State Spaces: The Locale Way

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December 12, 2021

Contents

1 Introduction

These theories introduce a new command called **statespace**. It's usage is similar to **record**s. However, the command does not introduce a new type but an abstract specification based on the locale infrastructure. This leads to extra flexibility in composing state space components, in particular multiple inheritance and renaming of components.

The state space infrastructure basically manages the following things:

- distinctness of field names
- projections / injections from / to an abstract *value* type
- syntax translations for lookup and update, hiding the projections and injections
- simplification procedure for lookups / updates, similar to records

Overview In Section 2 we define distinctness of the nodes in a binary tree and provide the basic prover tools to support efficient distinctness reasoning for field names managed by state spaces. The state is represented as a function from (abstract) names to (abstract) values as introduced in Section 3. The basic setup for state spaces is in Section 4. Some syntax for lookup and updates is added in Section 5. Finally Section 6 explains the usage of state spaces by examples.

2 Distinctness of Names in a Binary Tree

theory *DistinctTreeProver*
imports *Main*
begin

A state space manages a set of (abstract) names and assumes that the names are distinct. The names are stored as parameters of a locale and distinctness as an assumption. The most common request is to proof distinctness of two given names. We maintain the names in a balanced binary tree and formulate a predicate that all nodes in the tree have distinct names. This setup leads to logarithmic certificates.

2.1 The Binary Tree

datatype $'a\ tree = Node\ 'a\ tree\ 'a\ bool\ 'a\ tree \mid Tip$

The boolean flag in the node marks the content of the node as deleted, without having to build a new tree. We prefer the boolean flag to an option type, so that the ML-layer can still use the node content to facilitate binary search in the tree. The ML code keeps the nodes sorted using the term order. We do not have to push ordering to the HOL level.

2.2 Distinctness of Nodes

primrec $\textit{set-of} :: 'a\ tree \Rightarrow 'a\ set$
where
$\quad \textit{set-of}\ Tip = \{\}$
$\mid \textit{set-of}\ (Node\ l\ x\ d\ r) = (\textit{if}\ d\ \textit{then}\ \{\}\ \textit{else}\ \{x\}) \cup \textit{set-of}\ l \cup \textit{set-of}\ r$

primrec $\textit{all-distinct} :: 'a\ tree \Rightarrow bool$
where
$\quad \textit{all-distinct}\ Tip = True$
$\mid \textit{all-distinct}\ (Node\ l\ x\ d\ r) =$
$\quad ((d \lor (x \notin \textit{set-of}\ l \land x \notin \textit{set-of}\ r)) \land$
$\quad\quad \textit{set-of}\ l \cap \textit{set-of}\ r = \{\} \land$
$\quad\quad \textit{all-distinct}\ l \land \textit{all-distinct}\ r)$

Given a binary tree t for which *all-distinct* holds, given two different nodes contained in the tree, we want to write a ML function that generates a logarithmic certificate that the content of the nodes is distinct. We use the following lemmas to achieve this.

lemma *all-distinct-left*: $\textit{all-distinct}\ (\textit{Node}\ l\ x\ b\ r) \Longrightarrow \textit{all-distinct}\ l$
  by *simp*

lemma *all-distinct-right*: $\textit{all-distinct}\ (\textit{Node}\ l\ x\ b\ r) \Longrightarrow \textit{all-distinct}\ r$
  by *simp*

lemma *distinct-left*: $\textit{all-distinct}\ (\textit{Node}\ l\ x\ \textit{False}\ r) \Longrightarrow y \in \textit{set-of}\ l \Longrightarrow x \neq y$
  by *auto*

lemma *distinct-right*: $\textit{all-distinct}\ (\textit{Node}\ l\ x\ \textit{False}\ r) \Longrightarrow y \in \textit{set-of}\ r \Longrightarrow x \neq y$
  by *auto*

lemma *distinct-left-right*:
    $\textit{all-distinct}\ (\textit{Node}\ l\ z\ b\ r) \Longrightarrow x \in \textit{set-of}\ l \Longrightarrow y \in \textit{set-of}\ r \Longrightarrow x \neq y$
  by *auto*

lemma *in-set-root*: $x \in \textit{set-of}\ (\textit{Node}\ l\ x\ \textit{False}\ r)$
  by *simp*

lemma *in-set-left*: $y \in \textit{set-of}\ l \Longrightarrow\ y \in \textit{set-of}\ (\textit{Node}\ l\ x\ \textit{False}\ r)$
  by *simp*

lemma *in-set-right*: $y \in \textit{set-of}\ r \Longrightarrow\ y \in \textit{set-of}\ (\textit{Node}\ l\ x\ \textit{False}\ r)$
  by *simp*

lemma *swap-neq*: $x \neq y \Longrightarrow y \neq x$
  by *blast*

lemma *neq-to-eq-False*: $x{\neq}y \Longrightarrow (x{=}y){\equiv}\textit{False}$
  by *simp*

