Reducing the Key Size of Rainbow using Non-Commutative Rings

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MQ problem and key size

- **MQ equations**

$$\begin{cases} f_1(x_1,\ldots,x_n) = \sum_{1\le i,j\le n} a_{ij}^{(1)} x_i x_j + \sum_{1\le i\le n} b_i^{(1)} x_i + c^{(1)} = d_1 \\ f_2(x_1,\ldots,x_n) = \sum_{1\le i,j\le n} a_{ij}^{(2)} x_i x_j + \sum_{1\le i\le n} b_i^{(2)} x_i + c^{(2)} = d_2 \\ \vdots \\ f_m(x_1,\ldots,x_n) = \sum_{1\le i,j\le n} a_{ij}^{(m)} x_i x_j + \sum_{1\le i\le n} b_i^{(m)} x_i + c^{(m)} = d_m \end{cases}$$

- **Public key = set of coefficients of polynomials**

$$\textit{the number of coefficients} = \frac{m(n+1)(n+2)}{2} \Rightarrow$$ large key size

Rainbow

- Signature of a multilayer variant of UOV
- Secret key size is also large

We reduced by 75%

?

Rainbow	Secret key size	Public key size	Key size of RSA	Ratio (secret key)
R(17,13,13)	19.1kB	25.7kB	1369bits	116.6times
R(21,16,17)	36.5kB	50.8kB	1937bits	150.7times
R(27,19,19)	60.5kB	84.0kB	2560bits	189.0times

Reference: A.Petzoldt et al. "Selecting Parameters for the Rainbow Signature Scheme", PQCrypto'10, Springer LNCS vol. 6061 (2010)

Reduction by 62%

- Problem: reduction of key size
  - reduction of public key
    CyclicRainbow (INDOCRIPT' 10, SCC' 10, PKC' 11)
  - reduction of secret key TTS, TRMS

Proposed scheme
Rainbow using non-commutative rings

$$\mathrm{NC\text{-}Rainbow}(R; \tilde{v}_1, \tilde{o}_1, \tilde{o}_2, \ldots, \tilde{o}_s)$$

Non-commutative rings

- **R: non-commutative ring**
  - R: finite dimensional algebra over a finite field K (dimension=r)

  Fix a K-linear isomorphism $\phi : K^r \xrightarrow{\sim} R$

- **Example (quaternion algebra** Q_q (q: order of K))

$$\begin{cases} set) \quad Q_q = K \cdot 1 \oplus K \cdot i \oplus K \cdot j \oplus K \cdot ij, & (r = 4), \\ product) \quad i^2 = j^2 = -1, \quad ij = -ji. \end{cases}$$

There is a natural isomorphism $\phi : K^4 \xrightarrow{\sim} Q_q$

NC-Rainbow$(R;\tilde{v}_1,\tilde{o}_1,\tilde{o}_2,\ldots,\tilde{o}_s)$ (1/2)

$\tilde{n}$: positive number

$$0<\tilde{v}_1<\tilde{v}_2<\cdots<\tilde{v}_s<\tilde{v}_{s+1}=\tilde{n}$$

For $i = 1, \ldots, s$

- $\tilde{S}_i=\{1,\ldots,\tilde{v}_i\},\ \tilde{O}_i=\{\tilde{v}_i+1,\ldots,\tilde{v}_{i+1}\},$
- $\tilde{o}_i=\tilde{v}_{i+1}-\tilde{v}_i.$

$\boxed{\text{Central map}}$ $\tilde{G}=(\tilde{g}_{\tilde{v}_1+1},\ldots,\tilde{g}_n):R^{\tilde{n}}\to R^{\tilde{m}},\quad (\tilde{m}=\tilde{n}-\tilde{v}_1)$

$$\begin{aligned}\tilde{g}_k(x_1,\ldots,x_n)=&\sum_{i\in\tilde{O}_h,\,j\in\tilde{S}_h}(x_i\alpha_{ij}^{(k)}x_j+x_j\alpha_{ij}^{(k)}x_i)\\&+\sum_{i,j\in\tilde{S}_h}x_i\beta_{ij}^{(k)}x_j+\sum_{i\in\tilde{S}_{h+1}}(\gamma_i^{(k,1)}x_i+x_i\gamma_i^{(k,2)})\\&+\eta^{(k)}\qquad(k=\tilde{v}_1+1,\ldots,\tilde{n},\ \ \alpha_{ij}^{(k)},\ldots\in R).\end{aligned}$$

NC-Rainbow$(R;\tilde{v}_1,\tilde{o}_1,\tilde{o}_2,\ldots,\tilde{o}_s)$ (2/2)

- **Key Generation**
  - Secret key $(n=\tilde{n}r, m=\tilde{m}r)$
    - $\tilde{G}$, two affine transformations $A_1: K^m \to K^m$, $A_2: K^n \to K^n$.
  - Public key
    - $\tilde{F} = A_1 \circ \phi^{-\tilde{m}} \circ \tilde{G} \circ \phi^{\tilde{n}} \circ A_2 : K^n \to K^m$.
- **Signature Generation**

  For message $M \in K^m$, calculate

  $$(1)\ a=\phi^{\tilde{m}}(A_1^{-1}(M)),\ (2)\ b=\tilde{G}^{-1}(a),\ (3)\ c=\phi^{-\tilde{n}}(A_2^{-1}(b))$$

  in this order. *c* is a signature.
- **Verification**
  - If $\tilde{F}(c)=M$, the signature is accepted.

Original Rainbow

If *R=K* , then this becomes $\boxed{\mathrm{Rainbow}(K;\tilde{v}_1,\tilde{o}_1,\tilde{o}_2,\ldots,\tilde{o}_s)}$

Correspondence between NC-Rainbow and Rainbow

Theorem

There exists a correspondence

$$\text{NC-Rainbow}(R; \tilde{v}_1, \tilde{o}_1, \tilde{o}_2, \ldots, \tilde{o}_s)$$
$$\to \text{Rainbow}(K; r\tilde{v}_1, r\tilde{o}_1, r\tilde{o}_2, \ldots, r\tilde{o}_s)$$

which holds public key.

- **Secret key size of NC-Rainbow**

$$m(m+1) + n(n+1) + \sum_{h=1}^{s} r\tilde{o}_h (2\tilde{v}_h \tilde{o}_h + \tilde{v}_h^2 + 2\tilde{v}_{h+1} + 1) \text{ field elements}$$

- **Secret key size of corresponding Rainbow**

$$m(m+1) + n(n+1) + \sum_{h=1}^{s} r\tilde{o}_h \left(r^2 \tilde{v}_h \tilde{o}_h + \frac{r\tilde{v}_h (r\tilde{v}_h + 1)}{2} + r\tilde{v}_{h+1} + 1 \right) \text{ field elements}$$

Comparison of Secret key size

$K = GF(256)$,

$R = Q_{256}$ $(r = 4)$.

Comparison of NC-Rainbow$(Q_{256}; \tilde{v}_1, \tilde{o}_1, \tilde{o}_2)$ *and* Rainbow$(GF(256); 4\tilde{v}_1, 4\tilde{o}_1, 4\tilde{o}_2)$

$(\tilde{v}_1, \tilde{o}_1, \tilde{o}_2)$	NC-size	Corr. Rainbow	R-size	ratio
(4,3,3)	4.2kB	(16,12,12)	15.9kB	26.7%
(5,4,4)	8.0kB	(20,16,16)	33.6kB	23.9%
(7,5,5)	15.1kB	(28,20,20)	70.7kB	21.5%
(9,6,6)	25.5kB	(36,24,24)	128.2kB	19.9%

NC-size : Secret key size of NC-Rainbow

R-size : Secret key size of corresponding Rainbow

ratio = NC-size/R-size

Reason of reduction of key size

- **Property of "regular action"**
  - R is expressed by a subring of matrix algebra of size r.

$$Q_q \ni (\square \quad \square \quad \square \quad \square) \longrightarrow \begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \in M(4, K)$$

4 entries 16 entries

$$M(d, Q_q) \longrightarrow M(4d, K)$$

$4d^2$ entries $16d^2$ entries

$$\mathrm{NC\text{-}Rainbow}(Q_q; \tilde{v}_1, \tilde{o}_1, \tilde{o}_2, \ldots, \tilde{o}_s)$$

(Map in Theorem) $\rightarrow \mathrm{Rainbow}(K; 4\tilde{v}_1, 4\tilde{o}_1, 4\tilde{o}_2, \ldots, 4\tilde{o}_s)$

Attacks against Rainbow (1/2)

Need to analyze attacks against Rainbow to know whether or not they work efficiently against NC-Rainbow.

