Equivalents of the Axiom of Choice

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Abstract

This development [1] proves the equivalence of seven formulations of the well-ordering theorem and twenty formulations of the axiom of choice. It formalizes the first two chapters of the monograph *Equivalents of the Axiom of Choice* by Rubin and Rubin [2]. Some of this mmaterial involves extremely complex techniques.

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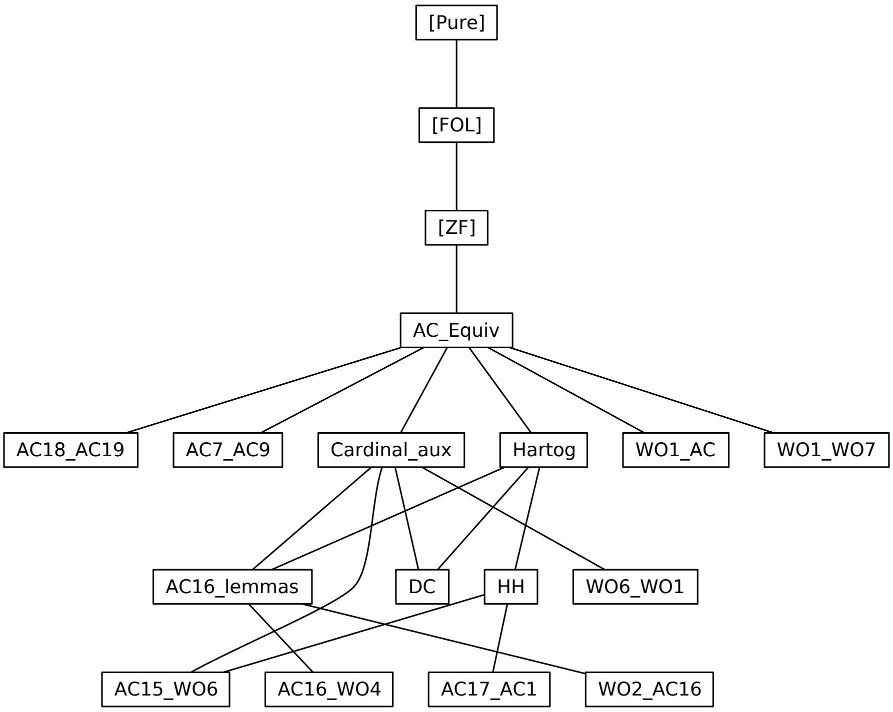
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theory *AC_Equiv*
imports *ZF*
begin

```
definition
    "WO1 == ∀A. ∃R. well_ord(A,R)"

definition
    "WO2 == ∀A. ∃a. Ord(a) & A≈a"

definition
    "WO3 == ∀A. ∃a. Ord(a) & (∃b. b ⊆ a & A≈b)"

definition
    "WO4(m) == ∀A. ∃a f. Ord(a) & domain(f)=a &
                         (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"

definition
    "WO5 == ∃m ∈ nat. 1≤m & WO4(m)"

definition
    "WO6 == ∀A. ∃m ∈ nat. 1≤m & (∃a f. Ord(a) & domain(f)=a
                              & (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m))"

definition
    "WO7 == ∀A. Finite(A) ⟷ (∀R. well_ord(A,R) ⟶ well_ord(A,converse(R)))"

definition
    "WO8 == ∀A. (∃f. f ∈ (∏X ∈ A. X)) ⟶ (∃R. well_ord(A,R))"


definition

  pairwise_disjoint :: "i => o"  where
    "pairwise_disjoint(A) == ∀A1 ∈ A. ∀A2 ∈ A. A1 ∩ A2 ≠ 0 ⟶ A1=A2"

definition
  sets_of_size_between :: "[i, i, i] => o"  where
    "sets_of_size_between(A,m,n) == ∀B ∈ A. m ≲ B & B ≲ n"


definition
    "AC0 == ∀A. ∃f. f ∈ (∏X ∈ Pow(A)-{0}. X)"
```

definition
```
"AC1 == ∀A. 0∉A ⟶ (∃f. f ∈ (∏X ∈ A. X))"
```

definition
```
"AC2 == ∀A. 0∉A & pairwise_disjoint(A)
               ⟶ (∃C. ∀B ∈ A. ∃y. B ∩ C = {y})"
```
definition
```
"AC3 == ∀A B. ∀f ∈ A->B. ∃g. g ∈ (∏x ∈ {a ∈ A. f‘a≠0}. f‘x)"
```

definition
```
"AC4 == ∀R A B. (R ⊆ A*B ⟶ (∃f. f ∈ (∏x ∈ domain(R). R‘‘{x})))"
```

definition
```
"AC5 == ∀A B. ∀f ∈ A->B. ∃g ∈ range(f)->A. ∀x ∈ domain(g). f‘(g‘x)
= x"
```

definition
```
"AC6 == ∀A. 0∉A ⟶ (∏B ∈ A. B)≠0"
```

definition
```
"AC7 == ∀A. 0∉A & (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) ⟶ (∏B ∈ A. B) ≠
0"
```

definition
```
"AC8 == ∀A. (∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1≈B2)
               ⟶ (∃f. ∀B ∈ A. f‘B ∈ bij(fst(B),snd(B)))"
```

definition
```
"AC9 == ∀A. (∀B1 ∈ A. ∀B2 ∈ A. B1≈B2) ⟶
               (∃f. ∀B1 ∈ A. ∀B2 ∈ A. f‘<B1,B2> ∈ bij(B1,B2))"
```

definition
```
"AC10(n) ==  ∀A. (∀B ∈ A. ~Finite(B)) ⟶
               (∃f. ∀B ∈ A. (pairwise_disjoint(f‘B) &
               sets_of_size_between(f‘B, 2, succ(n)) & ⋃(f‘B)=B))"
```

definition
```
"AC11 == ∃n ∈ nat. 1≤n & AC10(n)"
```

definition
```
"AC12 == ∀A. (∀B ∈ A. ~Finite(B)) ⟶
             (∃n ∈ nat. 1≤n & (∃f. ∀B ∈ A. (pairwise_disjoint(f‘B)
&
                  sets_of_size_between(f‘B, 2, succ(n)) & ⋃(f‘B)=B)))"
```

definition
```
"AC13(m) == ∀A. 0∉A ⟶ (∃f. ∀B ∈ A. f‘B≠0 & f‘B ⊆ B & f‘B ≲ m)"
```

definition

```
"AC14 == ∃m ∈ nat. 1≤m & AC13(m)"
```

definition

```
"AC15 == ∀A. 0∉A ⟶
              (∃m ∈ nat. 1≤m & (∃f. ∀B ∈ A. f`B≠0 & f`B ⊆ B & f`B
≲ m))"
```

definition

```
"AC16(n, k) ==
   ∀A. ~Finite(A) ⟶
       (∃T. T ⊆ {X ∈ Pow(A). X≈succ(n)} &
       (∀X ∈ {X ∈ Pow(A). X≈succ(k)}. ∃! Y. Y ∈ T & X ⊆ Y))"
```

definition

```
"AC17 == ∀A. ∀g ∈ (Pow(A)-{0} -> A) -> Pow(A)-{0}.
               ∃f ∈ Pow(A)-{0} -> A. f`(g`f) ∈ g`f"
```

locale AC18 =

assumes AC18: "A≠0 & (∀a ∈ A. B(a) ≠ 0) ⟶
```
  ((⋂a ∈ A. ⋃b ∈ B(a). X(a,b)) =
    (⋃f ∈ ∏a ∈ A. B(a). ⋂a ∈ A. X(a, f`a)))"
```
— AC18 cannot be expressed within the object-logic

definition

```
"AC19 == ∀A. A≠0 & 0∉A ⟶ ((⋂a ∈ A. ⋃b ∈ a. b) =
               (⋃f ∈ (∏B ∈ A. B). ⋂a ∈ A. f`a))"
```

lemma rvimage_id: "rvimage(A,id(A),r) = r ∩ A*A"
⟨*proof*⟩

lemma ordertype_Int:
```
"well_ord(A,r) ==> ordertype(A, r ∩ A*A) = ordertype(A,r)"
```
⟨*proof*⟩

lemma lam_sing_bij: "(λx ∈ A. {x}) ∈ bij(A, {{x}. x ∈ A})"
⟨*proof*⟩

lemma inj_strengthen_type:
```
"[| f ∈ inj(A, B);  !!a. a ∈ A ==> f`a ∈ C |] ==> f ∈ inj(A,C)"
```
⟨*proof*⟩

lemma *ex1_two_eq:* "[| ∃! x. P(x); P(x); P(y) |] ==> x=y"
⟨*proof*⟩

lemma *first_in_B:*
"[| well_ord(⋃(A),r); 0 ∉ A; B ∈ A |] ==> (THE b. first(b,B,r)) ∈ B"
⟨*proof*⟩

lemma *ex_choice_fun:* "[| well_ord(⋃(A), R); 0 ∉ A |] ==> ∃f. f ∈ (∏X ∈ A. X)"
⟨*proof*⟩

lemma *ex_choice_fun_Pow:* "well_ord(A, R) ==> ∃f. f ∈ (∏X ∈ Pow(A)-{0}. X)"
⟨*proof*⟩

lemma *lepoll_m_imp_domain_lepoll_m:*
"[| m ∈ nat; u ≲ m |] ==> domain(u) ≲ m"
⟨*proof*⟩

lemma *rel_domain_ex1:*
"[| succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat |] ==> function(r)"
⟨*proof*⟩

lemma *rel_is_fun:*
"[| succ(m) ≲ domain(r); r ≲ succ(m); m ∈ nat;
r ⊆ A*B; A=domain(r) |] ==> r ∈ A->B"
⟨*proof*⟩

end

theory *Cardinal_aux* **imports** *AC_Equiv* **begin**

lemma *Diff_lepoll:* "[| A $\lesssim$ succ(m); B $\subseteq$ A; B$\neq$0 |] ==> A-B $\lesssim$ m"
$\langle proof \rangle$

lemma *lepoll_imp_ex_le_eqpoll:*
"[| A $\lesssim$ i; Ord(i) |] ==> $\exists$j. j $\leq$ i & A $\approx$ j"
$\langle proof \rangle$

lemma *lesspoll_imp_ex_lt_eqpoll:*
"[| A $\prec$ i; Ord(i) |] ==> $\exists$j. j<i & A $\approx$ j"
$\langle proof \rangle$

lemma *Un_eqpoll_Inf_Ord:*
assumes *A:* "A $\approx$ i" **and** *B:* "B $\approx$ i" **and** *NFI:* "$\neg$ Finite(i)" **and** *i:* "Ord(i)"
shows "A $\cup$ B $\approx$ i"
$\langle proof \rangle$

schematic_goal *paired_bij:* "?f $\in$ bij({{y,z}. y $\in$ x}, x)"
$\langle proof \rangle$

lemma *paired_eqpoll:* "{{y,z}. y $\in$ x} $\approx$ x"
$\langle proof \rangle$

lemma *ex_eqpoll_disjoint:* "$\exists$B. B $\approx$ A & B $\cap$ C = 0"
$\langle proof \rangle$

lemma *Un_lepoll_Inf_Ord:*
"[| A $\lesssim$ i; B $\lesssim$ i; ~Finite(i); Ord(i) |] ==> A $\cup$ B $\lesssim$ i"
$\langle proof \rangle$

lemma *Least_in_Ord:* "[| P(i); i $\in$ j; Ord(j) |] ==> (μ i. P(i)) $\in$ j"
$\langle proof \rangle$

lemma *Diff_first_lepoll:*
"[| well_ord(x,r); y $\subseteq$ x; y $\lesssim$ succ(n); n $\in$ nat |]
==> y - {THE b. first(b,y,r)} $\lesssim$ n"
$\langle proof \rangle$

lemma *UN_subset_split:*
"$(\bigcup x \in X.\ P(x)) \subseteq (\bigcup x \in X.\ P(x)-Q(x)) \cup (\bigcup x \in X.\ Q(x))$"
$\langle proof \rangle$

lemma *UN_sing_lepoll:* "$Ord(a) \Longrightarrow (\bigcup x \in a.\ \{P(x)\}) \lesssim a$"
$\langle proof \rangle$