2.3 Containment of Trees

When deriving a state space from other ones, we create a new name tree which contains all the names of the parent state spaces and assume the predicate *all-distinct*. We then prove that the new locale interprets all parent locales. Hence we have to show that the new distinctness assumption on all names implies the distinctness assumptions of the parent locales. This proof is implemented in ML. We do this efficiently by defining a kind of containment check of trees by "subtraction". We subtract the parent tree from the new tree. If this succeeds we know that *all-distinct* of the new tree implies *all-distinct* of the parent tree. The resulting certificate is of the order $n * \textit{log}\ m$ where n is the size of the (smaller) parent tree and m the size of the (bigger) new tree.

```
primrec delete :: 'a ⇒ 'a tree ⇒ 'a tree option
where
  delete x Tip = None
| delete x (Node l y d r) = (case delete x l of
                               Some l' ⇒
                                (case delete x r of
                                   Some r' ⇒ Some (Node l' y (d ∨ (x=y)) r')
                                 | None ⇒ Some (Node l' y (d ∨ (x=y)) r))
                             | None ⇒
                                 (case delete x r of
                                    Some r' ⇒ Some (Node l y (d ∨ (x=y)) r')
                                  | None ⇒ if x=y ∧ ¬d then Some (Node l y True r)
                                           else None))
```

```
lemma delete-Some-set-of: delete x t = Some t' ⟹ set-of t' ⊆ set-of t
proof (induct t arbitrary: t')
  case Tip thus ?case by simp
next
  case (Node l y d r)
  have del: delete x (Node l y d r) = Some t' by fact
  show ?case
  proof (cases delete x l)
    case (Some l')
    note x-l-Some = this
    with Node.hyps
    have l'-l: set-of l' ⊆ set-of l
      by simp
    show ?thesis
    proof (cases delete x r)
      case (Some r')
      with Node.hyps
      have set-of r' ⊆ set-of r
        by simp
      with l'-l Some x-l-Some del
      show ?thesis
        by (auto split: if-split-asm)
    next
      case None
      with l'-l Some x-l-Some del
      show ?thesis
        by (fastforce split: if-split-asm)
    qed
  next
    case None
    note x-l-None = this
    show ?thesis
    proof (cases delete x r)
```

case (*Some r′*)
with *Node.hyps*
have *set-of r′* ⊆ *set-of r*
by *simp*
with *Some x-l-None del*
show *?thesis*
by (*fastforce split*: *if-split-asm*)
next
case *None*
with *x-l-None del*
show *?thesis*
by (*fastforce split*: *if-split-asm*)
qed
qed
qed

lemma *delete-Some-all-distinct*:
delete x t = Some t′ ⟹ *all-distinct t* ⟹ *all-distinct t′*
proof (*induct t arbitrary*: *t′*)
case *Tip* **thus** *?case* **by** *simp*
next
case (*Node l y d r*)
have *del*: *delete x* (*Node l y d r*) = *Some t′* **by** *fact*
have *all-distinct* (*Node l y d r*) **by** *fact*
then obtain
dist-l: *all-distinct l* **and**
dist-r: *all-distinct r* **and**
d: *d* ∨ (*y* ∉ *set-of l* ∧ *y* ∉ *set-of r*) **and**
dist-l-r: *set-of l* ∩ *set-of r* = {}
by *auto*
show *?case*
proof (*cases delete x l*)
case (*Some l′*)
note *x-l-Some* = *this*
from *Node.hyps* (*1*) [*OF Some dist-l*]
have *dist-l′*: *all-distinct l′*
by *simp*
from *delete-Some-set-of* [*OF x-l-Some*]
have *l′-l*: *set-of l′* ⊆ *set-of l*.
show *?thesis*
proof (*cases delete x r*)
case (*Some r′*)
from *Node.hyps* (*2*) [*OF Some dist-r*]
have *dist-r′*: *all-distinct r′*
by *simp*
from *delete-Some-set-of* [*OF Some*]
have *set-of r′* ⊆ *set-of r*.