- **Known attacks against Rainbow**
  - Direct attacks
    - Using XL and Grobner basis algorithm etc.
  - UOV attack
    - determine a simultaneous isotropic subspace (which coincides with Oil space in the last layer with high probability)
    - compute invariant spaces of certain matrices
  - UOV-Reconciliation(UOV-R) attack
    - determine a simultaneous isotropic subspace (which coincides with Oil space in the last layer with high probability)
    - Solve a system of equations w.r.t. coefficients of right affine transformation

Attacks against Rainbow (2/2)

- **Known attacks against Rainbow (continued)**
  - MinRank attack
    - determine a matrix with minimal rank among linear combinations of quadratic part of components of public key
  - HighRank attack
    - determine a matrix with the second highest rank among linear combinations of quadratic part of components of public key
  - Rainbow-Band-Separation(RBS) attack
    - transform public key to a form of central map of Rainbow
    - Solve a system of equations w.r.t. coefficients of both affine transformations

Security for NC−Rainbow

l −bit security　(K=GF(2^a))

1. UOV attack

$$n - 2r\tilde{o}_s \geq l/a + 1$$

2. MinRank attack

$$r(\tilde{o}_1 + \tilde{v}_1) \geq l/a$$

3. HighRank attack

$$r\tilde{o}_s \geq l/a$$

4. UOV−R attack

$\tilde{v}_1 \geq \tilde{o}_1 \Rightarrow$ same security level against direct attacks

5. Direct attacks, RBS attack

A.Petzoldt et al. "Selecting Parameters for the Rainbow Signature Scheme", PQCrypto'10, Springer LNCS vol. 6061 (2010)

Table of security and secret key size

$$\text{NC-Rainbow}(Q_{256}; \tilde{v}_1, \tilde{o}_1, \tilde{o}_2)\ \textit{and}\ \text{Rainbow}(GF(256); 4\tilde{v}_1, 4\tilde{o}_1, 4\tilde{o}_2)$$

NC-Rainbow	$(5,4,4)$	$(7,5,5)$	$(9,6,6)$
Security level	83bits	96bits	107bits
Secret key size	8.0kB	15.1kB	25.5kB
Corr. Rainbow	(20,16,16)	(28,20,20)	(36,24,24)
Secret key size	33.6kB	70.7kB	128.2kB
Ratio	23.9%	21.5%	19.9%

ratio＝Secret key size of NCRainbow/Secret key size of corr. Rainbow

Conclusion

- **Conclusion**
  - We proposed a scheme using non-commutative rings, which is regarded as another construction of Rainbow.
  - This scheme can reduce the secret key size in comparison with original Rainbow.
  - In paticular, the secret key size of the proposed NC-Rainbow is reduced by about 75% in the security level of 80 bits.

- **Future works**
  - Finding a non-commutative ring with efficient arithmetic operation.
    - ⇒ Speed up the signature generation

Dual Exponentiation Schemes

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The Problem

- *Motivation*: New algorithms are always useful as there are always so many different optimisations and conflicting pressures on resource-constrained platforms.
- *Aim*: Better exponentiation on space-limited chip. (Fast memory is expensive.)
- *Setting*: Mixed base representation for the exponent.
- *Solution*: Define a *dual* for the associated addition chain.
- *Benefits*: Derive new algorithms from existing ones; Better understanding of exponentiation.

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Outline

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Background

r-ary Exponentiation — L2R (Brauer, 1939)

Inputs: $g \in G$,
$D = ((d_{n-1}r+d_{n-2})r+\ldots+d_1)r+d_0 \in \mathbb{N}$ where $0 \le d_i < r$.

Output: $g^D \in G$

Initialise table: $T[d] \leftarrow g^d$ **for all** d, $0<d<r$.
$P \leftarrow 1_G$
for $i \leftarrow n-1$ **downto** 0 **do** {
  if $i \neq n-1$ **then** $P \leftarrow P^r$
  if $d_i \neq 0$ **then** $P \leftarrow P \times T[d_i]$ }
return P

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r-ary Exponentiation — R2L (Yao, 1976)

Inputs: $g \in G$,
$D = d_{n-1}r^{n-1} + d_{n-2}r^{n-2} + \ldots + d_1 r^1 + d_0$ where $0 \le d_i < r$.
Output: $g^D \in G$

Initialise table: $T[d] \leftarrow 1_G$ **for all** d, $0<d<r$.
$P \leftarrow g$
for $i \leftarrow 0$ **to** $n-1$ **do** $\{$
  if $d_i \neq 0$ **then** $T[d_i] \leftarrow T[d_i] \times P$
  if $i \neq n-1$ **then** $P \leftarrow P^r$ $\}$
return $\prod_{d:0<d<r} T[d]^d$

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Sliding Window — L2R

Inputs: $g \in G$,
$D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$, where
$d_i \in \{0, \pm1, \pm3, \ldots, \pm\frac{1}{2}(r-1)\}$, $r_i \in \{2, 2^w\}$ and $d_i=0$ if $r_i=2$.

Output: $g^D \in G$

Initialise table: $T[d] \leftarrow g^d$ **for all** $d \neq 0$.
$P \leftarrow 1_G$
for $i \leftarrow n-1$ **downto** 0 **do** {
  if $i \neq n-1$ **then** $P \leftarrow P^{r_i}$
  if $d_i \neq 0$ **then** $P \leftarrow P \times T[d_i]$ }
return P

Sliding Window — R2L

Inputs: $g \in G$,
$D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$, where
$d_i \in \{0, \pm1, \pm3, \ldots, \pm\frac{1}{2}(r-1)\}$, $r_i \in \{2, 2^w\}$ and $d_i=0$ if $r_i=2$.

Output: $g^D \in G$

Initialise table: $T[d] \leftarrow 1_G$ **for all** $d \neq 0$.
$P \leftarrow g$
for $i \leftarrow 0$ **to** $n-1$ **do** {
  if $d_i \neq 0$ **then** $T[d_i] \leftarrow T[d_i] \times P$
  if $i \neq n-1$ **then** $P \leftarrow P^{r_i}$ }
return $\prod_{d\neq0} T[d]^d$

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Mixed Base Exponentiation — L2R

Inputs: $g \in G$,
$D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$,
where $(r_i, d_i) \in \mathcal{R}\times\mathcal{D}$.

Output: $g^D \in G$

Initialise table: $T[d] \leftarrow g^d$ **for all** $d \in \mathcal{D} \setminus \{0\}$.
$P \leftarrow 1_G$
for $i \leftarrow n-1$ **downto** 0 **do** $\{$
  if $i \neq n-1$ **then** $P \leftarrow P^{r_i}$
  if $d_i \neq 0$ **then** $P \leftarrow P \times T[d_i]$ $\}$
return P

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Mixed Base Exponentiation — R2L

Inputs: $g \in G$,
$D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$,
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Output: $g^D \in G$

Initialise table: $T[d] \leftarrow 1_G$ **for all** $d \in \mathcal{D} \setminus \{0\}$.
$P \leftarrow g$
for $i \leftarrow 0$ **to** $n-1$ **do** {
  if $d_i \neq 0$ **then** $T[d_i] \leftarrow T[d_i]\times P$
  if $i \neq n-1$ **then** $P \leftarrow P^{r_i}$ }
return $\prod_{d\in\mathcal{D}\setminus\{0\}} T[d]^d$

A Compact Right-to-Left Algorithm (Arith13, 1997)

Inputs: $g \in G$,
$D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$,
where $(r_i, d_i) \in \mathcal{R}\times\mathcal{D}$.

Output: $g^D \in G$

$T \leftarrow 1_G$
$P \leftarrow g$
for $i \leftarrow 0$ **to** $n-1$ **do** $\{$
  if $d_i \neq 0$ **then** $T \leftarrow T \times P^{d_i}$
  if $i \neq n-1$ **then** $P \leftarrow P^{r_i}$ $\}$
return T

The loop body involves computing P^{d_i} en route to P^{r_i}.

The Transposition Method

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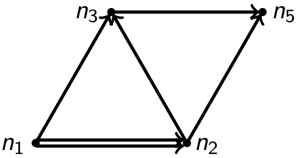
The Computational Di-Graph

An addition chain for D yields a computational, acyclic *di-graph*:

Here is that for

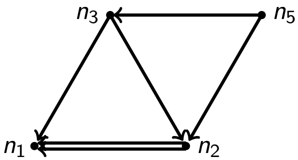
1+1=2; 1+2=3; 2+3=5.



For convenience, nodes are numbered so n_d represents g^d.

- Addition $i+j = k$ gives directed edges $n_i n_k$ and $n_j n_k$.
- It is *acyclic*, with a single root n_1 and a single leaf n_5.
- All nodes except root n_1 have input degree 2 as all op^s are binary.
- $\#\text{Ops} = \#\text{Nodes}-1 = \frac{1}{2}\#\text{Edges}$.
- By induction, $D = \#\text{paths from } n_1 \text{ to } n_D$.

Di-Graph for the Transpose Method



- Reverse the edges for the "*transposition*" method. Node inputs are again multiplied together.
- Path count is D, as before. So it again computes g^D.
- Nodes may need *merging* or *expanding* to restore in-degree 2. The #binary operations is not changed: $\frac{1}{2}$#edges.
- This reverses the addition chain in some sense.
- It *doesn't* preserve space requirements and without care, sq^{g} & mult^{n} counts may change.

Space Duality

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Space-Aware Addition Chains

Definition. For a given set of registers, take five classes of "atomic" op^s:

- *Copying* one register to another;
- *Copying* one register to another *& initialising* source register to 1_G;
- In-place *squaring* of the contents of one register;
- *Multiplying* two different registers into one of the input registers;
- *Multiplying* two different registers into one of the input registers, *& initialising* the other input to 1_G.

A **space-aware addition chain** is a sequence of such operations in which the registers are named.

Every addition chain can be written as a space-aware addition chain.

Matrix Representation — Space

For a device with two locations, matrix examples of each class are:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

They act on a column vector containing the values in each register.

By omitting more general op^{ns}, this set is *closed under transposition*.

- Copy (without initialise) becomes multiplication with initialise, and *vice versa*. (The *red* matrices.)
- Other operations stay in their class under transposition.

Definition. The *dual* of a space-aware chain is its transpose.
(The transposed operations are applied in reverse order.)

The dual uses the same space but may not have the same mult^{n} count.

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The Dual Chain — An Example

$R3 \leftarrow R2;\ R3 \leftarrow R2{+}R3;\ R1 \leftarrow_I R2;\ R2 \leftarrow_I R3;\ R2 \leftarrow_I R1{+}R2$

In matrices acting on a col^{mn} vector:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The dual (the transpose) is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. $R1 \leftarrow R2;\ R3 \leftarrow_I R2;\ R2 \leftarrow_I R1;\ R2 \leftarrow R2{+}R3;\ R2 \leftarrow_I R2{+}R3$

- Both have two multiplications and no squarings.
- Both compute g^3 from $g \in G$ with R_2 for I/O.

Extra Requirements

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The Main Problems

1. #Mults may not be preserved in the dual as copying becomes mult^n with initialisation.
2. The dual chain may not compute the same value unless the matrix product is symmetric.

