Independence

lemma (**in** *prob-space*) *char-distr-add*:
fixes *X1 X2* :: *'a* $\Rightarrow$ *real* **and** *t* :: *real*
assumes *indep-var borel X1 borel X2*
shows *char* (*distr M borel* ($\lambda\omega$. *X1* ω + *X2* ω)) *t* =
char (*distr M borel X1*) *t* $*$ *char* (*distr M borel X2*) *t*
⟨*proof*⟩

lemma (**in** *prob-space*) *char-distr-sum*:
indep-vars (λi. *borel*) *X A* $\Longrightarrow$
char (*distr M borel* ($\lambda\omega$. $\sum$ *i*$\in$*A*. *X i* ω)) *t* = ($\prod$ *i*$\in$*A*. *char* (*distr M borel* (*X i*)) *t*)
⟨*proof*⟩

14.4 Approximations to e^{ix}

Proofs from Billingsley, page 343.

lemma *CLBINT-I0c-power-mirror-iexp*:
fixes *x* :: *real* **and** *n* :: *nat*
defines *f s m* $\equiv$ *complex-of-real* ((*x* − *s*) ^ *m*)
shows (*CLBINT s=0..x. f s n* $*$ *iexp s*) =
x^*Suc n* / *Suc n* + (i / *Suc n*) $*$ (*CLBINT s=0..x. f s* (*Suc n*) $*$ *iexp s*)
⟨*proof*⟩

lemma *iexp-eq1*:
fixes *x* :: *real*
defines *f s m* $\equiv$ *complex-of-real* ((*x* − *s*) ^ *m*)
shows *iexp x* =
($\sum$ *k* $\le$ *n*. (i $*$ *x*)^*k* / (*fact k*)) + ((i ^ (*Suc n*)) / (*fact n*)) $*$ (*CLBINT s=0..x.* (*f s n*) $*$ (*iexp s*)) (**is** *?P n*)
⟨*proof*⟩

lemma *iexp-eq2*:
fixes *x* :: *real*
defines *f s m* $\equiv$ *complex-of-real* ((*x* − *s*) ^ *m*)
shows *iexp x* = ($\sum$ *k*$\le$*Suc n*. (i$*$*x*)^*k*/*fact k*) + i^*Suc n*/*fact n* $*$ (*CLBINT s=0..x. f s n*$*$(*iexp s* − *1*))
⟨*proof*⟩

lemma *abs-LBINT-I0c-abs-power-diff*:
|*LBINT s=0..x.* |(*x* − *s*)^*n*|| = |*x* ^ (*Suc n*) / (*Suc n*)|
⟨*proof*⟩

lemma *iexp-approx1*: *cmod* (*iexp* x − ($\sum k \le n$. (i ∗ x)^k / *fact* k)) ≤ |x|^(*Suc* n) / *fact* (*Suc* n)
⟨*proof*⟩

lemma *iexp-approx2*: *cmod* (*iexp* x − ($\sum k \le n$. (i ∗ x)^k / *fact* k)) ≤ *2* ∗ |x|^n / *fact* n
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx1*:
assumes *integrable-moments*: $\bigwedge k$. $k \le n \Longrightarrow$ *integrable* M (λx. x^k)
shows *cmod* (*char* M t − ($\sum k \le n$. ((i ∗ t)^k / *fact* k) ∗ *expectation* (λx. x^k))) ≤
(*2*∗|t|^n / *fact* n) ∗ *expectation* (λx. |x|^n) (**is** *cmod* (*char* M t − *?t1*) ≤ -)
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx2*:
assumes *integrable-moments*: $\bigwedge k$. $k \le n \Longrightarrow$ *integrable* M (λx. x ^ k)
shows *cmod* (*char* M t − ($\sum k \le n$. ((i ∗ t)^k / *fact* k) ∗ *expectation* (λx. x^k))) ≤
(|t|^n / *fact* (*Suc* n)) ∗ *expectation* (λx. *min* (*2* ∗ |x|^n ∗ *Suc* n) (|t| ∗ |x|^*Suc* n))
(**is** *cmod* (*char* M t − *?t1*) ≤ -)
⟨*proof*⟩

lemma (**in** *real-distribution*) *char-approx3*:
fixes t
assumes
integrable-1: *integrable* M (λx. x) **and**
integral-1: *expectation* (λx. x) = *0* **and**
integrable-2: *integrable* M (λx. x^*2*) **and**
integral-2: *variance* (λx. x) = σ*2*
shows *cmod* (*char* M t − (*1* − t^*2* ∗ σ*2* / *2*)) ≤
(t^*2* / *6*) ∗ *expectation* (λx. *min* (*6* ∗ x^*2*) (*abs* t ∗ (*abs* x)^*3*))
⟨*proof*⟩

This is a more familiar textbook formulation in terms of random variables, but we will use the previous version for the CLT.

lemma (**in** *prob-space*) *char-approx3'*:
fixes μ :: *real measure* **and** X
assumes *rv-X* [*simp*]: *random-variable borel* X
and [*simp*]: *integrable* M X *integrable* M (λx. (X x)^*2*) *expectation* X = *0*
and *var-X*: *variance* X = σ*2*
and μ*-def*: μ = *distr* M *borel* X
shows *cmod* (*char* μ t − (*1* − t^*2* ∗ σ*2* / *2*)) ≤
(t^*2* / *6*) ∗ *expectation* (λx. *min* (*6* ∗ (X x)^*2*) (|t| ∗ |X x|^*3*))
⟨*proof*⟩

this is the formulation in the book – in terms of a random variable *with* the distribution, rather the distribution itself. I don't know which is more useful, though in principal we can go back and forth between them.

lemma (**in** *prob-space*) *char-approx1'*:
fixes μ :: *real measure* **and** X
assumes *integrable-moments* : $\bigwedge k.\ k \leq n \Longrightarrow$ *integrable* M $(\lambda x.\ X\ x\ \hat{}\ k)$
and *rv-X*[*measurable*]: *random-variable borel* X
and μ-*distr* : *distr* M *borel* $X = \mu$
shows *cmod* $(char\ \mu\ t - (\sum k \leq n.\ ((\mathrm{i} * t)\hat{}k\ /\ fact\ k) * expectation\ (\lambda x.\ (X\ x)\hat{}k))) \leq$
$(2 * |t|\hat{}n\ /\ fact\ n) * expectation\ (\lambda x.\ |X\ x|\hat{}n)$
⟨*proof*⟩

14.5 Calculation of the Characteristic Function of the Standard Distribution

abbreviation
std-normal-distribution ≡ *density lborel std-normal-density*

lemma *real-dist-normal-dist*: *real-distribution std-normal-distribution*
⟨*proof*⟩

lemma *std-normal-distribution-even-moments*:
fixes k :: *nat*
shows $(LINT\ x|std\text{-}normal\text{-}distribution.\ x\hat{}(2 * k)) = fact\ (2 * k)\ /\ (2\hat{}k * fact\ k)$
and *integrable std-normal-distribution* $(\lambda x.\ x\hat{}(2 * k))$
⟨*proof*⟩

lemma *integrable-std-normal-distribution-moment*: *integrable std-normal-distribution* $(\lambda x.\ x\hat{}k)$
⟨*proof*⟩

lemma *integral-std-normal-distribution-moment-odd*:
odd $k \Longrightarrow integral^L$ *std-normal-distribution* $(\lambda x.\ x\hat{}k) = 0$
⟨*proof*⟩

lemma *std-normal-distribution-even-moments-abs*:
fixes k :: *nat*
shows $(LINT\ x|std\text{-}normal\text{-}distribution.\ |x|\hat{}(2 * k)) = fact\ (2 * k)\ /\ (2\hat{}k * fact\ k)$
⟨*proof*⟩

lemma *std-normal-distribution-odd-moments-abs*:
fixes k :: *nat*
shows $(LINT\ x|std\text{-}normal\text{-}distribution.\ |x|\hat{}(2 * k + 1)) = sqrt\ (2\ /\ pi) * 2\ \hat{}\ k * fact\ k$
⟨*proof*⟩

theorem *char-std-normal-distribution*:

char std-normal-distribution = (λt. *complex-of-real* (*exp* ($-$ (t^*2*) / *2*)))
⟨*proof*⟩

end

15 Helly's selection theorem

The set of bounded, monotone, right continuous functions is sequentially compact

theory *Helly-Selection*
imports *HOL−Library.Diagonal-Subsequence Weak-Convergence*
begin

lemma *minus-one-less*: $x - 1 < (x::real)$
⟨*proof*⟩

theorem *Helly-selection*:
fixes f :: *nat* ⇒ *real* ⇒ *real*
assumes *rcont*: ⋀n x. *continuous* (*at-right* x) (f n)
assumes *mono*: ⋀n. *mono* (f n)
assumes *bdd*: ⋀n x. $|f\ n\ x| \leq M$
shows ∃ s. *strict-mono* (s::*nat* ⇒ *nat*) ∧ (∃ F. (∀ x. *continuous* (*at-right* x) F) ∧ *mono* F ∧ (∀ x. $|F\ x| \leq M$) ∧
(∀ x. *continuous* (*at* x) F ⟶ (λn. f (s n) x) ⟶ F x))
⟨*proof*⟩

definition
tight :: (*nat* ⇒ *real measure*) ⇒ *bool*
where
tight μ ≡ (∀ n. *real-distribution* (μ n)) ∧ (∀ (ε::*real*)>*0*. ∃ a b::*real*. $a < b$ ∧ (∀ n. *measure* (μ n) {a<..b} > *1* − ε))

theorem *tight-imp-convergent-subsubsequence*:
assumes μ: *tight* μ *strict-mono* s
shows ∃ r M. *strict-mono* (r :: *nat* ⇒ *nat*) ∧ *real-distribution* M ∧ *weak-conv-m* ($\mu \circ s \circ r$) M
⟨*proof*⟩

corollary *tight-subseq-weak-converge*:
fixes μ :: *nat* ⇒ *real measure* **and** M :: *real measure*
assumes ⋀n. *real-distribution* (μ n) *real-distribution* M **and** *tight*: *tight* μ **and**
subseq: ⋀s ν. *strict-mono* s ⟹ *real-distribution* ν ⟹ *weak-conv-m* ($\mu \circ s$) ν ⟹ *weak-conv-m* ($\mu \circ s$) M
shows *weak-conv-m* μ M
⟨*proof*⟩

end

16 Integral of sinc

theory *Sinc-Integral*
imports *Distributions*
begin

16.1 Various preparatory integrals

Naming convention The theorem name consists of the following parts:

- Kind of integral: *has-bochner-integral* / *integrable* / *LBINT*
- Interval: Interval (0 / infinity / open / closed) (infinity / open / closed)
- Name of the occurring constants: power, exp, m (for minus), scale, sin, ...

lemma *has-bochner-integral-I0i-power-exp-m'*:
has-bochner-integral lborel ($\lambda x.\ x\hat{\ }k * exp\ (-x) * indicator\ \{0\ ..\}\ x{::}real$) (*fact k*)
⟨*proof*⟩

lemma *has-bochner-integral-I0i-power-exp-m*:
has-bochner-integral lborel ($\lambda x.\ x\hat{\ }k * exp\ (-x) * indicator\ \{0\ <..\}\ x{::}real$) (*fact k*)
⟨*proof*⟩

lemma *integrable-I0i-exp-mscale*: $0 < (u{::}real) \Longrightarrow$ *set-integrable lborel* $\{0\ <..\}$ $(\lambda x.\ exp\ (-(x * u)))$
⟨*proof*⟩

lemma *LBINT-I0i-exp-mscale*: $0 < (u{::}real) \Longrightarrow LBINT\ x{=}0..\infty.\ exp\ (-(x * u)) = 1\ /\ u$
⟨*proof*⟩

lemma *LBINT-I0c-exp-mscale-sin*:
$LBINT\ x{=}0..t.\ exp\ (-(u * x)) * sin\ x =$
$(1\ /\ (1 + u\hat{\ }2)) * (1 - exp\ (-(u * t)) * (u * sin\ t + cos\ t))$ (**is** - = *?F t*)
⟨*proof*⟩

lemma *LBINT-I0i-exp-mscale-sin*:
assumes $0 < x$
shows $LBINT\ u{=}0..\infty.\ |exp\ (-u * x) * sin\ x| = |sin\ x|\ /\ x$
⟨*proof*⟩

lemma

shows *integrable-inverse-1-plus-square*:
set-integrable lborel (*einterval* $(-\infty)$ ∞) ($\lambda x.$ *inverse* $(1 + x\hat{}2)$)
and *LBINT-inverse-1-plus-square*:
LBINT $x{=}-\infty..\infty.$ *inverse* $(1 + x\hat{}2) = pi$
⟨*proof*⟩

lemma
shows *integrable-I0i-1-div-plus-square*:
interval-lebesgue-integrable lborel 0 ∞ ($\lambda x.$ 1 / $(1 + x\hat{}2)$)
and *LBINT-I0i-1-div-plus-square*:
LBINT $x{=}0..\infty.$ 1 / $(1 + x\hat{}2) = pi$ / 2
⟨*proof*⟩

17 The sinc function, and the sine integral (Si)

abbreviation *sinc* :: *real* ⇒ *real* **where**
sinc ≡ ($\lambda x.$ *if* $x = 0$ *then* 1 *else sin* x / x)

lemma *sinc-at-0*: (($\lambda x.$ *sin* x / x::*real*) $\longrightarrow$ 1) (*at* 0)
⟨*proof*⟩

lemma *isCont-sinc*: *isCont sinc x*
⟨*proof*⟩

lemma *continuous-on-sinc*[*continuous-intros*]:
continuous-on S f ⟹ *continuous-on S* ($\lambda x.$ *sinc* (*f x*))
⟨*proof*⟩

lemma *borel-measurable-sinc*[*measurable*]: *sinc* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *sinc-AE*: *AE x in lborel. sin x* / *x* = *sinc x*
⟨*proof*⟩

definition *Si* :: *real* ⇒ *real* **where** *Si t* ≡ *LBINT* $x{=}0..t.$ *sin* x / x

lemma *sinc-neg* [*simp*]: *sinc* $(-\ x)$ = *sinc x*
⟨*proof*⟩

lemma *Si-alt-def* : *Si t* = *LBINT* $x{=}0..t.$ *sinc x*
⟨*proof*⟩

lemma *Si-neg*:
assumes $T \geq 0$ **shows** *Si* $(-\ T) = -$ *Si T*
⟨*proof*⟩

lemma *integrable-sinc′*:
interval-lebesgue-integrable lborel (*ereal* 0) (*ereal* T) ($\lambda t.$ *sin* $(t * \vartheta)$ / t)

⟨*proof*⟩

lemma *DERIV-Si*: (*Si has-real-derivative sinc x*) (*at x*)
⟨*proof*⟩

lemma *isCont-Si*: *isCont Si x*
⟨*proof*⟩

lemma *borel-measurable-Si*[*measurable*]: *Si* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *Si-at-top-LBINT*:
((λ*t*. (*LBINT x=0..∞. exp* (−(*x* ∗ *t*)) ∗ (*x* ∗ *sin t* + *cos t*) / (*1* + *x*^*2*))) ⟶ *0*) *at-top*
⟨*proof*⟩

lemma *Si-at-top-integrable*:
assumes *t* ≥ *0*
shows *interval-lebesgue-integrable lborel 0* ∞ (λ*x*. *exp* (− (*x* ∗ *t*)) ∗ (*x* ∗ *sin t* + *cos t*) / (*1* + x^2))
⟨*proof*⟩

lemma *Si-at-top*: (*Si* ⟶ *pi* / *2*) *at-top*
⟨*proof*⟩

17.1 The final theorems: boundedness and scalability

lemma *bounded-Si*: ∃ *B*. ∀ *T*. |*Si T*| ≤ *B*
⟨*proof*⟩

lemma *LBINT-I0c-sin-scale-divide*:
assumes *T* ≥ *0*
shows *LBINT t=0..T. sin* (*t* ∗ ϑ) / *t* = *sgn* ϑ ∗ *Si* (*T* ∗ |ϑ|)
⟨*proof*⟩

end

18 The Levy inversion theorem, and the Levy continuity theorem.

theory *Levy*
imports *Characteristic-Functions Helly-Selection Sinc-Integral*
begin

18.1 The Levy inversion theorem

lemma *Levy-Inversion-aux1*:
fixes *a b* :: *real*

assumes $a \leq b$
shows $((\lambda t.\ (iexp\ (-(t * a)) - iexp\ (-(t * b)))\ /\ (\mathrm{i} * t)) \longrightarrow b - a)\ (at\ 0)$
(**is** $(?F \longrightarrow \text{-})\ (at\ \text{-})$)
$\langle proof \rangle$

lemma *Levy-Inversion-aux2*:
fixes $a\ b\ t :: real$
assumes $a \leq b$ **and** $t \neq 0$
shows $cmod\ ((iexp\ (t * b) - iexp\ (t * a))\ /\ (\mathrm{i} * t)) \leq b - a$ (**is** $?F \leq \text{-}$)
$\langle proof \rangle$

theorem (**in** *real-distribution*) *Levy-Inversion*:
fixes $a\ b :: real$
assumes $a \leq b$
defines $\mu \equiv measure\ M$ **and** $\varphi \equiv char\ M$
assumes $\mu\ \{a\} = 0$ **and** $\mu\ \{b\} = 0$
shows $(\lambda T.\ 1\ /\ (2 * pi) * (CLBINT\ t{=}{-}T..T.\ (iexp\ (-(t * a)) - iexp\ (-(t * b)))\ /\ (\mathrm{i} * t) * \varphi\ t))$
$\longrightarrow \mu\ \{a{<}..b\}$
(**is** $(\lambda T.\ 1\ /\ (2 * pi) * (CLBINT\ t{=}{-}T..T.\ ?F\ t * \varphi\ t)) \longrightarrow of\text{-}real\ (\mu\ \{a{<}..b\})$)
$\langle proof \rangle$

theorem *Levy-uniqueness*:
fixes $M1\ M2 :: real\ measure$
assumes *real-distribution M1 real-distribution M2* **and**
char M1 = char M2
shows $M1 = M2$
$\langle proof \rangle$

18.2 The Levy continuity theorem

theorem *levy-continuity1*:
fixes $M :: nat \Rightarrow real\ measure$ **and** $M' :: real\ measure$
assumes $\bigwedge n.$ *real-distribution* $(M\ n)$ *real-distribution* M' *weak-conv-m* $M\ M'$
shows $(\lambda n.\ char\ (M\ n)\ t) \longrightarrow char\ M'\ t$
$\langle proof \rangle$

theorem *levy-continuity*:
fixes $M :: nat \Rightarrow real\ measure$ **and** $M' :: real\ measure$
assumes *real-distr-M* : $\bigwedge n.$ *real-distribution* $(M\ n)$
and *real-distr-M'*: *real-distribution* M'
and *char-conv*: $\bigwedge t.\ (\lambda n.\ char\ (M\ n)\ t) \longrightarrow char\ M'\ t$
shows *weak-conv-m* $M\ M'$
$\langle proof \rangle$

end

19 The Central Limit Theorem

theory *Central-Limit-Theorem*
imports *Levy*
begin

theorem (**in** *prob-space*) *central-limit-theorem-zero-mean*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: *indep-vars* ($\lambda i.\ borel$) *X UNIV*
and *X-mean-0*: $\bigwedge n.\ expectation\ (X\ n) = 0$
and *σ-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$
and *X-distrib*: $\bigwedge n.\ distr\ M\ borel\ (X\ n) = \mu$
defines $S\ n \equiv \lambda x.\ \sum i{<}n.\ X\ i\ x$
shows *weak-conv-m* ($\lambda n.\ distr\ M\ borel\ (\lambda x.\ S\ n\ x\ /\ sqrt\ (n * \sigma^2))$) *std-normal-distribution*
⟨*proof*⟩

theorem (**in** *prob-space*) *central-limit-theorem*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
and $\mu :: real\ measure$
and $\sigma :: real$
and $S :: nat \Rightarrow 'a \Rightarrow real$
assumes *X-indep*: *indep-vars* ($\lambda i.\ borel$) *X UNIV*
and *X-mean*: $\bigwedge n.\ expectation\ (X\ n) = m$
and *σ-pos*: $\sigma > 0$
and *X-square-integrable*: $\bigwedge n.\ integrable\ M\ (\lambda x.\ (X\ n\ x)^2)$
and *X-variance*: $\bigwedge n.\ variance\ (X\ n) = \sigma^2$
and *X-distrib*: $\bigwedge n.\ distr\ M\ borel\ (X\ n) = \mu$
defines $X'\ i\ x \equiv X\ i\ x - m$
shows *weak-conv-m* ($\lambda n.\ distr\ M\ borel\ (\lambda x.\ (\sum i{<}n.\ X'\ i\ x)\ /\ sqrt\ (n{*}\sigma^2))$)
std-normal-distribution
⟨*proof*⟩

end

theory *Discrete-Topology*
imports *HOL−Analysis.Analysis*
begin

Copy of discrete types with discrete topology. This space is polish.

typedef *'a discrete = UNIV::'a set*
morphisms *of-discrete discrete*
⟨*proof*⟩

instantiation *discrete* :: (*type*) *metric-space*
begin

definition *dist-discrete* :: *'a discrete* ⇒ *'a discrete* ⇒ *real*
where *dist-discrete n m* = (*if n* = *m then 0 else 1*)

definition *uniformity-discrete* :: (*'a discrete* × *'a discrete*) *filter* **where**
(*uniformity*::(*'a discrete* × *'a discrete*) *filter*) = (*INF e*∈{*0* <..}. *principal* {(*x*, *y*). *dist x y* < *e*})

definition *open-discrete* :: *'a discrete set* ⇒ *bool* **where**
(*open*::*'a discrete set* ⇒ *bool*) *U* ⟷ (∀*x*∈*U*. *eventually* (λ(*x'*, *y*). *x'* = *x* ⟶ *y* ∈ *U*) *uniformity*)

instance ⟨*proof*⟩
end

lemma *open-discrete*: *open* (*S* :: *'a discrete set*)
⟨*proof*⟩

instance *discrete* :: (*type*) *complete-space*
⟨*proof*⟩

instance *discrete* :: (*countable*) *countable*
⟨*proof*⟩

instance *discrete* :: (*countable*) *second-countable-topology*
⟨*proof*⟩

instance *discrete* :: (*countable*) *polish-space* ⟨*proof*⟩

end

20 Probability mass function

theory *Probability-Mass-Function*
imports
Giry-Monad
HOL−Library.Multiset
begin

lemma *AE-emeasure-singleton*:
assumes *x*: *emeasure M* {*x*} ≠ *0* **and** *ae*: *AE x in M*. *P x* **shows** *P x*
⟨*proof*⟩

lemma *AE-measure-singleton*: *measure M* {*x*} ≠ *0* ⟹ *AE x in M*. *P x* ⟹ *P x*
⟨*proof*⟩

lemma (**in** *finite-measure*) *AE-support-countable*:

assumes [*simp*]: *sets M = UNIV*
shows $(AE\ x\ in\ M.\ measure\ M\ \{x\} \neq 0) \longleftrightarrow (\exists S.\ countable\ S \wedge (AE\ x\ in\ M.\ x \in S))$
⟨*proof*⟩

20.1 PMF as measure

typedef $'a\ pmf = \{M :: 'a\ measure.\ prob\text{-}space\ M \wedge sets\ M = UNIV \wedge (AE\ x\ in\ M.\ measure\ M\ \{x\} \neq 0)\}$
morphisms *measure-pmf Abs-pmf*
⟨*proof*⟩

declare [[*coercion measure-pmf*]]

lemma *prob-space-measure-pmf*: *prob-space* (*measure-pmf p*)
⟨*proof*⟩

interpretation *measure-pmf*: *prob-space measure-pmf M* **for** *M*
⟨*proof*⟩

interpretation *measure-pmf*: *subprob-space measure-pmf M* **for** *M*
⟨*proof*⟩

lemma *subprob-space-measure-pmf*: *subprob-space* (*measure-pmf x*)
⟨*proof*⟩

locale *pmf-as-measure*
begin

setup-lifting *type-definition-pmf*

end

context
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lemma *sets-measure-pmf*[*simp*]: *sets* (*measure-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *sets-measure-pmf-count-space*[*measurable-cong*]:
sets (*measure-pmf M*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *space-measure-pmf*[*simp*]: *space* (*measure-pmf p*) = *UNIV*
⟨*proof*⟩

lemma *measure-pmf-UNIV* [*simp*]: *measure* (*measure-pmf p*) *UNIV* = *1*

⟨*proof*⟩

lemma *measure-pmf-in-subprob-algebra*[*measurable* (*raw*)]: *measure-pmf x* ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *measurable-pmf-measure1* [*simp*]: *measurable* (M :: *'a pmf*) N = *UNIV* → *space* N
⟨*proof*⟩

lemma *measurable-pmf-measure2* [*simp*]: *measurable* N (M :: *'a pmf*) = *measurable* N (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-pair-restrict-pmf2*:
assumes *countable* A
assumes [*measurable*]: $\bigwedge y.\ y \in A \Longrightarrow (\lambda x.\ f\ (x,\ y)) \in$ *measurable* M L
shows $f \in$ *measurable* ($M \bigotimes_M$ *restrict-space* (*measure-pmf* N) A) L (**is** $f \in$ *measurable ?M* -)
⟨*proof*⟩

lemma *measurable-pair-restrict-pmf1*:
assumes *countable* A
assumes [*measurable*]: $\bigwedge x.\ x \in A \Longrightarrow (\lambda y.\ f\ (x,\ y)) \in$ *measurable* N L
shows $f \in$ *measurable* (*restrict-space* (*measure-pmf* M) $A \bigotimes_M N$) L
⟨*proof*⟩

lift-definition *pmf* :: *'a pmf* ⇒ *'a* ⇒ *real* **is** $\lambda M\ x.$ *measure* M $\{x\}$ ⟨*proof*⟩

lift-definition *set-pmf* :: *'a pmf* ⇒ *'a set* **is** $\lambda M.\ \{x.$ *measure* M $\{x\} \neq 0\}$ ⟨*proof*⟩
declare [[*coercion set-pmf*]]

lemma *AE-measure-pmf*: *AE x in* (M::*'a pmf*). $x \in M$
⟨*proof*⟩

lemma *emeasure-pmf-single-eq-zero-iff*:
fixes M :: *'a pmf*
shows *emeasure* M $\{y\} = 0 \longleftrightarrow y \notin M$
⟨*proof*⟩

lemma *AE-measure-pmf-iff*: (*AE x in measure-pmf M*. $P\ x$) ⟷ ($\forall y \in M.\ P\ y$)
⟨*proof*⟩

lemma *AE-pmfI*: ($\bigwedge y.\ y \in$ *set-pmf* $M \Longrightarrow P\ y$) ⟹ *almost-everywhere* (*measure-pmf* M) P
⟨*proof*⟩

lemma *countable-set-pmf* [*simp*]: *countable* (*set-pmf p*)
⟨*proof*⟩

lemma *pmf-positive*: $x \in$ *set-pmf p* $\Longrightarrow$ *0* $<$ *pmf p x*
⟨*proof*⟩

lemma *pmf-nonneg*[*simp*]: *0* $\leq$ *pmf p x*
⟨*proof*⟩

lemma *pmf-not-neg* [*simp*]: $\neg$*pmf p x* $<$ *0*
⟨*proof*⟩

lemma *pmf-pos* [*simp*]: *pmf p x* $\neq$ *0* $\Longrightarrow$ *pmf p x* $>$ *0*
⟨*proof*⟩

lemma *pmf-le-1*: *pmf p x* $\leq$ *1*
⟨*proof*⟩

lemma *set-pmf-not-empty*: *set-pmf M* $\neq$ {}
⟨*proof*⟩

lemma *set-pmf-iff*: $x \in$ *set-pmf M* $\longleftrightarrow$ *pmf M x* $\neq$ *0*
⟨*proof*⟩

lemma *pmf-positive-iff*: *0* $<$ *pmf p x* $\longleftrightarrow$ $x \in$ *set-pmf p*
⟨*proof*⟩

lemma *set-pmf-eq*: *set-pmf M* = {*x. pmf M x* $\neq$ *0*}
⟨*proof*⟩

lemma *set-pmf-eq'*: *set-pmf p* = {*x. pmf p x* $>$ *0*}
⟨*proof*⟩

lemma *emeasure-pmf-single*:
fixes *M* :: *'a pmf*
shows *emeasure M* {*x*} = *pmf M x*
⟨*proof*⟩

lemma *measure-pmf-single*: *measure* (*measure-pmf M*) {*x*} = *pmf M x*
⟨*proof*⟩

lemma *emeasure-measure-pmf-finite*: *finite S* $\Longrightarrow$ *emeasure* (*measure-pmf M*) *S* = ($\sum$ *s*$\in$*S. pmf M s*)
⟨*proof*⟩

lemma *measure-measure-pmf-finite*: *finite S* $\Longrightarrow$ *measure* (*measure-pmf M*) *S* = *sum* (*pmf M*) *S*
⟨*proof*⟩

lemma *sum-pmf-eq-1*:
assumes *finite A set-pmf p* $\subseteq$ *A*

shows $(\sum x{\in}A.\ pmf\ p\ x) = 1$
⟨*proof*⟩

lemma *nn-integral-measure-pmf-support*:
fixes $f :: 'a \Rightarrow ennreal$
assumes f: *finite* A **and** nn: $\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x\ \bigwedge x.\ x \in set\text{-}pmf\ M \Longrightarrow x \notin A \Longrightarrow f\ x = 0$
shows $(\int^{+} x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\sum x{\in}A.\ f\ x * pmf\ M\ x)$
⟨*proof*⟩

lemma *nn-integral-measure-pmf-finite*:
fixes $f :: 'a \Rightarrow ennreal$
assumes f: *finite* $(set\text{-}pmf\ M)$ **and** nn: $\bigwedge x.\ x \in set\text{-}pmf\ M \Longrightarrow 0 \le f\ x$
shows $(\int^{+} x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\sum x{\in}set\text{-}pmf\ M.\ f\ x * pmf\ M\ x)$
⟨*proof*⟩

lemma *integrable-measure-pmf-finite*:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows *finite* $(set\text{-}pmf\ M) \Longrightarrow integrable\ M\ f$
⟨*proof*⟩

lemma *integral-measure-pmf-real*:
assumes [*simp*]: *finite* A **and** $\bigwedge a.\ a \in set\text{-}pmf\ M \Longrightarrow f\ a \ne 0 \Longrightarrow a \in A$
shows $(\int x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\sum a{\in}A.\ f\ a * pmf\ M\ a)$
⟨*proof*⟩

lemma *integrable-pmf*: $integrable\ (count\text{-}space\ X)\ (pmf\ M)$
⟨*proof*⟩

lemma *integral-pmf*: $(\int x.\ pmf\ M\ x\ \partial count\text{-}space\ X) = measure\ M\ X$
⟨*proof*⟩

lemma *integral-pmf-restrict*:
$(f::'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}) \in borel\text{-}measurable\ (count\text{-}space\ UNIV) \Longrightarrow$
$(\int x.\ f\ x\ \partial measure\text{-}pmf\ M) = (\int x.\ f\ x\ \partial restrict\text{-}space\ M\ M)$
⟨*proof*⟩

lemma *emeasure-pmf*: $emeasure\ (M::'a\ pmf)\ M = 1$
⟨*proof*⟩

lemma *emeasure-pmf-UNIV* [*simp*]: $emeasure\ (measure\text{-}pmf\ M)\ UNIV = 1$
⟨*proof*⟩

lemma *in-null-sets-measure-pmfI*:
$A \cap set\text{-}pmf\ p = \{\} \Longrightarrow A \in null\text{-}sets\ (measure\text{-}pmf\ p)$
⟨*proof*⟩

lemma *measure-subprob*: $measure\text{-}pmf\ M \in space\ (subprob\text{-}algebra\ (count\text{-}space$

UNIV))
⟨*proof*⟩

20.2 Monad Interpretation

lemma *measurable-measure-pmf*[*measurable*]:
(λx. *measure-pmf* (*M x*)) $\in$ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *bind-measure-pmf-cong*:
assumes $\bigwedge x$. *A x* $\in$ *space* (*subprob-algebra N*) $\bigwedge x$. *B x* $\in$ *space* (*subprob-algebra N*)
assumes $\bigwedge i$. *i* $\in$ *set-pmf x* $\Longrightarrow$ *A i* = *B i*
shows *bind* (*measure-pmf x*) *A* = *bind* (*measure-pmf x*) *B*
⟨*proof*⟩

lift-definition *bind-pmf* :: *'a pmf* $\Rightarrow$ (*'a* $\Rightarrow$ *'b pmf*) $\Rightarrow$ *'b pmf* **is** *bind*
⟨*proof*⟩

adhoc-overloading *Monad-Syntax.bind bind-pmf*

lemma *ennreal-pmf-bind*: *pmf* (*bind-pmf N f*) *i* = ($\int^+ x$. *pmf* (*f x*) *i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *pmf-bind*: *pmf* (*bind-pmf N f*) *i* = ($\int x$. *pmf* (*f x*) *i* ∂*measure-pmf N*)
⟨*proof*⟩

lemma *bind-pmf-const*[*simp*]: *bind-pmf M* (λx. *c*) = *c*
⟨*proof*⟩

lemma *set-bind-pmf*[*simp*]: *set-pmf* (*bind-pmf M N*) = ($\bigcup$ *M*$\in$*set-pmf M*. *set-pmf* (*N M*))
⟨*proof*⟩

lemma *bind-pmf-cong* [*fundef-cong*]:
assumes *p* = *q*
shows ($\bigwedge x$. *x* $\in$ *set-pmf q* $\Longrightarrow$ *f x* = *g x*) $\Longrightarrow$ *bind-pmf p f* = *bind-pmf q g*
⟨*proof*⟩

lemma *bind-pmf-cong-simp*:
p = *q* $\Longrightarrow$ ($\bigwedge x$. *x* $\in$ *set-pmf q* =*simp*=> *f x* = *g x*) $\Longrightarrow$ *bind-pmf p f* = *bind-pmf q g*
⟨*proof*⟩

lemma *measure-pmf-bind*: *measure-pmf* (*bind-pmf M f*) = (*measure-pmf M* $\gg=$ (λx. *measure-pmf* (*f x*)))
⟨*proof*⟩

lemma *nn-integral-bind-pmf*[*simp*]: $(\int^+ x.\ f\ x\ \partial bind\text{-}pmf\ M\ N) = (\int^+ x.\ \int^+ y.\ f\ y\ \partial N\ x\ \partial M)$
⟨*proof*⟩

lemma *emeasure-bind-pmf*[*simp*]: $emeasure\ (bind\text{-}pmf\ M\ N)\ X = (\int^+ x.\ emeasure\ (N\ x)\ X\ \partial M)$
⟨*proof*⟩

lift-definition *return-pmf* :: $'a \Rightarrow\ 'a\ pmf$ **is** *return* (*count-space UNIV*)
⟨*proof*⟩

lemma *bind-return-pmf*: *bind-pmf* (*return-pmf x*) *f* = *f x*
⟨*proof*⟩

lemma *set-return-pmf*[*simp*]: *set-pmf* (*return-pmf x*) = $\{x\}$
⟨*proof*⟩

lemma *bind-return-pmf'*: *bind-pmf N return-pmf* = *N*
⟨*proof*⟩

lemma *bind-assoc-pmf*: *bind-pmf* (*bind-pmf A B*) *C* = *bind-pmf A* ($\lambda x.$ *bind-pmf* (*B x*) *C*)
⟨*proof*⟩

definition *map-pmf f M* = *bind-pmf M* ($\lambda x.$ *return-pmf* (*f x*))

lemma *map-bind-pmf*: *map-pmf f* (*bind-pmf M g*) = *bind-pmf M* ($\lambda x.$ *map-pmf f* (*g x*))
⟨*proof*⟩

lemma *bind-map-pmf*: *bind-pmf* (*map-pmf f M*) *g* = *bind-pmf M* ($\lambda x.$ *g* (*f x*))
⟨*proof*⟩

lemma *map-pmf-transfer*[*transfer-rule*]:
rel-fun (=) (*rel-fun cr-pmf cr-pmf*) ($\lambda f\ M.$ *distr M* (*count-space UNIV*) *f*) *map-pmf*
⟨*proof*⟩

lemma *map-pmf-rep-eq*:
measure-pmf (*map-pmf f M*) = *distr* (*measure-pmf M*) (*count-space UNIV*) *f*
⟨*proof*⟩

lemma *map-pmf-id*[*simp*]: *map-pmf id* = *id*
⟨*proof*⟩

lemma *map-pmf-ident*[*simp*]: *map-pmf* ($\lambda x.\ x$) = ($\lambda x.\ x$)
⟨*proof*⟩

lemma *map-pmf-compose*: $map\text{-}pmf\ (f \circ g) = map\text{-}pmf\ f \circ map\text{-}pmf\ g$
⟨*proof*⟩

lemma *map-pmf-comp*: $map\text{-}pmf\ f\ (map\text{-}pmf\ g\ M) = map\text{-}pmf\ (\lambda x.\ f\ (g\ x))\ M$
⟨*proof*⟩

lemma *map-pmf-cong*: $p = q \Longrightarrow (\bigwedge x.\ x \in set\text{-}pmf\ q \Longrightarrow f\ x = g\ x) \Longrightarrow map\text{-}pmf\ f\ p = map\text{-}pmf\ g\ q$
⟨*proof*⟩

lemma *pmf-set-map*: $set\text{-}pmf \circ map\text{-}pmf\ f = (`)\ f \circ set\text{-}pmf$
⟨*proof*⟩

lemma *set-map-pmf*[*simp*]: $set\text{-}pmf\ (map\text{-}pmf\ f\ M) = f`set\text{-}pmf\ M$
⟨*proof*⟩

lemma *emeasure-map-pmf*[*simp*]: $emeasure\ (map\text{-}pmf\ f\ M)\ X = emeasure\ M\ (f\ -`\ X)$
⟨*proof*⟩

lemma *measure-map-pmf*[*simp*]: $measure\ (map\text{-}pmf\ f\ M)\ X = measure\ M\ (f\ -`\ X)$
⟨*proof*⟩

lemma *nn-integral-map-pmf*[*simp*]: $(\int^+ x.\ f\ x\ \partial map\text{-}pmf\ g\ M) = (\int^+ x.\ f\ (g\ x)\ \partial M)$
⟨*proof*⟩

lemma *ennreal-pmf-map*: $pmf\ (map\text{-}pmf\ f\ p)\ x = (\int^+\ y.\ indicator\ (f\ -`\ \{x\})\ y\ \partial measure\text{-}pmf\ p)$
⟨*proof*⟩

lemma *pmf-map*: $pmf\ (map\text{-}pmf\ f\ p)\ x = measure\ p\ (f\ -`\ \{x\})$
⟨*proof*⟩

lemma *nn-integral-pmf*: $(\int^+\ x.\ pmf\ p\ x\ \partial count\text{-}space\ A) = emeasure\ (measure\text{-}pmf\ p)\ A$
⟨*proof*⟩

lemma *integral-map-pmf*[*simp*]:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows $integral^L\ (map\text{-}pmf\ g\ p)\ f = integral^L\ p\ (\lambda x.\ f\ (g\ x))$
⟨*proof*⟩

lemma *integrable-map-pmf-eq* [*simp*]:
fixes $g :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows $integrable\ (map\text{-}pmf\ f\ p)\ g \longleftrightarrow integrable\ (measure\text{-}pmf\ p)\ (\lambda x.\ g\ (f\ x))$