lemma *UN_fun_lepoll_lemma [rule_format]:*
"[| well_ord(T, R); ~Finite(a); Ord(a); $n \in nat$ |]
==> $\forall f.\ (\forall b \in a.\ f`b \lesssim n\ \&\ f`b \subseteq T) \longrightarrow (\bigcup b \in a.\ f`b) \lesssim a$"
$\langle proof \rangle$

lemma *UN_fun_lepoll:*
"[| $\forall b \in a.\ f`b \lesssim n\ \&\ f`b \subseteq T$; well_ord(T, R);
~Finite(a); Ord(a); $n \in nat$ |] ==> $(\bigcup b \in a.\ f`b) \lesssim a$"
$\langle proof \rangle$

lemma *UN_lepoll:*
"[| $\forall b \in a.\ F(b) \lesssim n\ \&\ F(b) \subseteq T$; well_ord(T, R);
~Finite(a); Ord(a); $n \in nat$ |]
==> $(\bigcup b \in a.\ F(b)) \lesssim a$"
$\langle proof \rangle$

lemma *UN_eq_UN_Diffs:*
"$Ord(a) \Longrightarrow (\bigcup b \in a.\ F(b)) = (\bigcup b \in a.\ F(b) - (\bigcup c \in b.\ F(c)))$"
$\langle proof \rangle$

lemma *lepoll_imp_eqpoll_subset:*
"$a \lesssim X \Longrightarrow \exists Y.\ Y \subseteq X\ \&\ a \approx Y$"
$\langle proof \rangle$

lemma *Diff_lesspoll_eqpoll_Card_lemma:*
"[| $A \approx a$; ~Finite(a); Card(a); $B \prec a$; $A-B \prec a$ |] ==> P"
$\langle proof \rangle$

lemma *Diff_lesspoll_eqpoll_Card:*
"[| $A \approx a$; ~Finite(a); Card(a); $B \prec a$ |] ==> $A - B \approx a$"
$\langle proof \rangle$

end

theory *WO6_WO1*
imports *Cardinal_aux*
begin

definition
```
  NN  :: "i => i"  where
    "NN(y) == {m ∈ nat. ∃a. ∃f. Ord(a) & domain(f)=a  &
                         (⋃b<a. f`b) = y & (∀b<a. f`b ≲ m)}"
```

definition
```
  uu  :: "[i, i, i, i] => i"  where
    "uu(f, beta, gamma, delta) == (f`beta * f`gamma) ∩ f`delta"
```

definition
```
  vv1 :: "[i, i, i] => i"  where
     "vv1(f,m,b) ==
           let g = μ g. (∃d. Ord(d) & (domain(uu(f,b,g,d)) ≠ 0 &
                                 domain(uu(f,b,g,d)) ≲ m));
               d = μ d. domain(uu(f,b,g,d)) ≠ 0 &
                             domain(uu(f,b,g,d)) ≲ m
           in  if f`b ≠ 0 then domain(uu(f,b,g,d)) else 0"
```

definition
```
  ww1 :: "[i, i, i] => i"  where
     "ww1(f,m,b) == f`b - vv1(f,m,b)"
```

definition
```
  gg1 :: "[i, i, i] => i"  where
     "gg1(f,a,m) == λb ∈ a++a. if b<a then vv1(f,m,b) else ww1(f,m,b--a)"
```

definition
```
  vv2 :: "[i, i, i, i] => i"  where
     "vv2(f,b,g,s) ==
           if f`g ≠ 0 then {uu(f, b, g, μ d. uu(f,b,g,d) ≠ 0)`s} else
0"
```

definition
```
  ww2 :: "[i, i, i, i] => i"  where
     "ww2(f,b,g,s) == f`g - vv2(f,b,g,s)"
```

definition
```
  gg2 :: "[i, i, i, i] => i"  where
     "gg2(f,a,b,s) ==
              λg ∈ a++a. if g<a then vv2(f,b,g,s) else ww2(f,b,g--a,s)"
```

lemma `WO2_WO3: "WO2 ==> WO3"`

⟨*proof*⟩

lemma *WO3_WO1: "WO3 ==> WO1"*
⟨*proof*⟩

lemma *WO1_WO2: "WO1 ==> WO2"*
⟨*proof*⟩

lemma *lam_sets: "f ∈ A->B ==> (λx ∈ A. {f‘x}): A -> {{b}. b ∈ B}"*
⟨*proof*⟩

lemma *surj_imp_eq': "f ∈ surj(A,B) ==> (⋃a ∈ A. {f‘a}) = B"*
⟨*proof*⟩

lemma *surj_imp_eq: "[| f ∈ surj(A,B); Ord(A) |] ==> (⋃a<A. {f‘a}) = B"*
⟨*proof*⟩

lemma *WO1_WO4: "WO1 ==> WO4(1)"*
⟨*proof*⟩

lemma *WO4_mono: "[| m≤n; WO4(m) |] ==> WO4(n)"*
⟨*proof*⟩

lemma *WO4_WO5: "[| m ∈ nat; 1≤m; WO4(m) |] ==> WO5"*
⟨*proof*⟩

lemma *WO5_WO6: "WO5 ==> WO6"*
⟨*proof*⟩

lemma *lt_oadd_odiff_disj:*

```
      "[| k < i++j;  Ord(i);  Ord(j) |]
       ==> k < i  |  (~ k<i & k = i ++ (k--i) & (k--i)<j)"
```

⟨*proof*⟩

lemma *domain_uu_subset:* "domain(uu(f,b,g,d)) ⊆ f`b"
⟨*proof*⟩

lemma *quant_domain_uu_lepoll_m:*
"∀ b<a. f`b ≲ m ==> ∀ b<a. ∀ g<a. ∀ d<a. domain(uu(f,b,g,d)) ≲ m"
⟨*proof*⟩

lemma *uu_subset1:* "uu(f,b,g,d) ⊆ f`b * f`g"
⟨*proof*⟩

lemma *uu_subset2:* "uu(f,b,g,d) ⊆ f`d"
⟨*proof*⟩

lemma *uu_lepoll_m:* "[| ∀ b<a. f`b ≲ m; d<a |] ==> uu(f,b,g,d) ≲ m"
⟨*proof*⟩

lemma *cases:*

```
"∀ b<a. ∀ g<a. ∀ d<a. u(f,b,g,d) ≲ m
 ==> (∀ b<a. f`b ≠ 0 ⟶
              (∃ g<a. ∃ d<a. u(f,b,g,d) ≠ 0 & u(f,b,g,d) ≺ m))
   | (∃ b<a. f`b ≠ 0 & (∀ g<a. ∀ d<a. u(f,b,g,d) ≠ 0 ⟶
                                 u(f,b,g,d) ≈ m))"
```

⟨*proof*⟩

lemma *UN_oadd:* "Ord(a) ==> (⋃ b<a++a. C(b)) = (⋃ b<a. C(b) ∪ C(a++b))"
⟨*proof*⟩

lemma *vv1_subset:* "vv1(f,m,b) ⊆ f`b"

⟨proof⟩

```
lemma UN_gg1_eq:
  "[| Ord(a);  m ∈ nat |] ==> (⋃b<a++a. gg1(f,a,m)`b) = (⋃b<a. f`b)"
```
⟨proof⟩

```
lemma domain_gg1: "domain(gg1(f,a,m)) = a++a"
```
⟨proof⟩

```
lemma nested_LeastI:
     "[| P(a, b);  Ord(a);  Ord(b);
         Least_a = (μ a. ∃x. Ord(x) & P(a, x)) |]
      ==> P(Least_a, μ b. P(Least_a, b))"
```
⟨proof⟩

```
lemmas nested_Least_instance =
       nested_LeastI [of "%g d. domain(uu(f,b,g,d)) ≠ 0 &
                                domain(uu(f,b,g,d)) ≲ m"] for f b m
```

```
lemma gg1_lepoll_m:
     "[| Ord(a);  m ∈ nat;
         ∀b<a. f`b ≠0 ⟶
             (∃g<a. ∃d<a. domain(uu(f,b,g,d)) ≠ 0  &
                          domain(uu(f,b,g,d)) ≲ m);
         ∀b<a. f`b ≲ succ(m);  b<a++a |]
      ==> gg1(f,a,m)`b ≲ m"
```
⟨proof⟩

```
lemma ex_d_uu_not_empty:
     "[| b<a;  g<a;  f`b≠0;  f`g≠0;
         y*y ⊆ y;  (⋃b<a. f`b)=y |]
      ==> ∃d<a. uu(f,b,g,d) ≠ 0"
```
⟨proof⟩

```
lemma uu_not_empty:
```

"[| b<a; g<a; f`b≠0; f`g≠0; y*y ⊆ y; (⋃b<a. f`b)=y |]
==> uu(f,b,g,μ d. (uu(f,b,g,d) ≠ 0)) ≠ 0"

⟨*proof*⟩

lemma *not_empty_rel_imp_domain: "[| r ⊆ A*B; r≠0 |] ==> domain(r)≠0"*

⟨*proof*⟩

lemma *Least_uu_not_empty_lt_a:*

"[| b<a; g<a; f`b≠0; f`g≠0; y*y ⊆ y; (⋃b<a. f`b)=y |]
==> (μ d. uu(f,b,g,d) ≠ 0) < a"

⟨*proof*⟩

lemma *subset_Diff_sing: "[| B ⊆ A; a∉B |] ==> B ⊆ A-{a}"*

⟨*proof*⟩

lemma *supset_lepoll_imp_eq:*

"[| A ≲ m; m ≲ B; B ⊆ A; m ∈ nat |] ==> A=B"

⟨*proof*⟩

lemma *uu_Least_is_fun:*

"[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 ⟶
     domain(uu(f, b, g, d)) ≈ succ(m);
     ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
     (⋃b<a. f`b)=y; b<a; g<a; d<a;
     f`b≠0;  f`g≠0; m ∈ nat; s ∈ f`b |]
==> uu(f, b, g, μ d. uu(f,b,g,d)≠0) ∈ f`b -> f`g"

⟨*proof*⟩

lemma *vv2_subset:*

"[| ∀g<a. ∀d<a. domain(uu(f, b, g, d))≠0 ⟶
          domain(uu(f, b, g, d)) ≈ succ(m);
    ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
    (⋃b<a. f`b)=y;  b<a;  g<a;  m ∈ nat;  s ∈ f`b |]
==> vv2(f,b,g,s) ⊆ f`g"

⟨*proof*⟩

lemma *UN_gg2_eq:*

"[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
        domain(uu(f,b,g,d)) ≈ succ(m);
    ∀b<a. f`b ≲ succ(m); y*y ⊆ y;
    (⋃b<a. f`b)=y;  Ord(a);  m ∈ nat;  s ∈ f`b; b<a |]
==> (⋃g<a++a. gg2(f,a,b,s) ` g) = y"

⟨*proof*⟩

lemma *domain_gg2: "domain(gg2(f,a,b,s)) = a++a"*

⟨proof⟩

lemma *vv2_lepoll: "[| m ∈ nat; m≠0 |] ==> vv2(f,b,g,s) ≲ m"*
⟨proof⟩

lemma *ww2_lepoll:*
"[| ∀b<a. f`b ≲ succ(m); g<a; m ∈ nat; vv2(f,b,g,d) ⊆ f`g |]

==> ww2(f,b,g,d) ≲ m"
⟨proof⟩

lemma *gg2_lepoll_m:*
"[| ∀g<a. ∀d<a. domain(uu(f,b,g,d)) ≠ 0 ⟶
domain(uu(f,b,g,d)) ≈ succ(m);
*∀b<a. f`b ≲ succ(m); y*y ⊆ y;*
(⋃b<a. f`b)=y; b<a; s ∈ f`b; m ∈ nat; m≠ 0; g<a++a |]

==> gg2(f,a,b,s) ` g ≲ m"
⟨proof⟩

lemma *lemma_ii: "[| succ(m) ∈ NN(y); y*y ⊆ y; m ∈ nat; m≠0 |] ==> m ∈ NN(y)"*
⟨proof⟩

lemma *z_n_subset_z_succ_n:*
*"∀n ∈ nat. rec(n, x, %k r. r ∪ r*r) ⊆ rec(succ(n), x, %k r. r ∪ r*r)"*
⟨proof⟩

lemma *le_subsets:*
"[| ∀n ∈ nat. f(n)<=f(succ(n)); n≤m; n ∈ nat; m ∈ nat |]
==> f(n)<=f(m)"