with *dist-l′ dist-r′ l′-l Some x-l-Some del d dist-l-r*

```
      show ?thesis
        by fastforce
    next
      case None
      with l'-l dist-l'  x-l-Some del d dist-l-r dist-r
      show ?thesis
        by fastforce
    qed
  next
    case None
    note x-l-None = this
    show ?thesis
    proof (cases delete x r)
      case (Some r')
      with Node.hyps (2) [OF Some dist-r]
      have dist-r': all-distinct r'
        by simp
      from delete-Some-set-of [OF Some]
      have set-of r' ⊆ set-of r.
      with Some dist-r' x-l-None del dist-l d dist-l-r
      show ?thesis
        by fastforce
    next
      case None
      with x-l-None del dist-l dist-r d dist-l-r
      show ?thesis
        by (fastforce split: if-split-asm)
    qed
  qed
qed

lemma delete-None-set-of-conv: delete x t = None = (x ∉ set-of t)
proof (induct t)
  case Tip thus ?case by simp
next
  case (Node l y d r)
  thus ?case
    by (auto split: option.splits)
qed

lemma delete-Some-x-set-of:
  delete x t = Some t' ⟹ x ∈ set-of t ∧ x ∉ set-of t'
proof (induct t arbitrary: t')
  case Tip thus ?case by simp
next
  case (Node l y d r)
  have del: delete x (Node l y d r) = Some t' by fact
  show ?case
  proof (cases delete x l)
```

```
    case (Some l′)
    note x-l-Some = this
    from Node.hyps (1) [OF Some]
    obtain x-l: x ∈ set-of l x ∉ set-of l′
      by simp
    show ?thesis
    proof (cases delete x r)
      case (Some r′)
      from Node.hyps (2) [OF Some]
      obtain x-r: x ∈ set-of r x ∉ set-of r′
        by simp
      from x-r x-l Some x-l-Some del
      show ?thesis
        by (clarsimp split: if-split-asm)
    next
      case None
      then have x ∉ set-of r
        by (simp add: delete-None-set-of-conv)
      with x-l None x-l-Some del
      show ?thesis
        by (clarsimp split: if-split-asm)
    qed
  next
    case None
    note x-l-None = this
    then have x-notin-l: x ∉ set-of l
      by (simp add: delete-None-set-of-conv)
    show ?thesis
    proof (cases delete x r)
      case (Some r′)
      from Node.hyps (2) [OF Some]
      obtain x-r: x ∈ set-of r x ∉ set-of r′
        by simp
      from x-r x-notin-l Some x-l-None del
      show ?thesis
        by (clarsimp split: if-split-asm)
    next
      case None
      then have x ∉ set-of r
        by (simp add: delete-None-set-of-conv)
      with None x-l-None x-notin-l del
      show ?thesis
        by (clarsimp split: if-split-asm)
    qed
  qed
qed
```

```
primrec subtract :: ′a tree ⇒ ′a tree ⇒ ′a tree option
```

```
where
  subtract Tip t = Some t
| subtract (Node l x b r) t =
    (case delete x t of
       Some t′ ⇒ (case subtract l t′ of
                    Some t′′ ⇒ subtract r t′′
                  | None ⇒ None)
     | None ⇒ None)

lemma subtract-Some-set-of-res:
  subtract t1 t2 = Some t =⇒ set-of t ⊆ set-of t2
proof (induct t1 arbitrary: t2 t)
  case Tip thus ?case by simp
next
  case (Node l x b r)
  have sub: subtract (Node l x b r) t2 = Some t by fact
  show ?case
  proof (cases delete x t2)
    case (Some t2′)
    note del-x-Some = this
    from delete-Some-set-of [OF Some]
    have t2′-t2: set-of t2′ ⊆ set-of t2 .
    show ?thesis
    proof (cases subtract l t2′)
      case (Some t2′′)
      note sub-l-Some = this
      from Node.hyps (1) [OF Some]
      have t2′′-t2′: set-of t2′′ ⊆ set-of t2′ .
      show ?thesis
      proof (cases subtract r t2′′)
        case (Some t2′′′)
        from Node.hyps (2) [OF Some ]
        have set-of t2′′′ ⊆ set-of t2′′ .
        with Some sub-l-Some del-x-Some sub t2′′-t2′ t2′-t2
        show ?thesis
          by simp
      next
        case None
        with del-x-Some sub-l-Some sub
        show ?thesis
          by simp
      qed
    next
      case None
      with del-x-Some sub
      show ?thesis
        by simp
    qed
  next
```

```
    case None
    with sub show ?thesis by simp
  qed
qed

lemma subtract-Some-set-of:
  subtract t₁ t₂ = Some t ⟹ set-of t₁ ⊆ set-of t₂
proof (induct t₁ arbitrary: t₂ t)
  case Tip thus ?case by simp
next
  case (Node l x d r)
  have sub: subtract (Node l x d r) t₂ = Some t by fact
  show ?case
  proof (cases delete x t₂)
    case (Some t₂')
    note del-x-Some = this
    from delete-Some-set-of [OF Some]
    have t2'-t2: set-of t₂' ⊆ set-of t₂ .
    from delete-None-set-of-conv [of x t₂] Some
    have x-t2: x ∈ set-of t₂
      by simp
    show ?thesis
    proof (cases subtract l t₂')
      case (Some t₂'')
      note sub-l-Some = this
      from Node.hyps (1) [OF Some]
      have l-t2': set-of l ⊆ set-of t₂' .
      from subtract-Some-set-of-res [OF Some]
      have t2''-t2': set-of t₂'' ⊆ set-of t₂' .
      show ?thesis
      proof (cases subtract r t₂'')
        case (Some t₂''')
        from Node.hyps (2) [OF Some ]
        have r-t₂'': set-of r ⊆ set-of t₂'' .
        from Some sub-l-Some del-x-Some sub r-t₂'' l-t2' t2'-t2 t2''-t2' x-t2
        show ?thesis
          by auto
      next
        case None
        with del-x-Some sub-l-Some sub
        show ?thesis
          by simp
      qed
    next
      case None
      with del-x-Some sub
      show ?thesis
        by simp
    qed
```

```
  next
    case None
    with sub show ?thesis by simp
  qed
qed

lemma subtract-Some-all-distinct-res:
  subtract t₁ t₂ = Some t ⟹ all-distinct t₂ ⟹ all-distinct t
proof (induct t₁ arbitrary: t₂ t)
  case Tip thus ?case by simp
next
  case (Node l x d r)
  have sub: subtract (Node l x d r) t₂ = Some t by fact
  have dist-t2: all-distinct t₂ by fact
  show ?case
  proof (cases delete x t₂)
    case (Some t₂')
    note del-x-Some = this
    from delete-Some-all-distinct [OF Some dist-t2]
    have dist-t2': all-distinct t₂' .
    show ?thesis
    proof (cases subtract l t₂')
      case (Some t₂'')
      note sub-l-Some = this
      from Node.hyps (1) [OF Some dist-t2']
      have dist-t2'': all-distinct t₂'' .
      show ?thesis
      proof (cases subtract r t₂'')
        case (Some t₂''')
        from Node.hyps (2) [OF Some dist-t2'']
        have dist-t2''': all-distinct t₂''' .
        from Some sub-l-Some del-x-Some sub
             dist-t2'''
        show ?thesis
          by simp
      next
        case None
        with del-x-Some sub-l-Some sub
        show ?thesis
          by simp
      qed
    next
      case None
      with del-x-Some sub
      show ?thesis
        by simp
    qed
  next
    case None
```

```
    with sub show ?thesis by simp
  qed
qed


lemma subtract-Some-dist-res:
  subtract t₁ t₂ = Some t ⟹ set-of t₁ ∩ set-of t = {}
proof (induct t₁ arbitrary: t₂ t)
  case Tip thus ?case by simp
next
  case (Node l x d r)
  have sub: subtract (Node l x d r) t₂ = Some t by fact
  show ?case
  proof (cases delete x t₂)
    case (Some t₂')
    note del-x-Some = this
    from delete-Some-x-set-of [OF Some]
    obtain x-t2: x ∈ set-of t₂ and x-not-t2': x ∉ set-of t₂'
      by simp
    from delete-Some-set-of [OF Some]
    have t2'-t2: set-of t₂' ⊆ set-of t₂ .
    show ?thesis
    proof (cases subtract l t₂')
      case (Some t₂'')
      note sub-l-Some = this
      from Node.hyps (1) [OF Some ]
      have dist-l-t2'': set-of l ∩ set-of t₂'' = {}.
      from subtract-Some-set-of-res [OF Some]
      have t2''-t2': set-of t₂'' ⊆ set-of t₂' .
      show ?thesis
      proof (cases subtract r t₂'')
        case (Some t₂''')
        from Node.hyps (2) [OF Some]
        have dist-r-t2''': set-of r ∩ set-of t₂''' = {} .
        from subtract-Some-set-of-res [OF Some]
        have t2'''-t2'': set-of t₂''' ⊆ set-of t₂''.

        from Some sub-l-Some del-x-Some sub t2'''-t2'' dist-l-t2'' dist-r-t2'''
             t2''-t2' t2'-t2 x-not-t2'
        show ?thesis
          by auto
      next
        case None
        with del-x-Some sub-l-Some sub
        show ?thesis
          by simp
      qed
    next
      case None
```

```
      with del-x-Some sub
      show ?thesis
        by simp
    qed
  next
    case None
    with sub show ?thesis by simp
  qed
qed

lemma subtract-Some-all-distinct:
  subtract t1 t2 = Some t ⟹ all-distinct t2 ⟹ all-distinct t1
proof (induct t1 arbitrary: t2 t)
  case Tip thus ?case by simp
next
  case (Node l x d r)
  have sub: subtract (Node l x d r) t2 = Some t by fact
  have dist-t2: all-distinct t2 by fact
  show ?case
  proof (cases delete x t2)
    case (Some t2')
    note del-x-Some = this
    from delete-Some-all-distinct [OF Some dist-t2 ]
    have dist-t2': all-distinct t2' .
    from delete-Some-set-of [OF Some]
    have t2'-t2: set-of t2' ⊆ set-of t2 .
    from delete-Some-x-set-of [OF Some]
    obtain x-t2: x ∈ set-of t2 and x-not-t2': x ∉ set-of t2'
      by simp

    show ?thesis
    proof (cases subtract l t2')
      case (Some t2'')
      note sub-l-Some = this
      from Node.hyps (1) [OF Some dist-t2' ]
      have dist-l: all-distinct l .
      from subtract-Some-all-distinct-res [OF Some dist-t2']
      have dist-t2'': all-distinct t2'' .
      from subtract-Some-set-of [OF Some]
      have l-t2': set-of l ⊆ set-of t2' .
      from subtract-Some-set-of-res [OF Some]
      have t2''-t2': set-of t2'' ⊆ set-of t2' .
      from subtract-Some-dist-res [OF Some]
      have dist-l-t2'': set-of l ∩ set-of t2'' = {}.
      show ?thesis
      proof (cases subtract r t2'')
        case (Some t2''')
        from Node.hyps (2) [OF Some dist-t2'']
        have dist-r: all-distinct r .
```