⟨*proof*⟩

lemma *integrable-map-pmf* [*intro*]:
fixes g :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
shows *integrable* (*measure-pmf p*) (λx. g (f x)) $\Longrightarrow$ *integrable* (*map-pmf f p*) g
⟨*proof*⟩

lemma *pmf-abs-summable* [*intro*]: *pmf p abs-summable-on A*
⟨*proof*⟩

lemma *measure-pmf-conv-infsetsum*: *measure* (*measure-pmf p*) A = *infsetsum* (*pmf p*) A
⟨*proof*⟩

lemma *infsetsum-pmf-eq-1*:
assumes *set-pmf p* $\subseteq A$
shows *infsetsum* (*pmf p*) $A = 1$
⟨*proof*⟩

lemma *map-return-pmf* [*simp*]: *map-pmf f* (*return-pmf x*) = *return-pmf* (f x)
⟨*proof*⟩

lemma *map-pmf-const*[*simp*]: *map-pmf* (λ-. c) M = *return-pmf c*
⟨*proof*⟩

lemma *pmf-return* [*simp*]: *pmf* (*return-pmf x*) y = *indicator* $\{y\}$ x
⟨*proof*⟩

lemma *nn-integral-return-pmf*[*simp*]: $0 \leq f\ x \Longrightarrow (\int^{+} x.\ f\ x\ \partial return\text{-}pmf\ x) = f\ x$
⟨*proof*⟩

lemma *emeasure-return-pmf*[*simp*]: *emeasure* (*return-pmf x*) X = *indicator* X x
⟨*proof*⟩

lemma *measure-return-pmf* [*simp*]: *measure-pmf.prob* (*return-pmf x*) A = *indicator* A x
⟨*proof*⟩

lemma *return-pmf-inj*[*simp*]: *return-pmf x* = *return-pmf y* $\longleftrightarrow x = y$
⟨*proof*⟩

lemma *map-pmf-eq-return-pmf-iff*:
map-pmf f p = *return-pmf x* $\longleftrightarrow$ ($\forall\, y \in$ *set-pmf p*. f $y = x$)
⟨*proof*⟩

definition *pair-pmf A B* = *bind-pmf A* (λx. *bind-pmf B* (λy. *return-pmf* (x, y)))

lemma *pmf-pair*: *pmf* (*pair-pmf M N*) (a, b) = *pmf M a* $*$ *pmf N b*
⟨*proof*⟩

lemma *set-pair-pmf*[*simp*]: *set-pmf* (*pair-pmf A B*) = *set-pmf A* × *set-pmf B*
⟨*proof*⟩

lemma *measure-pmf-in-subprob-space*[*measurable* (*raw*)]:
measure-pmf M ∈ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-pair-pmf′*: ($\int^+ x.\ f\ x\ \partial$*pair-pmf A B*) = ($\int^+ a.\ \int^+ b.\ f\ (a,\ b)$ $\partial B\ \partial A$)
⟨*proof*⟩

lemma *bind-pair-pmf*:
assumes *M*[*measurable*]: *M* ∈ *measurable* (*count-space UNIV* $\bigotimes_M$ *count-space UNIV*) (*subprob-algebra N*)
shows *measure-pmf* (*pair-pmf A B*) ⤜ *M* = (*measure-pmf A* ⤜ (λ*x*. *measure-pmf B* ⤜ (λ*y*. *M* (*x*, *y*))))
(**is** *?L* = *?R*)
⟨*proof*⟩

lemma *map-fst-pair-pmf*: *map-pmf fst* (*pair-pmf A B*) = *A*
⟨*proof*⟩

lemma *map-snd-pair-pmf*: *map-pmf snd* (*pair-pmf A B*) = *B*
⟨*proof*⟩

lemma *nn-integral-pmf′*:
inj-on f A ⟹ ($\int^+ x.$ *pmf p* (*f x*) ∂*count-space A*) = *emeasure p* (*f* ‘ *A*)
⟨*proof*⟩

lemma *pmf-le-0-iff*[*simp*]: *pmf M p* ≤ *0* ⟷ *pmf M p* = *0*
⟨*proof*⟩

lemma *min-pmf-0*[*simp*]: *min* (*pmf M p*) *0* = *0 min 0* (*pmf M p*) = *0*
⟨*proof*⟩

lemma *pmf-eq-0-set-pmf*: *pmf M p* = *0* ⟷ *p* ∉ *set-pmf M*
⟨*proof*⟩

lemma *pmf-map-inj*: *inj-on f* (*set-pmf M*) ⟹ *x* ∈ *set-pmf M* ⟹ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
⟨*proof*⟩

lemma *pair-return-pmf* [*simp*]: *pair-pmf* (*return-pmf x*) (*return-pmf y*) = *return-pmf* (*x*, *y*)
⟨*proof*⟩

lemma *pmf-map-inj′*: *inj f* ⟹ *pmf* (*map-pmf f M*) (*f x*) = *pmf M x*
⟨*proof*⟩

lemma *expectation-pair-pmf-fst* [*simp*]:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows *measure-pmf.expectation* (*pair-pmf p q*) ($\lambda x.\ f\ (fst\ x)$) = *measure-pmf.expectation p f*
⟨*proof*⟩

lemma *expectation-pair-pmf-snd* [*simp*]:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows *measure-pmf.expectation* (*pair-pmf p q*) ($\lambda x.\ f\ (snd\ x)$) = *measure-pmf.expectation q f*
⟨*proof*⟩

lemma *pmf-map-outside*: $x \notin f$ ‘ *set-pmf M* $\Longrightarrow$ *pmf* (*map-pmf f M*) $x = 0$
⟨*proof*⟩

lemma *measurable-set-pmf*[*measurable*]: *Measurable.pred* (*count-space UNIV*) ($\lambda x.\ x \in$ *set-pmf M*)
⟨*proof*⟩

20.3 PMFs as function

context
fixes $f :: 'a \Rightarrow real$
assumes *nonneg*: $\bigwedge x.\ 0 \le f\ x$
assumes *prob*: ($\int^+ x.\ f\ x\ \partial$*count-space UNIV*) = *1*
begin

lift-definition *embed-pmf* :: *'a pmf* **is** *density* (*count-space UNIV*) (*ennreal* $\circ$ *f*)
⟨*proof*⟩

lemma *pmf-embed-pmf*: *pmf embed-pmf* $x = f\ x$
⟨*proof*⟩

lemma *set-embed-pmf*: *set-pmf embed-pmf* = $\{x.\ f\ x \neq 0\}$
⟨*proof*⟩

end

lemma *embed-pmf-transfer*:
rel-fun (*eq-onp* ($\lambda f.$ ($\forall x.\ 0 \le f\ x$) $\wedge$ ($\int^+ x.$ *ennreal* ($f\ x$) ∂*count-space UNIV*) = *1*)) *pmf-as-measure.cr-pmf* ($\lambda f.$ *density* (*count-space UNIV*) (*ennreal* $\circ$ *f*)) *embed-pmf*
⟨*proof*⟩

lemma *measure-pmf-eq-density*: *measure-pmf p* = *density* (*count-space UNIV*) (*pmf p*)
⟨*proof*⟩

lemma *td-pmf-embed-pmf*:

type-definition pmf embed-pmf $\{f::'a \Rightarrow real.\ (\forall x.\ 0 \leq f\ x) \wedge (\int^+ x.\ ennreal\ (f\ x)\ \partial count\text{-}space\ UNIV) = 1\}$
⟨*proof*⟩

end

lemma *nn-integral-measure-pmf*: $(\int^+ x.\ f\ x\ \partial measure\text{-}pmf\ p) = \int^+ x.\ ennreal\ (pmf\ p\ x) * f\ x\ \partial count\text{-}space\ UNIV$
⟨*proof*⟩

lemma *integral-measure-pmf*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes A: *finite* A
shows $(\bigwedge a.\ a \in set\text{-}pmf\ M \Longrightarrow f\ a \neq 0 \Longrightarrow a \in A) \Longrightarrow (LINT\ x|M.\ f\ x) = (\sum a{\in}A.\ pmf\ M\ a *_R f\ a)$
⟨*proof*⟩

lemma *expectation-return-pmf* [*simp*]:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
shows $measure\text{-}pmf.expectation\ (return\text{-}pmf\ x)\ f = f\ x$
⟨*proof*⟩

lemma *pmf-expectation-bind*:
fixes $p :: 'a\ pmf$ **and** $f :: 'a \Rightarrow 'b\ pmf$
and $h :: 'b \Rightarrow 'c::\{banach, second\text{-}countable\text{-}topology\}$
assumes *finite* A $\bigwedge x.\ x \in A \Longrightarrow finite\ (set\text{-}pmf\ (f\ x))$ $set\text{-}pmf\ p \subseteq A$
shows $measure\text{-}pmf.expectation\ (p \gg= f)\ h =$
$(\sum a{\in}A.\ pmf\ p\ a *_R measure\text{-}pmf.expectation\ (f\ a)\ h)$
⟨*proof*⟩

lemma *continuous-on-LINT-pmf*: — This is dominated convergence!?
fixes $f :: 'i \Rightarrow 'a::topological\text{-}space \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes f: $\bigwedge i.\ i \in set\text{-}pmf\ M \Longrightarrow continuous\text{-}on\ A\ (f\ i)$
and *bnd*: $\bigwedge a\ i.\ a \in A \Longrightarrow i \in set\text{-}pmf\ M \Longrightarrow norm\ (f\ i\ a) \leq B$
shows $continuous\text{-}on\ A\ (\lambda a.\ LINT\ i|M.\ f\ i\ a)$
⟨*proof*⟩

lemma *continuous-on-LBINT*:
fixes $f :: real \Rightarrow real$
assumes f: $\bigwedge b.\ a \leq b \Longrightarrow set\text{-}integrable\ lborel\ \{a..b\}\ f$
shows $continuous\text{-}on\ UNIV\ (\lambda b.\ LBINT\ x{:}\{a..b\}.\ f\ x)$
⟨*proof*⟩

locale *pmf-as-function*
begin

setup-lifting *td-pmf-embed-pmf*

lemma *set-pmf-transfer*[*transfer-rule*]:

assumes *bi-total A*
shows *rel-fun* (*pcr-pmf A*) (*rel-set A*) (λf. {x. $f\ x \neq 0$}) *set-pmf*
⟨*proof*⟩

end

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *pmf-eqI*: ($\bigwedge i$. *pmf M i* = *pmf N i*) $\Longrightarrow$ $M = N$
⟨*proof*⟩

lemma *pmf-eq-iff*: $M = N \longleftrightarrow$ ($\forall$ *i*. *pmf M i* = *pmf N i*)
⟨*proof*⟩

lemma *pmf-neq-exists-less*:
assumes $M \neq N$
shows $\exists$ *x*. *pmf M x* < *pmf N x*
⟨*proof*⟩

lemma *bind-commute-pmf*: *bind-pmf A* (λx. *bind-pmf B* (*C x*)) = *bind-pmf B* (λy. *bind-pmf A* (λx. *C x y*))
⟨*proof*⟩

lemma *pair-map-pmf1*: *pair-pmf* (*map-pmf f A*) *B* = *map-pmf* (*apfst f*) (*pair-pmf A B*)
⟨*proof*⟩

lemma *pair-map-pmf2*: *pair-pmf A* (*map-pmf f B*) = *map-pmf* (*apsnd f*) (*pair-pmf A B*)
⟨*proof*⟩

lemma *map-pair*: *map-pmf* (λ(*a*, *b*). (*f a*, *g b*)) (*pair-pmf A B*) = *pair-pmf* (*map-pmf f A*) (*map-pmf g B*)
⟨*proof*⟩

end

lemma *pair-return-pmf1*: *pair-pmf* (*return-pmf x*) *y* = *map-pmf* (*Pair x*) *y*
⟨*proof*⟩

lemma *pair-return-pmf2*: *pair-pmf x* (*return-pmf y*) = *map-pmf* (λx. (*x*, *y*)) *x*
⟨*proof*⟩

lemma *pair-pair-pmf*: *pair-pmf* (*pair-pmf u v*) *w* = *map-pmf* (λ(*x*, (*y*, *z*)). ((*x*, *y*), *z*)) (*pair-pmf u* (*pair-pmf v w*))
⟨*proof*⟩

lemma *pair-commute-pmf*: *pair-pmf x y* = *map-pmf* $(\lambda(x, y). (y, x))$ (*pair-pmf y x*)
⟨*proof*⟩

lemma *set-pmf-subset-singleton*: *set-pmf p* ⊆ $\{x\}$ ⟷ p = *return-pmf x*
⟨*proof*⟩

lemma *bind-eq-return-pmf*:
bind-pmf p f = *return-pmf x* ⟷ (∀ *y*∈*set-pmf p*. *f y* = *return-pmf x*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *pmf-False-conv-True*: *pmf p False* = *1* − *pmf p True*
⟨*proof*⟩

lemma *pmf-True-conv-False*: *pmf p True* = *1* − *pmf p False*
⟨*proof*⟩

20.4 Conditional Probabilities

lemma *measure-pmf-zero-iff*: *measure* (*measure-pmf p*) *s* = *0* ⟷ *set-pmf p* ∩ *s* = {}
⟨*proof*⟩

context
fixes *p* :: *'a pmf* **and** *s* :: *'a set*
assumes *not-empty*: *set-pmf p* ∩ *s* ≠ {}
begin

interpretation *pmf-as-measure* ⟨*proof*⟩

lemma *emeasure-measure-pmf-not-zero*: *emeasure* (*measure-pmf p*) *s* ≠ *0*
⟨*proof*⟩

lemma *measure-measure-pmf-not-zero*: *measure* (*measure-pmf p*) *s* ≠ *0*
⟨*proof*⟩

lift-definition *cond-pmf* :: *'a pmf* **is**
uniform-measure (*measure-pmf p*) *s*
⟨*proof*⟩

lemma *pmf-cond*: *pmf cond-pmf x* = (*if x* ∈ *s then pmf p x* / *measure p s else 0*)
⟨*proof*⟩

lemma *set-cond-pmf*[*simp*]: *set-pmf cond-pmf* = *set-pmf p* ∩ *s*
⟨*proof*⟩

end

lemma *measure-pmf-posI*: $x \in \textit{set-pmf } p \Longrightarrow x \in A \Longrightarrow \textit{measure-pmf.prob } p\ A > 0$
⟨*proof*⟩

lemma *cond-map-pmf*:
assumes $\textit{set-pmf } p \cap f -` s \neq \{\}$
shows $\textit{cond-pmf } (\textit{map-pmf } f\ p)\ s = \textit{map-pmf } f\ (\textit{cond-pmf } p\ (f -` s))$
⟨*proof*⟩

lemma *bind-cond-pmf-cancel*:
assumes [*simp*]: $\bigwedge x.\ x \in \textit{set-pmf } p \Longrightarrow \textit{set-pmf } q \cap \{y.\ R\ x\ y\} \neq \{\}$
assumes [*simp*]: $\bigwedge y.\ y \in \textit{set-pmf } q \Longrightarrow \textit{set-pmf } p \cap \{x.\ R\ x\ y\} \neq \{\}$
assumes [*simp*]: $\bigwedge x\ y.\ x \in \textit{set-pmf } p \Longrightarrow y \in \textit{set-pmf } q \Longrightarrow R\ x\ y \Longrightarrow \textit{measure } q\ \{y.\ R\ x\ y\} = \textit{measure } p\ \{x.\ R\ x\ y\}$
shows $\textit{bind-pmf } p\ (\lambda x.\ \textit{cond-pmf } q\ \{y.\ R\ x\ y\}) = q$
⟨*proof*⟩

20.5 Relator

inductive *rel-pmf* :: $('a \Rightarrow 'b \Rightarrow \textit{bool}) \Rightarrow 'a\ \textit{pmf} \Rightarrow 'b\ \textit{pmf} \Rightarrow \textit{bool}$
for $R\ p\ q$
where
$[\![\bigwedge x\ y.\ (x, y) \in \textit{set-pmf } pq \Longrightarrow R\ x\ y;$
$\textit{map-pmf fst } pq = p;\ \textit{map-pmf snd } pq = q\ ]\!]$
$\Longrightarrow \textit{rel-pmf } R\ p\ q$

lemma *rel-pmfI*:
assumes *R*: $\textit{rel-set } R\ (\textit{set-pmf } p)\ (\textit{set-pmf } q)$
assumes *eq*: $\bigwedge x\ y.\ x \in \textit{set-pmf } p \Longrightarrow y \in \textit{set-pmf } q \Longrightarrow R\ x\ y \Longrightarrow$
$\textit{measure } p\ \{x.\ R\ x\ y\} = \textit{measure } q\ \{y.\ R\ x\ y\}$
shows $\textit{rel-pmf } R\ p\ q$
⟨*proof*⟩

lemma *rel-pmf-imp-rel-set*: $\textit{rel-pmf } R\ p\ q \Longrightarrow \textit{rel-set } R\ (\textit{set-pmf } p)\ (\textit{set-pmf } q)$
⟨*proof*⟩

lemma *rel-pmfD-measure*:
assumes *rel-R*: $\textit{rel-pmf } R\ p\ q$ **and** *R*: $\bigwedge a\ b.\ R\ a\ b \Longrightarrow R\ a\ y \longleftrightarrow R\ x\ b$
assumes $x \in \textit{set-pmf } p\ y \in \textit{set-pmf } q$
shows $\textit{measure } p\ \{x.\ R\ x\ y\} = \textit{measure } q\ \{y.\ R\ x\ y\}$
⟨*proof*⟩

lemma *rel-pmf-measureD*:
assumes $\textit{rel-pmf } R\ p\ q$
shows $\textit{measure } (\textit{measure-pmf } p)\ A \leq \textit{measure } (\textit{measure-pmf } q)\ \{y.\ \exists x{\in}A.\ R\ x\ y\}$ (**is** $?lhs \leq ?rhs$)
⟨*proof*⟩

lemma *rel-pmf-iff-measure*:
assumes *symp R transp R*
shows *rel-pmf R p q* $\longleftrightarrow$
rel-set R (*set-pmf p*) (*set-pmf q*) $\wedge$
($\forall$ *x*$\in$*set-pmf p*. $\forall$ *y*$\in$*set-pmf q*. *R x y* $\longrightarrow$ *measure p* {*x*. *R x y*} = *measure q* {*y*. *R x y*})
⟨*proof*⟩

lemma *quotient-rel-set-disjoint*:
equivp R $\Longrightarrow$ *C* $\in$ *UNIV* // {(*x*, *y*). *R x y*} $\Longrightarrow$ *rel-set R A B* $\Longrightarrow$ *A* $\cap$ *C* = {} $\longleftrightarrow$ *B* $\cap$ *C* = {}
⟨*proof*⟩

lemma *quotientD*: *equiv X R* $\Longrightarrow$ *A* $\in$ *X* // *R* $\Longrightarrow$ *x* $\in$ *A* $\Longrightarrow$ *A* = *R* ‘‘ {*x*}
⟨*proof*⟩

lemma *rel-pmf-iff-equivp*:
assumes *equivp R*
shows *rel-pmf R p q* $\longleftrightarrow$ ($\forall$ *C*$\in$*UNIV* // {(*x*, *y*). *R x y*}. *measure p C* = *measure q C*)
(**is** - $\longleftrightarrow$ ($\forall$ *C*$\in$-//*?R*. -))
⟨*proof*⟩

bnf *pmf*: *'a pmf map*: *map-pmf sets*: *set-pmf bd* : *natLeq rel*: *rel-pmf*
⟨*proof*⟩

lemma *map-pmf-idI*: ($\bigwedge$*x*. *x* $\in$ *set-pmf p* $\Longrightarrow$ *f x* = *x*) $\Longrightarrow$ *map-pmf f p* = *p*
⟨*proof*⟩

lemma *rel-pmf-conj*[*simp*]:
rel-pmf (λ*x y*. *P* $\wedge$ *Q x y*) *x y* $\longleftrightarrow$ *P* $\wedge$ *rel-pmf Q x y*
rel-pmf (λ*x y*. *Q x y* $\wedge$ *P*) *x y* $\longleftrightarrow$ *P* $\wedge$ *rel-pmf Q x y*
⟨*proof*⟩

lemma *rel-pmf-top*[*simp*]: *rel-pmf top* = *top*
⟨*proof*⟩

lemma *rel-pmf-return-pmf1*: *rel-pmf R* (*return-pmf x*) *M* $\longleftrightarrow$ ($\forall$ *a*$\in$*M*. *R x a*)
⟨*proof*⟩

lemma *rel-pmf-return-pmf2*: *rel-pmf R M* (*return-pmf x*) $\longleftrightarrow$ ($\forall$ *a*$\in$*M*. *R a x*)
⟨*proof*⟩

lemma *rel-return-pmf*[*simp*]: *rel-pmf R* (*return-pmf x1*) (*return-pmf x2*) = *R x1 x2*
⟨*proof*⟩

lemma *rel-pmf-False*[*simp*]: *rel-pmf* (λ*x y*. *False*) *x y* = *False*
⟨*proof*⟩

lemma *rel-pmf-rel-prod*:
$rel\text{-}pmf\ (rel\text{-}prod\ R\ S)\ (pair\text{-}pmf\ A\ A')\ (pair\text{-}pmf\ B\ B') \longleftrightarrow rel\text{-}pmf\ R\ A\ B \wedge rel\text{-}pmf\ S\ A'\ B'$
$\langle proof \rangle$

lemma *rel-pmf-reflI*:
assumes $\bigwedge x.\ x \in set\text{-}pmf\ p \Longrightarrow P\ x\ x$
shows $rel\text{-}pmf\ P\ p\ p$
$\langle proof \rangle$

lemma *rel-pmf-bij-betw*:
assumes f: $bij\text{-}betw\ f\ (set\text{-}pmf\ p)\ (set\text{-}pmf\ q)$
and eq: $\bigwedge x.\ x \in set\text{-}pmf\ p \Longrightarrow pmf\ p\ x = pmf\ q\ (f\ x)$
shows $rel\text{-}pmf\ (\lambda x\ y.\ f\ x = y)\ p\ q$
$\langle proof \rangle$

context
begin

interpretation *pmf-as-measure* $\langle proof \rangle$

definition $join\text{-}pmf\ M = bind\text{-}pmf\ M\ (\lambda x.\ x)$

lemma *bind-eq-join-pmf*: $bind\text{-}pmf\ M\ f = join\text{-}pmf\ (map\text{-}pmf\ f\ M)$
$\langle proof \rangle$

lemma *join-eq-bind-pmf*: $join\text{-}pmf\ M = bind\text{-}pmf\ M\ id$
$\langle proof \rangle$

lemma *pmf-join*: $pmf\ (join\text{-}pmf\ N)\ i = (\int M.\ pmf\ M\ i\ \partial measure\text{-}pmf\ N)$
$\langle proof \rangle$

lemma *ennreal-pmf-join*: $ennreal\ (pmf\ (join\text{-}pmf\ N)\ i) = (\int^+ M.\ pmf\ M\ i\ \partial measure\text{-}pmf\ N)$
$\langle proof \rangle$

lemma *set-pmf-join-pmf*[*simp*]: $set\text{-}pmf\ (join\text{-}pmf\ f) = (\bigcup p \in set\text{-}pmf\ f.\ set\text{-}pmf\ p)$
$\langle proof \rangle$

lemma *join-return-pmf*: $join\text{-}pmf\ (return\text{-}pmf\ M) = M$
$\langle proof \rangle$

lemma *map-join-pmf*: $map\text{-}pmf\ f\ (join\text{-}pmf\ AA) = join\text{-}pmf\ (map\text{-}pmf\ (map\text{-}pmf\ f)\ AA)$
$\langle proof \rangle$

lemma *join-map-return-pmf*: $join\text{-}pmf\ (map\text{-}pmf\ return\text{-}pmf\ A) = A$

⟨*proof*⟩

end

lemma *rel-pmf-joinI*:
assumes *rel-pmf* (*rel-pmf P*) *p q*
shows *rel-pmf P* (*join-pmf p*) (*join-pmf q*)
⟨*proof*⟩

lemma *rel-pmf-bindI*:
assumes *pq*: *rel-pmf R p q*
and *fg*: $\bigwedge x\ y.\ R\ x\ y \Longrightarrow$ *rel-pmf P* $(f\ x)$ $(g\ y)$
shows *rel-pmf P* (*bind-pmf p f*) (*bind-pmf q g*)
⟨*proof*⟩

Proof that *rel-pmf* preserves orders. Antisymmetry proof follows Thm. 1 in N. Saheb-Djahromi, Cpo's of measures for nondeterminism, Theoretical Computer Science 12(1):19–37, 1980, https://doi.org/10.1016/0304-3975(80)90003-1

lemma
assumes $*$: *rel-pmf R p q*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *measure-Ici*: *measure p* $\{y.\ R\ x\ y\} \leq$ *measure q* $\{y.\ R\ x\ y\}$ (**is** *?thesis1*)
and *measure-Ioi*: *measure p* $\{y.\ R\ x\ y \wedge \neg\ R\ y\ x\} \leq$ *measure q* $\{y.\ R\ x\ y \wedge \neg\ R\ y\ x\}$ (**is** *?thesis2*)
⟨*proof*⟩

lemma *rel-pmf-inf*:
fixes *p q* :: *'a pmf*
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R*
shows *rel-pmf* (*inf R* R^{-1-1}) *p q*
⟨*proof*⟩

lemma *rel-pmf-antisym*:
fixes *p q* :: *'a pmf*
assumes *1*: *rel-pmf R p q*
assumes *2*: *rel-pmf R q p*
and *refl*: *reflp R* **and** *trans*: *transp R* **and** *antisym*: *antisymp R*
shows $p = q$
⟨*proof*⟩

lemma *reflp-rel-pmf*: *reflp R* $\Longrightarrow$ *reflp* (*rel-pmf R*)
⟨*proof*⟩

lemma *antisymp-rel-pmf*:
⟦ *reflp R*; *transp R*; *antisymp R* ⟧
$\Longrightarrow$ *antisymp* (*rel-pmf R*)

⟨*proof*⟩

lemma *transp-rel-pmf*:
assumes *transp R*
shows *transp* (*rel-pmf R*)
⟨*proof*⟩

20.6 Distributions

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

20.6.1 Bernoulli Distribution

lift-definition *bernoulli-pmf* :: *real* ⇒ *bool pmf* **is**
$\lambda p\ b.\ ((\lambda p.\ \mathit{if}\ b\ \mathit{then}\ p\ \mathit{else}\ 1 - p) \circ \mathit{min}\ 1 \circ \mathit{max}\ 0)\ p$
⟨*proof*⟩

lemma *pmf-bernoulli-True*[*simp*]: $0 \le p \Longrightarrow p \le 1 \Longrightarrow \mathit{pmf}\ (\mathit{bernoulli\text{-}pmf}\ p)\ \mathit{True} = p$
⟨*proof*⟩

lemma *pmf-bernoulli-False*[*simp*]: $0 \le p \Longrightarrow p \le 1 \Longrightarrow \mathit{pmf}\ (\mathit{bernoulli\text{-}pmf}\ p)\ \mathit{False} = 1 - p$
⟨*proof*⟩

lemma *set-pmf-bernoulli*[*simp*]: $0 < p \Longrightarrow p < 1 \Longrightarrow \mathit{set\text{-}pmf}\ (\mathit{bernoulli\text{-}pmf}\ p) = \mathit{UNIV}$
⟨*proof*⟩

lemma *nn-integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: $0 \le p\ p \le 1\ \bigwedge x.\ 0 \le f\ x$
shows $(\int^{+} x.\ f\ x\ \partial \mathit{bernoulli\text{-}pmf}\ p) = f\ \mathit{True} * p + f\ \mathit{False} * (1 - p)$
⟨*proof*⟩

lemma *integral-bernoulli-pmf*[*simp*]:
assumes [*simp*]: $0 \le p\ p \le 1$
shows $(\int x.\ f\ x\ \partial \mathit{bernoulli\text{-}pmf}\ p) = f\ \mathit{True} * p + f\ \mathit{False} * (1 - p)$
⟨*proof*⟩

lemma *pmf-bernoulli-half* [*simp*]: $\mathit{pmf}\ (\mathit{bernoulli\text{-}pmf}\ (1\ /\ 2))\ x = 1\ /\ 2$
⟨*proof*⟩

lemma *measure-pmf-bernoulli-half*: $\mathit{measure\text{-}pmf}\ (\mathit{bernoulli\text{-}pmf}\ (1\ /\ 2)) = \mathit{uniform\text{-}count\text{-}measure}\ \mathit{UNIV}$
⟨*proof*⟩

20.6.2 Geometric Distribution

context
  fixes p :: *real* **assumes** $p[arith]$: $0 < p$ $p \leq 1$
begin

lift-definition *geometric-pmf* :: *nat pmf* **is** $\lambda n.\ (1 - p)\hat{}n * p$
⟨*proof*⟩

lemma *pmf-geometric*[*simp*]: *pmf geometric-pmf* $n = (1 - p)\hat{}n * p$
  ⟨*proof*⟩

end

lemma *geometric-pmf-1* [*simp*]: *geometric-pmf* 1 = *return-pmf* 0
  ⟨*proof*⟩

lemma *set-pmf-geometric*: $0 < p \Longrightarrow p < 1 \Longrightarrow$ *set-pmf* (*geometric-pmf* p) = *UNIV*
  ⟨*proof*⟩

lemma *geometric-sums-times-n*:
  fixes c::'a::{*banach*,*real-normed-field*}
  assumes *norm* $c < 1$
  shows ($\lambda n.\ c\hat{}n$ * *of-nat* n) *sums* (c / $(1 - c)^2$)
⟨*proof*⟩

lemma *geometric-sums-times-norm*:
  fixes c::'a::{*banach*,*real-normed-field*}
  assumes *norm* $c < 1$
  shows ($\lambda n.$ *norm* ($c\hat{}n$ * *of-nat* n)) *sums* (*norm* c / $(1 -$ *norm* $c)^2$)
⟨*proof*⟩

lemma *integrable-real-geometric-pmf*:
  assumes $p \in \{0<..1\}$
  shows *integrable* (*geometric-pmf* p) *real*
⟨*proof*⟩

lemma *expectation-geometric-pmf*:
  assumes $p \in \{0<..1\}$
  shows *measure-pmf.expectation* (*geometric-pmf* p) *real* = $(1 - p) / p$
⟨*proof*⟩

lemma *geometric-bind-pmf-unfold*:
  assumes $p \in \{0<..1\}$
  shows *geometric-pmf* p =
    do {$b \leftarrow$ *bernoulli-pmf* p;
      if b *then return-pmf* 0 *else map-pmf Suc* (*geometric-pmf* p)}
⟨*proof*⟩

20.6.3 Uniform Multiset Distribution

context
fixes M :: *'a multiset* **assumes** *M-not-empty*: $M \neq \{\#\}$
begin

lift-definition *pmf-of-multiset* :: *'a pmf* **is** $\lambda x.$ *count M x / size M*
⟨*proof*⟩

lemma *pmf-of-multiset*[*simp*]: *pmf pmf-of-multiset x = count M x / size M*
⟨*proof*⟩

lemma *set-pmf-of-multiset*[*simp*]: *set-pmf pmf-of-multiset = set-mset M*
⟨*proof*⟩

end

20.6.4 Uniform Distribution

context
fixes S :: *'a set* **assumes** *S-not-empty*: $S \neq \{\}$ **and** *S-finite*: *finite S*
begin

lift-definition *pmf-of-set* :: *'a pmf* **is** $\lambda x.$ *indicator S x / card S*
⟨*proof*⟩

lemma *pmf-of-set*[*simp*]: *pmf pmf-of-set x = indicator S x / card S*
⟨*proof*⟩

lemma *set-pmf-of-set*[*simp*]: *set-pmf pmf-of-set = S*
⟨*proof*⟩

lemma *emeasure-pmf-of-set-space*[*simp*]: *emeasure pmf-of-set S = 1*
⟨*proof*⟩

lemma *nn-integral-pmf-of-set*: *nn-integral* (*measure-pmf pmf-of-set*) *f = sum f S / card S*
⟨*proof*⟩

lemma *integral-pmf-of-set*: $integral^L$ (*measure-pmf pmf-of-set*) *f = sum f S / card S*
⟨*proof*⟩

lemma *emeasure-pmf-of-set*: *emeasure* (*measure-pmf pmf-of-set*) *A = card* ($S \cap A$) / *card S*
⟨*proof*⟩

lemma *measure-pmf-of-set*: *measure* (*measure-pmf pmf-of-set*) *A = card* ($S \cap A$) / *card S*
⟨*proof*⟩

end

lemma *pmf-expectation-bind-pmf-of-set*:
fixes A :: *'a set* **and** f :: *'a* ⇒ *'b pmf*
and h :: *'b* ⇒ *'c*::{*banach*, *second-countable-topology*}
assumes $A \neq \{\}$ *finite* A $\bigwedge x.\ x \in A \Longrightarrow$ *finite* (*set-pmf* (f x))
shows *measure-pmf.expectation* (*pmf-of-set* $A \gg= f$) h =
($\sum a{\in}A.$ *measure-pmf.expectation* (f a) h $/_R$ *real* (*card* A))
⟨*proof*⟩

lemma *map-pmf-of-set*:
assumes *finite* A $A \neq \{\}$
shows *map-pmf* f (*pmf-of-set* A) = *pmf-of-multiset* (*image-mset* f (*mset-set* A))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pmf-bind-pmf-of-set*:
assumes $A \neq \{\}$ *finite* A
shows *pmf* (*bind-pmf* (*pmf-of-set* A) f) x =
($\sum xa{\in}A.$ *pmf* (f *xa*) x) / *real-of-nat* (*card* A) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pmf-of-set-singleton*: *pmf-of-set* $\{x\}$ = *return-pmf* x
⟨*proof*⟩

lemma *map-pmf-of-set-inj*:
assumes f: *inj-on* f A
and [*simp*]: $A \neq \{\}$ *finite* A
shows *map-pmf* f (*pmf-of-set* A) = *pmf-of-set* (f ‘ A) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-pmf-of-set-bij-betw*:
assumes *bij-betw* f A B $A \neq \{\}$ *finite* A
shows *map-pmf* f (*pmf-of-set* A) = *pmf-of-set* B
⟨*proof*⟩

Choosing an element uniformly at random from the union of a disjoint family of finite non-empty sets with the same size is the same as first choosing a set from the family uniformly at random and then choosing an element from the chosen set uniformly at random.

lemma *pmf-of-set-UN*:
assumes *finite* ($\bigcup$(f ‘ A)) $A \neq \{\}$ $\bigwedge x.\ x \in A \Longrightarrow f\ x \neq \{\}$
$\bigwedge x.\ x \in A \Longrightarrow$ *card* (f x) = n *disjoint-family-on* f A
shows *pmf-of-set* ($\bigcup$(f ‘ A)) = *do* {$x \leftarrow$ *pmf-of-set* A; *pmf-of-set* (f x)}
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bernoulli-pmf-half-conv-pmf-of-set*: *bernoulli-pmf* (*1* / *2*) = *pmf-of-set UNIV*
⟨*proof*⟩

20.6.5 Poisson Distribution

context
fixes *rate* :: *real* **assumes** *rate-pos*: *0* < *rate*
begin

lift-definition *poisson-pmf* :: *nat pmf* **is** λk. *rate* ^ *k* / *fact k* ∗ *exp* (−*rate*)
⟨*proof*⟩

lemma *pmf-poisson*[*simp*]: *pmf poisson-pmf k* = *rate* ^ *k* / *fact k* ∗ *exp* (−*rate*)
⟨*proof*⟩

lemma *set-pmf-poisson*[*simp*]: *set-pmf poisson-pmf* = *UNIV*
⟨*proof*⟩

end

20.6.6 Binomial Distribution

context
fixes *n* :: *nat* **and** *p* :: *real* **assumes** *p-nonneg*: *0* ≤ *p* **and** *p-le-1*: *p* ≤ *1*
begin

lift-definition *binomial-pmf* :: *nat pmf* **is** λk. (*n choose k*) ∗ *p*^*k* ∗ (*1* − *p*)^(*n* − *k*)
⟨*proof*⟩

lemma *pmf-binomial*[*simp*]: *pmf binomial-pmf k* = (*n choose k*) ∗ *p*^*k* ∗ (*1* − *p*)^(*n* − *k*)
⟨*proof*⟩

lemma *set-pmf-binomial-eq*: *set-pmf binomial-pmf* = (*if p* = *0 then* {*0*} *else if p* = *1 then* {*n*} *else* {.. *n*})
⟨*proof*⟩

end

end

lemma *set-pmf-binomial-0*[*simp*]: *set-pmf* (*binomial-pmf n 0*) = {*0*}
⟨*proof*⟩

lemma *set-pmf-binomial-1*[*simp*]: *set-pmf* (*binomial-pmf n 1*) = {*n*}
⟨*proof*⟩

lemma *set-pmf-binomial*[*simp*]: *0* < *p* ⟹ *p* < *1* ⟹ *set-pmf* (*binomial-pmf n p*) = {..*n*}

⟨proof⟩

lemma *finite-set-pmf-binomial-pmf* [*intro*]: $p \in \{0..1\} \Longrightarrow$ *finite* (*set-pmf* (*binomial-pmf* *n p*))
⟨proof⟩

lemma *expectation-binomial-pmf'*:
fixes f :: *nat* ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes *p*: $p \in \{0..1\}$
shows *measure-pmf.expectation* (*binomial-pmf n p*) f =
$(\sum k \le n.\ (real\ (n\ choose\ k) * p\ \hat{}\ k * (1 - p)\ \hat{}\ (n - k)) *_R f\ k)$
⟨proof⟩

lemma *integrable-binomial-pmf* [*simp*, *intro*]:
fixes f :: *nat* ⇒ *'a* :: {*banach*, *second-countable-topology*}
assumes *p*: $p \in \{0..1\}$
shows *integrable* (*binomial-pmf n p*) *f*
⟨proof⟩

context includes *lifting-syntax*
begin

lemma *bind-pmf-parametric* [*transfer-rule*]:
(*rel-pmf A* ===> (*A* ===> *rel-pmf B*) ===> *rel-pmf B*) *bind-pmf bind-pmf*
⟨proof⟩

lemma *return-pmf-parametric* [*transfer-rule*]: (*A* ===> *rel-pmf A*) *return-pmf return-pmf*
⟨proof⟩

end

primrec *replicate-pmf* :: *nat* ⇒ *'a pmf* ⇒ *'a list pmf* **where**
replicate-pmf 0 - = return-pmf []
| *replicate-pmf* (*Suc n*) *p* = *do* {*x* ← *p*; *xs* ← *replicate-pmf n p*; *return-pmf* (*x#xs*)}

lemma *replicate-pmf-1*: *replicate-pmf 1 p = map-pmf* (λ*x*. [*x*]) *p*
⟨proof⟩

lemma *set-replicate-pmf*:
set-pmf (*replicate-pmf n p*) = {*xs*∈*lists* (*set-pmf p*). *length xs* = *n*}
⟨proof⟩

lemma *replicate-pmf-distrib*:
replicate-pmf (*m* + *n*) *p* =
do {*xs* ← *replicate-pmf m p*; *ys* ← *replicate-pmf n p*; *return-pmf* (*xs* @ *ys*)}
⟨proof⟩

lemma *power-diff'*:
assumes $b \leq a$
shows $x \mathbin{\hat{}} (a - b) = (\mathit{if}\ x = 0 \wedge a = b\ \mathit{then}\ 1\ \mathit{else}\ x \mathbin{\hat{}} a\ /\ (x::'a::\mathit{field}) \mathbin{\hat{}} b)$
⟨*proof*⟩

lemma *binomial-pmf-Suc*:
assumes $p \in \{0..1\}$
shows $\mathit{binomial\text{-}pmf}\ (\mathit{Suc}\ n)\ p =$
$\mathit{do}\ \{b \leftarrow \mathit{bernoulli\text{-}pmf}\ p;$
$k \leftarrow \mathit{binomial\text{-}pmf}\ n\ p;$
$\mathit{return\text{-}pmf}\ ((\mathit{if}\ b\ \mathit{then}\ 1\ \mathit{else}\ 0) + k)\}$ (**is** - = *?rhs*)
⟨*proof*⟩

lemma *binomial-pmf-0*: $p \in \{0..1\} \Longrightarrow \mathit{binomial\text{-}pmf}\ 0\ p = \mathit{return\text{-}pmf}\ 0$
⟨*proof*⟩

lemma *binomial-pmf-altdef*:
assumes $p \in \{0..1\}$
shows $\mathit{binomial\text{-}pmf}\ n\ p = \mathit{map\text{-}pmf}\ (\mathit{length} \circ \mathit{filter}\ \mathit{id})\ (\mathit{replicate\text{-}pmf}\ n\ (\mathit{bernoulli\text{-}pmf}\ p))$
⟨*proof*⟩

20.7 Negative Binomial distribution

The negative binomial distribution counts the number of times a weighted coin comes up tails before having come up heads n times. In other words: how many failures do we see before seeing the n-th success?

An alternative view is that the negative binomial distribution is the sum of n i.i.d. geometric variables (this is the definition that we use).