To overcome the first of these, extra conditions are required:

- Select the initialising op^n when possible.
- Eliminate 1_G as an operand.
- Remove operations whose output is not used.
- Fix a subset of registers for I/O.
  (An I/O register *must* read input *and* write non-trivial output.)

Definition. A space-aware chain is *normalised* if the above hold.

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Counting Ones

Instances of 1_G or $\perp$ arise from:

a) Initial value of a non-input register.

b) Initialised by copy or mult^n op^n.

Instances of 1_G or $\perp$ finish their lives as:

c) Final value in a non-output register.

d) Overwritten by a copy op^n.

Since $\#a = \#c$, we conclude $\#b = \#d$.
Subtracting the $\#\{\text{copies with init}^n\}$ from $\#b$ and $\#d$, we have

$$\#\text{Mult}^{ns} \text{ with init}^n = \#\text{Copies without init}^n$$

These op^n types are swapped in the dual & others stay as they are. So:

- **Theorem.** For a normalised space-aware chain,
  $\#\text{Mult}^{ns}$ & $\#\text{Sq}^{res}$ are the same for the dual.

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Instances of 1_G or $\perp$ arise from:

a) Initial value of a non-input register.

b) Initialised by copy or mult^n op^n.

Instances of 1_G or $\perp$ finish their lives as:

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Since $\#a = \#c$, we conclude $\#b = \#d$.
Subtracting the $\#\{\text{copies with init}^n\}$ from $\#b$ and $\#d$, we have

$$\#\text{Mult}^{ns} \text{ with init}^n = \#\text{Copies without init}^n$$

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Symmetric Cases

If the action of a (multi-) exponentiation function f on registers is described by matrix M then a dual f^* is described by the transpose M^{T}.

Theorem a) f^* computes the same values as f iff its matrix is symmetric.
b) In particular, it uses the same registers for output as input.

- In the normalised case, unused registers give columns of zeros.
- Used, non-output registers are over-written with 1_G: more zeros.
- Used, non-input registers are initialised to 1_G: more zeros.
- So only the sub-matrix M_{IO} on I/O registers need be symmetric.

Theorem A normalised space-aware chain for a *single* exponentiation and its dual compute the same values.

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New Algorithms

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High Level Algorithms

Question: When is an algorithm *dualisable* if its steps are more complex than the atomic operations?

We want to be able to decompose steps independently into atomic op^{ns} yet obtain the normalised property when all steps are concatenated.

Solution: For each step the values initially in its non-input registers must not be used and its used non-output registers must be reset to 1_G.

The output registers for one step must be the input registers for the next. (Include unused registers in the I/O set for convenience here.)

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An "Old" Algorithm (Arith13, 1997)

Inputs: $g \in G, \quad D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$
Output: $g^D \in G$

$T \leftarrow 1_G$
$P \leftarrow g$
for $i \leftarrow 0$ **to** $n-1$ **do** $\{$
  if $d_i \neq 0$ **then** $T \leftarrow T \times P^{d_i}$
  if $i \neq n-1$ **then** $P \leftarrow P^{r_i}$ $\}$
return T

The loop body involves computing P^{d_i} *en route* to P^{r_i}.

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One Iteration

Base/digit pairs (r, d) are chosen for compact, fast performance.
Specifically at most one register in addition to P and T.

e.g. $r = 2^i \pm 1, d = 2^j$ will involve i squarings & 2 mult^s.

It avoids a table entry for each d.

There is now a dual algorithm using the same space – only three registers.

The step $T \leftarrow TP^d$, $P \leftarrow P^r$ is achieved by $\begin{bmatrix} r & 0 \\ d & 1 \end{bmatrix} = \begin{bmatrix} r & d \\ 0 & 1 \end{bmatrix}^{\mathsf{T}}$.

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A New Compact Left-to-Right Algorithm

Inputs: $g \in G, \quad D = ((d_{n-1}r_{n-2}+d_{n-2})r_{n-3}+\ldots+d_1)r_0+d_0 \in \mathbb{N}$
Output: $g^D \in G$

$T \leftarrow g$
$P \leftarrow 1_G$
for $i \leftarrow n-1$ **downto** 0 **do**
    $P \leftarrow P^{r_i} \times T^{d_i}$
return P

Loop iterations are computed as described on last slide.

It is the dual of the previous R2L algorithm, as just derived.

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The Value of the Algorithm

- "Table-less" exponentiation – useful in constrained environments.
- If space for only three registers and division has the same cost as mult$^{\mathsf{n}}$, the compact algorithms are faster.
- A left-to-right version allows better use of composite op$^{\mathsf{s}}$, e.g. double-and-add, triple-and-add, quintuple-and-add.
- Recoding is done on-the-fly for R2L exp$^{\mathrm{n}}$; in advance for L2R exp$^{\mathrm{n}}$. The recoding typically needs up to 3 times the storage space of D.

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Conclusion

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Summary & Final Remarks

- A general setting enabling most exp^n algorithms to be described naturally, namely a mixed base recoding.
- A new space- and time-preserving duality between left-to-right and right-to-left exp^n algorithms.
- A new tableless exp^n algorithm.
  It enables new speed records to be set in certain environments.
- New understanding of exp^n is possible,
  e.g. a comparison of R2L initialisation with L2R finalisation steps.
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CT-RSA 2012 — February 29th, 2012

Optimal Eta Pairing on Supersingular Genus-2 Binary Hyperelliptic Curves

Nicolas Estibals

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Jean-Luc Beuchat	Graduate School of Systems and Information Engineering, University of Tsukuba, Japan
Jérémie Detrey	CARAMEL project-team, LORIA, INRIA / Université de Lorraine / CNRS, France

Pairings and cryptology

- used as a primitive in many protocols and devices
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- implementations needed for various targets
  - online server $\rightarrow$ high-speed software
  - smart card $\rightarrow$ low-resource hardware
- reach 128 bits of security (equivalent to AES)

What's a cryptographic pairing

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \longrightarrow \mathbb{G}_T$$

- where $(\mathbb{G}_1, +)$, $(\mathbb{G}_2, +)$ and $(\mathbb{G}_T, \times)$ are cyclic groups of order ℓ
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- Choice of the groups:
  - $\mathbb{G}_1$, $\mathbb{G}_2$: related to an algebraic curve
  - $\mathbb{G}_T$: related to the field of definition of the curve

Classical choice of curves

Barreto–Naehrig curves

+ Lots of literature
+ Huge optimization efforts
+ Suited for 128 bits of security

– Arithmetic modulo $p \approx 256$ bits
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  Larger base field: $\mathbb{F}_{2^{1223}}$, $\mathbb{F}_{3^{509}}$

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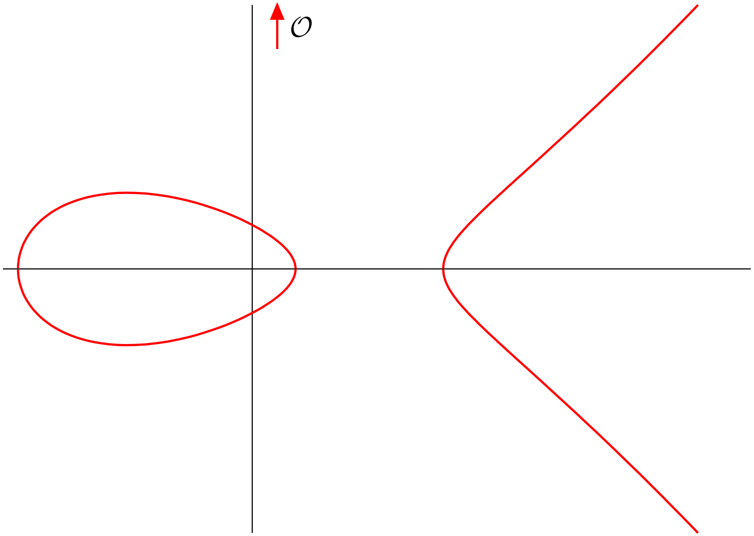
  - (This work) Use genus-2 hyperelliptic curves: base field will be $\mathbb{F}_{2^{367}}$

Elliptic curves

$$E/K : y^2 + h(x) \cdot y = f(x)$$
with $\deg h \leq 1$ and $\deg f = 3$



Elliptic curves

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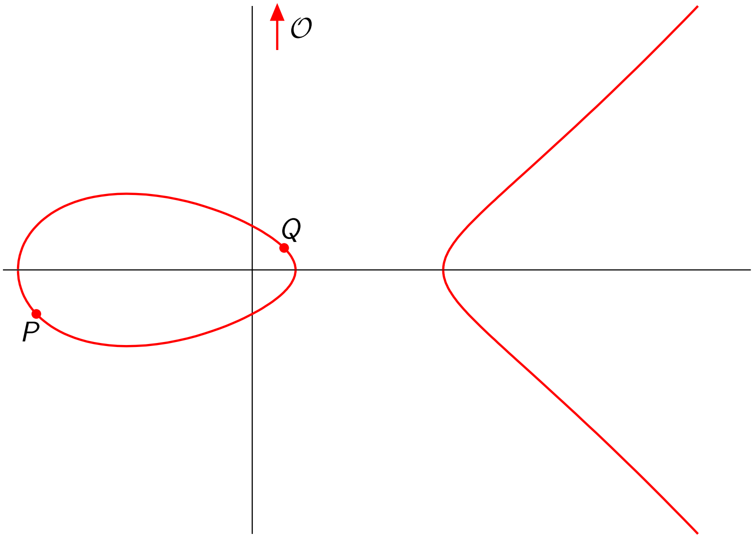

