Fundamental Properties of Lambda-calculus

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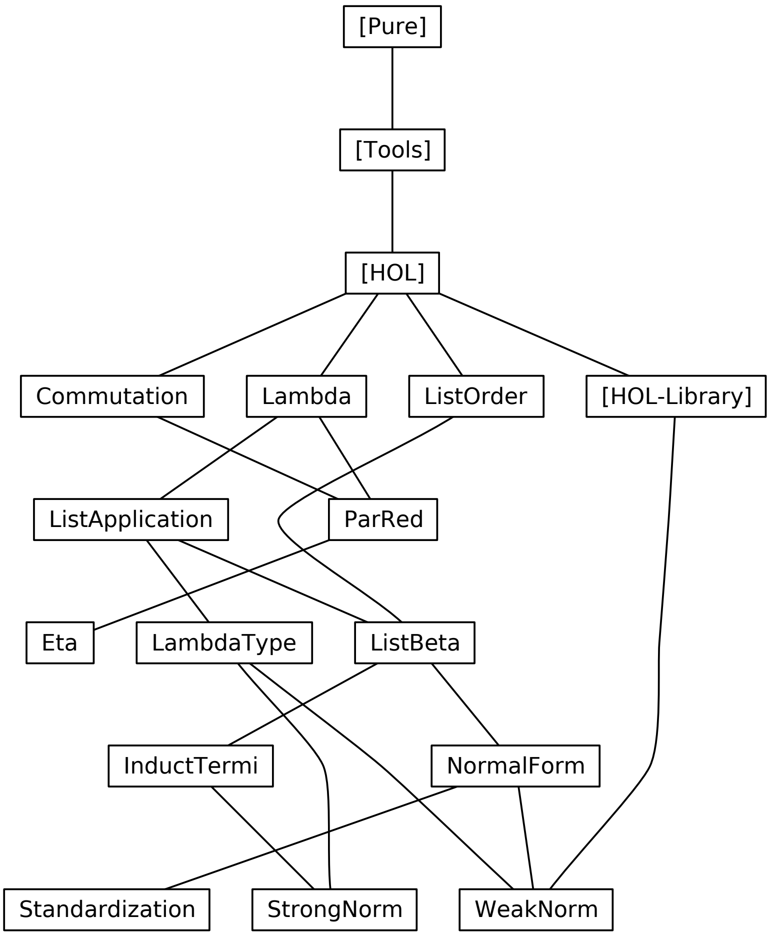
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1 Basic definitions of Lambda-calculus

```
theory Lambda
imports Main
begin

declare [[syntax-ambiguity-warning = false]]
```

1.1 Lambda-terms in de Bruijn notation and substitution

```
datatype dB =
    Var nat
  | App dB dB (infixl ° 200)
  | Abs dB

primrec
  lift :: [dB, nat] => dB
where
    lift (Var i) k = (if i < k then Var i else Var (i + 1))
  | lift (s ° t) k = lift s k ° lift t k
  | lift (Abs s) k = Abs (lift s (k + 1))

primrec
  subst :: [dB, dB, nat] => dB  (-[-'/-] [300, 0, 0] 300)
where
    subst-Var: (Var i)[s/k] =
      (if k < i then Var (i − 1) else if i = k then s else Var i)
  | subst-App: (t ° u)[s/k] = t[s/k] ° u[s/k]
  | subst-Abs: (Abs t)[s/k] = Abs (t[lift s 0 / k+1])

declare subst-Var [simp del]
```

Optimized versions of *subst* and *lift*.

```
primrec
  liftn :: [nat, dB, nat] => dB
where
    liftn n (Var i) k = (if i < k then Var i else Var (i + n))
  | liftn n (s ° t) k = liftn n s k ° liftn n t k
  | liftn n (Abs s) k = Abs (liftn n s (k + 1))

primrec
  substn :: [dB, dB, nat] => dB
where
    substn (Var i) s k =
      (if k < i then Var (i − 1) else if i = k then liftn k s 0 else Var i)
  | substn (t ° u) s k = substn t s k ° substn u s k
  | substn (Abs t) s k = Abs (substn t s (k + 1))
```

1.2 Beta-reduction

inductive *beta* :: [*dB*, *dB*] => *bool* (**infixl** $\to_\beta$ *50*)
where
beta [*simp*, *intro*!]: *Abs* s ° $t \to_\beta s[t/0]$
| *appL* [*simp*, *intro*!]: $s \to_\beta t$ ==> s ° $u \to_\beta t$ ° u
| *appR* [*simp*, *intro*!]: $s \to_\beta t$ ==> u ° $s \to_\beta u$ ° t
| *abs* [*simp*, *intro*!]: $s \to_\beta t$ ==> *Abs* $s \to_\beta$ *Abs* t

abbreviation
beta-reds :: [*dB*, *dB*] => *bool* (**infixl** $\to_\beta{}^*$ *50*) **where**
$s \to_\beta{}^* t$ == *beta*** s t

inductive-cases *beta-cases* [*elim*!]:
Var $i \to_\beta t$
Abs $r \to_\beta s$
s ° $t \to_\beta u$

declare *if-not-P* [*simp*] *not-less-eq* [*simp*]
— don't add *r-into-rtrancl*[*intro*!]

1.3 Congruence rules

lemma *rtrancl-beta-Abs* [*intro*!]:
$s \to_\beta{}^* s'$ ==> *Abs* $s \to_\beta{}^*$ *Abs* s'
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-AppL*:
$s \to_\beta{}^* s'$ ==> s ° $t \to_\beta{}^* s'$ ° t
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-AppR*:
$t \to_\beta{}^* t'$ ==> s ° $t \to_\beta{}^* s$ ° t'
by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-beta-App* [*intro*]:
[| $s \to_\beta{}^* s'$; $t \to_\beta{}^* t'$ |] ==> s ° $t \to_\beta{}^* s'$ ° t'
by (*blast intro*!: *rtrancl-beta-AppL rtrancl-beta-AppR intro*: *rtranclp-trans*)

1.4 Substitution-lemmas

lemma *subst-eq* [*simp*]: (*Var* k)[u/k] = u
by (*simp add*: *subst-Var*)

lemma *subst-gt* [*simp*]: $i < j$ ==> (*Var* j)[u/i] = *Var* (j − *1*)
by (*simp add*: *subst-Var*)

lemma *subst-lt* [*simp*]: $j < i$ ==> (*Var* j)[u/i] = *Var* j
by (*simp add*: *subst-Var*)

lemma *lift-lift*:
$i < k + 1 \Longrightarrow lift\ (lift\ t\ i)\ (Suc\ k) = lift\ (lift\ t\ k)\ i$
by (*induct t arbitrary*: *i k*) *auto*

lemma *lift-subst* [*simp*]:
$j < i + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ (i + 1))\ [lift\ s\ i\ /\ j]$
by (*induct t arbitrary*: *i j s*)
(*simp-all add*: *diff-Suc subst-Var lift-lift split*: *nat.split*)

lemma *lift-subst-lt*:
$i < j + 1 \Longrightarrow lift\ (t[s/j])\ i = (lift\ t\ i)\ [lift\ s\ i\ /\ j + 1]$
by (*induct t arbitrary*: *i j s*) (*simp-all add*: *subst-Var lift-lift*)

lemma *subst-lift* [*simp*]:
$(lift\ t\ k)[s/k] = t$
by (*induct t arbitrary*: *k s*) *simp-all*

lemma *subst-subst*:
$i < j + 1 \Longrightarrow t[lift\ v\ i\ /\ Suc\ j][u[v/j]/i] = t[u/i][v/j]$
by (*induct t arbitrary*: *i j u v*)
(*simp-all add*: *diff-Suc subst-Var lift-lift* [*symmetric*] *lift-subst-lt*
split: *nat.split*)

1.5 Equivalence proof for optimized substitution

lemma *liftn-0* [*simp*]: $liftn\ 0\ t\ k = t$
by (*induct t arbitrary*: *k*) (*simp-all add*: *subst-Var*)

lemma *liftn-lift* [*simp*]: $liftn\ (Suc\ n)\ t\ k = lift\ (liftn\ n\ t\ k)\ k$
by (*induct t arbitrary*: *k*) (*simp-all add*: *subst-Var*)

lemma *substn-subst-n* [*simp*]: $substn\ t\ s\ n = t[liftn\ n\ s\ 0\ /\ n]$
by (*induct t arbitrary*: *n*) (*simp-all add*: *subst-Var*)

theorem *substn-subst-0*: $substn\ t\ s\ 0 = t[s/0]$
by *simp*

1.6 Preservation theorems

Not used in Church-Rosser proof, but in Strong Normalization.

theorem *subst-preserves-beta* [*simp*]:
$r \rightarrow_\beta s$ ==> $r[t/i] \rightarrow_\beta s[t/i]$
by (*induct arbitrary*: *t i set*: *beta*) (*simp-all add*: *subst-subst* [*symmetric*])

theorem *subst-preserves-beta'*: $r \rightarrow_\beta{}^* s$ ==> $r[t/i] \rightarrow_\beta{}^* s[t/i]$
apply (*induct set*: *rtranclp*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*erule rtranclp.rtrancl-into-rtrancl*)

apply (*erule subst-preserves-beta*)
done

theorem *lift-preserves-beta* [*simp*]:
$r \to_\beta s$ ==> *lift r i* $\to_\beta$ *lift s i*
by (*induct arbitrary*: *i set*: *beta*) *auto*

theorem *lift-preserves-beta'*: $r \to_\beta{}^* s$ ==> *lift r i* $\to_\beta{}^*$ *lift s i*
apply (*induct set*: *rtranclp*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*erule rtranclp.rtrancl-into-rtrancl*)
apply (*erule lift-preserves-beta*)
done

theorem *subst-preserves-beta2* [*simp*]: $r \to_\beta s$ ==> $t[r/i] \to_\beta{}^* t[s/i]$
apply (*induct t arbitrary*: *r s i*)
apply (*simp add*: *subst-Var r-into-rtranclp*)
apply (*simp add*: *rtrancl-beta-App*)
apply (*simp add*: *rtrancl-beta-Abs*)
done

theorem *subst-preserves-beta2'*: $r \to_\beta{}^* s$ ==> $t[r/i] \to_\beta{}^* t[s/i]$
apply (*induct set*: *rtranclp*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*erule rtranclp-trans*)
apply (*erule subst-preserves-beta2*)
done

end

2 Abstract commutation and confluence notions

theory *Commutation*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

2.1 Basic definitions

definition
square :: ['a => 'a => *bool*, 'a => 'a => *bool*, 'a => 'a => *bool*, 'a => 'a => *bool*] => *bool* **where**
square R S T U =
$(\forall x\, y.\ R\ x\ y \dashrightarrow (\forall z.\ S\ x\ z \dashrightarrow (\exists u.\ T\ y\ u \wedge U\ z\ u)))$

definition
commute :: ['a => 'a => *bool*, 'a => 'a => *bool*] => *bool* **where**
commute R S = *square R S S R*

definition
diamond :: $('a => 'a => bool) => bool$ **where**
diamond R = *commute* R R

definition
Church-Rosser :: $('a => 'a => bool) => bool$ **where**
Church-Rosser R =
$(\forall x\ y.\ (sup\ R\ (R^{-1-1}))^{**}\ x\ y \longrightarrow (\exists z.\ R^{**}\ x\ z \wedge R^{**}\ y\ z))$

abbreviation
confluent :: $('a => 'a => bool) => bool$ **where**
confluent R == *diamond* (R^{**})

2.2 Basic lemmas

square

lemma *square-sym*: *square* $R\ S\ T\ U$ ==> *square* $S\ R\ U\ T$
apply (*unfold square-def*)
apply *blast*
done

lemma *square-subset*:
[| *square* $R\ S\ T\ U$; $T \leq T'$ |] ==> *square* $R\ S\ T'\ U$
apply (*unfold square-def*)
apply (*blast dest*: *predicate2D*)
done

lemma *square-reflcl*:
[| *square* $R\ S\ T\ (R^{==})$; $S \leq T$ |] ==> *square* $(R^{==})\ S\ T\ (R^{==})$
apply (*unfold square-def*)
apply (*blast dest*: *predicate2D*)
done

lemma *square-rtrancl*:
square $R\ S\ S\ T$ ==> *square* $(R^{**})\ S\ S\ (T^{**})$
apply (*unfold square-def*)
apply (*intro strip*)
apply (*erule rtranclp-induct*)
apply *blast*
apply (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)
done

lemma *square-rtrancl-reflcl-commute*:
square $R\ S\ (S^{**})\ (R^{==})$ ==> *commute* $(R^{**})\ (S^{**})$
apply (*unfold commute-def*)
apply (*fastforce dest*: *square-reflcl square-sym* [*THEN square-rtrancl*])
done

commute

lemma *commute-sym*: *commute R S* ==> *commute S R*
  apply (*unfold commute-def*)
  apply (*blast intro*: *square-sym*)
  done

lemma *commute-rtrancl*: *commute R S* ==> *commute* (R^{**}) (S^{**})
  apply (*unfold commute-def*)
  apply (*blast intro*: *square-rtrancl square-sym*)
  done

lemma *commute-Un*:
    [| *commute R T*; *commute S T* |] ==> *commute* (*sup R S*) *T*
  apply (*unfold commute-def square-def*)
  apply *blast*
  done

diamond, *confluence*, and *union*

lemma *diamond-Un*:
    [| *diamond R*; *diamond S*; *commute R S* |] ==> *diamond* (*sup R S*)
  apply (*unfold diamond-def*)
  apply (*blast intro*: *commute-Un commute-sym*)
  done

lemma *diamond-confluent*: *diamond R* ==> *confluent R*
  apply (*unfold diamond-def*)
  apply (*erule commute-rtrancl*)
  done

lemma *square-reflcl-confluent*:
    square R R ($R^{==}$) ($R^{==}$) ==> *confluent R*
  apply (*unfold diamond-def*)
  apply (*fast intro*: *square-rtrancl-reflcl-commute elim*: *square-subset*)
  done

lemma *confluent-Un*:
    [| *confluent R*; *confluent S*; *commute* (R^{**}) (S^{**}) |] ==> *confluent* (*sup R S*)
  apply (*rule rtranclp-sup-rtranclp* [*THEN subst*])
  apply (*blast dest*: *diamond-Un intro*: *diamond-confluent*)
  done

lemma *diamond-to-confluence*:
    [| *diamond R*; $T \leq R$; $R \leq T^{**}$ |] ==> *confluent T*
  apply (*force intro*: *diamond-confluent*
    dest: *rtranclp-subset* [*symmetric*])
  done

2.3 Church-Rosser

lemma *Church-Rosser-confluent*: *Church-Rosser R* = *confluent R*
apply (*unfold square-def commute-def diamond-def Church-Rosser-def*)
apply (*tactic* ‹*safe-tac* (*put-claset HOL-cs* ***context***)›)
apply (*tactic* ‹
blast-tac (*put-claset HOL-cs* ***context*** *addIs*
[@{*thm sup-ge2*} *RS* @{*thm rtranclp-mono*} *RS* @{*thm predicate2D*} *RS* @{*thm rtranclp-trans*},
@{*thm rtranclp-converseI*}, @{*thm conversepI*},
@{*thm sup-ge1*} *RS* @{*thm rtranclp-mono*} *RS* @{*thm predicate2D*}]) *1*›)
apply (*erule rtranclp-induct*)
apply *blast*
apply (*blast del*: *rtranclp.rtrancl-refl intro*: *rtranclp-trans*)
done

2.4 Newman's lemma

Proof by Stefan Berghofer

theorem *newman*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
using *wf*
proof *induct*
case (*less x b c*)
have *xc*: $R^{**}\ x\ c$ **by** *fact*
have *xb*: $R^{**}\ x\ b$ **by** *fact* **thus** *?case*
proof (*rule converse-rtranclpE*)
assume $x = b$
with *xc* **have** $R^{**}\ b\ c$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix *y*
assume *xy*: $R\ x\ y$
assume *yb*: $R^{**}\ y\ b$
from *xc* **show** *?thesis*
proof (*rule converse-rtranclpE*)
assume $x = c$
with *xb* **have** $R^{**}\ c\ b$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix y'
assume $y'c$: $R^{**}\ y'\ c$
assume xy': $R\ x\ y'$
with *xy* **have** $\exists\, u.\ R^{**}\ y\ u \wedge R^{**}\ y'\ u$ **by** (*rule lc*)
then obtain *u* **where** *yu*: $R^{**}\ y\ u$ **and** $y'u$: $R^{**}\ y'\ u$ **by** *iprover*

from *xy* **have** $R^{-1-1}\ y\ x$ **..**
from *this* **and** *yb yu* **have** $\exists\, d.\ R^{**}\ b\ d \land R^{**}\ u\ d$ **by** (*rule less*)
then obtain v **where** *bv*: $R^{**}\ b\ v$ **and** *uv*: $R^{**}\ u\ v$ **by** *iprover*
from *xy'* **have** $R^{-1-1}\ y'\ x$ **..**
moreover from *y'u* **and** *uv* **have** $R^{**}\ y'\ v$ **by** (*rule rtranclp-trans*)
moreover note *y'c*
ultimately have $\exists\, d.\ R^{**}\ v\ d \land R^{**}\ c\ d$ **by** (*rule less*)
then obtain w **where** *vw*: $R^{**}\ v\ w$ **and** *cw*: $R^{**}\ c\ w$ **by** *iprover*
from *bv vw* **have** $R^{**}\ b\ w$ **by** (*rule rtranclp-trans*)
with *cw* **show** *?thesis* **by** *iprover*
qed
qed
qed

Alternative version. Partly automated by Tobias Nipkow. Takes 2 minutes (2002).