Note that there are sometimes different conventions for this distributions in the literature; for instance, sometimes the number of *attempts* is counted instead of the number of failures. This only shifts the entire distribution by a constant number and is thus not a big difference. I think that the convention we use is the most natural one since the support of the distribution starts at 0, whereas for the other convention it starts at n.

primrec *neg-binomial-pmf* :: $\mathit{nat} \Rightarrow \mathit{real} \Rightarrow \mathit{nat}\ \mathit{pmf}$ **where**
$\mathit{neg\text{-}binomial\text{-}pmf}\ 0\ p = \mathit{return\text{-}pmf}\ 0$
$|\ \mathit{neg\text{-}binomial\text{-}pmf}\ (\mathit{Suc}\ n)\ p =$
$\mathit{map\text{-}pmf}\ (\lambda(x,y).\ (x + y))\ (\mathit{pair\text{-}pmf}\ (\mathit{geometric\text{-}pmf}\ p)\ (\mathit{neg\text{-}binomial\text{-}pmf}\ n\ p))$

lemma *neg-binomial-pmf-Suc-0* [*simp*]: $\mathit{neg\text{-}binomial\text{-}pmf}\ (\mathit{Suc}\ 0)\ p = \mathit{geometric\text{-}pmf}\ p$
⟨*proof*⟩

lemmas *neg-binomial-pmf-Suc* [*simp del*] = *neg-binomial-pmf.simps*(*2*)

lemma *neg-binomial-prob-1* [*simp*]: *neg-binomial-pmf n 1 = return-pmf 0*
⟨*proof*⟩

We can now show the aforementioned intuition about counting the failures before the n-th success with the following recurrence:

lemma *neg-binomial-pmf-unfold*:
assumes *p*: $p \in \{0<..1\}$
shows *neg-binomial-pmf* (*Suc n*) *p* =
do {*b* ← *bernoulli-pmf p*;
if b then neg-binomial-pmf n p else map-pmf Suc (*neg-binomial-pmf* (*Suc n*) *p*)}
(**is** - = *?rhs*)
⟨*proof*⟩

Next, we show an explicit formula for the probability mass function of the negative binomial distribution:

lemma *pmf-neg-binomial*:
assumes *p*: $p \in \{0<..1\}$
shows *pmf* (*neg-binomial-pmf n p*) *k* = *real* ((*k* + *n* − *1*) *choose k*) ∗ *p* ^ *n* ∗ (*1* − *p*) ^ *k*
⟨*proof*⟩

lemma *gbinomial-0-left*: *0 gchoose k* = (*if k* = *0 then 1 else 0*)
⟨*proof*⟩

The following alternative formula highlights why it is called 'negative binomial distribution':

lemma *pmf-neg-binomial′*:
assumes *p*: $p \in \{0<..1\}$
shows *pmf* (*neg-binomial-pmf n p*) *k* = (−*1*) ^ *k* ∗ ((−*real n*) *gchoose k*) ∗ *p* ^ *n* ∗ (*1* − *p*) ^ *k*
⟨*proof*⟩

The cumulative distribution function of the negative binomial distribution can be expressed in terms of that of the 'normal' binomial distribution.

lemma *prob-neg-binomial-pmf-atMost*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) {..*k*} =
measure-pmf.prob (*binomial-pmf* (*n* + *k*) (*1* − *p*)) {..*k*}
⟨*proof*⟩

lemma *prob-neg-binomial-pmf-lessThan*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) {..<*k*} =
measure-pmf.prob (*binomial-pmf* (*n* + *k* − *1*) (*1* − *p*)) {..<*k*}
⟨*proof*⟩

The expected value of the negative binomial distribution is $n(1-p)/p$:

lemma *nn-integral-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *nn-integral* (*measure-pmf* (*neg-binomial-pmf n p*)) *of-nat* = *ennreal* ($n *$ ($1 - p$) / p)
⟨*proof*⟩

lemma *integrable-neg-binomial-pmf-real*:
assumes *p*: $p \in \{0<..1\}$
shows *integrable* (*measure-pmf* (*neg-binomial-pmf n p*)) *real*
⟨*proof*⟩

lemma *expectation-neg-binomial-pmf*:
assumes *p*: $p \in \{0<..1\}$
shows *measure-pmf.expectation* (*neg-binomial-pmf n p*) *real* = $n * (1 - p) / p$
⟨*proof*⟩

20.8 PMFs from association lists

definition *pmf-of-list* :: ($'a \times$ *real*) *list* $\Rightarrow$ $'a$ *pmf* **where**
pmf-of-list xs = *embed-pmf* (λx. *sum-list* (*map snd* (*filter* (λz. *fst z* = *x*) *xs*)))

definition *pmf-of-list-wf* **where**
pmf-of-list-wf xs $\longleftrightarrow$ ($\forall x \in$*set* (*map snd xs*) . $x \geq 0$) $\wedge$ *sum-list* (*map snd xs*) = 1

lemma *pmf-of-list-wfI*:
($\bigwedge x$. $x \in$ *set* (*map snd xs*) $\Longrightarrow x \geq 0$) $\Longrightarrow$ *sum-list* (*map snd xs*) = 1 $\Longrightarrow$ *pmf-of-list-wf xs*
⟨*proof*⟩

context
begin

private lemma *pmf-of-list-aux*:
assumes $\bigwedge x$. $x \in$ *set* (*map snd xs*) $\Longrightarrow x \geq 0$
assumes *sum-list* (*map snd xs*) = 1
shows ($\int^+ x$. *ennreal* (*sum-list* (*map snd* [$z \leftarrow xs$. *fst z* = *x*])) ∂*count-space UNIV*) = 1
⟨*proof*⟩

lemma *pmf-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *pmf* (*pmf-of-list xs*) *x* = *sum-list* (*map snd* (*filter* (λz. *fst z* = *x*) *xs*))
⟨*proof*⟩

end

lemma *set-pmf-of-list*:

assumes *pmf-of-list-wf xs*
shows *set-pmf* (*pmf-of-list xs*) $\subseteq$ *set* (*map fst xs*)
⟨*proof*⟩

lemma *finite-set-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *finite* (*set-pmf* (*pmf-of-list xs*))
⟨*proof*⟩

lemma *emeasure-Int-set-pmf*:
emeasure (*measure-pmf p*) ($A \cap$ *set-pmf p*) = *emeasure* (*measure-pmf p*) A
⟨*proof*⟩

lemma *measure-Int-set-pmf*:
measure (*measure-pmf p*) ($A \cap$ *set-pmf p*) = *measure* (*measure-pmf p*) A
⟨*proof*⟩

lemma *measure-prob-cong-0*:
assumes $\bigwedge x.\ x \in A - B \Longrightarrow$ *pmf p x* = *0*
assumes $\bigwedge x.\ x \in B - A \Longrightarrow$ *pmf p x* = *0*
shows *measure* (*measure-pmf p*) A = *measure* (*measure-pmf p*) B
⟨*proof*⟩

lemma *emeasure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *emeasure* (*pmf-of-list xs*) A = *ennreal* (*sum-list* (*map snd* (*filter* ($\lambda x.$ *fst* $x \in A$) *xs*)))
⟨*proof*⟩

lemma *measure-pmf-of-list*:
assumes *pmf-of-list-wf xs*
shows *measure* (*pmf-of-list xs*) A = *sum-list* (*map snd* (*filter* ($\lambda x.$ *fst* $x \in A$) *xs*))
⟨*proof*⟩

lemma *sum-list-nonneg-eq-zero-iff*:
fixes *xs* :: *'a* :: *linordered-ab-group-add list*
shows ($\bigwedge x.\ x \in$ *set xs* $\Longrightarrow x \geq 0$) $\Longrightarrow$ *sum-list xs* = *0* $\longleftrightarrow$ *set xs* $\subseteq \{0\}$
⟨*proof*⟩

lemma *sum-list-filter-nonzero*:
sum-list (*filter* ($\lambda x.\ x \neq 0$) *xs*) = *sum-list xs*
⟨*proof*⟩

lemma *set-pmf-of-list-eq*:
assumes *pmf-of-list-wf xs* $\bigwedge x.\ x \in$ *snd* ` *set xs* $\Longrightarrow x > 0$
shows *set-pmf* (*pmf-of-list xs*) = *fst* ` *set xs*

⟨*proof*⟩

lemma *pmf-of-list-remove-zeros*:
assumes *pmf-of-list-wf xs*
defines $xs' \equiv filter\ (\lambda z.\ snd\ z \neq 0)\ xs$
shows $pmf\text{-}of\text{-}list\text{-}wf\ xs'$ $pmf\text{-}of\text{-}list\ xs' = pmf\text{-}of\text{-}list\ xs$
⟨*proof*⟩

end

21 Code generation for PMFs

theory *PMF-Impl*
imports *Probability-Mass-Function HOL−Library.AList-Mapping*
begin

21.1 General code generation setup

definition *pmf-of-mapping* :: $('a,\ real)\ mapping \Rightarrow 'a\ pmf$ **where**
$pmf\text{-}of\text{-}mapping\ m = embed\text{-}pmf\ (Mapping.lookup\text{-}default\ 0\ m)$

lemma *nn-integral-lookup-default*:
fixes $m :: ('a,\ real)\ mapping$
assumes $finite\ (Mapping.keys\ m)$ $All\text{-}mapping\ m\ (\lambda\text{-}\ x.\ x \geq 0)$
shows $nn\text{-}integral\ (count\text{-}space\ UNIV)\ (\lambda k.\ ennreal\ (Mapping.lookup\text{-}default\ 0\ m\ k)) =$
$ennreal\ (\sum k \in Mapping.keys\ m.\ Mapping.lookup\text{-}default\ 0\ m\ k)$
⟨*proof*⟩

lemma *pmf-of-mapping*:
assumes $finite\ (Mapping.keys\ m)$ $All\text{-}mapping\ m\ (\lambda\text{-}\ p.\ p \geq 0)$
assumes $(\sum x \in Mapping.keys\ m.\ Mapping.lookup\text{-}default\ 0\ m\ x) = 1$
shows $pmf\ (pmf\text{-}of\text{-}mapping\ m)\ x = Mapping.lookup\text{-}default\ 0\ m\ x$
⟨*proof*⟩

lemma *pmf-of-set-pmf-of-mapping*:
assumes $A \neq \{\}$ $set\ xs = A$ $distinct\ xs$
shows $pmf\text{-}of\text{-}set\ A = pmf\text{-}of\text{-}mapping\ (Mapping.tabulate\ xs\ (\lambda\text{-}.\ 1\ /\ real\ (length\ xs)))$
(**is** $?lhs = ?rhs$)
⟨*proof*⟩

lift-definition *mapping-of-pmf* :: $'a\ pmf \Rightarrow ('a,\ real)\ mapping$ **is**
$\lambda p\ x.\ if\ pmf\ p\ x = 0\ then\ None\ else\ Some\ (pmf\ p\ x)$ ⟨*proof*⟩

lemma *lookup-default-mapping-of-pmf*:
$Mapping.lookup\text{-}default\ 0\ (mapping\text{-}of\text{-}pmf\ p)\ x = pmf\ p\ x$
⟨*proof*⟩

context
begin

interpretation *pmf-as-function* ⟨*proof*⟩

lemma *nn-integral-pmf-eq-1*: $(\int^+ x.\ ennreal\ (pmf\ p\ x)\ \partial count\text{-}space\ UNIV) = 1$
⟨*proof*⟩
end

lemma *pmf-of-mapping-mapping-of-pmf* [*code abstype*]:
pmf-of-mapping (*mapping-of-pmf p*) = *p*
⟨*proof*⟩

lemma *mapping-of-pmfI*:
assumes $\bigwedge x.\ x \in Mapping.keys\ m \Longrightarrow Mapping.lookup\ m\ x = Some\ (pmf\ p\ x)$
assumes *Mapping.keys m* = *set-pmf p*
shows *mapping-of-pmf p* = *m*
⟨*proof*⟩

lemma *mapping-of-pmfI'*:
assumes $\bigwedge x.\ x \in Mapping.keys\ m \Longrightarrow Mapping.lookup\text{-}default\ 0\ m\ x = pmf\ p\ x$

assumes *Mapping.keys m* = *set-pmf p*
shows *mapping-of-pmf p* = *m*
⟨*proof*⟩

lemma *return-pmf-code* [*code abstract*]:
mapping-of-pmf (*return-pmf x*) = *Mapping.update x 1 Mapping.empty*
⟨*proof*⟩

lemma *pmf-of-set-code-aux*:
assumes $A \neq \{\}$ *set xs* = *A distinct xs*
shows *mapping-of-pmf* (*pmf-of-set A*) = *Mapping.tabulate xs* (λ-. *1* / *real* (*length xs*))
⟨*proof*⟩

definition *pmf-of-set-impl* **where**
pmf-of-set-impl A = *mapping-of-pmf* (*pmf-of-set A*)

lemma *pmf-of-set-impl-code-alt*:
assumes $A \neq \{\}$ *finite A*
shows *pmf-of-set-impl A* =
(*let p* = *1* / *real* (*card A*)
in Finite-Set.fold (λ*x. Mapping.update x p*) *Mapping.empty A*)
⟨*proof*⟩

lemma *pmf-of-set-impl-code* [*code*]:
pmf-of-set-impl (*set xs*) =

(*if* $xs = []$ *then*
Code.abort (*STR* ''*pmf-of-set of empty set*'') (λ-. *mapping-of-pmf* (*pmf-of-set* (*set xs*)))
else let $xs' =$ *remdups xs*; $p = 1$ / *real* (*length* xs') *in*
Mapping.tabulate xs' (λ-. *p*))
⟨*proof*⟩

lemma *pmf-of-set-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-set A*) = *pmf-of-set-impl A*
⟨*proof*⟩

lemma *pmf-of-multiset-pmf-of-mapping*:
assumes $A \neq \{\#\}$ *set xs* = *set-mset A* *distinct xs*
shows *mapping-of-pmf* (*pmf-of-multiset A*) = *Mapping.tabulate xs* (λx. *count A x* / *real* (*size A*))
⟨*proof*⟩

definition *pmf-of-multiset-impl* **where**
pmf-of-multiset-impl A = *mapping-of-pmf* (*pmf-of-multiset A*)

lemma *pmf-of-multiset-impl-code-alt*:
assumes $A \neq \{\#\}$
shows *pmf-of-multiset-impl A* =
(*let* $p = 1$ / *real* (*size A*)
in *fold-mset* (λx. *Mapping.map-default x 0* ((+) *p*)) *Mapping.empty A*)
⟨*proof*⟩

lemma *pmf-of-multiset-impl-code* [*code*]:
pmf-of-multiset-impl (*mset xs*) =
(*if* $xs = []$ *then*
Code.abort (*STR* ''*pmf-of-multiset of empty multiset*'')
(λ-. *mapping-of-pmf* (*pmf-of-multiset* (*mset xs*)))
else let $xs' =$ *remdups xs*; $p = 1$ / *real* (*length xs*) *in*
Mapping.tabulate xs' (λx. *real* (*count* (*mset xs*) *x*) $*$ *p*))
⟨*proof*⟩

lemma *pmf-of-multiset-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-multiset A*) = *pmf-of-multiset-impl A*
⟨*proof*⟩

lemma *bernoulli-pmf-code* [*code abstract*]:
mapping-of-pmf (*bernoulli-pmf p*) =
(*if* $p \leq 0$ *then Mapping.update False 1 Mapping.empty*
else if $p \geq 1$ *then Mapping.update True 1 Mapping.empty*
else Mapping.update False $(1 - p)$ (*Mapping.update True p Mapping.empty*))
⟨*proof*⟩

lemma *pmf-code* [*code*]: *pmf p x* = *Mapping.lookup-default 0* (*mapping-of-pmf p*) *x*
⟨*proof*⟩

lemma *set-pmf-code* [*code*]: *set-pmf p* = *Mapping.keys* (*mapping-of-pmf p*)
⟨*proof*⟩

lemma *keys-mapping-of-pmf* [*simp*]: *Mapping.keys* (*mapping-of-pmf p*) = *set-pmf p*
⟨*proof*⟩

definition *fold-combine-plus* **where**
fold-combine-plus = *comm-monoid-set.F* (*Mapping.combine* ((+) :: *real* ⇒ -)) *Mapping.empty*

context
begin

interpretation *fold-combine-plus*: *combine-mapping-abel-semigroup* (+) :: *real* ⇒ -
⟨*proof*⟩ **lemma** *lookup-default-fold-combine-plus*:
fixes *A* :: *'b set* **and** *f* :: *'b* ⇒ (*'a*, *real*) *mapping*
assumes *finite A*
shows *Mapping.lookup-default 0* (*fold-combine-plus f A*) *x* =
($\sum$ *y*∈*A*. *Mapping.lookup-default 0* (*f y*) *x*)
⟨*proof*⟩ **lemma** *keys-fold-combine-plus*:
finite A ⟹ *Mapping.keys* (*fold-combine-plus f A*) = ($\bigcup$ *x*∈*A*. *Mapping.keys* (*f x*))
⟨*proof*⟩ **lemma** *fold-combine-plus-code* [*code*]:
fold-combine-plus g (*set xs*) = *foldr* (λ*x*. *Mapping.combine* (+) (*g x*)) (*remdups xs*) *Mapping.empty*
⟨*proof*⟩ **lemma** *lookup-default-0-map-values*:
assumes *f x 0* = *0*
shows *Mapping.lookup-default 0* (*Mapping.map-values f m*) *x* = *f x* (*Mapping.lookup-default 0 m x*)
⟨*proof*⟩ **lemma** *mapping-of-bind-pmf*:
assumes *finite* (*set-pmf p*)
shows *mapping-of-pmf* (*bind-pmf p f*) =
fold-combine-plus (λ*x*. *Mapping.map-values* (λ-. (∗) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) (*set-pmf p*)
⟨*proof*⟩

lift-definition *bind-pmf-aux* :: *'a pmf* ⇒ (*'a* ⇒ *'b pmf*) ⇒ *'a set* ⇒ (*'b*, *real*) *mapping* **is**
λ(*p* :: *'a pmf*) (*f* :: *'a* ⇒ *'b pmf*) (*A*::*'a set*) (*x*::*'b*).
if x ∈ ($\bigcup$ *y*∈*A*. *set-pmf* (*f y*)) *then*

Some (*measure-pmf.expectation p* (λy. *indicator A y* $*$ *pmf* (*f y*) *x*))
else None ⟨*proof*⟩

lemma *keys-bind-pmf-aux* [*simp*]:
Mapping.keys (*bind-pmf-aux p f A*) = ($\bigcup x \in A$. *set-pmf* (*f x*))
⟨*proof*⟩

lemma *lookup-default-bind-pmf-aux*:
Mapping.lookup-default 0 (*bind-pmf-aux p f A*) *x* =
(*if x* $\in$ ($\bigcup y \in A$. *set-pmf* (*f y*)) *then*
measure-pmf.expectation p (λy. *indicator A y* $*$ *pmf* (*f y*) *x*) *else 0*)
⟨*proof*⟩

lemma *lookup-default-bind-pmf-aux'* [*simp*]:
Mapping.lookup-default 0 (*bind-pmf-aux p f* (*set-pmf p*)) *x* = *pmf* (*bind-pmf p f*) *x*
⟨*proof*⟩

lemma *bind-pmf-aux-correct*:
mapping-of-pmf (*bind-pmf p f*) = *bind-pmf-aux p f* (*set-pmf p*)
⟨*proof*⟩

lemma *bind-pmf-aux-code-aux*:
assumes *finite A*
shows *bind-pmf-aux p f A* =
fold-combine-plus (λx. *Mapping.map-values* (λ-. ($*$) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) *A* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bind-pmf-aux-code* [*code*]:
bind-pmf-aux p f (*set xs*) =
fold-combine-plus (λx. *Mapping.map-values* (λ-. ($*$) (*pmf p x*))
(*mapping-of-pmf* (*f x*))) (*set xs*)
⟨*proof*⟩

lemmas *bind-pmf-code* [*code abstract*] = *bind-pmf-aux-correct*

end

hide-const (**open**) *fold-combine-plus*

lift-definition *cond-pmf-impl* :: *'a pmf* $\Rightarrow$ *'a set* $\Rightarrow$ (*'a, real*) *mapping option* **is**
λp *A. if A* $\cap$ *set-pmf p* = {} *then None else*
Some (λx. *if x* $\in$ *A* $\cap$ *set-pmf p then Some* (*pmf p x* / *measure-pmf.prob p A*) *else None*) ⟨*proof*⟩

lemma *cond-pmf-impl-code-alt*:
assumes *finite A*

shows *cond-pmf-impl p A = (*
let C = A ∩ set-pmf p;
prob = (∑x∈C. pmf p x)
in if prob = 0 then
None
else
Some (Mapping.map-values (λ- y. y / prob)
(Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))

⟨*proof*⟩

lemma *cond-pmf-impl-code* [*code*]:
cond-pmf-impl p (set xs) = (
let C = set xs ∩ set-pmf p;
prob = (∑x∈C. pmf p x)
in if prob = 0 then
None
else
Some (Mapping.map-values (λ- y. y / prob)
(Mapping.filter (λk -. k ∈ C) (mapping-of-pmf p))))
⟨*proof*⟩

lemma *cond-pmf-code* [*code abstract*]:
mapping-of-pmf (cond-pmf p A) =
(case cond-pmf-impl p A of
None ⇒ Code.abort (STR ''cond-pmf with set of probability 0'')
(λ-. mapping-of-pmf (cond-pmf p A))
| Some m ⇒ m)

⟨*proof*⟩

lemma *binomial-pmf-code* [*code abstract*]:
mapping-of-pmf (binomial-pmf n p) = (
if p < 0 ∨ p > 1 then
Code.abort (STR ''binomial-pmf with invalid probability'')
(λ-. mapping-of-pmf (binomial-pmf n p))
else if p = 0 then Mapping.update 0 1 Mapping.empty
else if p = 1 then Mapping.update n 1 Mapping.empty
else Mapping.tabulate [0..<Suc n] (λk. real (n choose k) ∗ p ^ k ∗ (1 − p) ^ (n − k)))
⟨*proof*⟩

lemma *pred-pmf-code* [*code*]:
pred-pmf P p = (∀x∈set-pmf p. P x)
⟨*proof*⟩

lemma *mapping-of-pmf-pmf-of-list*:
assumes ⋀*x. x ∈ snd ‘ set xs ⟹ x > 0 sum-list (map snd xs) = 1*

shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs*))
($\lambda x.$ *sum-list* (*map snd* (*filter* ($\lambda z.$ *fst* $z = x$) *xs*)))
⟨*proof*⟩

lemma *mapping-of-pmf-pmf-of-list′*:
assumes *pmf-of-list-wf xs*
defines $xs' \equiv$ *filter* ($\lambda z.$ *snd* $z \neq 0$) *xs*
shows *mapping-of-pmf* (*pmf-of-list xs*) =
Mapping.tabulate (*remdups* (*map fst xs′*))
($\lambda x.$ *sum-list* (*map snd* (*filter* ($\lambda z.$ *fst* $z = x$) *xs′*))) (**is** - = *?rhs*)
⟨*proof*⟩

lemma *pmf-of-list-wf-code* [*code*]:
pmf-of-list-wf xs $\longleftrightarrow$ *list-all* ($\lambda z.$ *snd* $z \geq 0$) *xs* $\wedge$ *sum-list* (*map snd xs*) = *1*
⟨*proof*⟩

lemma *pmf-of-list-code* [*code abstract*]:
mapping-of-pmf (*pmf-of-list xs*) = (
if pmf-of-list-wf xs then
let xs' = *filter* ($\lambda z.$ *snd* $z \neq 0$) *xs*
in *Mapping.tabulate* (*remdups* (*map fst xs′*))
($\lambda x.$ *sum-list* (*map snd* (*filter* ($\lambda z.$ *fst* $z = x$) *xs′*)))
else
Code.abort (*STR ′′Invalid list for pmf-of-list′′*) (λ-. *mapping-of-pmf* (*pmf-of-list xs*)))
⟨*proof*⟩

lemma *mapping-of-pmf-eq-iff* [*simp*]:
mapping-of-pmf p = *mapping-of-pmf q* $\longleftrightarrow$ $p = (q :: {}'a\ pmf)$
⟨*proof*⟩

21.2 Code abbreviations for integrals and probabilities

Integrals and probabilities are defined for general measures, so we cannot give any code equations directly. We can, however, specialise these constants them to PMFs, give code equations for these specialised constants, and tell the code generator to unfold the original constants to the specialised ones whenever possible.

definition *pmf-integral* **where**
pmf-integral p f = *lebesgue-integral* (*measure-pmf p*) (f :: - $\Rightarrow$ *real*)

definition *pmf-set-integral* **where**
pmf-set-integral p f A = *lebesgue-integral* (*measure-pmf p*) ($\lambda x.$ *indicator* $A\ x * f\ x$ :: *real*)

definition *pmf-prob* **where**
pmf-prob p A = *measure-pmf.prob p A*

lemma *pmf-prob-compl*: *pmf-prob p* $(-A)$ = *1* $-$ *pmf-prob p A*
⟨*proof*⟩

lemma *pmf-integral-pmf-set-integral* [*code*]:
pmf-integral p f = *pmf-set-integral p f* (*set-pmf p*)
⟨*proof*⟩

lemma *pmf-prob-pmf-set-integral*:
pmf-prob p A = *pmf-set-integral p* (λ-. *1*) *A*
⟨*proof*⟩

lemma *pmf-set-integral-code-alt-finite*:
finite A ⟹ *pmf-set-integral p f A* = ($\sum$ *x*∈*A*. *pmf p x* ∗ *f x*)
⟨*proof*⟩

lemma *pmf-set-integral-code* [*code*]:
pmf-set-integral p f (*set xs*) = ($\sum$ *x*∈*set xs*. *pmf p x* ∗ *f x*)
⟨*proof*⟩

lemma *pmf-prob-code-alt-finite*:
finite A ⟹ *pmf-prob p A* = ($\sum$ *x*∈*A*. *pmf p x*)
⟨*proof*⟩

lemma *pmf-prob-code* [*code*]:
pmf-prob p (*set xs*) = ($\sum$ *x*∈*set xs*. *pmf p x*)
pmf-prob p (*List.coset xs*) = *1* $-$ ($\sum$ *x*∈*set xs*. *pmf p x*)
⟨*proof*⟩

lemma *pmf-prob-code-unfold* [*code-abbrev*]: *pmf-prob p* = *measure-pmf.prob p*
⟨*proof*⟩

lemma *pmf-integral-code-unfold* [*code-abbrev*]: *pmf-integral p* = *measure-pmf.expectation p*
⟨*proof*⟩

definition *pmf-of-alist xs* = *embed-pmf* (λ*x*. *case map-of xs x of Some p* ⇒ *p* | *None* ⇒ *0*)

lemma *pmf-of-mapping-Mapping* [*code-post*]:
pmf-of-mapping (*Mapping xs*) = *pmf-of-alist xs*
⟨*proof*⟩

instantiation *pmf* :: (*equal*) *equal*
begin

definition *equal-pmf p q* = (*mapping-of-pmf p* = *mapping-of-pmf* (*q* :: *'a pmf*))

instance ⟨*proof*⟩
end

definition *single* :: *'a* ⇒ *'a multiset* **where**
single s = {#*s*#}

instantiation *pmf* :: (*random*) *random*
begin

context
  includes *state-combinator-syntax term-syntax*
begin

definition
  pmfify :: (*'b*::*typerep multiset* × (*unit* ⇒ *Code-Evaluation.term*)) ⇒
            'b × (*unit* ⇒ *Code-Evaluation.term*) ⇒
            'b pmf × (*unit* ⇒ *Code-Evaluation.term*) **where**
  [*code-unfold*]: *pmfify A x* =
    Code-Evaluation.valtermify pmf-of-multiset {·}
      (*Code-Evaluation.valtermify* (+) {·} *A* {·}
       (*Code-Evaluation.valtermify single* {·} *x*))

definition
  Quickcheck-Random.random i =
     Quickcheck-Random.random i ∘→ (λ*A*.
       Quickcheck-Random.random i ∘→ (λ*x*. *Pair* (*pmfify A x*)))

instance ⟨*proof*⟩

end

end

instantiation *pmf* :: (*full-exhaustive*) *full-exhaustive*
begin

definition *full-exhaustive-pmf* :: (*'a pmf* × (*unit* ⇒ *term*) ⇒ (*bool* × *term list*) *option*) ⇒ *natural* ⇒ (*bool* × *term list*) *option*
where
  full-exhaustive-pmf f i =
     Quickcheck-Exhaustive.full-exhaustive (λ*A*.
       Quickcheck-Exhaustive.full-exhaustive (λ*x*. *f* (*pmfify A x*)) *i*) *i*

instance ⟨*proof*⟩

end

end

22 Finite Maps

theory *Fin-Map*
imports *HOL−Analysis.Finite-Product-Measure HOL−Library.Finite-Map*
begin

The *fmap* type can be instantiated to *polish-space*, needed for the proof of projective limit. *extensional* functions are used for the representation in order to stay close to the developments of (finite) products Pi_E and their sigma-algebra Pi_M.

type-notation *fmap* ((- $\Rightarrow_F$ /-) [*22*, *21*] *21*)

unbundle *fmap.lifting*

22.1 Domain and Application

lift-definition *domain*::($'i \Rightarrow_F 'a$) $\Rightarrow$ *'i set* **is** *dom* ⟨*proof*⟩

lemma *finite-domain*[*simp*, *intro*]: *finite* (*domain P*)
⟨*proof*⟩

lift-definition *proj* :: ($'i \Rightarrow_F 'a$) $\Rightarrow$ $'i \Rightarrow 'a$ ($'((-)')_F$ [*0*] *1000*) **is**
$\lambda f\ x.$ *if* $x \in$ *dom f then the* (*f x*) *else undefined* ⟨*proof*⟩

declare [[*coercion proj*]]

lemma *extensional-proj*[*simp*, *intro*]: $(P)_F \in$ *extensional* (*domain P*)
⟨*proof*⟩

lemma *proj-undefined*[*simp*, *intro*]: $i \notin$ *domain P* $\Longrightarrow$ *P i* = *undefined*
⟨*proof*⟩

lemma *finmap-eq-iff*: *P* = *Q* $\longleftrightarrow$ (*domain P* = *domain Q* $\wedge$ ($\forall$ *i*$\in$*domain P*. *P i* = *Q i*))
⟨*proof*⟩

22.2 Constructor of Finite Maps

lift-definition *finmap-of*::*'i set* $\Rightarrow$ ($'i \Rightarrow 'a$) $\Rightarrow$ ($'i \Rightarrow_F 'a$) **is**
$\lambda I\ f\ x.$ *if* $x \in I \wedge$ *finite I then Some* (*f x*) *else None*
⟨*proof*⟩

lemma *proj-finmap-of*[*simp*]:

assumes *finite inds*
shows $(\mathit{finmap\text{-}of}\ \mathit{inds}\ f)_F = \mathit{restrict}\ f\ \mathit{inds}$
$\langle \mathit{proof} \rangle$

lemma *domain-finmap-of* [*simp*]:
assumes *finite inds*
shows $\mathit{domain}\ (\mathit{finmap\text{-}of}\ \mathit{inds}\ f) = \mathit{inds}$
$\langle \mathit{proof} \rangle$

lemma *finmap-of-eq-iff* [*simp*]:
assumes *finite i finite j*
shows $\mathit{finmap\text{-}of}\ i\ m = \mathit{finmap\text{-}of}\ j\ n \longleftrightarrow i = j \wedge (\forall\, k{\in}i.\ m\ k{=}\ n\ k)$
$\langle \mathit{proof} \rangle$

lemma *finmap-of-inj-on-extensional-finite*:
assumes *finite K*
assumes $S \subseteq \mathit{extensional}\ K$
shows $\mathit{inj\text{-}on}\ (\mathit{finmap\text{-}of}\ K)\ S$
$\langle \mathit{proof} \rangle$

22.3 Product set of Finite Maps

This is *Pi* for Finite Maps, most of this is copied

definition $\mathit{Pi}' :: {'i}\ \mathit{set} \Rightarrow ({'i} \Rightarrow {'a}\ \mathit{set}) \Rightarrow ({'i} \Rightarrow_F {'a})\ \mathit{set}$ **where**
$\mathit{Pi}'\ I\ A = \{\ P.\ \mathit{domain}\ P = I \wedge (\forall\, i.\ i \in I \longrightarrow (P)_F\ \ i \in A\ i)\ \}$

syntax
$\text{-}\mathit{Pi}' :: [\mathit{pttrn},\ {'a}\ \mathit{set},\ {'b}\ \mathit{set}] \Rightarrow ({'a} \Rightarrow {'b})\ \mathit{set}\ \ ((3\Pi''\ \text{-}{\in}\text{-./ -})\ \ \ 10)$
translations
$\Pi'\ x{\in}A.\ B == \mathit{CONST}\ \mathit{Pi}'\ A\ (\lambda x.\ B)$

22.3.1 Basic Properties of Pi'

lemma *Pi'-I* [*intro*!]: $\mathit{domain}\ f = A \Longrightarrow (\bigwedge x.\ x \in A \Longrightarrow f\ x \in B\ x) \Longrightarrow f \in \mathit{Pi}'\ A\ B$
$\langle \mathit{proof} \rangle$

lemma *Pi'-I'* [*simp*]: $\mathit{domain}\ f = A \Longrightarrow (\bigwedge x.\ x \in A \longrightarrow f\ x \in B\ x) \Longrightarrow f \in \mathit{Pi}'\ A\ B$
$\langle \mathit{proof} \rangle$

lemma *Pi'-mem*: $f{\in}\ \mathit{Pi}'\ A\ B \Longrightarrow x \in A \Longrightarrow f\ x \in B\ x$
$\langle \mathit{proof} \rangle$

lemma *Pi'-iff*: $f \in \mathit{Pi}'\ I\ X \longleftrightarrow \mathit{domain}\ f = I \wedge (\forall\, i{\in}I.\ f\ i \in X\ i)$
$\langle \mathit{proof} \rangle$

lemma *Pi'E* [*elim*]:

$f \in Pi'\ A\ B \Longrightarrow (f\ x \in B\ x \Longrightarrow domain\ f = A \Longrightarrow Q) \Longrightarrow (x \notin A \Longrightarrow Q) \Longrightarrow Q$
⟨*proof*⟩

lemma *in-Pi'-cong*:
$domain\ f = domain\ g \Longrightarrow (\bigwedge w.\ w \in A \Longrightarrow f\ w = g\ w) \Longrightarrow f \in Pi'\ A\ B \longleftrightarrow g \in Pi'\ A\ B$
⟨*proof*⟩

lemma *Pi'-eq-empty*[*simp*]:
assumes *finite* A **shows** $(Pi'\ A\ B) = \{\} \longleftrightarrow (\exists x{\in}A.\ B\ x = \{\})$
⟨*proof*⟩

lemma *Pi'-mono*: $(\bigwedge x.\ x \in A \Longrightarrow B\ x \subseteq C\ x) \Longrightarrow Pi'\ A\ B \subseteq Pi'\ A\ C$
⟨*proof*⟩

lemma *Pi-Pi'*: *finite* $A \Longrightarrow (Pi_E\ A\ B) = proj\ `\ Pi'\ A\ B$
⟨*proof*⟩

22.4 Topological Space of Finite Maps

instantiation *fmap* :: (*type*, *topological-space*) *topological-space*
begin

definition *open-fmap* :: $('a \Rightarrow_F 'b)\ set \Rightarrow bool$ **where**
[*code del*]: $open\text{-}fmap = generate\text{-}topology\ \{Pi'\ a\ b | a\ b.\ \forall i{\in}a.\ open\ (b\ i)\}$

lemma *open-Pi'I*: $(\bigwedge i.\ i \in I \Longrightarrow open\ (A\ i)) \Longrightarrow open\ (Pi'\ I\ A)$
⟨*proof*⟩

instance ⟨*proof*⟩

end

lemma *open-restricted-space*:
shows $open\ \{m.\ P\ (domain\ m)\}$
⟨*proof*⟩

lemma *closed-restricted-space*:
shows $closed\ \{m.\ P\ (domain\ m)\}$
⟨*proof*⟩

lemma *tendsto-proj*: $((\lambda x.\ x) \longrightarrow a)\ F \Longrightarrow ((\lambda x.\ (x)_F\ i) \longrightarrow (a)_F\ i)\ F$
⟨*proof*⟩

lemma *continuous-proj*:
shows $continuous\text{-}on\ s\ (\lambda x.\ (x)_F\ i)$
⟨*proof*⟩

instance *fmap* :: (*type*, *first-countable-topology*) *first-countable-topology*
⟨*proof*⟩

22.5 Metric Space of Finite Maps

instantiation *fmap* :: (*type*, *metric-space*) *dist*
begin

definition *dist-fmap* **where**
$dist\ P\ Q = Max\ (range\ (\lambda i.\ dist\ ((P)_F\ i)\ ((Q)_F\ i))) + (if\ domain\ P = domain\ Q\ then\ 0\ else\ 1)$

instance ⟨*proof*⟩
end

instantiation *fmap* :: (*type*, *metric-space*) *uniformity-dist*
begin

definition [*code del*]:
$(uniformity :: (('a, 'b)\ fmap \times ('a \Rightarrow_F 'b))\ filter) =$
$(INF\ e\in\{0 <..\}.\ principal\ \{(x, y).\ dist\ x\ y < e\})$

instance
⟨*proof*⟩
end

declare *uniformity-Abort*[**where** $'a=('a \Rightarrow_F 'b::metric\text{-}space)$, *code*]

instantiation *fmap* :: (*type*, *metric-space*) *metric-space*
begin

lemma *finite-proj-image'*: $x \notin domain\ P \Longrightarrow finite\ ((P)_F\ \text{`}\ S)$
⟨*proof*⟩

lemma *finite-proj-image*: $finite\ ((P)_F\ \text{`}\ S)$
⟨*proof*⟩

lemma *finite-proj-diag*: $finite\ ((\lambda i.\ d\ ((P)_F\ i)\ ((Q)_F\ i))\ \text{`}\ S)$
⟨*proof*⟩

lemma *dist-le-1-imp-domain-eq*:
shows $dist\ P\ Q < 1 \Longrightarrow domain\ P = domain\ Q$
⟨*proof*⟩

lemma *dist-proj*:
shows $dist\ ((x)_F\ i)\ ((y)_F\ i) \leq dist\ x\ y$
⟨*proof*⟩

lemma *dist-finmap-lessI*:

assumes *domain P* = *domain Q*
assumes $0 < e$
assumes $\bigwedge i.\ i \in domain\ P \Longrightarrow dist\ (P\ i)\ (Q\ i) < e$
shows $dist\ P\ Q < e$
⟨*proof*⟩

instance
⟨*proof*⟩

end

22.6 Complete Space of Finite Maps

lemma *tendsto-finmap*:
fixes f::$nat \Rightarrow ('i \Rightarrow_F ('a$::$metric\text{-}space))$
assumes *ind-f*: $\bigwedge n.\ domain\ (f\ n) = domain\ g$
assumes *proj-g*: $\bigwedge i.\ i \in domain\ g \Longrightarrow (\lambda n.\ (f\ n)\ i) \longrightarrow g\ i$
shows $f \longrightarrow g$
⟨*proof*⟩

instance *fmap* :: (*type*, *complete-space*) *complete-space*
⟨*proof*⟩

22.7 Second Countable Space of Finite Maps

instantiation *fmap* :: (*countable*, *second-countable-topology*) *second-countable-topology*
begin

definition *basis-proj*::*'b set set*
where *basis-proj* = (*SOME B. countable B* ∧ *topological-basis B*)

lemma *countable-basis-proj*: *countable basis-proj* **and** *basis-proj*: *topological-basis basis-proj*
⟨*proof*⟩

definition *basis-finmap*::$('a \Rightarrow_F 'b)$ *set set*
where *basis-finmap* = $\{Pi'\ I\ S | I\ S.\ finite\ I \wedge (\forall i \in I.\ S\ i \in basis\text{-}proj)\}$

lemma *in-basis-finmapI*:
assumes *finite I* **assumes** $\bigwedge i.\ i \in I \Longrightarrow S\ i \in basis\text{-}proj$
shows $Pi'\ I\ S \in basis\text{-}finmap$
⟨*proof*⟩

lemma *basis-finmap-eq*:
assumes *basis-proj* ≠ {}
shows *basis-finmap* = $(\lambda f.\ Pi'\ (domain\ f)\ (\lambda i.\ from\text{-}nat\text{-}into\ basis\text{-}proj\ ((f)_F\ i)))$ ‘
($UNIV$::$('a \Rightarrow_F nat)$ *set*) (**is** - = *?f* ‘ -)
⟨*proof*⟩

lemma *basis-finmap-eq-empty*: *basis-proj* $=$ $\{\}$ $\Longrightarrow$ *basis-finmap* $=$ $\{Pi'$ $\{\}$ *undefined*$\}$
  ⟨*proof*⟩

lemma *countable-basis-finmap*: *countable basis-finmap*
  ⟨*proof*⟩

lemma *finmap-topological-basis*:
  topological-basis basis-finmap
⟨*proof*⟩

lemma *range-enum-basis-finmap-imp-open*:
  assumes $x \in$ *basis-finmap*
  shows *open x*
  ⟨*proof*⟩

instance ⟨*proof*⟩

end

22.8 Polish Space of Finite Maps

instance *fmap* :: (*countable*, *polish-space*) *polish-space* ⟨*proof*⟩

22.9 Product Measurable Space of Finite Maps

definition *PiF I M* $\equiv$
  sigma $(\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$ $\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$

abbreviation
  $Pi_F\ I\ M \equiv PiF\ I\ M$

syntax
  -PiF :: *pttrn* $\Rightarrow$ *'i set* $\Rightarrow$ *'a measure* $\Rightarrow$ (*'i => 'a*) *measure* $((3\Pi_F$ -∈-./ -) *10*)
translations
  $\Pi_F\ x{\in}I.\ M$ == *CONST PiF I* (%x. *M*)

lemma *PiF-gen-subset*: $\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$ $\subseteq$
    Pow $(\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
  ⟨*proof*⟩

lemma *space-PiF*: *space* (*PiF I M*) $= (\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
  ⟨*proof*⟩

lemma *sets-PiF*:
  sets (*PiF I M*) $=$ *sigma-sets* $(\bigcup J \in I.\ (\Pi'\ j{\in}J.\ space\ (M\ j)))$
    $\{(\Pi'\ j{\in}J.\ X\ j)\ |X\ J.\ J \in I \wedge X \in (\Pi\ j{\in}J.\ sets\ (M\ j))\}$
  ⟨*proof*⟩

lemma *sets-PiF-singleton*:
  $sets\ (PiF\ \{I\}\ M) = sigma\text{-}sets\ (\Pi'\ j{\in}I.\ space\ (M\ j))$
    $\{(\Pi'\ j{\in}I.\ X\ j)\ |X.\ X \in (\Pi\ j{\in}I.\ sets\ (M\ j))\}$
  ⟨*proof*⟩

lemma *in-sets-PiFI*:
  assumes $X = (Pi'\ J\ S)\ J \in I\ \bigwedge i.\ i{\in}J \Longrightarrow S\ i \in sets\ (M\ i)$
  shows $X \in sets\ (PiF\ I\ M)$
  ⟨*proof*⟩

lemma *product-in-sets-PiFI*:
  assumes $J \in I\ \bigwedge i.\ i{\in}J \Longrightarrow S\ i \in sets\ (M\ i)$
  shows $(Pi'\ J\ S) \in sets\ (PiF\ I\ M)$
  ⟨*proof*⟩

lemma *singleton-space-subset-in-sets*:
  fixes J
  assumes $J \in I$
  assumes *finite* J
  shows $space\ (PiF\ \{J\}\ M) \in sets\ (PiF\ I\ M)$
  ⟨*proof*⟩

lemma *singleton-subspace-set-in-sets*:
  assumes A: $A \in sets\ (PiF\ \{J\}\ M)$
  assumes *finite* J
  assumes $J \in I$
  shows $A \in sets\ (PiF\ I\ M)$
  ⟨*proof*⟩

lemma *finite-measurable-singletonI*:
  assumes *finite* I
  assumes $\bigwedge J.\ J \in I \Longrightarrow finite\ J$
  assumes MN: $\bigwedge J.\ J \in I \Longrightarrow A \in measurable\ (PiF\ \{J\}\ M)\ N$
  shows $A \in measurable\ (PiF\ I\ M)\ N$
  ⟨*proof*⟩

lemma *countable-finite-comprehension*:
  fixes $f :: 'a{::}countable\ set \Rightarrow \_$
  assumes $\bigwedge s.\ P\ s \Longrightarrow finite\ s$
  assumes $\bigwedge s.\ P\ s \Longrightarrow f\ s \in sets\ M$
  shows $\bigcup\{f\ s|s.\ P\ s\} \in sets\ M$
⟨*proof*⟩

lemma *space-subset-in-sets*:
  fixes $J{::}'a{::}countable\ set\ set$
  assumes $J \subseteq I$
  assumes $\bigwedge j.\ j \in J \Longrightarrow finite\ j$
  shows $space\ (PiF\ J\ M) \in sets\ (PiF\ I\ M)$