from *subtract-Some-set-of* [*OF Some*]
have *r-t2''*: *set-of* $r \subseteq$ *set-of* t_2'' .
from *subtract-Some-dist-res* [*OF Some*]
have *dist-r-t2'''*: *set-of* $r \cap$ *set-of* $t_2''' = \{\}$.

from *dist-l dist-r Some sub-l-Some del-x-Some r-t2'' l-t2' x-t2 x-not-t2' t2''-t2' dist-l-t2'' dist-r-t2'''*
show *?thesis*
by *auto*
next
case *None*
with *del-x-Some sub-l-Some sub*
show *?thesis*
by *simp*
qed
next
case *None*
with *del-x-Some sub*
show *?thesis*
by *simp*
qed
next
case *None*
with *sub* **show** *?thesis* **by** *simp*
qed
qed

lemma *delete-left*:
assumes *dist*: *all-distinct* (*Node l y d r*)
assumes *del-l*: *delete x l* = *Some l'*
shows *delete x* (*Node l y d r*) = *Some* (*Node l' y d r*)
proof –
from *delete-Some-x-set-of* [*OF del-l*]
obtain *x*: $x \in$ *set-of l*
by *simp*
with *dist*
have *delete x r* = *None*
by (*cases delete x r*) (*auto dest*:*delete-Some-x-set-of*)

with *x*
show *?thesis*
using *del-l dist*
by (*auto split*: *option.splits*)
qed

lemma *delete-right*:
assumes *dist*: *all-distinct* (*Node l y d r*)
assumes *del-r*: *delete x r* = *Some r'*

```
  shows delete x (Node l y d r) = Some (Node l y d r')
proof -
  from delete-Some-x-set-of [OF del-r]
  obtain x: x ∈ set-of r
    by simp
  with dist
  have delete x l = None
    by (cases delete x l) (auto dest:delete-Some-x-set-of)

  with x
  show ?thesis
    using del-r dist
    by (auto split: option.splits)
qed

lemma delete-root:
  assumes dist: all-distinct (Node l x False r)
  shows delete x (Node l x False r) = Some (Node l x True r)
proof -
  from dist have delete x r = None
    by (cases delete x r) (auto dest:delete-Some-x-set-of)
  moreover
  from dist have delete x l = None
    by (cases delete x l) (auto dest:delete-Some-x-set-of)
  ultimately show ?thesis
    using dist
      by (auto split: option.splits)
qed

lemma subtract-Node:
 assumes del: delete x t = Some t'
 assumes sub-l: subtract l t' = Some t''
 assumes sub-r: subtract r t'' = Some t'''
 shows subtract (Node l x False r) t = Some t'''
using del sub-l sub-r
by simp

lemma subtract-Tip: subtract Tip t = Some t
  by simp
```

Now we have all the theorems in place that are needed for the certificate generating ML functions.

ML-file ‹*distinct-tree-prover.ML*›

end

3 State Space Representation as Function

theory *StateFun* **imports** *DistinctTreeProver*

begin

The state space is represented as a function from names to values. We neither fix the type of names nor the type of values. We define lookup and update functions and provide simprocs that simplify expressions containing these, similar to HOL-records.

The lookup and update function get constructor/destructor functions as parameters. These are used to embed various HOL-types into the abstract value type. Conceptually the abstract value type is a sum of all types that we attempt to store in the state space.