⟨*proof*⟩

lemma *le_imp_rec_subset:*
"[| n≤m; m ∈ nat |]
*==> rec(n, x, %k r. r ∪ r*r) ⊆ rec(m, x, %k r. r ∪ r*r)"*
⟨*proof*⟩

lemma *lemma_iv: "∃y. x ∪ y*y ⊆ y"*
⟨*proof*⟩

lemma *WO6_imp_NN_not_empty: "WO6 ==> NN(y) ≠ 0"*
⟨*proof*⟩

lemma *lemma1:*
"[| (⋃b<a. f‘b)=y; x ∈ y; ∀b<a. f‘b ≲ 1; Ord(a) |] ==> ∃c<a. f‘c = {x}"
⟨*proof*⟩

lemma *lemma2:*
"[| (⋃b<a. f‘b)=y; x ∈ y; ∀b<a. f‘b ≲ 1; Ord(a) |]
==> f‘ (μ i. f‘i = {x}) = {x}"
⟨*proof*⟩

lemma *NN_imp_ex_inj: "1 ∈ NN(y) ==> ∃a f. Ord(a) & f ∈ inj(y, a)"*
⟨*proof*⟩

lemma *y_well_ord: "[| y*y ⊆ y; 1 ∈ NN(y) |] ==> ∃r. well_ord(y, r)"*
⟨*proof*⟩

```
lemma rev_induct_lemma [rule_format]:
      "[| n ∈ nat; !!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
       ==> n≠0 ⟶ P(n) ⟶ P(1)"
⟨proof⟩

lemma rev_induct:
      "[| n ∈ nat;  P(n);  n≠0;
          !!m. [| m ∈ nat; m≠0; P(succ(m)) |] ==> P(m) |]
       ==> P(1)"
⟨proof⟩

lemma NN_into_nat: "n ∈ NN(y) ==> n ∈ nat"
⟨proof⟩

lemma lemma3: "[| n ∈ NN(y); y*y ⊆ y; n≠0 |] ==> 1 ∈ NN(y)"
⟨proof⟩

lemma NN_y_0: "0 ∈ NN(y) ==> y=0"
⟨proof⟩

lemma WO6_imp_WO1: "WO6 ==> WO1"
⟨proof⟩

end

theory WO1_WO7
imports AC_Equiv
begin

definition
     "LEMMA ==
      ∀X. ~Finite(X) ⟶ (∃R. well_ord(X,R) & ~well_ord(X,converse(R)))"

lemma WO7_iff_LEMMA: "WO7 ⟷ LEMMA"
⟨proof⟩
```

lemma *LEMMA_imp_WO1: "LEMMA ==> WO1"*
⟨*proof*⟩

lemma *converse_Memrel_not_wf_on:*
```
"[| Ord(a); ~Finite(a) |] ==> ~wf[a](converse(Memrel(a)))"
```
⟨*proof*⟩

lemma *converse_Memrel_not_well_ord:*
```
"[| Ord(a); ~Finite(a) |] ==> ~well_ord(a,converse(Memrel(a)))"
```
⟨*proof*⟩

lemma *well_ord_rvimage_ordertype:*
```
"well_ord(A,r) ⟹
  rvimage (ordertype(A,r), converse(ordermap(A,r)),r) =
  Memrel(ordertype(A,r))"
```
⟨*proof*⟩

lemma *well_ord_converse_Memrel:*
```
"[| well_ord(A,r); well_ord(A,converse(r)) |]
 ==> well_ord(ordertype(A,r), converse(Memrel(ordertype(A,r))))"
```

⟨*proof*⟩

lemma *WO1_imp_LEMMA: "WO1 ==> LEMMA"*
⟨*proof*⟩

lemma *WO1_iff_WO7: "WO1 ⟷ WO7"*
⟨*proof*⟩

lemma *WO1_WO8: "WO1 ==> WO8"*
⟨*proof*⟩

lemma *WO8_WO1: "WO8 ==> WO1"*
⟨*proof*⟩

end

theory *AC7_AC9*
imports *AC_Equiv*
begin

lemma *Sigma_fun_space_not0: "[| 0∉A; B ∈ A |] ==> (nat->⋃(A)) * B ≠ 0"*
⟨*proof*⟩

lemma *inj_lemma:*

```
"C ∈ A ==> (λg ∈ (nat->⋃(A))*C.
        (λn ∈ nat. if(n=0, snd(g), fst(g)`(n #- 1))))
        ∈ inj((nat->⋃(A))*C, (nat->⋃(A)) ) "
```

⟨*proof*⟩

lemma *Sigma_fun_space_eqpoll:*

```
"[| C ∈ A; 0∉A |] ==> (nat->⋃(A)) * C ≈ (nat->⋃(A))"
```

⟨*proof*⟩

lemma *AC6_AC7: "AC6 ==> AC7"*
⟨*proof*⟩

lemma *lemma1_1:* "y ∈ (∏B ∈ A. Y*B) ==> (λB ∈ A. snd(y‘B)) ∈ (∏B ∈ A. B)"
⟨*proof*⟩

lemma *lemma1_2:*
"y ∈ (∏B ∈ {Y*C. C ∈ A}. B) ==> (λB ∈ A. y‘(Y*B)) ∈ (∏B ∈ A. Y*B)"
⟨*proof*⟩

lemma *AC7_AC6_lemma1:*
"(∏B ∈ {(nat->⋃(A))*C. C ∈ A}. B) ≠ 0 ==> (∏B ∈ A. B) ≠ 0"
⟨*proof*⟩

lemma *AC7_AC6_lemma2:* "0 ∉ A ==> 0 ∉ {(nat -> ⋃(A)) * C. C ∈ A}"
⟨*proof*⟩

lemma *AC7_AC6:* "AC7 ==> AC6"
⟨*proof*⟩

lemma *AC1_AC8_lemma1:*
"∀B ∈ A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2
==> 0 ∉ { bij(fst(B),snd(B)). B ∈ A }"
⟨*proof*⟩

lemma *AC1_AC8_lemma2:*
"[| f ∈ (∏X ∈ RepFun(A,p). X); D ∈ A |] ==> (λx ∈ A. f‘p(x))‘D ∈ p(D)"
⟨*proof*⟩

lemma *AC1_AC8:* "AC1 ==> AC8"
⟨*proof*⟩

lemma *AC8_AC9_lemma:*
"∀B1 ∈ A. ∀B2 ∈ A. B1 ≈ B2

==> ∀B ∈ A*A. ∃B1 B2. B=<B1,B2> & B1 ≈ B2"
⟨*proof*⟩

lemma *AC8_AC9: "AC8 ==> AC9"*
⟨*proof*⟩

lemma *snd_lepoll_SigmaI: "b ∈ B ⟹ X ≲ B × X"*
⟨*proof*⟩

lemma *nat_lepoll_lemma:*
"[|0 ∉ A; B ∈ A|] ==> nat ≲ ((nat → ⋃(A)) × B) × nat"
⟨*proof*⟩

lemma *AC9_AC1_lemma1:*
"[| 0∉A; A≠0;
*C = {((nat->⋃(A))*B)*nat. B ∈ A} ∪*
*{cons(0,((nat->⋃(A))*B)*nat). B ∈ A};*
B1 ∈ C; B2 ∈ C |]
==> B1 ≈ B2"
⟨*proof*⟩

lemma *AC9_AC1_lemma2:*
*"∀B1 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0,(F*B)*N). B ∈ A}.*
*∀B2 ∈ {(F*B)*N. B ∈ A} ∪ {cons(0,(F*B)*N). B ∈ A}.*
f‘<B1,B2> ∈ bij(B1, B2)
*==> (λB ∈ A. snd(fst((f‘<cons(0,(F*B)*N),(F*B)*N>)‘0))) ∈ (∏X ∈ A. X)"*
⟨*proof*⟩

lemma *AC9_AC1: "AC9 ==> AC1"*
⟨*proof*⟩

end

theory *WO1_AC*
imports *AC_Equiv*
begin

theorem *WO1_AC1: "WO1 ==> AC1"*
⟨*proof*⟩

lemma *lemma1: "[| WO1; ∀B ∈ A. ∃C ∈ D(B). P(C,B) |] ==> ∃f. ∀B ∈ A. P(f‘B,B)"*
⟨*proof*⟩

lemma *lemma2_1: "[| ~Finite(B); WO1 |] ==> |B| + |B| ≈ B"*
⟨*proof*⟩

lemma *lemma2_2:*
"f ∈ bij(D+D, B) ==> {{f‘Inl(i), f‘Inr(i)}. i ∈ D} ∈ Pow(Pow(B))"
⟨*proof*⟩

lemma *lemma2_3:*
"f ∈ bij(D+D, B) ==> pairwise_disjoint({{f‘Inl(i), f‘Inr(i)}. i ∈ D})"
⟨*proof*⟩

lemma *lemma2_4:*
"[| f ∈ bij(D+D, B); 1≤n |]
==> sets_of_size_between({{f‘Inl(i), f‘Inr(i)}. i ∈ D}, 2, succ(n))"
⟨*proof*⟩

lemma *lemma2_5:*
"f ∈ bij(D+D, B) ==> ⋃({{f‘Inl(i), f‘Inr(i)}. i ∈ D})=B"
⟨*proof*⟩

lemma *lemma2:*
"[| WO1; ~Finite(B); 1≤n |]
==> ∃C ∈ Pow(Pow(B)). pairwise_disjoint(C) &
sets_of_size_between(C, 2, succ(n)) &
⋃(C)=B"
⟨*proof*⟩

theorem *WO1_AC10: "[| WO1; 1≤n |] ==> AC10(n)"*
⟨*proof*⟩

end

```
theory Hartog
imports AC_Equiv
begin

definition
  Hartog :: "i => i"  where
   "Hartog(X) == μ i. ~ i ≲ X"

lemma Ords_in_set: "∀a. Ord(a) ⟶ a ∈ X ==> P"
⟨proof⟩

lemma Ord_lepoll_imp_ex_well_ord:
     "[| Ord(a); a ≲ X |]
      ==> ∃Y. Y ⊆ X & (∃R. well_ord(Y,R) & ordertype(Y,R)=a)"
⟨proof⟩

lemma Ord_lepoll_imp_eq_ordertype:
     "[| Ord(a); a ≲ X |] ==> ∃Y. Y ⊆ X & (∃R. R ⊆ X*X & ordertype(Y,R)=a)"
⟨proof⟩

lemma Ords_lepoll_set_lemma:
     "(∀a. Ord(a) ⟶ a ≲ X) ==>
       ∀a. Ord(a) ⟶
        a ∈ {b. Z ∈ Pow(X)*Pow(X*X), ∃Y R. Z=<Y,R> & ordertype(Y,R)=b}"
⟨proof⟩

lemma Ords_lepoll_set: "∀a. Ord(a) ⟶ a ≲ X ==> P"
⟨proof⟩

lemma ex_Ord_not_lepoll: "∃a. Ord(a) & ~a ≲ X"
⟨proof⟩

lemma not_Hartog_lepoll_self: "~ Hartog(A) ≲ A"
⟨proof⟩

lemmas Hartog_lepoll_selfE = not_Hartog_lepoll_self [THEN notE]

lemma Ord_Hartog: "Ord(Hartog(A))"
⟨proof⟩

lemma less_HartogE1: "[| i < Hartog(A); ~ i ≲ A |] ==> P"
⟨proof⟩

lemma less_HartogE: "[| i < Hartog(A); i ≈ Hartog(A) |] ==> P"
⟨proof⟩

lemma Card_Hartog: "Card(Hartog(A))"
⟨proof⟩
```

end

```
theory HH
imports AC_Equiv Hartog
begin
```

```
definition
  HH :: "[i, i, i] => i"  where
    "HH(f,x,a) == transrec(a, %b r. let z = x - (⋃c ∈ b. r`c)
                                    in  if f`z ∈ Pow(z)-{0} then f`z else
{x})"
```