⟨*proof*⟩

lemma *subspace-set-in-sets*:
fixes $J::'a::countable\ set\ set$
assumes A: $A \in sets\ (PiF\ J\ M)$
assumes $J \subseteq I$
assumes $\bigwedge j.\ j \in J \Longrightarrow finite\ j$
shows $A \in sets\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *countable-measurable-PiFI*:
fixes $I::'a::countable\ set\ set$
assumes MN: $\bigwedge J.\ J \in I \Longrightarrow finite\ J \Longrightarrow A \in measurable\ (PiF\ \{J\}\ M)\ N$
shows $A \in measurable\ (PiF\ I\ M)\ N$
⟨*proof*⟩

lemma *measurable-PiF*:
assumes f: $\bigwedge x.\ x \in space\ N \Longrightarrow domain\ (f\ x) \in I \land (\forall i{\in}domain\ (f\ x).\ (f\ x)\ i \in space\ (M\ i))$
assumes S: $\bigwedge J\ S.\ J \in I \Longrightarrow (\bigwedge i.\ i \in J \Longrightarrow S\ i \in sets\ (M\ i)) \Longrightarrow$
$f\ -`\ (Pi'\ J\ S) \cap space\ N \in sets\ N$
shows $f \in measurable\ N\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *restrict-sets-measurable*:
assumes A: $A \in sets\ (PiF\ I\ M)$ **and** $J \subseteq I$
shows $A \cap \{m.\ domain\ m \in J\} \in sets\ (PiF\ J\ M)$
⟨*proof*⟩

lemma *measurable-finmap-of*:
assumes f: $\bigwedge i.\ (\exists x \in space\ N.\ i \in J\ x) \Longrightarrow (\lambda x.\ f\ x\ i) \in measurable\ N\ (M\ i)$
assumes J: $\bigwedge x.\ x \in space\ N \Longrightarrow J\ x \in I\ \bigwedge x.\ x \in space\ N \Longrightarrow finite\ (J\ x)$
assumes JN: $\bigwedge S.\ \{x.\ J\ x = S\} \cap space\ N \in sets\ N$
shows $(\lambda x.\ finmap\text{-}of\ (J\ x)\ (f\ x)) \in measurable\ N\ (PiF\ I\ M)$
⟨*proof*⟩

lemma *measurable-PiM-finmap-of*:
assumes *finite J*
shows $finmap\text{-}of\ J \in measurable\ (Pi_M\ J\ M)\ (PiF\ \{J\}\ M)$
⟨*proof*⟩

lemma *proj-measurable-singleton*:
assumes $A \in sets\ (M\ i)$
shows $(\lambda x.\ (x)_F\ i)\ -`\ A \cap space\ (PiF\ \{I\}\ M) \in sets\ (PiF\ \{I\}\ M)$
⟨*proof*⟩

lemma *measurable-proj-singleton*:
assumes $i \in I$
shows $(\lambda x.\ (x)_F\ i) \in measurable\ (PiF\ \{I\}\ M)\ (M\ i)$

⟨*proof*⟩

lemma *measurable-proj-countable*:
fixes $I::'a::countable\ set\ set$
assumes $y \in space\ (M\ i)$
shows $(\lambda x.\ if\ i \in domain\ x\ then\ (x)_F\ i\ else\ y) \in measurable\ (PiF\ I\ M)\ (M\ i)$
⟨*proof*⟩

lemma *measurable-restrict-proj*:
assumes $J \in II$ *finite* J
shows *finmap-of* $J \in measurable\ (PiM\ J\ M)\ (PiF\ II\ M)$
⟨*proof*⟩

lemma *measurable-proj-PiM*:
fixes $J\ K\ ::'a::countable\ set$ **and** $I::'a\ set\ set$
assumes *finite* J $J \in I$
assumes $x \in space\ (PiM\ J\ M)$
shows $proj \in measurable\ (PiF\ \{J\}\ M)\ (PiM\ J\ M)$
⟨*proof*⟩

lemma *space-PiF-singleton-eq-product*:
assumes *finite* I
shows $space\ (PiF\ \{I\}\ M) = (\Pi'\ i{\in}I.\ space\ (M\ i))$
⟨*proof*⟩

adapted from $sets\ (Pi_M\ ?I\ ?M) = sigma\text{-}sets\ (\Pi_E\ i{\in}?I.\ space\ (?M\ i))\ \{\{f \in \Pi_E\ i{\in}?I.\ space\ (?M\ i).\ f\ i \in A\}\ |i\ A.\ i \in ?I \wedge A \in sets\ (?M\ i)\}$

lemma *sets-PiF-single*:
assumes *finite* I $I \neq \{\}$
shows $sets\ (PiF\ \{I\}\ M) =$
$sigma\text{-}sets\ (\Pi'\ i{\in}I.\ space\ (M\ i))$
$\{\{f{\in}\Pi'\ i{\in}I.\ space\ (M\ i).\ f\ i \in A\}\ |\ i\ A.\ i \in I \wedge A \in sets\ (M\ i)\}$
(**is** - $= sigma\text{-}sets\ ?\Omega\ ?R$)
⟨*proof*⟩

adapted from $(\bigwedge i.\ i \in ?I \Longrightarrow ?A\ i = ?B\ i) \Longrightarrow Pi_E\ ?I\ ?A = Pi_E\ ?I\ ?B$

lemma *Pi'-cong*:
assumes *finite* I
assumes $\bigwedge i.\ i \in I \Longrightarrow f\ i = g\ i$
shows $Pi'\ I\ f = Pi'\ I\ g$
⟨*proof*⟩

adapted from $[\![finite\ ?I;\ \bigwedge i\ n\ m.\ [\![i \in ?I;\ n \le m]\!] \Longrightarrow ?A\ n\ i \subseteq ?A\ m\ i]\!] \Longrightarrow (\bigcup_n Pi\ ?I\ (?A\ n)) = (\Pi\ i{\in}?I.\ \bigcup_n ?A\ n\ i)$

lemma *Pi'-UN*:
fixes $A :: nat \Rightarrow 'i \Rightarrow 'a\ set$
assumes *finite* I
assumes *mono*: $\bigwedge i\ n\ m.\ i \in I \Longrightarrow n \le m \Longrightarrow A\ n\ i \subseteq A\ m\ i$

shows $(\bigcup n.\ Pi'\ I\ (A\ n)) = Pi'\ I\ (\lambda i.\ \bigcup n.\ A\ n\ i)$
⟨*proof*⟩

adapted from $[\![\bigwedge i.\ i \in\ ?I \Longrightarrow \exists S \subseteq ?E\ i.\ countable\ S \wedge\ ?\Omega\ i = \bigcup\ S; \bigwedge i.\ i \in\ ?I \Longrightarrow\ ?E\ i \subseteq Pow\ (?\Omega\ i); \bigwedge j.\ j \in\ ?J \Longrightarrow finite\ j; \bigcup\ ?J =\ ?I]\!] \Longrightarrow sets\ (Pi_M\ ?I\ (\lambda i.\ sigma\ (?\Omega\ i)\ (?E\ i))) = sets\ (sigma\ (Pi_E\ ?I\ ?\Omega)\ \{\{f \in Pi_E\ ?I\ ?\Omega.\ \forall i{\in}j.\ f\ i \in A\ i\}\ |A\ j.\ j \in\ ?J \wedge A \in Pi\ j\ ?E\})$

lemma *sigma-fprod-algebra-sigma-eq*:
fixes $E :: 'i \Rightarrow 'a\ set\ set$ **and** $S :: 'i \Rightarrow nat \Rightarrow 'a\ set$
assumes [*simp*]: $finite\ I\ I \neq \{\}$
and *S-union*: $\bigwedge i.\ i \in I \Longrightarrow (\bigcup j.\ S\ i\ j) = space\ (M\ i)$
and *S-in-E*: $\bigwedge i.\ i \in I \Longrightarrow range\ (S\ i) \subseteq E\ i$
assumes *E-closed*: $\bigwedge i.\ i \in I \Longrightarrow E\ i \subseteq Pow\ (space\ (M\ i))$
and *E-generates*: $\bigwedge i.\ i \in I \Longrightarrow sets\ (M\ i) = sigma\text{-}sets\ (space\ (M\ i))\ (E\ i)$
defines $P == \{\ Pi'\ I\ F\ |\ F.\ \forall i{\in}I.\ F\ i \in E\ i\ \}$
shows $sets\ (PiF\ \{I\}\ M) = sigma\text{-}sets\ (space\ (PiF\ \{I\}\ M))\ P$
⟨*proof*⟩

lemma *product-open-generates-sets-PiF-single*:
assumes $I \neq \{\}$
assumes [*simp*]: *finite I*
shows $sets\ (PiF\ \{I\}\ (\lambda\text{-}.\ borel::'b::second\text{-}countable\text{-}topology\ measure)) = sigma\text{-}sets\ (space\ (PiF\ \{I\}\ (\lambda\text{-}.\ borel)))\ \{Pi'\ I\ F\ |F.\ (\forall i{\in}I.\ F\ i \in Collect\ open)\}$
⟨*proof*⟩

lemma *finmap-UNIV*[*simp*]: $(\bigcup J{\in}Collect\ finite.\ \Pi'\ j{\in}J.\ UNIV) = UNIV$ ⟨*proof*⟩

lemma *borel-eq-PiF-borel*:
shows $(borel :: ('i::countable \Rightarrow_F 'a::polish\text{-}space)\ measure) = PiF\ (Collect\ finite)\ (\lambda\text{-}.\ borel :: 'a\ measure)$
⟨*proof*⟩

22.10 Isomorphism between Functions and Finite Maps

lemma *measurable-finmap-compose*:
shows $(\lambda m.\ compose\ J\ m\ f) \in measurable\ (PiM\ (f\ `\ J)\ (\lambda\text{-}.\ M))\ (PiM\ J\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *measurable-compose-inv*:
assumes *inj*: $\bigwedge j.\ j \in J \Longrightarrow f'\ (f\ j) = j$
shows $(\lambda m.\ compose\ (f\ `\ J)\ m\ f') \in measurable\ (PiM\ J\ (\lambda\text{-}.\ M))\ (PiM\ (f\ `\ J)\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

locale *function-to-finmap* =
fixes $J::'a\ set$ **and** $f :: 'a \Rightarrow 'b::countable$ **and** f'
assumes [*simp*]: *finite J*

assumes *inv*: $i \in J \Longrightarrow f'\ (f\ i) = i$
begin

to measure finmaps

definition $fm = (finmap\text{-}of\ (f\ `\ J))\ o\ (\lambda g.\ compose\ (f\ `\ J)\ g\ f')$

lemma *domain-fm*[*simp*]: $domain\ (fm\ x) = f\ `\ J$
⟨*proof*⟩

lemma *fm-restrict*[*simp*]: $fm\ (restrict\ y\ J) = fm\ y$
⟨*proof*⟩

lemma *fm-product*:
assumes $\bigwedge i.\ space\ (M\ i) = UNIV$
shows $fm\ -`\ Pi'\ (f\ `\ J)\ S \cap space\ (Pi_M\ J\ M) = (\Pi_E\ j \in J.\ S\ (f\ j))$
⟨*proof*⟩

lemma *fm-measurable*:
assumes $f\ `\ J \in N$
shows $fm \in measurable\ (Pi_M\ J\ (\lambda\text{-}.\ M))\ (Pi_F\ N\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *proj-fm*:
assumes $x \in J$
shows $fm\ m\ (f\ x) = m\ x$
⟨*proof*⟩

lemma *inj-on-compose-f'*: $inj\text{-}on\ (\lambda g.\ compose\ (f\ `\ J)\ g\ f')\ (extensional\ J)$
⟨*proof*⟩

lemma *inj-on-fm*:
assumes $\bigwedge i.\ space\ (M\ i) = UNIV$
shows $inj\text{-}on\ fm\ (space\ (Pi_M\ J\ M))$
⟨*proof*⟩

to measure functions

definition $mf = (\lambda g.\ compose\ J\ g\ f)\ o\ proj$

lemma *mf-fm*:
assumes $x \in space\ (Pi_M\ J\ (\lambda\text{-}.\ M))$
shows $mf\ (fm\ x) = x$
⟨*proof*⟩

lemma *mf-measurable*:
assumes $space\ M = UNIV$
shows $mf \in measurable\ (PiF\ \{f\ `\ J\}\ (\lambda\text{-}.\ M))\ (PiM\ J\ (\lambda\text{-}.\ M))$
⟨*proof*⟩

lemma *fm-image-measurable*:

assumes *space M* = *UNIV*
assumes $X \in$ *sets* (*Pi*$_M$ *J* (λ-. *M*))
shows *fm* ' $X \in$ *sets* (*PiF* {*f* ' *J*} (λ-. *M*))
⟨*proof*⟩

lemma *fm-image-measurable-finite*:
assumes *space M* = *UNIV*
assumes $X \in$ *sets* (*Pi*$_M$ *J* (λ-. *M*::'*c measure*))
shows *fm* ' $X \in$ *sets* (*PiF* (*Collect finite*) (λ-. *M*::'*c measure*))
⟨*proof*⟩

measure on finmaps

definition *mapmeasure M N* = *distr M* (*PiF* (*Collect finite*) *N*) (*fm*)

lemma *sets-mapmeasure*[*simp*]: *sets* (*mapmeasure M N*) = *sets* (*PiF* (*Collect finite*) *N*)
⟨*proof*⟩

lemma *space-mapmeasure*[*simp*]: *space* (*mapmeasure M N*) = *space* (*PiF* (*Collect finite*) *N*)
⟨*proof*⟩

lemma *mapmeasure-PiF*:
assumes *s1*: *space M* = *space* (*Pi*$_M$ *J* (λ-. *N*))
assumes *s2*: *sets M* = *sets* (*Pi*$_M$ *J* (λ-. *N*))
assumes *space N* = *UNIV*
assumes $X \in$ *sets* (*PiF* (*Collect finite*) (λ-. *N*))
shows *emeasure* (*mapmeasure M* (λ-. *N*)) *X* = *emeasure M* ((*fm* −' $X \cap$ *extensional J*))
⟨*proof*⟩

lemma *mapmeasure-PiM*:
fixes *N*::'*c measure*
assumes *s1*: *space M* = *space* (*Pi*$_M$ *J* (λ-. *N*))
assumes *s2*: *sets M* = (*Pi*$_M$ *J* (λ-. *N*))
assumes *N*: *space N* = *UNIV*
assumes *X*: $X \in$ *sets M*
shows *emeasure M X* = *emeasure* (*mapmeasure M* (λ-. *N*)) (*fm* ' *X*)
⟨*proof*⟩

end

end

23 Projective Limit

theory *Projective-Limit*
imports
Fin-Map

Infinite-Product-Measure
HOL–Library.Diagonal-Subsequence
begin

23.1 Sequences of Finite Maps in Compact Sets

locale *finmap-seqs-into-compact* =
fixes K::*nat* ⇒ (*nat* $\Rightarrow_F$ *'a*::*metric-space*) *set* **and** f::*nat* ⇒ (*nat* $\Rightarrow_F$ *'a*) **and** M
assumes *compact*: $\bigwedge n.$ *compact* (K n)
assumes *f-in-K*: $\bigwedge n.$ $K\ n \neq \{\}$
assumes *domain-K*: $\bigwedge n.\ k \in K\ n \Longrightarrow$ *domain* k = *domain* (f n)
assumes *proj-in-K*:
$\bigwedge t\ n\ m.\ m \geq n \Longrightarrow t \in$ *domain* (f n) $\Longrightarrow (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)$ ' K n
begin

lemma *proj-in-K'*: $(\exists n.\ \forall m \geq n.\ (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)$ ' $K\ n)$
⟨*proof*⟩

lemma *proj-in-KE*:
obtains n **where** $\bigwedge m.\ m \geq n \Longrightarrow (f\ m)_F\ t \in (\lambda k.\ (k)_F\ t)$ ' $K\ n$
⟨*proof*⟩

lemma *compact-projset*:
shows *compact* $((\lambda k.\ (k)_F\ i)$ ' $K\ n)$
⟨*proof*⟩

end

lemma *compactE'*:
fixes S :: *'a* :: *metric-space set*
assumes *compact* S $\forall n \geq m.\ f\ n \in S$
obtains l r **where** $l \in S$ *strict-mono* (r::*nat*⇒*nat*) $((f \circ r) \longrightarrow l)$ *sequentially*
⟨*proof*⟩

sublocale *finmap-seqs-into-compact* ⊆ *subseqs* $\lambda n\ s.\ (\exists l.\ (\lambda i.\ ((f\ o\ s)\ i)_F\ n) \longrightarrow l)$
⟨*proof*⟩

lemma (**in** *finmap-seqs-into-compact*) *diagonal-tendsto*: $\exists l.\ (\lambda i.\ (f\ (diagseq\ i))_F\ n) \longrightarrow l$
⟨*proof*⟩

23.2 Daniell-Kolmogorov Theorem

Existence of Projective Limit

locale *polish-projective* = *projective-family* I P λ-. *borel*::*'a*::*polish-space measure*
for I::*'i set* **and** P
begin

lemma *emeasure-lim-emb*:
assumes X: $J \subseteq I$ *finite* J $X \in$ *sets* (Π_M $i{\in}J$. *borel*)
shows *lim* (*emb* I J X) = P J X
⟨*proof*⟩

lemma *measure-lim-emb*:
$J \subseteq I \Longrightarrow$ *finite* $J \Longrightarrow X \in$ *sets* (Π_M $i{\in}J$. *borel*) $\Longrightarrow$ *measure lim* (*emb* I J X) = *measure* (P J) X
⟨*proof*⟩

end

hide-const (**open**) *PiF*
hide-const (**open**) Pi_F
hide-const (**open**) Pi'
hide-const (**open**) *finmap-of*
hide-const (**open**) *proj*
hide-const (**open**) *domain*
hide-const (**open**) *basis-finmap*

sublocale *polish-projective* $\subseteq$ P: *prob-space lim*
⟨*proof*⟩

locale *polish-product-prob-space* =
product-prob-space λ-. *borel*::('*a*::*polish-space*) *measure* I **for** I::'*i set*

sublocale *polish-product-prob-space* $\subseteq$ P: *polish-projective* I λJ. *PiM* J (λ-. *borel*::('*a*) *measure*)
⟨*proof*⟩

lemma (**in** *polish-product-prob-space*) *limP-eq-PiM*: *lim* = *PiM* I (λ-. *borel*)
⟨*proof*⟩

end

24 Random Permutations

theory *Random-Permutations*
imports
HOL−Combinatorics.Multiset-Permutations
Probability-Mass-Function
begin

Choosing a set permutation (i.e. a distinct list with the same elements as the set) uniformly at random is the same as first choosing the first element of the list and then choosing the rest of the list as a permutation of the remaining set.

lemma *random-permutation-of-set*:
assumes *finite A A* $\neq$ {}
shows *pmf-of-set* (*permutations-of-set A*) =
do {
$x \leftarrow$ *pmf-of-set A*;
$xs \leftarrow$ *pmf-of-set* (*permutations-of-set* ($A - \{x\}$));
return-pmf ($x\#xs$)
} (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

A generic fold function that takes a function, an initial state, and a set and chooses a random order in which it then traverses the set in the same fashion as a left fold over a list. We first give a recursive definition.

function *fold-random-permutation* :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'b\ pmf$
where
fold-random-permutation f x {} = *return-pmf x*
| $\neg$*finite A* $\Longrightarrow$ *fold-random-permutation f x A* = *return-pmf x*
| *finite A* $\Longrightarrow$ $A \neq \{\}$ $\Longrightarrow$
fold-random-permutation f x A =
pmf-of-set A $\gg=$ (λa. *fold-random-permutation f* ($f\ a\ x$) ($A - \{a\}$))
⟨*proof*⟩
termination ⟨*proof*⟩

We can now show that the above recursive definition is equivalent to choosing a random set permutation and folding over it (in any direction).

lemma *fold-random-permutation-foldl*:
assumes *finite A*
shows *fold-random-permutation f x A* =
map-pmf (*foldl* ($\lambda x\ y.\ f\ y\ x$) x) (*pmf-of-set* (*permutations-of-set A*))
⟨*proof*⟩

lemma *fold-random-permutation-foldr*:
assumes *finite A*
shows *fold-random-permutation f x A* =
map-pmf (λxs. *foldr f xs x*) (*pmf-of-set* (*permutations-of-set A*))
⟨*proof*⟩

lemma *fold-random-permutation-fold*:
assumes *finite A*
shows *fold-random-permutation f x A* =
map-pmf (λxs. *fold f xs x*) (*pmf-of-set* (*permutations-of-set A*))
⟨*proof*⟩

lemma *fold-random-permutation-code* [*code*]:
fold-random-permutation f x (*set xs*) =
map-pmf (*foldl* ($\lambda x\ y.\ f\ y\ x$) x) (*pmf-of-set* (*permutations-of-set* (*set xs*)))
⟨*proof*⟩

We now introduce a slightly generalised version of the above fold operation that does not simply return the result in the end, but applies a monadic bind to it. This may seem somewhat arbitrary, but it is a common use case, e.g. in the Social Decision Scheme of Random Serial Dictatorship, where voters narrow down a set of possible winners in a random order and the winner is chosen from the remaining set uniformly at random.

function *fold-bind-random-permutation*
  :: $('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow ('b \Rightarrow 'c\ pmf) \Rightarrow 'b \Rightarrow 'a\ set \Rightarrow 'c\ pmf$ **where**
 fold-bind-random-permutation f g x $\{\} = g\ x$
| $\neg$*finite A* $\Longrightarrow$ *fold-bind-random-permutation f g x A* $= g\ x$
| *finite A* $\Longrightarrow A \neq \{\} \Longrightarrow$
  fold-bind-random-permutation f g x A =
   pmf-of-set A $\gg=$ (λa. *fold-bind-random-permutation f g* $(f\ a\ x)$ $(A - \{a\})$)
 ⟨*proof*⟩
termination ⟨*proof*⟩

We now show that the recursive definition is equivalent to a random fold followed by a monadic bind.

lemma *fold-bind-random-permutation-altdef* [*code*]:
 fold-bind-random-permutation f g x A = *fold-random-permutation f x A* $\gg=$ *g*
⟨*proof*⟩

We can now derive the following nice monadic representations of the combined fold-and-bind:

lemma *fold-bind-random-permutation-foldl*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
  do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldl* ($\lambda x\ y.\ f\ y\ x$) *x xs*)}
 ⟨*proof*⟩

lemma *fold-bind-random-permutation-foldr*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
  do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*foldr f xs x*)}
 ⟨*proof*⟩

lemma *fold-bind-random-permutation-fold*:
 assumes *finite A*
 shows *fold-bind-random-permutation f g x A* =
  do {*xs* ← *pmf-of-set* (*permutations-of-set A*); *g* (*fold f xs x*)}
 ⟨*proof*⟩

The following useful lemma allows us to swap partitioning a set w. r. t. a predicate and drawing a random permutation of that set.

lemma *partition-random-permutations*:
 assumes *finite A*
 shows *map-pmf* (*partition P*) (*pmf-of-set* (*permutations-of-set A*)) =
  pair-pmf (*pmf-of-set* (*permutations-of-set* $\{x \in A.\ P\ x\}$))

$(pmf\text{-}of\text{-}set\ (permutations\text{-}of\text{-}set\ \{x{\in}A.\ \neg P\ x\}))$ (**is** $?lhs = ?rhs$)
⟨*proof*⟩

end

25 Discrete subprobability distribution

theory *SPMF* **imports**
Probability-Mass-Function
HOL−Library.Complete-Partial-Order2
HOL−Library.Rewrite
begin

25.1 Auxiliary material

lemma *cSUP-singleton* [*simp*]: $(SUP\ x{\in}\{x\}.\ f\ x :: \text{-} :: conditionally\text{-}complete\text{-}lattice) = f\ x$
⟨*proof*⟩

25.1.1 More about extended reals

lemma [*simp*]:
shows *ennreal-max-0*: $ennreal\ (max\ 0\ x) = ennreal\ x$
and *ennreal-max-0'*: $ennreal\ (max\ x\ 0) = ennreal\ x$
⟨*proof*⟩

lemma *e2ennreal-0* [*simp*]: $e2ennreal\ 0 = 0$
⟨*proof*⟩

lemma *enn2real-bot* [*simp*]: $enn2real\ \bot = 0$
⟨*proof*⟩

lemma *continuous-at-ennreal*[*continuous-intros*]: $continuous\ F\ f \Longrightarrow continuous\ F\ (\lambda x.\ ennreal\ (f\ x))$
⟨*proof*⟩

lemma *ennreal-Sup*:
assumes $*$: $(SUP\ a{\in}A.\ ennreal\ a) \neq \top$
and $A \neq \{\}$
shows $ennreal\ (Sup\ A) = (SUP\ a{\in}A.\ ennreal\ a)$
⟨*proof*⟩

lemma *ennreal-SUP*:
$[\![\ (SUP\ a{\in}A.\ ennreal\ (f\ a)) \neq \top;\ A \neq \{\}\ ]\!] \Longrightarrow ennreal\ (SUP\ a{\in}A.\ f\ a) = (SUP\ a{\in}A.\ ennreal\ (f\ a))$
⟨*proof*⟩

lemma *ennreal-lt-0*: $x < 0 \Longrightarrow ennreal\ x = 0$
⟨*proof*⟩

25.1.2 More about $'a$ *option*

lemma *None-in-map-option-image* [*simp*]: $None \in map\text{-}option\ f\ `\ A \longleftrightarrow None \in A$
⟨*proof*⟩

lemma *Some-in-map-option-image* [*simp*]: $Some\ x \in map\text{-}option\ f\ `\ A \longleftrightarrow (\exists\, y.\ x = f\ y \wedge Some\ y \in A)$
⟨*proof*⟩

lemma *case-option-collapse*: $case\text{-}option\ x\ (\lambda\text{-}.\ x) = (\lambda\text{-}.\ x)$
⟨*proof*⟩

lemma *case-option-id*: $case\text{-}option\ None\ Some = id$
⟨*proof*⟩

inductive *ord-option* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ option \Rightarrow 'b\ option \Rightarrow bool$
 for *ord* :: $'a \Rightarrow 'b \Rightarrow bool$
where
 None: *ord-option ord None x*
| *Some*: $ord\ x\ y \Longrightarrow ord\text{-}option\ ord\ (Some\ x)\ (Some\ y)$

inductive-simps *ord-option-simps* [*simp*]:
 ord-option ord None x
 ord-option ord x None
 ord-option ord (*Some x*) (*Some y*)
 ord-option ord (*Some x*) *None*

inductive-simps *ord-option-eq-simps* [*simp*]:
 ord-option (=) *None y*
 ord-option (=) (*Some x*) *y*

lemma *ord-option-reflI*: $(\bigwedge y.\ y \in set\text{-}option\ x \Longrightarrow ord\ y\ y) \Longrightarrow ord\text{-}option\ ord\ x\ x$
⟨*proof*⟩

lemma *reflp-ord-option*: $reflp\ ord \Longrightarrow reflp\ (ord\text{-}option\ ord)$
⟨*proof*⟩

lemma *ord-option-trans*:
 $[\![\ ord\text{-}option\ ord\ x\ y;\ ord\text{-}option\ ord\ y\ z;$
 $\bigwedge a\ b\ c.\ [\![\ a \in set\text{-}option\ x;\ b \in set\text{-}option\ y;\ c \in set\text{-}option\ z;\ ord\ a\ b;\ ord\ b\ c\ ]\!] \Longrightarrow ord\ a\ c\ ]\!]$
 $\Longrightarrow ord\text{-}option\ ord\ x\ z$
⟨*proof*⟩

lemma *transp-ord-option*: $transp\ ord \Longrightarrow transp\ (ord\text{-}option\ ord)$
⟨*proof*⟩

lemma *antisymp-ord-option*: $antisymp\ ord \Longrightarrow antisymp\ (ord\text{-}option\ ord)$

⟨*proof*⟩

lemma *ord-option-chainD*:
Complete-Partial-Order.chain (*ord-option ord*) Y
$\Longrightarrow$ *Complete-Partial-Order.chain ord* $\{x.$ *Some* $x \in Y\}$
⟨*proof*⟩

definition *lub-option* :: $('a\ set \Rightarrow 'b) \Rightarrow 'a\ option\ set \Rightarrow 'b\ option$
where *lub-option lub* Y = (*if* $Y \subseteq \{None\}$ *then None else Some* (*lub* $\{x.$ *Some* $x \in Y\}$))

lemma *map-lub-option*: *map-option f* (*lub-option lub* Y) = *lub-option* ($f \circ lub$) Y
⟨*proof*⟩

lemma *lub-option-upper*:
assumes *Complete-Partial-Order.chain* (*ord-option ord*) Y $x \in Y$
and *lub-upper*: $\bigwedge Y\ x.$ ⟦ *Complete-Partial-Order.chain ord* Y; $x \in Y$ ⟧ $\Longrightarrow$ *ord* x (*lub* Y)
shows *ord-option ord* x (*lub-option lub* Y)
⟨*proof*⟩

lemma *lub-option-least*:
assumes Y: *Complete-Partial-Order.chain* (*ord-option ord*) Y
and *upper*: $\bigwedge x.\ x \in Y \Longrightarrow$ *ord-option ord* $x\ y$
assumes *lub-least*: $\bigwedge Y\ y.$ ⟦ *Complete-Partial-Order.chain ord* Y; $\bigwedge x.\ x \in Y \Longrightarrow$ *ord* $x\ y$ ⟧ $\Longrightarrow$ *ord* (*lub* Y) y
shows *ord-option ord* (*lub-option lub* Y) y
⟨*proof*⟩

lemma *lub-map-option*: *lub-option lub* (*map-option f* ‘ Y) = *lub-option* (*lub* ∘ (‘) *f*) Y
⟨*proof*⟩

lemma *ord-option-mono*: ⟦ *ord-option* $A\ x\ y$; $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y$ ⟧ $\Longrightarrow$ *ord-option* $B\ x\ y$
⟨*proof*⟩

lemma *ord-option-mono'* [*mono*]:
($\bigwedge x\ y.\ A\ x\ y \longrightarrow B\ x\ y$) $\Longrightarrow$ *ord-option* $A\ x\ y \longrightarrow$ *ord-option* $B\ x\ y$
⟨*proof*⟩

lemma *ord-option-compp*: *ord-option* (A *OO* B) = *ord-option* A *OO ord-option* B
⟨*proof*⟩

lemma *ord-option-inf*: *inf* (*ord-option* A) (*ord-option* B) = *ord-option* (*inf* A B) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *ord-option-map2*: *ord-option ord* x (*map-option f* y) = *ord-option* ($\lambda x\ y.$

ord x $(f\ y))$ x y
⟨*proof*⟩

lemma *ord-option-map1*: *ord-option ord* (*map-option* f x) y = *ord-option* ($\lambda x\ y.$ *ord* (f x) y) x y
⟨*proof*⟩

lemma *option-ord-Some1-iff*: *option-ord* (*Some* x) $y \longleftrightarrow y$ = *Some* x
⟨*proof*⟩

25.1.3 A relator for sets that treats sets like predicates

context includes *lifting-syntax*
begin

definition *rel-pred* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ set \Rightarrow 'b\ set \Rightarrow bool$
where *rel-pred* R A B = (R ===> (=)) ($\lambda x.\ x \in A$) ($\lambda y.\ y \in B$)

lemma *rel-predI*: (R ===> (=)) ($\lambda x.\ x \in A$) ($\lambda y.\ y \in B$) $\Longrightarrow$ *rel-pred* R A B
⟨*proof*⟩

lemma *rel-predD*: ⟦ *rel-pred* R A B; R x y ⟧ $\Longrightarrow x \in A \longleftrightarrow y \in B$
⟨*proof*⟩

lemma *Collect-parametric*: ((A ===> (=)) ===> *rel-pred* A) *Collect Collect*
— Declare this rule as *transfer-rule* only locally because it blows up the search space for *transfer* (in combination with *Collect-transfer*)
⟨*proof*⟩

end

25.1.4 Monotonicity rules

lemma *monotone-gfp-eadd1*: *monotone* ($\geq$) ($\geq$) ($\lambda x.\ x + y$ :: *enat*)
⟨*proof*⟩

lemma *monotone-gfp-eadd2*: *monotone* ($\geq$) ($\geq$) ($\lambda y.\ x + y$ :: *enat*)
⟨*proof*⟩

lemma *mono2mono-gfp-eadd*[*THEN gfp.mono2mono2*, *cont-intro*, *simp*]:
shows *monotone-eadd*: *monotone* (*rel-prod* ($\geq$) ($\geq$)) ($\geq$) ($\lambda(x, y).\ x + y$ :: *enat*)
⟨*proof*⟩

lemma *eadd-gfp-partial-function-mono* [*partial-function-mono*]:
⟦ *monotone* (*fun-ord* ($\geq$)) ($\geq$) f; *monotone* (*fun-ord* ($\geq$)) ($\geq$) g ⟧
$\Longrightarrow$ *monotone* (*fun-ord* ($\geq$)) ($\geq$) ($\lambda x.\ f\ x + g\ x$ :: *enat*)
⟨*proof*⟩

lemma *mono2mono-ereal*[*THEN lfp.mono2mono*]:
shows *monotone-ereal*: *monotone* ($\leq$) ($\leq$) *ereal*

⟨*proof*⟩

lemma *mono2mono-ennreal*[*THEN lfp.mono2mono*]:
shows *monotone-ennreal*: *monotone* $(\leq)$ $(\leq)$ *ennreal*
⟨*proof*⟩

25.1.5 Bijections

lemma *bi-unique-rel-set-bij-betw*:
assumes *unique*: *bi-unique R*
and *rel*: *rel-set R A B*
shows $\exists f.$ *bij-betw f A B* $\wedge$ $(\forall x \in A.\ R\ x\ (f\ x))$
⟨*proof*⟩

lemma *bij-betw-rel-setD*: *bij-betw f A B* $\Longrightarrow$ *rel-set* $(\lambda x\ y.\ y = f\ x)$ *A B*
⟨*proof*⟩

25.2 Subprobability mass function

type-synonym *'a spmf* = *'a option pmf*
translations (*type*) *'a spmf* ↽ (*type*) *'a option pmf*

definition *measure-spmf* :: *'a spmf* ⇒ *'a measure*
where *measure-spmf p* = *distr* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*) *the*

abbreviation *spmf* :: *'a spmf* ⇒ *'a* ⇒ *real*
where *spmf p x* ≡ *pmf p* (*Some x*)

lemma *space-measure-spmf*: *space* (*measure-spmf p*) = *UNIV*
⟨*proof*⟩

lemma *sets-measure-spmf* [*simp*, *measurable-cong*]: *sets* (*measure-spmf p*) = *sets* (*count-space UNIV*)
⟨*proof*⟩

lemma *measure-spmf-not-bot* [*simp*]: *measure-spmf p* $\neq \bot$
⟨*proof*⟩

lemma *measurable-the-measure-pmf-Some* [*measurable*, *simp*]:
the $\in$ *measurable* (*restrict-space* (*measure-pmf p*) (*range Some*)) (*count-space UNIV*)
⟨*proof*⟩

lemma *measurable-spmf-measure1*[*simp*]: *measurable* (*measure-spmf M*) *N* = *UNIV* → *space N*
⟨*proof*⟩

lemma *measurable-spmf-measure2*[*simp*]: *measurable N* (*measure-spmf M*) = *measurable N* (*count-space UNIV*)

⟨*proof*⟩

lemma *subprob-space-measure-spmf* [*simp*, *intro*!]: *subprob-space* (*measure-spmf p*)
⟨*proof*⟩

interpretation *measure-spmf*: *subprob-space measure-spmf p* **for** *p*
⟨*proof*⟩

lemma *finite-measure-spmf* [*simp*]: *finite-measure* (*measure-spmf p*)
⟨*proof*⟩

lemma *spmf-conv-measure-spmf*: *spmf p x* = *measure* (*measure-spmf p*) {*x*}
⟨*proof*⟩

lemma *emeasure-measure-spmf-conv-measure-pmf*:
  emeasure (*measure-spmf p*) *A* = *emeasure* (*measure-pmf p*) (*Some* ‘ *A*)
⟨*proof*⟩

lemma *measure-measure-spmf-conv-measure-pmf*:
  measure (*measure-spmf p*) *A* = *measure* (*measure-pmf p*) (*Some* ‘ *A*)
⟨*proof*⟩

lemma *emeasure-spmf-map-pmf-Some* [*simp*]:
  emeasure (*measure-spmf* (*map-pmf Some p*)) *A* = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *measure-spmf-map-pmf-Some* [*simp*]:
  measure (*measure-spmf* (*map-pmf Some p*)) *A* = *measure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *nn-integral-measure-spmf*: ($\int^+$ *x*. *f x* ∂*measure-spmf p*) = $\int^+$ *x*. *ennreal* (*spmf p x*) ∗ *f x* ∂*count-space UNIV*
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *integral-measure-spmf*:
  assumes *integrable* (*measure-spmf p*) *f*
  shows ($\int$ *x*. *f x* ∂*measure-spmf p*) = $\int$ *x*. *spmf p x* ∗ *f x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *emeasure-spmf-single*: *emeasure* (*measure-spmf p*) {*x*} = *spmf p x*
⟨*proof*⟩

lemma *measurable-measure-spmf*[*measurable*]:
  (λ*x*. *measure-spmf* (*M x*)) ∈ *measurable* (*count-space UNIV*) (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-measure-spmf-conv-measure-pmf*:

assumes [*measurable*]: $f \in$ *borel-measurable* (*count-space UNIV*)
shows *nn-integral* (*measure-spmf p*) $f =$ *nn-integral* (*restrict-space* (*measure-pmf p*) (*range Some*)) ($f \circ$ *the*)
⟨*proof*⟩

lemma *measure-spmf-in-space-subprob-algebra* [*simp*]:
measure-spmf p $\in$ *space* (*subprob-algebra* (*count-space UNIV*))
⟨*proof*⟩

lemma *nn-integral-spmf-neq-top*: ($\int^+$ *x. spmf p x* ∂*count-space UNIV*) $\neq \top$
⟨*proof*⟩

lemma *SUP-spmf-neq-top'*: (*SUP p*$\in$*Y. ennreal* (*spmf p x*)) $\neq \top$
⟨*proof*⟩

lemma *SUP-spmf-neq-top*: (*SUP i. ennreal* (*spmf* (*Y i*) *x*)) $\neq \top$
⟨*proof*⟩

lemma *SUP-emeasure-spmf-neq-top*: (*SUP p*$\in$*Y. emeasure* (*measure-spmf p*) *A*) $\neq \top$
⟨*proof*⟩

25.3 Support

definition *set-spmf* :: *'a spmf* $\Rightarrow$ *'a set*
where *set-spmf p* = *set-pmf p* $\gg=$ *set-option*

lemma *set-spmf-rep-eq*: *set-spmf p* = {*x. measure* (*measure-spmf p*) {*x*} $\neq$ *0*}
⟨*proof*⟩

lemma *in-set-spmf*: $x \in$ *set-spmf p* $\longleftrightarrow$ *Some x* $\in$ *set-pmf p*
⟨*proof*⟩

lemma *AE-measure-spmf-iff* [*simp*]: (*AE x in measure-spmf p. P x*) $\longleftrightarrow$ ($\forall$ *x*$\in$*set-spmf p. P x*)
⟨*proof*⟩

lemma *spmf-eq-0-set-spmf*: *spmf p x* = *0* $\longleftrightarrow$ $x \notin$ *set-spmf p*
⟨*proof*⟩

lemma *in-set-spmf-iff-spmf*: $x \in$ *set-spmf p* $\longleftrightarrow$ *spmf p x* $\neq$ *0*
⟨*proof*⟩

lemma *set-spmf-return-pmf-None* [*simp*]: *set-spmf* (*return-pmf None*) = {}
⟨*proof*⟩

lemma *countable-set-spmf* [*simp*]: *countable* (*set-spmf p*)
⟨*proof*⟩

lemma *spmf-eqI*:
  assumes $\bigwedge i.\ spmf\ p\ i = spmf\ q\ i$
  shows $p = q$
⟨*proof*⟩

lemma *integral-measure-spmf-restrict*:
  fixes $f :: \ 'a \Rightarrow 'b :: \{banach,\ second\text{-}countable\text{-}topology\}$ **shows**
  $(\int\ x.\ f\ x\ \partial measure\text{-}spmf\ M) = (\int\ x.\ f\ x\ \partial restrict\text{-}space\ (measure\text{-}spmf\ M)\ (set\text{-}spmf\ M))$
⟨*proof*⟩

lemma *nn-integral-measure-spmf'*:
  $(\int^+\ x.\ f\ x\ \partial measure\text{-}spmf\ p) = \int^+\ x.\ ennreal\ (spmf\ p\ x) * f\ x\ \partial count\text{-}space\ (set\text{-}spmf\ p)$
⟨*proof*⟩