This is the maximal amount of automation possible using *blast*.

theorem *newman'*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \land R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \land R^{**}\ c\ d$
using *wf*
proof *induct*
case (*less x b c*)
note $IH = \langle \bigwedge y\ b\ c.\ [\![R^{-1-1}\ y\ x;\ R^{**}\ y\ b;\ R^{**}\ y\ c]\!]$
$\Longrightarrow \exists\, d.\ R^{**}\ b\ d \land R^{**}\ c\ d\rangle$
have *xc*: $R^{**}\ x\ c$ **by** *fact*
have *xb*: $R^{**}\ x\ b$ **by** *fact*
thus *?case*
proof (*rule converse-rtranclpE*)
assume $x = b$
with *xc* **have** $R^{**}\ b\ c$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix y
assume *xy*: $R\ x\ y$
assume *yb*: $R^{**}\ y\ b$
from *xc* **show** *?thesis*
proof (*rule converse-rtranclpE*)
assume $x = c$
with *xb* **have** $R^{**}\ c\ b$ **by** *simp*
thus *?thesis* **by** *iprover*
next
fix y'
assume *y'c*: $R^{**}\ y'\ c$
assume *xy'*: $R\ x\ y'$
with *xy* **obtain** u **where** u: $R^{**}\ y\ u\ R^{**}\ y'\ u$

by (*blast dest*: *lc*)
from *yb u y′c* **show** *?thesis*
by (*blast del*: *rtranclp.rtrancl-refl*
intro: *rtranclp-trans*
dest: *IH* [*OF conversepI*, *OF xy*] *IH* [*OF conversepI*, *OF xy′*])
qed
qed
qed

Using the coherent logic prover, the proof of the induction step is completely automatic.

lemma *eq-imp-rtranclp*: $x = y \Longrightarrow r^{**}\ x\ y$
by *simp*

theorem *newman′′*:
assumes *wf*: *wfP* (R^{-1-1})
and *lc*: $\bigwedge a\ b\ c.\ R\ a\ b \Longrightarrow R\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
shows $\bigwedge b\ c.\ R^{**}\ a\ b \Longrightarrow R^{**}\ a\ c \Longrightarrow$
$\exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d$
using *wf*
proof *induct*
case (*less x b c*)
note $IH = ‹\bigwedge y\ b\ c.\ [\![R^{-1-1}\ y\ x;\ R^{**}\ y\ b;\ R^{**}\ y\ c]\!]$
$\Longrightarrow \exists\, d.\ R^{**}\ b\ d \wedge R^{**}\ c\ d›$
show *?case*
by (*coherent*
$‹R^{**}\ x\ c›\ ‹R^{**}\ x\ b›$
refl [**where** *′a=′a*] *sym*
eq-imp-rtranclp
r-into-rtranclp [*of R*]
rtranclp-trans
lc IH [*OF conversepI*]
converse-rtranclpE)
qed

end

3 Parallel reduction and a complete developments

theory *ParRed* **imports** *Lambda Commutation* **begin**

3.1 Parallel reduction

inductive *par-beta* :: [*dB*, *dB*] => *bool* (**infixl** => *50*)
where
var [*simp*, *intro*!]: *Var n* => *Var n*
| *abs* [*simp*, *intro*!]: *s* => *t* ==> *Abs s* => *Abs t*
| *app* [*simp*, *intro*!]: [| *s* => *s′*; *t* => *t′* |] ==> *s* ° *t* => *s′* ° *t′*

| *beta* [*simp*, *intro*!]: [| $s \Rightarrow s'$; $t \Rightarrow t'$ |] ==> (*Abs s*) ° t => $s'[t'/0]$

inductive-cases *par-beta-cases* [*elim*!]:
Var n => *t*
Abs s => *Abs t*
(*Abs s*) ° *t* => *u*
s ° *t* => *u*
Abs s => *t*

3.2 Inclusions

beta ⊆ *par-beta* ⊆ *beta**

lemma *par-beta-varL* [*simp*]:
(*Var n* => *t*) = (*t* = *Var n*)
by *blast*

lemma *par-beta-refl* [*simp*]: *t* => *t*
by (*induct t*) *simp-all*

lemma *beta-subset-par-beta*: *beta* <= *par-beta*
apply (*rule predicate2I*)
apply (*erule beta.induct*)
apply (*blast intro*!: *par-beta-refl*)+
done

lemma *par-beta-subset-beta*: *par-beta* ≤ *beta***
apply (*rule predicate2I*)
apply (*erule par-beta.induct*)
apply *blast*
apply (*blast del*: *rtranclp.rtrancl-refl intro*: *rtranclp.rtrancl-into-rtrancl*)+
— *rtrancl-refl* complicates the proof by increasing the branching factor
done

3.3 Misc properties of *par-beta*

lemma *par-beta-lift* [*simp*]:
$t \Rightarrow t' \Longrightarrow$ *lift t n* => *lift t' n*
by (*induct t arbitrary*: *t' n*) *fastforce*+

lemma *par-beta-subst*:
$s \Rightarrow s' \Longrightarrow t \Rightarrow t' \Longrightarrow t[s/n] \Rightarrow t'[s'/n]$
apply (*induct t arbitrary*: *s s' t' n*)
apply (*simp add*: *subst-Var*)
apply (*erule par-beta-cases*)
apply *simp*
apply (*simp add*: *subst-subst* [*symmetric*])
apply (*fastforce intro*!: *par-beta-lift*)
apply *fastforce*
done

3.4 Confluence (directly)

```
lemma diamond-par-beta: diamond par-beta
  apply (unfold diamond-def commute-def square-def)
  apply (rule impI [THEN allI [THEN allI]])
  apply (erule par-beta.induct)
     apply (blast intro!: par-beta-subst)+
  done
```

3.5 Complete developments

```
fun
  cd :: dB => dB
where
  cd (Var n) = Var n
| cd (Var n ° t) = Var n ° cd t
| cd ((s1 ° s2) ° t) = cd (s1 ° s2) ° cd t
| cd (Abs u ° t) = (cd u)[cd t/0]
| cd (Abs s) = Abs (cd s)

lemma par-beta-cd: s => t ⟹ t => cd s
  apply (induct s arbitrary: t rule: cd.induct)
      apply auto
  apply (fast intro!: par-beta-subst)
  done
```

3.6 Confluence (via complete developments)

```
lemma diamond-par-beta2: diamond par-beta
  apply (unfold diamond-def commute-def square-def)
  apply (blast intro: par-beta-cd)
  done

theorem beta-confluent: confluent beta
  apply (rule diamond-par-beta2 diamond-to-confluence
    par-beta-subset-beta beta-subset-par-beta)+
  done

end
```

4 Eta-reduction

```
theory Eta imports ParRed begin
```

4.1 Definition of eta-reduction and relatives

```
primrec
  free :: dB => nat => bool
where
    free (Var j) i = (j = i)
```

```
  | free (s ° t) i = (free s i ∨ free t i)
  | free (Abs s) i = free s (i + 1)

inductive
  eta :: [dB, dB] => bool  (infixl →η 50)
where
    eta [simp, intro]: ¬ free s 0 ==> Abs (s ° Var 0) →η s[dummy/0]
  | appL [simp, intro]: s →η t ==> s ° u →η t ° u
  | appR [simp, intro]: s →η t ==> u ° s →η u ° t
  | abs [simp, intro]: s →η t ==> Abs s →η Abs t

abbreviation
  eta-reds :: [dB, dB] => bool   (infixl →η* 50) where
  s →η* t == eta** s t

abbreviation
  eta-red0 :: [dB, dB] => bool   (infixl →η= 50) where
  s →η= t == eta== s t

inductive-cases eta-cases [elim!]:
  Abs s →η z
  s ° t →η u
  Var i →η t
```

4.2 Properties of *eta*, *subst* and *free*

```
lemma subst-not-free [simp]: ¬ free s i ⟹ s[t/i] = s[u/i]
  by (induct s arbitrary: i t u) (simp-all add: subst-Var)

lemma free-lift [simp]:
    free (lift t k) i = (i < k ∧ free t i ∨ k < i ∧ free t (i − 1))
  apply (induct t arbitrary: i k)
  apply (auto cong: conj-cong)
  done

lemma free-subst [simp]:
    free (s[t/k]) i =
      (free s k ∧ free t i ∨ free s (if i < k then i else i + 1))
  apply (induct s arbitrary: i k t)
    prefer 2
    apply simp
    apply blast
   prefer 2
   apply simp
  apply (simp add: diff-Suc subst-Var split: nat.split)
  done

lemma free-eta: s →η t ==> free t i = free s i
  by (induct arbitrary: i set: eta) (simp-all cong: conj-cong)
```

lemma *not-free-eta*:
  [| $s \to_\eta t$; ¬ *free s i* |] ==> ¬ *free t i*
 by (*simp add*: *free-eta*)

lemma *eta-subst* [*simp*]:
  $s \to_\eta t$ ==> $s[u/i] \to_\eta t[u/i]$
 by (*induct arbitrary*: *u i set*: *eta*) (*simp-all add*: *subst-subst* [*symmetric*])

theorem *lift-subst-dummy*: ¬ *free s i* ⟹ *lift* (*s*[*dummy*/*i*]) *i* = *s*
 by (*induct s arbitrary*: *i dummy*) *simp-all*

4.3 Confluence of *eta*

lemma *square-eta*: *square eta eta* ($eta^{==}$) ($eta^{==}$)
 apply (*unfold square-def id-def*)
 apply (*rule impI* [*THEN allI* [*THEN allI*]])
 apply (*erule eta.induct*)
   apply (*slowsimp intro*: *subst-not-free eta-subst free-eta* [*THEN iffD1*])
  apply *safe*
    prefer *5*
    apply (*blast intro*!: *eta-subst intro*: *free-eta* [*THEN iffD1*])
   apply *blast+*
 done

theorem *eta-confluent*: *confluent eta*
 apply (*rule square-eta* [*THEN square-reflcl-confluent*])
 done

4.4 Congruence rules for eta^*

lemma *rtrancl-eta-Abs*: $s \to_\eta{}^* s'$ ==> *Abs* $s \to_\eta{}^*$ *Abs* s'
 by (*induct set*: *rtranclp*)
  (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-AppL*: $s \to_\eta{}^* s'$ ==> $s \circ t \to_\eta{}^* s' \circ t$
 by (*induct set*: *rtranclp*)
  (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-AppR*: $t \to_\eta{}^* t'$ ==> $s \circ t \to_\eta{}^* s \circ t'$
 by (*induct set*: *rtranclp*) (*blast intro*: *rtranclp.rtrancl-into-rtrancl*)+

lemma *rtrancl-eta-App*:
  [| $s \to_\eta{}^* s'$; $t \to_\eta{}^* t'$ |] ==> $s \circ t \to_\eta{}^* s' \circ t'$
 by (*blast intro*!: *rtrancl-eta-AppL rtrancl-eta-AppR intro*: *rtranclp-trans*)

4.5 Commutation of *beta* and *eta*

lemma *free-beta*:
  $s \to_\beta t$ ==> *free t i* ⟹ *free s i*

by (*induct arbitrary*: *i set*: *beta*) *auto*

lemma *beta-subst* [*intro*]: $s \to_\beta t ==> s[u/i] \to_\beta t[u/i]$
by (*induct arbitrary*: *u i set*: *beta*) (*simp-all add*: *subst-subst* [*symmetric*])

lemma *subst-Var-Suc* [*simp*]: $t[Var\ i/i] = t[Var(i)/i + 1]$
by (*induct t arbitrary*: *i*) (*auto elim*!: *linorder-neqE simp*: *subst-Var*)

lemma *eta-lift* [*simp*]: $s \to_\eta t ==> lift\ s\ i \to_\eta lift\ t\ i$
by (*induct arbitrary*: *i set*: *eta*) *simp-all*

lemma *rtrancl-eta-subst*: $s \to_\eta t \Longrightarrow u[s/i] \to_\eta{}^* u[t/i]$
apply (*induct u arbitrary*: *s t i*)
apply (*simp-all add*: *subst-Var*)
apply *blast*
apply (*blast intro*: *rtrancl-eta-App*)
apply (*blast intro*!: *rtrancl-eta-Abs eta-lift*)
done

lemma *rtrancl-eta-subst'*:
fixes *s t* :: *dB*
assumes *eta*: $s \to_\eta{}^* t$
shows $s[u/i] \to_\eta{}^* t[u/i]$ **using** *eta*
by *induct* (*iprover intro*: *eta-subst*)+

lemma *rtrancl-eta-subst''*:
fixes *s t* :: *dB*
assumes *eta*: $s \to_\eta{}^* t$
shows $u[s/i] \to_\eta{}^* u[t/i]$ **using** *eta*
by *induct* (*iprover intro*: *rtrancl-eta-subst rtranclp-trans*)+

lemma *square-beta-eta*: *square beta eta* (eta^{**}) ($beta^{==}$)
apply (*unfold square-def*)
apply (*rule impI* [*THEN allI* [*THEN allI*]])
apply (*erule beta.induct*)
apply (*slowsimp intro*: *rtrancl-eta-subst eta-subst*)
apply (*blast intro*: *rtrancl-eta-AppL*)
apply (*blast intro*: *rtrancl-eta-AppR*)
apply *simp*
apply (*slowsimp intro*: *rtrancl-eta-Abs free-beta*
iff del: *dB.distinct simp*: *dB.distinct*)
done

lemma *confluent-beta-eta*: *confluent* (*sup beta eta*)
apply (*assumption* |
rule square-rtrancl-reflcl-commute confluent-Un
beta-confluent eta-confluent square-beta-eta)+
done

4.6 Implicit definition of *eta*

Abs (*lift s 0* ° *Var 0*) $\to_\eta$ *s*

```
lemma not-free-iff-lifted:
   (¬ free s i) = (∃ t. s = lift t i)
  apply (induct s arbitrary: i)
    apply simp
    apply (rule iffI)
     apply (erule linorder-neqE)
      apply (rename-tac nat a, rule-tac x = Var nat in exI)
      apply simp
     apply (rename-tac nat a, rule-tac x = Var (nat − 1) in exI)
     apply simp
    apply clarify
    apply (rule notE)
     prefer 2
     apply assumption
    apply (erule thin-rl)
    apply (case-tac t)
      apply simp
     apply simp
    apply simp
   apply simp
   apply (erule thin-rl)
   apply (erule thin-rl)
   apply (rule iffI)
    apply (elim conjE exE)
    apply (rename-tac u1 u2)
    apply (rule-tac x = u1 ° u2 in exI)
    apply simp
   apply (erule exE)
   apply (erule rev-mp)
   apply (case-tac t)
     apply simp
    apply simp
    apply blast
   apply simp
  apply simp
  apply (erule thin-rl)
  apply (rule iffI)
   apply (erule exE)
   apply (rule-tac x = Abs t in exI)
   apply simp
  apply (erule exE)
  apply (erule rev-mp)
  apply (case-tac t)
    apply simp
   apply simp
  apply simp
```

apply *blast*
done

theorem *explicit-is-implicit*:
$(\forall s\ u.\ (\neg\ free\ s\ 0) \ {-}{-}{>}\ R\ (Abs\ (s\ {}^\circ\ Var\ 0))\ (s[u/0])) =$
$(\forall s.\ R\ (Abs\ (lift\ s\ 0\ {}^\circ\ Var\ 0))\ s)$
by (*auto simp add*: *not-free-iff-lifted*)

4.7 Eta-postponement theorem

Based on a paper proof due to Andreas Abel. Unlike the proof by Masako Takahashi [4], it does not use parallel eta reduction, which only seems to complicate matters unnecessarily.

theorem *eta-case*:
fixes $s :: dB$
assumes *free*: $\neg\ free\ s\ 0$
and *s*: $s[dummy/0] => u$
shows $\exists t'.\ Abs\ (s\ {}^\circ\ Var\ 0) => t' \wedge t' \to_\eta{}^*\ u$
proof −
from *s* **have** $lift\ (s[dummy/0])\ 0 => lift\ u\ 0$ **by** (*simp del*: *lift-subst*)
with *free* **have** $s => lift\ u\ 0$ **by** (*simp add*: *lift-subst-dummy del*: *lift-subst*)
hence $Abs\ (s\ {}^\circ\ Var\ 0) => Abs\ (lift\ u\ 0\ {}^\circ\ Var\ 0)$ **by** *simp*
moreover have $\neg\ free\ (lift\ u\ 0)\ 0$ **by** *simp*
hence $Abs\ (lift\ u\ 0\ {}^\circ\ Var\ 0) \to_\eta lift\ u\ 0[dummy/0]$
by (*rule eta.eta*)
hence $Abs\ (lift\ u\ 0\ {}^\circ\ Var\ 0) \to_\eta{}^*\ u$ **by** *simp*
ultimately show *?thesis* **by** *iprover*
qed

theorem *eta-par-beta*:
assumes *st*: $s \to_\eta t$
and *tu*: $t => u$
shows $\exists t'.\ s => t' \wedge t' \to_\eta{}^*\ u$ **using** *tu st*
proof (*induct arbitrary*: *s*)
case (*var n*)
thus *?case* **by** (*iprover intro*: *par-beta-refl*)
next
case (*abs s' t*)
note $abs' = this$
from ‹$s \to_\eta Abs\ s'$› **show** *?case*
proof *cases*
case (*eta s'' dummy*)
from *abs* **have** $Abs\ s' => Abs\ t$ **by** *simp*
with *eta* **have** $s''[dummy/0] => Abs\ t$ **by** *simp*
with ‹$\neg\ free\ s''\ 0$› **have** $\exists t'.\ Abs\ (s''\ {}^\circ\ Var\ 0) => t' \wedge t' \to_\eta{}^*\ Abs\ t$
by (*rule eta-case*)
with *eta* **show** *?thesis* **by** *simp*
next
case (*abs r*)