0.1 Lemmas useful in each of the three proofs

```
lemma HH_def_satisfies_eq:
     "HH(f,x,a) = (let z = x - (⋃b ∈ a. HH(f,x,b))
                   in  if f`z ∈ Pow(z)-{0} then f`z else {x})"
⟨proof⟩
```

```
lemma HH_values: "HH(f,x,a) ∈ Pow(x)-{0} | HH(f,x,a)={x}"
⟨proof⟩
```

```
lemma subset_imp_Diff_eq:
     "B ⊆ A ==> X-(⋃a ∈ A. P(a)) = X-(⋃a ∈ A-B. P(a))-(⋃b ∈ B. P(b))"
⟨proof⟩
```

```
lemma Ord_DiffE: "[| c ∈ a-b; b<a |] ==> c=b | b<c & c<a"
⟨proof⟩
```

```
lemma Diff_UN_eq_self: "(!!y. y∈A ==> P(y) = {x}) ==> x - (⋃y ∈ A.
P(y)) = x"
⟨proof⟩
```

```
lemma HH_eq: "x - (⋃b ∈ a. HH(f,x,b)) = x - (⋃b ∈ a1. HH(f,x,b))
              ==> HH(f,x,a) = HH(f,x,a1)"
⟨proof⟩
```

```
lemma HH_is_x_gt_too: "[| HH(f,x,b)={x}; b<a |] ==> HH(f,x,a)={x}"
⟨proof⟩
```

```
lemma HH_subset_x_lt_too:
     "[| HH(f,x,a) ∈ Pow(x)-{0}; b<a |] ==> HH(f,x,b) ∈ Pow(x)-{0}"
⟨proof⟩
```

```
lemma HH_subset_x_imp_subset_Diff_UN:
    "HH(f,x,a) ∈ Pow(x)-{0} ==> HH(f,x,a) ∈ Pow(x - (⋃b ∈ a. HH(f,x,b)))-{0}"
⟨proof⟩
```

lemma *HH_eq_arg_lt:*
```
"[| HH(f,x,v)=HH(f,x,w); HH(f,x,v) ∈ Pow(x)-{0}; v ∈ w |] ==> P"
```
⟨*proof*⟩

lemma *HH_eq_imp_arg_eq:*
```
"[| HH(f,x,v)=HH(f,x,w); HH(f,x,w) ∈ Pow(x)-{0}; Ord(v); Ord(w) |] ==>
v=w"
```
⟨*proof*⟩

lemma *HH_subset_x_imp_lepoll:*
```
"[| HH(f, x, i) ∈ Pow(x)-{0}; Ord(i) |] ==> i ≲ Pow(x)-{0}"
```
⟨*proof*⟩

lemma *HH_Hartog_is_x:* `"HH(f, x, Hartog(Pow(x)-{0})) = {x}"`
⟨*proof*⟩

lemma *HH_Least_eq_x:* `"HH(f, x, μ i. HH(f, x, i) = {x}) = {x}"`
⟨*proof*⟩

lemma *less_Least_subset_x:*
```
"a ∈ (μ i. HH(f,x,i)={x}) ==> HH(f,x,a) ∈ Pow(x)-{0}"
```
⟨*proof*⟩

0.2 Lemmas used in the proofs of AC1 ==> WO2 and AC17 ==> AC1

lemma *lam_Least_HH_inj_Pow:*
```
"(λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))
 ∈ inj(μ i. HH(f,x,i)={x}, Pow(x)-{0})"
```
⟨*proof*⟩

lemma *lam_Least_HH_inj:*
```
"∀a ∈ (μ i. HH(f,x,i)={x}). ∃z ∈ x. HH(f,x,a) = {z}
 ==> (λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))
     ∈ inj(μ i. HH(f,x,i)={x}, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma *lam_surj_sing:*
```
"[| x - (⋃a ∈ A. F(a)) = 0;  ∀a ∈ A. ∃z ∈ x. F(a) = {z} |]

 ==> (λa ∈ A. F(a)) ∈ surj(A, {{y}. y ∈ x})"
```
⟨*proof*⟩

lemma *not_emptyI2:* `"y ∈ Pow(x)-{0} ==> x ≠ 0"`
⟨*proof*⟩

lemma *f_subset_imp_HH_subset:*
```
"f`(x - (⋃j ∈ i. HH(f,x,j))) ∈ Pow(x - (⋃j ∈ i. HH(f,x,j)))-{0}
```

```
      ==> HH(f, x, i) ∈ Pow(x) - {0}"
```

⟨*proof*⟩

```
lemma f_subsets_imp_UN_HH_eq_x:
     "∀z ∈ Pow(x)-{0}. f`z ∈ Pow(z)-{0}
      ==> x - (⋃j ∈ (μ i. HH(f,x,i)={x}). HH(f,x,j)) = 0"
```

⟨*proof*⟩

```
lemma HH_values2: "HH(f,x,i) = f`(x - (⋃j ∈ i. HH(f,x,j))) | HH(f,x,i)={x}"
```

⟨*proof*⟩

```
lemma HH_subset_imp_eq:
     "HH(f,x,i): Pow(x)-{0} ==> HH(f,x,i)=f`(x - (⋃j ∈ i. HH(f,x,j)))"
```

⟨*proof*⟩

```
lemma f_sing_imp_HH_sing:
     "[| f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x};
         a ∈ (μ i. HH(f,x,i)={x}) |] ==> ∃z ∈ x. HH(f,x,a) = {z}"
```

⟨*proof*⟩

```
lemma f_sing_lam_bij:
     "[| x - (⋃j ∈ (μ i. HH(f,x,i)={x}). HH(f,x,j)) = 0;
         f ∈ (Pow(x)-{0}) -> {{z}. z ∈ x} |]
      ==> (λa ∈ (μ i. HH(f,x,i)={x}). HH(f,x,a))
          ∈ bij(μ i. HH(f,x,i)={x}, {{y}. y ∈ x})"
```

⟨*proof*⟩

```
lemma lam_singI:
     "f ∈ (∏X ∈ Pow(x)-{0}. F(X))
      ==> (λX ∈ Pow(x)-{0}. {f`X}) ∈ (∏X ∈ Pow(x)-{0}. {{z}. z ∈ F(X)})"
```

⟨*proof*⟩

```
lemmas bij_Least_HH_x =
    comp_bij [OF f_sing_lam_bij [OF _ lam_singI]
              lam_sing_bij [THEN bij_converse_bij]]
```

0.3 The proof of AC1 ==> WO2

```
lemma bijection:
     "f ∈ (∏X ∈ Pow(x) - {0}. X)
      ==> ∃g. g ∈ bij(x, μ i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) = {x})"
```

⟨*proof*⟩

```
lemma AC1_WO2: "AC1 ==> WO2"
```

⟨*proof*⟩

end

theory *AC15_WO6*
imports *HH Cardinal_aux*
begin

lemma *lepoll_Sigma: "A≠0 ==> B ≲ A*B"*
⟨*proof*⟩

lemma *cons_times_nat_not_Finite:*
*"0∉A ==> ∀B ∈ {cons(0,x*nat). x ∈ A}. ~Finite(B)"*
⟨*proof*⟩

lemma *lemma1: "[| ⋃(C)=A; a ∈ A |] ==> ∃B ∈ C. a ∈ B & B ⊆ A"*
⟨*proof*⟩

lemma *lemma2:*
"[| pairwise_disjoint(A); B ∈ A; C ∈ A; a ∈ B; a ∈ C |] ==> B=C"
⟨*proof*⟩

lemma *lemma3:*
*"∀B ∈ {cons(0, x*nat). x ∈ A}. pairwise_disjoint(f‘B) &*
sets_of_size_between(f‘B, 2, n) & ⋃(f‘B)=B
*==> ∀B ∈ A. ∃! u. u ∈ f‘cons(0, B*nat) & u ⊆ cons(0, B*nat) &*
0 ∈ u & 2 ≲ u & u ≲ n"
⟨*proof*⟩

lemma *lemma4: "[| A ≲ i; Ord(i) |] ==> {P(a). a ∈ A} ≲ i"*
⟨*proof*⟩

lemma *lemma5_1:*
"[| B ∈ A; 2 ≲ u(B) |] ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ≠ 0"
⟨*proof*⟩

lemma *lemma5_2:*
*"[| B ∈ A; u(B) ⊆ cons(0, B*nat) |]*
==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})‘B ⊆ B"
⟨*proof*⟩

lemma *lemma5_3:*