25.4 Functorial structure

abbreviation $map\text{-}spmf :: ('a \Rightarrow 'b) \Rightarrow 'a\ spmf \Rightarrow 'b\ spmf$
where $map\text{-}spmf\ f \equiv map\text{-}pmf\ (map\text{-}option\ f)$

context begin
⟨*ML*⟩

lemma *map-comp*: $map\text{-}spmf\ f\ (map\text{-}spmf\ g\ p) = map\text{-}spmf\ (f \circ g)\ p$
⟨*proof*⟩

lemma *map-id0*: $map\text{-}spmf\ id = id$
⟨*proof*⟩

lemma *map-id* [*simp*]: $map\text{-}spmf\ id\ p = p$
⟨*proof*⟩

lemma *map-ident* [*simp*]: $map\text{-}spmf\ (\lambda x.\ x)\ p = p$
⟨*proof*⟩

end

lemma *set-map-spmf* [*simp*]: $set\text{-}spmf\ (map\text{-}spmf\ f\ p) = f\ `\ set\text{-}spmf\ p$
⟨*proof*⟩

lemma *map-spmf-cong*:
  $[\![\ p = q;\ \bigwedge x.\ x \in set\text{-}spmf\ q \Longrightarrow f\ x = g\ x\ ]\!]$
  $\Longrightarrow map\text{-}spmf\ f\ p = map\text{-}spmf\ g\ q$
⟨*proof*⟩

lemma *map-spmf-cong-simp*:
  $[\![\ p = q;\ \bigwedge x.\ x \in set\text{-}spmf\ q =simp=> f\ x = g\ x\ ]\!]$
  $\Longrightarrow map\text{-}spmf\ f\ p = map\text{-}spmf\ g\ q$

$\langle proof \rangle$

lemma *map-spmf-idI*: $(\bigwedge x.\ x \in set\text{-}spmf\ p \Longrightarrow f\ x = x) \Longrightarrow map\text{-}spmf\ f\ p = p$
$\langle proof \rangle$

lemma *emeasure-map-spmf*:
$emeasure\ (measure\text{-}spmf\ (map\text{-}spmf\ f\ p))\ A = emeasure\ (measure\text{-}spmf\ p)\ (f\ -`\ A)$
$\langle proof \rangle$

lemma *measure-map-spmf*: $measure\ (measure\text{-}spmf\ (map\text{-}spmf\ f\ p))\ A = measure\ (measure\text{-}spmf\ p)\ (f\ -`\ A)$
$\langle proof \rangle$

lemma *measure-map-spmf-conv-distr*:
$measure\text{-}spmf\ (map\text{-}spmf\ f\ p) = distr\ (measure\text{-}spmf\ p)\ (count\text{-}space\ UNIV)\ f$
$\langle proof \rangle$

lemma *spmf-map-pmf-Some* [*simp*]: $spmf\ (map\text{-}pmf\ Some\ p)\ i = pmf\ p\ i$
$\langle proof \rangle$

lemma *spmf-map-inj*: $[\![\ inj\text{-}on\ f\ (set\text{-}spmf\ M);\ x \in set\text{-}spmf\ M\ ]\!] \Longrightarrow spmf\ (map\text{-}spmf\ f\ M)\ (f\ x) = spmf\ M\ x$
$\langle proof \rangle$

lemma *spmf-map-inj'*: $inj\ f \Longrightarrow spmf\ (map\text{-}spmf\ f\ M)\ (f\ x) = spmf\ M\ x$
$\langle proof \rangle$

lemma *spmf-map-outside*: $x \notin f\ `\ set\text{-}spmf\ M \Longrightarrow spmf\ (map\text{-}spmf\ f\ M)\ x = 0$
$\langle proof \rangle$

lemma *ennreal-spmf-map*: $ennreal\ (spmf\ (map\text{-}spmf\ f\ p)\ x) = emeasure\ (measure\text{-}spmf\ p)\ (f\ -`\ \{x\})$
$\langle proof \rangle$

lemma *spmf-map*: $spmf\ (map\text{-}spmf\ f\ p)\ x = measure\ (measure\text{-}spmf\ p)\ (f\ -`\ \{x\})$
$\langle proof \rangle$

lemma *ennreal-spmf-map-conv-nn-integral*:
$ennreal\ (spmf\ (map\text{-}spmf\ f\ p)\ x) = integral^N\ (measure\text{-}spmf\ p)\ (indicator\ (f\ -`\ \{x\}))$
$\langle proof \rangle$

25.5 Monad operations

25.5.1 Return

abbreviation $return\text{-}spmf :: 'a \Rightarrow 'a\ spmf$
where $return\text{-}spmf\ x \equiv return\text{-}pmf\ (Some\ x)$

lemma *pmf-return-spmf*: $pmf\ (return\text{-}spmf\ x)\ y = indicator\ \{y\}\ (Some\ x)$
⟨*proof*⟩

lemma *measure-spmf-return-spmf*: $measure\text{-}spmf\ (return\text{-}spmf\ x) = Giry\text{-}Monad.return\ (count\text{-}space\ UNIV)\ x$
⟨*proof*⟩

lemma *measure-spmf-return-pmf-None* [*simp*]: $measure\text{-}spmf\ (return\text{-}pmf\ None) = null\text{-}measure\ (count\text{-}space\ UNIV)$
⟨*proof*⟩

lemma *set-return-spmf* [*simp*]: $set\text{-}spmf\ (return\text{-}spmf\ x) = \{x\}$
⟨*proof*⟩

25.5.2 Bind

definition $bind\text{-}spmf :: 'a\ spmf \Rightarrow ('a \Rightarrow 'b\ spmf) \Rightarrow 'b\ spmf$
where $bind\text{-}spmf\ x\ f = bind\text{-}pmf\ x\ (\lambda a.\ case\ a\ of\ None \Rightarrow return\text{-}pmf\ None \mid Some\ a' \Rightarrow f\ a')$

adhoc-overloading *Monad-Syntax.bind bind-spmf*

lemma *return-None-bind-spmf* [*simp*]: $return\text{-}pmf\ None \gg= (f :: 'a \Rightarrow \text{-}) = return\text{-}pmf\ None$
⟨*proof*⟩

lemma *return-bind-spmf* [*simp*]: $return\text{-}spmf\ x \gg= f = f\ x$
⟨*proof*⟩

lemma *bind-return-spmf* [*simp*]: $x \gg= return\text{-}spmf = x$
⟨*proof*⟩

lemma *bind-spmf-assoc* [*simp*]:
fixes $x :: 'a\ spmf$ **and** $f :: 'a \Rightarrow 'b\ spmf$ **and** $g :: 'b \Rightarrow 'c\ spmf$
shows $(x \gg= f) \gg= g = x \gg= (\lambda y.\ f\ y \gg= g)$
⟨*proof*⟩

lemma *pmf-bind-spmf-None*: $pmf\ (p \gg= f)\ None = pmf\ p\ None + \int\ x.\ pmf\ (f\ x)\ None\ \partial measure\text{-}spmf\ p$
(**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *spmf-bind*: $spmf\ (p \gg= f)\ y = \int\ x.\ spmf\ (f\ x)\ y\ \partial measure\text{-}spmf\ p$
⟨*proof*⟩

lemma *ennreal-spmf-bind*: $ennreal\ (spmf\ (p \gg= f)\ x) = \int^{+}\ y.\ spmf\ (f\ y)\ x\ \partial measure\text{-}spmf\ p$
⟨*proof*⟩

lemma *measure-spmf-bind-pmf*: *measure-spmf* $(p \gg= f) =$ *measure-pmf* $p \gg=$ *measure-spmf* $\circ f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *measure-spmf-bind*: *measure-spmf* $(p \gg= f) =$ *measure-spmf* $p \gg=$ *measure-spmf* $\circ f$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-spmf-bind-spmf*: *map-spmf* f (*bind-spmf* p g) = *bind-spmf* p (*map-spmf* $f \circ g$)
⟨*proof*⟩

lemma *bind-map-spmf*: *map-spmf* f $p \gg= g = p \gg= g \circ f$
⟨*proof*⟩

lemma *spmf-bind-leI*:
assumes $\bigwedge y.\ y \in$ *set-spmf* $p \Longrightarrow$ *spmf* $(f\ y)\ x \leq r$
and $0 \leq r$
shows *spmf* (*bind-spmf* p f) $x \leq r$
⟨*proof*⟩

lemma *map-spmf-conv-bind-spmf*: *map-spmf* f $p = (p \gg= (\lambda x.$ *return-spmf* $(f\ x)))$
⟨*proof*⟩

lemma *bind-spmf-cong*:
$[\![\ p = q;\ \bigwedge x.\ x \in$ *set-spmf* $q \Longrightarrow f\ x = g\ x\ ]\!]$
$\Longrightarrow$ *bind-spmf* p f = *bind-spmf* q g
⟨*proof*⟩

lemma *bind-spmf-cong-simp*:
$[\![\ p = q;\ \bigwedge x.\ x \in$ *set-spmf* q =*simp*=> $f\ x = g\ x\ ]\!]$
$\Longrightarrow$ *bind-spmf* p f = *bind-spmf* q g
⟨*proof*⟩

lemma *set-bind-spmf*: *set-spmf* $(M \gg= f) =$ *set-spmf* $M \gg=$ (*set-spmf* $\circ f$)
⟨*proof*⟩

lemma *bind-spmf-const-return-None* [*simp*]: *bind-spmf* p (λ-. *return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *bind-commute-spmf*:
bind-spmf p $(\lambda x.$ *bind-spmf* q $(f\ x))$ = *bind-spmf* q $(\lambda y.$ *bind-spmf* p $(\lambda x.\ f\ x\ y))$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

25.6 Relator

abbreviation *rel-spmf* :: $('a \Rightarrow 'b \Rightarrow bool) \Rightarrow 'a\ spmf \Rightarrow 'b\ spmf \Rightarrow bool$
where *rel-spmf R* $\equiv$ *rel-pmf* (*rel-option R*)

lemma *rel-pmf-mono*:
$[\![rel\text{-}pmf\ A\ f\ g;\ \bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y\ ]\!] \Longrightarrow rel\text{-}pmf\ B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-mono*:
$[\![rel\text{-}spmf\ A\ f\ g;\ \bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y\ ]\!] \Longrightarrow rel\text{-}spmf\ B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-mono-strong*:
$[\![\ rel\text{-}spmf\ A\ f\ g;\ \bigwedge x\ y.\ [\![\ A\ x\ y;\ x \in set\text{-}spmf\ f;\ y \in set\text{-}spmf\ g\ ]\!] \Longrightarrow B\ x\ y\ ]\!] \Longrightarrow$
$rel\text{-}spmf\ B\ f\ g$
⟨*proof*⟩

lemma *rel-spmf-reflI*: $(\bigwedge x.\ x \in set\text{-}spmf\ p \Longrightarrow P\ x\ x) \Longrightarrow rel\text{-}spmf\ P\ p\ p$
⟨*proof*⟩

lemma *rel-spmfI* [*intro?*]:
$[\![\ \bigwedge x\ y.\ (x,\ y) \in set\text{-}spmf\ pq \Longrightarrow P\ x\ y;\ map\text{-}spmf\ fst\ pq = p;\ map\text{-}spmf\ snd\ pq$
$= q\ ]\!]$
$\Longrightarrow rel\text{-}spmf\ P\ p\ q$
⟨*proof*⟩

lemma *rel-spmfE* [*elim?*, *consumes 1*, *case-names rel-spmf*]:
assumes *rel-spmf P p q*
obtains *pq* **where**
$\bigwedge x\ y.\ (x,\ y) \in set\text{-}spmf\ pq \Longrightarrow P\ x\ y$
$p = map\text{-}spmf\ fst\ pq$
$q = map\text{-}spmf\ snd\ pq$
⟨*proof*⟩

lemma *rel-spmf-simps*:
$rel\text{-}spmf\ R\ p\ q \longleftrightarrow (\exists pq.\ (\forall (x,\ y) \in set\text{-}spmf\ pq.\ R\ x\ y) \wedge map\text{-}spmf\ fst\ pq = p\ \wedge$
$map\text{-}spmf\ snd\ pq = q)$
⟨*proof*⟩

lemma *spmf-rel-map*:
shows *spmf-rel-map1*: $\bigwedge R\ f\ x.\ rel\text{-}spmf\ R\ (map\text{-}spmf\ f\ x) = rel\text{-}spmf\ (\lambda x.\ R\ (f$
$x))\ x$
and *spmf-rel-map2*: $\bigwedge R\ x\ g\ y.\ rel\text{-}spmf\ R\ x\ (map\text{-}spmf\ g\ y) = rel\text{-}spmf\ (\lambda x\ y.\ R$
$x\ (g\ y))\ x\ y$
⟨*proof*⟩

lemma *spmf-rel-conversep*: $rel\text{-}spmf\ R^{-1-1} = (rel\text{-}spmf\ R)^{-1-1}$
⟨*proof*⟩

lemma *spmf-rel-eq*: *rel-spmf* (=) = (=)
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *bind-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> (*A* ===> *rel-spmf B*) ===> *rel-spmf B*) *bind-spmf bind-spmf*
⟨*proof*⟩

lemma *return-spmf-parametric*: (*A* ===> *rel-spmf A*) *return-spmf return-spmf*
⟨*proof*⟩

lemma *map-spmf-parametric*: ((*A* ===> *B*) ===> *rel-spmf A* ===> *rel-spmf B*) *map-spmf map-spmf*
⟨*proof*⟩

lemma *rel-spmf-parametric*:
((*A* ===> *B* ===> (=)) ===> *rel-spmf A* ===> *rel-spmf B* ===> (=)) *rel-spmf rel-spmf*
⟨*proof*⟩

lemma *set-spmf-parametric* [*transfer-rule*]:
(*rel-spmf A* ===> *rel-set A*) *set-spmf set-spmf*
⟨*proof*⟩

lemma *return-spmf-None-parametric*:
(*rel-spmf A*) (*return-pmf None*) (*return-pmf None*)
⟨*proof*⟩

end

lemma *rel-spmf-bindI*:
$[\![$ *rel-spmf R p q*; $\bigwedge x\ y.\ R\ x\ y \Longrightarrow$ *rel-spmf P* $(f\ x)\ (g\ y)$ $]\!]$
$\Longrightarrow$ *rel-spmf P* $(p \ggg f)\ (q \ggg g)$
⟨*proof*⟩

lemma *rel-spmf-bind-reflI*:
$(\bigwedge x.\ x \in$ *set-spmf p* $\Longrightarrow$ *rel-spmf P* $(f\ x)\ (g\ x)) \Longrightarrow$ *rel-spmf P* $(p \ggg f)\ (p \ggg g)$
⟨*proof*⟩

lemma *rel-pmf-return-pmfI*: $P\ x\ y \Longrightarrow$ *rel-pmf P* (*return-pmf x*) (*return-pmf y*)
⟨*proof*⟩

context includes *lifting-syntax*
begin

We do not yet have a relator for *'a measure*, so we combine *Sigma-Algebra.measure* and *measure-pmf*

lemma *measure-pmf-parametric*:
(*rel-pmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-pmf p*)) (λ*q*. *measure* (*measure-pmf q*))
⟨*proof*⟩

lemma *measure-spmf-parametric*:
(*rel-spmf A* ===> *rel-pred A* ===> (=)) (λ*p*. *measure* (*measure-spmf p*)) (λ*q*. *measure* (*measure-spmf q*))
⟨*proof*⟩

end

25.7 From *'a pmf* to *'a spmf*

definition *spmf-of-pmf* :: *'a pmf* ⇒ *'a spmf*
where *spmf-of-pmf* = *map-pmf Some*

lemma *set-spmf-spmf-of-pmf* [*simp*]: *set-spmf* (*spmf-of-pmf p*) = *set-pmf p*
⟨*proof*⟩

lemma *spmf-spmf-of-pmf* [*simp*]: *spmf* (*spmf-of-pmf p*) *x* = *pmf p x*
⟨*proof*⟩

lemma *pmf-spmf-of-pmf-None* [*simp*]: *pmf* (*spmf-of-pmf p*) *None* = *0*
⟨*proof*⟩

lemma *emeasure-spmf-of-pmf* [*simp*]: *emeasure* (*measure-spmf* (*spmf-of-pmf p*)) *A* = *emeasure* (*measure-pmf p*) *A*
⟨*proof*⟩

lemma *measure-spmf-spmf-of-pmf* [*simp*]: *measure-spmf* (*spmf-of-pmf p*) = *measure-pmf p*
⟨*proof*⟩

lemma *map-spmf-of-pmf* [*simp*]: *map-spmf f* (*spmf-of-pmf p*) = *spmf-of-pmf* (*map-pmf f p*)
⟨*proof*⟩

lemma *rel-spmf-spmf-of-pmf* [*simp*]: *rel-spmf R* (*spmf-of-pmf p*) (*spmf-of-pmf q*) = *rel-pmf R p q*
⟨*proof*⟩

lemma *spmf-of-pmf-return-pmf* [*simp*]: *spmf-of-pmf* (*return-pmf x*) = *return-spmf x*
⟨*proof*⟩

lemma *bind-spmf-of-pmf* [*simp*]: *bind-spmf* (*spmf-of-pmf p*) *f* = *bind-pmf p f*
⟨*proof*⟩

lemma *set-spmf-bind-pmf*: *set-spmf* (*bind-pmf p f*) = *Set.bind* (*set-pmf p*) (*set-spmf* ∘ *f*)
⟨*proof*⟩

lemma *spmf-of-pmf-bind*: *spmf-of-pmf* (*bind-pmf p f*) = *bind-pmf p* (λ*x*. *spmf-of-pmf* (*f x*))
⟨*proof*⟩

lemma *bind-pmf-return-spmf*: *p* ⤜ (λ*x*. *return-spmf* (*f x*)) = *spmf-of-pmf* (*map-pmf f p*)
⟨*proof*⟩

25.8 Weight of a subprobability

abbreviation *weight-spmf* :: ′*a spmf* ⇒ *real*
where *weight-spmf p* ≡ *measure* (*measure-spmf p*) (*space* (*measure-spmf p*))

lemma *weight-spmf-def*: *weight-spmf p* = *measure* (*measure-spmf p*) *UNIV*
⟨*proof*⟩

lemma *weight-spmf-le-1*: *weight-spmf p* ≤ *1*
⟨*proof*⟩

lemma *weight-return-spmf* [*simp*]: *weight-spmf* (*return-spmf x*) = *1*
⟨*proof*⟩

lemma *weight-return-pmf-None* [*simp*]: *weight-spmf* (*return-pmf None*) = *0*
⟨*proof*⟩

lemma *weight-map-spmf* [*simp*]: *weight-spmf* (*map-spmf f p*) = *weight-spmf p*
⟨*proof*⟩

lemma *weight-spmf-of-pmf* [*simp*]: *weight-spmf* (*spmf-of-pmf p*) = *1*
⟨*proof*⟩

lemma *weight-spmf-nonneg*: *weight-spmf p* ≥ *0*
⟨*proof*⟩

lemma (**in** *finite-measure*) *integrable-weight-spmf* [*simp*]:
(λ*x*. *weight-spmf* (*f x*)) ∈ *borel-measurable M* ⟹ *integrable M* (λ*x*. *weight-spmf* (*f x*))
⟨*proof*⟩

lemma *weight-spmf-eq-nn-integral-spmf*: *weight-spmf p* = ∫⁺ *x*. *spmf p x* ∂*count-space UNIV*
⟨*proof*⟩

lemma *weight-spmf-eq-nn-integral-support*:

$\mathit{weight\text{-}spmf}\ p = \int^+ x.\ \mathit{spmf}\ p\ x\ \partial\mathit{count\text{-}space}\ (\mathit{set\text{-}spmf}\ p)$
⟨*proof*⟩

lemma *pmf-None-eq-weight-spmf*: $\mathit{pmf}\ p\ \mathit{None} = 1 - \mathit{weight\text{-}spmf}\ p$
⟨*proof*⟩

lemma *weight-spmf-conv-pmf-None*: $\mathit{weight\text{-}spmf}\ p = 1 - \mathit{pmf}\ p\ \mathit{None}$
⟨*proof*⟩

lemma *weight-spmf-le-0*: $\mathit{weight\text{-}spmf}\ p \le 0 \longleftrightarrow \mathit{weight\text{-}spmf}\ p = 0$
⟨*proof*⟩

lemma *weight-spmf-lt-0*: $\neg\ \mathit{weight\text{-}spmf}\ p < 0$
⟨*proof*⟩

lemma *spmf-le-weight*: $\mathit{spmf}\ p\ x \le \mathit{weight\text{-}spmf}\ p$
⟨*proof*⟩

lemma *weight-spmf-eq-0*: $\mathit{weight\text{-}spmf}\ p = 0 \longleftrightarrow p = \mathit{return\text{-}pmf}\ \mathit{None}$
⟨*proof*⟩

lemma *weight-bind-spmf*: $\mathit{weight\text{-}spmf}\ (x \gg= f) = \mathit{lebesgue\text{-}integral}\ (\mathit{measure\text{-}spmf}\ x)\ (\mathit{weight\text{-}spmf} \circ f)$
⟨*proof*⟩

lemma *rel-spmf-weightD*: $\mathit{rel\text{-}spmf}\ A\ p\ q \Longrightarrow \mathit{weight\text{-}spmf}\ p = \mathit{weight\text{-}spmf}\ q$
⟨*proof*⟩

lemma *rel-spmf-bij-betw*:
  assumes *f*: $\mathit{bij\text{-}betw}\ f\ (\mathit{set\text{-}spmf}\ p)\ (\mathit{set\text{-}spmf}\ q)$
  and *eq*: $\bigwedge x.\ x \in \mathit{set\text{-}spmf}\ p \Longrightarrow \mathit{spmf}\ p\ x = \mathit{spmf}\ q\ (f\ x)$
  shows $\mathit{rel\text{-}spmf}\ (\lambda x\ y.\ f\ x = y)\ p\ q$
⟨*proof*⟩

25.9 From density to spmfs

context fixes $f :: 'a \Rightarrow \mathit{real}$ **begin**

definition $\mathit{embed\text{-}spmf} :: 'a\ \mathit{spmf}$
where $\mathit{embed\text{-}spmf} = \mathit{embed\text{-}pmf}\ (\lambda x.\ \mathit{case}\ x\ \mathit{of}\ \mathit{None} \Rightarrow 1 - \mathit{enn2real}\ (\int^+ x.\ \mathit{ennreal}\ (f\ x)\ \partial\mathit{count\text{-}space}\ \mathit{UNIV})\ |\ \mathit{Some}\ x' \Rightarrow \mathit{max}\ 0\ (f\ x'))$

context
  assumes *prob*: $(\int^+ x.\ \mathit{ennreal}\ (f\ x)\ \partial\mathit{count\text{-}space}\ \mathit{UNIV}) \le 1$
begin

lemma *nn-integral-embed-spmf-eq-1*:
  $(\int^+ x.\ \mathit{ennreal}\ (\mathit{case}\ x\ \mathit{of}\ \mathit{None} \Rightarrow 1 - \mathit{enn2real}\ (\int^+ x.\ \mathit{ennreal}\ (f\ x)\ \partial\mathit{count\text{-}space}\ \mathit{UNIV})\ |\ \mathit{Some}\ x' \Rightarrow \mathit{max}\ 0\ (f\ x'))\ \partial\mathit{count\text{-}space}\ \mathit{UNIV}) = 1$

(**is** *?lhs* = - **is** ($\int^+ x.\ ?f\ x\ \partial ?M$) = -)
⟨*proof*⟩

lemma *pmf-embed-spmf-None*: *pmf embed-spmf None* = *1* − *enn2real* ($\int^+ x.$ *ennreal* (*f x*) ∂*count-space UNIV*)
⟨*proof*⟩

lemma *spmf-embed-spmf* [*simp*]: *spmf embed-spmf x* = *max 0* (*f x*)
⟨*proof*⟩

end

end

lemma *embed-spmf-K-0*[*simp*]: *embed-spmf* (λ-. *0*) = *return-pmf None* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

25.10 Ordering on spmfs

rel-pmf does not preserve a ccpo structure. Counterexample by Saheb-Djahromi: Take prefix order over *bool llist* and the set *range* (λ*n* :: *nat*. *uniform* (*llist-n n*)) where *llist-n* is the set of all *llist*s of length *n* and *uniform* returns a uniform distribution over the given set. The set forms a chain in *ord-pmf lprefix*, but it has not an upper bound. Any upper bound may contain only infinite lists in its support because otherwise it is not greater than the *n*+*1*-st element in the chain where *n* is the length of the finite list. Moreover its support must contain all infinite lists, because otherwise there is a finite list all of whose finite extensions are not in the support - a contradiction to the upper bound property. Hence, the support is uncountable, but pmf's only have countable support.

However, if all chains in the ccpo are finite, then it should preserve the ccpo structure.

abbreviation *ord-spmf* :: (*'a* ⇒ *'a* ⇒ *bool*) ⇒ *'a spmf* ⇒ *'a spmf* ⇒ *bool*
where *ord-spmf ord* ≡ *rel-pmf* (*ord-option ord*)

locale *ord-spmf-syntax* **begin**
notation *ord-spmf* (**infix** $\sqsubseteq_1$ *60*)
end

lemma *ord-spmf-map-spmf1*: *ord-spmf R* (*map-spmf f p*) = *ord-spmf* (λ*x*. *R* (*f x*)) *p*
⟨*proof*⟩

lemma *ord-spmf-map-spmf2*: *ord-spmf R p* (*map-spmf f q*) = *ord-spmf* (λ*x y*. *R x* (*f y*)) *p q*
⟨*proof*⟩

lemma *ord-spmf-map-spmf12*: *ord-spmf R* (*map-spmf f p*) (*map-spmf f q*) = *ord-spmf* ($\lambda x\ y.\ R\ (f\ x)\ (f\ y)$) *p q*
⟨*proof*⟩

lemmas *ord-spmf-map-spmf* = *ord-spmf-map-spmf1 ord-spmf-map-spmf2 ord-spmf-map-spmf12*

context fixes *ord* :: $'a \Rightarrow 'a \Rightarrow$ *bool* (**structure**) **begin**
interpretation *ord-spmf-syntax* ⟨*proof*⟩

lemma *ord-spmfI*:
$\llbracket \bigwedge x\ y.\ (x,\ y) \in$ *set-spmf pq* $\Longrightarrow$ *ord x y*; *map-spmf fst pq* = *p*; *map-spmf snd pq* = $q\ \rrbracket$
$\Longrightarrow p \sqsubseteq q$
⟨*proof*⟩

lemma *ord-spmf-None* [*simp*]: *return-pmf None* $\sqsubseteq x$
⟨*proof*⟩

lemma *ord-spmf-reflI*: ($\bigwedge x.\ x \in$ *set-spmf p* $\Longrightarrow$ *ord x x*) $\Longrightarrow p \sqsubseteq p$
⟨*proof*⟩

lemma *rel-spmf-inf*:
assumes $p \sqsubseteq q$
and $q \sqsubseteq p$
and *refl*: *reflp ord*
and *trans*: *transp ord*
shows *rel-spmf* (*inf ord* ord^{-1-1}) *p q*
⟨*proof*⟩

end

lemma *ord-spmf-return-spmf2*: *ord-spmf R p* (*return-spmf y*) $\longleftrightarrow$ ($\forall x\in$*set-spmf p*. *R x y*)
⟨*proof*⟩

lemma *ord-spmf-mono*: $\llbracket$ *ord-spmf A p q*; $\bigwedge x\ y.\ A\ x\ y \Longrightarrow B\ x\ y\ \rrbracket \Longrightarrow$ *ord-spmf B p q*
⟨*proof*⟩

lemma *ord-spmf-compp*: *ord-spmf* (*A OO B*) = *ord-spmf A OO ord-spmf B*
⟨*proof*⟩

lemma *ord-spmf-bindI*:
assumes *pq*: *ord-spmf R p q*
and *fg*: $\bigwedge x\ y.\ R\ x\ y \Longrightarrow$ *ord-spmf P* (*f x*) (*g y*)
shows *ord-spmf P* ($p \gg= f$) ($q \gg= g$)
⟨*proof*⟩

lemma *ord-spmf-bind-reflI*:
$(\bigwedge x.\ x \in \textit{set-spmf}\ p \Longrightarrow \textit{ord-spmf}\ R\ (f\ x)\ (g\ x))$
$\Longrightarrow \textit{ord-spmf}\ R\ (p \gg= f)\ (p \gg= g)$
⟨*proof*⟩

lemma *ord-pmf-increaseI*:
assumes *le*: $\bigwedge x.\ \textit{spmf}\ p\ x \le \textit{spmf}\ q\ x$
and *refl*: $\bigwedge x.\ x \in \textit{set-spmf}\ p \Longrightarrow R\ x\ x$
shows *ord-spmf R p q*
⟨*proof*⟩

lemma *ord-spmf-eq-leD*:
assumes *ord-spmf* (=) *p q*
shows $\textit{spmf}\ p\ x \le \textit{spmf}\ q\ x$
⟨*proof*⟩

lemma *ord-spmf-eqD-set-spmf*: $\textit{ord-spmf}\ (=)\ p\ q \Longrightarrow \textit{set-spmf}\ p \subseteq \textit{set-spmf}\ q$
⟨*proof*⟩

lemma *ord-spmf-eqD-emeasure*:
$\textit{ord-spmf}\ (=)\ p\ q \Longrightarrow \textit{emeasure}\ (\textit{measure-spmf}\ p)\ A \le \textit{emeasure}\ (\textit{measure-spmf}\ q)\ A$
⟨*proof*⟩

lemma *ord-spmf-eqD-measure-spmf*: $\textit{ord-spmf}\ (=)\ p\ q \Longrightarrow \textit{measure-spmf}\ p \le \textit{measure-spmf}\ q$
⟨*proof*⟩

25.11 CCPO structure for the flat ccpo *ord-option* (=)

context fixes $Y :: {'a}\ \textit{spmf set}$ **begin**

definition $\textit{lub-spmf} :: {'a}\ \textit{spmf}$
where $\textit{lub-spmf} = \textit{embed-spmf}\ (\lambda x.\ \textit{enn2real}\ (\textit{SUP}\ p \in Y.\ \textit{ennreal}\ (\textit{spmf}\ p\ x)))$
— We go through *ennreal* to have a sensible definition even if Y is empty.

lemma *lub-spmf-empty* [*simp*]: *SPMF.lub-spmf* {} = *return-pmf None*
⟨*proof*⟩

context assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) Y **begin**

lemma *chain-ord-spmf-eqD*: $\textit{Complete-Partial-Order.chain}\ (\le)\ ((\lambda p\ x.\ \textit{ennreal}\ (\textit{spmf}\ p\ x))\ `\ Y)$
(**is** *Complete-Partial-Order.chain* - (*?f* ‘ -))
⟨*proof*⟩

lemma *ord-spmf-eq-pmf-None-eq*:
assumes *le*: *ord-spmf* (=) *p q*
and *None*: *pmf p None* = *pmf q None*

shows $p = q$
⟨*proof*⟩

lemma *ord-spmf-eqD-pmf-None*:
assumes *ord-spmf* $(=)$ x y
shows *pmf* x *None* $\geq$ *pmf* y *None*
⟨*proof*⟩

Chains on *'a spmf* maintain countable support. Thanks to Johannes Hölzl for the proof idea.

lemma *spmf-chain-countable*: *countable* $(\bigcup p{\in}Y.$ *set-spmf* $p)$
⟨*proof*⟩

lemma *lub-spmf-subprob*: $(\int^+ x.$ $(SUP\ p \in Y.$ *ennreal* $($*spmf* p $x))$ ∂*count-space* *UNIV*$) \leq 1$
⟨*proof*⟩

lemma *spmf-lub-spmf*:
assumes $Y \neq \{\}$
shows *spmf lub-spmf* $x = (SUP\ p \in Y.$ *spmf* p $x)$
⟨*proof*⟩

lemma *ennreal-spmf-lub-spmf*: $Y \neq \{\} \Longrightarrow$ *ennreal* $($*spmf lub-spmf* $x) = (SUP$ $p{\in}Y.$ *ennreal* $($*spmf* p $x))$
⟨*proof*⟩

lemma *lub-spmf-upper*:
assumes p: $p \in Y$
shows *ord-spmf* $(=)$ p *lub-spmf*
⟨*proof*⟩

lemma *lub-spmf-least*:
assumes z: $\bigwedge x.$ $x \in Y \Longrightarrow$ *ord-spmf* $(=)$ x z
shows *ord-spmf* $(=)$ *lub-spmf* z
⟨*proof*⟩

lemma *set-lub-spmf*: *set-spmf lub-spmf* $= (\bigcup p{\in}Y.$ *set-spmf* $p)$ (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *emeasure-lub-spmf*:
assumes Y: $Y \neq \{\}$
shows *emeasure* $($*measure-spmf lub-spmf*$)$ $A = (SUP\ y{\in}Y.$ *emeasure* $($*measure-spmf* $y)$ $A)$
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *measure-lub-spmf*:
assumes Y: $Y \neq \{\}$
shows *measure* $($*measure-spmf lub-spmf*$)$ $A = (SUP\ y{\in}Y.$ *measure* $($*measure-spmf*

y) *A*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *weight-lub-spmf*:
  assumes *Y*: $Y \neq \{\}$
  shows *weight-spmf lub-spmf* = (*SUP* $y \in Y$. *weight-spmf y*)
⟨*proof*⟩

lemma *measure-spmf-lub-spmf*:
  assumes *Y*: $Y \neq \{\}$
  shows *measure-spmf lub-spmf* = (*SUP* $p \in Y$. *measure-spmf p*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

end

lemma *partial-function-definitions-spmf*: *partial-function-definitions* (*ord-spmf* (=)) *lub-spmf*
  (**is** *partial-function-definitions ?R* -)
⟨*proof*⟩

lemma *ccpo-spmf*: *class.ccpo lub-spmf* (*ord-spmf* (=)) (*mk-less* (*ord-spmf* (=)))
⟨*proof*⟩

interpretation *spmf*: *partial-function-definitions ord-spmf* (=) *lub-spmf*
  rewrites *lub-spmf* {} ≡ *return-pmf None*
⟨*proof*⟩

⟨*ML*⟩

declare *spmf.leq-refl*[*simp*]
declare *admissible-leI*[*OF ccpo-spmf*, *cont-intro*]

abbreviation *mono-spmf* ≡ *monotone* (*fun-ord* (*ord-spmf* (=))) (*ord-spmf* (=))

lemma *lub-spmf-const* [*simp*]: *lub-spmf* {*p*} = *p*
⟨*proof*⟩

lemma *bind-spmf-mono'*:
  assumes *fg*: *ord-spmf* (=) *f g*
  and *hk*: $\bigwedge x :: 'a.$ *ord-spmf* (=) (*h x*) (*k x*)
  shows *ord-spmf* (=) (*f* ⤜ *h*) (*g* ⤜ *k*)
⟨*proof*⟩

lemma *bind-spmf-mono* [*partial-function-mono*]:
  assumes *mf*: *mono-spmf B* **and** *mg*: $\bigwedge y.$ *mono-spmf* (λ*f*. *C y f*)
  shows *mono-spmf* (λ*f*. *bind-spmf* (*B f*) (λ*y*. *C y f*))
⟨*proof*⟩

lemma *monotone-bind-spmf1*: *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) (λy. *bind-spmf* *y g*)
⟨*proof*⟩

lemma *monotone-bind-spmf2*:
assumes *g*: $\bigwedge x$. *monotone ord* (*ord-spmf* (=)) (λy. *g y x*)
shows *monotone ord* (*ord-spmf* (=)) (λy. *bind-spmf p* (*g y*))
⟨*proof*⟩

lemma *bind-lub-spmf*:
assumes *chain*: *Complete-Partial-Order.chain* (*ord-spmf* (=)) *Y*
shows *bind-spmf* (*lub-spmf Y*) *f* = *lub-spmf* ((λp. *bind-spmf p f*) ‘ *Y*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-lub-spmf*:
Complete-Partial-Order.chain (*ord-spmf* (=)) *Y*
$\Longrightarrow$ *map-spmf f* (*lub-spmf Y*) = *lub-spmf* (*map-spmf f* ‘ *Y*)
⟨*proof*⟩

lemma *mcont-bind-spmf1*: *mcont lub-spmf* (*ord-spmf* (=)) *lub-spmf* (*ord-spmf* (=)) (λy. *bind-spmf y f*)
⟨*proof*⟩

lemma *bind-lub-spmf2*:
assumes *chain*: *Complete-Partial-Order.chain ord Y*
and *g*: $\bigwedge y$. *monotone ord* (*ord-spmf* (=)) (*g y*)
shows *bind-spmf x* (λy. *lub-spmf* (*g y* ‘ *Y*)) = *lub-spmf* ((λp. *bind-spmf x* (λy. *g y p*)) ‘ *Y*)
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *mcont-bind-spmf* [*cont-intro*]:
assumes *f*: *mcont luba orda lub-spmf* (*ord-spmf* (=)) *f*
and *g*: $\bigwedge y$. *mcont luba orda lub-spmf* (*ord-spmf* (=)) (*g y*)
shows *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λx. *bind-spmf* (*f x*) (λy. *g y x*))
⟨*proof*⟩

lemma *bind-pmf-mono* [*partial-function-mono*]:
($\bigwedge y$. *mono-spmf* (λf. *C y f*)) $\Longrightarrow$ *mono-spmf* (λf. *bind-pmf p* (λx. *C x f*))
⟨*proof*⟩

lemma *map-spmf-mono* [*partial-function-mono*]: *mono-spmf B* $\Longrightarrow$ *mono-spmf* (λg. *map-spmf f* (*B g*))
⟨*proof*⟩

lemma *mcont-map-spmf* [*cont-intro*]:

mcont luba orda lub-spmf (*ord-spmf* (=)) *g*
$\Longrightarrow$ *mcont luba orda lub-spmf* (*ord-spmf* (=)) (λx. *map-spmf f* (*g x*))
⟨*proof*⟩

lemma *monotone-set-spmf*: *monotone* (*ord-spmf* (=)) (⊆) *set-spmf*
⟨*proof*⟩

lemma *cont-set-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
⟨*proof*⟩

lemma *mcont2mcont-set-spmf*[*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-set-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Union* (⊆) *set-spmf*
⟨*proof*⟩

lemma *monotone-spmf*: *monotone* (*ord-spmf* (=)) (≤) (λp. *spmf p x*)
⟨*proof*⟩

lemma *cont-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λp. *spmf p x*)
⟨*proof*⟩

lemma *mcont-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λp. *spmf p x*)
⟨*proof*⟩

lemma *cont-ennreal-spmf*: *cont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λp. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *mcont2mcont-ennreal-spmf* [*THEN mcont2mcont*, *cont-intro*]:
shows *mcont-ennreal-spmf*: *mcont lub-spmf* (*ord-spmf* (=)) *Sup* (≤) (λp. *ennreal* (*spmf p x*))
⟨*proof*⟩

lemma *nn-integral-map-spmf* [*simp*]: *nn-integral* (*measure-spmf* (*map-spmf f p*)) *g* = *nn-integral* (*measure-spmf p*) (*g* ∘ *f*)
⟨*proof*⟩

25.11.1 Admissibility of *rel-spmf*

lemma *rel-spmf-measureD*:
assumes *rel-spmf R p q*
shows *measure* (*measure-spmf p*) *A* ≤ *measure* (*measure-spmf q*) {*y*. ∃ *x*∈*A*. *R x y*} (**is** *?lhs* ≤ *?rhs*)
⟨*proof*⟩

locale *rel-spmf-characterisation* =
assumes *rel-pmf-measureI*:
$\bigwedge$(*R* :: *'a option* ⇒ *'b option* ⇒ *bool*) *p q*.
($\bigwedge$*A*. *measure* (*measure-pmf p*) *A* ≤ *measure* (*measure-pmf q*) {*y*. ∃ *x*∈*A*. *R x y*})