from $\langle r \rightarrow_\eta s' \rangle$
obtain t' **where** r: r => t' **and** t': $t' \rightarrow_\eta{}^* t$ **by** (*iprover dest*: *abs'*)
from r **have** $Abs\ r$ => $Abs\ t'$ **..**
moreover from t' **have** $Abs\ t' \rightarrow_\eta{}^* Abs\ t$ **by** (*rule rtrancl-eta-Abs*)
ultimately show *?thesis* **using** *abs* **by** *simp iprover*
qed
next
case (*app* $u\ u'\ t\ t'$)
from $\langle s \rightarrow_\eta u \circ t \rangle$ **show** *?case*
proof *cases*
case (*eta* s' *dummy*)
from *app* **have** $u \circ t$ => $u' \circ t'$ **by** *simp*
with *eta* **have** $s'[dummy/0]$ => $u' \circ t'$ **by** *simp*
with $\langle \neg\ free\ s'\ 0 \rangle$ **have** $\exists r.\ Abs\ (s' \circ Var\ 0)$ => $r \wedge r \rightarrow_\eta{}^* u' \circ t'$
by (*rule eta-case*)
with *eta* **show** *?thesis* **by** *simp*
next
case (*appL* s')
from $\langle s' \rightarrow_\eta u \rangle$
obtain r **where** s': s' => r **and** r: $r \rightarrow_\eta{}^* u'$ **by** (*iprover dest*: *app*)
from s' **and** *app* **have** $s' \circ t$ => $r \circ t'$ **by** *simp*
moreover from r **have** $r \circ t' \rightarrow_\eta{}^* u' \circ t'$ **by** (*simp add*: *rtrancl-eta-AppL*)
ultimately show *?thesis* **using** *appL* **by** *simp iprover*
next
case (*appR* s')
from $\langle s' \rightarrow_\eta t \rangle$
obtain r **where** s': s' => r **and** r: $r \rightarrow_\eta{}^* t'$ **by** (*iprover dest*: *app*)
from s' **and** *app* **have** $u \circ s'$ => $u' \circ r$ **by** *simp*
moreover from r **have** $u' \circ r \rightarrow_\eta{}^* u' \circ t'$ **by** (*simp add*: *rtrancl-eta-AppR*)
ultimately show *?thesis* **using** *appR* **by** *simp iprover*
qed
next
case (*beta* $u\ u'\ t\ t'$)
from $\langle s \rightarrow_\eta Abs\ u \circ t \rangle$ **show** *?case*
proof *cases*
case (*eta* s' *dummy*)
from *beta* **have** $Abs\ u \circ t$ => $u'[t'/0]$ **by** *simp*
with *eta* **have** $s'[dummy/0]$ => $u'[t'/0]$ **by** *simp*
with $\langle \neg\ free\ s'\ 0 \rangle$ **have** $\exists r.\ Abs\ (s' \circ Var\ 0)$ => $r \wedge r \rightarrow_\eta{}^* u'[t'/0]$
by (*rule eta-case*)
with *eta* **show** *?thesis* **by** *simp*
next
case (*appL* s')
from $\langle s' \rightarrow_\eta Abs\ u \rangle$ **show** *?thesis*
proof *cases*
case (*eta* s'' *dummy*)
have $Abs\ (lift\ u\ 1) = lift\ (Abs\ u)\ 0$ **by** *simp*
also from *eta* **have** $\ldots = s''$ **by** (*simp add*: *lift-subst-dummy del*: *lift-subst*)
finally have s: $s = Abs\ (Abs\ (lift\ u\ 1) \circ Var\ 0) \circ t$ **using** *appL* **and** *eta* **by**

```
simp
      from beta have lift u 1 => lift u' 1 by simp
      hence Abs (lift u 1) ° Var 0 => lift u' 1[Var 0/0]
        using par-beta.var ..
      hence Abs (Abs (lift u 1) ° Var 0) ° t => lift u' 1[Var 0/0][t'/0]
        using ‹t => t'› ..
      with s have s => u'[t'/0] by simp
      thus ?thesis by iprover
    next
      case (abs r)
      from ‹r →η u›
      obtain r'' where r: r => r'' and r'': r'' →η* u' by (iprover dest: beta)
      from r and beta have Abs r ° t => r''[t'/0] by simp
      moreover from r'' have r''[t'/0] →η* u'[t'/0]
        by (rule rtrancl-eta-subst')
      ultimately show ?thesis using abs and appL by simp iprover
    qed
  next
    case (appR s')
    from ‹s' →η t›
    obtain r where s': s' => r and r: r →η* t' by (iprover dest: beta)
    from s' and beta have Abs u ° s' => u'[r/0] by simp
    moreover from r have u'[r/0] →η* u'[t'/0]
      by (rule rtrancl-eta-subst'')
    ultimately show ?thesis using appR by simp iprover
  qed
qed

theorem eta-postponement':
  assumes eta: s →η* t and beta: t => u
  shows ∃ t'. s => t' ∧ t' →η* u using eta beta
proof (induct arbitrary: u)
  case base
  thus ?case by blast
next
  case (step s' s'' s''')
  then obtain t' where s': s' => t' and t': t' →η* s'''
    by (auto dest: eta-par-beta)
  from s' obtain t'' where s: s => t'' and t'': t'' →η* t' using step
    by blast
  from t'' and t' have t'' →η* s''' by (rule rtranclp-trans)
  with s show ?case by iprover
qed

theorem eta-postponement:
  assumes (sup beta eta)** s t
  shows (beta** OO eta**) s t using assms
proof induct
  case base
```

```
  show ?case by blast
next
  case (step s' s'')
  from step(3) obtain t' where s: s →β* t' and t': t' →η* s' by blast
  from step(2) show ?case
  proof
    assume s' →β s''
    with beta-subset-par-beta have s' => s'' ..
    with t' obtain t'' where st: t' => t'' and tu: t'' →η* s''
      by (auto dest: eta-postponement')
    from par-beta-subset-beta st have t' →β* t'' ..
    with s have s →β* t'' by (rule rtranclp-trans)
    thus ?thesis using tu ..
  next
    assume s' →η s''
    with t' have t' →η* s'' ..
    with s show ?thesis ..
  qed
qed

end
```

5 Application of a term to a list of terms

```
theory ListApplication imports Lambda begin

abbreviation
  list-application :: dB => dB list => dB  (infixl °° 150) where
  t °° ts == foldl (°) t ts

lemma apps-eq-tail-conv [iff]: (r °° ts = s °° ts) = (r = s)
  by (induct ts rule: rev-induct) auto

lemma Var-eq-apps-conv [iff]: (Var m = s °° ss) = (Var m = s ∧ ss = [])
  by (induct ss arbitrary: s) auto

lemma Var-apps-eq-Var-apps-conv [iff]:
    (Var m °° rs = Var n °° ss) = (m = n ∧ rs = ss)
  apply (induct rs arbitrary: ss rule: rev-induct)
   apply simp
   apply blast
  apply (induct-tac ss rule: rev-induct)
   apply auto
  done

lemma App-eq-foldl-conv:
  (r ° s = t °° ts) =
    (if ts = [] then r ° s = t
    else (∃ ss. ts = ss @ [s] ∧ r = t °° ss))
```

apply (*rule-tac xs* = *ts* **in** *rev-exhaust*)
apply *auto*
done

lemma *Abs-eq-apps-conv* [*iff*]:
($Abs\ r = s\ {}^{\circ\circ}\ ss$) = ($Abs\ r = s \wedge ss = []$)
by (*induct ss rule*: *rev-induct*) *auto*

lemma *apps-eq-Abs-conv* [*iff*]: ($s\ {}^{\circ\circ}\ ss = Abs\ r$) = ($s = Abs\ r \wedge ss = []$)
by (*induct ss rule*: *rev-induct*) *auto*

lemma *Abs-apps-eq-Abs-apps-conv* [*iff*]:
($Abs\ r\ {}^{\circ\circ}\ rs = Abs\ s\ {}^{\circ\circ}\ ss$) = ($r = s \wedge rs = ss$)
apply (*induct rs arbitrary*: *ss rule*: *rev-induct*)
apply *simp*
apply *blast*
apply (*induct-tac ss rule*: *rev-induct*)
apply *auto*
done

lemma *Abs-App-neq-Var-apps* [*iff*]:
$Abs\ s\ {}^{\circ}\ t \neq Var\ n\ {}^{\circ\circ}\ ss$
by (*induct ss arbitrary*: *s t rule*: *rev-induct*) *auto*

lemma *Var-apps-neq-Abs-apps* [*iff*]:
$Var\ n\ {}^{\circ\circ}\ ts \neq Abs\ r\ {}^{\circ\circ}\ ss$
apply (*induct ss arbitrary*: *ts rule*: *rev-induct*)
apply *simp*
apply (*induct-tac ts rule*: *rev-induct*)
apply *auto*
done

lemma *ex-head-tail*:
$\exists\, ts\ h.\ t = h\ {}^{\circ\circ}\ ts \wedge ((\exists\, n.\ h = Var\ n) \vee (\exists\, u.\ h = Abs\ u))$
apply (*induct t*)
apply (*rule-tac x* = [] **in** *exI*)
apply *simp*
apply *clarify*
apply (*rename-tac ts1 ts2 h1 h2*)
apply (*rule-tac x* = $ts1$ @ [$h2\ {}^{\circ\circ}\ ts2$] **in** *exI*)
apply *simp*
apply *simp*
done

lemma *size-apps* [*simp*]:
$size\ (r\ {}^{\circ\circ}\ rs) = size\ r + foldl\ (+)\ 0\ (map\ size\ rs) + length\ rs$
by (*induct rs rule*: *rev-induct*) *auto*

lemma *lem0*: [| (*0*::*nat*) < *k*; *m* <= *n* |] ==> *m* < *n* + *k*

by *simp*

lemma *lift-map* [*simp*]:
    $lift\ (t\ {}^{\circ\circ}\ ts)\ i = lift\ t\ i\ {}^{\circ\circ}\ map\ (\lambda t.\ lift\ t\ i)\ ts$
  by (*induct ts arbitrary*: *t*) *simp-all*

lemma *subst-map* [*simp*]:
    $subst\ (t\ {}^{\circ\circ}\ ts)\ u\ i = subst\ t\ u\ i\ {}^{\circ\circ}\ map\ (\lambda t.\ subst\ t\ u\ i)\ ts$
  by (*induct ts arbitrary*: *t*) *simp-all*

lemma *app-last*: $(t\ {}^{\circ\circ}\ ts)\ {}^{\circ}\ u = t\ {}^{\circ\circ}\ (ts\ @\ [u])$
  by *simp*

A customized induction schema for $^{\circ\circ}$.

lemma *lem*:
  assumes $!!n\ ts.\ \forall t \in set\ ts.\ P\ t ==> P\ (Var\ n\ {}^{\circ\circ}\ ts)$
    and $!!u\ ts.\ [|\ P\ u;\ \forall t \in set\ ts.\ P\ t\ |] ==> P\ (Abs\ u\ {}^{\circ\circ}\ ts)$
  shows $size\ t = n \Longrightarrow P\ t$
  apply (*induct n arbitrary*: *t rule*: *nat-less-induct*)
  apply (*cut-tac t* = *t* **in** *ex-head-tail*)
  apply *clarify*
  apply (*erule disjE*)
   apply *clarify*
   apply (*rule assms*)
   apply *clarify*
   apply (*erule allE*, *erule impE*)
    prefer *2*
    apply (*erule allE*, *erule mp*, *rule refl*)
   apply *simp*
   apply (*simp only*: *foldl-conv-fold add.commute fold-plus-sum-list-rev*)
   apply (*fastforce simp add*: *sum-list-map-remove1*)
  apply *clarify*
  apply (*rule assms*)
   apply (*erule allE*, *erule impE*)
    prefer *2*
    apply (*erule allE*, *erule mp*, *rule refl*)
   apply *simp*
  apply *clarify*
  apply (*erule allE*, *erule impE*)
   prefer *2*
   apply (*erule allE*, *erule mp*, *rule refl*)
  apply *simp*
  apply (*rule le-imp-less-Suc*)
  apply (*rule trans-le-add1*)
  apply (*rule trans-le-add2*)
  apply (*simp only*: *foldl-conv-fold add.commute fold-plus-sum-list-rev*)
  apply (*simp add*: *member-le-sum-list*)
  done

theorem *Apps-dB-induct*:
  assumes !!*n ts*. ∀ *t* ∈ *set ts*. *P t* ==> *P* (*Var n* °° *ts*)
    and !!*u ts*. [| *P u*; ∀ *t* ∈ *set ts*. *P t* |] ==> *P* (*Abs u* °° *ts*)
  shows *P t*
  apply (*rule-tac t* = *t* **in** *lem*)
    prefer *3*
    apply (*rule refl*)
    using *assms* **apply** *iprover*+
  done

end

6 Simply-typed lambda terms

theory *LambdaType* **imports** *ListApplication* **begin**

6.1 Environments

definition
  shift :: (*nat* ⇒ ′*a*) ⇒ *nat* ⇒ ′*a* ⇒ *nat* ⇒ ′*a* (-⟨-:-⟩ [*90*, *0*, *0*] *91*) **where**
  e⟨*i*:*a*⟩ = (λ*j*. *if j* < *i then e j else if j* = *i then a else e* (*j* − *1*))

lemma *shift-eq* [*simp*]: *i* = *j* ⟹ (*e*⟨*i*:*T*⟩) *j* = *T*
  by (*simp add*: *shift-def*)

lemma *shift-gt* [*simp*]: *j* < *i* ⟹ (*e*⟨*i*:*T*⟩) *j* = *e j*
  by (*simp add*: *shift-def*)

lemma *shift-lt* [*simp*]: *i* < *j* ⟹ (*e*⟨*i*:*T*⟩) *j* = *e* (*j* − *1*)
  by (*simp add*: *shift-def*)

lemma *shift-commute* [*simp*]: *e*⟨*i*:*U*⟩⟨*0*:*T*⟩ = *e*⟨*0*:*T*⟩⟨*Suc i*:*U*⟩
  by (*rule ext*) (*simp-all add*: *shift-def split*: *nat.split*)

6.2 Types and typing rules

datatype *type* =
    Atom nat
  | *Fun type type*   (**infixr** ⇒ *200*)

inductive *typing* :: (*nat* ⇒ *type*) ⇒ *dB* ⇒ *type* ⇒ *bool* (- ⊢ - : - [*50*, *50*, *50*] *50*)
  where
    Var [*intro*!]: *env x* = *T* ⟹ *env* ⊢ *Var x* : *T*
  | *Abs* [*intro*!]: *env*⟨*0*:*T*⟩ ⊢ *t* : *U* ⟹ *env* ⊢ *Abs t* : (*T* ⇒ *U*)
  | *App* [*intro*!]: *env* ⊢ *s* : *T* ⇒ *U* ⟹ *env* ⊢ *t* : *T* ⟹ *env* ⊢ (*s* ° *t*) : *U*

inductive-cases *typing-elims* [*elim*!]:
  e ⊢ *Var i* : *T*
  e ⊢ *t* ° *u* : *T*

$e \vdash Abs\ t : T$

primrec
typings :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
where
typings e [] *Ts* = (*Ts* = [])
| *typings e* (*t* # *ts*) *Ts* =
(*case Ts of*
[] ⇒ *False*
| *T* # *Ts* ⇒ *e* ⊢ *t* : *T* ∧ *typings e ts Ts*)

abbreviation
typings-rel :: (*nat* ⇒ *type*) ⇒ *dB list* ⇒ *type list* ⇒ *bool*
(- ⊩ - : - [*50*, *50*, *50*] *50*) **where**
env ⊩ *ts* : *Ts* == *typings env ts Ts*

abbreviation
funs :: *type list* ⇒ *type* ⇒ *type* (**infixr** ⇛ *200*) **where**
Ts ⇛ *T* == *foldr Fun Ts T*

6.3 Some examples

schematic-goal *e* ⊢ *Abs* (*Abs* (*Abs* (*Var 1* ° (*Var 2* ° *Var 1* ° *Var 0*)))) : *?T*
by *force*

schematic-goal *e* ⊢ *Abs* (*Abs* (*Abs* (*Var 2* ° *Var 0* ° (*Var 1* ° *Var 0*)))) : *?T*
by *force*

6.4 Lists of types

lemma *lists-typings*:
e ⊩ *ts* : *Ts* ⟹ *listsp* (λ*t*. ∃ *T*. *e* ⊢ *t* : *T*) *ts*
apply (*induct ts arbitrary*: *Ts*)
apply (*case-tac Ts*)
apply *simp*
apply (*rule listsp.Nil*)
apply *simp*
apply (*case-tac Ts*)
apply *simp*
apply *simp*
apply (*rule listsp.Cons*)
apply *blast*
apply *blast*
done

lemma *types-snoc*: *e* ⊩ *ts* : *Ts* ⟹ *e* ⊢ *t* : *T* ⟹ *e* ⊩ *ts* @ [*t*] : *Ts* @ [*T*]
apply (*induct ts arbitrary*: *Ts*)
apply *simp*
apply (*case-tac Ts*)
apply *simp+*

done

lemma *types-snoc-eq*: $e \Vdash ts$ @ $[t]$: Ts @ $[T]$ =
($e \Vdash ts : Ts \wedge e \vdash t : T$)
apply (*induct ts arbitrary*: *Ts*)
apply (*case-tac Ts*)
apply *simp+*
apply (*case-tac Ts*)
apply (*case-tac ts* @ $[t]$)
apply *simp+*
done

lemma *rev-exhaust2* [*extraction-expand*]:
obtains (*Nil*) $xs = []$ | (*snoc*) ys y **where** $xs = ys$ @ $[y]$
— Cannot use *rev-exhaust* from the *List* theory, since it is not constructive
apply (*subgoal-tac* $\forall ys.\ xs = rev\ ys \longrightarrow thesis$)
apply (*erule-tac* x=*rev xs* **in** *allE*)
apply *simp*
apply (*rule allI*)
apply (*rule impI*)
apply (*case-tac ys*)
apply *simp*
apply *simp*
done

lemma *types-snocE*:
assumes ‹$e \Vdash ts$ @ $[t]$: Ts›
obtains Us **and** U **where** ‹$Ts = Us$ @ $[U]$› ‹$e \Vdash ts : Us$› ‹$e \vdash t : U$›
proof (*cases Ts rule*: *rev-exhaust2*)
case *Nil*
with *assms* **show** *?thesis*
by (*cases ts* @ $[t]$) *simp-all*
next
case (*snoc Us U*)
with *assms* **have** $e \Vdash ts$ @ $[t]$: Us @ $[U]$ **by** *simp*
then have $e \Vdash ts : Us$ $e \vdash t : U$ **by** (*simp-all add*: *types-snoc-eq*)
with *snoc* **show** *?thesis* **by** (*rule that*)
qed

6.5 n-ary function types

lemma *list-app-typeD*:
$e \vdash t \circ\circ ts : T \Longrightarrow \exists Ts.\ e \vdash t : Ts \Rrightarrow T \wedge e \Vdash ts : Ts$
apply (*induct ts arbitrary*: t T)
apply *simp*
apply (*rename-tac* a b t T)
apply *atomize*
apply *simp*
apply (*erule-tac* $x = t \circ a$ **in** *allE*)

apply (*erule-tac* $x = T$ **in** *allE*)
apply (*erule impE*)
 apply *assumption*
apply (*elim exE conjE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply (*rule-tac* $x = Ta \# Ts$ **in** *exI*)
apply *simp*
done

lemma *list-app-typeE*:
$e \vdash t \circ\circ\ ts : T \Longrightarrow (\bigwedge Ts.\ e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow C) \Longrightarrow C$
by (*insert list-app-typeD*) *fast*

lemma *list-app-typeI*:
$e \vdash t : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow e \vdash t \circ\circ\ ts : T$
apply (*induct ts arbitrary*: $t\ T\ Ts$)
 apply *simp*
apply (*rename-tac* $a\ b\ t\ T\ Ts$)
apply *atomize*
apply (*case-tac* Ts)
 apply *simp*
apply *simp*
apply (*erule-tac* $x = t \circ a$ **in** *allE*)
apply (*erule-tac* $x = T$ **in** *allE*)
apply (*rename-tac list*)
apply (*erule-tac* $x = list$ **in** *allE*)
apply (*erule impE*)
 apply (*erule conjE*)
 apply (*erule typing.App*)
 apply *assumption*
apply *blast*
done