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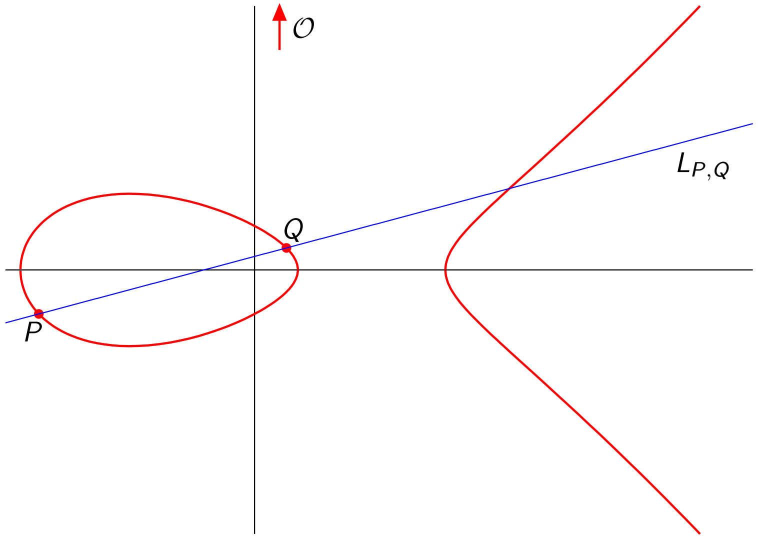

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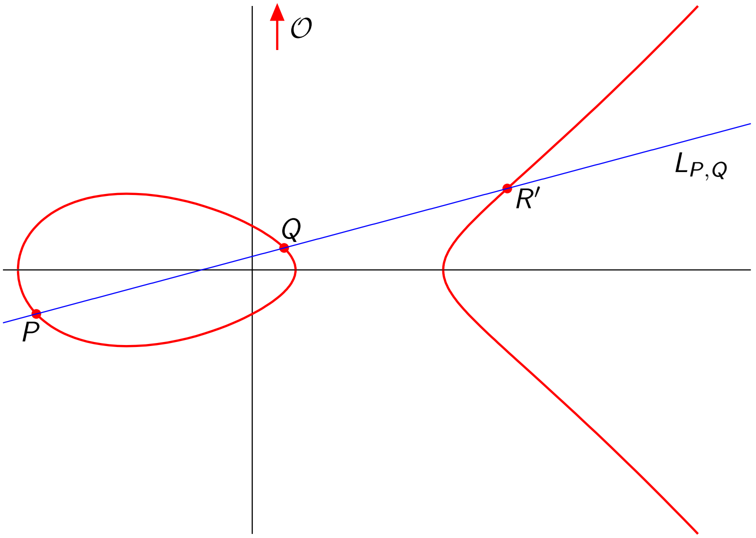

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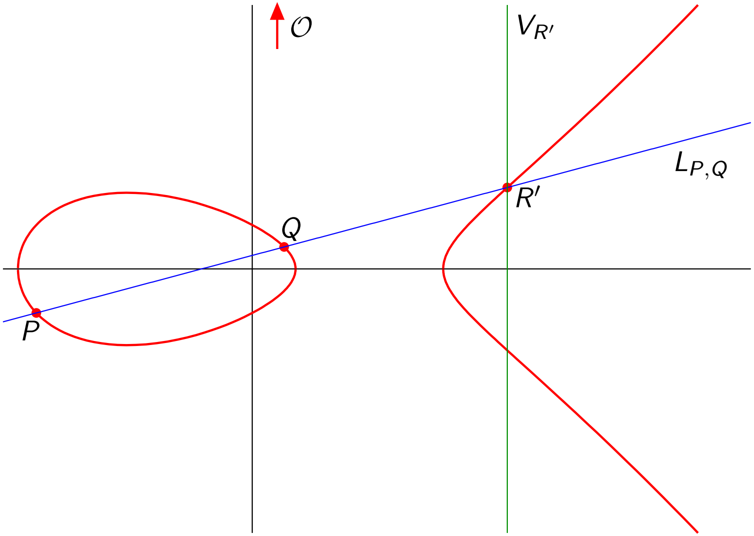

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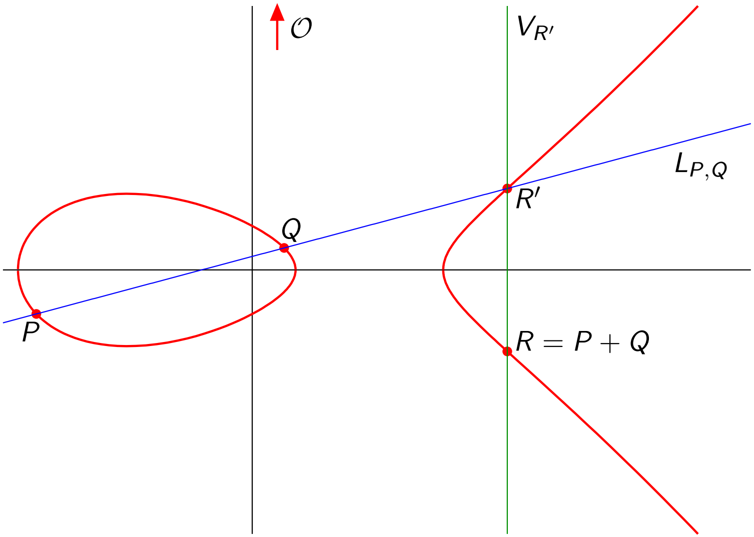

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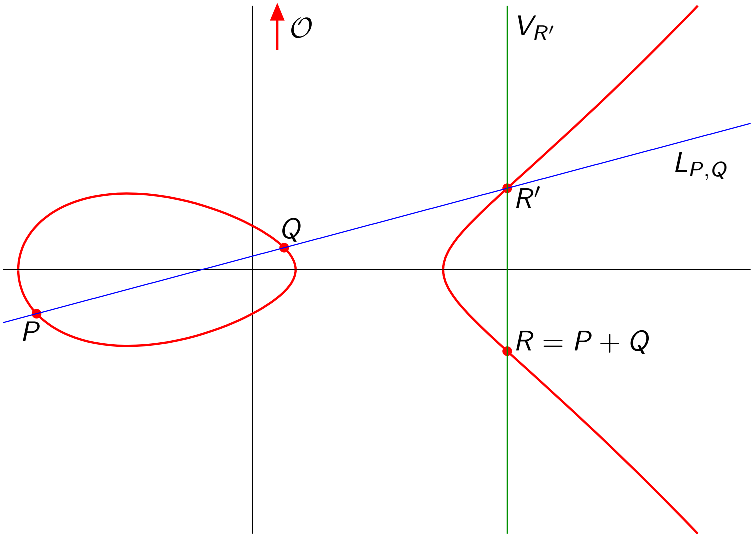

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- $E(\mathbb{F}_q)$ is a finite group
- ℓ: a large prime dividing $\#E(\mathbb{F}_q)$
- Use the cyclic subgroup

$$E(\mathbb{F}_q)[\ell] = \{P \mid [\ell]P = \mathcal{O}\}$$

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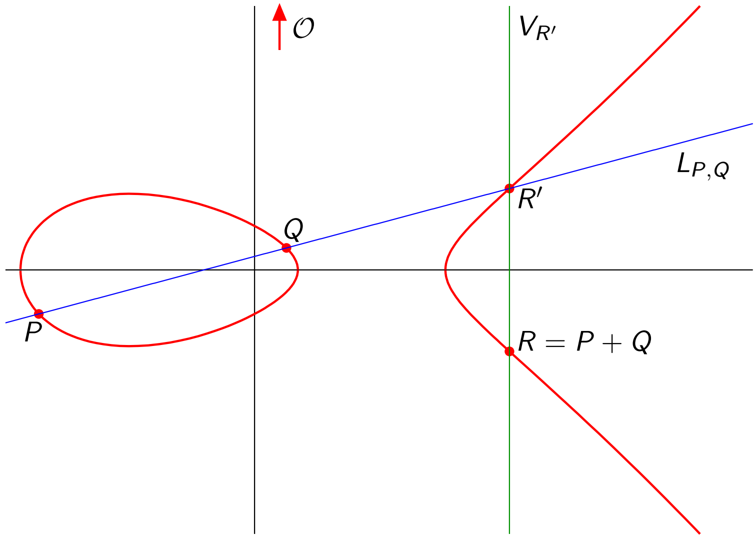

Genus-2 hyperelliptic curves

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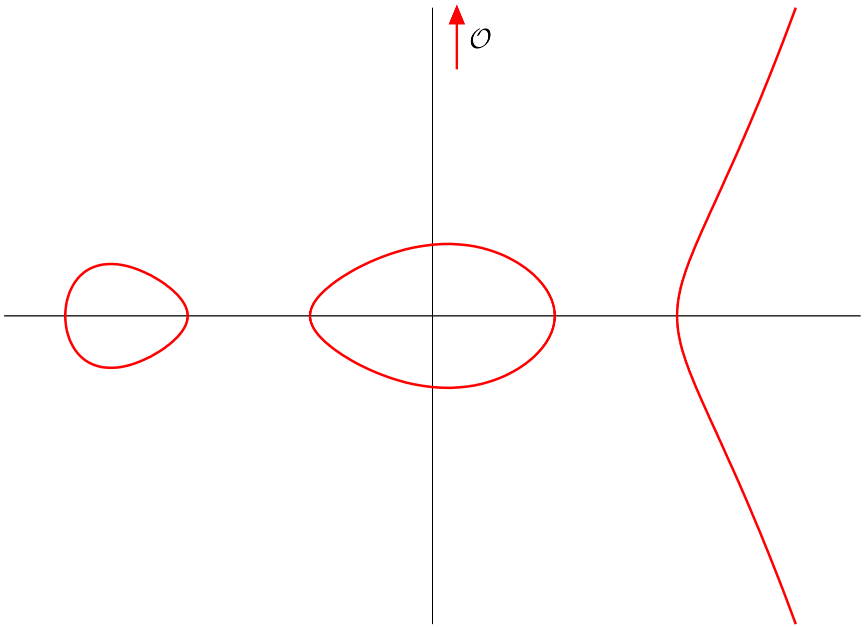