$\Longrightarrow$ *rel-pmf R p q*
— This assumption is shown to hold in general in the AFP entry *MFMC-Countable*.
begin

context fixes $R :: 'a \Rightarrow 'b \Rightarrow bool$ **begin**

lemma *rel-spmf-measureI*:
assumes *eq1*: $\bigwedge A.$ *measure* (*measure-spmf p*) $A \leq$ *measure* (*measure-spmf q*) $\{y.\ \exists x \in A.\ R\ x\ y\}$
assumes *eq2*: *weight-spmf q* $\leq$ *weight-spmf p*
shows *rel-spmf R p q*
⟨*proof*⟩

lemma *admissible-rel-spmf*:
ccpo.admissible (*prod-lub lub-spmf lub-spmf*) (*rel-prod* (*ord-spmf* (=)) (*ord-spmf* (=))) (*case-prod* (*rel-spmf R*))
(**is** *ccpo.admissible ?lub ?ord ?P*)
⟨*proof*⟩

lemma *admissible-rel-spmf-mcont* [*cont-intro*]:
⟦ *mcont lub ord lub-spmf* (*ord-spmf* (=)) *f*; *mcont lub ord lub-spmf* (*ord-spmf* (=)) *g* ⟧
$\Longrightarrow$ *ccpo.admissible lub ord* ($\lambda x.$ *rel-spmf R* (*f x*) (*g x*))
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *fixp-spmf-parametric'*:
assumes *f*: $\bigwedge x.$ *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) *F*
and *g*: $\bigwedge x.$ *monotone* (*ord-spmf* (=)) (*ord-spmf* (=)) *G*
and *param*: (*rel-spmf R* ===> *rel-spmf R*) *F G*
shows (*rel-spmf R*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) *F*) (*ccpo.fixp lub-spmf* (*ord-spmf* (=)) *G*)
⟨*proof*⟩

lemma *fixp-spmf-parametric*:
assumes *f*: $\bigwedge x.$ *mono-spmf* ($\lambda f.$ *F f x*)
and *g*: $\bigwedge x.$ *mono-spmf* ($\lambda f.$ *G f x*)
and *param*: ((*A* ===> *rel-spmf R*) ===> *A* ===> *rel-spmf R*) *F G*
shows (*A* ===> *rel-spmf R*) (*spmf.fixp-fun F*) (*spmf.fixp-fun G*)
⟨*proof*⟩

end

end

end

25.12 Restrictions on spmfs

definition *restrict-spmf* :: $'a$ *spmf* $\Rightarrow$ $'a$ *set* $\Rightarrow$ $'a$ *spmf* (**infixl** $\upharpoonright$ *110*)
where $p \upharpoonright A$ = *map-pmf* ($\lambda x.$ $x \gg=$ ($\lambda y.$ *if* $y \in A$ *then Some* y *else None*)) p

lemma *set-restrict-spmf* [*simp*]: *set-spmf* ($p \upharpoonright A$) = *set-spmf* $p \cap A$
⟨*proof*⟩

lemma *restrict-map-spmf*: *map-spmf* f $p \upharpoonright A$ = *map-spmf* f ($p \upharpoonright$ (f −' A))
⟨*proof*⟩

lemma *restrict-restrict-spmf* [*simp*]: $p \upharpoonright A \upharpoonright B = p \upharpoonright (A \cap B)$
⟨*proof*⟩

lemma *restrict-spmf-empty* [*simp*]: $p \upharpoonright \{\}$ = *return-pmf None*
⟨*proof*⟩

lemma *restrict-spmf-UNIV* [*simp*]: $p \upharpoonright$ *UNIV* = p
⟨*proof*⟩

lemma *spmf-restrict-spmf-outside* [*simp*]: $x \notin A \Longrightarrow$ *spmf* ($p \upharpoonright A$) x = *0*
⟨*proof*⟩

lemma *emeasure-restrict-spmf* [*simp*]:
emeasure (*measure-spmf* ($p \upharpoonright A$)) X = *emeasure* (*measure-spmf* p) ($X \cap A$)
⟨*proof*⟩

lemma *measure-restrict-spmf* [*simp*]:
measure (*measure-spmf* ($p \upharpoonright A$)) X = *measure* (*measure-spmf* p) ($X \cap A$)
⟨*proof*⟩

lemma *spmf-restrict-spmf*: *spmf* ($p \upharpoonright A$) x = (*if* $x \in A$ *then spmf* p x *else 0*)
⟨*proof*⟩

lemma *spmf-restrict-spmf-inside* [*simp*]: $x \in A \Longrightarrow$ *spmf* ($p \upharpoonright A$) x = *spmf* p x
⟨*proof*⟩

lemma *pmf-restrict-spmf-None*: *pmf* ($p \upharpoonright A$) *None* = *pmf* p *None* + *measure* (*measure-spmf* p) (− A)
⟨*proof*⟩

lemma *restrict-spmf-trivial*: ($\bigwedge x.$ $x \in$ *set-spmf* $p \Longrightarrow x \in A$) $\Longrightarrow p \upharpoonright A = p$
⟨*proof*⟩

lemma *restrict-spmf-trivial'*: *set-spmf* $p \subseteq A \Longrightarrow p \upharpoonright A = p$
⟨*proof*⟩

lemma *restrict-return-spmf*: *return-spmf* $x \upharpoonright A$ = (*if* $x \in A$ *then return-spmf* x *else return-pmf None*)
⟨*proof*⟩

lemma *restrict-return-spmf-inside* [*simp*]: $x \in A \Longrightarrow$ *return-spmf* $x \upharpoonright A =$ *return-spmf* x
⟨*proof*⟩

lemma *restrict-return-spmf-outside* [*simp*]: $x \notin A \Longrightarrow$ *return-spmf* $x \upharpoonright A =$ *return-pmf None*
⟨*proof*⟩

lemma *restrict-spmf-return-pmf-None* [*simp*]: *return-pmf None* $\upharpoonright A =$ *return-pmf None*
⟨*proof*⟩

lemma *restrict-bind-pmf*: *bind-pmf* $p\ g \upharpoonright A = p \gg= (\lambda x.\ g\ x \upharpoonright A)$
⟨*proof*⟩

lemma *restrict-bind-spmf*: *bind-spmf* $p\ g \upharpoonright A = p \gg= (\lambda x.\ g\ x \upharpoonright A)$
⟨*proof*⟩

lemma *bind-restrict-pmf*: *bind-pmf* $(p \upharpoonright A)\ g = p \gg= (\lambda x.$ *if* $x \in$ *Some* ‘ A *then* $g\ x$ *else* g *None*$)$
⟨*proof*⟩

lemma *bind-restrict-spmf*: *bind-spmf* $(p \upharpoonright A)\ g = p \gg= (\lambda x.$ *if* $x \in A$ *then* $g\ x$ *else return-pmf None*$)$
⟨*proof*⟩

lemma *spmf-map-restrict*: *spmf* (*map-spmf fst* $(p \upharpoonright (snd\ -`\ \{y\})))\ x =$ *spmf* $p\ (x, y)$
⟨*proof*⟩

lemma *measure-eqI-restrict-spmf*:
  assumes *rel-spmf* R (*restrict-spmf* p A) (*restrict-spmf* q B)
  shows *measure* (*measure-spmf* p) $A =$ *measure* (*measure-spmf* q) B
⟨*proof*⟩

25.13 Subprobability distributions of sets

definition *spmf-of-set* :: $'a$ *set* $\Rightarrow$ $'a$ *spmf*
where
  spmf-of-set $A =$ (*if finite* $A \wedge A \neq \{\}$ *then spmf-of-pmf* (*pmf-of-set* A) *else return-pmf None*)

lemma *spmf-of-set*: *spmf* (*spmf-of-set* A) $x =$ *indicator* $A\ x$ / *card* A
⟨*proof*⟩

lemma *pmf-spmf-of-set-None* [*simp*]: *pmf* (*spmf-of-set* A) *None* $=$ *indicator* $\{A.$ *infinite* $A \vee A = \{\}\}\ A$
⟨*proof*⟩

lemma *set-spmf-of-set*: *set-spmf* (*spmf-of-set* A) = (*if finite* A *then* A *else* {})
⟨*proof*⟩

lemma *set-spmf-of-set-finite* [*simp*]: *finite* $A \Longrightarrow$ *set-spmf* (*spmf-of-set* A) = A
⟨*proof*⟩

lemma *spmf-of-set-singleton*: *spmf-of-set* $\{x\}$ = *return-spmf* x
⟨*proof*⟩

lemma *map-spmf-of-set-inj-on* [*simp*]:
inj-on f $A \Longrightarrow$ *map-spmf* f (*spmf-of-set* A) = *spmf-of-set* (f ‘ A)
⟨*proof*⟩

lemma *spmf-of-pmf-pmf-of-set* [*simp*]:
⟦ *finite* A; $A \neq \{\}$ ⟧ $\Longrightarrow$ *spmf-of-pmf* (*pmf-of-set* A) = *spmf-of-set* A
⟨*proof*⟩

lemma *weight-spmf-of-set*:
weight-spmf (*spmf-of-set* A) = (*if finite* $A \wedge A \neq \{\}$ *then* *1* *else* *0*)
⟨*proof*⟩

lemma *weight-spmf-of-set-finite* [*simp*]: ⟦ *finite* A; $A \neq \{\}$ ⟧ $\Longrightarrow$ *weight-spmf* (*spmf-of-set* A) = *1*
⟨*proof*⟩

lemma *weight-spmf-of-set-infinite* [*simp*]: *infinite* $A \Longrightarrow$ *weight-spmf* (*spmf-of-set* A) = *0*
⟨*proof*⟩

lemma *measure-spmf-spmf-of-set*:
measure-spmf (*spmf-of-set* A) = (*if finite* $A \wedge A \neq \{\}$ *then measure-pmf* (*pmf-of-set* A) *else null-measure* (*count-space UNIV*))
⟨*proof*⟩

lemma *emeasure-spmf-of-set*:
emeasure (*measure-spmf* (*spmf-of-set* S)) A = *card* ($S \cap A$) / *card* S
⟨*proof*⟩

lemma *measure-spmf-of-set*:
measure (*measure-spmf* (*spmf-of-set* S)) A = *card* ($S \cap A$) / *card* S
⟨*proof*⟩

lemma *nn-integral-spmf-of-set*: *nn-integral* (*measure-spmf* (*spmf-of-set* A)) f = *sum* f A / *card* A
⟨*proof*⟩

lemma *integral-spmf-of-set*: *integral*^{L} (*measure-spmf* (*spmf-of-set* A)) f = *sum* f A / *card* A

⟨*proof*⟩

notepad begin — *pmf-of-set* is not fully parametric.
⟨*proof*⟩
end

lemma *rel-pmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *A*: $A \neq \{\}$ *finite A*
and *B*: $B \neq \{\}$ *finite B*
and *R*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$
shows *rel-pmf R* (*pmf-of-set A*) (*pmf-of-set B*)
⟨*proof*⟩

lemma *rel-spmf-of-set-bij*:
assumes *f*: *bij-betw f A B*
and *R*: $\bigwedge x.\ x \in A \Longrightarrow R\ x\ (f\ x)$
shows *rel-spmf R* (*spmf-of-set A*) (*spmf-of-set B*)
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *rel-spmf-of-set*:
assumes *bi-unique R*
shows (*rel-set R* ===> *rel-spmf R*) *spmf-of-set spmf-of-set*
⟨*proof*⟩

end

lemma *map-mem-spmf-of-set*:
assumes *finite B* $B \neq \{\}$
shows *map-spmf* ($\lambda x.\ x \in A$) (*spmf-of-set B*) = *spmf-of-pmf* (*bernoulli-pmf* (*card* ($A \cap B$) / *card B*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

abbreviation *coin-spmf* :: *bool spmf*
where *coin-spmf* ≡ *spmf-of-set UNIV*

lemma *map-eq-const-coin-spmf*: *map-spmf* ((=) *c*) *coin-spmf* = *coin-spmf*
⟨*proof*⟩

lemma *bind-coin-spmf-eq-const*: *coin-spmf* ⤜ (λx :: *bool*. *return-spmf* ($b = x$)) = *coin-spmf*
⟨*proof*⟩

lemma *bind-coin-spmf-eq-const'*: *coin-spmf* ⤜ (λx :: *bool*. *return-spmf* ($x = b$)) = *coin-spmf*

⟨*proof*⟩

25.14 Losslessness

definition *lossless-spmf* :: *'a spmf* ⇒ *bool*
where *lossless-spmf p* ⟷ *weight-spmf p* = *1*

lemma *lossless-iff-pmf-None*: *lossless-spmf p* ⟷ *pmf p None* = *0*
⟨*proof*⟩

lemma *lossless-return-spmf* [*iff*]: *lossless-spmf* (*return-spmf x*)
⟨*proof*⟩

lemma *lossless-return-pmf-None* [*iff*]: ¬ *lossless-spmf* (*return-pmf None*)
⟨*proof*⟩

lemma *lossless-map-spmf* [*simp*]: *lossless-spmf* (*map-spmf f p*) ⟷ *lossless-spmf p*
⟨*proof*⟩

lemma *lossless-bind-spmf* [*simp*]:
lossless-spmf (*p* ⤜ *f*) ⟷ *lossless-spmf p* ∧ (∀ *x*∈*set-spmf p*. *lossless-spmf* (*f x*))
⟨*proof*⟩

lemma *lossless-weight-spmfD*: *lossless-spmf p* ⟹ *weight-spmf p* = *1*
⟨*proof*⟩

lemma *lossless-iff-set-pmf-None*:
lossless-spmf p ⟷ *None* ∉ *set-pmf p*
⟨*proof*⟩

lemma *lossless-spmf-of-set* [*simp*]: *lossless-spmf* (*spmf-of-set A*) ⟷ *finite A* ∧ *A* ≠ {}
⟨*proof*⟩

lemma *lossless-spmf-spmf-of-spmf* [*simp*]: *lossless-spmf* (*spmf-of-pmf p*)
⟨*proof*⟩

lemma *lossless-spmf-bind-pmf* [*simp*]:
lossless-spmf (*bind-pmf p f*) ⟷ (∀ *x*∈*set-pmf p*. *lossless-spmf* (*f x*))
⟨*proof*⟩

lemma *lossless-spmf-conv-spmf-of-pmf*: *lossless-spmf p* ⟷ (∃ *p'*. *p* = *spmf-of-pmf p'*)
⟨*proof*⟩

lemma *spmf-False-conv-True*: *lossless-spmf p* ⟹ *spmf p False* = *1* − *spmf p True*
⟨*proof*⟩

lemma *spmf-True-conv-False*: *lossless-spmf p* $\Longrightarrow$ *spmf p True* = *1* − *spmf p False*
⟨*proof*⟩

lemma *bind-eq-return-spmf*:
bind-spmf p f = *return-spmf x* $\longleftrightarrow$ ($\forall$ *y*$\in$*set-spmf p*. *f y* = *return-spmf x*) $\wedge$ *lossless-spmf p*
⟨*proof*⟩

lemma *rel-spmf-return-spmf2*:
rel-spmf R p (*return-spmf x*) $\longleftrightarrow$ *lossless-spmf p* $\wedge$ ($\forall$ *a*$\in$*set-spmf p*. *R a x*)
⟨*proof*⟩

lemma *rel-spmf-return-spmf1*:
rel-spmf R (*return-spmf x*) *p* $\longleftrightarrow$ *lossless-spmf p* $\wedge$ ($\forall$ *a*$\in$*set-spmf p*. *R x a*)
⟨*proof*⟩

lemma *rel-spmf-bindI1*:
assumes *f*: $\bigwedge$*x*. *x* $\in$ *set-spmf p* $\Longrightarrow$ *rel-spmf R* (*f x*) *q*
and *p*: *lossless-spmf p*
shows *rel-spmf R* (*bind-spmf p f*) *q*
⟨*proof*⟩

lemma *rel-spmf-bindI2*:
⟦ $\bigwedge$*x*. *x* $\in$ *set-spmf q* $\Longrightarrow$ *rel-spmf R p* (*f x*); *lossless-spmf q* ⟧
$\Longrightarrow$ *rel-spmf R p* (*bind-spmf q f*)
⟨*proof*⟩

25.15 Scaling

definition *scale-spmf* :: *real* $\Rightarrow$ *'a spmf* $\Rightarrow$ *'a spmf*
where
scale-spmf r p = *embed-spmf* (λ*x*. *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗ *spmf p x*)

lemma *scale-spmf-le-1*:
($\int^+$ *x*. *min* (*inverse* (*weight-spmf p*)) (*max 0 r*) ∗ *spmf p x* ∂*count-space UNIV*) $\leq$ *1* (**is** *?lhs* $\leq$ -)
⟨*proof*⟩

lemma *spmf-scale-spmf*: *spmf* (*scale-spmf r p*) *x* = *max 0* (*min* (*inverse* (*weight-spmf p*)) *r*) ∗ *spmf p x* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *real-inverse-le-1-iff*: **fixes** *x* :: *real*
shows ⟦ *0* $\leq$ *x*; *x* $\leq$ *1* ⟧ $\Longrightarrow$ *1* / *x* $\leq$ *1* $\longleftrightarrow$ *x* = *1* $\vee$ *x* = *0*
⟨*proof*⟩

lemma *spmf-scale-spmf'*: *r* $\leq$ *1* $\Longrightarrow$ *spmf* (*scale-spmf r p*) *x* = *max 0 r* ∗ *spmf p*

x
⟨*proof*⟩

lemma *scale-spmf-neg*: $r \le 0 \Longrightarrow$ *scale-spmf r p* = *return-pmf None*
⟨*proof*⟩

lemma *scale-spmf-return-None* [*simp*]: *scale-spmf r* (*return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *scale-spmf-conv-bind-bernoulli*:
assumes $r \le 1$
shows *scale-spmf r p* = *bind-pmf* (*bernoulli-pmf r*) (λb. *if b then p else return-pmf None*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *nn-integral-spmf*: ($\int^+ x$. *spmf p x* ∂*count-space A*) = *emeasure* (*measure-spmf p*) *A*
⟨*proof*⟩

lemma *measure-spmf-scale-spmf*: *measure-spmf* (*scale-spmf r p*) = *scale-measure* (*min* (*inverse* (*weight-spmf p*)) *r*) (*measure-spmf p*)
⟨*proof*⟩

lemma *measure-spmf-scale-spmf'*:
$r \le 1 \Longrightarrow$ *measure-spmf* (*scale-spmf r p*) = *scale-measure r* (*measure-spmf p*)
⟨*proof*⟩

lemma *scale-spmf-1* [*simp*]: *scale-spmf 1 p* = *p*
⟨*proof*⟩

lemma *scale-spmf-0* [*simp*]: *scale-spmf 0 p* = *return-pmf None*
⟨*proof*⟩

lemma *bind-scale-spmf*:
assumes *r*: $r \le 1$
shows *bind-spmf* (*scale-spmf r p*) *f* = *bind-spmf p* (λx. *scale-spmf r* (*f x*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *scale-bind-spmf*:
assumes $r \le 1$
shows *scale-spmf r* (*bind-spmf p f*) = *bind-spmf p* (λx. *scale-spmf r* (*f x*))
(**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *bind-spmf-const*: *bind-spmf p* (λx. *q*) = *scale-spmf* (*weight-spmf p*) *q* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *map-scale-spmf*: *map-spmf f* (*scale-spmf r p*) = *scale-spmf r* (*map-spmf f p*) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *set-scale-spmf*: *set-spmf* (*scale-spmf r p*) = (*if r* > *0 then set-spmf p else* {})
⟨*proof*⟩

lemma *set-scale-spmf'* [*simp*]: *0* < *r* ⟹ *set-spmf* (*scale-spmf r p*) = *set-spmf p*
⟨*proof*⟩

lemma *rel-spmf-scaleI*:
  assumes *r* > *0* ⟹ *rel-spmf A p q*
  shows *rel-spmf A* (*scale-spmf r p*) (*scale-spmf r q*)
⟨*proof*⟩

lemma *weight-scale-spmf*: *weight-spmf* (*scale-spmf r p*) = *min 1* (*max 0 r* ∗ *weight-spmf p*)
⟨*proof*⟩

lemma *weight-scale-spmf'* [*simp*]:
  ⟦ *0* ≤ *r*; *r* ≤ *1* ⟧ ⟹ *weight-spmf* (*scale-spmf r p*) = *r* ∗ *weight-spmf p*
⟨*proof*⟩

lemma *pmf-scale-spmf-None*:
  pmf (*scale-spmf k p*) *None* = *1* − *min 1* (*max 0 k* ∗ (*1* − *pmf p None*))
⟨*proof*⟩

lemma *scale-scale-spmf*:
  scale-spmf r (*scale-spmf r' p*) = *scale-spmf* (*r* ∗ *max 0* (*min* (*inverse* (*weight-spmf p*)) *r'*)) *p*
  (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *scale-scale-spmf'* [*simp*]:
  ⟦ *0* ≤ *r*; *r* ≤ *1*; *0* ≤ *r'*; *r'* ≤ *1* ⟧
  ⟹ *scale-spmf r* (*scale-spmf r' p*) = *scale-spmf* (*r* ∗ *r'*) *p*
⟨*proof*⟩

lemma *scale-spmf-eq-same*: *scale-spmf r p* = *p* ⟷ *weight-spmf p* = *0* ∨ *r* = *1* ∨ *r* ≥ *1* ∧ *weight-spmf p* = *1*
  (**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *map-const-spmf-of-set*:
  ⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *map-spmf* (λ-. *c*) (*spmf-of-set A*) = *return-spmf c*
⟨*proof*⟩

25.16 Conditional spmfs

lemma *set-pmf-Int-Some*: $set\text{-}pmf\ p \cap Some\ `\ A = \{\} \longleftrightarrow set\text{-}spmf\ p \cap A = \{\}$
⟨*proof*⟩

lemma *measure-spmf-zero-iff*: $measure\ (measure\text{-}spmf\ p)\ A = 0 \longleftrightarrow set\text{-}spmf\ p \cap A = \{\}$
⟨*proof*⟩

definition $cond\text{-}spmf :: 'a\ spmf \Rightarrow 'a\ set \Rightarrow 'a\ spmf$
where $cond\text{-}spmf\ p\ A = (if\ set\text{-}spmf\ p \cap A = \{\}\ then\ return\text{-}pmf\ None\ else\ cond\text{-}pmf\ p\ (Some\ `\ A))$

lemma *set-cond-spmf* [*simp*]: $set\text{-}spmf\ (cond\text{-}spmf\ p\ A) = set\text{-}spmf\ p \cap A$
⟨*proof*⟩

lemma *cond-map-spmf* [*simp*]: $cond\text{-}spmf\ (map\text{-}spmf\ f\ p)\ A = map\text{-}spmf\ f\ (cond\text{-}spmf\ p\ (f -`\ A))$
⟨*proof*⟩

lemma *spmf-cond-spmf* [*simp*]:
$spmf\ (cond\text{-}spmf\ p\ A)\ x = (if\ x \in A\ then\ spmf\ p\ x\ /\ measure\ (measure\text{-}spmf\ p)\ A\ else\ 0)$
⟨*proof*⟩

lemma *bind-eq-return-pmf-None*:
$bind\text{-}spmf\ p\ f = return\text{-}pmf\ None \longleftrightarrow (\forall x \in set\text{-}spmf\ p.\ f\ x = return\text{-}pmf\ None)$
⟨*proof*⟩

lemma *return-pmf-None-eq-bind*:
$return\text{-}pmf\ None = bind\text{-}spmf\ p\ f \longleftrightarrow (\forall x \in set\text{-}spmf\ p.\ f\ x = return\text{-}pmf\ None)$
⟨*proof*⟩

25.17 Product spmf

definition $pair\text{-}spmf :: 'a\ spmf \Rightarrow 'b\ spmf \Rightarrow ('a \times 'b)\ spmf$
where $pair\text{-}spmf\ p\ q = bind\text{-}pmf\ (pair\text{-}pmf\ p\ q)\ (\lambda xy.\ case\ xy\ of\ (Some\ x,\ Some\ y) \Rightarrow return\text{-}spmf\ (x,\ y)\ |\ \text{-} \Rightarrow return\text{-}pmf\ None)$

lemma *map-fst-pair-spmf* [*simp*]: $map\text{-}spmf\ fst\ (pair\text{-}spmf\ p\ q) = scale\text{-}spmf\ (weight\text{-}spmf\ q)\ p$
⟨*proof*⟩

lemma *map-snd-pair-spmf* [*simp*]: $map\text{-}spmf\ snd\ (pair\text{-}spmf\ p\ q) = scale\text{-}spmf\ (weight\text{-}spmf\ p)\ q$
⟨*proof*⟩

lemma *set-pair-spmf* [*simp*]: $set\text{-}spmf\ (pair\text{-}spmf\ p\ q) = set\text{-}spmf\ p \times set\text{-}spmf\ q$
⟨*proof*⟩

lemma *spmf-pair* [*simp*]: *spmf* (*pair-spmf p q*) (*x*, *y*) = *spmf p x* ∗ *spmf q y* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *pair-map-spmf2*: *pair-spmf p* (*map-spmf f q*) = *map-spmf* (*apsnd f*) (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-map-spmf1*: *pair-spmf* (*map-spmf f p*) *q* = *map-spmf* (*apfst f*) (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-map-spmf*: *pair-spmf* (*map-spmf f p*) (*map-spmf g q*) = *map-spmf* (*map-prod f g*) (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-spmf-alt-def*: *pair-spmf p q* = *bind-spmf p* (λ*x*. *bind-spmf q* (λ*y*. *return-spmf* (*x*, *y*)))
⟨*proof*⟩

lemma *weight-pair-spmf* [*simp*]: *weight-spmf* (*pair-spmf p q*) = *weight-spmf p* ∗ *weight-spmf q*
⟨*proof*⟩

lemma *pair-scale-spmf1*:
  r ≤ *1* ⟹ *pair-spmf* (*scale-spmf r p*) *q* = *scale-spmf r* (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-scale-spmf2*:
  r ≤ *1* ⟹ *pair-spmf p* (*scale-spmf r q*) = *scale-spmf r* (*pair-spmf p q*)
⟨*proof*⟩

lemma *pair-spmf-return-None1* [*simp*]: *pair-spmf* (*return-pmf None*) *p* = *return-pmf None*
⟨*proof*⟩

lemma *pair-spmf-return-None2* [*simp*]: *pair-spmf p* (*return-pmf None*) = *return-pmf None*
⟨*proof*⟩

lemma *pair-spmf-return-spmf1*: *pair-spmf* (*return-spmf x*) *q* = *map-spmf* (*Pair x*) *q*
⟨*proof*⟩

lemma *pair-spmf-return-spmf2*: *pair-spmf p* (*return-spmf y*) = *map-spmf* (λ*x*. (*x*, *y*)) *p*
⟨*proof*⟩

lemma *pair-spmf-return-spmf* [*simp*]: *pair-spmf* (*return-spmf x*) (*return-spmf y*)

= *return-spmf* (x, y)
⟨*proof*⟩

lemma *rel-pair-spmf-prod*:
rel-spmf (*rel-prod A B*) (*pair-spmf p q*) (*pair-spmf p′ q′*) ⟷
rel-spmf A (*scale-spmf* (*weight-spmf q*) *p*) (*scale-spmf* (*weight-spmf q′*) *p′*) ∧
rel-spmf B (*scale-spmf* (*weight-spmf p*) *q*) (*scale-spmf* (*weight-spmf p′*) *q′*)
(**is** *?lhs* ⟷ *?rhs* **is** - ⟷ *?A* ∧ *?B* **is** - ⟷ *rel-spmf* - *?p* *?p′* ∧ *rel-spmf* - *?q* *?q′*)
⟨*proof*⟩

lemma *pair-pair-spmf*:
pair-spmf (*pair-spmf p q*) *r* = *map-spmf* $(\lambda(x, (y, z)).\ ((x, y), z))$ (*pair-spmf p* (*pair-spmf q r*))
⟨*proof*⟩

lemma *pair-commute-spmf*:
pair-spmf p q = *map-spmf* $(\lambda(y, x).\ (x, y))$ (*pair-spmf q p*)
⟨*proof*⟩

25.18 Assertions

definition *assert-spmf* :: *bool* ⇒ *unit spmf*
where *assert-spmf b* = (*if b then return-spmf* () *else return-pmf None*)

lemma *assert-spmf-simps* [*simp*]:
assert-spmf True = *return-spmf* ()
assert-spmf False = *return-pmf None*
⟨*proof*⟩

lemma *in-set-assert-spmf* [*simp*]: *x* ∈ *set-spmf* (*assert-spmf p*) ⟷ *p*
⟨*proof*⟩

lemma *set-spmf-assert-spmf-eq-empty* [*simp*]: *set-spmf* (*assert-spmf b*) = {} ⟷ ¬ *b*
⟨*proof*⟩

lemma *lossless-assert-spmf* [*iff*]: *lossless-spmf* (*assert-spmf b*) ⟷ *b*
⟨*proof*⟩

25.19 Try

definition *try-spmf* :: *′a spmf* ⇒ *′a spmf* ⇒ *′a spmf* (*TRY* - *ELSE* - [*0*,*60*] *59*)
where *try-spmf p q* = *bind-pmf p* (λ*x*. *case x of None* ⇒ *q* | *Some y* ⇒ *return-spmf y*)

lemma *try-spmf-lossless* [*simp*]:
assumes *lossless-spmf p*
shows *TRY p ELSE q* = *p*
⟨*proof*⟩

lemma *try-spmf-return-spmf1*: *TRY return-spmf x ELSE q = return-spmf x*
⟨*proof*⟩

lemma *try-spmf-return-None* [*simp*]: *TRY return-pmf None ELSE q = q*
⟨*proof*⟩

lemma *try-spmf-return-pmf-None2* [*simp*]: *TRY p ELSE return-pmf None = p*
⟨*proof*⟩

lemma *map-try-spmf*: *map-spmf f (try-spmf p q) = try-spmf (map-spmf f p) (map-spmf f q)*
⟨*proof*⟩

lemma *try-spmf-bind-pmf*: *TRY (bind-pmf p f) ELSE q = bind-pmf p* (λx. *TRY (f x) ELSE q)*
⟨*proof*⟩

lemma *try-spmf-bind-spmf-lossless*:
lossless-spmf p $\Longrightarrow$ *TRY (bind-spmf p f) ELSE q = bind-spmf p* (λx. *TRY (f x) ELSE q)*
⟨*proof*⟩

lemma *try-spmf-bind-out*:
lossless-spmf p $\Longrightarrow$ *bind-spmf p* (λx. *TRY (f x) ELSE q) = TRY (bind-spmf p f) ELSE q*
⟨*proof*⟩

lemma *lossless-try-spmf* [*simp*]:
lossless-spmf (TRY p ELSE q) $\longleftrightarrow$ *lossless-spmf p* $\vee$ *lossless-spmf q*
⟨*proof*⟩

context includes *lifting-syntax*
begin

lemma *try-spmf-parametric* [*transfer-rule*]:
(rel-spmf A ===> rel-spmf A ===> rel-spmf A) try-spmf try-spmf
⟨*proof*⟩

end

lemma *try-spmf-cong*:
$[\![$ *p = p'*; $\neg$ *lossless-spmf p'* $\Longrightarrow$ *q = q'* $]\!]$ $\Longrightarrow$ *TRY p ELSE q = TRY p' ELSE q'*
⟨*proof*⟩

lemma *rel-spmf-try-spmf*:
$[\![$ *rel-spmf R p p'*; $\neg$ *lossless-spmf p'* $\Longrightarrow$ *rel-spmf R q q'* $]\!]$
$\Longrightarrow$ *rel-spmf R (TRY p ELSE q) (TRY p' ELSE q')*
⟨*proof*⟩

lemma *spmf-try-spmf*:
spmf (*TRY p ELSE q*) x = *spmf p x* + *pmf p None* ∗ *spmf q x*
⟨*proof*⟩

lemma *try-scale-spmf-same* [*simp*]: *lossless-spmf p* ⟹ *TRY scale-spmf k p ELSE p* = *p*
⟨*proof*⟩

lemma *pmf-try-spmf-None* [*simp*]: *pmf* (*TRY p ELSE q*) *None* = *pmf p None* ∗ *pmf q None* (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

lemma *try-bind-spmf-lossless2*:
lossless-spmf q ⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* ≫= (λ*x*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
⟨*proof*⟩

lemma *try-bind-spmf-lossless2'*:
fixes *f* :: '*a* ⇒ '*b spmf* **shows**
⟦ *NO-MATCH* (λ*x* :: '*a*. *try-spmf* (*g x* :: '*b spmf*) (*h x*)) *f*; *lossless-spmf q* ⟧
⟹ *TRY* (*bind-spmf p f*) *ELSE q* = *TRY* (*p* ≫= (λ*x* :: '*a*. *TRY* (*f x*) *ELSE q*)) *ELSE q*
⟨*proof*⟩

lemma *try-bind-assert-spmf*:
TRY (*assert-spmf b* ≫= *f*) *ELSE q* = (*if b then TRY* (*f* ()) *ELSE q else q*)
⟨*proof*⟩

25.20 Miscellaneous

lemma assumes *rel-spmf* (λ*x y*. *bad1 x* = *bad2 y* ∧ (¬ *bad2 y* ⟶ *A x* ⟷ *B y*)) *p q* (**is** *rel-spmf ?A - -*)
shows *fundamental-lemma-bad*: *measure* (*measure-spmf p*) {*x*. *bad1 x*} = *measure* (*measure-spmf q*) {*y*. *bad2 y*} (**is** *?bad*)
and *fundamental-lemma*: |*measure* (*measure-spmf p*) {*x*. *A x*} − *measure* (*measure-spmf q*) {*y*. *B y*}| ≤
measure (*measure-spmf p*) {*x*. *bad1 x*} (**is** *?fundamental*)
⟨*proof*⟩

end

26 Indexed products of PMFs

theory *Product-PMF*
imports *Probability-Mass-Function Independent-Family*
begin

26.1 Preliminaries

lemma *pmf-expectation-eq-infsetsum*: *measure-pmf.expectation p f* = *infsetsum* (λx. *pmf p x* $*$ *f x*) *UNIV*
⟨*proof*⟩

lemma *measure-pmf-prob-product*:
assumes *countable A countable B*
shows *measure-pmf.prob* (*pair-pmf M N*) ($A \times B$) = *measure-pmf.prob M A* $*$ *measure-pmf.prob N B*
⟨*proof*⟩

26.2 Definition

In analogy to Pi_M, we define an indexed product of PMFs. In the literature, this is typically called taking a vector of independent random variables. Note that the components do not have to be identically distributed.

The operation takes an explicit index set A and a function f that maps each element from A to a PMF and defines the product measure $\bigotimes_{i \in A} f(i)$, which is represented as a ($'a \Rightarrow 'b$) *pmf*.

Note that unlike Pi_M, this only works for *finite* index sets. It could be extended to countable sets and beyond, but the construction becomes somewhat more involved.

definition *Pi-pmf* :: $'a$ *set* $\Rightarrow$ $'b$ $\Rightarrow$ ($'a \Rightarrow 'b$ *pmf*) $\Rightarrow$ ($'a \Rightarrow 'b$) *pmf* **where**
Pi-pmf A dflt p =
embed-pmf (λf. *if* ($\forall x.\ x \notin A \longrightarrow f\ x = dflt$) *then* $\prod x \in A$. *pmf* (*p x*) (*f x*) *else 0*)

A technical subtlety that needs to be addressed is this: Intuitively, the functions in the support of a product distribution have domain A. However, since HOL is a total logic, these functions must still return *some* value for inputs outside A. The product measure Pi_M simply lets these functions return *undefined* in these cases. We chose a different solution here, which is to supply a default value *dflt* that is returned in these cases.

As one possible application, one could model the result of n different independent coin tosses as *Pi-pmf* {*0*::$'a$..<*n*} *False* (λ-. *bernoulli-pmf* (*1* / *2*)). This returns a function of type *nat* $\Rightarrow$ *bool* that maps every natural number below n to the result of the corresponding coin toss, and every other natural number to *False*.

lemma *pmf-Pi*:
assumes *A*: *finite A*
shows *pmf* (*Pi-pmf A dflt p*) *f* =
(*if* ($\forall x.\ x \notin A \longrightarrow f\ x = dflt$) *then* $\prod x \in A$. *pmf* (*p x*) (*f x*) *else 0*)
⟨*proof*⟩

lemma *Pi-pmf-cong*:

assumes $A = A'$ $dflt = dflt'$ $\bigwedge x.\ x \in A \Longrightarrow f\ x = f'\ x$
shows $Pi\text{-}pmf\ A\ dflt\ f = Pi\text{-}pmf\ A'\ dflt'\ f'$
⟨*proof*⟩

lemma *pmf-Pi'*:
assumes $finite\ A$ $\bigwedge x.\ x \notin A \Longrightarrow f\ x = dflt$
shows $pmf\ (Pi\text{-}pmf\ A\ dflt\ p)\ f = (\prod x{\in}A.\ pmf\ (p\ x)\ (f\ x))$
⟨*proof*⟩

lemma *pmf-Pi-outside*:
assumes $finite\ A$ $\exists x.\ x \notin A \land f\ x \neq dflt$
shows $pmf\ (Pi\text{-}pmf\ A\ dflt\ p)\ f = 0$
⟨*proof*⟩

lemma *pmf-Pi-empty* [*simp*]: $Pi\text{-}pmf\ \{\}\ dflt\ p = return\text{-}pmf\ (\lambda\text{-}.\ dflt)$
⟨*proof*⟩

lemma *set-Pi-pmf-subset*: $finite\ A \Longrightarrow set\text{-}pmf\ (Pi\text{-}pmf\ A\ dflt\ p) \subseteq \{f.\ \forall x.\ x \notin A \longrightarrow f\ x = dflt\}$
⟨*proof*⟩

26.3 Dependent product sets with a default

The following describes a dependent product of sets where the functions are required to return the default value *dflt* outside their domain, in analogy to Pi_E, which uses *undefined*.

definition *PiE-dflt*
where $PiE\text{-}dflt\ A\ dflt\ B = \{f.\ \forall x.\ (x \in A \longrightarrow f\ x \in B\ x) \land (x \notin A \longrightarrow f\ x = dflt)\}$

lemma *restrict-PiE-dflt*: $(\lambda h.\ restrict\ h\ A)$ ` $PiE\text{-}dflt\ A\ dflt\ B = PiE\ A\ B$
⟨*proof*⟩

lemma *dflt-image-PiE*: $(\lambda h\ x.\ if\ x \in A\ then\ h\ x\ else\ dflt)$ ` $PiE\ A\ B = PiE\text{-}dflt\ A\ dflt\ B$
(**is** ?f ` ?X = ?Y)
⟨*proof*⟩

lemma *finite-PiE-dflt* [*intro*]:
assumes $finite\ A$ $\bigwedge x.\ x \in A \Longrightarrow finite\ (B\ x)$
shows $finite\ (PiE\text{-}dflt\ A\ d\ B)$
⟨*proof*⟩

lemma *card-PiE-dflt*:
assumes $finite\ A$ $\bigwedge x.\ x \in A \Longrightarrow finite\ (B\ x)$
shows $card\ (PiE\text{-}dflt\ A\ d\ B) = (\prod x{\in}A.\ card\ (B\ x))$
⟨*proof*⟩

lemma *PiE-dflt-empty-iff* [*simp*]: $PiE\text{-}dflt\ A\ dflt\ B = \{\} \longleftrightarrow (\exists x{\in}A.\ B\ x = \{\})$

⟨*proof*⟩

lemma *set-Pi-pmf-subset'*:
assumes *finite A*
shows $set\text{-}pmf\ (Pi\text{-}pmf\ A\ dflt\ p) \subseteq PiE\text{-}dflt\ A\ dflt\ (set\text{-}pmf \circ p)$
⟨*proof*⟩

lemma *set-Pi-pmf*:
assumes *finite A*
shows $set\text{-}pmf\ (Pi\text{-}pmf\ A\ dflt\ p) = PiE\text{-}dflt\ A\ dflt\ (set\text{-}pmf \circ p)$
⟨*proof*⟩

The probability of an independent combination of events is precisely the product of the probabilities of each individual event.

lemma *measure-Pi-pmf-PiE-dflt*:
assumes [*simp*]: *finite A*
shows $measure\text{-}pmf.prob\ (Pi\text{-}pmf\ A\ dflt\ p)\ (PiE\text{-}dflt\ A\ dflt\ B) = (\prod x{\in}A.\ measure\text{-}pmf.prob\ (p\ x)\ (B\ x))$
⟨*proof*⟩

lemma *measure-Pi-pmf-Pi*:
fixes *t*::*nat*
assumes [*simp*]: *finite A*
shows $measure\text{-}pmf.prob\ (Pi\text{-}pmf\ A\ dflt\ p)\ (Pi\ A\ B) = (\prod x{\in}A.\ measure\text{-}pmf.prob\ (p\ x)\ (B\ x))$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

26.4 Common PMF operations on products

Pi-pmf distributes over the 'bind' operation in the Giry monad:

lemma *Pi-pmf-bind*:
assumes *finite A*
shows $Pi\text{-}pmf\ A\ d\ (\lambda x.\ bind\text{-}pmf\ (p\ x)\ (q\ x)) = do\ \{f \leftarrow Pi\text{-}pmf\ A\ d'\ p;\ Pi\text{-}pmf\ A\ d\ (\lambda x.\ q\ x\ (f\ x))\}$ (**is** *?lhs = ?rhs*)
⟨*proof*⟩

lemma *Pi-pmf-return-pmf* [*simp*]:
assumes *finite A*
shows $Pi\text{-}pmf\ A\ dflt\ (\lambda x.\ return\text{-}pmf\ (f\ x)) = return\text{-}pmf\ (\lambda x.\ if\ x \in A\ then\ f\ x\ else\ dflt)$
⟨*proof*⟩

Analogously any componentwise mapping can be pulled outside the product:

lemma *Pi-pmf-map*:
assumes [*simp*]: *finite A* **and** $f\ dflt = dflt'$
shows $Pi\text{-}pmf\ A\ dflt'\ (\lambda x.\ map\text{-}pmf\ f\ (g\ x)) = map\text{-}pmf\ (\lambda h.\ f \circ h)\ (Pi\text{-}pmf\ A\ dflt\ g)$
⟨*proof*⟩

We can exchange the default value in a product of PMFs like this:

lemma *Pi-pmf-default-swap*:
assumes *finite A*
shows $map\text{-}pmf\ (\lambda f\ x.\ if\ x \in A\ then\ f\ x\ else\ dflt')\ (Pi\text{-}pmf\ A\ dflt\ p) = Pi\text{-}pmf\ A\ dflt'\ p$ (**is** $?lhs = ?rhs$)
⟨*proof*⟩