The update is actually generalized to a map function. The map supplies better compositionality, especially if you think of nested state spaces.

definition *K-statefun* :: $'a \Rightarrow 'b \Rightarrow 'a$ **where** *K-statefun c x* $\equiv$ *c*

lemma *K-statefun-apply* [*simp*]: *K-statefun c x* = *c*
  by (*simp add*: *K-statefun-def*)

lemma *K-statefun-comp* [*simp*]: (*K-statefun c* $\circ$ *f*) = *K-statefun c*
  by (*rule ext*) (*simp add*: *comp-def*)

lemma *K-statefun-cong* [*cong*]: *K-statefun c x* = *K-statefun c x*
  by (*rule refl*)

definition *lookup* :: $('v \Rightarrow 'a) \Rightarrow 'n \Rightarrow ('n \Rightarrow 'v) \Rightarrow 'a$
  where *lookup destr n s* = *destr* (*s n*)

definition *update* ::
  $('v \Rightarrow 'a1) \Rightarrow ('a2 \Rightarrow 'v) \Rightarrow 'n \Rightarrow ('a1 \Rightarrow 'a2) \Rightarrow ('n \Rightarrow 'v) \Rightarrow ('n \Rightarrow 'v)$
  where *update destr constr n f s* = *s*(*n* := *constr* (*f* (*destr* (*s n*))))

lemma *lookup-update-same*:
  $(\bigwedge v.\ destr\ (constr\ v) = v) \Longrightarrow lookup\ destr\ n\ (update\ destr\ constr\ n\ f\ s) =$
    $f\ (destr\ (s\ n))$
  by (*simp add*: *lookup-def update-def*)

lemma *lookup-update-id-same*:
  $lookup\ destr\ n\ (update\ destr'\ id\ n\ (K\text{-}statefun\ (lookup\ id\ n\ s'))\ s) =$
    $lookup\ destr\ n\ s'$
  by (*simp add*: *lookup-def update-def*)

lemma *lookup-update-other*:
  $n \neq m \Longrightarrow lookup\ destr\ n\ (update\ destr'\ constr\ m\ f\ s) = lookup\ destr\ n\ s$
  by (*simp add*: *lookup-def update-def*)

lemma *id-id-cancel*: *id* (*id x*) = *x*
  by (*simp add*: *id-def*)

lemma *destr-contstr-comp-id*: $(\bigwedge v.\ destr\ (constr\ v) = v) \Longrightarrow destr \circ constr = id$
by (*rule ext*) *simp*

lemma *block-conj-cong*: $(P \wedge Q) = (P \wedge Q)$
by *simp*

lemma *conj1-False*: $P \equiv False \Longrightarrow (P \wedge Q) \equiv False$
by *simp*

lemma *conj2-False*: $Q \equiv False \Longrightarrow (P \wedge Q) \equiv False$
by *simp*

lemma *conj-True*: $P \equiv True \Longrightarrow Q \equiv True \Longrightarrow (P \wedge Q) \equiv True$
by *simp*

lemma *conj-cong*: $P \equiv P' \Longrightarrow Q \equiv Q' \Longrightarrow (P \wedge Q) \equiv (P' \wedge Q')$
by *simp*

lemma *update-apply*: $(update\ destr\ constr\ n\ f\ s\ x) =$
$(if\ x{=}n\ then\ constr\ (f\ (destr\ (s\ n)))\ else\ s\ x)$
by (*simp add*: *update-def*)

lemma *ex-id*: $\exists x.\ id\ x = y$
by (*simp add*: *id-def*)

lemma *swap-ex-eq*:
$\exists s.\ f\ s = x \equiv True \Longrightarrow$
$\exists s.\ x = f\ s \equiv True$
apply (*rule eq-reflection*)
apply *auto*
done

lemmas *meta-ext* = *eq-reflection* [*OF ext*]

lemma $update\ d\ c\ n\ (K\text{-}statespace\ (lookup\ d\ n\ s))\ s = s$
apply (*simp add*: *update-def lookup-def*)
apply (*rule ext*)
apply *simp*
oops

end

4 Setup for State Space Locales

theory *StateSpaceLocale* **imports** *StateFun*

keywords *statespace* :: *thy-defn*
begin

ML-file ‹*state-space.ML*›
ML-file ‹*state-fun.ML*›

For every type that is to be stored in a state space, an instance of this locale is imported in order convert the abstract and concrete values.

locale *project-inject* =
 fixes *project* :: *'value* ⇒ *'a*
  and *inject* :: *'a* ⇒ *'value*
 assumes *project-inject-cancel* [*statefun-simp*]: *project* (*inject x*) = *x*
begin

lemma *ex-project* [*statefun-simp*]: ∃ *v*. *project v* = *x*
proof
  show *project* (*inject x*) = *x*
    by (*rule project-inject-cancel*)
qed

lemma *project-inject-comp-id* [*statefun-simp*]: *project* ∘ *inject* = *id*
  by (*rule ext*) (*simp add*: *project-inject-cancel*)

lemma *project-inject-comp-cancel*[*statefun-simp*]: *f* ∘ *project* ∘ *inject* = *f*
  by (*rule ext*) (*simp add*: *project-inject-cancel*)

end

end

5 Syntax for State Space Lookup and Update

theory *StateSpaceSyntax*
imports *StateSpaceLocale*
begin

The state space syntax is kept in an extra theory so that you can choose if you want to use it or not.

syntax
  -statespace-lookup :: (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'c* (-·- [*60*, *60*] *60*)
  -statespace-update :: (*'a* ⇒ *'b*) ⇒ *'a* ⇒ *'c* ⇒ (*'a* ⇒ *'b*)
  -statespace-updates :: (*'a* ⇒ *'b*) ⇒ *updbinds* ⇒ (*'a* ⇒ *'b*) (-<-> [*900*, *0*] *900*)

translations
  -statespace-updates f (*-updbinds b bs*) ==
    -statespace-updates (*-statespace-updates f b*) *bs*
  s<*x*:=*y*> == *-statespace-update s x y*

parse-translation
‹
 [(***syntax-const*** ‹-*statespace-lookup*›, *StateSpace.lookup-tr*),
  (***syntax-const*** ‹-*statespace-update*›, *StateSpace.update-tr*)]
›

print-translation
‹
 [(***const-syntax*** ‹*lookup*›, *StateSpace.lookup-tr′*),
  (***const-syntax*** ‹*update*›, *StateSpace.update-tr′*)]
›

end

6 Examples

theory *StateSpaceEx*
imports *StateSpaceLocale StateSpaceSyntax*
begin

Did you ever dream about records with multiple inheritance? Then you should definitely have a look at statespaces. They may be what you are dreaming of. Or at least almost ...