For the specific case where the head of the term is a variable, the following theorems allow to infer the types of the arguments without analyzing the typing derivation. This is crucial for program extraction.

theorem *var-app-type-eq*:
$e \vdash Var\ i \circ\circ\ ts : T \Longrightarrow e \vdash Var\ i \circ\circ\ ts : U \Longrightarrow T = U$
apply (*induct ts arbitrary*: $T\ U$ *rule*: *rev-induct*)
apply *simp*
apply (*ind-cases* $e \vdash Var\ i : T$ **for** T)
apply (*ind-cases* $e \vdash Var\ i : T$ **for** T)
apply *simp*
apply *simp*
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply *atomize*
apply (*erule-tac* $x = Ta \Rightarrow T$ **in** *allE*)
apply (*erule-tac* $x = Tb \Rightarrow U$ **in** *allE*)

apply (*erule impE*)
apply *assumption*
apply (*erule impE*)
apply *assumption*
apply *simp*
done

lemma *var-app-types*: $e \vdash Var\ i\ {}^{\circ\circ}\ ts\ {}^{\circ\circ}\ us : T \Longrightarrow e \Vdash ts : Ts \Longrightarrow$
$e \vdash Var\ i\ {}^{\circ\circ}\ ts : U \Longrightarrow \exists\, Us.\ U = Us \Rrightarrow T \wedge e \Vdash us : Us$
apply (*induct us arbitrary*: *ts Ts U*)
apply *simp*
apply (*erule var-app-type-eq*)
apply *assumption*
apply *simp*
apply (*rename-tac a b ts Ts U*)
apply *atomize*
apply (*case-tac U*)
apply (*rule FalseE*)
apply *simp*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** *t u T*)
apply (*rename-tac nat Tsa Ta*)
apply (*drule-tac* T=*Atom nat* **and** U=$Ta \Rightarrow Tsa \Rrightarrow T$ **in** *var-app-type-eq*)
apply *assumption*
apply *simp*
apply (*rename-tac nat type1 type2*)
apply (*erule-tac* x=*ts* @ [*a*] **in** *allE*)
apply (*erule-tac* x=*Ts* @ [*type1*] **in** *allE*)
apply (*erule-tac* x=*type2* **in** *allE*)
apply *simp*
apply (*erule impE*)
apply (*rule types-snoc*)
apply *assumption*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** *t u T*)
apply (*drule-tac* T=$type1 \Rightarrow type2$ **and** U=$Ta \Rightarrow Tsa \Rrightarrow T$ **in** *var-app-type-eq*)
apply *assumption*
apply *simp*
apply (*erule impE*)
apply (*rule typing.App*)
apply *assumption*
apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** *t u T*)
apply (*frule-tac* T=$type1 \Rightarrow type2$ **and** U=$Ta \Rightarrow Tsa \Rrightarrow T$ **in** *var-app-type-eq*)
apply *assumption*
apply *simp*
apply (*erule exE*)
apply (*rule-tac* x=*type1* # *Us* **in** *exI*)
apply *simp*

apply (*erule list-app-typeE*)
apply (*ind-cases* $e \vdash t \circ u : T$ **for** $t\ u\ T$)
apply (*frule-tac* T=*type1* $\Rightarrow$ *Us* $\Rrightarrow$ T **and** U=*Ta* $\Rightarrow$ *Tsa* $\Rrightarrow$ T **in** *var-app-type-eq*)
apply *assumption*
apply *simp*
done

lemma *var-app-typesE*: $e \vdash Var\ i \circ\circ ts : T \Longrightarrow$
($\bigwedge Ts.\ e \vdash Var\ i : Ts \Rrightarrow T \Longrightarrow e \Vdash ts : Ts \Longrightarrow P) \Longrightarrow P$
apply (*drule var-app-types* [*of* - - [], *simplified*])
apply (*iprover intro*: *typing.Var*)+
done

lemma *abs-typeE*: $e \vdash Abs\ t : T \Longrightarrow (\bigwedge U\ V.\ e\langle 0{:}U\rangle \vdash t : V \Longrightarrow P) \Longrightarrow P$
apply (*cases T*)
apply (*rule FalseE*)
apply (*erule typing.cases*)
apply *simp-all*
apply *atomize*
apply (*rename-tac type1 type2*)
apply (*erule-tac x=type1* **in** *allE*)
apply (*erule-tac x=type2* **in** *allE*)
apply (*erule mp*)
apply (*erule typing.cases*)
apply *simp-all*
done

6.6 Lifting preserves well-typedness

lemma *lift-type* [*intro*!]: $e \vdash t : T \Longrightarrow e\langle i{:}U\rangle \vdash lift\ t\ i : T$
by (*induct arbitrary*: *i U set*: *typing*) *auto*

lemma *lift-types*:
$e \Vdash ts : Ts \Longrightarrow e\langle i{:}U\rangle \Vdash (map\ (\lambda t.\ lift\ t\ i)\ ts) : Ts$
apply (*induct ts arbitrary*: *Ts*)
apply *simp*
apply (*case-tac Ts*)
apply *auto*
done

6.7 Substitution lemmas

lemma *subst-lemma*:
$e \vdash t : T \Longrightarrow e' \vdash u : U \Longrightarrow e = e'\langle i{:}U\rangle \Longrightarrow e' \vdash t[u/i] : T$
apply (*induct arbitrary*: *e' i U u set*: *typing*)
apply (*rule-tac* $x = x$ **and** $y = i$ **in** *linorder-cases*)
apply *auto*
apply *blast*
done

lemma *substs-lemma*:
  $e \vdash u : T \Longrightarrow e\langle i{:}T\rangle \Vdash ts : Ts \Longrightarrow$
    $e \Vdash (map\ (\lambda t.\ t[u/i])\ ts) : Ts$
  apply (*induct ts arbitrary*: *Ts*)
   apply (*case-tac Ts*)
    apply *simp*
   apply *simp*
  apply *atomize*
  apply (*case-tac Ts*)
   apply *simp*
  apply *simp*
  apply (*erule conjE*)
  apply (*erule* (*1*) *subst-lemma*)
  apply (*rule refl*)
  done

6.8 Subject reduction

lemma *subject-reduction*: $e \vdash t : T \Longrightarrow t \to_\beta t' \Longrightarrow e \vdash t' : T$
  apply (*induct arbitrary*: t' *set*: *typing*)
    apply *blast*
   apply *blast*
  apply *atomize*
  apply (*ind-cases* $s \circ t \to_\beta t'$ **for** $s\ t\ t'$)
    apply *hypsubst*
    apply (*ind-cases* $env \vdash Abs\ t : T \Rightarrow U$ **for** $env\ t\ T\ U$)
    apply (*rule subst-lemma*)
      apply *assumption*
     apply *assumption*
    apply (*rule ext*)
    apply (*case-tac x*)
     apply *auto*
  done

theorem *subject-reduction'*: $t \to_\beta{}^* t' \Longrightarrow e \vdash t : T \Longrightarrow e \vdash t' : T$
  by (*induct set*: *rtranclp*) (*iprover intro*: *subject-reduction*)+

6.9 Alternative induction rule for types

lemma *type-induct* [*induct type*]:
  assumes
  $(\bigwedge T.\ (\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T1) \Longrightarrow$
    $(\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow P\ T2) \Longrightarrow P\ T)$
  shows *P T*
proof (*induct T*)
  case *Atom*
  show *?case* **by** (*rule assms*) *simp-all*
next
  case *Fun*
  show *?case* **by** (*rule assms*) (*insert Fun*, *simp-all*)

qed

end

7 Lifting an order to lists of elements

theory *ListOrder*
imports *Main*
begin

declare [[*syntax-ambiguity-warning* = *false*]]

Lifting an order to lists of elements, relating exactly one element.

definition
  step1 :: ($'a$ => $'a$ => *bool*) => $'a$ *list* => $'a$ *list* => *bool* **where**
  step1 r =
    (λ*ys xs*. $\exists$ *us z z'* *vs*. *xs* = *us* @ *z* # *vs* $\wedge$ *r z' z* $\wedge$ *ys* =
      us @ *z'* # *vs*)

lemma *step1-converse* [*simp*]: *step1* (r^{-1-1}) = $(step1\ r)^{-1-1}$
  apply (*unfold step1-def*)
  apply (*blast intro*!: *order-antisym*)
  done

lemma *in-step1-converse* [*iff*]: (*step1* (r^{-1-1}) *x y*) = (($step1\ r)^{-1-1}$ *x y*)
  apply *auto*
  done

lemma *not-Nil-step1* [*iff*]: $\neg$ *step1 r* [] *xs*
  apply (*unfold step1-def*)
  apply *blast*
  done

lemma *not-step1-Nil* [*iff*]: $\neg$ *step1 r xs* []
  apply (*unfold step1-def*)
  apply *blast*
  done

lemma *Cons-step1-Cons* [*iff*]:
    (*step1 r* (*y* # *ys*) (*x* # *xs*)) =
      (*r y x* $\wedge$ *xs* = *ys* $\vee$ *x* = *y* $\wedge$ *step1 r ys xs*)
  apply (*unfold step1-def*)
  apply (*rule iffI*)
   apply (*erule exE*)
   apply (*rename-tac ts*)
   apply (*case-tac ts*)
    apply *fastforce*
   apply *force*

apply (*erule disjE*)
 apply *blast*
apply (*blast intro*: *Cons-eq-appendI*)
done

lemma *append-step1I*:
step1 r ys xs $\wedge$ *vs* = *us* $\vee$ *ys* = *xs* $\wedge$ *step1 r vs us*
==> *step1 r* (*ys* @ *vs*) (*xs* @ *us*)
apply (*unfold step1-def*)
apply *auto*
 apply *blast*
apply (*blast intro*: *append-eq-appendI*)
done

lemma *Cons-step1E* [*elim*!]:
assumes *step1 r ys* (*x* # *xs*)
and !!*y*. *ys* = *y* # *xs* $\Longrightarrow$ *r y x* $\Longrightarrow$ *R*
and !!*zs*. *ys* = *x* # *zs* $\Longrightarrow$ *step1 r zs xs* $\Longrightarrow$ *R*
shows *R*
using *assms*
apply (*cases ys*)
 apply (*simp add*: *step1-def*)
apply *blast*
done

lemma *Snoc-step1-SnocD*:
step1 r (*ys* @ [*y*]) (*xs* @ [*x*])
==> (*step1 r ys xs* $\wedge$ *y* = *x* $\vee$ *ys* = *xs* $\wedge$ *r y x*)
apply (*unfold step1-def*)
apply (*clarify del*: *disjCI*)
apply (*rename-tac vs*)
apply (*rule-tac xs* = *vs* **in** *rev-exhaust*)
 apply *force*
apply *simp*
apply *blast*
done

lemma *Cons-acc-step1I* [*intro*!]:
Wellfounded.accp r x ==> *Wellfounded.accp* (*step1 r*) *xs* $\Longrightarrow$ *Wellfounded.accp* (*step1 r*) (*x* # *xs*)
apply (*induct arbitrary*: *xs set*: *Wellfounded.accp*)
apply (*erule thin-rl*)
apply (*erule accp-induct*)
apply (*rule accp.accI*)
apply *blast*
done

lemma *lists-accD*: *listsp* (*Wellfounded.accp r*) *xs* ==> *Wellfounded.accp* (*step1 r*) *xs*

apply (*induct set*: *listsp*)
apply (*rule accp.accI*)
apply *simp*
apply (*rule accp.accI*)
apply (*fast dest*: *accp-downward*)
done

lemma *ex-step1I*:
[| $x \in$ *set xs*; *r y x* |]
==> $\exists$ *ys*. *step1 r ys xs* $\wedge$ $y \in$ *set ys*
apply (*unfold step1-def*)
apply (*drule in-set-conv-decomp* [*THEN iffD1*])
apply *force*
done

lemma *lists-accI*: *Wellfounded.accp* (*step1 r*) *xs* ==> *listsp* (*Wellfounded.accp r*) *xs*
apply (*induct set*: *Wellfounded.accp*)
apply *clarify*
apply (*rule accp.accI*)
apply (*drule-tac r=r* **in** *ex-step1I*, *assumption*)
apply *blast*
done

end

8 Lifting beta-reduction to lists

theory *ListBeta* **imports** *ListApplication ListOrder* **begin**

Lifting beta-reduction to lists of terms, reducing exactly one element.

abbreviation
list-beta :: *dB list* => *dB list* => *bool* (**infixl** => *50*) **where**
rs => *ss* == *step1 beta rs ss*

lemma *head-Var-reduction*:
Var n °° *rs* $\rightarrow_\beta$ *v* $\Longrightarrow$ $\exists$ *ss*. *rs* => *ss* $\wedge$ *v* = *Var n* °° *ss*
apply (*induct u* == *Var n* °° *rs v arbitrary*: *rs set*: *beta*)
apply *simp*
apply (*rule-tac xs* = *rs* **in** *rev-exhaust*)
apply *simp*
apply (*atomize*, *force intro*: *append-step1I*)
apply (*rule-tac xs* = *rs* **in** *rev-exhaust*)
apply *simp*
apply (*auto 0 3 intro*: *disjI2* [*THEN append-step1I*])
done

lemma *apps-betasE* [*elim*!]:
assumes *major*: *r* °° *rs* $\rightarrow_\beta$ *s*

```
    and cases: !!r'. [| r →β r'; s = r' °° rs |] ==> R
      !!rs'. [| rs => rs'; s = r °° rs' |] ==> R
      !!t u us. [| r = Abs t; rs = u # us; s = t[u/0] °° us |] ==> R
  shows R
proof -
  from major have
   (∃ r'. r →β r' ∧ s = r' °° rs) ∨
    (∃ rs'. rs => rs' ∧ s = r °° rs') ∨
    (∃ t u us. r = Abs t ∧ rs = u # us ∧ s = t[u/0] °° us)
    apply (induct u == r °° rs s arbitrary: r rs set: beta)
       apply (case-tac r)
         apply simp
        apply (simp add: App-eq-foldl-conv)
        apply (split if-split-asm)
         apply simp
         apply blast
        apply simp
       apply (simp add: App-eq-foldl-conv)
       apply (split if-split-asm)
        apply simp
       apply simp
      apply (drule App-eq-foldl-conv [THEN iffD1])
      apply (split if-split-asm)
       apply simp
       apply blast
      apply (force intro!: disjI1 [THEN append-step1I])
     apply (drule App-eq-foldl-conv [THEN iffD1])
     apply (split if-split-asm)
      apply simp
      apply blast
     apply (clarify, auto 0 3 intro!: exI intro: append-step1I)
    done
  with cases show ?thesis by blast
qed

lemma apps-preserves-beta [simp]:
    r →β s ==> r °° ss →β s °° ss
  by (induct ss rule: rev-induct) auto

lemma apps-preserves-beta2 [simp]:
    r →β* s ==> r °° ss →β* s °° ss
  apply (induct set: rtranclp)
   apply blast
  apply (blast intro: apps-preserves-beta rtranclp.rtrancl-into-rtrancl)
  done

lemma apps-preserves-betas [simp]:
    rs => ss ⟹ r °° rs →β r °° ss
  apply (induct rs arbitrary: ss rule: rev-induct)
```

```
   apply simp
  apply simp
  apply (rule-tac xs = ss in rev-exhaust)
   apply simp
  apply simp
  apply (drule Snoc-step1-SnocD)
  apply blast
  done

end
```

9 Inductive characterization of terminating lambda terms

```
theory InductTermi imports ListBeta begin
```

9.1 Terminating lambda terms

```
inductive IT :: dB => bool
  where
    Var [intro]: listsp IT rs ==> IT (Var n °° rs)
  | Lambda [intro]: IT r ==> IT (Abs r)
  | Beta [intro]: IT ((r[s/0]) °° ss) ==> IT s ==> IT ((Abs r ° s) °° ss)
```

9.2 Every term in *IT* terminates

```
lemma double-induction-lemma [rule-format]:
  termip beta s ==> ∀ t. termip beta t -->
    (∀ r ss. t = r[s/0] °° ss --> termip beta (Abs r ° s °° ss))
  apply (erule accp-induct)
  apply (rule allI)
  apply (rule impI)
  apply (erule thin-rl)
  apply (erule accp-induct)
  apply clarify
  apply (rule accp.accI)
  apply (safe elim!: apps-betasE)
    apply (blast intro: subst-preserves-beta apps-preserves-beta)
   apply (blast intro: apps-preserves-beta2 subst-preserves-beta2 rtranclp-converseI
     dest: accp-downwards)
  apply (blast dest: apps-preserves-betas)
  done

lemma IT-implies-termi: IT t ==> termip beta t
  apply (induct set: IT)
    apply (drule rev-predicate1D [OF - listsp-mono [where B=termip beta]])
    apply (fast intro!: predicate1I)
    apply (drule lists-accD)
    apply (erule accp-induct)
```

apply (*rule accp.accI*)
apply (*blast dest*: *head-Var-reduction*)
apply (*erule accp-induct*)
apply (*rule accp.accI*)
apply *blast*
apply (*blast intro*: *double-induction-lemma*)
done