The following rule allows reindexing the product:

lemma *Pi-pmf-bij-betw*:
assumes $finite\ A\ bij\text{-}betw\ h\ A\ B\ \bigwedge x.\ x \notin A \Longrightarrow h\ x \notin B$
shows $Pi\text{-}pmf\ A\ dflt\ (\lambda\text{-}.\ f) = map\text{-}pmf\ (\lambda g.\ g \circ h)\ (Pi\text{-}pmf\ B\ dflt\ (\lambda\text{-}.\ f))$
(**is** $?lhs = ?rhs$)
⟨*proof*⟩

A product of uniform random choices is again a uniform distribution.

lemma *Pi-pmf-of-set*:
assumes $finite\ A\ \bigwedge x.\ x \in A \Longrightarrow finite\ (B\ x)\ \bigwedge x.\ x \in A \Longrightarrow B\ x \neq \{\}$
shows $Pi\text{-}pmf\ A\ d\ (\lambda x.\ pmf\text{-}of\text{-}set\ (B\ x)) = pmf\text{-}of\text{-}set\ (PiE\text{-}dflt\ A\ d\ B)$ (**is** $?lhs = ?rhs$)
⟨*proof*⟩

26.5 Merging and splitting PMF products

The following lemma shows that we can add a single PMF to a product:

lemma *Pi-pmf-insert*:
assumes $finite\ A\ x \notin A$
shows $Pi\text{-}pmf\ (insert\ x\ A)\ dflt\ p = map\text{-}pmf\ (\lambda(y,f).\ f(x:=y))\ (pair\text{-}pmf\ (p\ x)\ (Pi\text{-}pmf\ A\ dflt\ p))$
⟨*proof*⟩

lemma *Pi-pmf-insert'*:
assumes $finite\ A\ \ x \notin A$
shows $Pi\text{-}pmf\ (insert\ x\ A)\ dflt\ p =$
$do\ \{y \leftarrow p\ x;\ f \leftarrow Pi\text{-}pmf\ A\ dflt\ p;\ return\text{-}pmf\ (f(x := y))\}$
⟨*proof*⟩

lemma *Pi-pmf-singleton*:
$Pi\text{-}pmf\ \{x\}\ dflt\ p = map\text{-}pmf\ (\lambda a\ b.\ if\ b = x\ then\ a\ else\ dflt)\ (p\ x)$
⟨*proof*⟩

Projecting a product of PMFs onto a component yields the expected result:

lemma *Pi-pmf-component*:
assumes *finite A*
shows $map\text{-}pmf\ (\lambda f.\ f\ x)\ (Pi\text{-}pmf\ A\ dflt\ p) = (if\ x \in A\ then\ p\ x\ else\ return\text{-}pmf\ dflt)$
⟨*proof*⟩

We can take merge two PMF products on disjoint sets like this:

lemma *Pi-pmf-union*:
  assumes *finite A finite B* $A \cap B = \{\}$
  shows *Pi-pmf* $(A \cup B)$ *dflt p* =
    map-pmf ($\lambda(f,g)$ *x. if* $x \in A$ *then f x else g x*)
    (*pair-pmf* (*Pi-pmf A dflt p*) (*Pi-pmf B dflt p*)) (**is** - = *map-pmf* (*?h A*) (*?q A*))
  ⟨*proof*⟩

We can also project a product to a subset of the indices by mapping all the other indices to the default value:

lemma *Pi-pmf-subset*:
  assumes *finite A* $A' \subseteq A$
  shows *Pi-pmf* A' *dflt p* = *map-pmf* (λf *x. if* $x \in A'$ *then f x else dflt*) (*Pi-pmf A dflt p*)
⟨*proof*⟩

lemma *Pi-pmf-subset'*:
  fixes $f :: 'a \Rightarrow 'b$ *pmf*
  assumes *finite A* $B \subseteq A$ $\bigwedge x.\ x \in A - B \Longrightarrow f\ x =$ *return-pmf dflt*
  shows *Pi-pmf A dflt f* = *Pi-pmf B dflt f*
⟨*proof*⟩

lemma *Pi-pmf-if-set*:
  assumes *finite A*
  shows *Pi-pmf A dflt* (λx. *if b x then f x else return-pmf dflt*) =
    Pi-pmf $\{x \in A.\ b\ x\}$ *dflt f*
⟨*proof*⟩

lemma *Pi-pmf-if-set'*:
  assumes *finite A*
  shows *Pi-pmf A dflt* (λx. *if b x then return-pmf dflt else f x*) =
    Pi-pmf $\{x \in A.\ \neg b\ x\}$ *dflt f*
⟨*proof*⟩

Lastly, we can delete a single component from a product:

lemma *Pi-pmf-remove*:
  assumes *finite A*
  shows *Pi-pmf* $(A - \{x\})$ *dflt p* = *map-pmf* ($\lambda f.\ f(x := dflt)$) (*Pi-pmf A dflt p*)
⟨*proof*⟩

26.6 Additional properties

lemma *nn-integral-prod-Pi-pmf*:
  assumes *finite A*
  shows *nn-integral* (*Pi-pmf A dflt p*) ($\lambda y.\ \prod x \in A.\ f\ x\ (y\ x)$) = ($\prod x \in A.$ *nn-integral* (*p x*) (*f x*))
  ⟨*proof*⟩

lemma *integrable-prod-Pi-pmf*:
fixes $f :: 'a \Rightarrow 'b \Rightarrow 'c :: \{real\text{-}normed\text{-}field,\ second\text{-}countable\text{-}topology,\ banach\}$
assumes *finite A* **and** $\bigwedge x.\ x \in A \Longrightarrow integrable\ (measure\text{-}pmf\ (p\ x))\ (f\ x)$
shows $integrable\ (measure\text{-}pmf\ (Pi\text{-}pmf\ A\ dflt\ p))\ (\lambda h.\ \prod x{\in}A.\ f\ x\ (h\ x))$
⟨*proof*⟩

lemma *expectation-prod-Pi-pmf*:
fixes $f :: \text{-} \Rightarrow \text{-} \Rightarrow real$
assumes *finite A*
assumes $\bigwedge x.\ x \in A \Longrightarrow integrable\ (measure\text{-}pmf\ (p\ x))\ (f\ x)$
assumes $\bigwedge x\ y.\ x \in A \Longrightarrow y \in set\text{-}pmf\ (p\ x) \Longrightarrow f\ x\ y \geq 0$
shows $measure\text{-}pmf.expectation\ (Pi\text{-}pmf\ A\ dflt\ p)\ (\lambda y.\ \prod x{\in}A.\ f\ x\ (y\ x)) =$
$(\prod x{\in}A.\ measure\text{-}pmf.expectation\ (p\ x)\ (\lambda v.\ f\ x\ v))$
⟨*proof*⟩

lemma *indep-vars-Pi-pmf*:
assumes *fin*: *finite I*
shows $prob\text{-}space.indep\text{-}vars\ (measure\text{-}pmf\ (Pi\text{-}pmf\ I\ dflt\ p))$
$(\lambda\text{-}.\ count\text{-}space\ UNIV)\ (\lambda x\ f.\ f\ x)\ I$
⟨*proof*⟩

lemma
fixes $h :: 'a :: comm\text{-}monoid\text{-}add \Rightarrow 'b{::}\{banach,\ second\text{-}countable\text{-}topology\}$
assumes *fin*: *finite I*
assumes *integrable*: $\bigwedge i.\ i \in I \Longrightarrow integrable\ (measure\text{-}pmf\ (D\ i))\ h$
shows *integrable-sum-Pi-pmf*: $integrable\ (Pi\text{-}pmf\ I\ dflt\ D)\ (\lambda g.\ \sum i{\in}I.\ h\ (g\ i))$
and *expectation-sum-Pi-pmf*:
$measure\text{-}pmf.expectation\ (Pi\text{-}pmf\ I\ dflt\ D)\ (\lambda g.\ \sum i{\in}I.\ h\ (g\ i)) =$
$(\sum i{\in}I.\ measure\text{-}pmf.expectation\ (D\ i)\ h)$
⟨*proof*⟩

26.7 Applications

Choosing a subset of a set uniformly at random is equivalent to tossing a fair coin independently for each element and collecting all the elements that came up heads.

lemma *pmf-of-set-Pow-conv-bernoulli*:
assumes $finite\ (A :: 'a\ set)$
shows $map\text{-}pmf\ (\lambda b.\ \{x{\in}A.\ b\ x\})\ (Pi\text{-}pmf\ A\ P\ (\lambda\text{-}.\ bernoulli\text{-}pmf\ (1/2))) = pmf\text{-}of\text{-}set\ (Pow\ A)$
⟨*proof*⟩

A binomial distribution can be seen as the number of successes in n independent coin tosses.

lemma *binomial-pmf-altdef'*:
fixes $A :: 'a\ set$
assumes *finite A* **and** $card\ A = n$ **and** p: $p \in \{0..1\}$
shows $binomial\text{-}pmf\ n\ p =$

map-pmf (λf. *card* $\{x{\in}A.\ f\ x\}$) (*Pi-pmf A dflt* (λ-. *bernoulli-pmf p*)) (**is** *?lhs* = *?rhs*)
⟨*proof*⟩

end

27 Hoeffding's Lemma and Hoeffding's Inequality

theory *Hoeffding*
imports *Product-PMF Independent-Family*
begin

Hoeffding's inequality shows that a sum of bounded independent random variables is concentrated around its mean, with an exponential decay of the tail probabilities.

27.1 Hoeffding's Lemma

lemma *convex-on-exp*:
fixes l :: *real*
assumes $l \geq 0$
shows *convex-on UNIV* (λx. *exp*($l{*}x$))
⟨*proof*⟩

lemma *mult-const-minus-self-real-le*:
fixes x :: *real*
shows $x * (c - x) \leq c^2 / 4$
⟨*proof*⟩

lemma *Hoeffdings-lemma-aux*:
fixes $h\ p$ :: *real*
assumes $h \geq 0$ **and** $p \geq 0$
defines $L \equiv (\lambda h.\ -h * p + ln\ (1 + p * (exp\ h - 1)))$
shows $L\ h \leq h^2 / 8$
⟨*proof*⟩

locale *interval-bounded-random-variable* = *prob-space* +
fixes f :: $'a \Rightarrow$ *real* **and** $a\ b$ :: *real*
assumes *random-variable* [*measurable*]: *random-variable borel f*
assumes *AE-in-interval*: *AE x in M*. $f\ x \in \{a..b\}$
begin

lemma *integrable* [*intro*]: *integrable M f*
⟨*proof*⟩

We first show Hoeffding's lemma for distributions whose expectation is 0. The general case will easily follow from this later.

lemma *Hoeffdings-lemma-nn-integral-0*:
assumes $l > 0$ **and** *E0*: *expectation* $f = 0$
shows *nn-integral* M $(\lambda x.\ exp\ (l * f\ x)) \leq ennreal\ (exp\ (l^2 * (b - a)^2\ /\ 8))$
⟨*proof*⟩

context
begin

interpretation *shift*: *interval-bounded-random-variable* M $\lambda x.\ f\ x - \mu\ a - \mu\ b - \mu$
rewrites $b - \mu - (a - \mu) \equiv b - a$
⟨*proof*⟩

lemma *expectation-shift*: *expectation* $(\lambda x.\ f\ x - expectation\ f) = 0$
⟨*proof*⟩

lemmas *Hoeffdings-lemma-nn-integral* = *shift.Hoeffdings-lemma-nn-integral-0*[*OF* - *expectation-shift*]

end

end

27.2 Hoeffding's Inequality

Consider n independent real random variables $X_1, \ldots, X_n$ that each almost surely lie in a compact interval $[a_i, b_i]$. Hoeffding's inequality states that the distribution of the sum of the X_i is tightly concentrated around the sum of the expected values: the probability of it being above or below the sum of the expected values by more than some ε decreases exponentially with ε.

locale *indep-interval-bounded-random-variables* = *prob-space* +
fixes I :: $'b$ *set* **and** X :: $'b \Rightarrow 'a \Rightarrow real$
fixes $a\ b$ :: $'b \Rightarrow real$
assumes *fin*: *finite* I
assumes *indep*: *indep-vars* $(\lambda$-. *borel*) X I
assumes *AE-in-interval*: $\bigwedge i.\ i \in I \Longrightarrow AE\ x\ in\ M.\ X\ i\ x \in \{a\ i..b\ i\}$
begin

lemma *random-variable* [*measurable*]:
assumes i: $i \in I$
shows *random-variable borel* $(X\ i)$
⟨*proof*⟩

lemma *bounded-random-variable* [*intro*]:
assumes i: $i \in I$
shows *interval-bounded-random-variable* M $(X\ i)$ $(a\ i)$ $(b\ i)$
⟨*proof*⟩

```
end


locale Hoeffding-ineq = indep-interval-bounded-random-variables +
  fixes μ :: real
  defines μ ≡ (∑ i∈I. expectation (X i))
begin

theorem Hoeffding-ineq-ge:
  assumes ε ≥ 0
  assumes (∑ i∈I. (b i − a i)^2) > 0
  shows   prob {x∈space M. (∑ i∈I. X i x) ≥ μ + ε} ≤ exp (−2 ∗ ε^2 / (∑ i∈I.
(b i − a i)^2))
proof (cases ε = 0)
  case [simp]: True
  have prob {x∈space M. (∑ i∈I. X i x) ≥ μ + ε} ≤ 1
    by simp
  thus ?thesis by simp
next
  case False
  with ‹ε ≥ 0› have ε: ε > 0
    by auto

  define d where d = (∑ i∈I. (b i − a i)^2)
  define l :: real where l = 4 ∗ ε / d
  have d: d > 0
    using assms by (simp add: d-def)
  have l: l > 0
    using ε d by (simp add: l-def)
  define μ' where μ' = (λi. expectation (X i))

  have {x∈space M. (∑ i∈I. X i x) ≥ μ + ε} = {x∈space M. (∑ i∈I. X i x) −
μ ≥ ε}
    by (simp add: algebra-simps)
  hence ennreal (prob {x∈space M. (∑ i∈I. X i x) ≥ μ + ε}) = emeasure M ...
    by (simp add: emeasure-eq-measure)
  also have ... ≤ ennreal (exp (−l∗ε)) ∗ (∫⁺x∈space M. exp (l ∗ ((∑ i∈I. X i
x) − μ)) ∂M)
    by (intro Chernoff-ineq-nn-integral-ge l) auto
  also have (λx. (∑ i∈I. X i x) − μ) = (λx. (∑ i∈I. X i x − μ' i))
    by (simp add: μ-def sum-subtractf μ'-def)
  also have (∫⁺x∈space M. exp (l ∗ ((∑ i∈I. X i x − μ' i))) ∂M) =
              (∫⁺x. (∏ i∈I. ennreal (exp (l ∗ (X i x − μ' i)))) ∂M)
    by (intro nn-integral-cong)
      (simp-all add: sum-distrib-left ring-distribs exp-diff exp-sum fin prod-ennreal)
  also have ... = (∏ i∈I. ∫⁺x. ennreal (exp (l ∗ (X i x − μ' i))) ∂M)
    by (intro indep-vars-nn-integral fin indep-vars-compose2[OF indep]) auto
  also have ennreal (exp (−l ∗ ε)) ∗ ... ≤
               ennreal (exp (−l ∗ ε)) ∗ (∏ i∈I. ennreal (exp (l^2 ∗ (b i − a i)^2 / 8)))
```

proof (*intro mult-left-mono prod-mono-ennreal*)
fix *i* **assume** *i*: $i \in I$
from *i* **interpret** *interval-bounded-random-variable M X i a i b i* **..**
show $(\int^+ x.\ ennreal\ (exp\ (l * (X\ i\ x - \mu'\ i)))\ \partial M) \le ennreal\ (exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8))$
unfolding μ'*-def* **by** (*rule Hoeffdings-lemma-nn-integral*) *fact+*
qed *auto*
also have $\ldots = ennreal\ (exp\ (-l{*}\varepsilon) * (\prod i{\in}I.\ exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)))$
by (*simp add*: *prod-ennreal prod-nonneg flip*: *ennreal-mult*)
also have $exp\ (-l{*}\varepsilon) * (\prod i{\in}I.\ exp\ (l^2 * (b\ i - a\ i)^2\ /\ 8)) = exp\ (d * l^2\ /\ 8 - l * \varepsilon)$
by (*simp add*: *exp-diff exp-minus sum-divide-distrib sum-distrib-left sum-distrib-right exp-sum fin divide-simps mult-ac d-def*)
also have $d * l^2\ /\ 8 - l * \varepsilon = -2 * \varepsilon^2\ /\ d$
using *d* **by** (*simp add*: *l-def field-simps power2-eq-square*)
finally show *?thesis*
by (*subst* (*asm*) *ennreal-le-iff*) (*simp-all add*: *d-def*)
qed

corollary *Hoeffding-ineq-le*:
assumes ε: $\varepsilon \ge 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows $prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x) \le \mu - \varepsilon\} \le exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
$\langle proof\rangle$

corollary *Hoeffding-ineq-abs-ge*:
assumes ε: $\varepsilon \ge 0$
assumes $(\sum i{\in}I.\ (b\ i - a\ i)^2) > 0$
shows $prob\ \{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x) - \mu| \ge \varepsilon\} \le 2 * exp\ (-2 * \varepsilon^2\ /\ (\sum i{\in}I.\ (b\ i - a\ i)^2))$
$\langle proof\rangle$

end

27.3 Hoeffding's inequality for i.i.d. bounded random variables

If we have n even identically-distributed random variables, the statement of Hoeffding's lemma simplifies a bit more: it shows that the probability that the average of the X_i is more than ε above the expected value is no greater than $e^{\frac{-2ny^2}{(b-a)^2}}$.

This essentially gives us a more concrete version of the weak law of large numbers: the law states that the probability vanishes for $n \to \infty$ for any $\varepsilon > 0$. Unlike Hoeffding's inequality, it does not assume the variables to have bounded support, but it does not provide concrete bounds.

locale *iid-interval-bounded-random-variables* = *prob-space* +

fixes I :: $'b$ *set* **and** X :: $'b \Rightarrow 'a \Rightarrow$ *real* **and** Y :: $'a \Rightarrow$ *real*
fixes a b :: *real*
assumes *fin*: *finite I*
assumes *indep*: *indep-vars* (λ-. *borel*) X I
assumes *distr-X*: $i \in I \Longrightarrow$ *distr M borel* $(X\ i)$ = *distr M borel Y*
assumes *rv-Y* [*measurable*]: *random-variable borel Y*
assumes *AE-in-interval*: *AE x in M*. $Y\ x \in \{a..b\}$
begin

lemma *random-variable* [*measurable*]:
assumes *i*: $i \in I$
shows *random-variable borel* $(X\ i)$
⟨*proof*⟩

sublocale *X*: *indep-interval-bounded-random-variables M I X* λ-. *a* λ-. *b*
⟨*proof*⟩

lemma *expectation-X* [*simp*]:
assumes *i*: $i \in I$
shows *expectation* $(X\ i)$ = *expectation Y*
⟨*proof*⟩

end

locale *Hoeffding-ineq-iid* = *iid-interval-bounded-random-variables* +
fixes μ :: *real*
defines $\mu \equiv$ *expectation Y*
begin

sublocale *X*: *Hoeffding-ineq M I X* λ-. *a* λ-. *b real* (*card I*) $* \mu$
⟨*proof*⟩

corollary
assumes ε: $\varepsilon \geq 0$
assumes $a < b$ $I \neq \{\}$
defines $n \equiv$ *card I*
shows *Hoeffding-ineq-ge*:
prob $\{x \in space\ M.\ (\sum i \in I.\ X\ i\ x) \geq n * \mu + \varepsilon\} \leq$
$exp\ (-2 * \varepsilon^2 / (n * (b - a)^2))$ (**is** *?le*)
and *Hoeffding-ineq-le*:
prob $\{x \in space\ M.\ (\sum i \in I.\ X\ i\ x) \leq n * \mu - \varepsilon\} \leq$
$exp\ (-2 * \varepsilon^2 / (n * (b - a)^2))$ (**is** *?ge*)
and *Hoeffding-ineq-abs-ge*:
prob $\{x \in space\ M.\ |(\sum i \in I.\ X\ i\ x) - n * \mu| \geq \varepsilon\} \leq$
$2 * exp\ (-2 * \varepsilon^2 / (n * (b - a)^2))$ (**is** *?abs-ge*)
⟨*proof*⟩

lemma

assumes ε: $\varepsilon \geq 0$
assumes $a < b$ $I \neq \{\}$
defines $n \equiv card\ I$
shows *Hoeffding-ineq-ge'*:
$prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x)\ /\ n \geq \mu + \varepsilon\} \leq$
$exp\ (-2 * n * \varepsilon^2\ /\ (b - a)^2)$ (**is** *?ge*)
and *Hoeffding-ineq-le'*:
$prob\ \{x{\in}space\ M.\ (\sum i{\in}I.\ X\ i\ x)\ /\ n \leq \mu - \varepsilon\} \leq$
$exp\ (-2 * n * \varepsilon^2\ /\ (b - a)^2)$ (**is** *?le*)
and *Hoeffding-ineq-abs-ge'*:
$prob\ \{x{\in}space\ M.\ |(\sum i{\in}I.\ X\ i\ x)\ /\ n - \mu| \geq \varepsilon\} \leq$
$2 * exp\ (-2 * n * \varepsilon^2\ /\ (b - a)^2)$ (**is** *?abs-ge*)
⟨*proof*⟩

end

27.4 Hoeffding's Inequality for the Binomial distribution

We can now apply Hoeffding's inequality to the Binomial distribution, which can be seen as the sum of n i.i.d. coin flips (the support of each of which is contained in $[0, 1]$).

locale *binomial-distribution* =
fixes n :: *nat* **and** p :: *real*
assumes p: $p \in \{0..1\}$
begin

context
fixes *coins* :: (*nat* ⇒ *bool*) *pmf* **and** μ
assumes n: $n > 0$
defines *coins* ≡ *Pi-pmf* {..<*n*} *False* (λ-. *bernoulli-pmf p*)
begin

lemma *coins-component*:
assumes i: $i < n$
shows *distr* (*measure-pmf coins*) *borel* (λ*f*. *if f i then 1 else 0*) =
distr (*measure-pmf* (*bernoulli-pmf p*)) *borel* (λ*b*. *if b then 1 else 0*)
⟨*proof*⟩

lemma *prob-binomial-pmf-conv-coins*:
measure-pmf.prob (*binomial-pmf n p*) {*x*. *P* (*real x*)} =
measure-pmf.prob coins {*x*. *P* (∑ *i*<*n*. *if x i then 1 else 0*)}
⟨*proof*⟩

interpretation *Hoeffding-ineq-iid*
coins {..<*n*} λ*i f*. *if f i then 1 else 0* λ*f*. *if f 0 then 1 else 0 0 1 p*
⟨*proof*⟩

corollary
fixes ε :: *real*

assumes ε: $\varepsilon \geq 0$
shows *prob-ge*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x \geq n * p + \varepsilon\} \leq exp$ $(-2 * \varepsilon^2 / n)$
and *prob-le*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x \leq n * p - \varepsilon\} \leq exp$ $(-2 * \varepsilon^2 / n)$
and *prob-abs-ge*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x - n * p| \geq \varepsilon\} \leq 2 * exp\ (-2 * \varepsilon^2 / n)$
⟨*proof*⟩

corollary
fixes ε :: *real*
assumes ε: $\varepsilon \geq 0$
shows *prob-ge'*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x / n \geq p + \varepsilon\} \leq exp$ $(-2 * n * \varepsilon^2)$
and *prob-le'*: *measure-pmf.prob* (*binomial-pmf n p*) $\{x.\ x / n \leq p - \varepsilon\} \leq exp$ $(-2 * n * \varepsilon^2)$
and *prob-abs-ge'*:
measure-pmf.prob (*binomial-pmf n p*) $\{x.\ |x / n - p| \geq \varepsilon\} \leq 2 * exp\ (-2 * n * \varepsilon^2)$
⟨*proof*⟩

end

end

27.5 Tail bounds for the negative binomial distribution

Since the tail probabilities of a negative Binomial distribution are equal to the tail probabilities of some Binomial distribution, we can obtain tail bounds for the negative Binomial distribution through the Hoeffding tail bounds for the Binomial distribution.

context
fixes *p q* :: *real*
assumes *p*: $p \in \{0<..<1\}$
defines $q \equiv 1 - p$
begin

lemma *prob-neg-binomial-pmf-ge-bound*:
fixes *n* :: *nat* **and** *k* :: *real*
defines $\mu \equiv real\ n * q / p$
assumes *k*: $k \geq 0$
shows *measure-pmf.prob* (*neg-binomial-pmf n p*) $\{x.\ real\ x \geq \mu + k\}$
$\leq exp\ (-\ 2 * p\ \hat{}\ 3 * k^2 / (n + p * k))$
⟨*proof*⟩

lemma *prob-neg-binomial-pmf-le-bound*:
fixes *n* :: *nat* **and** *k* :: *real*

defines $\mu \equiv real\ n * q\ /\ p$
assumes k: $k \geq 0$
shows *measure-pmf.prob* (*neg-binomial-pmf* $n\ p$) $\{x.\ real\ x \leq \mu - k\}$
$\leq exp\ (-2 * p\ \hat{}\ 3 * k^2\ /\ (n - p * k))$
⟨*proof*⟩

Due to the function $exp(-l/x)$ being concave for $x \geq \frac{l}{2}$, the above two bounds can be combined into the following one for moderate values of k. (cf. https://math.stackexchange.com/questions/1565559)

lemma *prob-neg-binomial-pmf-abs-ge-bound*:
fixes n :: *nat* **and** k :: *real*
defines $\mu \equiv real\ n * q\ /\ p$
assumes $k \geq 0$ **and** *n-ge*: $n \geq p * k * (p^2 * k + 1)$
shows *measure-pmf.prob* (*neg-binomial-pmf* $n\ p$) $\{x.\ |real\ x - \mu| \geq k\} \leq$
$2 * exp\ (-2 * p\ \hat{}\ 3 * k\ \hat{}\ 2\ /\ n)$
⟨*proof*⟩

end

end

theory *Stream-Space*
imports
Infinite-Product-Measure
HOL−Library.Stream
HOL−Library.Linear-Temporal-Logic-on-Streams
begin

lemma *stream-eq-Stream-iff*: $s = x\ \#\#\ t \longleftrightarrow (shd\ s = x \wedge stl\ s = t)$
⟨*proof*⟩

lemma *Stream-snth*: $(x\ \#\#\ s)\ !!\ n = (case\ n\ of\ 0 \Rightarrow x \mid Suc\ n \Rightarrow s\ !!\ n)$
⟨*proof*⟩

definition *to-stream* :: $(nat \Rightarrow 'a) \Rightarrow 'a\ stream$ **where**
to-stream X = smap X nats

lemma *to-stream-nat-case*: *to-stream* (*case-nat x X*) $= x\ \#\#$ *to-stream X*
⟨*proof*⟩

lemma *to-stream-in-streams*: *to-stream* $X \in$ *streams* $S \longleftrightarrow (\forall n.\ X\ n \in S)$
⟨*proof*⟩

definition *stream-space* :: $'a\ measure \Rightarrow 'a\ stream\ measure$ **where**
stream-space M =
distr $(\Pi_M\ i{\in}UNIV.\ M)$ (*vimage-algebra* (*streams* (*space M*)) *snth* $(\Pi_M\ i{\in}UNIV.\ M)$) *to-stream*

lemma *space-stream-space*: $space\ (stream\text{-}space\ M) = streams\ (space\ M)$
⟨*proof*⟩

lemma *streams-stream-space*[*intro*]: $streams\ (space\ M) \in sets\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *stream-space-Stream*:
$x\ \#\#\ \omega \in space\ (stream\text{-}space\ M) \longleftrightarrow x \in space\ M \wedge \omega \in space\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *stream-space-eq-distr*: $stream\text{-}space\ M = distr\ (\Pi_M\ i{\in}UNIV.\ M)\ (stream\text{-}space\ M)\ to\text{-}stream$
⟨*proof*⟩

lemma *sets-stream-space-cong*[*measurable-cong*]:
$sets\ M = sets\ N \Longrightarrow sets\ (stream\text{-}space\ M) = sets\ (stream\text{-}space\ N)$
⟨*proof*⟩

lemma *measurable-snth-PiM*: $(\lambda\omega\ n.\ \omega\ !!\ n) \in measurable\ (stream\text{-}space\ M)\ (\Pi_M\ i{\in}UNIV.\ M)$
⟨*proof*⟩

lemma *measurable-snth*[*measurable*]: $(\lambda\omega.\ \omega\ !!\ n) \in measurable\ (stream\text{-}space\ M)\ M$
⟨*proof*⟩

lemma *measurable-shd*[*measurable*]: $shd \in measurable\ (stream\text{-}space\ M)\ M$
⟨*proof*⟩

lemma *measurable-stream-space2*:
assumes *f-snth*: $\bigwedge n.\ (\lambda x.\ f\ x\ !!\ n) \in measurable\ N\ M$
shows $f \in measurable\ N\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-stream-coinduct*[*consumes 1*, *case-names shd stl*, *coinduct set*: *measurable*]:
assumes $F\ f$
assumes *h*: $\bigwedge f.\ F\ f \Longrightarrow (\lambda x.\ shd\ (f\ x)) \in measurable\ N\ M$
assumes *t*: $\bigwedge f.\ F\ f \Longrightarrow F\ (\lambda x.\ stl\ (f\ x))$
shows $f \in measurable\ N\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-sdrop*[*measurable*]: $sdrop\ n \in measurable\ (stream\text{-}space\ M)\ (stream\text{-}space\ M)$
⟨*proof*⟩

lemma *measurable-stl*[*measurable*]: $(\lambda\omega.\ stl\ \omega) \in measurable\ (stream\text{-}space\ M)\ (stream\text{-}space\ M)$

⟨*proof*⟩

lemma *measurable-to-stream*[*measurable*]: *to-stream* $\in$ *measurable* (Π_M *i*$\in$*UNIV*. *M*) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-Stream*[*measurable* (*raw*)]:
assumes *f*[*measurable*]: *f* $\in$ *measurable N M*
assumes *g*[*measurable*]: *g* $\in$ *measurable N* (*stream-space M*)
shows (λx. *f x* ## *g x*) $\in$ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-smap*[*measurable*]:
assumes *X*[*measurable*]: *X* $\in$ *measurable N M*
shows *smap X* $\in$ *measurable* (*stream-space N*) (*stream-space M*)
⟨*proof*⟩

lemma *measurable-stake*[*measurable*]:
stake i $\in$ *measurable* (*stream-space* (*count-space UNIV*)) (*count-space* (*UNIV* :: *'a*::*countable list set*))
⟨*proof*⟩

lemma *measurable-shift*[*measurable*]:
assumes *f*: *f* $\in$ *measurable N* (*stream-space M*)
assumes [*measurable*]: *g* $\in$ *measurable N* (*stream-space M*)
shows (λx. *stake n* (*f x*) @− *g x*) $\in$ *measurable N* (*stream-space M*)
⟨*proof*⟩

lemma *measurable-case-stream-replace*[*measurable* (*raw*)]:
(λx. *f x* (*shd* (*g x*)) (*stl* (*g x*))) $\in$ *measurable M N* $\Longrightarrow$ (λx. *case-stream* (*f x*) (*g x*)) $\in$ *measurable M N*
⟨*proof*⟩

lemma *measurable-ev-at*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*ev-at P n*)
⟨*proof*⟩

lemma *measurable-alw*[*measurable*]:
Measurable.pred (*stream-space M*) *P* $\Longrightarrow$ *Measurable.pred* (*stream-space M*) (*alw P*)
⟨*proof*⟩

lemma *measurable-ev*[*measurable*]:
Measurable.pred (*stream-space M*) *P* $\Longrightarrow$ *Measurable.pred* (*stream-space M*) (*ev P*)
⟨*proof*⟩

lemma *measurable-until*:

assumes [*measurable*]: *Measurable.pred* (*stream-space M*) φ *Measurable.pred* (*stream-space M*) ψ
shows *Measurable.pred* (*stream-space M*) (φ *until* ψ)
⟨*proof*⟩

lemma *measurable-holds* [*measurable*]: *Measurable.pred M P* $\Longrightarrow$ *Measurable.pred* (*stream-space M*) (*holds P*)
⟨*proof*⟩

lemma *measurable-hld*[*measurable*]: **assumes** [*measurable*]: $t \in$ *sets M* **shows** *Measurable.pred* (*stream-space M*) (*HLD t*)
⟨*proof*⟩

lemma *measurable-nxt*[*measurable* (*raw*)]:
Measurable.pred (*stream-space M*) *P* $\Longrightarrow$ *Measurable.pred* (*stream-space M*) (*nxt P*)
⟨*proof*⟩

lemma *measurable-suntil*[*measurable*]:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *Q* *Measurable.pred* (*stream-space M*) *P*
shows *Measurable.pred* (*stream-space M*) (*Q suntil P*)
⟨*proof*⟩

lemma *measurable-szip*:
($\lambda(\omega 1, \omega 2)$. *szip* $\omega 1$ $\omega 2$) $\in$ *measurable* (*stream-space M* $\bigotimes_M$ *stream-space N*) (*stream-space* (*M* $\bigotimes_M$ *N*))
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-space-stream-space*: *prob-space* (*stream-space M*)
⟨*proof*⟩

lemma (**in** *prob-space*) *nn-integral-stream-space*:
assumes [*measurable*]: $f \in$ *borel-measurable* (*stream-space M*)
shows ($\int^+ X.\ f\ X\ \partial$*stream-space M*) = ($\int^+ x.$ ($\int^+ X.\ f\ (x\ \#\#\ X)\ \partial$*stream-space M*) ∂M)
⟨*proof*⟩

lemma (**in** *prob-space*) *emeasure-stream-space*:
assumes *X*[*measurable*]: $X \in$ *sets* (*stream-space M*)
shows *emeasure* (*stream-space M*) *X* = ($\int^+ t.$ *emeasure* (*stream-space M*) {$x \in$*space* (*stream-space M*). $t\ \#\#\ x \in X$ } ∂M)
⟨*proof*⟩

lemma (**in** *prob-space*) *prob-stream-space*:
assumes *P*[*measurable*]: {$x \in$*space* (*stream-space M*). *P x*} $\in$ *sets* (*stream-space M*)
shows $\mathcal{P}$(*x in stream-space M*. *P x*) = ($\int^+ t.$ $\mathcal{P}$(*x in stream-space M*. *P* ($t\ \#\#\ x$)) ∂M)

⟨*proof*⟩

lemma (**in** *prob-space*) *AE-stream-space*:
assumes [*measurable*]: *Measurable.pred* (*stream-space M*) *P*
shows (*AE X in stream-space M. P X*) = (*AE x in M. AE X in stream-space M. P* (*x ## X*))
⟨*proof*⟩

lemma (**in** *prob-space*) *AE-stream-all*:
assumes [*measurable*]: *Measurable.pred M P* **and** *P*: *AE x in M. P x*
shows *AE x in stream-space M. stream-all P x*
⟨*proof*⟩

lemma *streams-sets*:
assumes *X*[*measurable*]: $X \in$ *sets M* **shows** *streams X* $\in$ *sets* (*stream-space M*)
⟨*proof*⟩

lemma *sets-stream-space-in-sets*:
assumes *space*: *space N* = *streams* (*space M*)
assumes *sets*: $\bigwedge i.\ (\lambda x.\ x\ !!\ i) \in$ *measurable N M*
shows *sets* (*stream-space M*) $\subseteq$ *sets N*
⟨*proof*⟩

lemma *sets-stream-space-eq*: *sets* (*stream-space M*) =
sets (*SUP i*$\in$*UNIV. vimage-algebra* (*streams* (*space M*)) ($\lambda s.\ s\ !!\ i$) *M*)
⟨*proof*⟩

lemma *sets-restrict-stream-space*:
assumes *S*[*measurable*]: $S \in$ *sets M*
shows *sets* (*restrict-space* (*stream-space M*) (*streams S*)) = *sets* (*stream-space* (*restrict-space M S*))
⟨*proof*⟩

primrec *sstart* :: *'a set* $\Rightarrow$ *'a list* $\Rightarrow$ *'a stream set* **where**
sstart S [] = *streams S*
| [*simp del*]: *sstart S* (*x # xs*) = (*##*) *x* ‘ *sstart S xs*

lemma *in-sstart*[*simp*]: $s \in$ *sstart S* (*x # xs*) $\longleftrightarrow$ *shd s* = $x \wedge$ *stl s* $\in$ *sstart S xs*
⟨*proof*⟩

lemma *sstart-in-streams*: *xs* $\in$ *lists S* $\Longrightarrow$ *sstart S xs* $\subseteq$ *streams S*
⟨*proof*⟩

lemma *sstart-eq*: $x \in$ *streams S* $\Longrightarrow$ $x \in$ *sstart S xs* = ($\forall$ *i*<*length xs. x* !! *i* = *xs* ! *i*)
⟨*proof*⟩

lemma *sstart-sets*: *sstart S xs* $\in$ *sets* (*stream-space* (*count-space UNIV*))
⟨*proof*⟩

lemma *sigma-sets-singletons*:
assumes *countable S*
shows *sigma-sets S* $((\lambda s.\ \{s\})$ *'S*$)$ = *Pow S*
⟨*proof*⟩

lemma *sets-count-space-eq-sigma*:
countable S $\Longrightarrow$ *sets* (*count-space S*) = *sets* (*sigma S* $((\lambda s.\ \{s\})$ *'S*))
⟨*proof*⟩

lemma *sets-stream-space-sstart*:
assumes *S*[*simp*]: *countable S*
shows *sets* (*stream-space* (*count-space S*)) = *sets* (*sigma* (*streams S*) (*sstart S'lists S* ∪ {{}}))
⟨*proof*⟩

lemma *Int-stable-sstart*: *Int-stable* (*sstart S'lists S* ∪ {{}})
⟨*proof*⟩

lemma *stream-space-eq-sstart*:
assumes *S*[*simp*]: *countable S*
assumes *P*: *prob-space M prob-space N*
assumes *ae*: *AE x in M*. *x* ∈ *streams S AE x in N*. *x* ∈ *streams S*
assumes *sets-M*: *sets M* = *sets* (*stream-space* (*count-space UNIV*))
assumes *sets-N*: *sets N* = *sets* (*stream-space* (*count-space UNIV*))
assumes ∗: $\bigwedge$*xs*. *xs* ≠ [] $\Longrightarrow$ *xs* ∈ *lists S* $\Longrightarrow$ *emeasure M* (*sstart S xs*) = *emeasure N* (*sstart S xs*)
shows *M* = *N*
⟨*proof*⟩

lemma *sets-sstart*[*measurable*]: *sstart* Ω *xs* ∈ *sets* (*stream-space* (*count-space UNIV*))
⟨*proof*⟩

primrec *scylinder* :: *'a set* ⇒ *'a set list* ⇒ *'a stream set*
where
scylinder S [] = *streams S*
| *scylinder S* (*A* # *As*) = {ω∈*streams S*. *shd* ω ∈ *A* ∧ *stl* ω ∈ *scylinder S As*}

lemma *scylinder-streams*: *scylinder S xs* ⊆ *streams S*
⟨*proof*⟩

lemma *sets-scylinder*: (∀ *x*∈*set xs*. *x* ∈ *sets S*) $\Longrightarrow$ *scylinder* (*space S*) *xs* ∈ *sets* (*stream-space S*)
⟨*proof*⟩

lemma *stream-space-eq-scylinder*:
assumes *P*: *prob-space M prob-space N*
assumes *Int-stable G* **and** *S*: *sets S* = *sets* (*sigma* (*space S*) *G*)
and *C*: *countable C C* ⊆ *G* $\bigcup$ *C* = *space S* **and** *G*: *G* ⊆ *Pow* (*space S*)

assumes *sets-M*: $sets\ M = sets\ (stream\text{-}space\ S)$
assumes *sets-N*: $sets\ N = sets\ (stream\text{-}space\ S)$
assumes $*$: $\bigwedge xs.\ xs \neq [] \Longrightarrow xs \in lists\ G \Longrightarrow emeasure\ M\ (scylinder\ (space\ S)\ xs) = emeasure\ N\ (scylinder\ (space\ S)\ xs)$
shows $M = N$
⟨*proof*⟩

lemma *stream-space-coinduct*:
fixes $R :: 'a\ stream\ measure \Rightarrow 'a\ stream\ measure \Rightarrow bool$
assumes $R\ A\ B$
assumes R: $\bigwedge A\ B.\ R\ A\ B \Longrightarrow \exists K{\in}space\ (prob\text{-}algebra\ M).$
$\exists A'{\in}M \rightarrow_M prob\text{-}algebra\ (stream\text{-}space\ M).\ \exists B'{\in}M \rightarrow_M prob\text{-}algebra\ (stream\text{-}space\ M).$
$(AE\ y\ in\ K.\ R\ (A'\ y)\ (B'\ y) \vee A'\ y = B'\ y) \wedge$
$A = do\ \{\ y \leftarrow K;\ \omega \leftarrow A'\ y;\ return\ (stream\text{-}space\ M)\ (y\ \#\#\ \omega)\ \} \wedge$
$B = do\ \{\ y \leftarrow K;\ \omega \leftarrow B'\ y;\ return\ (stream\text{-}space\ M)\ (y\ \#\#\ \omega)\ \}$
shows $A = B$
⟨*proof*⟩

end

theory *Tree-Space*
imports *HOL−Analysis.Analysis HOL−Library.Tree*
begin

lemma *countable-lfp*:
assumes *step*: $\bigwedge Y.\ countable\ Y \Longrightarrow countable\ (F\ Y)$
and *cont*: *Order-Continuity.sup-continuous F*
shows $countable\ (lfp\ F)$
⟨*proof*⟩

lemma *countable-lfp-apply*:
assumes *step*: $\bigwedge Y\ x.\ (\bigwedge x.\ countable\ (Y\ x)) \Longrightarrow countable\ (F\ Y\ x)$
and *cont*: *Order-Continuity.sup-continuous F*
shows $countable\ (lfp\ F\ x)$
⟨*proof*⟩

inductive-set $trees :: 'a\ set \Rightarrow 'a\ tree\ set$ **for** $S :: 'a\ set$ **where**
[*intro*!]: $Leaf \in trees\ S$
| $l \in trees\ S \Longrightarrow r \in trees\ S \Longrightarrow v \in S \Longrightarrow Node\ l\ v\ r \in trees\ S$

lemma *Node-in-trees-iff*[*simp*]: $Node\ l\ v\ r \in trees\ S \longleftrightarrow (l \in trees\ S \wedge v \in S \wedge r \in trees\ S)$
⟨*proof*⟩

lemma *trees-sub-lfp*: $trees\ S \subseteq lfp\ (\lambda T.\ T \cup \{Leaf\} \cup (\bigcup l{\in}T.\ (\bigcup v{\in}S.\ (\bigcup r{\in}T.\ \{Node\ l\ v\ r\}))))$
⟨*proof*⟩

lemma *countable-trees*: $countable\ A \Longrightarrow countable\ (trees\ A)$
⟨*proof*⟩

lemma *trees-UNIV*[*simp*]: $trees\ UNIV = UNIV$
⟨*proof*⟩

instance $tree :: (countable)\ countable$
⟨*proof*⟩

lemma *map-in-trees*[*intro*]: $(\bigwedge x.\ x \in set\text{-}tree\ t \Longrightarrow f\ x \in S) \Longrightarrow map\text{-}tree\ f\ t \in trees\ S$
⟨*proof*⟩

primrec $trees\text{-}cyl :: 'a\ set\ tree \Rightarrow 'a\ tree\ set$ **where**
$trees\text{-}cyl\ Leaf = \{Leaf\}$
$|\ trees\text{-}cyl\ (Node\ l\ v\ r) = (\bigcup l' \in trees\text{-}cyl\ l.\ (\bigcup v' \in v.\ (\bigcup r' \in trees\text{-}cyl\ r.\ \{Node\ l'\ v'\ r'\})))$

definition $tree\text{-}sigma :: 'a\ measure \Rightarrow 'a\ tree\ measure$
where
$tree\text{-}sigma\ M = sigma\ (trees\ (space\ M))\ (trees\text{-}cyl\ `\ trees\ (sets\ M))$

lemma *Node-in-trees-cyl*: $Node\ l'\ v'\ r' \in trees\text{-}cyl\ t \longleftrightarrow$
$(\exists\ l\ v\ r.\ t = Node\ l\ v\ r \wedge l' \in trees\text{-}cyl\ l \wedge r' \in trees\text{-}cyl\ r \wedge v' \in v)$
⟨*proof*⟩

lemma *trees-cyl-sub-trees*:
assumes $t \in trees\ A\ A \subseteq Pow\ B$ **shows** $trees\text{-}cyl\ t \subseteq trees\ B$
⟨*proof*⟩

lemma *trees-cyl-sets-in-space*: $trees\text{-}cyl\ `\ trees\ (sets\ M) \subseteq Pow\ (trees\ (space\ M))$
⟨*proof*⟩

lemma *space-tree-sigma*: $space\ (tree\text{-}sigma\ M) = trees\ (space\ M)$
⟨*proof*⟩

lemma *sets-tree-sigma-eq*: $sets\ (tree\text{-}sigma\ M) = sigma\text{-}sets\ (trees\ (space\ M))\ (trees\text{-}cyl\ `\ trees\ (sets\ M))$
⟨*proof*⟩

lemma *Leaf-in-space-tree-sigma* [*measurable*, *simp*, *intro*]: $Leaf \in space\ (tree\text{-}sigma\ M)$
⟨*proof*⟩

lemma *Leaf-in-tree-sigma* [*measurable*, *simp*, *intro*]: $\{Leaf\} \in sets\ (tree\text{-}sigma\ M)$
⟨*proof*⟩

lemma *trees-cyl-map-treeI*: $t \in trees\text{-}cyl\ (map\text{-}tree\ (\lambda x.\ A)\ t)$ **if** $*$: $t \in trees\ A$

