The Existential Uniqueness Quantifier

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Existential uniqueness; "at most one"

Unless otherwise specified, φ has free variable x (and no others).

"There exists exactly one x such that φ"

$$\exists! x \varphi \stackrel{\text{def}}{\leftrightarrow} \exists y \forall x (\varphi \leftrightarrow x = y) \tag{1}$$

"There exists at most one x such that φ"

$$\exists^* x \varphi \stackrel{\text{def}}{\leftrightarrow} \exists y \forall x (\varphi \rightarrow x = y) \tag{2}$$

Other definitions for existential uniqueness

$$\begin{aligned}
\exists! x\varphi &\leftrightarrow \exists x(\varphi \land \forall y([y/x]\varphi \to x = y)) && (3)\\
&\leftrightarrow (\exists x\varphi \land \forall x \forall y((\varphi \land [y/x]\varphi) \to x = y)) && (4)\\
&\leftrightarrow (\exists x\varphi \land \exists y \forall x(\varphi \to x = y)) && (5)\\
&\leftrightarrow (\exists x\varphi \land \exists^* x\varphi) && (6)
\end{aligned}$$

Other definitions for “at most one”

$$\begin{aligned}
\exists^* x\varphi &\leftrightarrow (\exists x\varphi \to \exists! x\varphi) && (7)\\
&\leftrightarrow \forall x \forall y((\varphi \land [y/x]\varphi) \to x = y) && (8)
\end{aligned}$$

Double existential uniqueness

Assume φ has free variables x and y. An idiom frequently used in literature is $\exists ! x \exists ! y \varphi$ to denote "there exists exactly one x and exactly one y such that φ is true." But formally it is false:

$$\nvdash \quad \exists ! x \exists ! y \varphi \leftrightarrow (\exists x \exists y \varphi \wedge \exists z \exists w \forall x \forall y (\varphi \rightarrow (x = z \wedge y = w))) \tag{9}$$

However, we do have the following equivalences:

$$\begin{aligned}
&(\exists x \exists y \varphi \wedge \exists z \exists w \forall x \forall y (\varphi \rightarrow (x = z \wedge y = w))) \\
&\leftrightarrow \quad \exists ! x \exists ! y \varphi \wedge \forall x \exists^* y \varphi) && (10) \\
&\leftrightarrow \quad \exists z \exists w \forall x \forall y (\varphi \leftrightarrow (x = z \wedge y = w)) && (11) \\
&\leftrightarrow \quad \exists ! x \exists ! y (\exists x \varphi \wedge \exists y \varphi) && (12) \\
&\leftrightarrow \quad \exists ! x \exists ! y (\exists ! x \varphi \wedge \exists y \varphi) && (13) \\
&\leftrightarrow \quad (\exists ! x \exists y \varphi \wedge \exists ! y \exists x \varphi) && (14)
\end{aligned}$$

Uniqueness theorems (1 of 3)

Assume that φ and ψ have x free and that χ does not have x free.

$$
\begin{aligned}
(\neg\chi \wedge \exists! x\psi) &\rightarrow \exists! x(\chi \vee \psi) && (15)\\
\exists x\varphi &\leftrightarrow (\exists^* x\varphi \rightarrow \exists! x\varphi) && (16)\\
(\forall x(\varphi \rightarrow \psi) &\rightarrow (\exists^* x\psi \rightarrow \exists^* x\varphi) && (17)\\
\exists^* x(\chi \rightarrow \psi) &\rightarrow (\chi \rightarrow \exists^* x\psi) && (18)\\
\forall x(\varphi \rightarrow \psi) &\rightarrow (\exists! x\psi \rightarrow \exists^* x\varphi) && (19)\\
\exists^* x\varphi &\rightarrow \exists^* x(\psi \wedge \varphi) && (20)\\
\exists^* x(\varphi \vee \psi) &\rightarrow \exists^* x\varphi && (21)\\
(\exists^* x\varphi \vee \exists^* x\psi) &\rightarrow \exists^* x(\varphi \wedge \psi) && (22)\\
\exists^* x(\varphi \vee \psi) &\rightarrow (\exists^* x\varphi \wedge \exists^* x\psi) && (23)
\end{aligned}
$$

Uniqueness theorems (2 of 3)

Assume that φ and ψ have x free and that χ does not have x free.

$$\exists^* x(\chi \wedge \psi) \leftrightarrow (\chi \rightarrow \exists^* x\psi) \tag{24}$$

$$\exists! x(\chi \wedge \psi) \leftrightarrow (\chi \wedge \exists! x\psi) \tag{25}$$

$$(\exists^* x\varphi \wedge \exists x(\varphi \wedge \psi)) \rightarrow (\varphi \rightarrow \psi) \tag{26}$$

$$(\exists! x\varphi \wedge \exists x(\varphi \wedge \psi)) \rightarrow (\varphi \rightarrow \psi) \tag{27}$$

$$(\exists! x\varphi \wedge \exists! x\psi \wedge \exists x(\varphi \wedge \psi)) \rightarrow (\varphi \leftrightarrow \psi) \tag{28}$$

$$\neg \exists! x\, x = x \leftrightarrow \neg \forall x\, x = y \tag{29}$$

$$(\exists x\varphi \wedge \exists x \neg \varphi) \rightarrow \neg \exists! x\, x = x \tag{30}$$