9.3 Every terminating term is in *IT*

declare *Var-apps-neq-Abs-apps* [*symmetric*, *simp*]

lemma [*simp*, *THEN not-sym*, *simp*]: $Var\ n\ {}^{\circ\circ}\ ss \neq Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ts$
by (*simp add*: *foldl-Cons* [*symmetric*] *del*: *foldl-Cons*)

lemma [*simp*]:
$(Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss = Abs\ r'\ {}^{\circ}\ s'\ {}^{\circ\circ}\ ss') = (r = r' \wedge s = s' \wedge ss = ss')$
by (*simp add*: *foldl-Cons* [*symmetric*] *del*: *foldl-Cons*)

inductive-cases [*elim*!]:
$IT\ (Var\ n\ {}^{\circ\circ}\ ss)$
$IT\ (Abs\ t)$
$IT\ (Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ts)$

theorem *termi-implies-IT*: *termip beta r* ==> *IT r*
apply (*erule accp-induct*)
apply (*rename-tac r*)
apply (*erule thin-rl*)
apply (*erule rev-mp*)
apply *simp*
apply (*rule-tac* $t = r$ **in** *Apps-dB-induct*)
apply *clarify*
apply (*rule IT.intros*)
apply *clarify*
apply (*drule bspec*, *assumption*)
apply (*erule mp*)
apply *clarify*
apply (*drule-tac r*=*beta* **in** *conversepI*)
apply (*drule-tac r*=$beta^{-1-1}$ **in** *ex-step1I*, *assumption*)
apply *clarify*
apply (*rename-tac us*)
apply (*erule-tac* $x = Var\ n\ {}^{\circ\circ}\ us$ **in** *allE*)
apply *force*
apply (*rename-tac u ts*)
apply (*case-tac ts*)
apply *simp*
apply *blast*
apply (*rename-tac s ss*)
apply *simp*

```
  apply clarify
  apply (rule IT.intros)
   apply (blast intro: apps-preserves-beta)
  apply (erule mp)
  apply clarify
  apply (rename-tac t)
  apply (erule-tac x = Abs u ° t °° ss in allE)
  apply force
  done
```

end

10 Strong normalization for simply-typed lambda calculus

theory *StrongNorm* **imports** *LambdaType InductTermi* **begin**

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

10.1 Properties of *IT*

```
lemma lift-IT [intro!]: IT t ⟹ IT (lift t i)
 apply (induct arbitrary: i set: IT)
   apply (simp (no-asm))
   apply (rule conjI)
    apply
     (rule impI,
      rule IT.Var,
      erule listsp.induct,
      simp (no-asm),
      simp (no-asm),
      rule listsp.Cons,
      blast,
      assumption)+
    apply auto
  done

lemma lifts-IT: listsp IT ts ⟹ listsp IT (map (λt. lift t 0) ts)
 by (induct ts) auto

lemma subst-Var-IT: IT r ⟹ IT (r[Var i/j])
 apply (induct arbitrary: i j set: IT)
```

Case *Var*:

```
   apply (simp (no-asm) add: subst-Var)
   apply
   ((rule conjI impI)+,
```

```
      rule IT.Var,
      erule listsp.induct,
      simp (no-asm),
      simp (no-asm),
      rule listsp.Cons,
      fast,
      assumption)+
```

Case *Lambda*:

```
   apply atomize
   apply simp
   apply (rule IT.Lambda)
   apply fast
```

Case *Beta*:

```
  apply atomize
  apply (simp (no-asm-use) add: subst-subst [symmetric])
  apply (rule IT.Beta)
   apply auto
  done

lemma Var-IT: IT (Var n)
  apply (subgoal-tac IT (Var n °° []))
   apply simp
  apply (rule IT.Var)
  apply (rule listsp.Nil)
  done

lemma app-Var-IT: IT t ⟹ IT (t ° Var i)
  apply (induct set: IT)
    apply (subst app-last)
    apply (rule IT.Var)
    apply simp
    apply (rule listsp.Cons)
     apply (rule Var-IT)
    apply (rule listsp.Nil)
   apply (rule IT.Beta [where ?ss = [], unfolded foldl-Nil [THEN eq-reflection]])
    apply (erule subst-Var-IT)
   apply (rule Var-IT)
  apply (subst app-last)
  apply (rule IT.Beta)
   apply (subst app-last [symmetric])
   apply assumption
  apply assumption
  done
```

10.2 Well-typed substitution preserves termination

lemma *subst-type-IT*:

```
 ⋀t e T u i. IT t ⟹ e⟨i:U⟩ ⊢ t : T ⟹
   IT u ⟹ e ⊢ u : U ⟹ IT (t[u/i])
 (is PROP ?P U is ⋀t e T u i. - ⟹ PROP ?Q t e T u i U)
proof (induct U)
  fix T t
  assume MI1: ⋀T1 T2. T = T1 ⇒ T2 ⟹ PROP ?P T1
  assume MI2: ⋀T1 T2. T = T1 ⇒ T2 ⟹ PROP ?P T2
  assume IT t
  thus ⋀e T' u i. PROP ?Q t e T' u i T
  proof induct
    fix e T' u i
    assume uIT: IT u
    assume uT: e ⊢ u : T
    {
      case (Var rs n e1 T'1 u1 i1)
      assume nT: e⟨i:T⟩ ⊢ Var n °° rs : T'
      let ?ty = λt. ∃ T'. e⟨i:T⟩ ⊢ t : T'
      let ?R = λt. ∀ e T' u i.
        e⟨i:T⟩ ⊢ t : T' ⟶ IT u ⟶ e ⊢ u : T ⟶ IT (t[u/i])
      show IT ((Var n °° rs)[u/i])
      proof (cases n = i)
        case True
        show ?thesis
        proof (cases rs)
          case Nil
          with uIT True show ?thesis by simp
        next
          case (Cons a as)
          with nT have e⟨i:T⟩ ⊢ Var n ° a °° as : T' by simp
          then obtain Ts
              where headT: e⟨i:T⟩ ⊢ Var n ° a : Ts ⇛ T'
              and argsT: e⟨i:T⟩ ⊩ as : Ts
            by (rule list-app-typeE)
          from headT obtain T''
              where varT: e⟨i:T⟩ ⊢ Var n : T'' ⇒ Ts ⇛ T'
              and argT: e⟨i:T⟩ ⊢ a : T''
            by cases simp-all
          from varT True have T: T = T'' ⇒ Ts ⇛ T'
            by cases auto
          with uT have uT': e ⊢ u : T'' ⇒ Ts ⇛ T' by simp
          from T have IT ((Var 0 °° map (λt. lift t 0)
            (map (λt. t[u/i]) as))[(u ° a[u/i])/0])
          proof (rule MI2)
            from T have IT ((lift u 0 ° Var 0)[a[u/i]/0])
            proof (rule MI1)
              have IT (lift u 0) by (rule lift-IT [OF uIT])
              thus IT (lift u 0 ° Var 0) by (rule app-Var-IT)
              show e⟨0:T''⟩ ⊢ lift u 0 ° Var 0 : Ts ⇛ T'
              proof (rule typing.App)
```

show $e\langle 0{:}T''\rangle \vdash \mathit{lift}\ u\ 0 : T'' \Rightarrow Ts \Rrightarrow T'$
by (*rule lift-type*) (*rule* uT')
show $e\langle 0{:}T''\rangle \vdash \mathit{Var}\ 0 : T''$
by (*rule typing.Var*) *simp*
qed
from *Var* **have** *?R a* **by** *cases* (*simp-all add*: *Cons*)
with *argT uIT uT* **show** $IT\ (a[u/i])$ **by** *simp*
from *argT uT* **show** $e \vdash a[u/i] : T''$
by (*rule subst-lemma*) *simp*
qed
thus $IT\ (u \circ a[u/i])$ **by** *simp*
from *Var* **have** *listsp ?R as*
by *cases* (*simp-all add*: *Cons*)
moreover from *argsT* **have** *listsp ?ty as*
by (*rule lists-typings*)
ultimately have $\mathit{listsp}\ (\lambda t.\ ?R\ t \wedge ?ty\ t)\ \mathit{as}$
by *simp*
hence $\mathit{listsp}\ IT\ (\mathit{map}\ (\lambda t.\ \mathit{lift}\ t\ 0)\ (\mathit{map}\ (\lambda t.\ t[u/i])\ \mathit{as}))$
(**is** *listsp IT* (*?ls as*))
proof *induct*
case *Nil*
show *?case* **by** *fastforce*
next
case (*Cons b bs*)
hence *I*: *?R b* **by** *simp*
from *Cons* **obtain** *U* **where** $e\langle i{:}T\rangle \vdash b : U$ **by** *fast*
with *uT uIT I* **have** $IT\ (b[u/i])$ **by** *simp*
hence $IT\ (\mathit{lift}\ (b[u/i])\ 0)$ **by** (*rule lift-IT*)
hence $\mathit{listsp}\ IT\ (\mathit{lift}\ (b[u/i])\ 0\ \#\ ?ls\ bs)$
by (*rule listsp.Cons*) (*rule Cons*)
thus *?case* **by** *simp*
qed
thus $IT\ (\mathit{Var}\ 0 \circ\circ ?ls\ \mathit{as})$ **by** (*rule IT.Var*)
have $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash \mathit{Var}\ 0 : Ts \Rrightarrow T'$
by (*rule typing.Var*) *simp*
moreover from *uT argsT* **have** $e \Vdash \mathit{map}\ (\lambda t.\ t[u/i])\ \mathit{as} : Ts$
by (*rule substs-lemma*)
hence $e\langle 0{:}Ts \Rrightarrow T'\rangle \Vdash ?ls\ \mathit{as} : Ts$
by (*rule lift-types*)
ultimately show $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash \mathit{Var}\ 0 \circ\circ ?ls\ \mathit{as} : T'$
by (*rule list-app-typeI*)
from *argT uT* **have** $e \vdash a[u/i] : T''$
by (*rule subst-lemma*) (*rule refl*)
with uT' **show** $e \vdash u \circ a[u/i] : Ts \Rrightarrow T'$
by (*rule typing.App*)
qed
with *Cons True* **show** *?thesis*
by (*simp add*: *comp-def*)
qed

```
    next
      case False
      from Var have listsp ?R rs by simp
      moreover from nT obtain Ts where e⟨i:T⟩ ⊩ rs : Ts
        by (rule list-app-typeE)
      hence listsp ?ty rs by (rule lists-typings)
      ultimately have listsp (λt. ?R t ∧ ?ty t) rs
        by simp
      hence listsp IT (map (λx. x[u/i]) rs)
      proof induct
        case Nil
        show ?case by fastforce
      next
        case (Cons a as)
        hence I: ?R a by simp
        from Cons obtain U where e⟨i:T⟩ ⊢ a : U by fast
        with uT uIT I have IT (a[u/i]) by simp
        hence listsp IT (a[u/i] # map (λt. t[u/i]) as)
          by (rule listsp.Cons) (rule Cons)
        thus ?case by simp
      qed
      with False show ?thesis by (auto simp add: subst-Var)
    qed
  next
    case (Lambda r e1 T'1 u1 i1)
    assume e⟨i:T⟩ ⊢ Abs r : T'
      and ⋀e T' u i. PROP ?Q r e T' u i T
    with uIT uT show IT (Abs r[u/i])
      by fastforce
  next
    case (Beta r a as e1 T'1 u1 i1)
    assume T: e⟨i:T⟩ ⊢ Abs r ° a °° as : T'
    assume SI1: ⋀e T' u i. PROP ?Q (r[a/0] °° as) e T' u i T
    assume SI2: ⋀e T' u i. PROP ?Q a e T' u i T
    have IT (Abs (r[lift u 0/Suc i]) ° a[u/i] °° map (λt. t[u/i]) as)
    proof (rule IT.Beta)
      have Abs r ° a °° as →β r[a/0] °° as
        by (rule apps-preserves-beta) (rule beta.beta)
      with T have e⟨i:T⟩ ⊢ r[a/0] °° as : T'
        by (rule subject-reduction)
      hence IT ((r[a/0] °° as)[u/i])
        using uIT uT by (rule SI1)
      thus IT (r[lift u 0/Suc i][a[u/i]/0] °° map (λt. t[u/i]) as)
        by (simp del: subst-map add: subst-subst subst-map [symmetric])
      from T obtain U where e⟨i:T⟩ ⊢ Abs r ° a : U
        by (rule list-app-typeE) fast
      then obtain T'' where e⟨i:T⟩ ⊢ a : T'' by cases simp-all
      thus IT (a[u/i]) using uIT uT by (rule SI2)
    qed
```

thus IT $((Abs\ r \circ a \circ\circ as)[u/i])$ **by** *simp*
}
qed
qed

10.3 Well-typed terms are strongly normalizing

lemma *type-implies-IT*:
assumes $e \vdash t : T$
shows $IT\ t$
using *assms*
proof *induct*
case *Var*
show *?case* **by** (*rule Var-IT*)
next
case *Abs*
show *?case* **by** (*rule IT.Lambda*) (*rule Abs*)
next
case (*App e s T U t*)
have IT $((Var\ 0 \circ lift\ t\ 0)[s/0])$
proof (*rule subst-type-IT*)
have IT $(lift\ t\ 0)$ **using** ‹$IT\ t$› **by** (*rule lift-IT*)
hence *listsp* IT $[lift\ t\ 0]$ **by** (*rule listsp.Cons*) (*rule listsp.Nil*)
hence IT $(Var\ 0 \circ\circ [lift\ t\ 0])$ **by** (*rule IT.Var*)
also have $Var\ 0 \circ\circ [lift\ t\ 0] = Var\ 0 \circ lift\ t\ 0$ **by** *simp*
finally show IT ... **.**
have $e\langle 0{:}T \Rightarrow U\rangle \vdash Var\ 0 : T \Rightarrow U$
by (*rule typing.Var*) *simp*
moreover have $e\langle 0{:}T \Rightarrow U\rangle \vdash lift\ t\ 0 : T$
by (*rule lift-type*) (*rule App.hyps*)
ultimately show $e\langle 0{:}T \Rightarrow U\rangle \vdash Var\ 0 \circ lift\ t\ 0 : U$
by (*rule typing.App*)
show $IT\ s$ **by** *fact*
show $e \vdash s : T \Rightarrow U$ **by** *fact*
qed
thus *?case* **by** *simp*
qed

theorem *type-implies-termi*: $e \vdash t : T \Longrightarrow termip\ beta\ t$
proof −
assume $e \vdash t : T$
hence $IT\ t$ **by** (*rule type-implies-IT*)
thus *?thesis* **by** (*rule IT-implies-termi*)
qed

end

11 Inductive characterization of lambda terms in normal form

theory *NormalForm*
imports *ListBeta*
begin

11.1 Terms in normal form

definition
  $listall :: ('a \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow bool$ **where**
  $listall\ P\ xs \equiv (\forall\, i.\ i < length\ xs \longrightarrow P\ (xs\ !\ i))$

declare *listall-def* [*extraction-expand-def*]

theorem *listall-nil*: *listall P* []
  by (*simp add*: *listall-def*)

theorem *listall-nil-eq* [*simp*]: *listall P* [] = *True*
  by (*iprover intro*: *listall-nil*)

theorem *listall-cons*: $P\ x \Longrightarrow listall\ P\ xs \Longrightarrow listall\ P\ (x\ \#\ xs)$
  apply (*simp add*: *listall-def*)
  apply (*rule allI impI*)+
  apply (*case-tac i*)
  apply *simp*+
  done

theorem *listall-cons-eq* [*simp*]: $listall\ P\ (x\ \#\ xs) = (P\ x \wedge listall\ P\ xs)$
  apply (*rule iffI*)
  prefer *2*
  apply (*erule conjE*)
  apply (*erule listall-cons*)
  apply *assumption*
  apply (*unfold listall-def*)
  apply (*rule conjI*)
  apply (*erule-tac x=0* **in** *allE*)
  apply *simp*
  apply *simp*
  apply (*rule allI*)
  apply (*erule-tac x=Suc i* **in** *allE*)
  apply *simp*
  done

lemma *listall-conj1*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ P\ xs$
  by (*induct xs*) *simp-all*

lemma *listall-conj2*: $listall\ (\lambda x.\ P\ x \wedge Q\ x)\ xs \Longrightarrow listall\ Q\ xs$
  by (*induct xs*) *simp-all*

lemma *listall-app*: *listall P* (*xs* @ *ys*) = (*listall P xs* ∧ *listall P ys*)
  apply (*induct xs*)
   apply (*rule iffI*, *simp*, *simp*)
  apply (*rule iffI*, *simp*, *simp*)
  done

lemma *listall-snoc* [*simp*]: *listall P* (*xs* @ [*x*]) = (*listall P xs* ∧ *P x*)
  apply (*rule iffI*)
  apply (*simp add*: *listall-app*)+
  done

lemma *listall-cong* [*cong*, *extraction-expand*]:
  xs = *ys* ⟹ *listall P xs* = *listall P ys*
  — Currently needed for strange technical reasons
  by (*unfold listall-def*) *simp*

listsp is equivalent to *listall*, but cannot be used for program extraction.

lemma *listall-listsp-eq*: *listall P xs* = *listsp P xs*
  by (*induct xs*) (*auto intro*: *listsp.intros*)

inductive *NF* :: *dB* ⇒ *bool*
where
  App: *listall NF ts* ⟹ *NF* (*Var x* °° *ts*)
| *Abs*: *NF t* ⟹ *NF* (*Abs t*)
monos *listall-def*

lemma *nat-eq-dec*: $\bigwedge$*n*::*nat*. *m* = *n* ∨ *m* ≠ *n*
  apply (*induct m*)
  apply (*case-tac n*)
  apply (*case-tac* [*3*] *n*)
  apply (*simp only*: *nat.simps*, *iprover?*)+
  done

lemma *nat-le-dec*: $\bigwedge$*n*::*nat*. *m* < *n* ∨ ¬ (*m* < *n*)
  apply (*induct m*)
  apply (*case-tac n*)
    apply (*case-tac* [*3*] *n*)
  apply (*simp del*: *simp-thms subst-all*, *iprover?*)+
  done

lemma *App-NF-D*: **assumes** *NF*: *NF* (*Var n* °° *ts*)
  shows *listall NF ts* **using** *NF*
  by *cases simp-all*