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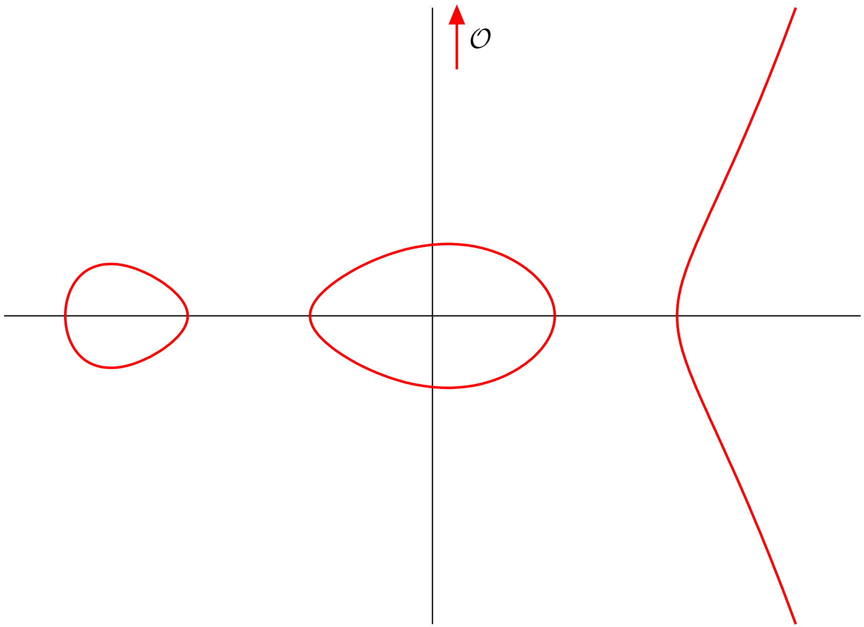

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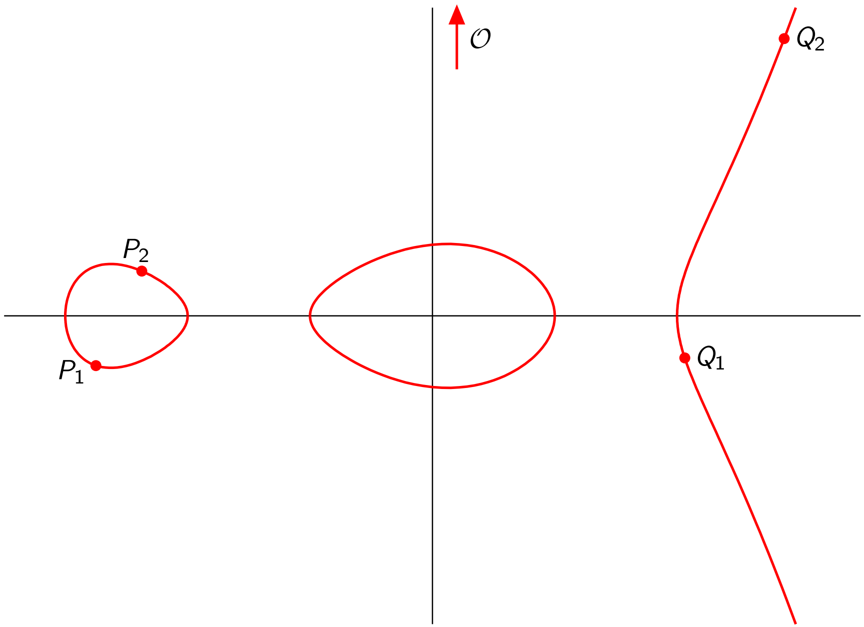

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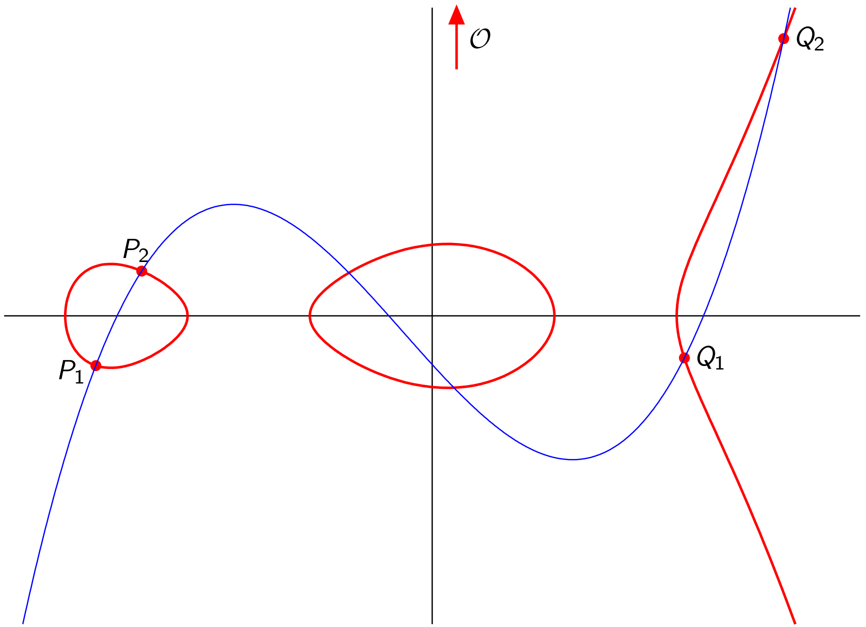

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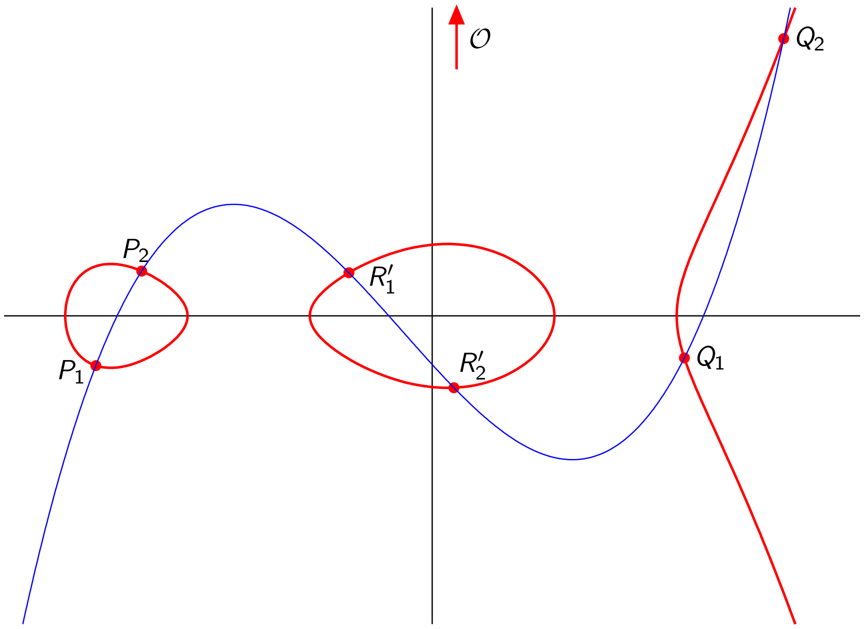

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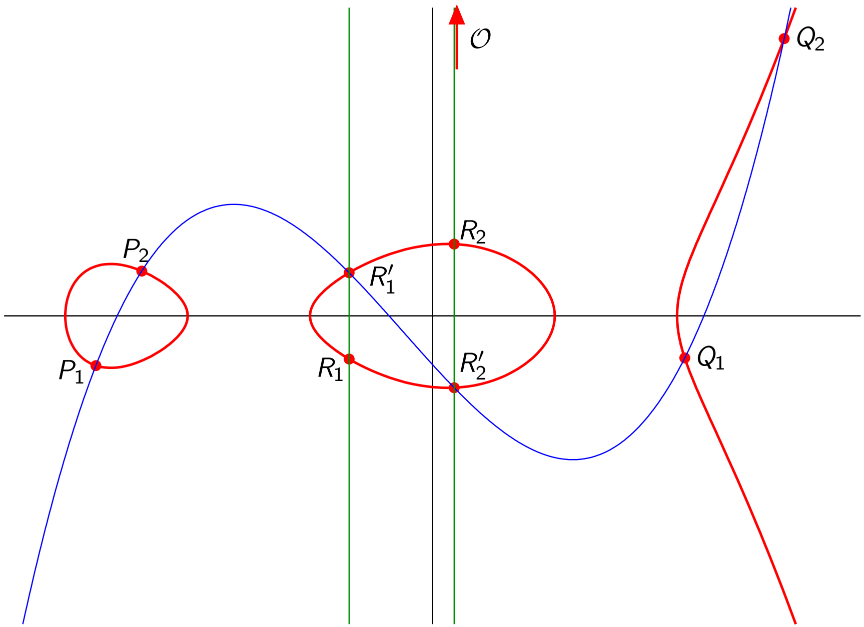

$$\{P_1, P_2\} + \{Q_1, Q_2\} = \{R_1, R_2\}$$

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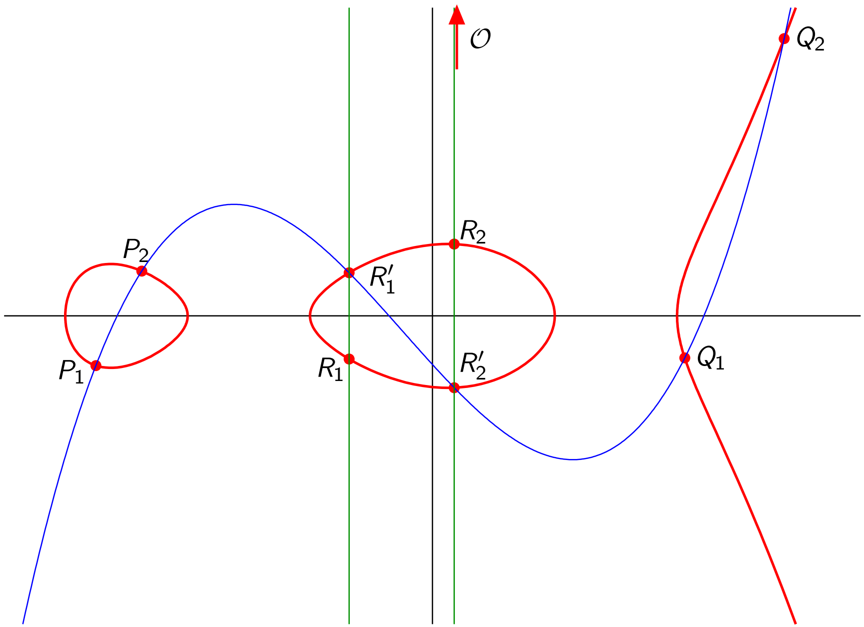