```
"[| n ∈ nat; B ∈ A; 0 ∈ u(B); u(B) ≲ succ(n) |]
 ==> (λx ∈ A. {fst(x). x ∈ u(x)-{0}})`B ≲ n"
```
⟨*proof*⟩

lemma *ex_fun_AC13_AC15:*
```
"[| ∀B ∈ {cons(0, x*nat). x ∈ A}.
          pairwise_disjoint(f`B) &
          sets_of_size_between(f`B, 2, succ(n)) & ⋃(f`B)=B;
    n ∈ nat |]
 ==> ∃f. ∀B ∈ A. f`B ≠ 0 & f`B ⊆ B & f`B ≲ n"
```
⟨*proof*⟩

theorem *AC10_AC11:* `"[| n ∈ nat; 1≤n; AC10(n) |] ==> AC11"`
⟨*proof*⟩

theorem *AC11_AC12:* `"AC11 ==> AC12"`
⟨*proof*⟩

theorem *AC12_AC15:* `"AC12 ==> AC15"`
⟨*proof*⟩

lemma *OUN_eq_UN:* `"Ord(x) ==> (⋃a<x. F(a)) = (⋃a ∈ x. F(a))"`
⟨*proof*⟩

lemma *AC15_WO6_aux1:*
```
"∀x ∈ Pow(A)-{0}. f`x≠0 & f`x ⊆ x & f`x ≲ m
 ==> (⋃i<μ x. HH(f,A,x)={A}. HH(f,A,i)) = A"
```
⟨*proof*⟩

lemma *AC15_WO6_aux2:*
"$\forall$ x $\in$ Pow(A)-{0}. f`x$\neq$0 & f`x $\subseteq$ x & f`x $\lesssim$ m
==> $\forall$ x < (μ x. HH(f,A,x)={A}). HH(f,A,x) $\lesssim$ m"
$\langle$*proof*$\rangle$

theorem *AC15_WO6: "AC15 ==> WO6"*
$\langle$*proof*$\rangle$

theorem *AC10_AC13: "[| n $\in$ nat; 1$\leq$n; AC10(n) |] ==> AC13(n)"*
$\langle$*proof*$\rangle$

lemma *AC1_AC13: "AC1 ==> AC13(1)"*
$\langle$*proof*$\rangle$

lemma *AC13_mono: "[| m$\leq$n; AC13(m) |] ==> AC13(n)"*
$\langle$*proof*$\rangle$

theorem *AC13_AC14: "[| n ∈ nat; 1≤n; AC13(n) |] ==> AC14"*
⟨*proof*⟩

theorem *AC14_AC15: "AC14 ==> AC15"*
⟨*proof*⟩

lemma *lemma_aux: "[| A≠0; A ≲ 1 |] ==> ∃a. A={a}"*
⟨*proof*⟩

lemma *AC13_AC1_lemma:*
"∀B ∈ A. f(B)≠0 & f(B)<=B & f(B) ≲ 1
==> (λx ∈ A. THE y. f(x)={y}) ∈ (∏X ∈ A. X)"
⟨*proof*⟩

theorem *AC13_AC1: "AC13(1) ==> AC1"*
⟨*proof*⟩

theorem *AC11_AC14: "AC11 ==> AC14"*
⟨*proof*⟩

end

theory *AC16_lemmas*
imports *AC_Equiv Hartog Cardinal_aux*
begin

lemma *cons_Diff_eq: "a∉A ==> cons(a,A)-{a}=A"*
⟨*proof*⟩

lemma *nat_1_lepoll_iff: "1≲X ⟷ (∃x. x ∈ X)"*

⟨proof⟩

lemma *eqpoll_1_iff_singleton: "X≈1 ⟷ (∃x. X={x})"*
⟨proof⟩

lemma *cons_eqpoll_succ: "[| x≈n; y∉x |] ==> cons(y,x)≈succ(n)"*
⟨proof⟩

lemma *subsets_eqpoll_1_eq: "{Y ∈ Pow(X). Y≈1} = {{x}. x ∈ X}"*
⟨proof⟩

lemma *eqpoll_RepFun_sing: "X≈{{x}. x ∈ X}"*
⟨proof⟩

lemma *subsets_eqpoll_1_eqpoll: "{Y ∈ Pow(X). Y≈1}≈X"*
⟨proof⟩

lemma *InfCard_Least_in:*
"[| InfCard(x); y ⊆ x; y ≈ succ(z) |] ==> (μ i. i ∈ y) ∈ y"
⟨proof⟩

lemma *subsets_lepoll_lemma1:*
"[| InfCard(x); n ∈ nat |]
==> {y ∈ Pow(x). y≈succ(succ(n))} ≲ x{y ∈ Pow(x). y≈succ(n)}"*
⟨proof⟩

lemma *set_of_Ord_succ_Union: "(∀y ∈ z. Ord(y)) ==> z ⊆ succ(⋃(z))"*
⟨proof⟩

lemma *subset_not_mem: "j ⊆ i ==> i ∉ j"*
⟨proof⟩

lemma *succ_Union_not_mem:*
"(!!y. y ∈ z ==> Ord(y)) ==> succ(⋃(z)) ∉ z"
⟨proof⟩

lemma *Union_cons_eq_succ_Union:*
"⋃(cons(succ(⋃(z)),z)) = succ(⋃(z))"
⟨proof⟩

lemma *Un_Ord_disj: "[| Ord(i); Ord(j) |] ==> i ∪ j = i | i ∪ j = j"*
⟨proof⟩

lemma *Union_eq_Un: "x ∈ X ==> ⋃(X) = x ∪ ⋃(X-{x})"*
⟨proof⟩

lemma *Union_in_lemma [rule_format]:*
"n ∈ nat ==> ∀z. (∀y ∈ z. Ord(y)) & z≈n & z≠0 ⟶ ⋃(z) ∈ z"
⟨proof⟩

lemma `Union_in: "[| ∀x ∈ z. Ord(x); z≈n; z≠0; n ∈ nat |] ==> ⋃(z) ∈ z"`
⟨proof⟩

lemma `succ_Union_in_x:`
`"[| InfCard(x); z ∈ Pow(x); z≈n; n ∈ nat |] ==> succ(⋃(z)) ∈ x"`
⟨proof⟩

lemma `succ_lepoll_succ_succ:`
`"[| InfCard(x); n ∈ nat |]`
`==> {y ∈ Pow(x). y≈succ(n)} ≲ {y ∈ Pow(x). y≈succ(succ(n))}"`
⟨proof⟩

lemma `subsets_eqpoll_X:`
`"[| InfCard(X); n ∈ nat |] ==> {Y ∈ Pow(X). Y≈succ(n)} ≈ X"`
⟨proof⟩

lemma `image_vimage_eq:`
`"[| f ∈ surj(A,B); y ⊆ B |] ==> f``(converse(f)``y) = y"`
⟨proof⟩

lemma `vimage_image_eq: "[| f ∈ inj(A,B); y ⊆ A |] ==> converse(f)``(f``y) = y"`
⟨proof⟩

lemma `subsets_eqpoll:`
`"A≈B ==> {Y ∈ Pow(A). Y≈n}≈{Y ∈ Pow(B). Y≈n}"`
⟨proof⟩

lemma `WO2_imp_ex_Card: "WO2 ==> ∃a. Card(a) & X≈a"`
⟨proof⟩

lemma `lepoll_infinite: "[| X≲Y; ~Finite(X) |] ==> ~Finite(Y)"`
⟨proof⟩

lemma `infinite_Card_is_InfCard: "[| ~Finite(X); Card(X) |] ==> InfCard(X)"`
⟨proof⟩

lemma `WO2_infinite_subsets_eqpoll_X: "[| WO2; n ∈ nat; ~Finite(X) |]`

`==> {Y ∈ Pow(X). Y≈succ(n)}≈X"`
⟨proof⟩

lemma `well_ord_imp_ex_Card: "well_ord(X,R) ==> ∃a. Card(a) & X≈a"`
⟨proof⟩

lemma `well_ord_infinite_subsets_eqpoll_X:`
`"[| well_ord(X,R); n ∈ nat; ~Finite(X) |] ==> {Y ∈ Pow(X). Y≈succ(n)}≈X"`

⟨proof⟩

end

theory *WO2_AC16* **imports** *AC_Equiv AC16_lemmas Cardinal_aux* **begin**

definition
```
  recfunAC16 :: "[i,i,i,i] => i"  where
    "recfunAC16(f,h,i,a) ==
         transrec2(i, 0,
              %g r. if (∃y ∈ r. h`g ⊆ y) then r
                    else r ∪ {f`(μ i. h`g ⊆ f`i &
                         (∀b<a. (h`b ⊆ f`i ⟶ (∀t ∈ r. ~ h`b ⊆ t))))})"
```

lemma *recfunAC16_0: "recfunAC16(f,h,0,a) = 0"*
⟨proof⟩

lemma *recfunAC16_succ:*
```
     "recfunAC16(f,h,succ(i),a) =
      (if (∃y ∈ recfunAC16(f,h,i,a). h ` i ⊆ y) then recfunAC16(f,h,i,a)

       else recfunAC16(f,h,i,a) ∪
            {f ` (μ j. h ` i ⊆ f ` j &
             (∀b<a. (h`b ⊆ f`j
              ⟶ (∀t ∈ recfunAC16(f,h,i,a). ~ h`b ⊆ t))))})"
```
⟨proof⟩

lemma *recfunAC16_Limit: "Limit(i)*
```
        ==> recfunAC16(f,h,i,a) = (⋃j<i. recfunAC16(f,h,j,a))"
```
⟨proof⟩

lemma *transrec2_mono_lemma [rule_format]:*
```
     "[| !!g r. r ⊆ B(g,r);  Ord(i) |]
      ==> j<i ⟶ transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
```
⟨proof⟩

lemma *transrec2_mono:*
```
     "[| !!g r. r ⊆ B(g,r); j≤i |]
```

```
      ==> transrec2(j, 0, B) ⊆ transrec2(i, 0, B)"
```
⟨*proof*⟩

lemma *recfunAC16_mono:*
```
        "i≤j ==> recfunAC16(f, g, i, a) ⊆ recfunAC16(f, g, j, a)"
```
⟨*proof*⟩

lemma *lemma3_1:*
```
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) ⟶ (∃! Y. Y ∈ F(y) &
f(z)<=Y);
        ∀i j. i≤j ⟶ F(i) ⊆ F(j); j≤i; i<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
     ==> V = W"
```
⟨*proof*⟩

lemma *lemma3:*
```
    "[| ∀y<x. ∀z<a. z<y | (∃Y ∈ F(y). f(z)<=Y) ⟶ (∃! Y. Y ∈ F(y) &
f(z)<=Y);
        ∀i j. i≤j ⟶ F(i) ⊆ F(j); i<x; j<x; z<a;
        V ∈ F(i); f(z)<=V; W ∈ F(j); f(z)<=W |]
     ==> V = W"
```
⟨*proof*⟩

lemma *lemma4:*
```
     "[| ∀y<x. F(y) ⊆ X &
                (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) ⟶
                 (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a |]
      ==> ∀y<x. ∀z<a. z < y | (∃Y ∈ F(y). h(z) ⊆ Y) ⟶
                      (∃! Y. Y ∈ F(y) & h(z) ⊆ Y)"
```
⟨*proof*⟩

lemma *lemma5:*
```
     "[| ∀y<x. F(y) ⊆ X &
               (∀x<a. x < y | (∃Y ∈ F(y). h(x) ⊆ Y) ⟶
                (∃! Y. Y ∈ F(y) & h(x) ⊆ Y));
         x < a; Limit(x); ∀i j. i≤j ⟶ F(i) ⊆ F(j) |]
      ==> (⋃x<x. F(x)) ⊆ X &
          (∀xa<a. xa < x | (∃x ∈ ⋃x<x. F(x). h(xa) ⊆ x)
```

$\longrightarrow$ (∃! Y. Y ∈ (⋃x<x. F(x)) & h(xa) ⊆ Y))"
⟨proof⟩

lemma *dbl_Diff_eqpoll_Card:*
"[| A≈a; Card(a); ~Finite(a); B≺a; C≺a |] ==> A - B - C≈a"
⟨proof⟩

lemma *Finite_lesspoll_infinite_Ord:*
"[| Finite(X); ~Finite(a); Ord(a) |] ==> X≺a"
⟨proof⟩

lemma *Union_lesspoll:*
"[| ∀x ∈ X. x ≲ n & x ⊆ T; well_ord(T, R); X ≲ b;
b<a; ~Finite(a); Card(a); n ∈ nat |]
==> ⋃(X)≺a"
⟨proof⟩

lemma *Un_sing_eq_cons:* "A ∪ {a} = cons(a, A)"
⟨proof⟩

lemma *Un_lepoll_succ:* "A ≲ B ==> A ∪ {a} ≲ succ(B)"
⟨proof⟩

lemma *Diff_UN_succ_empty:* "Ord(a) ==> F(a) - (⋃b<succ(a). F(b)) = 0"
⟨proof⟩

lemma *Diff_UN_succ_subset:* "Ord(a) ==> F(a) ∪ X - (⋃b<succ(a). F(b))
⊆ X"
⟨proof⟩

lemma *recfunAC16_Diff_lepoll_1:*
```
"Ord(x)
 ==> recfunAC16(f, g, x, a) - (⋃i<x. recfunAC16(f, g, i, a)) ≲ 1"
```
⟨*proof*⟩

lemma *in_Least_Diff:*
```
"[| z ∈ F(x); Ord(x) |]
 ==> z ∈ F(μ i. z ∈ F(i)) - (⋃j<(μ i. z ∈ F(i)). F(j))"
```
⟨*proof*⟩

lemma *Least_eq_imp_ex:*
```
"[| (μ i. w ∈ F(i)) = (μ i. z ∈ F(i));
    w ∈ (⋃i<a. F(i)); z ∈ (⋃i<a. F(i)) |]
 ==> ∃b<a. w ∈ (F(b) - (⋃c<b. F(c))) & z ∈ (F(b) - (⋃c<b. F(c)))"
```
⟨*proof*⟩

lemma *two_in_lepoll_1:* `"[| A ≲ 1; a ∈ A; b ∈ A |] ==> a=b"`
⟨*proof*⟩

lemma *UN_lepoll_index:*
```
"[| ∀i<a. F(i)-(⋃j<i. F(j)) ≲ 1; Limit(a) |]
 ==> (⋃x<a. F(x)) ≲ a"
```
⟨*proof*⟩

lemma *recfunAC16_lepoll_index:* `"Ord(y) ==> recfunAC16(f, h, y, a) ≲ y"`
⟨*proof*⟩

lemma *Union_recfunAC16_lesspoll:*
```
"[| recfunAC16(f,g,y,a) ⊆ {X ∈ Pow(A). X≈n};
    A≈a;  y<a;  ~Finite(a);  Card(a);  n ∈ nat |]
 ==> ⋃(recfunAC16(f,g,y,a))≺a"
```
⟨*proof*⟩

lemma *dbl_Diff_eqpoll:*
```
"[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
    Card(a); ~ Finite(a); A≈a;
    k ∈ nat;  y<a;
    h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)}) |]
 ==> A - ⋃(recfunAC16(f, h, y, a)) - h`y≈a"
```
⟨*proof*⟩

lemmas *disj_Un_eqpoll_nat_sum =*

```
    eqpoll_trans [THEN eqpoll_trans,
                 OF disj_Un_eqpoll_sum sum_eqpoll_cong nat_sum_eqpoll_sum]
```

lemma *Un_in_Collect:*

```
"[| x ∈ Pow(A - B - h`i); x≈m;
        h ∈ bij(a, {x ∈ Pow(A) . x≈k}); i<a; k ∈ nat; m ∈ nat |]
        ==> h ` i ∪ x ∈ {x ∈ Pow(A) . x≈k #+ m}"
```

⟨*proof*⟩

lemma *lemma6:*

```
     "[| ∀y<succ(j). F(y)<=X & (∀x<a. x<y | P(x,y) ⟶ Q(x,y)); succ(j)<a
|]
      ==> F(j)<=X & (∀x<a. x<j | P(x,j) ⟶ Q(x,j))"
```

⟨*proof*⟩

lemma *lemma7:*

```
     "[| ∀x<a. x<j | P(x,j) ⟶ Q(x,j);  succ(j)<a |]
      ==> P(j,j) ⟶ (∀x<a. x≤j | P(x,j) ⟶ Q(x,j))"
```

⟨*proof*⟩

lemma *ex_subset_eqpoll:*