Isabelle allows to add new top-level commands to the system. Building on the locale infrastructure, we provide a command **statespace** like this:

statespace *vars* =
  n::*nat*
  b::*bool*

print-locale *vars-namespace*
print-locale *vars-valuetypes*
print-locale *vars*

This resembles a **record** definition, but introduces sophisticated locale infrastructure instead of HOL type schemes. The resulting context postulates two distinct names *n* and *b* and projection / injection functions that convert from abstract values to *nat* and *bool*. The logical content of the locale is:

locale *vars′* =
  fixes *n*::*′name* **and** *b*::*′name*
  assumes *distinct* [*n*, *b*]

  fixes *project-nat*::*′value* ⇒ *nat* **and** *inject-nat*::*nat* ⇒ *′value*
  assumes $\bigwedge$*n*. *project-nat* (*inject-nat n*) = *n*

fixes *project-bool*::$'$*value* $\Rightarrow$ *bool* **and** *inject-bool*::*bool* $\Rightarrow$ $'$*value*
assumes $\bigwedge b$. *project-bool* (*inject-bool* b) $= b$

The HOL predicate *distinct* describes distinctness of all names in the context. Locale *vars$'$* defines the raw logical content that is defined in the state space locale. We also maintain non-logical context information to support the user:

- Syntax for state lookup and updates that automatically inserts the corresponding projection and injection functions.
- Setup for the proof tools that exploit the distinctness information and the cancellation of projections and injections in deductions and simplifications.

  This extra-logical information is added to the locale in form of declarations, which associate the name of a variable to the corresponding projection and injection functions to handle the syntax transformations, and a link from the variable name to the corresponding distinctness theorem. As state spaces are merged or extended there are multiple distinctness theorems in the context. Our declarations take care that the link always points to the strongest distinctness assumption. With these declarations in place, a lookup can be written as $s \cdot n$, which is translated to *project-nat* (s n), and an update as $s\langle n := 2\rangle$, which is translated to $s(n :=$ *inject-nat* $2)$. We can now establish the following lemma:

lemma (**in** *vars*) *foo*: $s{<}n := 2{>}\cdot b = s \cdot b$ **by** *simp*

Here the simplifier was able to refer to distinctness of b and n to solve the equation. The resulting lemma is also recorded in locale *vars* for later use and is automatically propagated to all its interpretations. Here is another example:

statespace $'a$ *varsX* = *NB*: *vars* [$n{=}N$, $b{=}B$] + *vars* + x::$'a$

The state space *varsX* imports two copies of the state space *vars*, where one has the variables renamed to upper-case letters, and adds another variable x of type $'a$. This type is fixed inside the state space but may get instantiated later on, analogous to type parameters of an ML-functor. The distinctness assumption is now *distinct* $[N, B, n, b, x]$, from this we can derive both *distinct* $[N, B]$ and *distinct* $[n, b]$, the distinction assumptions for the two versions of locale *vars* above. Moreover we have all necessary projection and injection assumptions available. These assumptions together allow us to establish state space *varsX* as an interpretation of both instances of locale *vars*. Hence we inherit both variants of theorem *foo*: $s\langle N := 2\rangle \cdot B = s \cdot B$ as well as $s\langle n := 2\rangle\cdot b = s\cdot b$. These are immediate consequences of the locale interpretation action.

The declarations for syntax and the distinctness theorems also observe the morphisms generated by the locale package due to the renaming $n = N$:

lemma (**in** *varsX*) *foo*: $s\langle N := 2\rangle\cdot x = s\cdot x$ **by** *simp*

To assure scalability towards many distinct names, the distinctness predicate is refined to operate on balanced trees. Thus we get logarithmic certificates for the distinctness of two names by the distinctness of the paths in the tree. Asked for the distinctness of two names, our tool produces the paths of the variables in the tree (this is implemented in Isabelle/ML, outside the logic) and returns a certificate corresponding to the different paths. Merging state spaces requires to prove that the combined distinctness assumption implies the distinctness assumptions of the components. Such a proof is of the order $m \cdot \log n$, where n and m are the number of nodes in the larger and smaller tree, respectively.

We continue with more examples.

statespace $'a$ *foo* =
  $f::nat \Rightarrow nat$
  $a::int$
  $b::nat$
  $c::'a$

lemma (**in** *foo*) *foo1*:
  shows $s\langle a := i\rangle\cdot a = i$
  by *simp*

lemma (**in** *foo*) *foo2*:
  shows $(s\langle a{:=}i\rangle)\cdot a = i$
  by *simp*

lemma (**in** *foo*) *foo3*:
  shows $(s\langle a{:=}i\rangle)\cdot b = s\cdot b$
  by *simp*

lemma (**in** *foo*) *foo4*:
  shows $(s\langle a{:=}i,b{:=}j,c{:=}k,a{:=}x\rangle) = (s\langle b{:=}j,c{:=}k,a{:=}x\rangle)$
  by *simp*

statespace *bar* =
  $b::bool$
  $c::string$

lemma (**in** *bar*) *bar1*:

shows $(s\langle b{:=}True\rangle)\cdot c = s\cdot c$
by *simp*

You can define a derived state space by inheriting existing state spaces, renaming of components if you like, and by declaring new components.

statespace $('a,'b)$ *loo* $=$ $'a$ *foo* $+$ *bar* $[b{=}B, c{=}C]$ $+$
$X{::}'b$

lemma (**in** *loo*) *loo1*:
shows $s\langle a{:=}i\rangle\cdot B = s\cdot B$
proof $-$
thm *foo1*