11.2 Properties of *NF*

lemma *Var-NF*: *NF* (*Var n*)
  apply (*subgoal-tac NF* (*Var n* °° []))

apply *simp*
apply (*rule NF.App*)
apply *simp*
done

lemma *Abs-NF*:
assumes *NF*: *NF* (*Abs t* °° *ts*)
shows *ts* = [] **using** *NF*
proof *cases*
case (*App us i*)
thus *?thesis* **by** (*simp add*: *Var-apps-neq-Abs-apps* [*THEN not-sym*])
next
case (*Abs u*)
thus *?thesis* **by** *simp*
qed

lemma *subst-terms-NF*: *listall NF ts* $\Longrightarrow$
listall ($\lambda t.\ \forall\, i\ j.\ NF\ (t[Var\ i/j])$) *ts* $\Longrightarrow$
listall NF (*map* ($\lambda t.\ t[Var\ i/j]$) *ts*)
by (*induct ts*) *simp-all*

lemma *subst-Var-NF*: *NF t* $\Longrightarrow$ *NF* ($t[Var\ i/j]$)
apply (*induct arbitrary*: *i j set*: *NF*)
apply *simp*
apply (*frule listall-conj1*)
apply (*drule listall-conj2*)
apply (*drule-tac i=i* **and** *j=j* **in** *subst-terms-NF*)
apply *assumption*
apply (*rule-tac m1=x* **and** *n1=j* **in** *nat-eq-dec* [*THEN disjE*])
apply *simp*
apply (*erule NF.App*)
apply (*rule-tac m1=j* **and** *n1=x* **in** *nat-le-dec* [*THEN disjE*])
apply *simp*
apply (*iprover intro*: *NF.App*)
apply *simp*
apply (*iprover intro*: *NF.App*)
apply *simp*
apply (*iprover intro*: *NF.Abs*)
done

lemma *app-Var-NF*: *NF t* $\Longrightarrow \exists\, t'.\ t\ ^\circ\ Var\ i \rightarrow_\beta{}^*\ t' \wedge NF\ t'$
apply (*induct set*: *NF*)
apply (*simplesubst app-last*) — Using *subst* makes extraction fail
apply (*rule exI*)
apply (*rule conjI*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*rule NF.App*)
apply (*drule listall-conj1*)
apply (*simp add*: *listall-app*)

```
  apply (rule Var-NF)
  apply (rule exI)
  apply (rule conjI)
  apply (rule rtranclp.rtrancl-into-rtrancl)
  apply (rule rtranclp.rtrancl-refl)
  apply (rule beta)
  apply (erule subst-Var-NF)
  done

lemma lift-terms-NF: listall NF ts ⟹
    listall (λt. ∀ i. NF (lift t i)) ts ⟹
    listall NF (map (λt. lift t i) ts)
  by (induct ts) simp-all

lemma lift-NF: NF t ⟹ NF (lift t i)
  apply (induct arbitrary: i set: NF)
  apply (frule listall-conj1)
  apply (drule listall-conj2)
  apply (drule-tac i=i in lift-terms-NF)
  apply assumption
  apply (rule-tac m1=x and n1=i in nat-le-dec [THEN disjE])
  apply simp
  apply (rule NF.App)
  apply assumption
  apply simp
  apply (rule NF.App)
  apply assumption
  apply simp
  apply (rule NF.Abs)
  apply simp
  done
```

NF characterizes exactly the terms that are in normal form.

```
lemma NF-eq: NF t = (∀ t'. ¬ t →β t')
proof
  assume NF t
  then have ⋀t'. ¬ t →β t'
  proof induct
    case (App ts t)
    show ?case
    proof
      assume Var t °° ts →β t'
      then obtain rs where ts => rs
        by (iprover dest: head-Var-reduction)
      with App show False
        by (induct rs arbitrary: ts) auto
    qed
  next
    case (Abs t)
```

```
    show ?case
    proof
      assume Abs t →β t′
      then show False using Abs by cases simp-all
    qed
  qed
  then show ∀ t′. ¬ t →β t′ ..
next
  assume H: ∀ t′. ¬ t →β t′
  then show NF t
  proof (induct t rule: Apps-dB-induct)
    case (1 n ts)
    then have ∀ ts′. ¬ ts => ts′
      by (iprover intro: apps-preserves-betas)
    with 1(1) have listall NF ts
      by (induct ts) auto
    then show ?case by (rule NF.App)
  next
    case (2 u ts)
    show ?case
    proof (cases ts)
      case Nil
      from 2 have ∀ u′. ¬ u →β u′
        by (auto intro: apps-preserves-beta)
      then have NF u by (rule 2)
      then have NF (Abs u) by (rule NF.Abs)
      with Nil show ?thesis by simp
    next
      case (Cons r rs)
      have Abs u ° r →β u[r/0] ..
      then have Abs u ° r °° rs →β u[r/0] °° rs
        by (rule apps-preserves-beta)
      with Cons have Abs u °° ts →β u[r/0] °° rs
        by simp
      with 2 show ?thesis by iprover
    qed
  qed
qed

end
```

12 Standardization

theory *Standardization*
imports *NormalForm*
begin

Based on lecture notes by Ralph Matthes [3], original proof idea due to Ralph Loader [2].

12.1 Standard reduction relation

declare *listrel-mono* [*mono-set*]

inductive
  sred :: *dB* ⇒ *dB* ⇒ *bool* (**infixl** $\rightarrow_s$ *50*)
  and *sredlist* :: *dB list* ⇒ *dB list* ⇒ *bool* (**infixl** $[\rightarrow_s]$ *50*)
where
  $s\ [\rightarrow_s]\ t \equiv listrelp\ (\rightarrow_s)\ s\ t$
| *Var*: $rs\ [\rightarrow_s]\ rs' \Longrightarrow Var\ x\ {}^{\circ\circ}\ rs \rightarrow_s Var\ x\ {}^{\circ\circ}\ rs'$
| *Abs*: $r \rightarrow_s r' \Longrightarrow ss\ [\rightarrow_s]\ ss' \Longrightarrow Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s Abs\ r'\ {}^{\circ\circ}\ ss'$
| *Beta*: $r[s/0]\ {}^{\circ\circ}\ ss \rightarrow_s t \Longrightarrow Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss \rightarrow_s t$

lemma *refl-listrelp*: $\forall x\in set\ xs.\ R\ x\ x \Longrightarrow listrelp\ R\ xs\ xs$
  by (*induct xs*) (*auto intro*: *listrelp.intros*)

lemma *refl-sred*: $t \rightarrow_s t$
  by (*induct t rule*: *Apps-dB-induct*) (*auto intro*: *refl-listrelp sred.intros*)

lemma *refl-sreds*: $ts\ [\rightarrow_s]\ ts$
  by (*simp add*: *refl-sred refl-listrelp*)

lemma *listrelp-conj1*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ R\ x\ y$
  by (*erule listrelp.induct*) (*auto intro*: *listrelp.intros*)

lemma *listrelp-conj2*: $listrelp\ (\lambda x\ y.\ R\ x\ y \wedge S\ x\ y)\ x\ y \Longrightarrow listrelp\ S\ x\ y$
  by (*erule listrelp.induct*) (*auto intro*: *listrelp.intros*)

lemma *listrelp-app*:
  assumes *xsys*: *listrelp R xs ys*
  shows $listrelp\ R\ xs'\ ys' \Longrightarrow listrelp\ R\ (xs\ @\ xs')\ (ys\ @\ ys')$ **using** *xsys*
  by (*induct arbitrary*: *xs' ys'*) (*auto intro*: *listrelp.intros*)

lemma *lemma1*:
  assumes *r*: $r \rightarrow_s r'$ **and** *s*: $s \rightarrow_s s'$
  shows $r\ {}^{\circ}\ s \rightarrow_s r'\ {}^{\circ}\ s'$ **using** *r*
proof *induct*
  case (*Var rs rs' x*)
  then have $rs\ [\rightarrow_s]\ rs'$ **by** (*rule listrelp-conj1*)
  moreover have $[s]\ [\rightarrow_s]\ [s']$ **by** (*iprover intro*: *s listrelp.intros*)
  ultimately have $rs\ @\ [s]\ [\rightarrow_s]\ rs'\ @\ [s']$ **by** (*rule listrelp-app*)
  hence $Var\ x\ {}^{\circ\circ}\ (rs\ @\ [s]) \rightarrow_s Var\ x\ {}^{\circ\circ}\ (rs'\ @\ [s'])$ **by** (*rule sred.Var*)
  thus *?case* **by** (*simp only*: *app-last*)
next
  case (*Abs r r' ss ss'*)
  from *Abs*(*3*) **have** $ss\ [\rightarrow_s]\ ss'$ **by** (*rule listrelp-conj1*)
  moreover have $[s]\ [\rightarrow_s]\ [s']$ **by** (*iprover intro*: *s listrelp.intros*)
  ultimately have $ss\ @\ [s]\ [\rightarrow_s]\ ss'\ @\ [s']$ **by** (*rule listrelp-app*)
  with ‹$r \rightarrow_s r'$› **have** $Abs\ r\ {}^{\circ\circ}\ (ss\ @\ [s]) \rightarrow_s Abs\ r'\ {}^{\circ\circ}\ (ss'\ @\ [s'])$
    by (*rule sred.Abs*)

  thus *?case* **by** (*simp only*: *app-last*)
next
  case (*Beta r u ss t*)
  hence $r[u/0] \circ\circ (ss\ @\ [s]) \rightarrow_s t \circ s'$ **by** (*simp only*: *app-last*)
  hence $Abs\ r \circ u \circ\circ (ss\ @\ [s]) \rightarrow_s t \circ s'$ **by** (*rule sred.Beta*)
  thus *?case* **by** (*simp only*: *app-last*)
qed

lemma *lemma1'*:
  assumes *ts*: $ts\ [\rightarrow_s]\ ts'$
  shows $r \rightarrow_s r' \Longrightarrow r \circ\circ ts \rightarrow_s r' \circ\circ ts'$ **using** *ts*
  by (*induct arbitrary*: $r\ r'$) (*auto intro*: *lemma1*)

lemma *lemma2-1*:
  assumes *beta*: $t \rightarrow_\beta u$
  shows $t \rightarrow_s u$ **using** *beta*
proof *induct*
  case (*beta s t*)
  have $Abs\ s \circ t \circ\circ [] \rightarrow_s s[t/0] \circ\circ []$ **by** (*iprover intro*: *sred.Beta refl-sred*)
  thus *?case* **by** *simp*
next
  case (*appL s t u*)
  thus *?case* **by** (*iprover intro*: *lemma1 refl-sred*)
next
  case (*appR s t u*)
  thus *?case* **by** (*iprover intro*: *lemma1 refl-sred*)
next
  case (*abs s t*)
  hence $Abs\ s \circ\circ [] \rightarrow_s Abs\ t \circ\circ []$ **by** (*iprover intro*: *sred.Abs listrelp.Nil*)
  thus *?case* **by** *simp*
qed

lemma *listrelp-betas*:
  assumes *ts*: *listrelp* $(\rightarrow_\beta{}^*)\ ts\ ts'$
  shows $\bigwedge t\ t'.\ t \rightarrow_\beta{}^* t' \Longrightarrow t \circ\circ ts \rightarrow_\beta{}^* t' \circ\circ ts'$ **using** *ts*
  by *induct auto*

lemma *lemma2-2*:
  assumes *t*: $t \rightarrow_s u$
  shows $t \rightarrow_\beta{}^* u$ **using** *t*
  by *induct* (*auto dest*: *listrelp-conj2*
    intro: *listrelp-betas apps-preserves-beta converse-rtranclp-into-rtranclp*)

lemma *sred-lift*:
  assumes *s*: $s \rightarrow_s t$
  shows *lift s i* $\rightarrow_s$ *lift t i* **using** *s*
proof (*induct arbitrary*: *i*)
  case (*Var rs rs' x*)
  hence *map* ($\lambda t.$ *lift t i*) *rs* $[\rightarrow_s]$ *map* ($\lambda t.$ *lift t i*) *rs'*

```
    by induct (auto intro: listrelp.intros)
  thus ?case by (cases x < i) (auto intro: sred.Var)
next
  case (Abs r r′ ss ss′)
  from Abs(3) have map (λt. lift t i) ss [→s] map (λt. lift t i) ss′
    by induct (auto intro: listrelp.intros)
  thus ?case by (auto intro: sred.Abs Abs)
next
  case (Beta r s ss t)
  thus ?case by (auto intro: sred.Beta)
qed

lemma lemma3:
  assumes r: r →s r′
  shows s →s s′ ⟹ r[s/x] →s r′[s′/x] using r
proof (induct arbitrary: s s′ x)
  case (Var rs rs′ y)
  hence map (λt. t[s/x]) rs [→s] map (λt. t[s′/x]) rs′
    by induct (auto intro: listrelp.intros Var)
  moreover have Var y[s/x] →s Var y[s′/x]
  proof (cases y < x)
    case True thus ?thesis by simp (rule refl-sred)
  next
    case False
    thus ?thesis
      by (cases y = x) (auto simp add: Var intro: refl-sred)
  qed
  ultimately show ?case by simp (rule lemma1′)
next
  case (Abs r r′ ss ss′)
  from Abs(4) have lift s 0 →s lift s′ 0 by (rule sred-lift)
  hence r[lift s 0/Suc x] →s r′[lift s′ 0/Suc x] by (fast intro: Abs.hyps)
  moreover from Abs(3) have map (λt. t[s/x]) ss [→s] map (λt. t[s′/x]) ss′
    by induct (auto intro: listrelp.intros Abs)
  ultimately show ?case by simp (rule sred.Abs)
next
  case (Beta r u ss t)
  thus ?case by (auto simp add: subst-subst intro: sred.Beta)
qed

lemma lemma4-aux:
  assumes rs: listrelp (λt u. t →s u ∧ (∀r. u →β r ⟶ t →s r)) rs rs′
  shows rs′ => ss ⟹ rs [→s] ss using rs
proof (induct arbitrary: ss)
  case Nil
  thus ?case by cases (auto intro: listrelp.Nil)
next
  case (Cons x y xs ys)
  note Cons′ = Cons
```

```
  show ?case
  proof (cases ss)
    case Nil with Cons show ?thesis by simp
  next
    case (Cons y' ys')
    hence ss: ss = y' # ys' by simp
    from Cons Cons' have y →_β y' ∧ ys' = ys ∨ y' = y ∧ ys => ys' by simp
    hence x # xs [→_s] y' # ys'
    proof
      assume H: y →_β y' ∧ ys' = ys
      with Cons' have x →_s y' by blast
      moreover from Cons' have xs [→_s] ys by (iprover dest: listrelp-conj1)
      ultimately have x # xs [→_s] y' # ys by (rule listrelp.Cons)
      with H show ?thesis by simp
    next
      assume H: y' = y ∧ ys => ys'
      with Cons' have x →_s y' by blast
      moreover from H have xs [→_s] ys' by (blast intro: Cons')
      ultimately show ?thesis by (rule listrelp.Cons)
    qed
    with ss show ?thesis by simp
  qed
qed

lemma lemma4:
  assumes r: r →_s r'
  shows r' →_β r'' ⟹ r →_s r'' using r
proof (induct arbitrary: r'')
  case (Var rs rs' x)
  then obtain ss where rs: rs' => ss and r'': r'' = Var x °° ss
    by (blast dest: head-Var-reduction)
  from Var(1) rs have rs [→_s] ss by (rule lemma4-aux)
  hence Var x °° rs →_s Var x °° ss by (rule sred.Var)
  with r'' show ?case by simp
next
  case (Abs r r' ss ss')
  from ‹Abs r' °° ss' →_β r''› show ?case
  proof
    fix s
    assume r'': r'' = s °° ss'
    assume Abs r' →_β s
    then obtain r''' where s: s = Abs r''' and r''': r' →_β r''' by cases auto
    from r''' have r →_s r''' by (blast intro: Abs)
    moreover from Abs have ss [→_s] ss' by (iprover dest: listrelp-conj1)
    ultimately have Abs r °° ss →_s Abs r''' °° ss' by (rule sred.Abs)
    with r'' s show Abs r °° ss →_s r'' by simp
  next
    fix rs'
    assume ss' => rs'
```

with $Abs(3)$ **have** $ss\ [\rightarrow_s]\ rs'$ **by** $(rule\ lemma4\text{-}aux)$
with $\langle r \rightarrow_s r'\rangle$ **have** $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s Abs\ r'\ {}^{\circ\circ}\ rs'$ **by** $(rule\ sred.Abs)$
moreover assume $r'' = Abs\ r'\ {}^{\circ\circ}\ rs'$
ultimately show $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s r''$ **by** *simp*
next
fix $t\ u'\ us'$
assume $ss' = u'\ \#\ us'$
with $Abs(3)$ **obtain** $u\ us$ **where**
ss: $ss = u\ \#\ us$ **and** u: $u \rightarrow_s u'$ **and** us: $us\ [\rightarrow_s]\ us'$
by *cases* $(auto\ dest!: listrelp\text{-}conj1)$
have $r[u/0] \rightarrow_s r'[u'/0]$ **using** $Abs(1)$ **and** u **by** $(rule\ lemma3)$
with us **have** $r[u/0]\ {}^{\circ\circ}\ us \rightarrow_s r'[u'/0]\ {}^{\circ\circ}\ us'$ **by** $(rule\ lemma1')$
hence $Abs\ r\ {}^{\circ}\ u\ {}^{\circ\circ}\ us \rightarrow_s r'[u'/0]\ {}^{\circ\circ}\ us'$ **by** $(rule\ sred.Beta)$
moreover assume $Abs\ r' = Abs\ t$ **and** $r'' = t[u'/0]\ {}^{\circ\circ}\ us'$
ultimately show $Abs\ r\ {}^{\circ\circ}\ ss \rightarrow_s r''$ **using** ss **by** *simp*
qed
next
case $(Beta\ r\ s\ ss\ t)$
show *?case*
by $(rule\ sred.Beta)\ (rule\ Beta)+$
qed

lemma *rtrancl-beta-sred*:
assumes r: $r \rightarrow_\beta{}^* r'$
shows $r \rightarrow_s r'$ **using** r
by *induct* $(iprover\ intro: refl\text{-}sred\ lemma4)+$