```
     "[| A≈a; ~ Finite(a); Ord(a); m ∈ nat |] ==> ∃X ∈ Pow(A). X≈m"
```

⟨*proof*⟩

lemma *subset_Un_disjoint:* `"[| A ⊆ B ∪ C; A ∩ C = 0 |] ==> A ⊆ B"`

⟨*proof*⟩

lemma *Int_empty:*

```
     "[| X ∈ Pow(A - ⋃(B) -C); T ∈ B; F ⊆ T |] ==> F ∩ X = 0"
```

⟨*proof*⟩

lemma subset_imp_eq_lemma:
```
    "m ∈ nat ==> ∀A B. A ⊆ B & m ≲ A & B ≲ m ⟶ A=B"
```
⟨*proof*⟩

lemma subset_imp_eq:
```
"[| A ⊆ B; m ≲ A; B ≲ m; m ∈ nat |] ==> A=B"
```
⟨*proof*⟩

lemma bij_imp_arg_eq:
```
    "[| f ∈ bij(a, {Y ∈ X. Y≈succ(k)}); k ∈ nat; f`b ⊆ f`y; b<a; y<a
|]
     ==> b=y"
```
⟨*proof*⟩

lemma ex_next_set:
```
    "[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
        Card(a); ~ Finite(a); A≈a;
        k ∈ nat; m ∈ nat; y<a;
        h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
        ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
     ==> ∃X ∈ {Y ∈ Pow(A). Y≈succ(k #+ m)}. h`y ⊆ X &
                (∀b<a. h`b ⊆ X ⟶
                (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
```
⟨*proof*⟩

lemma ex_next_Ord:
```
    "[| recfunAC16(f, h, y, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)};
        Card(a); ~ Finite(a); A≈a;
        k ∈ nat; m ∈ nat; y<a;
        h ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k)});
        f ∈ bij(a, {Y ∈ Pow(A). Y≈succ(k #+ m)});
        ~ (∃Y ∈ recfunAC16(f, h, y, a). h`y ⊆ Y) |]
     ==> ∃c<a. h`y ⊆ f`c &
                (∀b<a. h`b ⊆ f`c ⟶
                (∀T ∈ recfunAC16(f, h, y, a). ~ h`b ⊆ T))"
```
⟨*proof*⟩

```
lemma lemma8:
      "[| ∀x<a. x<j | (∃xa ∈ F(j). P(x, xa))
          ⟶ (∃! Y. Y ∈ F(j) & P(x, Y)); F(j) ⊆ X;
          L ∈ X; P(j, L) & (∀x<a. P(x, L) ⟶ (∀xa ∈ F(j). ~P(x, xa)))
|]
       ==> F(j) ∪ {L} ⊆ X &
           (∀x<a. x≤j | (∃xa ∈ (F(j) ∪ {L}). P(x, xa)) ⟶
                  (∃! Y. Y ∈ (F(j) ∪ {L}) & P(x, Y)))"
⟨proof⟩
```

```
lemma main_induct:
      "[| b < a; f ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k #+ m)});
          h ∈ bij(a, {Y ∈ Pow(A) . Y≈succ(k)});
          ~Finite(a); Card(a); A≈a; k ∈ nat; m ∈ nat |]
       ==> recfunAC16(f, h, b, a) ⊆ {X ∈ Pow(A) . X≈succ(k #+ m)} &
           (∀x<a. x < b | (∃Y ∈ recfunAC16(f, h, b, a). h ` x ⊆ Y) ⟶

           (∃! Y. Y ∈ recfunAC16(f, h, b, a) & h ` x ⊆ Y))"
⟨proof⟩
```

```
lemma lemma_simp_induct:
      "[| ∀b. b<a ⟶ F(b) ⊆ S & (∀x<a. (x<b | (∃Y ∈ F(b). f`x ⊆ Y))

                                  ⟶ (∃! Y. Y ∈ F(b) & f`x ⊆ Y));
          f ∈ a->f``(a); Limit(a);
          ∀i j. i≤j ⟶ F(i) ⊆ F(j) |]
         ==> (⋃j<a. F(j)) ⊆ S &
             (∀x ∈ f``a. ∃! Y. Y ∈ (⋃j<a. F(j)) & x ⊆ Y)"
⟨proof⟩
```

```
theorem WO2_AC16: "[| WO2; 0<m; k ∈ nat; m ∈ nat |] ==> AC16(k #+ m,k)"
⟨proof⟩

end
```

theory *AC16_WO4*
imports *AC16_lemmas*
begin

```
lemma lemma1:
      "[| Finite(A); 0<m; m ∈ nat |]
       ==> ∃a f. Ord(a) & domain(f) = a &
                  (⋃b<a. f`b) = A & (∀b<a. f`b ≲ m)"
⟨proof⟩
```

```
lemmas well_ord_paired = paired_bij [THEN bij_is_inj, THEN well_ord_rvimage]

lemma lepoll_trans1: "[| A ≲ B; ~ A ≲ C |] ==> ~ B ≲ C"
⟨proof⟩
```

```
lemmas lepoll_paired = paired_eqpoll [THEN eqpoll_sym, THEN eqpoll_imp_lepoll]

lemma lemma2: "∃y R. well_ord(y,R) & x ∩ y = 0 & ~y ≲ z & ~Finite(y)"
⟨proof⟩

lemma infinite_Un: "~Finite(B) ==> ~Finite(A ∪ B)"
⟨proof⟩
```

lemma *succ_not_lepoll_lemma:*
```
"[| ~(∃x ∈ A. f`x=y); f ∈ inj(A, B); y ∈ B |]
 ==> (λa ∈ succ(A). if(a=A, y, f`a)) ∈ inj(succ(A), B)"
```
⟨*proof*⟩

lemma *succ_not_lepoll_imp_eqpoll:* `"[| ~A ≈ B; A ≲ B |] ==> succ(A) ≲ B"`
⟨*proof*⟩

lemmas *ordertype_eqpoll =*
```
ordermap_bij [THEN exI [THEN eqpoll_def [THEN def_imp_iff, THEN iffD2]]]
```

lemma *cons_cons_subset:*
```
"[| a ⊆ y; b ∈ y-a; u ∈ x |] ==> cons(b, cons(u, a)) ∈ Pow(x ∪ y)"
```
⟨*proof*⟩

lemma *cons_cons_eqpoll:*
```
"[| a ≈ k; a ⊆ y; b ∈ y-a; u ∈ x; x ∩ y = 0 |]
 ==> cons(b, cons(u, a)) ≈ succ(succ(k))"
```
⟨*proof*⟩

lemma *set_eq_cons:*
```
"[| succ(k) ≈ A; k ≈ B; B ⊆ A; a ∈ A-B; k ∈ nat |] ==> A = cons(a, B)"
```
⟨*proof*⟩

lemma *cons_eqE:* `"[| cons(x,a) = cons(y,a); x ∉ a |] ==> x = y "`
⟨*proof*⟩

lemma *eq_imp_Int_eq:* `"A = B ==> A ∩ C = B ∩ C"`
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll_lemma [rule_format]:*
```
"[| k ∈ nat; m ∈ nat |]
 ==> ∀A B. A ≈ k #+ m & k ≲ B & B ⊆ A ⟶ A-B ≲ m"
```
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_lepoll:*
```
"[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≲ B;  k ∈ nat; m ∈ nat |]

 ==> A-B ≲ m"
```
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll_lemma [rule_format]:*
```
"[| k ∈ nat; m ∈ nat |]
 ==> ∀A B. A ≈ k #+ m & k ≈ B & B ⊆ A ⟶ A-B ≈ m"
```
⟨*proof*⟩

lemma *eqpoll_sum_imp_Diff_eqpoll:*
```
"[| A ≈ succ(k #+ m); B ⊆ A; succ(k) ≈ B; k ∈ nat; m ∈ nat |]

 ==> A-B ≈ m"
```
⟨*proof*⟩

lemma *subsets_lepoll_0_eq_unit: "{x ∈ Pow(X). x ≲ 0} = {0}"*
⟨*proof*⟩

lemma *subsets_lepoll_succ:*
```
"n ∈ nat ==> {z ∈ Pow(y). z ≲ succ(n)} =
             {z ∈ Pow(y). z ≲ n} ∪ {z ∈ Pow(y). z ≈ succ(n)}"
```
⟨*proof*⟩

lemma *Int_empty:*
```
"n ∈ nat ==> {z ∈ Pow(y). z ≲ n} ∩ {z ∈ Pow(y). z ≈ succ(n)} =
0"
```
⟨*proof*⟩

locale *AC16 =*
fixes *x* **and** *y* **and** *k* **and** *l* **and** *m* **and** *t_n* **and** *R* **and** *MM* **and** *LL* **and** *GG* **and** *s*
defines *k_def:* *"k == succ(l)"*
and *MM_def:* *"MM == {v ∈ t_n. succ(k) ≲ v ∩ y}"*
and *LL_def:* *"LL == {v ∩ y. v ∈ MM}"*
and *GG_def:* *"GG == λv ∈ LL. (THE w. w ∈ MM & v ⊆ w) - v"*
and *s_def:* *"s(u) == {v ∈ t_n. u ∈ v & k ≲ v ∩ y}"*
assumes *all_ex:* *"∀z ∈ {z ∈ Pow(x ∪ y) . z ≈ succ(k)}.*

```
                    ∃! w. w ∈ t_n & z ⊆ w "
    and disjoint[iff]:  "x ∩ y = 0"
    and "includes":  "t_n ⊆ {v ∈ Pow(x ∪ y). v ≈ succ(k #+ m)}"
    and WO_R[iff]:      "well_ord(y,R)"
    and lnat[iff]:      "l ∈ nat"
    and mnat[iff]:      "m ∈ nat"
    and mpos[iff]:      "0<m"
    and Infinite[iff]:  "~ Finite(y)"
    and noLepoll:  "~ y ≲ {v ∈ Pow(x). v ≈ m}"
begin

lemma knat [iff]: "k ∈ nat"
⟨proof⟩




lemma Diff_Finite_eqpoll: "[| l ≈ a; a ⊆ y |] ==> y - a ≈ y"
⟨proof⟩


lemma s_subset: "s(u) ⊆ t_n"
⟨proof⟩

lemma sI:
      "[| w ∈ t_n; cons(b,cons(u,a)) ⊆ w; a ⊆ y; b ∈ y-a; l ≈ a |]

       ==> w ∈ s(u)"
⟨proof⟩

lemma in_s_imp_u_in: "v ∈ s(u) ==> u ∈ v"
⟨proof⟩


lemma ex1_superset_a:
     "[| l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
      ==> ∃! c. c ∈ s(u) & a ⊆ c & b ∈ c"
⟨proof⟩

lemma the_eq_cons:
     "[| ∀v ∈ s(u). succ(l) ≈ v ∩ y;
         l ≈ a;  a ⊆ y;  b ∈ y - a;  u ∈ x |]
      ==> (THE c. c ∈ s(u) & a ⊆ c & b ∈ c) ∩ y = cons(b, a)"
⟨proof⟩