The Lemma *foo1* from the parent state space is also available here:

$$?s\langle a := ?i\rangle\cdot a = ?i$$

have $s{<}a{:=}i{>}\cdot a = i$
by (*rule foo1*)
thm *bar1*

Note the renaming of the parameters in Lemma *bar1*:

$$?s\langle B := True\rangle\cdot C = ?s\cdot C$$

have $s{<}B{:=}True{>}\cdot C = s\cdot C$
by (*rule bar1*)
show *?thesis*
by *simp*
qed

statespace $'a$ *dup* $=$ *FA*: $'a$ *foo* $[f{=}F,\ a{=}A]$ $+$ $'a$ *foo* $+$
$x{::}int$

lemma (**in** *dup*)
shows $s{<}a := i{>}\cdot x = s\cdot x$
by *simp*

lemma (**in** *dup*)
shows $s{<}A := i{>}\cdot a = s\cdot a$
by *simp*

lemma (**in** *dup*)
shows $s{<}A := i{>}\cdot x = s\cdot x$
by *simp*

There were known problems with syntax-declarations. They only worked when the context is already completely built. This is now overcome. e.g.:

locale *fooX* = *foo* +
 assumes $s<a{:=}i>\cdot b = k$

We can also put statespaces side-by-side by using ordinary **locale** expressions (instead of the **statespace**).

locale *side-by-side* = *foo* + *bar* **where** $b{=}B{::}'a$ **and** $c{=}C$ **for** B C

context *side-by-side*
begin

Simplification within one of the statespaces works as expected.

lemma $s<B := i>\cdot C = s\cdot C$
 by *simp*

lemma $s<a := i>\cdot b = s\cdot b$
 by *simp*

In contrast to the statespace *loo* there is no 'inter' statespace distinctness between the names of *foo* and *bar*.

end

Sharing of names in side-by-side statespaces is also possible as long as they are mapped to the same type.

statespace *vars1* = $n{::}nat$ $m{::}nat$
statespace *vars2* = $n{::}nat$ $k{::}nat$

locale *vars1-vars2* = *vars1* + *vars2*

context *vars1-vars2*
begin

Note that the distinctness theorem for *vars1* is selected here to do the proof.

lemma $s<n := i>\cdot m = s\cdot m$
 by *simp*

Note that the distinctness theorem for *vars2* is selected here to do the proof.

lemma $s<n := i>\cdot k = s\cdot k$
 by *simp*

Still there is no inter-statespace distinctness.

lemma $s<k := i>\cdot m = s\cdot m$

 oops
end

statespace *merge-vars1-vars2* = *vars1* + *vars2*

context *merge-vars1-vars2*
begin

When defining a statespace instead of a side-by-side locale we get the distinctness of all variables.

lemma $s<k := i>\cdot m = s\cdot m$
  by *simp*
end

6.1 Benchmarks

Here are some bigger examples for benchmarking.

ML ‹
  fun make-benchmark n =
    writeln (Active.sendback-markup-command
      (statespace benchmark ^ string-of-int n ^ =\n ^
        (cat-lines (map (fn i => A ^ string-of-int i ^::nat) (1 upto n)))));
›

0.2s

statespace *benchmark100 = A1::nat A2::nat A3::nat A4::nat A5::nat A6::nat A7::nat A8::nat A9::nat A10::nat A11::nat A12::nat A13::nat A14::nat A15::nat A16::nat A17::nat A18::nat A19::nat A20::nat A21::nat A22::nat A23::nat A24::nat A25::nat A26::nat A27::nat A28::nat A29::nat A30::nat A31::nat A32::nat A33::nat A34::nat A35::nat A36::nat A37::nat A38::nat A39::nat A40::nat A41::nat A42::nat A43::nat A44::nat A45::nat A46::nat A47::nat A48::nat A49::nat A50::nat A51::nat A52::nat A53::nat A54::nat A55::nat A56::nat A57::nat A58::nat A59::nat A60::nat A61::nat A62::nat A63::nat A64::nat A65::nat A66::nat A67::nat A68::nat A69::nat A70::nat A71::nat A72::nat A73::nat A74::nat A75::nat A76::nat A77::nat A78::nat A79::nat A80::nat A81::nat A82::nat A83::nat A84::nat A85::nat A86::nat A87::nat A88::nat A89::nat A90::nat A91::nat A92::nat A93::nat A94::nat A95::nat A96::nat A97::nat A98::nat A99::nat A100::nat*

2.4s

statespace *benchmark500 = A1::nat A2::nat A3::nat A4::nat A5::nat A6::nat A7::nat A8::nat A9::nat A10::nat A11::nat A12::nat A13::nat A14::nat A15::nat A16::nat A17::nat A18::nat A19::nat A20::nat A21::nat A22::nat A23::nat A24::nat A25::nat A26::nat A27::nat A28::nat A29::nat A30::nat A31::nat A32::nat A33::nat A34::nat A35::nat A36::nat A37::nat A38::nat A39::nat A40::nat A41::nat A42::nat A43::nat A44::nat A45::nat A46::nat A47::nat A48::nat A49::nat A50::nat A51::nat A52::nat A53::nat A54::nat A55::nat A56::nat A57::nat A58::nat A59::nat A60::nat A61::nat A62::nat A63::nat A64::nat A65::nat A66::nat A67::nat A68::nat A69::nat A70::nat A71::nat A72::nat A73::nat A74::nat A75::nat A76::nat*