12.2 Leftmost reduction and weakly normalizing terms

inductive
$lred :: dB \Rightarrow dB \Rightarrow bool$ (**infixl** $\rightarrow_l$ *50*)
and $lredlist :: dB\ list \Rightarrow dB\ list \Rightarrow bool$ (**infixl** $[\rightarrow_l]$ *50*)
where
$s\ [\rightarrow_l]\ t \equiv listrelp\ (\rightarrow_l)\ s\ t$
| Var: $rs\ [\rightarrow_l]\ rs' \Longrightarrow Var\ x\ {}^{\circ\circ}\ rs \rightarrow_l Var\ x\ {}^{\circ\circ}\ rs'$
| Abs: $r \rightarrow_l r' \Longrightarrow Abs\ r \rightarrow_l Abs\ r'$
| $Beta$: $r[s/0]\ {}^{\circ\circ}\ ss \rightarrow_l t \Longrightarrow Abs\ r\ {}^{\circ}\ s\ {}^{\circ\circ}\ ss \rightarrow_l t$

lemma *lred-imp-sred*:
assumes $lred$: $s \rightarrow_l t$
shows $s \rightarrow_s t$ **using** *lred*
proof *induct*
case $(Var\ rs\ rs'\ x)$
then have $rs\ [\rightarrow_s]\ rs'$
by *induct* $(iprover\ intro: listrelp.intros)+$
then show *?case* **by** $(rule\ sred.Var)$
next
case $(Abs\ r\ r')$
from $\langle r \rightarrow_s r'\rangle$

```
  have Abs r °° [] →s Abs r′ °° [] using listrelp.Nil
    by (rule sred.Abs)
  then show ?case by simp
next
  case (Beta r s ss t)
  from ‹r[s/0] °° ss →s t›
  show ?case by (rule sred.Beta)
qed

inductive WN :: dB => bool
  where
    Var: listsp WN rs ⟹ WN (Var n °° rs)
  | Lambda: WN r ⟹ WN (Abs r)
  | Beta: WN ((r[s/0]) °° ss) ⟹ WN ((Abs r ° s) °° ss)

lemma listrelp-imp-listsp1:
  assumes H: listrelp (λx y. P x) xs ys
  shows listsp P xs using H
  by induct auto

lemma listrelp-imp-listsp2:
  assumes H: listrelp (λx y. P y) xs ys
  shows listsp P ys using H
  by induct auto

lemma lemma5:
  assumes lred: r →l r′
  shows WN r and NF r′ using lred
  by induct
    (iprover dest: listrelp-conj1 listrelp-conj2
     listrelp-imp-listsp1 listrelp-imp-listsp2 intro: WN.intros
     NF.intros [simplified listall-listsp-eq])+

lemma lemma6:
  assumes wn: WN r
  shows ∃ r′. r →l r′ using wn
proof induct
  case (Var rs n)
  then have ∃ rs′. rs [→l] rs′
    by induct (iprover intro: listrelp.intros)+
  then show ?case by (iprover intro: lred.Var)
qed (iprover intro: lred.intros)+

lemma lemma7:
  assumes r: r →s r′
  shows NF r′ ⟹ r →l r′ using r
proof induct
  case (Var rs rs′ x)
  from ‹NF (Var x °° rs′)› have listall NF rs′
```

by *cases simp-all*
with *Var*(*1*) **have** $rs\ [\to_l]\ rs'$
proof *induct*
case *Nil*
show *?case* **by** (*rule listrelp.Nil*)
next
case (*Cons x y xs ys*)
hence $x \to_l y$ **and** $xs\ [\to_l]\ ys$ **by** *simp-all*
thus *?case* **by** (*rule listrelp.Cons*)
qed
thus *?case* **by** (*rule lred.Var*)
next
case (*Abs* $r\ r'\ ss\ ss'$)
from ‹*NF* (*Abs* r' °° ss')›
have ss': $ss' = []$ **by** (*rule Abs-NF*)
from *Abs*(*3*) **have** *ss*: $ss = []$ **using** ss'
by *cases simp-all*
from ss' *Abs* **have** *NF* (*Abs* r') **by** *simp*
hence *NF* r' **by** *cases simp-all*
with *Abs* **have** $r \to_l r'$ **by** *simp*
hence *Abs* $r \to_l$ *Abs* r' **by** (*rule lred.Abs*)
with *ss* ss' **show** *?case* **by** *simp*
next
case (*Beta r s ss t*)
hence $r[s/0]$ °° $ss \to_l t$ **by** *simp*
thus *?case* **by** (*rule lred.Beta*)
qed

lemma *WN-eq*: *WN* $t = (\exists t'.\ t \to_\beta{}^* t' \wedge$ *NF* $t')$
proof
assume *WN t*
then have $\exists t'.\ t \to_l t'$ **by** (*rule lemma6*)
then obtain t' **where** t': $t \to_l t'$ **..**
then have *NF*: *NF* t' **by** (*rule lemma5*)
from t' **have** $t \to_s t'$ **by** (*rule lred-imp-sred*)
then have $t \to_\beta{}^* t'$ **by** (*rule lemma2-2*)
with *NF* **show** $\exists t'.\ t \to_\beta{}^* t' \wedge$ *NF* t' **by** *iprover*
next
assume $\exists t'.\ t \to_\beta{}^* t' \wedge$ *NF* t'
then obtain t' **where** t': $t \to_\beta{}^* t'$ **and** *NF*: *NF* t'
by *iprover*
from t' **have** $t \to_s t'$ **by** (*rule rtrancl-beta-sred*)
then have $t \to_l t'$ **using** *NF* **by** (*rule lemma7*)
then show *WN t* **by** (*rule lemma5*)
qed

end

13 Weak normalization for simply-typed lambda calculus

theory *WeakNorm*
imports *LambdaType NormalForm HOL−Library.Realizers HOL−Library.Code-Target-Int*
begin

Formalization by Stefan Berghofer. Partly based on a paper proof by Felix Joachimski and Ralph Matthes [1].

13.1 Main theorems

lemma *norm-list*:
  assumes *f-compat*: $\bigwedge t\ t'.\ t \rightarrow_\beta{}^*\ t' \Longrightarrow f\ t \rightarrow_\beta{}^*\ f\ t'$
  and *f-NF*: $\bigwedge t.\ NF\ t \Longrightarrow NF\ (f\ t)$
  and *uNF*: $NF\ u$ **and** *uT*: $e \vdash u : T$
  shows $\bigwedge Us.\ e\langle i{:}T\rangle \Vdash as : Us \Longrightarrow$
    $listall\ (\lambda t.\ \forall\, e\ T'\ u\ i.\ e\langle i{:}T\rangle \vdash t : T' \longrightarrow$
      $NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \rightarrow_\beta{}^*\ t' \wedge NF\ t'))\ as \Longrightarrow$
    $\exists\, as'.\ \forall\, j.\ Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ as \rightarrow_\beta{}^*$
      $Var\ j\ {}^{\circ\circ}\ map\ f\ as' \wedge NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ as')$
  (**is** $\bigwedge Us.$ *-* $\Longrightarrow listall\ ?R\ as \Longrightarrow \exists\, as'.\ ?ex\ Us\ as\ as'$)
proof (*induct as rule*: *rev-induct*)
  case (*Nil Us*)
  with *Var-NF* **have** *?ex Us* [] [] **by** *simp*
  thus *?case* **..**
next
  case (*snoc b bs Us*)
  have $e\langle i{:}T\rangle \Vdash bs\ @\ [b] : Us$ **by** *fact*
  then obtain *Vs W* **where** *Us*: $Us = Vs\ @\ [W]$
    and *bs*: $e\langle i{:}T\rangle \Vdash bs : Vs$ **and** *bT*: $e\langle i{:}T\rangle \vdash b : W$
    by (*rule types-snocE*)
  from *snoc* **have** *listall ?R bs* **by** *simp*
  with *bs* **have** $\exists\, bs'.$ *?ex Vs bs bs'* **by** (*rule snoc*)
  then obtain *bs'* **where** *bsred*: $Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ bs \rightarrow_\beta{}^*\ Var\ j\ {}^{\circ\circ}\ map$
$f\ bs'$
    and *bsNF*: $NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ bs')$ **for** *j*
    by *iprover*
  from *snoc* **have** *?R b* **by** *simp*
  with *bT* **and** *uNF* **and** *uT* **have** $\exists\, b'.\ b[u/i] \rightarrow_\beta{}^*\ b' \wedge NF\ b'$
    by *iprover*
  then obtain *b'* **where** *bred*: $b[u/i] \rightarrow_\beta{}^*\ b'$ **and** *bNF*: $NF\ b'$
    by *iprover*
  from *bsNF* [*of 0*] **have** *listall NF* (*map f bs'*)
    by (*rule App-NF-D*)
  moreover have $NF\ (f\ b')$ **using** *bNF* **by** (*rule f-NF*)
  ultimately have *listall NF* (*map f* (*bs'* @ [*b'*]))
    by *simp*
  hence $\bigwedge j.\ NF\ (Var\ j\ {}^{\circ\circ}\ map\ f\ (bs'\ @\ [b']))$ **by** (*rule NF.App*)

moreover from *bred* **have** $f\ (b[u/i]) \to_\beta{}^* f\ b'$
by (*rule f-compat*)
with *bsred* **have**
$\bigwedge j.\ (Var\ j\ {}^{\circ\circ}\ map\ (\lambda t.\ f\ (t[u/i]))\ bs)\ {}^\circ\ f\ (b[u/i]) \to_\beta{}^*$
$(Var\ j\ {}^{\circ\circ}\ map\ f\ bs')\ {}^\circ\ f\ b'$ **by** (*rule rtrancl-beta-App*)
ultimately have *?ex Us* $(bs\ @\ [b])\ (bs'\ @\ [b'])$ **by** *simp*
thus *?case* **..**
qed

lemma *subst-type-NF*:
$\bigwedge t\ e\ T\ u\ i.\ NF\ t \Longrightarrow e\langle i{:}U\rangle \vdash t : T \Longrightarrow NF\ u \Longrightarrow e \vdash u : U \Longrightarrow \exists\, t'.\ t[u/i] \to_\beta{}^*\ t' \wedge NF\ t'$
(**is** *PROP ?P U* **is** $\bigwedge t\ e\ T\ u\ i.\ \text{-} \Longrightarrow$ *PROP ?Q t e T u i U*)
proof (*induct U*)
fix *T t*
let $?R = \lambda t.\ \forall\, e\ T'\ u\ i.$
$e\langle i{:}T\rangle \vdash t : T' \longrightarrow NF\ u \longrightarrow e \vdash u : T \longrightarrow (\exists\, t'.\ t[u/i] \to_\beta{}^*\ t' \wedge NF\ t')$
assume *MI1*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow$ *PROP ?P T1*
assume *MI2*: $\bigwedge T1\ T2.\ T = T1 \Rightarrow T2 \Longrightarrow$ *PROP ?P T2*
assume *NF t*
thus $\bigwedge e\ T'\ u\ i.$ *PROP ?Q t e* T' *u i T*
proof *induct*
fix $e\ T'\ u\ i$ **assume** *uNF*: *NF u* **and** *uT*: $e \vdash u : T$
{
case (*App ts x e1* $T'1$ *u1 i1*)
assume $e\langle i{:}T\rangle \vdash Var\ x\ {}^{\circ\circ}\ ts : T'$
then obtain *Us*
where *varT*: $e\langle i{:}T\rangle \vdash Var\ x : Us \Rrightarrow T'$
and *argsT*: $e\langle i{:}T\rangle \Vdash ts : Us$
by (*rule var-app-typesE*)
from *nat-eq-dec* **show** $\exists\, t'.\ (Var\ x\ {}^{\circ\circ}\ ts)[u/i] \to_\beta{}^*\ t' \wedge NF\ t'$
proof
assume *eq*: $x = i$
show *?thesis*
proof (*cases ts*)
case *Nil*
with *eq* **have** $(Var\ x\ {}^{\circ\circ}\ [])[u/i] \to_\beta{}^*\ u$ **by** *simp*
with *Nil* **and** *uNF* **show** *?thesis* **by** *simp iprover*
next
case (*Cons a as*)
with *argsT* **obtain** T'' *Ts* **where** *Us*: $Us = T''\ \#\ Ts$
by (*cases Us*) (*rule FalseE*, *simp*)
from *varT* **and** *Us* **have** *varT*: $e\langle i{:}T\rangle \vdash Var\ x : T'' \Rightarrow Ts \Rrightarrow T'$
by *simp*
from *varT eq* **have** *T*: $T = T'' \Rightarrow Ts \Rrightarrow T'$ **by** *cases auto*
with *uT* **have** *uT'*: $e \vdash u : T'' \Rightarrow Ts \Rrightarrow T'$ **by** *simp*
from *argsT Us Cons* **have** *argsT'*: $e\langle i{:}T\rangle \Vdash as : Ts$ **by** *simp*
from *argsT Us Cons* **have** *argT*: $e\langle i{:}T\rangle \vdash a : T''$ **by** *simp*
from *argT uT refl* **have** *aT*: $e \vdash a[u/i] : T''$ **by** (*rule subst-lemma*)

from *App* **and** *Cons* **have** *listall ?R as* **by** *simp* (*iprover dest*: *listall-conj2*)
with *lift-preserves-beta′ lift-NF uNF uT argsT′*
have $\exists$ *as′*. $\forall$ *j*. *Var j* $^{\circ\circ}$ *map* (λ*t*. *lift* (*t*[*u*/*i*]) *0*) *as* $\rightarrow_\beta{}^*$
Var j $^{\circ\circ}$ *map* (λ*t*. *lift t 0*) *as′* $\wedge$
NF (*Var j* $^{\circ\circ}$ *map* (λ*t*. *lift t 0*) *as′*) **by** (*rule norm-list*)
then obtain *as′* **where**
asred: *Var 0* $^{\circ\circ}$ *map* (λ*t*. *lift* (*t*[*u*/*i*]) *0*) *as* $\rightarrow_\beta{}^*$
Var 0 $^{\circ\circ}$ *map* (λ*t*. *lift t 0*) *as′*
and *asNF*: *NF* (*Var 0* $^{\circ\circ}$ *map* (λ*t*. *lift t 0*) *as′*) **by** *iprover*
from *App* **and** *Cons* **have** *?R a* **by** *simp*
with *argT* **and** *uNF* **and** *uT* **have** $\exists$ *a′*. *a*[*u*/*i*] $\rightarrow_\beta{}^*$ *a′* $\wedge$ *NF a′*
by *iprover*
then obtain *a′* **where** *ared*: *a*[*u*/*i*] $\rightarrow_\beta{}^*$ *a′* **and** *aNF*: *NF a′* **by** *iprover*
from *uNF* **have** *NF* (*lift u 0*) **by** (*rule lift-NF*)
hence $\exists$ *u′*. *lift u 0* $^\circ$ *Var 0* $\rightarrow_\beta{}^*$ *u′* $\wedge$ *NF u′* **by** (*rule app-Var-NF*)
then obtain *u′* **where** *ured*: *lift u 0* $^\circ$ *Var 0* $\rightarrow_\beta{}^*$ *u′* **and** *u′NF*: *NF u′*
by *iprover*
from *T* **and** *u′NF* **have** $\exists$ *ua*. *u′*[*a′*/*0*] $\rightarrow_\beta{}^*$ *ua* $\wedge$ *NF ua*
proof (*rule MI1*)
have $e\langle 0{:}T''\rangle \vdash$ *lift u 0* $^\circ$ *Var 0* : *Ts* $\Rrightarrow$ *T′*
proof (*rule typing.App*)
from *uT′* **show** $e\langle 0{:}T''\rangle \vdash$ *lift u 0* : $T'' \Rightarrow$ *Ts* $\Rrightarrow$ *T′* **by** (*rule lift-type*)
show $e\langle 0{:}T''\rangle \vdash$ *Var 0* : T'' **by** (*rule typing.Var*) *simp*
qed
with *ured* **show** $e\langle 0{:}T''\rangle \vdash u'$: *Ts* $\Rrightarrow$ *T′* **by** (*rule subject-reduction′*)
from *ared aT* **show** $e \vdash a'$: T'' **by** (*rule subject-reduction′*)
show *NF a′* **by** *fact*
qed
then obtain *ua* **where** *uared*: *u′*[*a′*/*0*] $\rightarrow_\beta{}^*$ *ua* **and** *uaNF*: *NF ua*
by *iprover*
from *ared* **have** (*lift u 0* $^\circ$ *Var 0*)[*a*[*u*/*i*]/*0*] $\rightarrow_\beta{}^*$ (*lift u 0* $^\circ$ *Var 0*)[*a′*/*0*]
by (*rule subst-preserves-beta2′*)
also from *ured* **have** (*lift u 0* $^\circ$ *Var 0*)[*a′*/*0*] $\rightarrow_\beta{}^*$ *u′*[*a′*/*0*]
by (*rule subst-preserves-beta′*)
also note *uared*
finally have (*lift u 0* $^\circ$ *Var 0*)[*a*[*u*/*i*]/*0*] $\rightarrow_\beta{}^*$ *ua* **.**
hence *uared′*: *u* $^\circ$ *a*[*u*/*i*] $\rightarrow_\beta{}^*$ *ua* **by** *simp*
from *T asNF - uaNF* **have** $\exists$ *r*. (*Var 0* $^{\circ\circ}$ *map* (λ*t*. *lift t 0*) *as′*)[*ua*/*0*]
$\rightarrow_\beta{}^*$ *r* $\wedge$ *NF r*
proof (*rule MI2*)
have $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash$ *Var 0* $^{\circ\circ}$ *map* (λ*t*. *lift* (*t*[*u*/*i*]) *0*) *as* : *T′*
proof (*rule list-app-typeI*)
show $e\langle 0{:}Ts \Rrightarrow T'\rangle \vdash$ *Var 0* : *Ts* $\Rrightarrow$ *T′* **by** (*rule typing.Var*) *simp*
from *uT argsT′* **have** $e \Vdash$ *map* (λ*t*. *t*[*u*/*i*]) *as* : *Ts*
by (*rule substs-lemma*)
hence $e\langle 0{:}Ts \Rrightarrow T'\rangle \Vdash$ *map* (λ*t*. *lift t 0*) (*map* (λ*t*. *t*[*u*/*i*]) *as*) : *Ts*
by (*rule lift-types*)
thus $e\langle 0{:}Ts \Rrightarrow T'\rangle \Vdash$ *map* (λ*t*. *lift* (*t*[*u*/*i*]) *0*) *as* : *Ts*
by (*simp-all add*: *o-def*)

```
          qed
          with asred show e⟨0:Ts ⇛ T'⟩ ⊢ Var 0 °° map (λt. lift t 0) as' : T'
            by (rule subject-reduction')
          from argT uT refl have e ⊢ a[u/i] : T'' by (rule subst-lemma)
          with uT' have e ⊢ u ° a[u/i] : Ts ⇛ T' by (rule typing.App)
          with uared' show e ⊢ ua : Ts ⇛ T' by (rule subject-reduction')
        qed
        then obtain r where rred: (Var 0 °° map (λt. lift t 0) as')[ua/0] →β* r
          and rnf: NF r by iprover
        from asred have
          (Var 0 °° map (λt. lift (t[u/i]) 0) as)[u ° a[u/i]/0] →β*
          (Var 0 °° map (λt. lift t 0) as')[u ° a[u/i]/0]
          by (rule subst-preserves-beta')
        also from uared' have (Var 0 °° map (λt. lift t 0) as')[u ° a[u/i]/0] →β*
          (Var 0 °° map (λt. lift t 0) as')[ua/0] by (rule subst-preserves-beta2')
        also note rred
        finally have (Var 0 °° map (λt. lift (t[u/i]) 0) as)[u ° a[u/i]/0] →β* r .
        with rnf Cons eq show ?thesis
          by (simp add: o-def) iprover
      qed
    next
      assume neq: x ≠ i
      from App have listall ?R ts by (iprover dest: listall-conj2)
      with uNF uT argsT
      have ∃ts'. ∀j. Var j °° map (λt. t[u/i]) ts →β* Var j °° ts' ∧
        NF (Var j °° ts') (is ∃ts'. ?ex ts')
        by (rule norm-list [of λt. t, simplified])
      then obtain ts' where NF: ?ex ts' ..
      from nat-le-dec show ?thesis
      proof
        assume i < x
        with NF show ?thesis by simp iprover
      next
        assume ¬ (i < x)
        with NF neq show ?thesis by (simp add: subst-Var) iprover
      qed
    qed
  next
    case (Abs r e1 T'1 u1 i1)
    assume absT: e⟨i:T⟩ ⊢ Abs r : T'
    then obtain R S where e⟨0:R⟩⟨Suc i:T⟩ ⊢ r : S by (rule abs-typeE) simp
    moreover have NF (lift u 0) using ‹NF u› by (rule lift-NF)
    moreover have e⟨0:R⟩ ⊢ lift u 0 : T using uT by (rule lift-type)
    ultimately have ∃t'. r[lift u 0/Suc i] →β* t' ∧ NF t' by (rule Abs)
    thus ∃t'. Abs r[u/i] →β* t' ∧ NF t'
      by simp (iprover intro: rtrancl-beta-Abs NF.Abs)
  }
 qed
qed
```