lemma y_lepoll_subset_s:
```

```
"[| ∀v ∈ s(u). succ(l) ≈ v ∩ y;
    l ≈ a;  a ⊆ y;  u ∈ x |]
 ==> y ≲ {v ∈ s(u). a ⊆ v}"
```
⟨*proof*⟩

lemma *x_imp_not_y [dest]: "a ∈ x ==> a ∉ y"*
⟨*proof*⟩

lemma *w_Int_eq_w_Diff:*
```
"w ⊆ x ∪ y ==> w ∩ (x - {u}) = w - cons(u, w ∩ y)"
```
⟨*proof*⟩

lemma *w_Int_eqpoll_m:*
```
"[| w ∈ {v ∈ s(u). a ⊆ v};
    l ≈ a;  u ∈ x;
    ∀v ∈ s(u). succ(l) ≈ v ∩ y |]
 ==> w ∩ (x - {u}) ≈ m"
```
⟨*proof*⟩

lemma *eqpoll_m_not_empty: "a ≈ m ==> a ≠ 0"*
⟨*proof*⟩

lemma *cons_cons_in:*
```
"[| z ∈ xa ∩ (x - {u}); l ≈ a; a ⊆ y; u ∈ x |]
 ==> ∃! w. w ∈ t_n & cons(z, cons(u, a)) ⊆ w"
```
⟨*proof*⟩

lemma *subset_s_lepoll_w:*
```
"[| ∀v ∈ s(u). succ(l) ≈ v ∩ y; a ⊆ y; l ≈ a; u ∈ x |]
 ==> {v ∈ s(u). a ⊆ v} ≲ {v ∈ Pow(x). v ≈ m}"
```
⟨*proof*⟩

lemma *well_ord_subsets_eqpoll_n:*
"n ∈ nat ==> ∃S. well_ord({z ∈ Pow(y) . z ≈ succ(n)}, S)"
⟨*proof*⟩

lemma *well_ord_subsets_lepoll_n:*
"n ∈ nat ==> ∃R. well_ord({z ∈ Pow(y). z ≲ n}, R)"
⟨*proof*⟩

lemma *LL_subset:* "LL ⊆ {z ∈ Pow(y). z ≲ succ(k #+ m)}"
⟨*proof*⟩

lemma *well_ord_LL:* "∃S. well_ord(LL,S)"
⟨*proof*⟩

lemma *unique_superset_in_MM:*
"v ∈ LL ==> ∃! w. w ∈ MM & v ⊆ w"
⟨*proof*⟩

lemma *Int_in_LL:* "w ∈ MM ==> w ∩ y ∈ LL"
⟨*proof*⟩

lemma *in_LL_eq_Int:*
"v ∈ LL ==> v = (THE x. x ∈ MM & v ⊆ x) ∩ y"
⟨*proof*⟩

lemma *unique_superset1:* "a ∈ LL ⟹ (THE x. x ∈ MM ∧ a ⊆ x) ∈ MM"
⟨*proof*⟩

lemma *the_in_MM_subset:*
"v ∈ LL ==> (THE x. x ∈ MM & v ⊆ x) ⊆ x ∪ y"
⟨*proof*⟩

lemma *GG_subset:* "v ∈ LL ==> GG ‘ v ⊆ x"
⟨*proof*⟩

lemma *nat_lepoll_ordertype: "nat ≲ ordertype(y, R)"*
⟨*proof*⟩

lemma *ex_subset_eqpoll_n: "n ∈ nat ==> ∃z. z ⊆ y & n ≈ z"*
⟨*proof*⟩

lemma *exists_proper_in_s: "u ∈ x ==> ∃v ∈ s(u). succ(k) ≲ v ∩ y"*
⟨*proof*⟩

lemma *exists_in_MM: "u ∈ x ==> ∃w ∈ MM. u ∈ w"*
⟨*proof*⟩

lemma *exists_in_LL: "u ∈ x ==> ∃w ∈ LL. u ∈ GG`w"*
⟨*proof*⟩

lemma *OUN_eq_x: "well_ord(LL,S) ==>*
(⋃b<ordertype(LL,S). GG ` (converse(ordermap(LL,S)) ` b)) = x"
⟨*proof*⟩

lemma *in_MM_eqpoll_n: "w ∈ MM ==> w ≈ succ(k #+ m)"*
⟨*proof*⟩

lemma *in_LL_eqpoll_n: "w ∈ LL ==> succ(k) ≲ w"*
⟨*proof*⟩

lemma *in_LL: "w ∈ LL ==> w ⊆ (THE x. x ∈ MM ∧ w ⊆ x)"*
⟨*proof*⟩

lemma *all_in_lepoll_m:*
"well_ord(LL,S) ==>
∀b<ordertype(LL,S). GG ` (converse(ordermap(LL,S)) ` b) ≲ m"
⟨*proof*⟩

lemma *"conclusion":*
"∃a f. Ord(a) & domain(f) = a & (⋃b<a. f ` b) = x & (∀b<a. f `
b ≲ m)"
⟨*proof*⟩

end

theorem *AC16_WO4:*
```
    "[| AC_Equiv.AC16(k #+ m, k); 0 < k; 0 < m; k ∈ nat; m ∈ nat |]
==> WO4(m)"
```
⟨*proof*⟩

end

theory *AC17_AC1*
imports *HH*
begin

lemma *AC0_AC1_lemma: "[| f:(∏X ∈ A. X); D ⊆ A |] ==> ∃g. g:(∏X ∈ D. X)"*
⟨*proof*⟩

lemma *AC0_AC1: "AC0 ==> AC1"*
⟨*proof*⟩

lemma *AC1_AC0: "AC1 ==> AC0"*
⟨*proof*⟩

lemma *AC1_AC17_lemma: "f ∈ (∏X ∈ Pow(A) - {0}. X) ==> f ∈ (Pow(A) - {0} -> A)"*
⟨*proof*⟩

lemma *AC1_AC17: "AC1 ==> AC17"*
⟨*proof*⟩

lemma *UN_eq_imp_well_ord:*
```
    "[| x - (⋃j ∈ μ i. HH(λX ∈ Pow(x)-{0}. {f`X}, x, i) = {x}.
       HH(λX ∈ Pow(x)-{0}. {f`X}, x, j)) = 0;
       f ∈ Pow(x)-{0} -> x |]
       ==> ∃r. well_ord(x,r)"
```

$\langle proof \rangle$

lemma *not_AC1_imp_ex:*

```
"~AC1 ==> ∃A. ∀f ∈ Pow(A)-{0} -> A. ∃u ∈ Pow(A)-{0}. f`u ∉ u"
```

$\langle proof \rangle$

lemma *AC17_AC1_aux1:*

```
"[| ∀f ∈ Pow(x) - {0} -> x. ∃u ∈ Pow(x) - {0}. f`u∉u;
    ∃f ∈ Pow(x)-{0}->x.
       x - (⋃a ∈ (μ i. HH(λX ∈ Pow(x)-{0}. {f`X},x,i)={x}).
       HH(λX ∈ Pow(x)-{0}. {f`X},x,a)) = 0 |]
   ==> P"
```

$\langle proof \rangle$

lemma *AC17_AC1_aux2:*

```
"~ (∃f ∈ Pow(x)-{0}->x. x - F(f) = 0)
  ==> (λf ∈ Pow(x)-{0}->x . x - F(f))
      ∈ (Pow(x) -{0} -> x) -> Pow(x) - {0}"
```

$\langle proof \rangle$

lemma *AC17_AC1_aux3:*

```
"[| f`Z ∈ Z; Z ∈ Pow(x)-{0} |]
 ==> (λX ∈ Pow(x)-{0}. {f`X})`Z ∈ Pow(Z)-{0}"
```

$\langle proof \rangle$

lemma *AC17_AC1_aux4:*

```
"∃f ∈ F. f`((λf ∈ F. Q(f))`f) ∈ (λf ∈ F. Q(f))`f
 ==> ∃f ∈ F. f`Q(f) ∈ Q(f)"
```

$\langle proof \rangle$

lemma *AC17_AC1:* *"AC17 ==> AC1"*

$\langle proof \rangle$

lemma *AC1_AC2_aux1:*

```
"[| f:(∏X ∈ A. X);  B ∈ A;  0∉A |] ==> {f`B} ⊆ B ∩ {f`C. C ∈
A}"
```

$\langle proof \rangle$

lemma *AC1_AC2_aux2:*
```
        "[| pairwise_disjoint(A); B ∈ A; C ∈ A; D ∈ B; D ∈ C |] ==>
f`B = f`C"
```
⟨*proof*⟩

lemma *AC1_AC2:* `"AC1 ==> AC2"`
⟨*proof*⟩

lemma *AC2_AC1_aux1:* `"0∉A ==> 0 ∉ {B*{B}. B ∈ A}"`
⟨*proof*⟩

lemma *AC2_AC1_aux2:*
```
"[| X*{X} ∩ C = {y}; X ∈ A |]
        ==> (THE y. X*{X} ∩ C = {y}): X*A"
```
⟨*proof*⟩

lemma *AC2_AC1_aux3:*
```
     "∀D ∈ {E*{E}. E ∈ A}. ∃y. D ∩ C = {y}
      ==> (λx ∈ A. fst(THE z. (x*{x} ∩ C = {z}))) ∈ (∏X ∈ A. X)"
```
⟨*proof*⟩

lemma *AC2_AC1:* `"AC2 ==> AC1"`
⟨*proof*⟩

lemma *empty_notin_images:* `"0 ∉ {R``{x}. x ∈ domain(R)}"`
⟨*proof*⟩

lemma *AC1_AC4:* `"AC1 ==> AC4"`
⟨*proof*⟩

lemma *AC4_AC3_aux1:* `"f ∈ A->B ==> (⋃z ∈ A. {z}*f`z) ⊆ A*⋃(B)"`
⟨*proof*⟩

lemma *AC4_AC3_aux2:* `"domain(⋃z ∈ A. {z}*f(z)) = {a ∈ A. f(a)≠0}"`
⟨*proof*⟩

lemma *AC4_AC3_aux3: "x ∈ A ==> (⋃z ∈ A. {z}*f(z))``{x} = f(x)"*
⟨*proof*⟩

lemma *AC4_AC3: "AC4 ==> AC3"*
⟨*proof*⟩

lemma *AC3_AC1_lemma:*
"b∉A ==> (∏x ∈ {a ∈ A. id(A)`a≠b}. id(A)`x) = (∏x ∈ A. x)"
⟨*proof*⟩

lemma *AC3_AC1: "AC3 ==> AC1"*
⟨*proof*⟩

lemma *AC4_AC5: "AC4 ==> AC5"*
⟨*proof*⟩

lemma *AC5_AC4_aux1: "R ⊆ A*B ==> (λx ∈ R. fst(x)) ∈ R -> A"*
⟨*proof*⟩

lemma *AC5_AC4_aux2: "R ⊆ A*B ==> range(λx ∈ R. fst(x)) = domain(R)"*
⟨*proof*⟩

lemma *AC5_AC4_aux3: "[| ∃f ∈ A->C. P(f,domain(f)); A=B |] ==> ∃f ∈ B->C. P(f,B)"*
⟨*proof*⟩

lemma *AC5_AC4_aux4: "[| R ⊆ A*B; g ∈ C->R; ∀x ∈ C. (λz ∈ R. fst(z))` (g`x) = x |]*
==> (λx ∈ C. snd(g`x)): (∏x ∈ C. R``{x})"
⟨*proof*⟩

lemma *AC5_AC4: "AC5 ==> AC4"*
⟨*proof*⟩

lemma *AC1_iff_AC6: "AC1 ⟷ AC6"*
⟨*proof*⟩

end

theory *AC18_AC19*
imports *AC_Equiv*
begin

definition
  uu   :: "i => i" **where**
    "uu(a) == {c ∪ {0}. c ∈ a}"

lemma *PROD_subsets:*
      "[| f ∈ (∏ b ∈ {P(a). a ∈ A}. b); ∀ a ∈ A. P(a)<=Q(a) |]
       ==> (λa ∈ A. f‘P(a)) ∈ (∏ a ∈ A. Q(a))"
⟨*proof*⟩

lemma *lemma_AC18:*
      "[| ∀ A. 0 ∉ A ⟶ (∃ f. f ∈ (∏ X ∈ A. X)); A ≠ 0 |]
       ==> (⋂ a ∈ A. ⋃ b ∈ B(a). X(a, b)) ⊆
             (⋃ f ∈ ∏ a ∈ A. B(a). ⋂ a ∈ A. X(a, f‘a))"
⟨*proof*⟩

lemma *AC1_AC18: "AC1 ==> PROP AC18"*
⟨*proof*⟩

theorem **(in** *AC18)* *AC19*
⟨*proof*⟩

lemma *RepRep_conj:*

"[| A ≠ 0; 0 ∉ A |] ==> {uu(a). a ∈ A} ≠ 0 & 0 ∉ {uu(a). a ∈ A}"
⟨proof⟩

lemma *lemma1_1:* "[|c ∈ a; x = c ∪ {0}; x ∉ a |] ==> x - {0} ∈ a"
⟨proof⟩

lemma *lemma1_2:*
"[| f‘(uu(a)) ∉ a; f ∈ (∏B ∈ {uu(a). a ∈ A}. B); a ∈ A |]
==> f‘(uu(a))-{0} ∈ a"
⟨proof⟩

lemma *lemma1:* "∃f. f ∈ (∏B ∈ {uu(a). a ∈ A}. B) ==> ∃f. f ∈ (∏B ∈ A. B)"
⟨proof⟩

lemma *lemma2_1:* "a≠0 ==> 0 ∈ (⋃b ∈ uu(a). b)"
⟨proof⟩

lemma *lemma2:* "[| A≠0; 0∉A |] ==> (⋂x ∈ {uu(a). a ∈ A}. ⋃b ∈ x. b) ≠ 0"
⟨proof⟩

lemma *AC19_AC1:* "AC19 ==> AC1"
⟨proof⟩

end

theory *DC*
imports *AC_Equiv Hartog Cardinal_aux*
begin

lemma *RepFun_lepoll:* "Ord(a) ==> {P(b). b ∈ a} ≲ a"
⟨proof⟩

Trivial in the presence of AC, but here we need a wellordering of X

lemma *image_Ord_lepoll:* "[| f ∈ X->Y; Ord(X) |] ==> f‘‘X ≲ X"
⟨proof⟩

lemma *range_subset_domain:*
"[| R ⊆ X*X; !!g. g ∈ X ==> ∃u. <g,u> ∈ R |]
==> range(R) ⊆ domain(R)"
⟨proof⟩

lemma *cons_fun_type:* "g ∈ n->X ==> cons(<n,x>, g) ∈ succ(n) -> cons(x, X)"
⟨proof⟩