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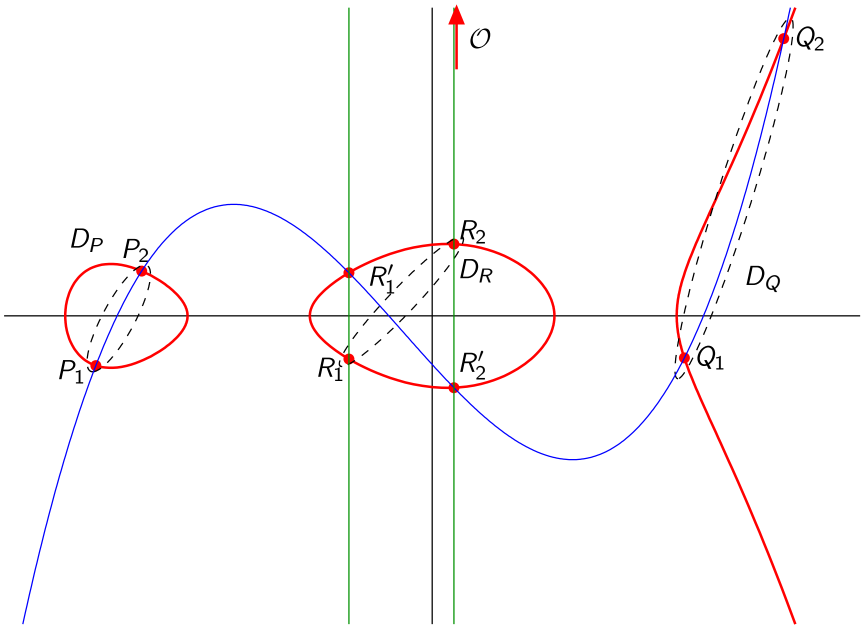

$$D_P + D_Q \qquad = D_R$$

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  - use the Jacobian
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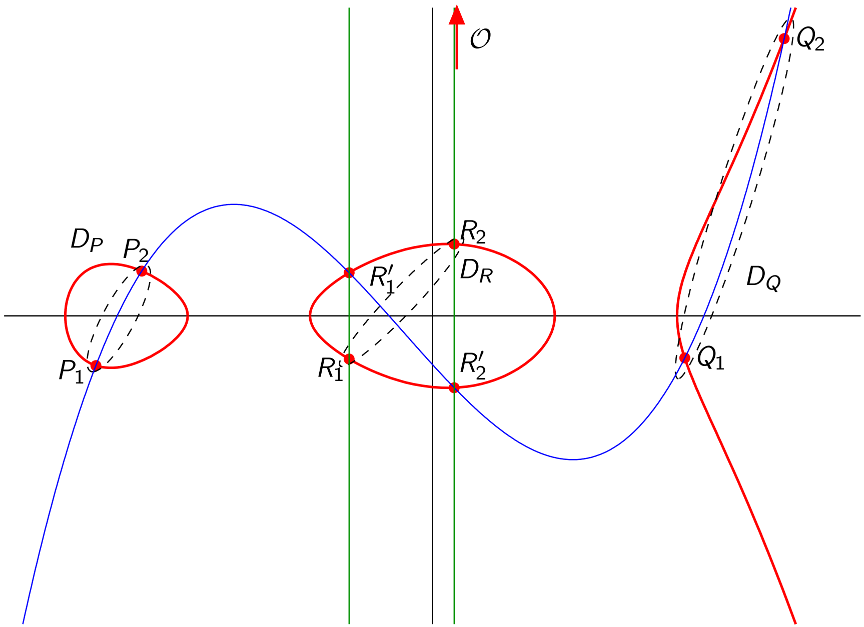
  - general form of the elements (called divisor)
    $D_P = (P_1) + (P_2) - 2(\mathcal{O})$

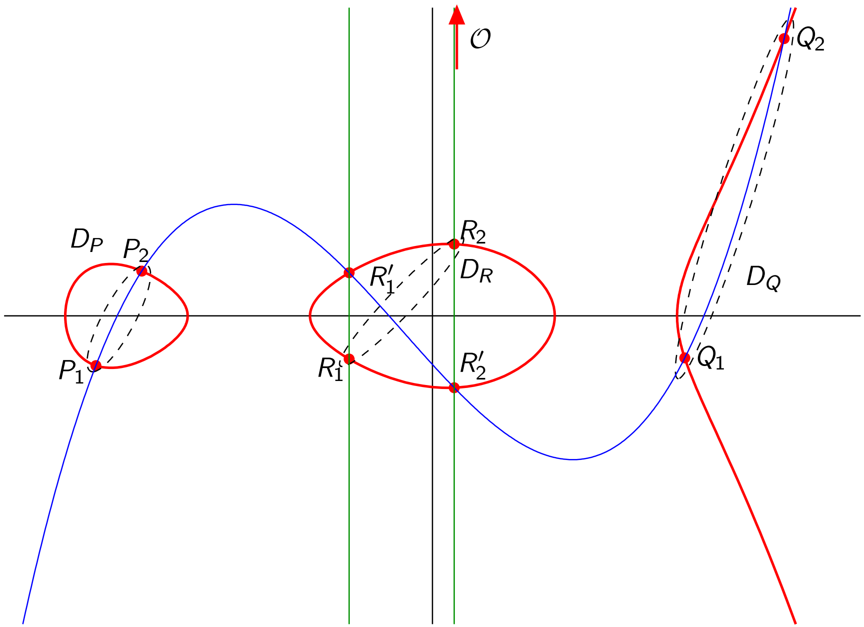


$$D_P + D_Q \qquad = D_R$$

Genus-2 hyperelliptic curves

- $C(K)$ not a group!
- But pairs of points
  $$\{P_1, P_2\}$$
- More formally
  - use the Jacobian
    $$\mathrm{Jac}_C(K)$$
  - general form of the elements (called divisor)
    $$D_P = (P_1) + (P_2) - 2(\mathcal{O})$$
  - degenerate form
    $$(P) - (\mathcal{O})$$

$$C/K : y^2 + h(x) \cdot y = f(x)$$
with $\deg h \leq 2$ and $\deg f = 5$



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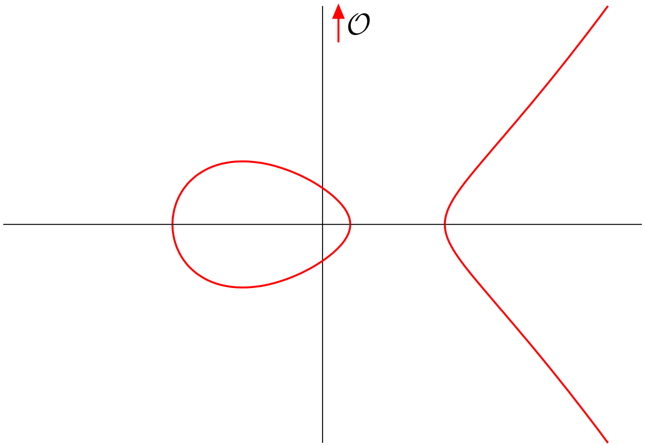
Computing the pairing: Miller's algorithm (elliptic case)

$$e : \quad \mathbb{G}_1 \times \mathbb{G}_2 \longrightarrow \mathbb{G}_T$$

- Reduced Tate pairing



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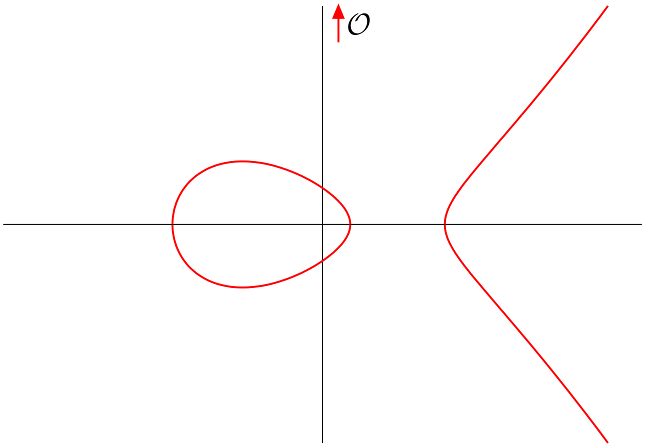

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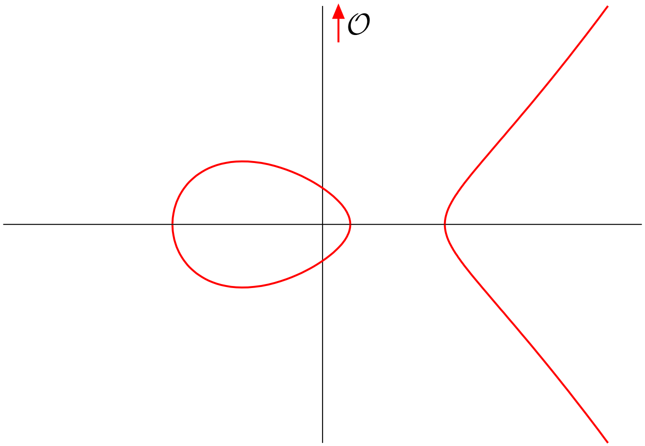
- k: embedding degree (curve parameter)