— A computationally relevant copy of $e \vdash t : T$
inductive *rtyping* :: (*nat* ⇒ *type*) ⇒ *dB* ⇒ *type* ⇒ *bool* (- $\vdash_R$ - : - [*50*, *50*, *50*] *50*)
  where
    Var: $e\ x = T \Longrightarrow e \vdash_R Var\ x : T$
  | *Abs*: $e\langle 0{:}T\rangle \vdash_R t : U \Longrightarrow e \vdash_R Abs\ t : (T \Rightarrow U)$
  | *App*: $e \vdash_R s : T \Rightarrow U \Longrightarrow e \vdash_R t : T \Longrightarrow e \vdash_R (s \circ t) : U$

lemma *rtyping-imp-typing*: $e \vdash_R t : T \Longrightarrow e \vdash t : T$
  apply (*induct set*: *rtyping*)
  apply (*erule typing.Var*)
  apply (*erule typing.Abs*)
  apply (*erule typing.App*)
  apply *assumption*
  done

theorem *type-NF*:
  assumes $e \vdash_R t : T$
  shows $\exists t'.\ t \rightarrow_\beta{}^* t' \wedge NF\ t'$ **using** *assms*
proof *induct*
  case *Var*
  show *?case* **by** (*iprover intro*: *Var-NF*)
next
  case *Abs*
  thus *?case* **by** (*iprover intro*: *rtrancl-beta-Abs NF.Abs*)
next
  case (*App e s T U t*)
  from *App* **obtain** s' t' **where**
    sred: $s \rightarrow_\beta{}^* s'$ **and** $NF\ s'$
    and *tred*: $t \rightarrow_\beta{}^* t'$ **and** *tNF*: $NF\ t'$ **by** *iprover*
  have $\exists u.\ (Var\ 0 \circ lift\ t'\ 0)[s'/0] \rightarrow_\beta{}^* u \wedge NF\ u$
  proof (*rule subst-type-NF*)
    have $NF\ (lift\ t'\ 0)$ **using** *tNF* **by** (*rule lift-NF*)
    hence $listall\ NF\ [lift\ t'\ 0]$ **by** (*rule listall-cons*) (*rule listall-nil*)
    hence $NF\ (Var\ 0 \circ\circ [lift\ t'\ 0])$ **by** (*rule NF.App*)
    thus $NF\ (Var\ 0 \circ lift\ t'\ 0)$ **by** *simp*
    show $e\langle 0{:}T \Rightarrow U\rangle \vdash Var\ 0 \circ lift\ t'\ 0 : U$
    proof (*rule typing.App*)
      show $e\langle 0{:}T \Rightarrow U\rangle \vdash Var\ 0 : T \Rightarrow U$
        by (*rule typing.Var*) *simp*
      from *tred* **have** $e \vdash t' : T$
        by (*rule subject-reduction'*) (*rule rtyping-imp-typing*, *rule App.hyps*)
      thus $e\langle 0{:}T \Rightarrow U\rangle \vdash lift\ t'\ 0 : T$
        by (*rule lift-type*)
    qed
    from *sred* **show** $e \vdash s' : T \Rightarrow U$

by (*rule subject-reduction′*) (*rule rtyping-imp-typing*, *rule App.hyps*)
show *NF* s' **by** *fact*
qed
then obtain u **where** *ured*: $s' \circ t' \rightarrow_\beta{}^* u$ **and** *unf*: *NF* u **by** *simp iprover*
from *sred tred* **have** $s \circ t \rightarrow_\beta{}^* s' \circ t'$ **by** (*rule rtrancl-beta-App*)
hence $s \circ t \rightarrow_\beta{}^* u$ **using** *ured* **by** (*rule rtranclp-trans*)
with *unf* **show** *?case* **by** *iprover*
qed

13.2 Extracting the program

declare *NF.induct* [*ind-realizer*]
declare *rtranclp.induct* [*ind-realizer irrelevant*]
declare *rtyping.induct* [*ind-realizer*]
lemmas [*extraction-expand*] = *conj-assoc listall-cons-eq subst-all equal-allI*

extract *type-NF*

lemma *rtranclR-rtrancl-eq*: *rtranclpR* $r\ a\ b = r^{**}\ a\ b$
apply (*rule iffI*)
apply (*erule rtranclpR.induct*)
apply (*rule rtranclp.rtrancl-refl*)
apply (*erule rtranclp.rtrancl-into-rtrancl*)
apply *assumption*
apply (*erule rtranclp.induct*)
apply (*rule rtranclpR.rtrancl-refl*)
apply (*erule rtranclpR.rtrancl-into-rtrancl*)
apply *assumption*
done

lemma *NFR-imp-NF*: *NFR nf t* $\Longrightarrow$ *NF t*
apply (*erule NFR.induct*)
apply (*rule NF.intros*)
apply (*simp add*: *listall-def*)
apply (*erule NF.intros*)
done

The program corresponding to the proof of the central lemma, which performs substitution and normalization, is shown in Figure 1. The correctness theorem corresponding to the program *subst-type-NF* is

$$
\begin{array}{l}
\bigwedge x.\ NFR\ x\ t \Longrightarrow \\
\quad e\langle i{:}U\rangle \vdash t : T \Longrightarrow \\
\quad (\bigwedge xa.\ NFR\ xa\ u \Longrightarrow \\
\qquad e \vdash u : U \Longrightarrow \\
\qquad t[u/i] \rightarrow_\beta{}^* fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa) \wedge \\
\qquad NFR\ (snd\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U\ T\ u\ x\ xa))\ (fst\ (subst\text{-}type\text{-}NF\ t\ e\ i\ U \\
T\ u\ x\ xa)))
\end{array}
$$

where *NFR* is the realizability predicate corresponding to the datatype *NFT*,

```
subst-type-NF ≡
λx xa xb xc xd xe H Ha.
   type-induct-P xc
    (λx H2 H2a xa xaa xb xc xd H.
        compat-NFT.rec-split-NFT default
         (λts xa xaa r xb xc xd xe H.
             var-app-typesE-P (xb⟨xe:x⟩) xa ts
              (λUs--. case nat-eq-dec xa xe of
                      Left ⇒ case ts of [] ⇒ (xd, H)
                              | a # list ⇒
                                  case Us-- of [] ⇒ default
                                  | T''-- # Ts-- ⇒
                                      let (x, y) =
                                            norm-list (λt. lift t 0) xd xb xe list Ts--
                                             (λt. lift-NF 0) H
                                             (listall-conj2-P-Q list (λi. (xaa (Suc i), r (Suc i))));
                                          (xa, ya) = snd (xaa 0, r 0) xb T''-- xd xe H;
                                          (xd, yb) = app-Var-NF 0 (lift-NF 0 H);
                                          (xa, ya) =
                                            H2 T''-- (Ts-- ⇛ xc) xd xb (Ts-- ⇛ xc) xa 0 yb ya;
                                          (x, y) =
                                            H2a T''-- (Ts-- ⇛ xc) (dB.Var 0 °° map (λt. lift t 0) x)
                                             xb xc xa 0 (y 0) ya
                                      in (x, y)
                      | Right ⇒
                          let (x, y) =
                                let (x, y) =
                                      norm-list (λt. t) xd xb xe ts Us-- (λx H. H) H
                                       (listall-conj2-P-Q ts (λz. (xaa z, r z)))
                                in (x, λx. y x)
                          in case nat-le-dec xe xa of
                              Left ⇒ (dB.Var (xa − Suc 0) °° x, y (xa − Suc 0))
                              | Right ⇒ (dB.Var xa °° x, y xa)))
         (λt x r xa xaa xb xc H.
             abs-typeE-P xaa
              (λU V. let (x, y) =
                           let (x, y) = r (λa. (xa⟨0:U⟩) a) V (lift xb 0) (Suc xc) (lift-NF 0 H)
                           in (dB.Abs x, NFT.Abs x y)
                       in (x, y)))
         H (λa. xaa a) xb xc xd)
    x xa xd xe xb H Ha
```

Figure 1: Program extracted from *subst-type-NF*

```
subst-Var-NF ≡
λx xa H.
   compat-NFT.rec-split-NFT default
    (λts x xa r xb xc.
        case nat-eq-dec x xc of
        Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) xb
                  (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z))))
        | Right ⇒
            case nat-le-dec xc x of
            Left ⇒ NFT.App (map (λt. t[dB.Var xb/xc]) ts) (x − Suc 0)
                      (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                        (listall-conj2-P-Q ts (λz. (xa z, r z))))
            | Right ⇒
                NFT.App (map (λt. t[dB.Var xb/xc]) ts) x
                 (subst-terms-NF ts xb xc (listall-conj1-P-Q ts (λz. (xa z, r z)))
                   (listall-conj2-P-Q ts (λz. (xa z, r z)))))
    (λt x r xa xaa. NFT.Abs (t[dB.Var (Suc xa)/Suc xaa]) (r (Suc xa) (Suc xaa))) H x xa

app-Var-NF ≡
λx. compat-NFT.rec-split-NFT default
      (λts xa xaa r.
          (dB.Var xa °° (ts @ [dB.Var x]),
           NFT.App (ts @ [dB.Var x]) xa
            (snd (listall-app-P ts)
              (listall-conj1-P-Q ts (λz. (xaa z, r z)),
               listall-cons-P (Var-NF x) listall-nil-eq-P))))
      (λt xa r. (t[dB.Var x/0], subst-Var-NF x 0 xa))

lift-NF ≡
λx H. compat-NFT.rec-split-NFT default
         (λts x xa r xb.
             case nat-le-dec x xb of
             Left ⇒ NFT.App (map (λt. lift t xb) ts) x
                       (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                         (listall-conj2-P-Q ts (λz. (xa z, r z))))
             | Right ⇒
                 NFT.App (map (λt. lift t xb) ts) (Suc x)
                  (lift-terms-NF ts xb (listall-conj1-P-Q ts (λz. (xa z, r z)))
                    (listall-conj2-P-Q ts (λz. (xa z, r z)))))
         (λt x r xa. NFT.Abs (lift t (Suc xa)) (r (Suc xa))) H x

type-NF ≡
λH. rec-rtypingT (λe x T. (dB.Var x, Var-NF x))
      (λe T t U x r. let (x, y) = r in (dB.Abs x, NFT.Abs x y))
      (λe s T U t x xa r ra.
          let (x, y) = r; (xa, ya) = ra;
              (x, y) =
                let (x, y) =
                      subst-type-NF (dB.Var 0 ° lift xa 0) e 0 (T ⇒ U) U x
                       (NFT.App [lift xa 0] 0 (listall-cons-P (lift-NF 0 ya) listall-nil-P)) y
                in (x, y)
          in (x, y))
      H
```

Figure 2: Program extracted from lemmas and main theorem

which is inductively defined by the rules

$\forall\, i{<}length\ ts.\ NFR\ (nfs\ i)\ (ts\ !\ i) \Longrightarrow NFR\ (NFT.App\ ts\ x\ nfs)\ (dB.Var\ x\ {}^{\circ\circ}\ ts)$
$NFR\ nf\ t \Longrightarrow NFR\ (NFT.Abs\ t\ nf)\ (dB.Abs\ t)$

The programs corresponding to the main theorem *type-NF*, as well as to some lemmas, are shown in Figure 2. The correctness statement for the main function *type-NF* is

$\bigwedge x.\ rtypingR\ x\ e\ t\ T \Longrightarrow t \to_\beta{}^*\ fst\ (type\text{-}NF\ x) \wedge NFR\ (snd\ (type\text{-}NF\ x))\ (fst\ (type\text{-}NF\ x))$

where the realizability predicate *rtypingR* corresponding to the computationally relevant version of the typing judgement is inductively defined by the rules

$e\ x = T \Longrightarrow rtypingR\ (rtypingT.Var\ e\ x\ T)\ e\ (dB.Var\ x)\ T$
$rtypingR\ ty\ (e\langle 0{:}T\rangle)\ t\ U \Longrightarrow rtypingR\ (rtypingT.Abs\ e\ T\ t\ U\ ty)\ e\ (dB.Abs\ t)\ (T \Rightarrow U)$
$rtypingR\ ty\ e\ s\ (T \Rightarrow U) \Longrightarrow$
$rtypingR\ ty'\ e\ t\ T \Longrightarrow rtypingR\ (rtypingT.App\ e\ s\ T\ U\ t\ ty\ ty')\ e\ (s\ {}^\circ\ t)\ U$

13.3 Generating executable code

instantiation *NFT* :: *default*
begin

definition *default* = *Dummy* ()

instance ..

end

instantiation *dB* :: *default*
begin

definition *default* = *dB.Var 0*

instance ..

end

instantiation *prod* :: (*default*, *default*) *default*
begin

definition *default* = (*default*, *default*)

instance ..

end

instantiation *list* :: (*type*) *default*
begin

definition *default* = []

instance ..

end

instantiation *fun* :: (*type*, *default*) *default*
begin

definition *default* = (λx. *default*)

instance ..

end

definition *int-of-nat* :: *nat* ⇒ *int* **where**
int-of-nat = *of-nat*

The following functions convert between Isabelle's built-in `term` datatype and the generated `dB` datatype. This allows to generate example terms using Isabelle's parser and inspect normalized terms using Isabelle's pretty printer.

ML ‹
```
val nat-of-integer = @{code nat} o @{code int-of-integer};

fun dBtype-of-typ (Type (fun, [T, U])) =
      @{code Fun} (dBtype-of-typ T, dBtype-of-typ U)
  | dBtype-of-typ (TFree (s, -)) = (case raw-explode s of
        [', a] => @{code Atom} (nat-of-integer (ord a − 97))
      | - => error dBtype-of-typ: variable name)
  | dBtype-of-typ - = error dBtype-of-typ: bad type;

fun dB-of-term (Bound i) = @{code dB.Var} (nat-of-integer i)
  | dB-of-term (t $ u) = @{code dB.App} (dB-of-term t, dB-of-term u)
  | dB-of-term (Abs (-, -, t)) = @{code dB.Abs} (dB-of-term t)
  | dB-of-term - = error dB-of-term: bad term;

fun term-of-dB Ts (Type (fun, [T, U])) (@{code dB.Abs} dBt) =
      Abs (x, T, term-of-dB (T :: Ts) U dBt)
  | term-of-dB Ts - dBt = term-of-dB' Ts dBt
and term-of-dB' Ts (@{code dB.Var} n) = Bound (@{code integer-of-nat} n)
  | term-of-dB' Ts (@{code dB.App} (dBt, dBu)) =
      let val t = term-of-dB' Ts dBt
      in case fastype-of1 (Ts, t) of
          Type (fun, [T, -]) => t $ term-of-dB Ts T dBu
```

```
    | - => error term-of-dB: function type expected
    end
  | term-of-dB' - - = error term-of-dB: term not in normal form;

fun typing-of-term Ts e (Bound i) =
      @{code Var} (e, nat-of-integer i, dBtype-of-typ (nth Ts i))
  | typing-of-term Ts e (t $ u) = (case fastype-of1 (Ts, t) of
        Type (fun, [T, U]) => @{code App} (e, dB-of-term t,
          dBtype-of-typ T, dBtype-of-typ U, dB-of-term u,
          typing-of-term Ts e t, typing-of-term Ts e u)
      | - => error typing-of-term: function type expected)
  | typing-of-term Ts e (Abs (-, T, t)) =
      let val dBT = dBtype-of-typ T
      in @{code Abs} (e, dBT, dB-of-term t,
        dBtype-of-typ (fastype-of1 (T :: Ts, t)),
        typing-of-term (T :: Ts) (@{code shift} e @{code 0::nat} dBT) t)
      end
  | typing-of-term - - - = error typing-of-term: bad term;

fun dummyf - = error dummy;

val ct1 = @{cterm %f. ((%f x. f (f (f x))) ((%f x. f (f (f (f x)))) f))};
val (dB1, -) = @{code type-NF} (typing-of-term [] dummyf (Thm.term-of ct1));
val ct1' = Thm.cterm-of @{context} (term-of-dB [] (Thm.typ-of-cterm ct1) dB1);

val ct2 = @{cterm %f x. (%x. f x x) ((%x. f x x) ((%x. f x x) ((%x. f x x) ((%x. f
x x) ((%x. f x x) x)))))};
val (dB2, -) = @{code type-NF} (typing-of-term [] dummyf (Thm.term-of ct2));
val ct2' = Thm.cterm-of @{context} (term-of-dB [] (Thm.typ-of-cterm ct2) dB2);
›
```

end

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