Uniqueness theorems (3 of 3)

Assume that φ has x and y free and that ψ has x but not y free.

$$(\exists^* x\psi \wedge \forall x\exists^* y\varphi) \;\rightarrow\; \exists^* y\exists x(\psi \wedge \varphi) \tag{31}$$

$$\exists^* x\exists y\varphi \;\rightarrow\; \forall y\exists^* x\varphi \tag{32}$$

$$\exists! x\exists y\varphi \;\rightarrow\; \exists y\exists! x\varphi \tag{33}$$

$$\exists! x\exists^* y\varphi \;\rightarrow\; \exists^* x\exists! y\varphi \tag{34}$$

$$\exists! x\exists! y\varphi \;\rightarrow\; \exists x\exists y\varphi \tag{35}$$

$$\forall x\exists^* y\varphi \;\rightarrow\; (\exists^* x\exists y\varphi \rightarrow \exists^* y\exists x\varphi) \tag{36}$$

$$\forall x\exists^* y\varphi \;\rightarrow\; (\exists! x\exists y\varphi \rightarrow \exists! y\exists x\varphi) \tag{37}$$

$$(\exists! x\exists y\varphi \wedge \exists! y\exists x\varphi) \;\rightarrow\; \exists! x\exists! y\varphi \tag{38}$$

Open Problem (?)

Is there a "finite" axiomatization (i.e. a finite number of axiom schemes) for extending predicate calculus (without equality) with $\exists!$, so that all theorems involving $\exists!$ but not involving equality can be proved?

Appendix - Equation references

The following list provides the hyperlinks to the formal proofs for most of the theorems.

Eq. 1—`http://us.metamath.org/mpegif/df-eu.html`
Eq. 2—`http://us.metamath.org/mpegif/mo2.html`
Eq. 3—`http://us.metamath.org/mpegif/eu1.html`
Eq. 4—`http://us.metamath.org/mpegif/eu2.html`
Eq. 5—`http://us.metamath.org/mpegif/eu3.html`
Eq. 6—`http://us.metamath.org/mpegif/eu5.html`
Eq. 7—`http://us.metamath.org/mpegif/df-mo.html`
Eq. 8—`http://us.metamath.org/mpegif/mo3.html`
Eq. 10—`http://us.metamath.org/mpegif/2eu5.html`
Eq. 11—`http://us.metamath.org/mpegif/2eu6.html`

Eq. 12—http://us.metamath.org/mpegif/2eu7.html
Eq. 13—http://us.metamath.org/mpegif/2eu8.html
Eq. 14—http://us.metamath.org/mpegif/2eu4.html
Eq. 15—http://us.metamath.org/mpegif/euorv.html
Eq. 16—http://us.metamath.org/mpegif/exmoeu.html
Eq. 17—http://us.metamath.org/mpegif/immo.html
Eq. 18—http://us.metamath.org/mpegif/moimv.html
Eq. 19—http://us.metamath.org/mpegif/euimmo.html
Eq. 20—http://us.metamath.org/mpegif/moan.html
Eq. 21—http://us.metamath.org/mpegif/moor.html
Eq. 22—http://us.metamath.org/mpegif/mooran1.html
Eq. 23—http://us.metamath.org/mpegif/mooran2.html
Eq. 24—http://us.metamath.org/mpegif/moanimv.html
Eq. 25—http://us.metamath.org/mpegif/euanv.html
Eq. 26—http://us.metamath.org/mpegif/mopick.html
Eq. 27—http://us.metamath.org/mpegif/eupick.html

Eq. 28—http://us.metamath.org/mpegif/eupickb.html

Eq. 29—http://us.metamath.org/mpegif/exists1.html

Eq. 30—http://us.metamath.org/mpegif/exists2.html

Eq. 31—http://us.metamath.org/mpegif/moexexv.html

Eq. 32—http://us.metamath.org/mpegif/2moex.html

Eq. 33—http://us.metamath.org/mpegif/2euex.html

Eq. 34—http://us.metamath.org/mpegif/2eumo.html

Eq. 35—http://us.metamath.org/mpegif/2eu2ex.html

Eq. 36—http://us.metamath.org/mpegif/2moswap.html

Eq. 37—http://us.metamath.org/mpegif/2euswap.html

Eq. 38—http://us.metamath.org/mpegif/2exeu.html