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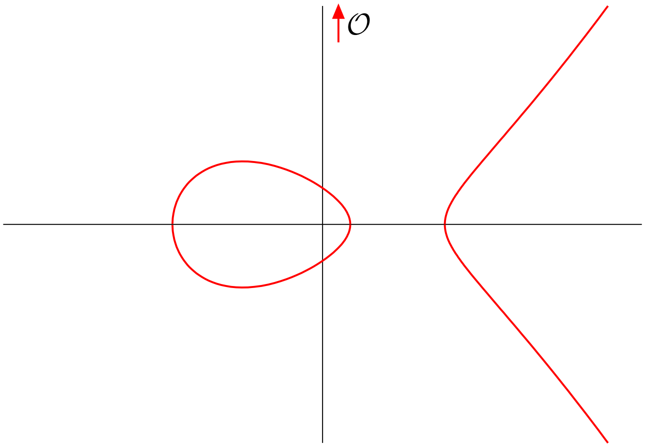

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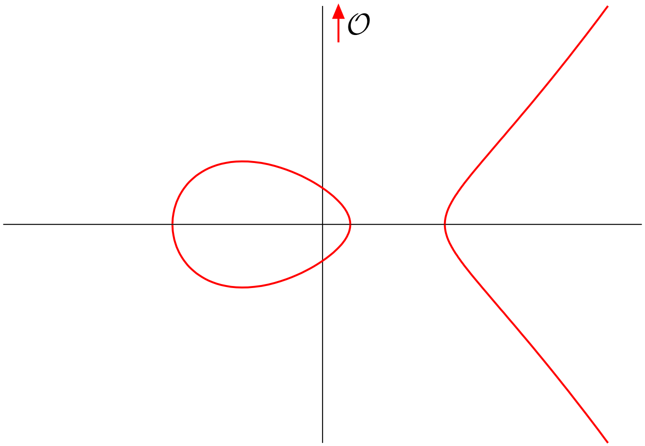

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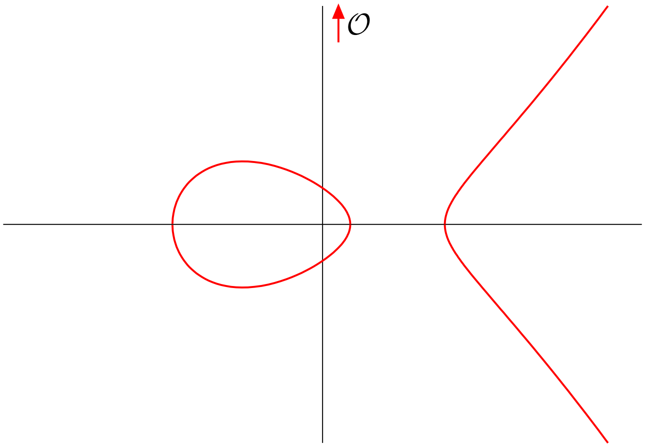

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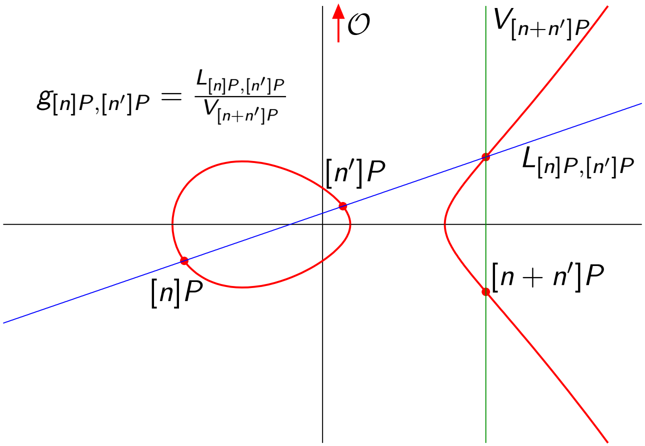

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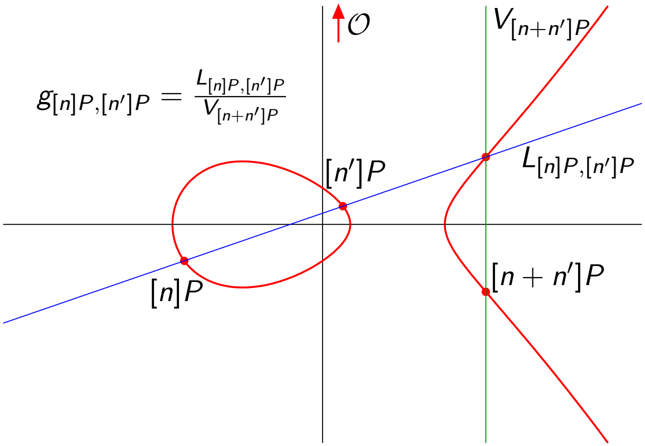
  - compute $f_{\ell,P}$ thanks to an addition chain
  - in practice: double-and-add

$\log_2 \ell$ iterations



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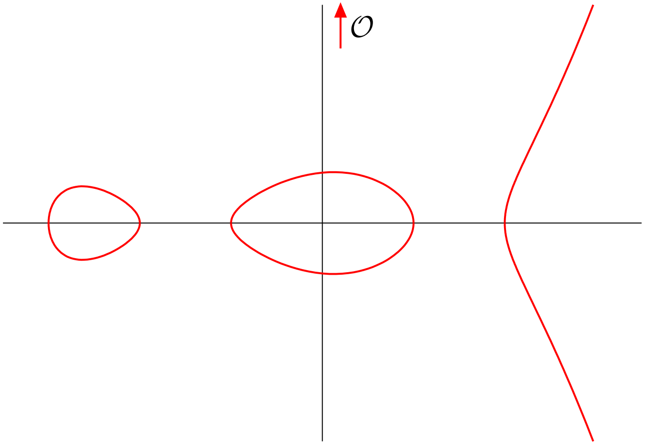

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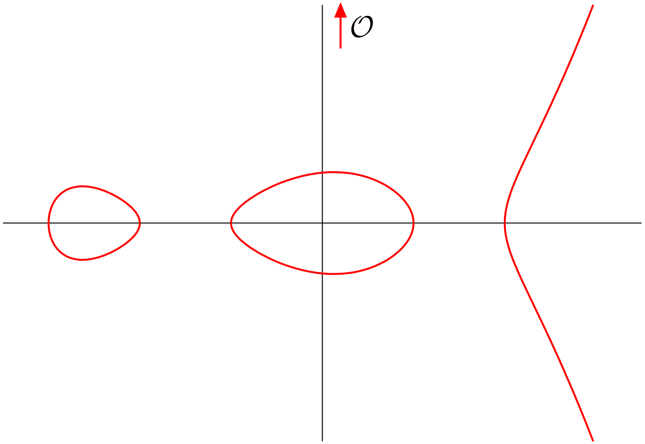

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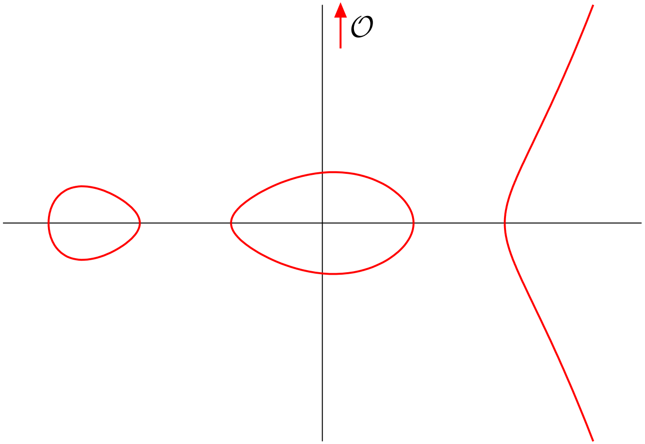

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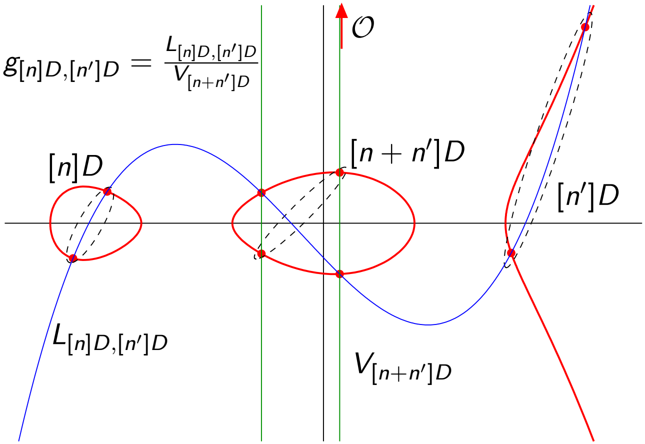

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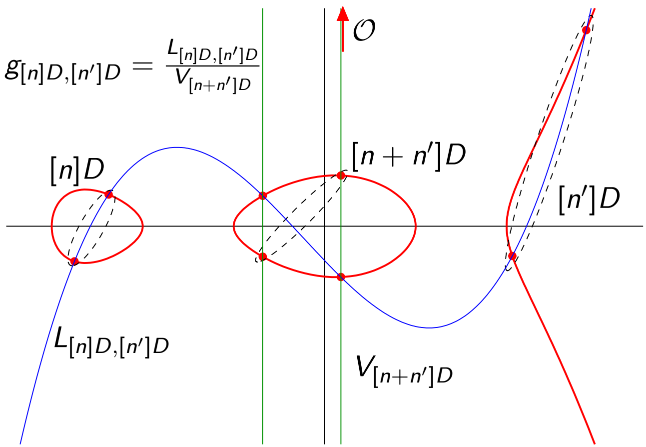