```
lemma cons_fun_type2:
    "[| g ∈ n->X; x ∈ X |] ==> cons(<n,x>, g) ∈ succ(n) -> X"
⟨proof⟩

lemma cons_image_n: "n ∈ nat ==> cons(<n,x>, g)``n = g``n"
⟨proof⟩

lemma cons_val_n: "g ∈ n->X ==> cons(<n,x>, g)`n = x"
⟨proof⟩

lemma cons_image_k: "k ∈ n ==> cons(<n,x>, g)``k = g``k"
⟨proof⟩

lemma cons_val_k: "[| k ∈ n; g ∈ n->X |] ==> cons(<n,x>, g)`k = g`k"
⟨proof⟩

lemma domain_cons_eq_succ: "domain(f)=x ==> domain(cons(<x,y>, f)) =
succ(x)"
⟨proof⟩

lemma restrict_cons_eq: "g ∈ n->X ==> restrict(cons(<n,x>, g), n) =
g"
⟨proof⟩

lemma succ_in_succ: "[| Ord(k); i ∈ k |] ==> succ(i) ∈ succ(k)"
⟨proof⟩

lemma restrict_eq_imp_val_eq:
     "[|restrict(f, domain(g)) = g; x ∈ domain(g)|]
      ==> f`x = g`x"
⟨proof⟩

lemma domain_eq_imp_fun_type: "[| domain(f)=A; f ∈ B->C |] ==> f ∈ A->C"
⟨proof⟩

lemma ex_in_domain: "[| R ⊆ A * B; R ≠ 0 |] ==> ∃x. x ∈ domain(R)"
⟨proof⟩


definition
  DC  :: "i => o"  where
    "DC(a) == ∀X R. R ⊆ Pow(X)*X  &
                    (∀Y ∈ Pow(X). Y ≺ a ⟶ (∃x ∈ X. <Y,x> ∈ R))
                    ⟶ (∃f ∈ a->X. ∀b<a. <f``b,f`b> ∈ R)"

definition
  DC0 :: o  where
    "DC0 == ∀A B R. R ⊆ A*B & R≠0 & range(R) ⊆ domain(R)
                    ⟶ (∃f ∈ nat->domain(R). ∀n ∈ nat. <f`n,f`succ(n)>:R)"
```

```
definition
  ff  :: "[i, i, i, i] => i"  where
    "ff(b, X, Q, R) ==
            transrec(b, %c r. THE x. first(x, {x ∈ X. <r``c, x> ∈ R},
Q))"


locale DC0_imp =
  fixes XX and RR and X and R

  assumes all_ex: "∀Y ∈ Pow(X). Y ≺ nat ⟶ (∃x ∈ X. <Y, x> ∈ R)"

  defines XX_def: "XX == (⋃n ∈ nat. {f ∈ n->X. ∀k ∈ n. <f``k, f`k> ∈
R})"
     and RR_def:  "RR == {<z1,z2>:XX*XX. domain(z2)=succ(domain(z1))
                                  & restrict(z2, domain(z1)) = z1}"
begin
```

```
lemma lemma1_1: "RR ⊆ XX*XX"
⟨proof⟩

lemma lemma1_2: "RR ≠ 0"
```

```
⟨proof⟩

lemma lemma1_3: "range(RR) ⊆ domain(RR)"
⟨proof⟩

lemma lemma2:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  n ∈ nat |]

      ==> ∃k ∈ nat. f‘succ(n) ∈ k -> X & n ∈ k
                  & <f‘succ(n)‘‘n, f‘succ(n)‘n> ∈ R"
⟨proof⟩

lemma lemma3_1:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]

      ==>  {f‘succ(x)‘x. x ∈ m} = {f‘succ(m)‘x. x ∈ m}"
⟨proof⟩

lemma lemma3:
     "[| ∀n ∈ nat. <f‘n, f‘succ(n)> ∈ RR;  f ∈ nat -> XX;  m ∈ nat |]

      ==> (λx ∈ nat. f‘succ(x)‘x) ‘‘ m = f‘succ(m)‘‘m"
⟨proof⟩

end

theorem DC0_imp_DC_nat: "DC0 ==> DC(nat)"
⟨proof⟩




lemma singleton_in_funs:
 "x ∈ X ==> {<0,x>} ∈
           (⋃n ∈ nat. {f ∈ succ(n)->X. ∀k ∈ n. <f‘k, f‘succ(k)> ∈
R})"
⟨proof⟩


locale imp_DC0 =
  fixes XX and RR and x and R and f and allRR
  defines XX_def: "XX == (⋃n ∈ nat.
                    {f ∈ succ(n)->domain(R). ∀k ∈ n. <f‘k, f‘succ(k)>
∈ R})"
      and RR_def:
         "RR == {<z1,z2>:Fin(XX)*XX.
                  (domain(z2)=succ(⋃f ∈ z1. domain(f))
                    & (∀f ∈ z1. restrict(z2, domain(f)) = f))
                  | (~ (∃g ∈ XX. domain(g)=succ(⋃f ∈ z1. domain(f))
```

```
                    & (∀f ∈ z1. restrict(g, domain(f)) = f)) & z2={<0,x>})}"
      and allRR_def:
        "allRR == ∀b<nat.
                   <f``b, f`b> ∈
                    {<z1,z2>∈Fin(XX)*XX. (domain(z2)=succ(⋃f ∈ z1. domain(f))
                                & (⋃f ∈ z1. domain(f)) = b
                                & (∀f ∈ z1. restrict(z2,domain(f))
= f))}"
begin
```

```
lemma lemma4:
     "[| range(R) ⊆ domain(R);  x ∈ domain(R) |]
      ==> RR ⊆ Pow(XX)*XX &
             (∀Y ∈ Pow(XX). Y ≺ nat ⟶ (∃x ∈ XX. <Y,x>:RR))"
⟨proof⟩
```

```
lemma UN_image_succ_eq:
     "[| f ∈ nat->X; n ∈ nat |]
      ==> (⋃x ∈ f``succ(n). P(x)) =  P(f`n) ∪ (⋃x ∈ f``n. P(x))"
⟨proof⟩
```

```
lemma UN_image_succ_eq_succ:
     "[| (⋃x ∈ f``n. P(x)) = y; P(f`n) = succ(y);
         f ∈ nat -> X; n ∈ nat |] ==> (⋃x ∈ f``succ(n). P(x)) = succ(y)"
⟨proof⟩
```

```
lemma apply_domain_type:
     "[| h ∈ succ(n) -> D;  n ∈ nat; domain(h)=succ(y) |] ==> h`y ∈ D"
⟨proof⟩
```

```
lemma image_fun_succ:
     "[| h ∈ nat -> X; n ∈ nat |] ==> h``succ(n) = cons(h`n, h``n)"
⟨proof⟩
```

```
lemma f_n_type:
     "[| domain(f`n) = succ(k); f ∈ nat -> XX;  n ∈ nat |]
      ==> f`n ∈ succ(k) -> domain(R)"
⟨proof⟩
```

```
lemma f_n_pairs_in_R [rule_format]:
     "[| h ∈ nat -> XX;  domain(h`n) = succ(k);  n ∈ nat |]
      ==> ∀i ∈ k. <h`n`i, h`n`succ(i)> ∈ R"
⟨proof⟩
```

```
lemma restrict_cons_eq_restrict:
     "[| restrict(h, domain(u))=u;  h ∈ n->X;  domain(u) ⊆ n |]
      ==> restrict(cons(<n, y>, h), domain(u)) = u"
⟨proof⟩
```

```
lemma all_in_image_restrict_eq:
     "[| ∀x ∈ f``n. restrict(f`n, domain(x))=x;
         f ∈ nat -> XX;
         n ∈ nat;  domain(f`n) = succ(n);
         (⋃x ∈ f``n. domain(x)) ⊆ n |]
      ==> ∀x ∈ f``succ(n). restrict(cons(<succ(n),y>, f`n), domain(x))
= x"
⟨proof⟩
```

```
lemma simplify_recursion:
     "[| ∀b<nat. <f``b, f`b> ∈ RR;
         f ∈ nat -> XX; range(R) ⊆ domain(R); x ∈ domain(R)|]
      ==> allRR"
⟨proof⟩
```

```
lemma lemma2:
     "[| allRR; f ∈ nat->XX; range(R) ⊆ domain(R); x ∈ domain(R); n ∈
nat |]
      ==> f`n ∈ succ(n) -> domain(R) & (∀i ∈ n. <f`n`i, f`n`succ(i)>:R)"
⟨proof⟩
```

```
lemma lemma3:
     "[| allRR; f ∈ nat->XX; n∈nat; range(R) ⊆ domain(R);  x ∈ domain(R)
|]
      ==> f`n`n = f`succ(n)`n"
⟨proof⟩
```

end

```
theorem DC_nat_imp_DC0: "DC(nat) ==> DC0"
⟨proof⟩
```

```
lemma fun_Ord_inj:
      "[| f ∈ a->X;  Ord(a);
          !!b c. [| b<c; c ∈ a |] ==> f`b≠f`c |]
       ==> f ∈ inj(a, X)"
⟨proof⟩
```

```
lemma value_in_image: "[| f ∈ X->Y; A ⊆ X; a ∈ A |] ==> f`a ∈ f``A"
⟨proof⟩
```

```
lemma lesspoll_lemma: "[| ~ A ≺ B; C ≺ B |] ==> A - C ≠ 0"
```

⟨*proof*⟩

theorem *DC_WO3: "(∀K. Card(K) ⟶ DC(K)) ==> WO3"*
⟨*proof*⟩

lemma *images_eq:*
```
"[| ∀x ∈ A. f`x=g`x; f ∈ Df->Cf; g ∈ Dg->Cg; A ⊆ Df; A ⊆ Dg |]

  ==> f``A = g``A"
```
⟨*proof*⟩

lemma *lam_images_eq:*
```
"[| Ord(a); b ∈ a |] ==> (λx ∈ a. h(x))``b = (λx ∈ b. h(x))``b"
```
⟨*proof*⟩

lemma *lam_type_RepFun: "(λb ∈ a. h(b)) ∈ a -> {h(b). b ∈ a}"*
⟨*proof*⟩

lemma *lemmaX:*
```
"[| ∀Y ∈ Pow(X). Y ≺ K ⟶ (∃x ∈ X. <Y, x> ∈ R);
    b ∈ K; Z ∈ Pow(X); Z ≺ K |]
 ==> {x ∈ X. <Z,x> ∈ R} ≠ 0"
```
⟨*proof*⟩

lemma *WO1_DC_lemma:*
```
"[| Card(K); well_ord(X,Q);
    ∀Y ∈ Pow(X). Y ≺ K ⟶ (∃x ∈ X. <Y, x> ∈ R); b ∈ K |]
 ==> ff(b, X, Q, R) ∈ {x ∈ X. <(λc ∈ b. ff(c, X, Q, R))``b, x> ∈
R}"
```
⟨*proof*⟩

theorem *WO1_DC_Card: "WO1 ==> ∀K. Card(K) ⟶ DC(K)"*
⟨*proof*⟩

end

References

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