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9.0s

statespace *benchmark1000* = *A1*::*nat* *A2*::*nat* *A3*::*nat* *A4*::*nat* *A5*::*nat* *A6*::*nat* *A7*::*nat* *A8*::*nat* *A9*::*nat* *A10*::*nat* *A11*::*nat* *A12*::*nat* *A13*::*nat* *A14*::*nat* *A15*::*nat* *A16*::*nat* *A17*::*nat* *A18*::*nat* *A19*::*nat* *A20*::*nat* *A21*::*nat* *A22*::*nat* *A23*::*nat* *A24*::*nat* *A25*::*nat* *A26*::*nat* *A27*::*nat* *A28*::*nat* *A29*::*nat* *A30*::*nat* *A31*::*nat* *A32*::*nat* *A33*::*nat* *A34*::*nat* *A35*::*nat* *A36*::*nat* *A37*::*nat* *A38*::*nat* *A39*::*nat* *A40*::*nat* *A41*::*nat* *A42*::*nat* *A43*::*nat* *A44*::*nat* *A45*::*nat* *A46*::*nat* *A47*::*nat* *A48*::*nat* *A49*::*nat* *A50*::*nat* *A51*::*nat* *A52*::*nat* *A53*::*nat* *A54*::*nat* *A55*::*nat* *A56*::*nat* *A57*::*nat* *A58*::*nat* *A59*::*nat* *A60*::*nat* *A61*::*nat* *A62*::*nat* *A63*::*nat* *A64*::*nat* *A65*::*nat* *A66*::*nat* *A67*::*nat* *A68*::*nat* *A69*::*nat* *A70*::*nat* *A71*::*nat* *A72*::*nat* *A73*::*nat* *A74*::*nat* *A75*::*nat* *A76*::*nat* *A77*::*nat* *A78*::*nat* *A79*::*nat* *A80*::*nat* *A81*::*nat* *A82*::*nat* *A83*::*nat* *A84*::*nat* *A85*::*nat* *A86*::*nat* *A87*::*nat* *A88*::*nat* *A89*::*nat* *A90*::*nat* *A91*::*nat* *A92*::*nat* *A93*::*nat* *A94*::*nat* *A95*::*nat* *A96*::*nat* *A97*::*nat* *A98*::*nat* *A99*::*nat* *A100*::*nat* *A101*::*nat* *A102*::*nat* *A103*::*nat* *A104*::*nat* *A105*::*nat* *A106*::*nat* *A107*::*nat* *A108*::*nat* *A109*::*nat* *A110*::*nat* *A111*::*nat* *A112*::*nat* *A113*::*nat* *A114*::*nat* *A115*::*nat* *A116*::*nat* *A117*::*nat* *A118*::*nat* *A119*::*nat* *A120*::*nat* *A121*::*nat* *A122*::*nat* *A123*::*nat* *A124*::*nat* *A125*::*nat* *A126*::*nat* *A127*::*nat* *A128*::*nat* *A129*::*nat* *A130*::*nat* *A131*::*nat* *A132*::*nat* *A133*::*nat* *A134*::*nat* *A135*::*nat* *A136*::*nat* *A137*::*nat* *A138*::*nat* *A139*::*nat* *A140*::*nat* *A141*::*nat* *A142*::*nat* *A143*::*nat* *A144*::*nat* *A145*::*nat* *A146*::*nat* *A147*::*nat* *A148*::*nat* *A149*::*nat* *A150*::*nat* *A151*::*nat* *A152*::*nat* *A153*::*nat* *A154*::*nat* *A155*::*nat* *A156*::*nat* *A157*::*nat* *A158*::*nat* *A159*::*nat* *A160*::*nat* *A161*::*nat* *A162*::*nat* *A163*::*nat* *A164*::*nat* *A165*::*nat* *A166*::*nat* *A167*::*nat* *A168*::*nat* *A169*::*nat* *A170*::*nat* *A171*::*nat* *A172*::*nat* *A173*::*nat* *A174*::*nat* *A175*::*nat* *A176*::*nat* *A177*::*nat* *A178*::*nat* *A179*::*nat* *A180*::*nat* *A181*::*nat* *A182*::*nat* *A183*::*nat* *A184*::*nat* *A185*::*nat* *A186*::*nat* *A187*::*nat* *A188*::*nat* *A189*::*nat* *A190*::*nat* *A191*::*nat* *A192*::*nat* *A193*::*nat* *A194*::*nat* *A195*::*nat* *A196*::*nat* *A197*::*nat* *A198*::*nat* *A199*::*nat* *A200*::*nat* *A201*::*nat* *A202*::*nat* *A203*::*nat* *A204*::*nat* *A205*::*nat* *A206*::*nat* *A207*::*nat* *A208*::*nat* *A209*::*nat* *A210*::*nat* *A211*::*nat* *A212*::*nat* *A213*::*nat* *A214*::*nat* *A215*::*nat* *A216*::*nat* *A217*::*nat* *A218*::*nat* *A219*::*nat* *A220*::*nat* *A221*::*nat* *A222*::*nat* *A223*::*nat* *A224*::*nat* *A225*::*nat* *A226*::*nat* *A227*::*nat* *A228*::*nat* *A229*::*nat* *A230*::*nat* *A231*::*nat* *A232*::*nat* *A233*::*nat* *A234*::*nat* *A235*::*nat* *A236*::*nat* *A237*::*nat* *A238*::*nat* *A239*::*nat* *A240*::*nat* *A241*::*nat* *A242*::*nat* *A243*::*nat* *A244*::*nat*

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A910::nat A911::nat A912::nat A913::nat A914::nat A915::nat A916::nat
A917::nat A918::nat A919::nat A920::nat A921::nat A922::nat A923::nat
A924::nat A925::nat A926::nat A927::nat A928::nat A929::nat A930::nat

A931::*nat* *A932*::*nat* *A933*::*nat* *A934*::*nat* *A935*::*nat* *A936*::*nat* *A937*::*nat*
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A994::*nat* *A995*::*nat* *A996*::*nat* *A997*::*nat* *A998*::*nat* *A999*::*nat* *A1000*::*nat*

lemma (**in** *benchmark100*) *test*: *s*<*A1* := *a*>·*A100* = *s*·*A100* **by** *simp*
lemma (**in** *benchmark500*) *test*: *s*<*A1* := *a*>·*A100* = *s*·*A100* **by** *simp*
lemma (**in** *benchmark1000*) *test*: *s*<*A1* := *a*>·*A100* = *s*·*A100* **by** *simp*

end