Genus-2 binary supersingular curve: our choice

$$C_d/\mathbb{F}_{2^m} : y^2 + y = x^5 + x^3 + d \text{ with } d \in \mathbb{F}_2$$

- A distortion map exists: symmetric pairing
- $\# \operatorname{Jac}_{C_d}(\mathbb{F}_{2^m}) = 2^{2m} \pm 2^{(3m+1)/2} + 2^m \pm 2^{(m+1)/2} + 1$
- Embedding degree of the curve: $k = 12$
- For 128 bits of security: $\mathbb{F}_{2^m} = \mathbb{F}_{2^{367}}$ and $d = 0$

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  - $f_{8,D}$ has a much simpler expression than $f_{2,D}$

Constructing the Optimal Eta pairing

Algorithm	Tate double & add			
#iterations	$2m$			

- Vanilla Tate pairing: $\log_2 \ell \approx \log_2 \# \operatorname{Jac}_C(\mathbb{F}_{2^m}) \approx 2m$ doublings

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  - but can use 2^{3m}-th power Verschiebung
  - 33% improvement compared to Barreto *et al.*'s work

Considering degenerate divisors

- Some protocols allow to choose the form of one or two input divisors
- Consider degenerate divisors of the form

$$(P) - (\mathcal{O})$$

  - only 2 coordinates in $\mathbb{F}_{2^m}$ to represent such a divisor (instead of 4 coordinates for a general one)
  - since octupling acts on the curve:

$$[8]((P) - (\mathcal{O})) = ([8]P) - (\mathcal{O})$$

  - we can work with a point!

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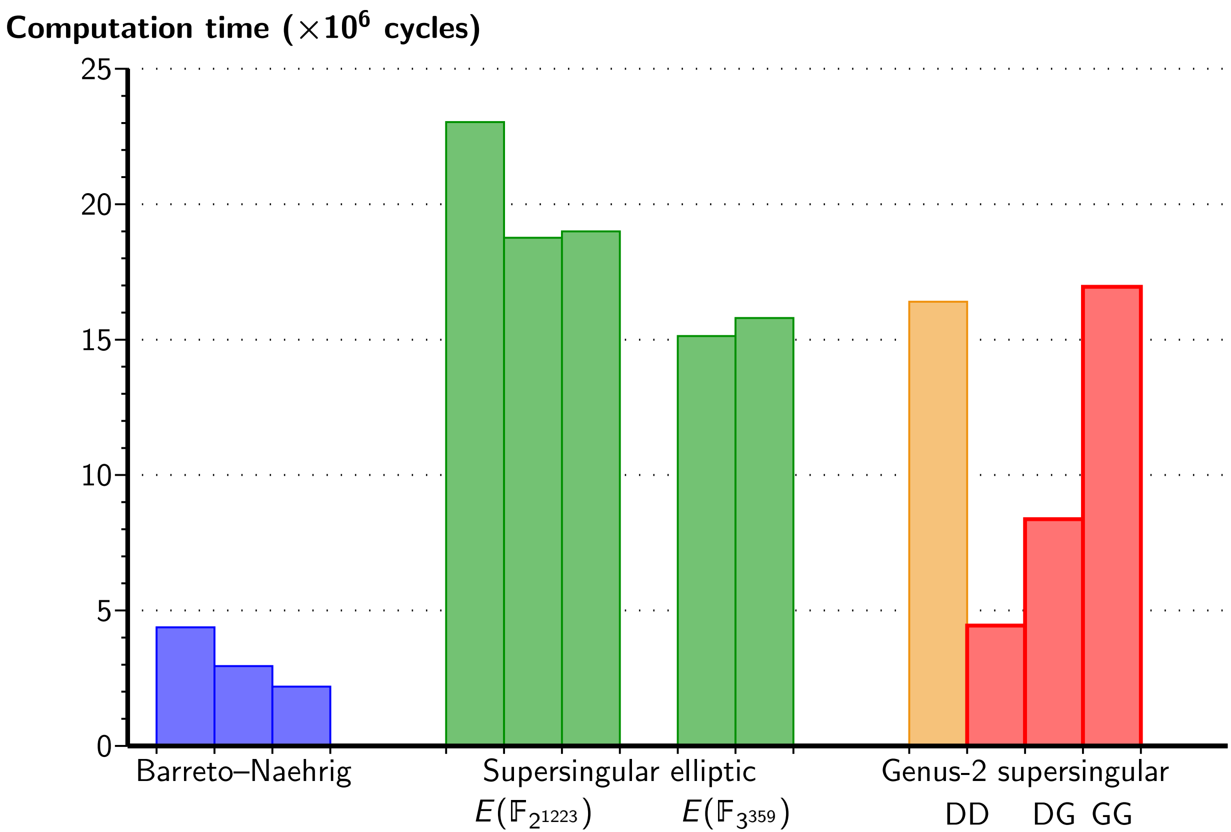
- We may compute the pairing of
  - two general divisors (GG)
  - one degenerate and one general divisor (DG)
    - ⋆ halves the amount of computation
    - ⋆ lot of protocols allow this
  - two degenerate divisors (DD)
    - ⋆ halves again the amount of computation
    - ⋆ some protocols still compatible

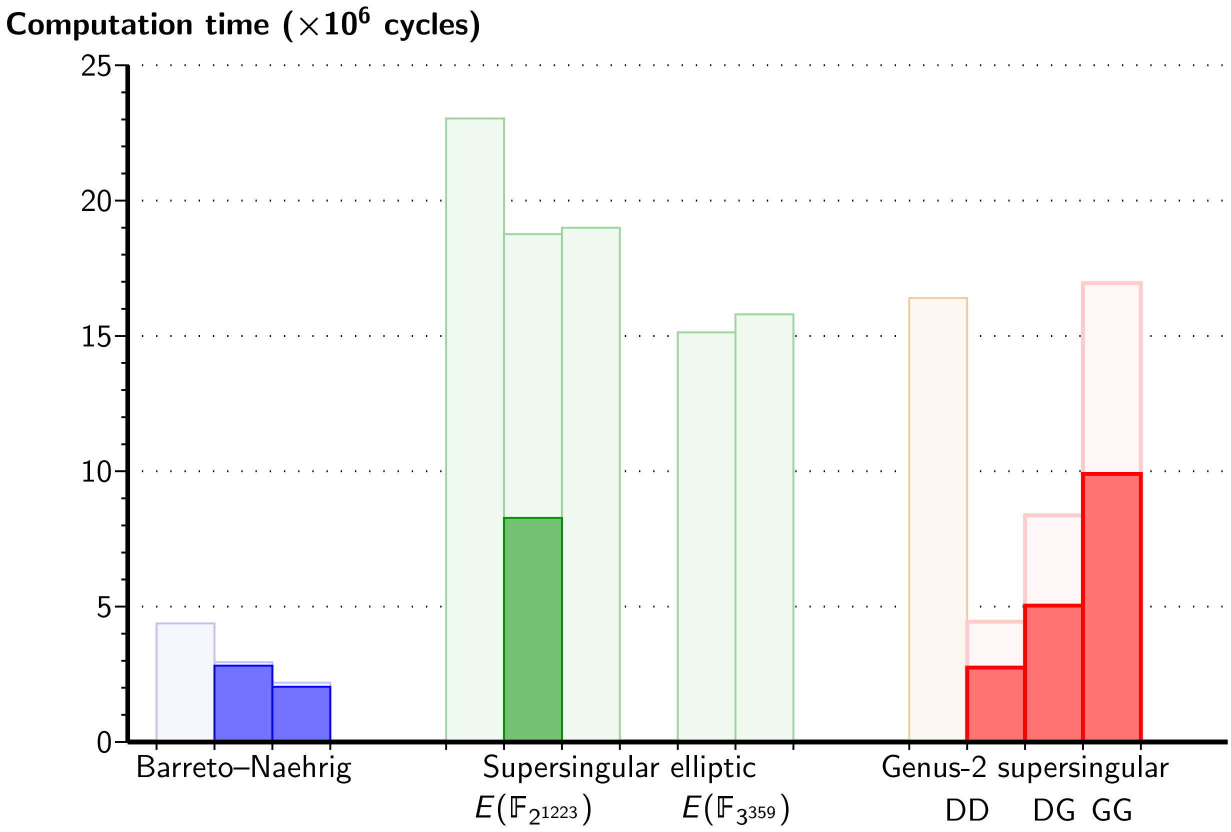
Software implementation

- Implementations for Intel Core 2



Software implementation

- Implementations for Intel Core 2 and Nehalem architecture
- Use of the native binary field multiplier on Nehalem



Hardware implementation

- Optimal Eta pairing on general divisors
- Implemented on a finite field coprocessor $\mathbb{F}_{2^{367}}$
  - addition
  - multiplication
  - Frobenius endomorphism
- Post place-and-route estimations on a Virtex 6-LX 130T results

Implementation	**Curve**	**Area (device usage)**	**Time (ms)**	**Area×time**
Cheung *et al.*	$E(\mathbb{F}_{p_{254}})$	35 %	0.57	4.03
Ghosh *et al.*	$E(\mathbb{F}_{2^{1223}})$	76 %	0.19	2.88
Estibals	$E(\mathbb{F}_{3^{5\cdot 97}})$	8 %	1.73	2.68
This work	$C_0(\mathbb{F}_{2^{367}})$ (GG)	7 %	3.09	4.30

Conclusion

- A novel pairing algorithm shortening Miller's loop
- Competitive timings compared to genus-1 pairings
- Comparable timings against non-symmetric pairings

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  - when at least one divisor is degenerate (DG and DD case)
- First hardware implementation of a genus-2 pairing reaching 128 bits of security
- Perspectives
  - Implement optimal Ate pairing on this curve (work in progress)
  - Use theta functions for faster curve arithmetic

Thank you for your attention!

Questions?