⟨*proof*⟩

lemma *trees-cyl-map-in-sets*:
$(\bigwedge x.\ x \in \textit{set-tree}\ t \Longrightarrow f\ x \in \textit{sets}\ M) \Longrightarrow \textit{trees-cyl}\ (\textit{map-tree}\ f\ t) \in \textit{sets}\ (\textit{tree-sigma}\ M)$
⟨*proof*⟩

lemma *Node-in-tree-sigma*:
assumes L: $X \in \textit{sets}\ (M \bigotimes_M (\textit{tree-sigma}\ M \bigotimes_M \textit{tree-sigma}\ M))$
shows $\{\textit{Node}\ l\ v\ r \mid l\ v\ r.\ (v, l, r) \in X\} \in \textit{sets}\ (\textit{tree-sigma}\ M)$
⟨*proof*⟩

lemma *measurable-left*[*measurable*]: $\textit{left} \in \textit{tree-sigma}\ M \rightarrow_M \textit{tree-sigma}\ M$
⟨*proof*⟩

lemma *measurable-right*[*measurable*]: $\textit{right} \in \textit{tree-sigma}\ M \rightarrow_M \textit{tree-sigma}\ M$
⟨*proof*⟩

lemma *measurable-value'*: $\textit{value} \in \textit{restrict-space}\ (\textit{tree-sigma}\ M)\ (-\{\textit{Leaf}\}) \rightarrow_M M$
⟨*proof*⟩

lemma *measurable-value*[*measurable* (*raw*)]:
assumes $f \in X \rightarrow_M \textit{tree-sigma}\ M$
and $\bigwedge x.\ x \in \textit{space}\ X \Longrightarrow f\ x \neq \textit{Leaf}$
shows $(\lambda\omega.\ \textit{value}\ (f\ \omega)) \in X \rightarrow_M M$
⟨*proof*⟩

lemma *measurable-Node* [*measurable*]:
$(\lambda(l,x,r).\ \textit{Node}\ l\ x\ r) \in \textit{tree-sigma}\ M \bigotimes_M M \bigotimes_M \textit{tree-sigma}\ M \rightarrow_M \textit{tree-sigma}\ M$
⟨*proof*⟩

lemma *measurable-Node'* [*measurable* (*raw*)]:
assumes [*measurable*]: $l \in B \rightarrow_M \textit{tree-sigma}\ A$
assumes [*measurable*]: $x \in B \rightarrow_M A$
assumes [*measurable*]: $r \in B \rightarrow_M \textit{tree-sigma}\ A$
shows $(\lambda y.\ \textit{Node}\ (l\ y)\ (x\ y)\ (r\ y)) \in B \rightarrow_M \textit{tree-sigma}\ A$
⟨*proof*⟩

lemma *measurable-rec-tree*[*measurable* (*raw*)]:
assumes t: $t \in B \rightarrow_M \textit{tree-sigma}\ M$
assumes l: $l \in B \rightarrow_M A$
assumes n: $(\lambda(x, l, v, r, al, ar).\ n\ x\ l\ v\ r\ al\ ar) \in$
$(B \bigotimes_M \textit{tree-sigma}\ M \bigotimes_M M \bigotimes_M \textit{tree-sigma}\ M \bigotimes_M A \bigotimes_M A) \rightarrow_M A$ (**is** $?N \in\ ?M \rightarrow_M A$)
shows $(\lambda x.\ \textit{rec-tree}\ (l\ x)\ (n\ x)\ (t\ x)) \in B \rightarrow_M A$
⟨*proof*⟩

lemma *measurable-case-tree* [*measurable* (*raw*)]:
  assumes $t \in B \rightarrow_M$ *tree-sigma* M
  assumes $l \in B \rightarrow_M A$
  assumes $(\lambda(x, l, v, r).\ n\ x\ l\ v\ r)$
    $\in B \bigotimes_M$ *tree-sigma* $M \bigotimes_M M \bigotimes_M$ *tree-sigma* $M \rightarrow_M A$
  shows $(\lambda x.$ *case-tree* $(l\ x)\ (n\ x)\ (t\ x)) \in B \rightarrow_M (A :: {}'a$ *measure*$)$
⟨*proof*⟩

hide-const (**open**) *left*
hide-const (**open**) *right*

end

28 Conditional Expectation

theory *Conditional-Expectation*
imports *Probability-Measure*
begin

28.1 Restricting a measure to a sub-sigma-algebra

definition *subalgebra*::$'a$ *measure* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ *bool* **where**
  subalgebra $M\ F = (($*space* $F =$ *space* $M) \wedge ($*sets* $F \subseteq$ *sets* $M))$

lemma *sub-measure-space*:
  assumes *i*: *subalgebra* $M\ F$
  shows *measure-space* (*space* M) (*sets* F) (*emeasure* M)
⟨*proof*⟩

definition *restr-to-subalg*::$'a$ *measure* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ $'a$ *measure* **where**
  restr-to-subalg $M\ F$ = *measure-of* (*space* M) (*sets* F) (*emeasure* M)

lemma *space-restr-to-subalg*:
  space (*restr-to-subalg* $M\ F$) = *space* M
⟨*proof*⟩

lemma *sets-restr-to-subalg* [*measurable-cong*]:
  assumes *subalgebra* $M\ F$
  shows *sets* (*restr-to-subalg* $M\ F$) = *sets* F
⟨*proof*⟩

lemma *emeasure-restr-to-subalg*:
  assumes *subalgebra* $M\ F$
    $A \in$ *sets* F
  shows *emeasure* (*restr-to-subalg* $M\ F$) A = *emeasure* $M\ A$
⟨*proof*⟩

lemma *null-sets-restr-to-subalg*:

assumes *subalgebra M F*
shows *null-sets* (*restr-to-subalg M F*) = *null-sets M* ∩ *sets F*
⟨*proof*⟩

lemma *AE-restr-to-subalg*:
assumes *subalgebra M F*
AE x in (*restr-to-subalg M F*). *P x*
shows *AE x in M*. *P x*
⟨*proof*⟩

lemma *AE-restr-to-subalg2*:
assumes *subalgebra M F*
AE x in M. *P x* **and** [*measurable*]: *P* ∈ *measurable F* (*count-space UNIV*)
shows *AE x in* (*restr-to-subalg M F*). *P x*
⟨*proof*⟩

lemma *prob-space-restr-to-subalg*:
assumes *subalgebra M F*
prob-space M
shows *prob-space* (*restr-to-subalg M F*)
⟨*proof*⟩

lemma *finite-measure-restr-to-subalg*:
assumes *subalgebra M F*
finite-measure M
shows *finite-measure* (*restr-to-subalg M F*)
⟨*proof*⟩

lemma *measurable-in-subalg*:
assumes *subalgebra M F*
f ∈ *measurable F N*
shows *f* ∈ *measurable* (*restr-to-subalg M F*) *N*
⟨*proof*⟩

lemma *measurable-in-subalg'*:
assumes *subalgebra M F*
f ∈ *measurable* (*restr-to-subalg M F*) *N*
shows *f* ∈ *measurable F N*
⟨*proof*⟩

lemma *measurable-from-subalg*:
assumes *subalgebra M F*
f ∈ *measurable F N*
shows *f* ∈ *measurable M N*
⟨*proof*⟩

The following is the direct transposition of `nn_integral_subalgebra` (from `Nonnegative_Lebesgue_Integration`) in the current notations, with the removal of the useless assumption $f \geq 0$.

lemma *nn-integral-subalgebra2*:
  assumes *subalgebra M F* **and** [*measurable*]: $f \in$ *borel-measurable F*
  shows $(\int^+ x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int^+ x.\ f\ x\ \partial M)$
⟨*proof*⟩

The following is the direct transposition of `integral_subalgebra` (from `Bochner_Integration`) in the current notations.

lemma *integral-subalgebra2*:
  fixes $f :: {'a} \Rightarrow {'b}::\{\textit{banach},\ \textit{second-countable-topology}\}$
  assumes *subalgebra M F* **and**
    [*measurable*]: $f \in$ *borel-measurable F*
  shows $(\int x.\ f\ x\ \partial(\textit{restr-to-subalg}\ M\ F)) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *integrable-from-subalg*:
  fixes $f :: {'a} \Rightarrow {'b}::\{\textit{banach},\ \textit{second-countable-topology}\}$
  assumes *subalgebra M F*
    integrable (*restr-to-subalg M F*) *f*
  shows *integrable M f*
⟨*proof*⟩

lemma *integrable-in-subalg*:
  fixes $f :: {'a} \Rightarrow {'b}::\{\textit{banach},\ \textit{second-countable-topology}\}$
  assumes [*measurable*]: *subalgebra M F*
    $f \in$ *borel-measurable F*
    integrable M f
  shows *integrable* (*restr-to-subalg M F*) *f*
⟨*proof*⟩

28.2 Nonnegative conditional expectation

The conditional expectation of a function f, on a measure space M, with respect to a sub sigma algebra F, should be a function g which is F-measurable whose integral on any F-set coincides with the integral of f. Such a function is uniquely defined almost everywhere. The most direct construction is to use the measure fdM, restrict it to the sigma-algebra F, and apply the Radon-Nikodym theorem to write it as $gdM_{|F}$ for some F-measurable function g. Another classical construction for L^2 functions is done by orthogonal projection on F-measurable functions, and then extending by density to L^1. The Radon-Nikodym point of view avoids the L^2 machinery, and works for all positive functions.

In this paragraph, we develop the definition and basic properties for nonnegative functions, as the basics of the general case. As in the definition of integrals, the nonnegative case is done with ennreal-valued functions, without any integrability assumption.

definition *nn-cond-exp* :: $'a$ *measure* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ $({'a} \Rightarrow \textit{ennreal}) \Rightarrow ({'a} \Rightarrow \textit{ennreal})$

where
nn-cond-exp M F f =
(*if f* ∈ *borel-measurable M* ∧ *subalgebra M F*
then RN-deriv (*restr-to-subalg M F*) (*restr-to-subalg* (*density M f*) *F*)
else (λ-. *0*))

lemma
shows *borel-measurable-nn-cond-exp* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable F*
and *borel-measurable-nn-cond-exp2* [*measurable*]: *nn-cond-exp M F f* ∈ *borel-measurable M*
⟨*proof*⟩

The good setting for conditional expectations is the situation where the subalgebra F gives rise to a sigma-finite measure space. To see what goes wrong if it is not sigma-finite, think of $\mathbb{R}$ with the trivial sigma-algebra $\{\emptyset, \mathbb{R}\}$. In this case, conditional expectations have to be constant functions, so they have integral 0 or ∞. This means that a positive integrable function can have no meaningful conditional expectation.

locale *sigma-finite-subalgebra* =
fixes *M F*::*'a measure*
assumes *subalg*: *subalgebra M F*
and *sigma-fin-subalg*: *sigma-finite-measure* (*restr-to-subalg M F*)

lemma *sigma-finite-subalgebra-is-sigma-finite*:
assumes *sigma-finite-subalgebra M F*
shows *sigma-finite-measure M*
⟨*proof*⟩

sublocale *sigma-finite-subalgebra* ⊆ *sigma-finite-measure*
⟨*proof*⟩

Conditional expectations are very often used in probability spaces. This is a special case of the previous one, as we prove now.

locale *finite-measure-subalgebra* = *finite-measure* +
fixes *F*::*'a measure*
assumes *subalg*: *subalgebra M F*

lemma *finite-measure-subalgebra-is-sigma-finite*:
assumes *finite-measure-subalgebra M F*
shows *sigma-finite-subalgebra M F*
⟨*proof*⟩

sublocale *finite-measure-subalgebra* ⊆ *sigma-finite-subalgebra*
⟨*proof*⟩

context *sigma-finite-subalgebra*
begin

The next lemma is arguably the most fundamental property of conditional expectation: when computing an expectation against an F-measurable function, it is equivalent to work with a function or with its F-conditional expectation.

This property (even for bounded test functions) characterizes conditional expectations, as the second lemma below shows. From this point on, we will only work with it, and forget completely about the definition using Radon-Nikodym derivatives.

lemma *nn-cond-exp-intg*:
assumes [*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
shows $(\int^+ x.\ f\ x * \textit{nn-cond-exp}\ M\ F\ g\ x\ \partial M) = (\int^+ x.\ f\ x * g\ x\ \partial M)$
⟨*proof*⟩

lemma *nn-cond-exp-charact*:
assumes $\bigwedge A.\ A \in \textit{sets}\ F \Longrightarrow (\int^+ x \in A.\ f\ x\ \partial M) = (\int^+ x \in A.\ g\ x\ \partial M)$ **and**
[*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* F
shows $AE\ x\ in\ M.\ g\ x = \textit{nn-cond-exp}\ M\ F\ f\ x$
⟨*proof*⟩

lemma *nn-cond-exp-F-meas*:
assumes $f \in$ *borel-measurable* F
shows $AE\ x\ in\ M.\ f\ x = \textit{nn-cond-exp}\ M\ F\ f\ x$
⟨*proof*⟩

lemma *nn-cond-exp-prod*:
assumes [*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
shows $AE\ x\ in\ M.\ f\ x * \textit{nn-cond-exp}\ M\ F\ g\ x = \textit{nn-cond-exp}\ M\ F\ (\lambda x.\ f\ x * g\ x)\ x$
⟨*proof*⟩

lemma *nn-cond-exp-sum*:
assumes [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows $AE\ x\ in\ M.\ \textit{nn-cond-exp}\ M\ F\ f\ x + \textit{nn-cond-exp}\ M\ F\ g\ x = \textit{nn-cond-exp}\ M\ F\ (\lambda x.\ f\ x + g\ x)\ x$
⟨*proof*⟩

lemma *nn-cond-exp-cong*:
assumes $AE\ x\ in\ M.\ f\ x = g\ x$
and [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows $AE\ x\ in\ M.\ \textit{nn-cond-exp}\ M\ F\ f\ x = \textit{nn-cond-exp}\ M\ F\ g\ x$
⟨*proof*⟩

lemma *nn-cond-exp-mono*:
assumes $AE\ x\ in\ M.\ f\ x \leq g\ x$
and [*measurable*]: $f \in$ *borel-measurable* M $g \in$ *borel-measurable* M
shows $AE\ x\ in\ M.\ \textit{nn-cond-exp}\ M\ F\ f\ x \leq \textit{nn-cond-exp}\ M\ F\ g\ x$
⟨*proof*⟩

lemma *nested-subalg-is-sigma-finite*:
assumes *subalgebra M G subalgebra G F*
shows *sigma-finite-subalgebra M G*
⟨*proof*⟩

lemma *nn-cond-exp-nested-subalg*:
assumes *subalgebra M G subalgebra G F*
and [*measurable*]: $f \in$ *borel-measurable M*
shows *AE x in M. nn-cond-exp M F f x* $=$ *nn-cond-exp M F* (*nn-cond-exp M G f*) *x*
⟨*proof*⟩

end

28.3 Real conditional expectation

Once conditional expectations of positive functions are defined, the definition for real-valued functions follows readily, by taking the difference of positive and negative parts. One could also define a conditional expectation of vector-space valued functions, as in `Bochner_Integral`, but since the real-valued case is the most important, and quicker to formalize, I concentrate on it. (It is also essential for the case of the most general Pettis integral.)

definition *real-cond-exp* :: $'a$ *measure* $\Rightarrow$ $'a$ *measure* $\Rightarrow$ ($'a \Rightarrow$ *real*) $\Rightarrow$ ($'a \Rightarrow$ *real*)
where
real-cond-exp M F f $=$
($\lambda x.$ *enn2real*(*nn-cond-exp M F* ($\lambda x.$ *ennreal* (*f x*)) *x*) $-$ *enn2real*(*nn-cond-exp M F* ($\lambda x.$ *ennreal* ($-f$ *x*)) *x*))

lemma
shows *borel-measurable-cond-exp* [*measurable*]: *real-cond-exp M F f* $\in$ *borel-measurable F*
and *borel-measurable-cond-exp2* [*measurable*]: *real-cond-exp M F f* $\in$ *borel-measurable M*
⟨*proof*⟩

context *sigma-finite-subalgebra*
begin

lemma *real-cond-exp-abs*:
assumes [*measurable*]: $f \in$ *borel-measurable M*
shows *AE x in M. abs*(*real-cond-exp M F f x*) $\leq$ *nn-cond-exp M F* ($\lambda x.$ *ennreal* (*abs*(*f x*))) *x*
⟨*proof*⟩

The next lemma shows that the conditional expectation is an F-measurable function whose average against an F-measurable function f coincides with the average of the original function against f. It is obtained as a consequence of the same property for the positive conditional expectation, taking the difference of the positive and the negative part. The proof is given first assuming $f \geq 0$ for simplicity, and then extended to the general case in the subsequent lemma. The idea of the proof is essentially trivial, but the implementation is slightly tedious as one should check all the integrability properties of the different parts, and go back and forth between positive integral and signed integrals, and between real-valued functions and ennreal-valued functions.

Once this lemma is available, we will use it to characterize the conditional expectation, and never come back to the original technical definition, as we did in the case of the nonnegative conditional expectation.

lemma *real-cond-exp-intg-fpos*:
  assumes *integrable* M $(\lambda x.\ f\ x * g\ x)$ **and** *f-pos*[*simp*]: $\bigwedge x.\ f\ x \geq 0$ **and**
    [*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
  shows *integrable* M $(\lambda x.\ f\ x *$ *real-cond-exp* $M\ F\ g\ x)$
    $(\int\ x.\ f\ x *$ *real-cond-exp* $M\ F\ g\ x\ \partial M) = (\int\ x.\ f\ x * g\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-intg*:
  assumes *integrable* M $(\lambda x.\ f\ x * g\ x)$ **and**
    [*measurable*]: $f \in$ *borel-measurable* F $g \in$ *borel-measurable* M
  shows *integrable* M $(\lambda x.\ f\ x *$ *real-cond-exp* $M\ F\ g\ x)$
    $(\int\ x.\ f\ x *$ *real-cond-exp* $M\ F\ g\ x\ \partial M) = (\int\ x.\ f\ x * g\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-intA*:
  assumes [*measurable*]: *integrable* $M\ f$ $A \in$ *sets* F
  shows $(\int\ x \in A.\ f\ x\ \partial M) = (\int x \in A.$ *real-cond-exp* $M\ F\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-int* [*intro*]:
  assumes *integrable* $M\ f$
  shows *integrable* M (*real-cond-exp* $M\ F\ f$) $(\int x.$ *real-cond-exp* $M\ F\ f\ x\ \partial M) = (\int x.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *real-cond-exp-charact*:
  assumes $\bigwedge A.\ A \in$ *sets* $F \Longrightarrow (\int\ x \in A.\ f\ x\ \partial M) = (\int\ x \in A.\ g\ x\ \partial M)$
    and [*measurable*]: *integrable* $M\ f$ *integrable* $M\ g$
      $g \in$ *borel-measurable* F
  shows *AE* x *in* M. *real-cond-exp* $M\ F\ f\ x = g\ x$
⟨*proof*⟩

lemma *real-cond-exp-F-meas* [*intro*, *simp*]:
  assumes *integrable* $M\ f$
    $f \in$ *borel-measurable* F
  shows *AE* x *in* M. *real-cond-exp* $M\ F\ f\ x = f\ x$

⟨*proof*⟩

lemma *real-cond-exp-mult*:
assumes [*measurable*]:*f* ∈ *borel-measurable F g* ∈ *borel-measurable M integrable M* (λx. *f x* ∗ *g x*)
shows *AE x in M*. *real-cond-exp M F* (λx. *f x* ∗ *g x*) *x* = *f x* ∗ *real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-add* [*intro*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-cond-exp M F* (λx. *f x* + *g x*) *x* = *real-cond-exp M F f x* + *real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-cong*:
assumes *ae*: *AE x in M*. *f x* = *g x* **and** [*measurable*]: *f* ∈ *borel-measurable M g* ∈ *borel-measurable M*
shows *AE x in M*. *real-cond-exp M F f x* = *real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-cmult* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M*. *real-cond-exp M F* (λx. *c* ∗ *f x*) *x* = *c* ∗ *real-cond-exp M F f x*
⟨*proof*⟩

lemma *real-cond-exp-cdiv* [*intro*, *simp*]:
fixes *c*::*real*
assumes *integrable M f*
shows *AE x in M*. *real-cond-exp M F* (λx. *f x* / *c*) *x* = *real-cond-exp M F f x* / *c*
⟨*proof*⟩

lemma *real-cond-exp-diff* [*intro*, *simp*]:
assumes [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-cond-exp M F* (λx. *f x* − *g x*) *x* = *real-cond-exp M F f x* − *real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-pos* [*intro*]:
assumes *AE x in M*. *f x* ≥ *0* **and** [*measurable*]: *f* ∈ *borel-measurable M*
shows *AE x in M*. *real-cond-exp M F f x* ≥ *0*
⟨*proof*⟩

lemma *real-cond-exp-mono*:
assumes *AE x in M*. *f x* ≤ *g x* **and** [*measurable*]: *integrable M f integrable M g*
shows *AE x in M*. *real-cond-exp M F f x* ≤ *real-cond-exp M F g x*

⟨*proof*⟩

lemma (**in** −) *measurable-P-restriction* [*measurable* (*raw*)]:
  assumes [*measurable*]: *Measurable.pred M P A* $\in$ *sets M*
  shows $\{x \in A.\ P\ x\} \in$ *sets M*
⟨*proof*⟩

lemma *real-cond-exp-gr-c*:
  assumes [*measurable*]: *integrable M f*
    and *AE*: *AE x in M*. $f\ x > c$
  shows *AE x in M*. *real-cond-exp M F f x* $> c$
⟨*proof*⟩

lemma *real-cond-exp-less-c*:
  assumes [*measurable*]: *integrable M f*
    and *AE x in M*. $f\ x < c$
  shows *AE x in M*. *real-cond-exp M F f x* $< c$
⟨*proof*⟩

lemma *real-cond-exp-ge-c*:
  assumes [*measurable*]: *integrable M f*
    and *AE x in M*. $f\ x \geq c$
  shows *AE x in M*. *real-cond-exp M F f x* $\geq c$
⟨*proof*⟩

lemma *real-cond-exp-le-c*:
  assumes [*measurable*]: *integrable M f*
    and *AE x in M*. $f\ x \leq c$
  shows *AE x in M*. *real-cond-exp M F f x* $\leq c$
⟨*proof*⟩

lemma *real-cond-exp-mono-strict*:
  assumes *AE x in M*. $f\ x < g\ x$ **and** [*measurable*]: *integrable M f integrable M g*
  shows *AE x in M*. *real-cond-exp M F f x* $<$ *real-cond-exp M F g x*
⟨*proof*⟩

lemma *real-cond-exp-nested-subalg* [*intro*, *simp*]:
  assumes *subalgebra M G subalgebra G F*
    and [*measurable*]: *integrable M f*
  shows *AE x in M*. *real-cond-exp M F* (*real-cond-exp M G f*) $x =$ *real-cond-exp M F f x*
⟨*proof*⟩

lemma *real-cond-exp-sum* [*intro*, *simp*]:
  fixes $f::'b \Rightarrow 'a \Rightarrow real$
  assumes [*measurable*]: $\bigwedge i.$ *integrable M* (*f i*)
  shows *AE x in M*. *real-cond-exp M F* ($\lambda x.\ \sum i \in I.\ f\ i\ x$) x = ($\sum i \in I.$ *real-cond-exp M F* (*f i*) *x*)
⟨*proof*⟩

Jensen's inequality, describing the behavior of the integral under a convex function, admits a version for the conditional expectation, as follows.

theorem *real-cond-exp-jensens-inequality*:
  fixes $q :: real \Rightarrow real$
  assumes X: *integrable M X AE x in M. X x* $\in I$
  assumes I: $I = \{a <..< b\} \vee I = \{a <..\} \vee I = \{..< b\} \vee I = UNIV$
  assumes q: *integrable M* ($\lambda x.$ *q* (*X x*)) *convex-on I q q* $\in$ *borel-measurable borel*
  shows *AE x in M. real-cond-exp M F X x* $\in I$
    AE x in M. q (*real-cond-exp M F X x*) $\leq$ *real-cond-exp M F* ($\lambda x.$ *q* (*X x*)) *x*
⟨*proof*⟩

Jensen's inequality does not imply that $q(E(X|F))$ is integrable, as it only proves an upper bound for it. Indeed, this is not true in general, as the following counterexample shows:

on $[1, \infty)$ with Lebesgue measure, let F be the sigma-algebra generated by the intervals $[n, n+1)$ for integer n. Let $q(x) = -\sqrt{x}$ for $x \geq 0$. Define X which is equal to $1/n$ over $[n, n+1/n)$ and 2^{-n} on $[n+1/n, n+1)$. Then X is integrable as $\sum 1/n^2 < \infty$, and $q(X)$ is integrable as $\sum 1/n^{3/2} < \infty$. On the other hand, $E(X|F)$ is essentially equal to $1/n^2$ on $[n, n+1)$ (we neglect the term 2^{-n}, we only put it there because X should take its values in $I = (0, \infty)$). Hence, $q(E(X|F))$ is equal to $-1/n$ on $[n, n+1)$, hence it is not integrable.

However, this counterexample is essentially the only situation where this function is not integrable, as shown by the next lemma.

lemma *integrable-convex-cond-exp*:
  fixes $q :: real \Rightarrow real$
  assumes X: *integrable M X AE x in M. X x* $\in I$
  assumes I: $I = \{a <..< b\} \vee I = \{a <..\} \vee I = \{..< b\} \vee I = UNIV$
  assumes q: *integrable M* ($\lambda x.$ *q* (*X x*)) *convex-on I q q* $\in$ *borel-measurable borel*
  assumes H: *emeasure M* (*space M*) $= \infty \Longrightarrow$ *0* $\in I$
  shows *integrable M* ($\lambda x.$ *q* (*real-cond-exp M F X x*))
⟨*proof*⟩

end

end

theory *Essential-Supremum*
imports *HOL−Analysis.Analysis*
begin

lemma *ae-filter-eq-bot-iff*: *ae-filter M = bot* $\longleftrightarrow$ *emeasure M* (*space M*) = *0*
  ⟨*proof*⟩

29 The essential supremum

In this paragraph, we define the essential supremum and give its basic properties. The essential supremum of a function is its maximum value if one is allowed to throw away a set of measure 0. It is convenient to define it to be infinity for non-measurable functions, as it allows for neater statements in general. This is a prerequisiste to define the space L^∞.

definition $esssup::'a\ measure \Rightarrow ('a \Rightarrow 'b::\{second\text{-}countable\text{-}topology,\ dense\text{-}linorder,\ linorder\text{-}topology,\ complete\text{-}linorder\}) \Rightarrow 'b$
where $esssup\ M\ f = (if\ f \in borel\text{-}measurable\ M\ then\ Limsup\ (ae\text{-}filter\ M)\ f\ else\ top)$

lemma *esssup-non-measurable*: $f \notin M \to_M borel \Longrightarrow esssup\ M\ f = top$
⟨*proof*⟩

lemma *esssup-eq-AE*:
assumes f: $f \in M \to_M borel$ **shows** $esssup\ M\ f = Inf\ \{z.\ AE\ x\ in\ M.\ f\ x \le z\}$
⟨*proof*⟩

lemma *esssup-eq*: $f \in M \to_M borel \Longrightarrow esssup\ M\ f = Inf\ \{z.\ emeasure\ M\ \{x \in space\ M.\ f\ x > z\} = 0\}$
⟨*proof*⟩

lemma *esssup-zero-measure*:
$emeasure\ M\ \{x \in space\ M.\ f\ x > esssup\ M\ f\} = 0$
⟨*proof*⟩

lemma *esssup-AE*: $AE\ x\ in\ M.\ f\ x \le esssup\ M\ f$
⟨*proof*⟩

lemma *esssup-pos-measure*:
$f \in borel\text{-}measurable\ M \Longrightarrow z < esssup\ M\ f \Longrightarrow emeasure\ M\ \{x \in space\ M.\ f\ x > z\} > 0$
⟨*proof*⟩

lemma *esssup-I* [*intro*]: $f \in borel\text{-}measurable\ M \Longrightarrow AE\ x\ in\ M.\ f\ x \le c \Longrightarrow esssup\ M\ f \le c$
⟨*proof*⟩

lemma *esssup-AE-mono*: $f \in borel\text{-}measurable\ M \Longrightarrow AE\ x\ in\ M.\ f\ x \le g\ x \Longrightarrow esssup\ M\ f \le esssup\ M\ g$
⟨*proof*⟩

lemma *esssup-mono*: $f \in borel\text{-}measurable\ M \Longrightarrow (\bigwedge x.\ f\ x \le g\ x) \Longrightarrow esssup\ M\ f \le esssup\ M\ g$
⟨*proof*⟩

lemma *esssup-AE-cong*:

f ∈ *borel-measurable M* ⟹ *g* ∈ *borel-measurable M* ⟹ *AE x in M. f x = g x* ⟹ *esssup M f = esssup M g*
⟨*proof*⟩

lemma *esssup-const*: *emeasure M (space M)* ≠ *0* ⟹ *esssup M (λx. c) = c*
⟨*proof*⟩

lemma *esssup-cmult*: **assumes** *c* > (*0*::*real*) **shows** *esssup M (λx. c ∗ f x::ereal) = c ∗ esssup M f*
⟨*proof*⟩

lemma *esssup-add*:
esssup M (λx. f x + g x::ereal) ≤ *esssup M f + esssup M g*
⟨*proof*⟩

lemma *esssup-zero-space*:
emeasure M (space M) = 0 ⟹ *f* ∈ *borel-measurable M* ⟹ *esssup M f* = (−∞::*ereal*)
⟨*proof*⟩

end

30 Stopping times

theory *Stopping-Time*
imports *HOL−Analysis.Analysis*
begin

30.1 Stopping Time

This is also called strong stopping time. Then stopping time is T with alternative is $T\ x < t$ measurable.

definition *stopping-time* :: (′*t*::*linorder* ⇒ ′*a measure*) ⇒ (′*a* ⇒ ′*t*) ⇒ *bool*
where
stopping-time F T = (∀ *t. Measurable.pred (F t) (λx. T x* ≤ *t*))

lemma *stopping-time-cong*: (⋀*t x. x* ∈ *space (F t)* ⟹ *T x = S x*) ⟹ *stopping-time F T = stopping-time F S*
⟨*proof*⟩

lemma *stopping-timeD*: *stopping-time F T* ⟹ *Measurable.pred (F t) (λx. T x* ≤ *t*)
⟨*proof*⟩

lemma *stopping-timeD2*: *stopping-time F T* ⟹ *Measurable.pred (F t) (λx. t* < *T x*)
⟨*proof*⟩

lemma *stopping-timeI*[*intro?*]: $(\bigwedge t.\ Measurable.pred\ (F\ t)\ (\lambda x.\ T\ x \le t)) \Longrightarrow$ *stopping-time F T*
⟨*proof*⟩

lemma *measurable-stopping-time*:
fixes $T :: 'a \Rightarrow 't::\{linorder\text{-}topology,\ second\text{-}countable\text{-}topology\}$
assumes T: *stopping-time F T*
and M: $\bigwedge t.\ sets\ (F\ t) \subseteq sets\ M\ \bigwedge t.\ space\ (F\ t) = space\ M$
shows $T \in M \to_M borel$
⟨*proof*⟩

lemma *stopping-time-const*: *stopping-time F* $(\lambda x.\ c)$
⟨*proof*⟩

lemma *stopping-time-min*:
stopping-time F T $\Longrightarrow$ *stopping-time F S* $\Longrightarrow$ *stopping-time F* $(\lambda x.\ min\ (T\ x)\ (S\ x))$
⟨*proof*⟩

lemma *stopping-time-max*:
stopping-time F T $\Longrightarrow$ *stopping-time F S* $\Longrightarrow$ *stopping-time F* $(\lambda x.\ max\ (T\ x)\ (S\ x))$
⟨*proof*⟩

31 Filtration

locale *filtration* =
fixes $\Omega :: 'a\ set$ **and** $F :: 't::\{linorder\text{-}topology,\ second\text{-}countable\text{-}topology\} \Rightarrow 'a\ measure$
assumes *space-F*: $\bigwedge i.\ space\ (F\ i) = \Omega$
assumes *sets-F-mono*: $\bigwedge i\ j.\ i \le j \Longrightarrow sets\ (F\ i) \le sets\ (F\ j)$
begin

31.1 σ-algebra of a Stopping Time

definition *pre-sigma* :: $('a \Rightarrow 't) \Rightarrow 'a\ measure$
where
pre-sigma $T = sigma\ \Omega\ \{A.\ \forall t.\ \{\omega\in A.\ T\ \omega \le t\} \in sets\ (F\ t)\}$

lemma *space-pre-sigma*: *space* (*pre-sigma* T) $= \Omega$
⟨*proof*⟩

lemma *measure-pre-sigma*[*simp*]: *emeasure* (*pre-sigma* T) $= (\lambda\text{-}.\ 0)$
⟨*proof*⟩

lemma *sigma-algebra-pre-sigma*:
assumes T: *stopping-time F T*
shows *sigma-algebra* $\Omega\ \{A.\ \forall t.\ \{\omega\in A.\ T\ \omega \le t\} \in sets\ (F\ t)\}$
⟨*proof*⟩

lemma *sets-pre-sigma*: *stopping-time* F $T \Longrightarrow$ *sets* (*pre-sigma* T) = $\{A.\ \forall\, t.\ \{\omega \in A.\ T\ \omega \le t\} \in$ *sets* $(F\ t)\}$
⟨*proof*⟩

lemma *sets-pre-sigmaI*: *stopping-time* F $T \Longrightarrow (\bigwedge t.\ \{\omega \in A.\ T\ \omega \le t\} \in$ *sets* $(F\ t)) \Longrightarrow A \in$ *sets* (*pre-sigma* T)
⟨*proof*⟩

lemma *pred-pre-sigmaI*:
assumes T: *stopping-time* F T
shows $(\bigwedge t.$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ P\ \omega \wedge T\ \omega \le t)) \Longrightarrow$ *Measurable.pred* (*pre-sigma* T) P
⟨*proof*⟩

lemma *sets-pre-sigmaD*: *stopping-time* F $T \Longrightarrow A \in$ *sets* (*pre-sigma* T) $\Longrightarrow \{\omega \in A.\ T\ \omega \le t\} \in$ *sets* $(F\ t)$
⟨*proof*⟩

lemma *stopping-time-le-const*: *stopping-time* F $T \Longrightarrow s \le t \Longrightarrow$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega \le s)$
⟨*proof*⟩

lemma *measurable-stopping-time-pre-sigma*:
assumes T: *stopping-time* F T **shows** $T \in$ *pre-sigma* $T \rightarrow_M$ *borel*
⟨*proof*⟩

lemma *mono-pre-sigma*:
assumes T: *stopping-time* F T **and** S: *stopping-time* F S
and *le*: $\bigwedge\omega.\ \omega \in \Omega \Longrightarrow T\ \omega \le S\ \omega$
shows *sets* (*pre-sigma* T) $\subseteq$ *sets* (*pre-sigma* S)
⟨*proof*⟩

lemma *stopping-time-less-const*:
assumes T: *stopping-time* F T **shows** *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega < t)$
⟨*proof*⟩

lemma *stopping-time-eq-const*: *stopping-time* F $T \Longrightarrow$ *Measurable.pred* $(F\ t)\ (\lambda\omega.\ T\ \omega = t)$
⟨*proof*⟩

lemma *stopping-time-less*:
assumes T: *stopping-time* F T **and** S: *stopping-time* F S
shows *Measurable.pred* (*pre-sigma* T) $(\lambda\omega.\ T\ \omega < S\ \omega)$
⟨*proof*⟩

end

lemma *stopping-time-SUP-enat*:

fixes T :: $nat \Rightarrow ('a \Rightarrow enat)$
shows $(\bigwedge i.\ stopping\text{-}time\ F\ (T\ i)) \Longrightarrow stopping\text{-}time\ F\ (SUP\ i.\ T\ i)$
$\langle proof \rangle$

lemma *less-eSuc-iff*: $a < eSuc\ b \longleftrightarrow (a \leq b \land a \neq \infty)$
$\langle proof \rangle$

lemma *stopping-time-Inf-enat*:
fixes F :: $enat \Rightarrow 'a\ measure$
assumes F: $filtration\ \Omega\ F$
assumes P: $\bigwedge i.\ Measurable.pred\ (F\ i)\ (P\ i)$
shows $stopping\text{-}time\ F\ (\lambda\omega.\ Inf\ \{i.\ P\ i\ \omega\})$
$\langle proof \rangle$

lemma *stopping-time-Inf-nat*:
fixes F :: $nat \Rightarrow 'a\ measure$
assumes F: $filtration\ \Omega\ F$
assumes P: $\bigwedge i.\ Measurable.pred\ (F\ i)\ (P\ i)$ **and** wf: $\bigwedge i\ \omega.\ \omega \in \Omega \Longrightarrow \exists\, n.\ P\ n$
ω
shows $stopping\text{-}time\ F\ (\lambda\omega.\ Inf\ \{i.\ P\ i\ \omega\})$
$\langle proof \rangle$

end

theory *Probability*
imports
Central-Limit-Theorem
Discrete-Topology
PMF-Impl
Projective-Limit
Random-Permutations
SPMF
Product-PMF
Hoeffding
Stream-Space
Tree-Space
Conditional-Expectation
Essential-Supremum
Stopping-Time
begin

end